MULTIPLICITY-FREE HAMILTONIAN ACTIONS NEED NOT BE KÄHLER

CHRIS WOODWARD

Abstract. Multiplicity-free actions are symplectic manifolds with a very high degree of symmetry. Delzant [2] showed that all compact multiplicity-free torus actions admit compatible Kähler structures, and are therefore toric varieties. In this note we show that Delzant's result does not generalize to the non-abelian case. Our examples are constructed by applying $U(2)$-equivariant symplectic surgery to the flag variety $U(3)/T^3$. We then show that these actions fail a criterion which Tolman [9] shows is necessary for the existence of a compatible Kähler structure.

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1. Introduction

In this note we investigate the equivariant version of the question "when is a symplectic manifold Kähler?" That is, suppose that a Lie group G acts on a symplectic manifold (M, ω). When does there exist an invariant complex structure J such that $\omega(\cdot, J\cdot)$ defines a Riemmanian metric on M?

The main point of this note, which builds on work by S. Tolman [9], is that even Hamiltonian actions which are *multiplicity-free* need not be Kähler. Multiplicity-free actions are maximal in the following sense: if the action of G has discrete principal isotropy subgroup, then $\dim M \leq \dim G + \operatorname{rank} G$. If equality holds, the action is called multiplicity-free. In the case that G is a torus, multiplicity-free actions are all Kähler: Delzant [2] proved that any compact multiplicity-free torus action admits a Kähler structure, and is a smooth projective toric variety. Because of Delzant's result, and the fact that transitive actions, which if the group is compact are coadjoint orbits, are all Kähler, it was thought that all compact multiplicity-free actions might be Kähler. In particular, it was thought that any compact multiplicity-free action might be the Hamiltonian action associated to a spherical variety (the non-abelian analogue of a toric variety; see e.g. [7].) We show here that this is not the case. F. Knop has independently constructed examples of non-Kähler multiplicity-free actions, by a different method.

The idea of proof is to construct an example related to Tolman's using "equivariant symplectic surgery." This type of surgery (see [10]) is a type of "symplectic cutting" as defined by Lerman [6], and uses the fact that any Hamiltonian group action has a densely-defined action of a maximal torus, which commutes with the original action.

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2. Tolman's example

Tolman proves that a symplectic gluing of two halves of two six-dimensional Hamiltonian T^2-spaces, M_1 and M_2, results in a non-Kähler Hamiltonian T^2-space M_3. In this note we will consider a closely related gluing which is represented in Figure 1. The pictures show the

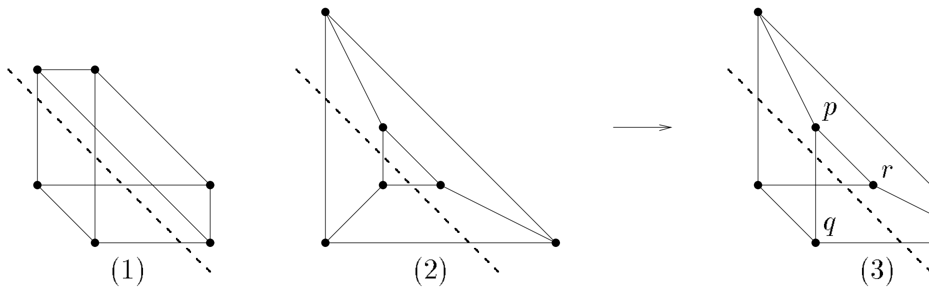

Figure 1: The X–rays for (1) M_1 (2) M_2 and (3) M_3

images under the moment map of the connected components of the non-principal orbit-type strata of M_1, M_2, and M_3, respectively, from left to right. That is, the points are the images of the fixed points, and the line segments are the images of the submanifolds fixed by circle subgroups of T^2. (In general, there are orbit-type strata with discrete isotropy groups, but in these examples it turns out that there aren't any.) Note that an intersection of two lines is not necessarily the image of a fixed point.

The manifold M_1 is a generic coadjoint orbit of $U(3)$, with the Hamiltonian action of T^2 obtained by restricting the action of $U(3)$ to $T^2 = U(1)^2 \times \mathrm{Id} \subset \mathrm{U}(3)$. The manifold M_2 is a toric variety with the action of T^2 obtained by restricting that of T^3 to $T^2 \times \mathrm{Id} \subset \mathrm{T}^3$. The polytope $P \subset (\mathfrak{t}^3)^* \cong \mathbb{R}^3$ associated to this toric variety is obtained by making the vertices of the outer triangle have z-coordinate 0 and those of the inner triangle z-coordinate 1, and giving each vertex x and y coordinates as drawn. The middle picture in Figure 1 is then the projection of P onto the x-y plane. The pictures are representations of an invariant which Tolman calls the X–ray.

To describe this invariant, let (M, ω) be a compact Hamiltonian T-space with moment map $\Phi : M \to \mathfrak{t}^*$. Recall that the orbit-type stratum M_H corresponding to a subgroup $H \subset T$ is the set of points $m \in M$ such that the isotropy subgroup T_m equals H. For some H, M_H is connected and dense; M_H is called the principal stratum. Let $\chi(M) = \{X_1, \ldots, X_k\}$ be the set of connected components of orbit-type strata, and let $H_1, \ldots, H_k$ be the corresponding isotropy subgroups. For each X_i the closure $\overline{X_i}$ is a component of the fixed point set of H_i, and by the equivariant Darboux theorem a symplectic submanifold. By the Atiyah-Guillemin-Sternberg convexity theorem, the image $\Phi(\overline{X_i})$ is a convex polytope which is the convex hull of the images of fixed points contained in $\overline{X_i}$. If the polytopes $\Phi(\overline{X_i})$ are distinct, we will define the X-ray to be the set $\{\Phi(\overline{X_i}), i \in \{1, \ldots, k\}\}$. This is slightly different from the definition in [9].

2.1. **X–ray computations.** The X–rays of actions considered in this note are computable from fixed point data (that is, the value of the moment map plus the weights of the action at each fixed point.) One begins by noting that each X_i contains T-fixed points in its closure. Let x be such a fixed point, and U a neighborhood of x. Let $\chi(U)$ (resp. $\chi(T_xM)$) denote the set of components of orbit-type strata of U (resp. T_xM). The set $X_i \cap U$ is then an element of $\chi(U)$. By linearizing the action at x, one can identify U with a neighborhood of 0 in T_xM, so that $\chi(U)$ becomes identified with $\chi(T_xM)$. Conversely, any element of $\chi(U)$ is contained in a unique element of $\chi(M)$. Let x and y be T-fixed points, and let us say that $X \in \chi(T_xM)$ and $Y \in \chi(T_xY)$ are identified if X and Y correspond to the same element of $\chi(M)$. By the convexity theorem, to determine the X–ray it suffices to know the sets $\chi(T_xY)$ and the images $\Phi(x)$ for each fixed point x, together with this identification. There is a simple fact which in some cases determines this identification: by definition of the moment map, the image $\Phi(X)$ is an open subset of an affine space orthogonal to $\mathfrak{h}$, where $\mathfrak{h}$ is the Lie algebra of the isotropy group H of any point in X. Thus, $X \in \chi(T_xM)$ and $Y \in \chi(T_yM)$ can be identified only if both sets have the same isotropy group, H, and if $\Phi(x) \in \Phi(y) + \mathfrak{h}^0$, where $\mathfrak{h}^0 \subset \mathfrak{t}^*$ is the annihilator of $\mathfrak{h}$.

Using this method we can compute the X–ray for the action of T^2 on the flag variety $M_1 = U(3)\lambda$ for a generic $\lambda \in \mathfrak{t}^*_+$. [1] Here T^2 denotes the subgroup $T^2 \times \mathrm{Id}$ of the maximal torus T^3 of $U(3)$. Since the center of $U(3)$ acts trivially on M_λ, the T^2-fixed points are exactly the T^3-fixed points. These are the elements of $W\lambda$, where W is the Weyl group of the maximal torus $T^3 \subset U(3)$. The weights of T^2 at a fixed point $w\lambda$ are $\pm\alpha_1, \pm\alpha_2$ and $\pm\alpha_3$, where α_1, α_2, and α_3 are the positive weights restricted to $\mathfrak{t}^2 \subset \mathfrak{t}^3$. With respect to the standard basis for the Cartan subalgebra of $U(3)$, the positive roots are

$$(1,-1,0), \quad (0,1,-1), \quad (1,0,-1)$$

and so

$$\alpha_1 = (1,-1), \quad \alpha_2 = (0,1), \quad \alpha_3 = (1,0).$$

From this list of weights we deduce that the there are only four non-principal isotropy groups of points in the neighborhood of each fixed point: the left factor, right factor and diagonal subgroups of T^2, and T^2 itself. For each one-dimensional isotropy subgroup H and fixed point $w\lambda$, there is only one other fixed point in the set $\Phi(w\lambda) + \mathfrak{h}^0$. This determines the X–ray, which is shown in Figure 1.

To compute the X–ray for M_2, we will apply the theory of toric varieties.

Lemma 2.1. *Let M be a smooth projective toric variety with moment map $\Phi : M \to \mathfrak{t}^*$ and moment polytope Δ, and let $F \subset \Delta$ be any open face. Then the isotropy subgroup of any point $m \in \Phi^{-1}(F)$ is connected and has Lie algebra equal to the annihilator F^0 of F.*

For a proof from the symplectic viewpoint, see [2]. By Lemma 2.1, if F is any face of P, then the isotropy subgroup for the action of T^2 at any point in $\Phi_3^{-1}(F)$ is the intersection $T^2 \cap T^3_F$. Furthermore, the moment map Φ_2 for T^2 is Φ_3 composed with projection onto $\mathrm{Lie}(T^2) \cong \mathbb{R}^2$. If F is a 0 or 1-dimensional face of P, then $\Phi_3^{-1}(F)$ is a component of a non-principal orbit-type stratum of the T^2 action whose whose image under Φ_2 is the projection of F onto $(\mathfrak{t}^2)^*$. If F

[1] We could also take M_1 to be an $SU(3)$-coadjoint orbit, and restrict to the action of the maximal torus. Our reason for not doing so will become apparent later.

is a 2-dimensional face, the intersection $T^2 \cap T^3_F$ is trivial, so $\Phi_3^{-1}(F)$ is part of the principal orbit-type stratum for the T^2-action. This implies that the X–ray is as shown in Figure 1.

2.2. **Non-existence of a compatible Kähler structure.** Let M_3 denote the Hamiltonian T-manifold constructed by gluing "half" of M_1 together with "half" of M_2, as in [9]. Tolman's extendibility criterion in [9] implies that M_3 does not admit any invariant compatible Kähler structure. Indeed, let C be the cone based at p in Figure 1, and generated by the line segments $\overline{pq}$ and $\overline{pr}$. If M admits a Kähler structure then there would exist an orbit Y of the complex torus $T_{\mathbb{C}}$ with the property that $\Phi(\overline{Y}) = C$ near p. In this case Y is the dense $T_{\mathbb{C}}$ orbit in the stable manifold of p with respect to the gradient flow of any Morse function of the form $\langle \Phi, v\rangle$, for a generic $v \in \mathfrak{t}$ such that $(v, q-p) < 0$ and $(v, r-p) < 0$. By Atiyah's theorem [1], $\Phi(\overline{Y})$ must be a convex polytope with vertices contained in $\Phi(M_3^T)$. From the X–ray one sees that such a polytope does not exist.[2]

3. Construction by $U(2)$-equivariant symplectic surgery

In this section we give an alternative construction of Tolman's example which uses E. Lerman's *symplectic cutting* technique [6]. The advantage is that (1) the construction is more explicit, and (2) the construction shows that the example has a multiplicity-free $U(2)$-action.

3.1. **Lerman's definition of symplectic cutting.** Symplectic cutting is a surgery operation which is closely related to Marsden-Weinstein symplectic reduction. To describe the simplest case of the construction, let (N, ω_N) be a Hamiltonian G-space and $\mu : N \to \mathbb{R}$ a moment map for a G-equivariant $U(1)$ action on N. Let $a \in \mathbb{R}$ be a regular value of μ, let $N_a = \mu^{-1}(a)/U(1)$ be the reduced space and let $N_{<a}$ be the subset $\mu^{-1}(-\infty, a) \subset N$.

Lemma 3.1 (Lerman). *The union $N_{\leq a} = N_a \cup N_{<a}$ has the structure of a Hamiltonian G-orbifold. Furthermore, if N_a is smooth then $N_{\leq a}$ is smooth as well.*

Proof. Let $N \times \mathbb{C}$ be the product with symplectic structure $\pi_1^*\omega_N + \pi_2^*(dz \wedge d\overline{z})/2i$, where π_1 and π_2 are the projections. Define $\nu : N \times \mathbb{C} \to \mathbb{R}$ by

$$\nu(n, z) = \mu(n) + |z|^2/2$$

so that ν is the moment map of the diagonal action of $U(1)$ on $N \times \mathbb{C}$, which is equivariant with respect to the action of G on the left factor. Let $N_{\leq a}$ be the reduction of $N \times \mathbb{C}$ at a. Then we can write

$$N_{\leq a} \cong \mu^{-1}(a)/U(1) \cup \mu^{-1}(-\infty, a)$$

as claimed. □

The space $N_{\leq a}$ is called the *symplectic cut of N at a*. The identification of a dense subset $(N_{\leq a})_{<a}$ of $N_{\leq a}$ with $N_{<a} \subset N$ is an equivariant symplectomorphism. This implies that $N_{\leq a}$ is defined even if μ is only smooth in a neighborhood U of $\mu^{-1}(a)$. (That is, μ only defines an $U(1)$ action locally.) Indeed, we can assume that $U = \mu^{-1}(b, c)$ for some $b, c \in \mathbb{R}$. The symplectic cut $U_{\leq a}$ is well-defined, and we define (see [6])

Definition 3.2. The symplectic cut $N_{\leq a}$ of N at a is the union of $U_{\leq a}$ and $N_{<a}$ modulo the identification of $(U_{\leq a})_{<a}$ with $N_{<a} \cap U$.

[2] There is an alternative proof (worked out by S. Tolman, S. Wu, and the author) that M_3 is not Kähler which uses Witten's equivariant holomorphic Morse inequalities. See Wu and Mathai [8].

3.2. **Equivariant symplectic surgery.** We begin by explaining the following statement: any Hamiltonian G-action has a canonical densely defined, G-equivariant action of T. This follows from a version of the symplectic cross-section theorem [5, Theorem 26.2]. Let $\mathfrak{t}^*_+$ be a closed positive Weyl chamber. For each Weyl wall $\sigma \subset \mathfrak{t}^*_+$ (not necessarily codimension 1) let G_σ be the isotropy subgroup of any point in σ.

Theorem 3.3. *Let M be any Hamiltonian G-space with moment map $\Phi : M \to \mathfrak{g}^*$, let $\sigma \subset \mathfrak{t}^*_+$ be a Weyl wall, and let $U_\sigma \subset \mathfrak{g}^*_\sigma$ be the maximal set such that $x \in U_\sigma$ implies $G_x \subset G_\sigma$. Then $\Phi^{-1}(U_\sigma)$ is a Hamiltonian G_σ-space, called the* symplectic cross-section *for σ.*

Note that $\Phi^{-1}(GU_\sigma) \cong G \times_{G_\sigma} \Phi^{-1}(U_\sigma)$. If Z_σ is the center of G_σ, then we can define a *new* action of Z_σ by requiring that the action agree with the old action of Z_σ on $\Phi^{-1}(U_\sigma)$, and that the action commute with the action of G. In particular if $\sigma = \operatorname{int} \mathfrak{t}^*_+$ then we have a new G-equivariant action of T on the (dense if non-empty) subset $G\Phi^{-1}(U_\sigma) = \Phi^{-1}(\mathfrak{g}^*_{\mathrm{reg}})$. We call the new action of Z_σ the *induced action*, and denote it by ρ.

Let $q : \mathfrak{g}^* \to \mathfrak{t}^*_+$ be the quotient map, and define $\tilde{\Phi} = q \circ \Phi$. For the following proposition, see e.g. [4],[10].

Proposition 3.4. *The composition of $\tilde{\Phi}$ with the projection $\pi_\sigma : \mathfrak{t}^* \to z^*_\sigma$ is a moment map for the induced action of Z_σ.*

Since symplectic cutting is local, if $a \in \mathbb{R}$ is such that $\mu = \langle \tilde{\Phi}, X \rangle$ is smooth at $\mu^{-1}(a)$, and if furthermore the induced action of $U(1) = \exp(tX) \subset T$ is free on $\mu^{-1}(a)$, then the symplectic cut of M at a is a Hamiltonian G-space $M_{\leq a}$ with moment polytope

$$\Delta_{\leq a} = \{v \in \Delta \mid \langle v, X \rangle \leq a\}.$$

Note that in order to check that the induced action of $U(1)$ is free, by equivariance it suffices to check that the left action of $U(1)$ is free on $\Phi^{-1}(\mathfrak{t}^*_+) \cap \mu^{-1}(a)$. Also, note that if the hyperplane $H = \{v \in \mathfrak{t}^* \mid \langle v, X \rangle = a\}$ meets perpendicularly every face σ of the positive chamber such that $\sigma \cap H \cap \Delta$ is non-empty, then μ is smooth at $\mu^{-1}(a)$.

3.3. **Construction of the example.** We will apply equivariant surgery to a flag variety. Let $M_1 = U(3)\lambda$ for some element $\lambda = \operatorname{diag}(\lambda_1, \lambda_2, \lambda_3)$ of $\mathfrak{u}(3)^*$ with $\lambda_1 > \lambda_2 > \lambda_3$. The action of $U(3)$ on M_1 restricts to a Hamiltonian action of $U(2)$ via the embedding of $U(2)$ in $U(3)$ given by $A \to \operatorname{diag}(A, 1)$. The moment map $\Phi_{U(2)} : M_1 \to \mathfrak{u}(2)^*$ is the projection of M_1 onto $\mathfrak{u}(2)^*$. Since $\dim M_1 = 6 = (\dim + \operatorname{rank}) U(2)$, and $U(2)$ acts freely on a dense subset, the action of $U(2)$ on M_1 is multiplicity-free. Let $T \subset U(2)$ be the diagonal maximal torus.

Proposition 3.5. *The moment polytope of the action of $U(2)$ on M_1 is $\Delta' = [\lambda_2, \lambda_1] \times [\lambda_3, \lambda_2]$.*

For a proof see [5, page 364] or [10]. We now apply $U(2)$-equivariant surgery. In this case the map $\tilde{\Phi} = q \circ \Phi_{U(2)}$ has two components, $\tilde{\Phi}_1$ and $\tilde{\Phi}_2$. Let μ be the function $\tilde{\Phi}_1 + 2\tilde{\Phi}_2$ and $U(1)_{1,2}$ the circle subgroup $\{(z, z^2) | z \in U(1)\} \subset U(1)^2$ whose induced action has moment map μ. Let $a \in \mathbb{R}$ be such that $\mu^{-1}(a)$ is the inverse image under $\tilde{\Phi}$ of the dotted line shown in Figure 2. Then μ is smooth at $\mu^{-1}(a)$, since $\mu^{-1}(a)$ lies entirely in GY_+. The symplectic cut $(M_1)_{\leq a}$ of M_1 at a using μ is therefore well-defined.

Proposition 3.6. *$(M_1)_{\leq a}$ is a smooth Hamiltonian $U(2)$-space, whose X–ray as a T^2-space is the same as M_3.*[3]

[3] Conjecturally, $(M_1)_{\leq a}$ is T^2-equivariantly symplectomorphic to M_3.

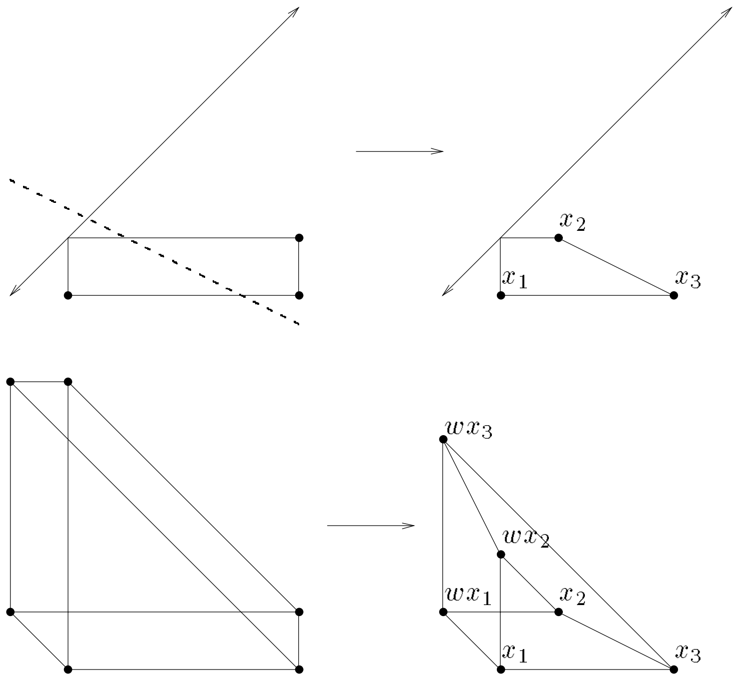

Figure 2: Construction by symplectic cutting

Proof. To see that $(M_1)_{\leq a}$ is smooth, it's enough to check that the induced action of $U(1)_{1,2}$ is free on $\mu^{-1}(a)$, or equivalently, since the induced action is $U(2)$-equivariant, that $U(1)_{1,2}$ acts freely on $\mu^{-1}(a) \cap \Phi^{-1}(\mathfrak{t}^*_+)$. This is shown either explicitly (see [10] page 6 or [5] page 364) or using the following lemma:

Lemma 3.7 (Delzant [3]). *Let M a multiplicity-free compact Hamiltonian G-space with moment map $\Phi : M \to \mathfrak{g}^*$, moment polytope Δ and trivial principal isotropy subgroup. Let F be a face of Δ contained in* int $\mathfrak{t}^*_+$*, and let m be a point in $\Phi^{-1}(F)$. Then the isotropy subgroup G_m of m is connected and its Lie algebra is $F^0 \subset \mathfrak{t}$.*

By Lemma 3.7 the isotropy subgroups of points in $\mu^{-1}(a) \cap Y_+$ are $\{\mathrm{Id}\} \times \mathrm{U}(1)$ and $\{\mathrm{Id}\}$. These intersect $U(1)_{1,2}$ trivially, so $U(1)_{1,2}$ acts freely on $\mu^{-1}(a)$. Therefore, $(M_1)_{\leq a}$ is a smooth Hamiltonian $U(2)$-space which has the moment polytope shown on the above right. Since T acts freely on a dense subset of Y_+, the group $U(2)$ acts freely on a dense subset of GY_+. Since $\dim((M_1)_{\leq a}) = 6 = (\dim + \mathrm{rank})U(2)$ the action of $U(2)$ on $(M_1)_{\leq a}$ is multiplicity-free.

Now consider the action of $T = U(1)^2 \subset U(2)$ on $(M_1)_{\leq a}$, and let $\Phi_T : (M_1)_{\leq a} \to \mathfrak{t}^*$ be the moment map. Let us compute the X–ray for $(M_1)_{\leq a}$. Since $(M_1)_{\leq a}$ is a symplectic cut, the T-fixed points are those lying in $\mu^{-1}(-\infty, a)$ and the "new" fixed points in $\mu^{-1}(a)/U(1)_{1,2}$. From Figure 2, we see that there is only one "old" fixed point q_1 with $\Phi_T(q_1) = x_1$, and by Delzant's lemma (or explicitly) there are T-fixed points q_2, q_3 whose images under Φ_T are x_2 and x_3 resp. The points $wq_i, i = 1, 2, 3$, where w is the non-trivial element of the Weyl group, are also T-fixed points, whose images under Φ_T are wx_i.

We now compute the other polytopes in the X–ray of $(M_1)_{\leq a}$. The splitting

$$T_{q_i}(M_1)_{\leq a} \cong T_{q_i}(Y_+)_{\leq a} \oplus \mathfrak{u}(2)^*/\mathfrak{t}^*$$

implies that there are two weights of T on $T_{q_i}(M_1)_{\leq a}$ which are weights of T acting on $T_{q_i}(Y_+)_{\leq a}$ and a third weight equal to $-\alpha_1$. The two weights of the action on $T_{q_i}(Y_+)_{\leq a}$ are proportional to the directions of the two edges of the moment polytope at x_i. The weights at the T-fixed points wq_i are obtained by Weyl reflection. By the reasoning similar to the computation of the X–ray of M_1, the X–ray can be computed from this fixed point data, and coincides with the X–ray of M_3 above. □

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Harvard University, Dept. of Mathematics, 1 Oxford Street, Cambridge, Massachusetts 02138
E-mail address: woodward@math.harvard.edu