INVARIANCE OF IMMERSED FLOER COHOMOLOGY THROUGH TRIPLE INTERSECTIONS

HADI AZIZI, JOSEPH PALMER, CHRIS WOODWARD

ABSTRACT. We fix an error in the proof of Lemma 7.9 (b) of Palmer-Woodward [1], which, charitably speaking, missed a case in the argument, as pointed out by the first author.

1. Statement of result

The claim in question is that immersed Floer cohomology is invariant under Maslov flow when the Maslov flow passes through a triple intersection; i.e. a time when three branches of the Lagrangian intersect. Generically, this happens only at a finite number of times for two-dimensional symplectic manifolds, and not at all in higher dimensions. To introduce notation, let X be a compact, oriented surface with symplectic form $\omega \in \Omega^2(X)$, and let $L_t \subset X$ be a Maslov flow of immersed Lagrangians (e.g. a curve shortening flow) as in Palmer-Woodward [1]. We assume that at time $t = 0$ the immersion L_t has a triple intersection, as shown in Figure 2. Choose small times $t_- < 0 < t_+$ and define

$$L_- = L_{t_-}, \quad L_+ = L_{t_+}$$

as shown. Let U be a neighborhood of the triple intersection so that there are three local branches $L_{\pm,1}, L_{\pm,2}, L_{\pm,3} \subset L_\pm$ of the Lagrangians as shown

$$U \cap L_\pm = L_{\pm,1} \cup L_{\pm,2} \cup L_{\pm,3}.$$

Assume that L_- is equipped with a weakly bounding cochain b_-.

Theorem 1.1. *(Lemma 7.9 (b) of Palmer-Woodward [1]) Under the above assumptions, there exists a weakly bounding cochain b_+ for L_+ so that*

(a) $W(b_+) = q^{2(t_+ - t_-)} W(b_-)$ *and*
(b) $HF(L_-, b_-) \cong HF(L_+, b_+)$.

We introduce the following notation. Let $R_\pm$ denote the small triangle at the center, that is, the bounded region in the complement of $L_{\pm,1} \cup L_{\pm,2} \cup L_{\pm,3}$. We assume that the areas of the small triangles R_-, R_+ are equal and given by some small real number ϵ, as in Figure 1.

By assumption, the self-intersection points of the Lagrangians are assumed to have $\mathbb{Z}_{2N}$ gradings for some N, and the $\mathbb{Z}_2$ grading by reducing the $\mathbb{Z}_{2n}$ grading is determined by the orientations as follows as in Seidel [2]: Suppose the corresponding corner $x \in L_{\pm,i} \cap L_{\pm,j}$ has orientations

$$o_{\pm,i} \in \pi_0(T_x L_{\pm,i} - \{0\}), \quad o_{\pm,j} \in \pi_0(T_x L_{\pm,j} - \{0\})$$

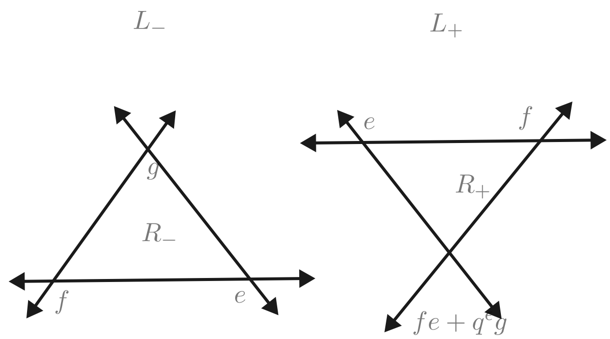

FIGURE 1. Moving past a triple intersection

on both Lagrangians. The following is immediate from the definitions:

Lemma 1.2. *(a) If both $o_{\pm,i}, o_{\pm,j}$ point outwards from x or both point inwards then the $\mathbb{Z}_2$ grading $|x| \in \mathbb{Z}_2$ is an odd as an ouput corner and even as an input corner, while*
(b) if one orientation points outwards and one points inwards then the associated grading $|x| \in \mathbb{Z}_2$ is even as an output corner and odd as an input corner.

By an interior angle, we mean an ordered self-intersection point arising as in Figure 2 or Figure 3, where the ordering is induced from the counter-clockwise ordering around the boundary of the triangle R_-.

Lemma 1.3. *There are either three even interior angles, or one even interior angle.*

Proof. For the symmetric case that the orientations of the three branches $L_{-,1}, L_{-,2}, L_{-,3}$ around the inner triangle R_- are all clockwise, all interior angles correspond to ordered self-intersection points that are even. Each change in orientation of one of the branches $L_{-,k}$ switches two of the self-intersection points, say $L_{-,k} \times_X L_{-,i}$ and $L_{-,k} \times_X L_{i,j}$, corresponding to interior angles from even to odd degree or vice-versa. So any sequence of switches of orientations of the branches of L_- produces a system of gradings where either one or three of the interior angles is even. □

For completeness we recall our notation for weakly bounding cochains. Let $L \to X$ be a compact, oriented, immersed Lagrangian equipped with an orientation, spin structure, and grading with transverse self-intersection points. The generators of $CF(L)$ are given by ordered self-intersection points, together with critical points of a Morse function, and two extra generators from the homotopy units construction, which may be ignored. Denote by

$$m_d : CF(L)^{\otimes d} \to CF(L)$$

the A_∞ operations for L obtained by counting holomorphic polygons. More precisely, if

$$x_1, \dots, x_d \in L \times_X L$$

are incoming ordered self-intersection points then

$$m_d(x_d, \ldots, x_1) = \sum_{u \text{ has output } x_0} q^{A(u)} \sigma(u)$$

where $\sigma(u) \in \{\pm 1\}$ is the orientation sign determined by the orientations on the Lagrangians, and the sum is over rigid holomorphic treed disks with incoming edges with limits $x_1, \ldots, x_d$ and outgoing edge with limit x_0. In the case of Morse generators one considers treed holomorphic polygons, as described in [1]. The condition for $b \in CF(L)$ to be a weakly bounding cochain is that $b_\pm$ solve the projective Maurer-Cartan equation

$$m_0^b(1) := m_0(1) + m_1(b) + m_2(b \otimes b) + \ldots = W(b) 1_L$$

for some element $W(b)$ in the Novikov ring called the *disk potential* of b; here $1_L \in CF(L)$ is the strict unit. In this case the deformed differential defined by

$$m_1^b(c) := \sum_{d_0, d_1} m_{1+d_0+d_1}(b^{\otimes d_1} \otimes c \otimes b^{\otimes d_0})$$

squares to zero. Denote by $HF(L, b)$ its cohomology. For a pair of Lagrangians L_0, L_1 intersecting cleanly and equipped with weakly bounding cochains b_0, b_1 there is a similar version $CF(L_0, L_1)$ with differential denoted $m_1^{b_0, b_1}$. Its cohomology is denoted $HF(L_0, L_1)$. In the special case $L = L_0 = L_1$ and $b_0 = b_1$ we have $HF(L) = HF(L_0, L_1)$. Furthermore, the Floer cohomology of a pair is invariant under Hamiltonian diffeomorphisms ϕ in the sense that $HF(L_0, L_1) \cong HF(L_0, \phi(L_1))$, assuming the intersections are clean.

The main result of Palmer-Woodward [1] was that the Floer cohomology is invariant under Maslov flow. Let

$$L_t, t \in [t_-, t_+]$$

be such a flow. Using notation from Palmer-Woodward [1] for $t \in [t_-, t_+]$ let

$$E_{t_-}^t : CF(L_{t_-}) \to CF(L_t)$$

denote the Euler flow from time t_- to time t from Definition 7.6 of [1] and

$$b_{-,t} = E_{t_-}^t b_- \in MC(L_t)$$

the flowed weakly bounding cochain, up to the time of the triple intersection. The Euler flow is defined so that, in particular, the coefficients of the curvature change by an overall power of q, so that, in the absence of a singularity or triple crossing, the element $b_{-,t}$ is a weakly bounding cochain for the flow L_t of L_-. We suppose for the rest of the note that L_+ is obtained by flowing L_- through a triple crossing, as shown in Figure 3.

2. One even interior angle

We first consider the case that the two lower interior angles in the small triangle R_- in the center are degree 0 ordered self-intersection points, and the top intersection point is degree 1. We label the interior angles of even degree in the Novikov ring as in Figure 2, giving coefficients of weakly bounding cochain

$$b_- = e\overline{x}_- + f\overline{y}_- + g\overline{z}_- + \ldots \in CF(L_-).$$

Notation 2.1. *As in Figure 2 denote by*

- u_1 *a general holomorphic disk with output corner labelled* f*, with arbitrary corners labelled* b_-*;*
- u_2^- *a general holomorphic disk with output corner with coefficient* f *and first input corner at* e*; and remaining corners labelled* b_-*;* u_2^+ *a general holomorphic disk with output corner at* e *and last input corner at* e*; Thus* u_2^-, u_2^+ *are geometrically the same disk up to permutation of inputs and outputs;*
- u_3 *with output corner* e*, and remaining corners labelled* b_-*;*
- u_2' *a general holomorphic disk with output corner* g*, not containing the interior triangle* R_-*, and remaining corners labelled* b_-*; and*
- u_2'' *a general holomorphic disk with output corner* g*, containing the interior triangle* R_- *with area* ϵ*, and remaining corners labelled* b_-*;*

We use the notation $\#u$ *to indicate the weighted count of disks with some specified type, contributing to the* A_∞ *structure maps. Thus by* $\#u_1$ *we mean the weighted count of disks* u_1 *with arbitrary numbers of corners labelled with the weakly bounding cochain, and output* f*:*

$$\#u_1 = \sum_{u_1 \ \ output f} (-1)^{\heartsuit} q^{A(u_1)} \sigma(u_1)$$

where $\sigma(u) \in \{\pm 1\}$ *is the orientation sign determined by the orientations on the Lagrangians, and* $\heartsuit$ *is a sign for which we follow the conventions of Seidel [3].*

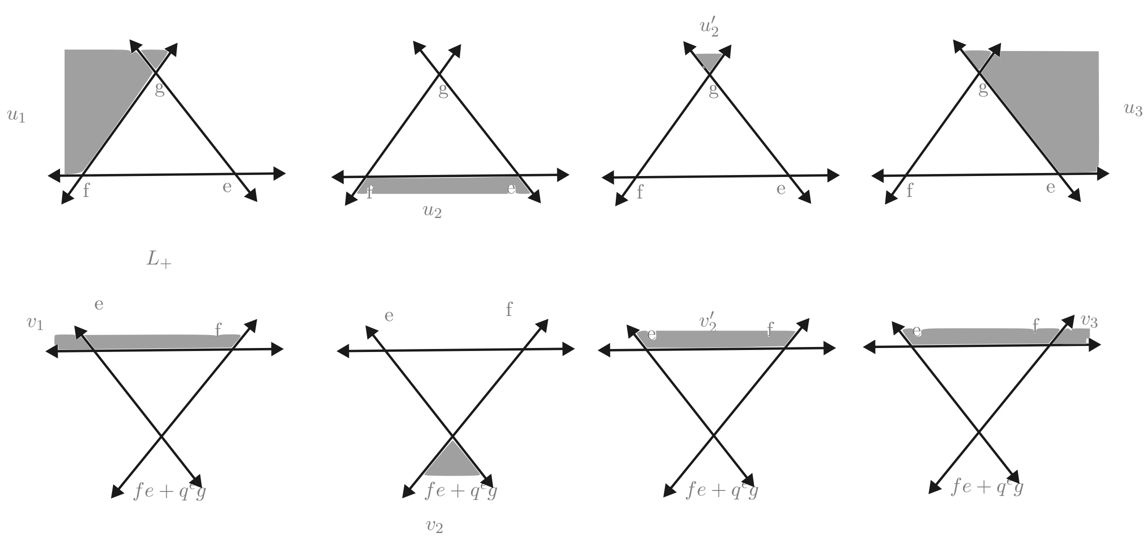

FIGURE 2. Disks bounding a triple intersection

Lemma 2.1. *With notation as above, the numbers of disks containing the bottom region in Figure 2 are related by*

(a) $\# u_2^- = -\# u_2^+$;
(b) $\# u_2^+ = \pm \# u_2'' = -\# u_2^-$, *with sign depending on the choice of orientations.*

Proof. For the first item, the disks u_2^-, u_2^+ are geometrically the same, but are related by naming the output of u_2^- the first input of u_2^+. In the construction of orientations in [4, (40)], this corresponds to the transposition of determinant lines (one of the outputs with one of the inputs) in the expression

$$\det(D^-_{\gamma_d}) \det(TL) \ldots \det(D^-_{\gamma_1}) \det(TL) \det(D^c) \det(TL) \det(D_{\gamma_0})$$

where D^c is a Cauchy-Riemann operator for a problem on a surface with boundary but without strip-like ends, from which the orientation on the moduli space of holomorphic disks is induced. Since these determinant lines are determinants of Freholm operators of odd index and the intervening expression

$$\det(TL) \ldots \det(D^-_{\gamma_1}) \det(TL) \det(D^c) \det(TL)$$

is even, this transposition produces a sign change as in [4, Remark 2.1.2 (4)].

For the second item, for any such disk u_2, one may take the union with the interior triangle u_2''. Vice-versa, given u_2'' removing the interior triangle gives a disk u_2 with corners f, e. The corresponding compactified boundary value problems are canonically isotopic and, as a result, the determinant lines are canonically isomorphic, and the isomorphism is orientation is orientation preserving for suitable choices of orientations at e, f, g. □

Unobstructedness of L_- implies the following equalities:

Lemma 2.2. *Suppose L_- is weakly unobstructed. Then*

$$\begin{aligned} \# u_1 + \# u_2^+ e &= 0 \\ \# u_3 + \# u_2^- f &= 0 \\ \# u_2 q^\epsilon + \# u_2' &= 0. \end{aligned}$$

Proof. Because holomorphic disks can come into an ordered self-intersection point we have, for example,

$$\# u_1 + \# u_2 e = \operatorname{coeff}(z_-, m_0^{b_-}(1))$$

where $\operatorname{coeff}(y_-, m_0^{b_-}(1))$ means the coefficient of y_- in the curvature $m_0^{b_-}(1)$, which must vanish since b_- is a weakly bounding cochain. The other equalities are equivalent to the vanishing of the curvature $\mu_0^{b_-}$ at the other degree zero intersection points in Figure 2. □

Definition 2.3. For $t > 0$, let $b_{+,t} \in CF(L_t)$ be the cochain equal to $b_{-,t}$ on the complement of the three self-intersection points shown in Figure 2, an given by the values $e_t, f_t, e_t f_t + q^\epsilon g_t$ at the degree one self-intersection points near the triple intersection shown in Figure 2, where e_t, f_t, g_t are the coefficients of $b_{-,t}$.

Proof of Theorem 1.1 (a), case of one even interior angle. We introduce the following notation. Denote by v_1 a general holomorphic disk bounding L_+ with a single corner at a, $v_2^{',+}$ a general holomorphic disk with output with coefficient f an last input at e, $v_2^{',-}$ a general holomorphic disk with output at e and first input with corner with coefficient f, v_3 a general holomorphic disk with f (so v_1, v_3 overlap at v_2') and v_2 a general holomorphic disk with corner labelled $fe + gq^\epsilon$.

The following relationship holds between the signed counts of disks before and after the triple intersection: As in Lemma 7.4 of Palmer-Woodward [1], for the family of bounding cochains b_t whose values are given there at intersection points outside of the diagram shown,

$$\#v_1 = \#u_1 q^{2t}, \quad \#v_2 = \#u_2 q^\epsilon q^{-2t}, \quad \#v_3 = \#u_3 q^{2t}.$$

To show unobstructedness we need to show the three equalities

$$\begin{aligned} \#v_2' q^\epsilon + \#v_2 &= 0 \\ \#v_1 + \#v_2^{',+} f_t &= 0 \\ \#v_3 + \#v_2^{',-} e_t &= 0 \end{aligned}$$

The signs in the second and third equations are in the opposite order of those in u because of the change in order of f_t, e_t in the ordering of punctures around the boundary of the disks. We have

$$\#v_2' q^\epsilon = \#u_2' q^{\epsilon+2t} = -\#u_2 q^{-\epsilon+2t} q^\epsilon = -\#v_2$$

as required. Similarly

$$\#v_1 = \#u_1 q^{2t} = -\#u_2^+ f_t q^{2t} = -\#u_2' f_t q^{-\epsilon+2t} = -\#v_2' q^\epsilon f_t q^{-\epsilon} = -\#v_2^{',+} f_t.$$

The last inequality is similar. This shows that L_+ is unobstructed.

It remains to show that the potentials agree up to a power of q. Let u_- be a disk with k_- corners at g_t from inside the triangle, and k_+ corners at g from outside the triangle, and no corners at e_t, f_t. From u_- we obtain a disk u_+ with k_- corners at the corner marked $f_t e_t + q^\epsilon g_t$ from outside the triangle and k_+ corners at $f_t e_t + q^\epsilon g_t$ from inside the triangle. There are also 2^{k_-} other disks u_-' with corners at either e_t, f_t or at g_t, and 2^{k_+} other disks u_+' with corners at e_t and f_t, or g_t. The curves u_-, u_+ are in bijection, but their weighted count differs by a factor of $q^{2t}(f_t e_t + q^\epsilon g_t)^{k_+ - k_-}$. Hence

$$\#u_-' = q^{2t}(f_t e_t + q^\epsilon g_t)^{k_-} \#u_- = \#u_+ q^{2t}(f_t e_t + q^\epsilon g_t)^{k_+} = q^{2t} \#u_+'$$

as desired. □

3. Three even interior angles

We next consider the case of three even interior angles, shown in Figure 3. Denote by u_1 resp. u_1' a general holomorphic disk with corner at e containing the inner triangle, resp. not containing the inner triangle; u_2 resp. u_2' a general holomorphic disk with corner at g containing the inner triangle, resp. not containing the inner triangle; and u_3 resp. u_3' a general holomorphic disk with corner at f containing the inner triangle, resp. not containing the inner triangle. Weak unobstructedness of L_- implies:

Lemma 3.1. *The equalities*

$$\#u_2 + q^\epsilon e_t f_t = -\#u_2', \ \#u_1 + q^\epsilon f_t g_t = -\#u_1', \ \#u_3 + q^\epsilon g_t e_t = -\#u_3'.$$

hold.

Proof. The equalities are the vanishing of the curvature at the intersection points with coefficients g_t, f_t, and e_t respectively. □

Proof of Theorem 1.1 (a), case of three even interior angles. From the previous Lemma we obtain

$$q^{2t}(\#v_2 + e_t f_t) = -q^{2t} v_2', \quad q^{2t}(\#v_1 + f_t g_t) = -q^{2t}\#v_1', \quad \#q^{2t}(v_3 + g_t e_t) = -q^{2t}\#v_3'$$

so

$$\#v_2 = -\#v_2' - e_t f_t, \quad \#v_1 = -\#v_1' - f_t g_t, \quad \#v_3 = -\#v_3' - g_t e_t.$$

These are the equations for the weak unobstructedness of L_+; the change in sign come from the fact that the small triangle R_- has induced orientation that reverses (with respect to the orientation induced from $L_\pm$) under the flow over the triple intersection. The computation of the potential is similar to the case of one even interior angle, treated above. □

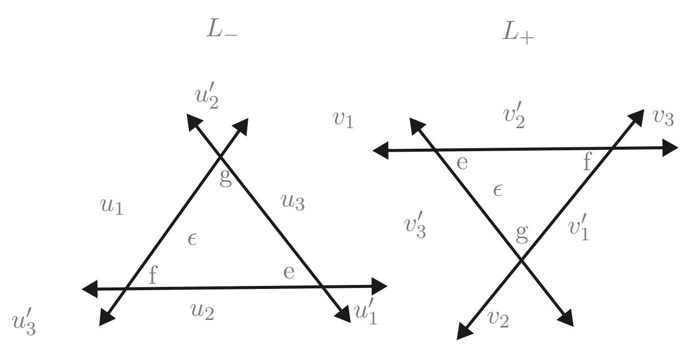

FIGURE 3. Moving past a triple intersection

4. ISOMORPHISM OF FLOER COHOMOLOGY

We finally prove the isomorphism of Floer cohomology. Following the philosophy suggested by the Yoneda lemma, we do this by comparing the Floer cohomology with a test Lagrangian, which may be taken to be a Hamiltonian isotopy of the original Lagrangian, whose Maslov flow has possibly a triple intersection at a different point in the symplectic manifold. Let $T_- \to X$ be an immersed Lagrangian with weakly bounding cochain $c_- \in CF(T_-)$ with the same potential w_- as L_-. Let T_t be a Maslov flow of T_- from t_- to t_+, with weakly bounding cochain c_t. As defined in [1], picking any Maslov flow T_t requires a choice of connection α_X on the anticanonical divisor K_X and a flat connection α_L on its restriction to the Lagrangian T. In order to define $HF(L_t, T_t)$, we need the Maslov flow T_t to be *compatible* with L_t, i.e., the chosen connections on K_X agree. T_t has the same potential as L_t, and so the Floer differential for the pair (L_t, T_t) squares to zero and the cohomology $HF(L_t, T_t)$ is well-defined. Let (T_+, c_+) be the specialization at small, positive t_+. Let $p \in X$ be the point of triple intersection of L.

Proposition 4.1. *Suppose there exist a Darboux neighborhood U of $p \in X$ so that*

- *(a) U is disjoint from T_t for all $t \in [t_-, t_+]$, and the intersection of L_t with U is as depicted in Figure 2 or 3;*
- *(b) T_t does not go through a triple intersection for $t \in [t_-, t_+]$.*
- *(c) the intersection points $T_t \cap L_t$ are transverse for all t.*

Then $HF((L_-, b_-), (T_-, c_-)) \cong HF((L_+, b_+), (T_+, c_+))$. Note that all three conditions are satisfied for a generic choice of compatible Maslov flow T_t and small time interval $[t_-, t_+]$.

Proof. This is proved in the same way as the equality of potentials: Suppose $x, y \in CF((L_-, b_-), (T_-, c_-))$ represent intersection points necessarily in the complement of the neighborhood U of the triple intersection. The count of contributions to the coefficient of y in $m_1^{b_-}(x)$ is equal, up to a factor of q, to the coefficient of y, by the same argument as in Theorem 1.1 (a). □

Proof of Theorem 1.1 (b). Let $\phi : X \to X$ be a Hamiltonian isotopy so that $\phi(L_t), t \in [t_-, t_+]$ is disjoint from a neighborhood of the triple intersection. Let $T_t := \phi(L_t)$. Then T_t is a Maslov flow and has a singularity at $t = 0$, at a point disjoint from the triple intersection of that of L_t. By choosing the neighborhood U sufficiently small, we may assume that U is disjoint from T_t, and $\phi(U)$ is disjoint from L_t. By Proposition 4.1 and invariance of Floer cohomology under Hamiltonian perturbation

$$\begin{aligned} HF(L_-, b_-) &\cong HF((L_-, b_-), (L_-, b_-)) \cong HF((L_-, b_-), (T_-, b_-)) \\ &\cong HF((L_+, b_+), (T_+, b_+)) \cong HF((L_+, b_+), (L_+, b_+)) \cong HF(L_+, b_+). \end{aligned} \tag{1}$$

□

References

[1] Joseph Palmer and Chris Woodward. Invariance of immersed Floer cohomology under Maslov flows. *Algebr. Geom. Topol.*, 21(5):2313–2410, 2021.

[2] Paul Seidel. Graded Lagrangian submanifolds. *Bulletin de la Société Mathématique de France*, 128(1):103–149, 2000.

[3] Paul Seidel. *Fukaya categories and Picard-Lefschetz theory.* Zurich Lectures in Advanced Mathematics. European Mathematical Society (EMS), Zurich, 2008.

[4] Katrin Wehrheim and Chris Woodward. Orientations for pseudoholomorphic quilts. https://arxiv.org/abs/1503.07803.