STABLE GAUGED MAPS

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ABSTRACT. We give an introduction to moduli stacks of gauged maps satisfying a stability condition introduced by Mundet [55] and Schmitt [61], and the associated integrals giving rise to gauged Gromov-Witten invariants. We survey various applications to cohomological and K-theoretic Gromov-Witten invariants.

CONTENTS

1. INTRODUCTION

The moduli stack of maps from a curve to the stack quotient of a smooth projective variety by the action of a complex reductive group has a natural stability condition introduced by Mundet in [55] and investigated further in Schmitt [61, 62]; the condition generalizes stability for bundles over a curve introduced by Mumford, Narasimhan-Seshadri and Ramanathan [60]. Let X be a smooth linearized projective G-variety such that the semi-stable locus is equal to the stable locus, X/G the quotient stack. By definition a map from a curve C to X/G is a pair that consists of a bundle $P \to C$ and a section u of the associated bundle $P \times_G X \to C$. We denote by $\pi : X/G \to \mathrm{pt}/G =: BG$ the projection to the classifying space. In case X is a point, a stability condition for $\mathrm{Hom}(C, X/G)$, bundles on C, was introduced by Ramanathan [60]. For X not a point, a stability condition that combines bundle and target stability was introduced by Mundet [55]. There is a compactified moduli stack $\overline{\mathcal{M}}_n^G(C, X, d)$ whose open locus consists of Mundet semistable maps of class $d \in H_2^G(X, \mathbb{Z})$ with markings

$$v : C \to X/G, \quad (z_1, \dots, z_n) \in C^n \text{ distinct},$$

and where the notion of semi-stability depends on a choice of linearization $\tilde{X} \to X$. The compactification uses the notion of Kontsevich stability for maps [68], [69],

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[70]. The stack admits evaluation maps to the quotient stack

$$\mathrm{ev} : \overline{\mathcal{M}}_n^G(C, X, d) \to (X/G)^n, \quad (\hat{C}, P, u, \underline{z}) \mapsto (z_i^* P, u \circ z_i).$$

In addition, assuming stable=semistable there is a virtual fundamental class constructed via the machinery of Behrend-Fantechi [6]. Let $\widehat{QH}_G(X)$ denote the formal completion of $QH_G(X)$ at 0. The *gauged Gromov-Witten trace* is the map

$$\tau_X^G : \widehat{QH}_G(X) \to \Lambda_X^G, \quad \alpha \mapsto \sum_{n,d} \frac{q^d}{n!} \int_{\overline{\mathcal{M}}_n^G(C,X,d)} \mathrm{ev}^*(\alpha, \dots, \alpha).$$

Here Λ_X^G is the (equivariant) field of Novikov variables (see Definition 4.4.) The derivatives of the potential will be called *gauged Gromov-Witten invariants*. For toric varieties, the potential τ_X^G already appears in Givental [26] and Lian-Liu-Yau [47] under the name of *quasimap potential*.[1] In those papers (following earlier work of Morrison-Plesser [53]) the gauged potential is explicitly computed in the toric case, and questions about Gromov-Witten invariants of toric varieties or complete intersections therein reduced to a computation of quasimap invariants. So what we are concerned with here is the generalization of that story to arbitrary git quotients, especially the non-abelian and non-Fano cases to cover situations such as arbitrary toric stacks and quiver moduli. Especially, we are interested in re-proving and extending the results of those papers in a uniform and geometric way that extends to quantum K-theory and non-abelian quotients and does not use any assumption such as the existence of a torus action with isolated fixed points. The splitting axiom for the gauged invariants is somewhat different than the usual splitting axiom in Gromov-Witten theory: the potential τ_X^G is a non-commutative version of a *trace* on the Frobenius manifold $QH_G(X)$. Note that there are several other notions of gauged Gromov-Witten invariants, for example, Ciocan-Fontanine-Kim-Maulik [15], Frenkel-Teleman-Tolland [23], as well as a growing body of work on gauged Gromov-Witten theory with potential [66], [22].

The gauged Gromov-Witten invariants so defined are closely related to, but different from in general, the graph Gromov-Witten invariants of the stack-theoretic geometric invariant theory quotient. The stack of marked maps to the git quotient

$$v : C \to X/\!\!/G, \quad (z_1, \dots, z_n) \in C^n \text{ distinct}$$

is compactified by the *graph space*

$$\overline{\mathcal{M}}_n(C, X/\!\!/G, d) := \overline{\mathcal{M}}_{g,n}(C \times X/\!\!/G, (1, d))$$

the moduli stack of stable maps to $C \times X/\!\!/G$ of class $(1, d)$; in case $X/\!\!/G$ is an orbifold the domain is allowed to have orbifold structures at the nodes and markings as in [12], [2]. The stack admits evaluation maps

$$\mathrm{ev} : \overline{\mathcal{M}}_n(C, X/\!\!/G, d) \to (\overline{\mathcal{I}}_{X/\!\!/G})^n$$

where $\overline{\mathcal{I}}_{X/\!\!/G}$ is the *rigidified inertia stack* of $X/\!\!/G$. The *graph trace* is the map

$$\tau_{X/\!\!/G} : \widehat{QH}_{\mathbb{C}^\times}(X/\!\!/G) \to \Lambda_X^G, \quad \alpha \mapsto \sum_{n,d} \frac{q^d}{n!} \int_{\overline{\mathcal{M}}_n(C,X/\!\!/G,d)} \mathrm{ev}^*(\alpha, \dots, \alpha)$$

[1] We are simplifying things a bit for the sake of exposition; actually the quasimap potentials in those papers involve an additional determinant line bundle in the integrals.

where the equivariant parameters are interpreted as Chern classes of the cotangent lines at the markings. The relationship between the graph Gromov-Witten invariants of $X/\!\!/G$ and Gromov-Witten invariants arising from stable maps to $X/\!\!/G$ in the toric case is studied in [26], [47], and other papers.

The goal of this paper is describe, from the point of view of algebraic geometry, a cobordism between the moduli stack of Mundet semistable maps and the moduli stack of stable maps to the git quotient with corrections coming from "affine gauged maps". Affine gauged maps are data

$$v : \mathbb{P}^1 \to X/G, \quad v(\infty) \in X^{\mathrm{ss}}/G, \quad z_1, \ldots, z_n \in \mathbb{P}^1 - \{\infty\} \text{ distinct}$$

where $\infty = [0,1] \in \mathbb{P}^1$ is the point "at infinity", modulo *affine* automorphisms, that is, automorphisms of $\mathbb{P}^1$ which preserve the standard affine structure on $\mathbb{P}^1 - \{0\}$. Denote by $\overline{\mathcal{M}}_{n,1}^G(\mathbb{A}, X)$ the compactified moduli stack of such affine gauged maps to X; we use the notation $\mathbb{A}$ to emphasize that the equivalence only uses affine automorphisms of the domains. Evaluation at the markings defines a morphism

$$\mathrm{ev} \times \mathrm{ev}_\infty : \overline{\mathcal{M}}_{n,1}^G(\mathbb{A}, X, d) \to (X/G)^n \times \overline{\mathcal{I}}_{X/\!\!/G}.$$

In the case $d = 0$, the moduli stack $\overline{\mathcal{M}}_{0,1}^G(\mathbb{A}, X, d)$ is isomorphic to $\overline{\mathcal{I}}_{X/\!\!/G}$ via evaluation at infinity. The *quantum Kirwan map* is the map

$$\kappa_X^G : \widehat{QH}_G(X) \to QH_{\mathbb{C}^\times}(X/\!\!/G)$$

defined as follows. Let $\mathrm{ev}_{\infty,d} : \overline{\mathcal{M}}_{n,1}^G(\mathbb{A}, X, d) \to \overline{\mathcal{I}}_{X/\!\!/G}$ be evaluation at infinity restricted to affine gauged maps of class d, and

$$\mathrm{ev}_{\infty,d,*} : H(\overline{\mathcal{M}}_{n,1}^G(\mathbb{A}, X, d)) \otimes_{\mathbb{Q}} \Lambda_X^G \to H_G(\overline{\mathcal{I}}_{X/\!\!/G}) \otimes_{\mathbb{Q}} \Lambda_X^G$$

push-forward using the virtual fundamental class. The quantum Kirwan map is

$$\kappa_X^G : \widehat{QH}_G(X) \to QH_{\mathbb{C}^\times}(X/\!\!/G), \quad \alpha \mapsto \sum_{n,d} \frac{q^d}{n!} \mathrm{ev}_{\infty,d,*} \mathrm{ev}^*(\alpha, \ldots, \alpha).$$

As a formal maps, each term in the Taylor series of κ_X^G and τ_X^G is well-defined on $QH_G(X)$, but in general the sums of the terms may have convergence issues. The $q = 0$ specialization of κ_X^G is the Kirwan map to the cohomology of a git quotient studied in [44].

The cobordism relating stable maps to the quotient with Mundet semistable maps is itself a moduli stack of gauged maps with extra structure, a scaling, defined by allowing the linearization to tend towards infinity, that is by considering Mundet semistability with respect to the linearization $\tilde{X}^k$ as k goes to infinity. In order to determine which stability condition to use, the source curves must be equipped with additional data of a *scaling*: a section

$$\delta : \hat{C} \to \mathbb{P}\left(\omega_{\hat{C}/(C \times S)} \oplus \mathcal{O}_{\hat{C}}\right)$$

of the projectivized relative dualizing sheaf. If the section is finite, one uses the Mundet semistability condition, while if infinite one uses the stability condition on the target. The possibility of constructing a cobordism in this way was suggested by a symplectic argument of Gaio-Salamon [25]. A *scaled gauged map* is a map to the quotient stack whose domain is a curve equipped with a section of the projectivized dualizing sheaf and a collection of distinct markings: A datum

$$\hat{C} \to S, \quad v : \hat{C} \to C \times X/G, \quad \delta : \hat{C} \to \mathbb{P}\left(\omega_{\hat{C}/(C\times S)} \oplus \mathcal{O}_{\hat{C}}\right), \quad z_1, \dots, z_n \in \hat{C}$$

where

- $\hat{C} \to S$ is a nodal curve of genus $g = \operatorname{genus} C$,
- $v = (P, u)$ is a morphism to the quotient stack X/G that consists of a principal G-bundle $P \to \hat{C}$ and a map $u : \hat{C} \to P \times_G X$ of whose class projects to $[C] \in H_2(C)$, so that a sub-curve of $\hat{C}$ is isomorphic to C and all other irreducible components map to points in C; and
- δ is a section of the projectivization of the relative dualizing sheaf $\omega_{\hat{C}/(C\times S)}$ satisfying certain properties.

In the case that $X/\!\!/G$ is an orbifold, the domain $\hat{C}$ is allowed to have orbifold singularities at the nodes and markings and the morphism is required to be representable. In particular, in the case X, G are points and $n = 0$, the stability condition requires $\hat{C} \cong C$ and the moduli space $\overline{\mathcal{M}}_{0,1} \cong \mathbb{P}^1$ is the projectivized space of sections δ of $\omega_{\hat{C}/C\times S} \cong \mathcal{O}_C$. The moduli stack of stable scaled gauged maps $\overline{\mathcal{M}}^G_{n,1}(C, X, d)$ with n markings and class $d \in H^G_2(X, \mathbb{Q})$ is equipped with a forgetful map

$$\rho : \overline{\mathcal{M}}^G_{n,1}(C, X, d) \to \overline{\mathcal{M}}_{0,1} \cong \mathbb{P}^1, \quad [\hat{C}, u, \delta, \underline{z}] \mapsto \delta.$$

The fibers of ρ over zero $0, \infty \in \mathbb{P}^1$ consist of either Mundet semistable gauged maps, in the case $\delta = 0$, or stable maps to the git quotient together with affine gauged maps, in the case $\delta = \infty$: In notation,

$$\begin{aligned} &\rho^{-1}(0) = \overline{\mathcal{M}}^G_n(C, X, d), \quad \rho^{-1}(\infty) = \bigcup_{d_0+\ldots+d_r=d} \ \bigcup_{I_1\cup\ldots\cup I_r=\{1,\ldots,n\}} \\ &\qquad (\overline{\mathcal{M}}^{\mathrm{fr}}_{g,r}(C \times X/\!\!/G, (1, d_0)) \times_{(\overline{\mathcal{I}}_{X/\!\!/G})^r} \prod_{j=1}^{r} \overline{\mathcal{M}}^G_{|I_j|,1}(\mathbb{A}, X, d_j))/(\mathbb{C}^\times)^r \end{aligned} \tag{1}$$

where we identify $H_2(X/\!\!/G)$ as a subspace of $H^G_2(X)$ via the inclusion $X/\!\!/G \subset X/G$, and $\overline{\mathcal{M}}^{\mathrm{fr}}_{g,r}(C \times X/\!\!/G, (1, d_0))$ denotes the moduli space of stable maps with framings of the tangent spaces at the markings. The properness of these moduli stacks was argued via symplectic geometry in [69]. We give an algebraic proof in [37].

The cobordism of the previous paragraph gives rise to a relationship between the gauged invariants and the invariants of the quotient that we call the quantum Witten formula. The formula expresses the failure of a diagram

$$\begin{array}{ccc} \widehat{QH}_G(X) & \xrightarrow{\kappa^G_X} & \widehat{QH}_{\mathbb{C}^\times}(X/\!\!/G) \\ & \searrow^{\tau^G_X} \quad \swarrow^{\tau_{X/\!\!/G}} & \\ & \Lambda^G_X & \end{array} \tag{2}$$

to commute as an explicit sum of contributions from wall-crossing terms. Here $\widehat{QH}_G(X), \widehat{QH}(X/\!\!/G)$ denote formal completions of the quantum cohomologies, Λ^G_X is the equivariant Novikov ring and the diagonal arrows are the potentials that arise from virtual integration over the certain moduli stacks of gauged maps. The wall-crossing terms vanish in the limit of large linearization and are not discussed in this paper. We explain how this gives rise to the diagram (2), at least in the large linearization limit.

Theorem 1.1. (Adiabatic limit theorem, [70]) *The diagram* (2) *commutes in the limit of large linearization* $\tilde{X}^k, k \to \infty$*, that is,*

$$\lim_{k \to \infty} \tau_X^G = \tau_{X/\!\!/G} \circ \kappa_X^G.$$

Before giving the proof perhaps we should begin by explaining what applications we have in mind for gauged Gromov-Witten invariants of this type and the adiabatic limit theorem in particular. Many interesting varieties in algebraic geometry have presentations as git quotients. The Grassmannian and projective toric varieties are obvious examples; well-studied also are quiver varieties such as the moduli of framed sheaves on the projective plane via the Atiyah-Drinfeld-Hitchin-Manin construction. In each of these cases, the moduli stacks of gauged maps are substantially simpler than the moduli stacks of maps to the git quotients. This is because the "upstairs spaces" are affine, and so the moduli spaces (at least in the case without markings) so consist simply of a bundle with section up to equivalence. In many cases this means that the gauged Gromov-Witten invariants can be explicitly computed, even though the Gromov-Witten invariants of the git quotient cannot. Sample applications of the quantum Kirwan map and the adiabatic limit theorem include presentations of the quantum cohomology rings of toric varieties (more generally toric stacks with projective coarse moduli spaces) [34] and formulas for quantum differential equations on, for example, the moduli space of framed sheaves on the projective plane [35]. More broadly, the gauged Gromov-Witten invariants often have better conceptual properties than the Gromov-Witten invariants of the git quotients. So for example, one obtains from the adiabatic limit theorem a wall-crossing formula for Gromov-Witten invariants under variation of git, which in particular shows invariance of the graph potentials in the case of a crepant wall-crossing [36].

Proof of Theorem. Consider the degree d contributions from $(\alpha, \ldots, \alpha)$ to τ_X^G and $\tau_{X/\!\!/G} \circ \kappa_X^G$. The former contribution is the integral of $\operatorname{ev}^*(\alpha, \ldots, \alpha)$ over $\overline{\mathcal{M}}_n^G(C, X, d)$. By (1), this integral is equal to the integral of $\operatorname{ev}^*(\alpha, \ldots, \alpha)$ over

$$\bigcup_{d_0 + \ldots + d_r = d} \bigcup_{I_1, \ldots, I_r} \left(\overline{\mathcal{M}}_{0,r}^{\mathrm{fr}}(C \times X/\!\!/G, (1, d_0)) \times_{\overline{\mathcal{I}}_{X/\!\!/G}^r} \prod_{j=1}^r \overline{\mathcal{M}}_{|I_j|,1}^G(\mathbb{A}, X, d_j) \right) / (\mathbb{C}^\times)^r.$$

With $i_j = |I_j|$ the integral of $\operatorname{ev}^*(\alpha, \ldots, \alpha)$ can be written as the push-forward of $\prod_{j=1}^r \operatorname{ev}^* \alpha^{\otimes i_j} / i_j!$ under the product of evaluation maps

$$\operatorname{ev}_{\infty, d_j, *} : \overline{\mathcal{M}}_{i_j,1}^G(\mathbb{A}, X, d_j) \to H(\overline{\mathcal{I}}_{X/\!\!/G})^r$$

followed by integration over $\overline{\mathcal{M}}_{0,r}^G(X/\!\!/G, d_0)$. Taking into account the number $n!/i_1! \ldots i_r! r!$ of unordered partitions $I_1, \ldots, I_r$ of the given sizes $i_1, \ldots, i_r$, this composition is equal to the degree d contribution from $(\alpha, \ldots, \alpha)$ to $\tau_{X/\!\!/G} \circ \kappa_X^G$. □

2. Scaled curves

Scaled curves are curves with a section of the projectivized dualizing sheaf incorporated, intended to give complex analogs of spaces introduced by Stasheff [63] such as the multiplihedron, cyclohedron etc. The commutativity of diagrams such as (2) will follow from divisor class relations in the moduli space of scaled curves, in a way similar to the proof of associativity of the quantum product via the divisor class relation in the moduli space of stable, 4-marked genus 0 curves.

Recall from Deligne-Mumford [20] and Behrend-Manin [7, Definition 2.1] the definition of stable and prestable curves. A *prestable curve* over the scheme S is a flat proper morphism $\pi : C \to S$ of schemes such that the geometric fibers of π are reduced, connected, one-dimensional and have at most ordinary double points (nodes) as singularities. A *marked prestable curve* over S is a prestable curve $\pi : C \to S$ equipped with a tuple $\underline{z} = (z_1, \ldots, z_n) : S \to C^n$ of distinct non-singular sections. A *morphism* $p : C \to D$ of prestable curves over S is an S-morphism of schemes, such that for every geometric point s of S we have (a) if η is the generic point of an irreducible component of D_s, then the fiber of p_s over η is a finite η-scheme of degree at most one, (b) if C' is the normalization of an irreducible component of C_s, then $p_s(C')$ is a single point only if C' is rational. A prestable curve is *stable* if it has finitely many automorphisms. Denote by $\overline{\mathcal{M}}_{g,n}$ the proper Deligne-Mumford stack of stable curves of genus g with n markings [20]. The stack $\overline{\mathfrak{M}}_{g,n}$ of prestable curves of genus g with n markings is an Artin stack locally of finite type [5, Proposition 2].

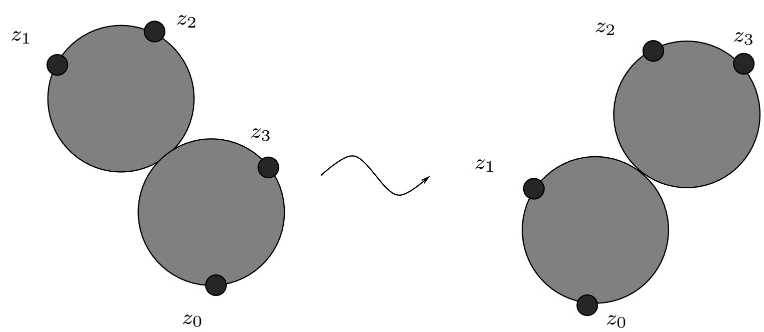


FIGURE 1. Associativity divisor relation

The following constructions give complex analogs of the spaces constructed in Stasheff [63]. For any family of possibly nodal curves $C \to S$ we denote by ω_C the relative dualizing sheaf defined for example in Arbarello-Cornalba-Griffiths [4, p. 97]. Similarly for any morphism $\hat{C} \to C$ we denote by $\omega_{\hat{C}/C}$ the relative dualizing sheaf and $\mathbb{P}(\omega_{\hat{C}/C} \oplus \mathcal{O}_{\hat{C}}) \to \hat{C}$ the projectivization. A *scaling* is a section

$$\delta : \hat{C} \to \mathbb{P}(\omega_{\hat{C}/C} \oplus \mathcal{O}_{\hat{C}}), \quad \mathbb{P}(\omega_{\hat{C}/C} \oplus \mathcal{O}_{\hat{C}}) = (\omega_{\hat{C}/C} \oplus \mathcal{O}_{\hat{C}})^\times / \mathbb{C}^\times.$$

If $\hat{C} \to C$ is an isomorphism then $\omega_{\hat{C}/C}$ is trivial:

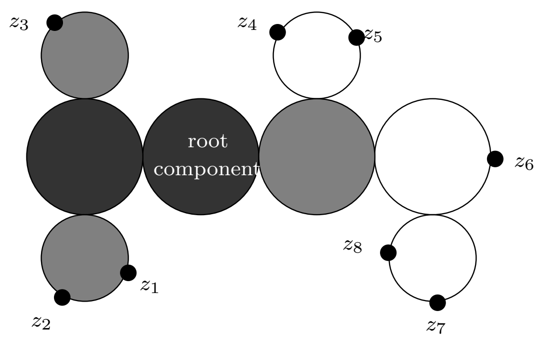
$$(\hat{C} \cong C) \implies (\mathbb{P}(\omega_{\hat{C}/C} \oplus \mathcal{O}_{\hat{C}}) \cong C \times \mathbb{P}^1).$$

In this case a scaling δ is a section $C \to \mathbb{P}^1$, and δ is required to be constant. Thus the space of scalings on an unmarked, irreducible curve is $\mathbb{P}^1$.

Scalings on nodal curves with markings are required to satisfy the following properties. First, δ should satisfy the *affinization* property that on any component $\hat{C}_i$ of $\hat{C}$ on which δ is finite and non-zero, δ has no zeroes. In particular, this implies that in the case $\hat{C} \cong C$, then δ is a constant section as in the last paragraph, while on any component $\hat{C}_i$ of $\hat{C}$ with finite non-zero scaling which maps to a point in C, δ has a single double pole and so defines an affine structure on the complement of the pole. To define the second property, note that any morphism $\hat{C} \to C$ of class $[C]$ defines a *rooted tree* whose vertices are components $\hat{C}_i$ of $\hat{C}$, whose edges are nodes $w_j \in \hat{C}$, and whose root vertex is the vertex corresponding to the component $\hat{C}_0$ that maps isomorphically to C. Let $\mathcal{T}$ denote the set of indices of *terminal* components $\hat{C}_i$ that meet only one other component of $\hat{C}$:

$$\mathcal{T} = \{i \mid \#\{j \neq i | \hat{C}_j \cap \hat{C}_i \neq \emptyset\} = 1\}$$

as in Figure 2. The *bubble components* are the components of $\hat{C}$ mapping to a point in C. For each terminal component $\hat{C}_i, i \in \mathcal{T}$ there is a canonical non-self-



expression $\tau((\kappa(z_1z_2)\kappa(z_3))\kappa((z_4z_5)(z_6(z_7z_8))))$

FIGURE 2. A scaled marked curve

crossing path of components $\hat{C}_{i,0} = \hat{C}_0, \ldots, \hat{C}_{i,k(i)} = \hat{C}_i$. Define a partial order on components by $\hat{C}_{i,j} \preceq \hat{C}_{i,k}$ for $j \leq k$. The *monotonicity property* requires that δ is finite and non-zero on at most one of these (gray shaded) components, say $\hat{C}_{i,f(i)}$, and

$$\delta | \hat{C}_{i,j} = \begin{cases} \infty & j < f(i) \\ 0 & j > f(i) \end{cases}. \tag{3}$$

We call $\hat{C}_{i,f(i)}$ a *transition component*. That is, the scaling δ is infinite on the components before the transition components and zero on the components after the transition components, in the ordering $\preceq$. See Figure 2. In addition the *marking property* requires that the scaling is finite at the markings:

$$\delta(z_i) < \infty, \quad \forall i = 1, \ldots, n.$$

Definition 2.1. *A* prestable scaled curve *with target a smooth projective curve C is a morphism from a prestable curve $\hat{C}$ to C of class $[C]$ equipped with a section δ and n markings $\underline{z} = (z_1, \ldots, z_n)$ satisfying the affinization, monotonicity and* *marking properties. Morphisms of prestable scaled curves are diagrams*

$$\begin{array}{ccc} \hat{C}_1 & \xrightarrow{\varphi} & \hat{C}_2 \\ \downarrow & & \downarrow \\ S_1 & \longrightarrow & S_2 \end{array}, \quad (D\varphi^*)\varphi^*(\delta_2) = \delta_1, \quad \varphi(z_{i,1}) = z_{i,2}, \quad \forall i = 1, \ldots, n$$

where the top arrow is a morphism of prestable curves and

$$D\varphi^* : \varphi^* \mathbb{P}(\omega_{\hat{C}_2/C} \oplus \mathcal{O}_{\hat{C}_2}) \to \mathbb{P}(\omega_{\hat{C}_1/C} \oplus \mathcal{O}_{\hat{C}_1})$$

is the associated morphism of projectivized relative dualizing sheaves. A scaled curve is stable *if on each bubble component $\hat{C}_i \subset \hat{C}$ (that is, component mapping to a point in C) there are at least three special points (markings or nodes),*

$$(\delta|\hat{C}_i \in \{0, \infty\}) \implies \#((\{z_i\} \cup \{w_j\}) \cap \hat{C}_i) \geq 3$$

or the scaling is finite and non-zero and there are least two special points

$$(\delta|\hat{C}_i \notin \{0, \infty\}) \implies \#((\{z_i\} \cup \{w_j\}) \cap \hat{C}_i) \geq 2.$$

Introduce the following notation for moduli spaces. Let $\overline{\mathfrak{M}}_{n,1}(C)$ denote the category of prestable n-marked scaled curves and $\overline{\mathcal{M}}_{n,1}(C)$ the subcategory of stable n-marked scaled curves.

The *combinatorial type* of a prestable marked scaled curve is defined as follows. Given such $(\hat{C}, u : \hat{C} \to C, \underline{z}, \delta)$. Let Γ be the graph whose vertex set $\mathrm{Vert}(\Gamma)$ is the set of irreducible components of C, finite edges $\mathrm{Edge}_{<\infty}(\Gamma)$ correspond to nodes, semi-infinite edges $\mathrm{Edge}_{\infty}(\Gamma)$ correspond to markings. The graph Γ equipped with the labelling of semi-infinite edges by $\{1, \ldots, n\}$ a distinguished *root vertex* $v_0 \in \mathrm{Vert}(\Gamma)$ corresponding to the root component and a set of *transition vertices* $\mathrm{Vert}^t(\Gamma) \subset \mathrm{Vert}(\Gamma)$ corresponding to the transition components. Graphically we represent a combinatorial type as a graph with transition vertices shaded by grey, and the vertices lying on three levels depending on whether they occur before or after the transition vertices. See Figure 3. Note that the combinatorial type is functorial; in particular any automorphism of prestable marked scaled curves induces an automorphism of the corresponding type, that is, an automorphism of the graph preserving the additional data.

We note that the graphical representation of the combinatorial type of a curve can be viewed as the graph of a Morse/height function on the curve. In general this gives a spider like figure with the principal component being the body of the spider. From this perspective the paths used in the monotonicity property of scalings are the legs of the spider.

Example 2.2. (a) For $n = 0$, no bubbling is possible and $\overline{\mathcal{M}}_{0,1}(C)$ is the projective line, $\overline{\mathcal{M}}_{0,1}(C) \cong \mathbb{P}^1$.
(b) For $n = 1$, $\overline{\mathcal{M}}_{1,1}(C)$ consists of configurations $\mathcal{M}_{1,1}(C) \cong C \times \mathbb{C}$ with irreducible domain and finite scaling; configurations $\overline{\mathcal{M}}_{1,1} - \mathcal{M}_{1,1}$ with one component $\hat{C}_0 \cong C$ with infinite scaling $\delta|\hat{C}_0$, and another component $\hat{C}_1$ mapping trivially to C, equipped with a one-form $\delta|\hat{C}_1$ with a double pole at the node and a marking $z_1 \in \hat{C}_1$. Thus $\overline{\mathcal{M}}_{1,1}(C) \cong C \times \mathbb{P}^1$.

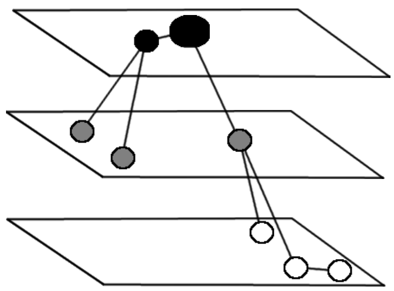


FIGURE 3. Combinatorial type of a scaled marked curve

(c) For $n = 2$, $\overline{\mathcal{M}}_{2,1}(C)$ consists of configurations $\mathcal{M}_{2,1}(C)$ with two distinct points $z_1, z_2 \in C$ and a scaling $\delta \in \mathbb{P}^1$; configurations $\mathcal{M}_{2,1,\Gamma_1}$ where the two points z_1, z_2 have come together and bubbled off onto a curve $z_1, z_2 \in \hat{C}_1$ with zero scaling $\delta|\hat{C}_1$, so that $\mathcal{M}_{2,1,\Gamma_1} \cong C \times \mathbb{P}^1$; configurations $\mathcal{M}_{2,1,\Gamma_2}$ with a root component $\hat{C}_0$ with infinite scaling $\delta|\hat{C}_0$, and two components $\hat{C}_1, \hat{C}_2$ with non-trivial scalings $\delta|\hat{C}_1, \delta|\hat{C}_2$ containing markings $z_1 \in \hat{C}_1, z_2 \in \hat{C}_2$; a stratum $\mathcal{M}_{2,1,\Gamma_3}$ of configurations with a component $\hat{C}_1$ containing two markings $z_1, z_2 \in \hat{C}_1$ and $\delta|\hat{C}_1$ non-zero; finally a stratum $\mathcal{M}_{2,1,\Gamma_3}$ containing with three components, one $\hat{C}_0$ mapping isomorphically to C; one $\hat{C}_1$ with two nodes and a one form $\delta|\hat{C}_1$ with a double pole at the node attaching to $\hat{C}_0$; and a component $\hat{C}_2$ with two markings $z_1, z_2 \in \hat{C}_2$, a node, and vanishing scaling $\delta|\hat{C}_2$. The two evaluation maps at the markings, together with the forgetful map to $\overline{\mathcal{M}}_{0,1}(C)$, define an isomorphism $\overline{\mathcal{M}}_{2,1}(C) \to C \times C \times \mathbb{P}^1$; a stratum $\mathcal{M}_{2,1,\Gamma_4}$ containing the root component $\hat{C}_0$, a component $\hat{C}_1$ with infinite scaling with three nodes, and two components $\hat{C}_2, \hat{C}_3$ with finite, non-zero scaling, each containing a node and a marking.

Remark 2.3. The extension of the one-form in a family of scaled curves may be explicitly described as follows. On each component of the limit, the one-form is determined by the limiting behavior of the product of deformation parameters for the nodes connecting that component to the root component of the limit. Let

$$\hat{C} \to S, \delta : \hat{C} \to \mathbb{P}(\omega_{\hat{C}/C\times S} \oplus \mathcal{O}_{\hat{C}}), \underline{z} : S \to \hat{C}^n$$

be a family of scaled curves over a punctured curve $S = \overline{S} - \{\infty\}$ and $\hat{C}_\infty$ a curve over ∞ extending the family $\hat{C}$. Let $\operatorname{Def}(\hat{C}_\infty)/\operatorname{Def}_\Gamma(\hat{C}_\infty)$ denote the deformation space of the curve $\hat{C}_\infty$ normal to the stratum of curves of the same combinatorial type Γ as $\hat{C}_\infty$. This normal deformation space is canonically identified with the sum of products of cotangent lines at the nodes

$$\operatorname{Def}(\hat{C}_\infty)/\operatorname{Def}_\Gamma(\hat{C}_\infty) = \sum_w T_w^\vee \hat{C}_{i_-(w)} \otimes T_w^\vee \hat{C}_{i_+(w)}$$

where $\hat{C}_{i\pm(w)}$ are components of $\hat{C}_\infty$ adjacent to w, see [4, p. 176]. Over the deformation space $\mathrm{Def}(\hat{C}_\infty)$ lives a semiversal family, universal if the curve is stable. Given family of curves $\hat{C} \to S$ as above the curve $\hat{C}$ is obtained by pull-back of the semiversal family by a map

$$S \to \sum_w T_w^\vee \hat{C}_{i_-(w)} \otimes T_w^\vee \hat{C}_{i_+(w)}, \quad z \mapsto (\delta_w(z))$$

describing the curves as local deformations (non-uniquely, since the curves themselves may be only prestable.) Let

$$\hat{C}_0 = \hat{C}_{i,0}, \ldots, \hat{C}_{i,l(i)} := \hat{C}_i$$

denote the path of components from the root component, and

$$w_{i,0}, \ldots, w_{i,l(i)-1} \in \hat{C}_\infty$$

the corresponding sequence of nodes. The nodes $w_{i,j}, w_{i,j+1}$ lie in the same component $C_{i,j+1}$ and we have a canonical isomorphism

$$T^\vee_{w_{i,j}} C_{i,j+1} \cong T_{w_{i,j+1}} C_{i,j+1}$$

corresponding to the relation of local coordinates $z_+ = 1/z_-$ near $w_{i,j}$. Deformation parameters for this chain lie in the space

$$\text{(4)} \quad \mathrm{Hom}(T^\vee_{w_{i,0}} \hat{C}_{i,0}, T^\vee_{w_{i,1}} \hat{C}_{i,1}) \oplus \mathrm{Hom}(T^\vee_{w_{i,1}} \hat{C}_{i,1}, T^\vee_{w_{i,2}} \hat{C}_{i,2}) \ldots \oplus \mathrm{Hom}(T^\vee_{w_{i,l(i)-2}} \hat{C}_{i,l(i)-2}, T^\vee_{w_{i,l(i)-1}} \hat{C}_{i,l(i)-1}).$$

In particular, the product of deformation parameters

$$\text{(5)} \quad \gamma_{w_{i,0}}(z) \cdot \ldots \cdot \gamma_{w_{i,l(i)-1}}(z) \in \mathrm{Hom}(T^\vee_{w_0} \hat{C}_{i,0}, T^\vee_{w_i} \hat{C}_{i,l(i)})$$

is well-defined. The product represents the *scale* at which the bubble component $\hat{C}_i$ forms in comparison with $\hat{C}_0 = \hat{C}_{i,0}$, that is, the ratio between the derivatives of local coordinates on $\hat{C}_i$ and $\hat{C}_0$. If z is a point in $\hat{C}_i$ then we also have a canonical isomorphism $T^\vee_z \hat{C}_i \to T_{w_{i,0}} \hat{C}_i$. The product (5) gives an isomorphism $T^\vee_z \hat{C}_i \to T^\vee_{w_0} \hat{C}_0$. The extension of δ over $\hat{C}_i$ is given by

$$\text{(6)} \quad \delta | \hat{C}_i = \lim_{z \to 0} \delta(z) (\gamma_{w_{i,0}}(z) \cdot \ldots \cdot \gamma_{w_{i,l(i)-1}}(z))$$

the ratio of the scale of the bubble component with the the parameter $\delta(z)^{-1}$. This ends the Remark.

One may view a scaled curve with infinite scaling on the root component as a nodal curve formed from the root component and a collection of bubble trees as follows.

Definition 2.4. *An* affine prestable scaled curve *consists of a tuple* $(C, \delta, \underline{z})$ *where* C *is a connected projective nodal curve,* $\delta : C \to \mathbb{P}(\omega_C \oplus \mathcal{O}_C)$ *a section of the projectivized dualizing sheaf, and* $\underline{z} = (z_0, \ldots, z_n)$ *non-singular, distinct points, such that*

(a) δ *is monotone in the following sense: For each terminal component* $\hat{C}_i, i \in \mathcal{T}$ *there is a canonical non-self-crossing path of components*

$$\hat{C}_{l(i),0} = \hat{C}_0, \ldots, \hat{C}_{i,k(i)} = \hat{C}_i.$$

For any such non-self-crossing path of components starting with a root component, that δ *is finite and non-zero on at most one of these* transition components, *say* $\hat{C}_{i,f(i)}$, *and the scaling is infinite for all components before the transition component and zero for components after the transition component:*

$$\delta|\hat{C}_{i,j} = \begin{cases} \infty & j < f(i) \\ 0 & j > f(i) \end{cases}.$$

(b) *δ is infinite at z_0, and finite at $z_1, \ldots, z_n$.*

A prestable affine scaled curve is stable *if it has finitely many automorphisms, or equivalently, if each component $C_i \subset C$ has at least three special points (markings or nodes),*

$$(\delta|C_i \in \{0, \infty\}) \implies \#((\{z_i\} \cup \{w_j\}) \cap C_i) \geq 3$$

or the scaling is finite and non-zero and there are least two special points

$$(\delta|C_i \notin \{0, \infty\}) \implies \#((\{z_i\} \cup \{w_j\}) \cap C_i) \geq 2.$$

We will see below in Theorem 2.5 that scaled marked curves have no automorphisms. Examples of stable affine scaled curves are shown in Figure 4. Denote the moduli stack of prestable affine scaled curves resp. stable affine n-marked scaled curves by $\overline{\mathfrak{M}}_{n,1}(\mathbb{A})$ resp. $\overline{\mathcal{M}}_{n,1}(\mathbb{A})$.

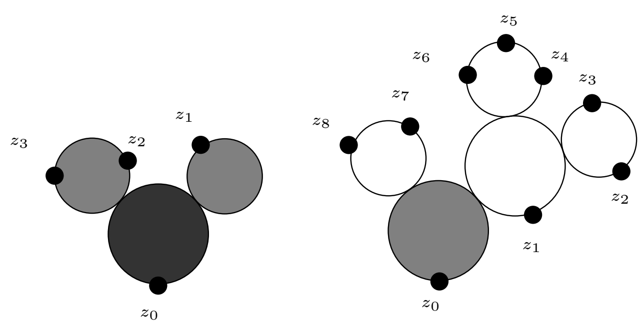


FIGURE 4. Examples of stable affine scaled curves

Theorem 2.5. *For each $n \geq 0$ and smooth projective curve C the moduli stack $\overline{\mathcal{M}}_{n,1}(C)$ resp. $\overline{\mathcal{M}}_{n,1}(\mathbb{A})$ of stable scaled affine curves is a proper scheme locally isomorphic to a product of a number of copies of C with a toric variety. The stack $\overline{\mathfrak{M}}_{n,1}(C)$ resp. $\overline{\mathfrak{M}}_{n,1}(\mathbb{A})$ of prestable scaled curves is an Artin stack of locally finite type.*

Proof. Standard arguments on imply that $\overline{\mathcal{M}}_{n,1}(C)$ and $\overline{\mathfrak{M}}_{n,1}(C)$ are stacks, that is, categories fibered in groupoids satisfying effective descent for objects and for which morphisms form a sheaf. An object $(\hat{C}, \underline{z}, \delta)$ of $\overline{\mathcal{M}}_{n,1}(C)$ over a scheme S is a family of curves with sections. Families of curves with markings and sections satisfy the gluing axioms for objects; similarly morphisms are determined uniquely by their pull-back under a covering. Standard results on hom-schemes imply that the diagonal for $\overline{\mathfrak{M}}_{n,1}(C)$, hence also $\overline{\mathcal{M}}_{n,1}(C)$, is representable, see for example [20, 1.11] for similar arguments, hence the stacks $\overline{\mathfrak{M}}_{n,1}(C)$ and $\overline{\mathcal{M}}_{n,1}(C)$ are algebraic.

In preparation for showing that $\overline{\mathcal{M}}_{n,1}(C)$ is a variety we claim that for any object $(\hat{C}, \underline{z}, \delta)$ of the moduli stack $\overline{\mathcal{M}}_{n,1}(C)$ the automorphism group is trivial.

Let Γ be the combinatorial type. The association of Γ to $(\hat{C}, \underline{z}, \delta)$ is functorial and any automorphism of $(\hat{C}, \underline{z}, \delta)$ induces an automorphism of Γ. Since the graph Γ is a tree with labelled semi-infinite edges, each vertex is determined uniquely by the partition of semi-infinite edges given by removing the vertex. Hence the automorphism acts trivially on the vertices of Γ. Each component has at least three special points, or two special points and a non-trivial scaling and so has trivial automorphism group fixing the special points. Thus the automorphism is trivial on each component of $\hat{C}$. The claim follows.

The moduli space of stable scaled curves has a canonical covering by varieties corresponding to the versal deformations of prestable curves constructed by gluing. Suppose that $(u : \hat{C} \to C, \underline{z}, \delta)$ is an object of $\overline{\mathcal{M}}_{n,1}(C)$ of combinatorial type Γ. Let $\rho : \overline{\mathcal{M}}_{n,1}(C) \to \overline{\mathcal{M}}_{0,1}(C) \cong \mathbb{P}$ denote the forgetful morphism. The locus $\rho^{-1}(\mathbb{C}) \subset \overline{\mathcal{M}}_{n,1}(C)$ of curves with finite scaling is isomorphic to $\overline{\mathcal{M}}_n(C) \times \mathbb{C}$, where the last factor denotes the scaling. In the case that the root component has infinite scaling, let $\Gamma_1, \ldots, \Gamma_k$ denote the (possibly empty) combinatorial types of the bubble trees attached at the special points. The stratum $\mathcal{M}_{n,1,\Gamma}(C)$ is the product of C^k with moduli stacks of scaled affine curves $\mathcal{M}_{n_i,1,\Gamma_i}(\mathbb{A})$ for $i = 1, \ldots, k$, each isomorphic to an affine space given by the number of markings and scalings minus the dimension of the automorphism group $(n_i+1)+1-\dim(\mathrm{Aut}(\mathbb{P}^1)) = n_i-1$ [51]. Let

$$\gamma_e \in T^\vee_{w(e)} \hat{C}_{i_-(e)} \otimes T^\vee_{w(e)} \hat{C}_{i_+(e)}, \quad e \in \mathrm{Edge}_{<\infty}(\Gamma)$$

be the deformation parameters for the nodes. A collection of deformation parameters $\underline{\gamma} = (\gamma_e)_{e \in \mathrm{Edge}(\Gamma)}$ is *balanced* if the signed product

$$\prod_{e \in P} \gamma_e^{\pm 1} \tag{7}$$

of parameters corresponding to any non-self-crossing path P between transition components is equal to 1, where the sign is positive for edges pointing towards the root vertex and equal to -1 if the edge is oriented away from it. Let Z_Γ denote the set of deformation parameters satisfying the condition (7). Then there is a morphism

$$\mathcal{M}_{n,1,\Gamma}(C) \times Z_\Gamma \to \overline{\mathcal{M}}_{n,1}(C)$$

described as follows. Choose local étale coordinates $z_e^\pm$ on the adjacent components to each node $w_e, \in \mathrm{Edge}_{<\infty}(\Gamma)$ and glue together the components using the identifications $z_e^+ \mapsto \gamma_e / z_e^-$, see for example [4, p. 176], [57, 2.2]. Set the scaling on the root component

$$\delta = \prod_{e \in P} \gamma_e$$

where P is a path of nodes from the root component to the transition component, independent of the choice of component by (7). This gives a family $(\hat{C}, u, \delta, \underline{z})$ of stable scaled curves over $\mathcal{M}_{n,1,\Gamma}(C) \times Z_\Gamma$ and hence a morphism to $\overline{\mathcal{M}}_{n,1}(C)$. The family $(\hat{C}, \underline{z}, u, \delta)$ defines a universal deformation of any curve of type Γ. Indeed, $(\hat{C}, \underline{z})$ is a versal deformation of any of its prestable fibers by [4], and it follows that the family $(\hat{C}, \underline{z}, u)$ is a versal deformation of any of its fibers since there is a unique extension of the stable map on the central fiber, up to automorphism. The equation (5) implies that any family of stable scaled curves satisfies the balanced relation (7) between the deformation parameters for any family of marked curves with scalings. This provides a cover of $\overline{\mathcal{M}}_{n,1}(C)$ by varieties. It follows that $\overline{\mathcal{M}}_{n,1}(C)$ is a variety.

The stack of prestable scaled curves $\overline{\mathfrak{M}}_{n,1}(C)$ is an Artin stack of locally finite type. Charts for the stack $\overline{\mathfrak{M}}_{n,1}(C)$, as in the case of prestable curves in [5], are given by using forgetful morphisms $\overline{\mathcal{M}}_{n+k,1}(C) \to \overline{\mathfrak{M}}_{n,1}(C)$. Since these morphisms admit sections locally, they provide a smooth covering of $\overline{\mathfrak{M}}_{n,1}(C)$ by varieties.

We check the valuative criterion for properness for $\overline{\mathcal{M}}_{n,1}(C)$. Given a family of stable scaled marked curves over a punctured curve S with finite scaling δ

$$(\hat{C}, u : \hat{C} \to C, \underline{z}, \delta) \to S = \overline{S} - \{\infty\}$$

we wish to construct there exists an extension over $\overline{S}$. We consider only the case $\hat{C} \cong C \times S$; the general case is similar. After forgetting the scaling δ and stabilizing we obtain a family of stable maps to C of degree $[C]$,

$$(\hat{C}^{\mathrm{st}}, u : \hat{C}^{\mathrm{st}} \to C, \underline{z}^{\mathrm{st}}) \to \overline{S} - \{\infty\}.$$

By properness of the stack $\overline{\mathcal{M}}_n(C)$ of stable maps to C, this family extends over the central fiber ∞ to give a family over $\overline{S}$. The section δ of $\omega_{\hat{C}^{\mathrm{st}}/C}$ defines an extension over $\overline{S}$ except possibly at the nodes. Here there are possible irremovable singularities corresponding to the following situation: suppose that $\hat{C}_0, \ldots \hat{C}_i$ is a chain of components in the curve at the central fiber, with $\hat{C}_0 \cong C$ the root component. Suppose that $\hat{C}_i, \hat{C}_{i+1}$ are adjacent component with δ infinite on $\hat{C}_i$ and zero on $\hat{C}_{i+1}$. Taking the closure of the graph of δ gives a family $\hat{C}$ of curves over C given by replacing some of the nodes of $\hat{C}^{\mathrm{st}}$ with fibers of $\mathbb{P}(\omega_{\hat{C}^{\mathrm{st}}/C} \oplus \mathcal{O}_{\hat{C}^{\mathrm{st}}})$ over the node. The relative cotangent bundle of $\hat{C}$ is related to that of $\hat{C}^{\mathrm{st}}$ by a twist at D_0, D_∞: If $\pi : \hat{C} \to \hat{C}^{\mathrm{st}}$ denotes the projection onto $\hat{C}$ then on the components of $\hat{C}$ collapsed by π we have

$$\omega_{\hat{C}/C \times S} = \pi^* \omega_{\hat{C}^{\mathrm{st}}/C}(-D_0 - D_\infty)$$

where D_0, D_∞ are the inverse images of the sections at zero and infinity in $\mathbb{P}(\omega_{\hat{C}^{\mathrm{st}}/C} \oplus \mathcal{O}_{\hat{C}^{\mathrm{st}}})$. Abusing notation $\omega_{\hat{C}_i^{\mathrm{st}}/C}(-D_0) = \omega_{\hat{C}_i^{\mathrm{st}}/C}$ resp. $\omega_{\hat{C}_i^{\mathrm{st}}/C}(-D_\infty) = \omega_{\hat{C}_i^{\mathrm{st}}/C}$ on components $\hat{C}_i^{\mathrm{st}}$ contained in D_0 resp. D_∞. The extension of δ to a rational section of $\pi^* \omega_{\hat{C}^{\mathrm{st}}/C}$ has, by definition a zero at $\delta^{-1}(D_0)$ and a pole at $\delta^{-1}(D_\infty)$. Hence the extension of δ to a section of $\pi^* \omega_{\hat{C}^{\mathrm{st}}/C}(-D_0 - D_\infty)$ has no zeroes at D_0 and a double pole at D_∞. This implies that δ extends uniquely as a section of $\mathbb{P}(\omega_{\hat{C}/C \times S} \oplus \mathcal{O}_{\hat{C}})$ to all of $\overline{S}$.

By the construction (6), the extension of δ satisfies the monotonicity condition (3). Indeed suppose that a component $\hat{C}_i$ is further away from a component $\hat{C}_j$ in the path of components from the root component $\hat{C}_0$. Since all deformation parameters $\gamma_{w_{i,k}}(z)$ are approaching zero, from (6), at most one of the limits $\delta | \hat{C}_i, \delta | \hat{C}_j$ can be finite, and

$$\begin{cases} \delta | \hat{C}_i \text{ finite} \implies \delta | \hat{C}_j \text{ infinite} \\ \delta | \hat{C}_j \text{ finite} \implies \delta | \hat{C}_i \text{ zero.} \end{cases}.$$

The condition (3) follows. ☐

Remark 2.6. Although we will not need it, a projective embedding of the moduli space of stable scaled curves is defined via canonical forgetful and evaluation morphisms for the moduli space:

$$f_I : \overline{\mathcal{M}}_{n,1}(C) \to \overline{\mathcal{M}}_{0,4}, \quad I \subset \{1,\ldots,n\}, \quad |I| = 4$$

and

$$\mathrm{ev}_i : \overline{\mathcal{M}}_{n,1}(C) \to C, \quad \rho : \overline{\mathcal{M}}_{n,1}(C) \to \overline{\mathcal{M}}_{0,1}(C) \cong \mathbb{P}^1.$$

Taking the product of these maps defines an embedding of $\overline{\mathcal{M}}_{n,1}(C)$ into a product of lower dimensional moduli spaces:

$$\overline{\mathcal{M}}_{n,1}(C) \to \left(\prod_{I \subset \{1,\ldots,n\}, |I|=4} \overline{\mathcal{M}}_{0,4} \right) \times C^n \times \mathbb{P}^1.$$

That this is an embedding may be checked as in [51, Theorem 10.3]. In particular, this implies that the moduli space is proper without checking the valuative criterion. This ends the Remark.

Remark 2.7. The basic divisor equivalences used in the proof of the cobordism (1) are the following. Recall the proof of associativity of the quantum product via divisor relations in $\overline{\mathcal{M}}_{0,n+1}$. The moduli stack of genus 0 curves with $n+1$ markings

$$\overline{\mathcal{M}}_{0,n+1} = \bigcup_{\Gamma} \mathcal{M}_{0,n+1,\Gamma}$$

admits a stratification into strata indexed by parenthesized expressions in n variables $a_1, \ldots, a_n$ in a commutative algebra A. The basic divisor class relation implying associativity in $\overline{\mathcal{M}}_{0,4}$ corresponds to an expression $(a_1a_2)a_3 \sim a_1(a_2a_3)$ corresponding to the picture in Figure 1. This relation pulls back to a relation

$$\begin{aligned} (8) \quad & \left[\cup_{d_1+d_2=d} \cup_{1,2\in I_1, 0,3 \in I_2} \overline{\mathcal{M}}_{0,I_1}(X,d_1) \times_X \overline{\mathcal{M}}_{0,I_2}(X,d_2)\right] \\ & \qquad \sim \left[\cup_{d_1+d_2=d} \cup_{2,3\in I_1, 0,1 \in I_2} \overline{\mathcal{M}}_{0,I_1}(X,d_1) \times_X \overline{\mathcal{M}}_{0,I_2}(X,d_2)\right] \end{aligned}$$

(where $I_1 \cup I_2$ is a partition of $\{0,\ldots,n\}$) in the moduli stack of stable maps $\overline{\mathcal{M}}_{0,n+1}(X,d)$. A standard argument using this relation implies the associativity of the quantum product [5, Theorem 6].

The divisor class relations imply the commutativity of the diagram (2) as follows. In the moduli space of scaled curves there is a stratification

$$\overline{\mathcal{M}}_{n,1}(C) = \bigcup_{\Gamma} \mathcal{M}_{n,1,\Gamma}(C)$$

in which the strata correspond to parenthesized expressions in symbols τ, κ and $a_1, \ldots, a_n$, such as $\tau(\kappa(a_1a_2)\kappa(a_3))$ etc [68]. In particular, there is a divisor class relation in $\overline{\mathcal{M}}_{0,1}(C)$ corresponding to the relation $(\tau\kappa)(a) \sim \tau(\kappa(a))$ which pulls back to a divisor class relation in $\overline{\mathcal{M}}_{n,1}(C)$. The trivialization of the relative canonical sheaf defines a canonical isomorphism $\overline{\mathcal{M}}_{0,1}(C) \cong \mathbb{P}^1$. The quantum Witten formula is obtained from the equivalence between the divisors corresponding to the configurations in Figure 5.

There is a similar stratification in the moduli space of scaled affine curves which gives a homomorphism property for the quantum Kirwan map. The stratification by combinatorial type is

$$\overline{\mathcal{M}}_{n,1}(\mathbb{A}) = \bigcup_{\Gamma} \mathcal{M}_{n,1,\Gamma}(\mathbb{A})$$

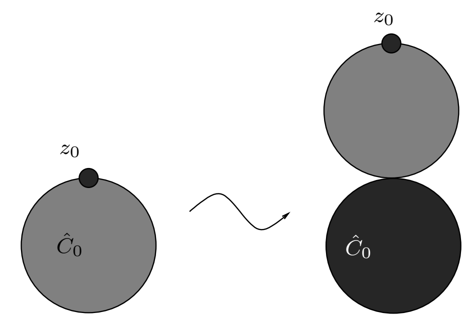


FIGURE 5. Linear equivalence in $\overline{\mathcal{M}}_{1,1}(C)$

and the strata correspond to parenthesized expressions in symbols κ and $a_1, \ldots, a_n$ [68]. There is a canonical identification

$$\overline{\mathcal{M}}_{2,1}(\mathbb{A}) \cong \mathbb{P}^1, \quad [z_1, z_2, \delta] \mapsto \delta(z_2 - z_1)$$

that is, the difference $z_2 - z_1$ in the affine coordinate defined by the scaling δ. The two singular points in $\overline{\mathcal{M}}_{2,1}(\mathbb{A})$ corresponding to the bubble trees in Figure 6 give linearly equivalent divisors corresponding to an expression $\kappa(a_1a_2) \sim \kappa(a_1)\kappa(a_2)$. The divisor class relation in $\overline{\mathcal{M}}_{2,1}(\mathbb{C})$ pulls back to a divisor class relation in any $\overline{\mathcal{M}}_{n,1}(\mathbb{A})$ implying the homomorphism property of the linearized quantum Kirwan map [68]. This ends the Remark.

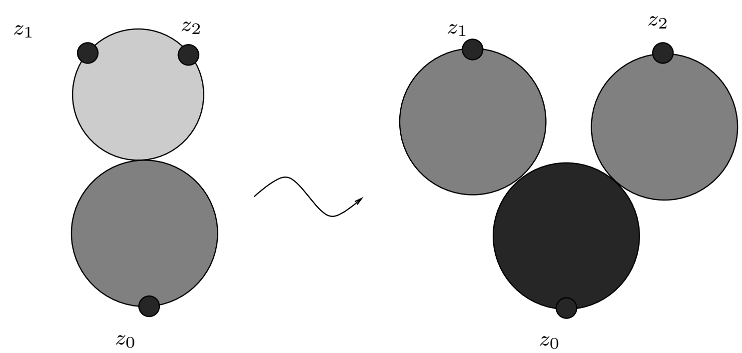


FIGURE 6. Strata in affine scaled curves with two markings

3. MUMFORD STABILITY

In this section we review the relationship between the stack-theoretic quotient and Mumford's geometric invariant theory quotient [54]. First we introduce various Lie-theoretic notation. Let G be a connected complex reductive group with Lie algebra $\mathfrak{g}$. If G is a torus denote by

$$\mathfrak{g}_{\mathbb{Z}} = \{D\phi(1) \in \mathfrak{g} \mid \phi \in \operatorname{Hom}(\mathbb{C}^\times, G)\}, \quad \mathfrak{g}_{\mathbb{Q}} = \mathfrak{g}_{\mathbb{Z}} \otimes_{\mathbb{Z}} \mathbb{Q}$$

the *coweight lattice* of derivatives of one-parameter subgroups resp. rational one-parameter subgroups. Dually denote by

$$\mathfrak{g}_{\mathbb{Z}}^{\vee} = \{D\chi \in \mathfrak{g}^{\vee} \mid \chi \in \operatorname{Hom}(G, \mathbb{C}^{\times})\}, \quad \mathfrak{g}_{\mathbb{Q}}^{\vee} = \mathfrak{g}_{\mathbb{Z}}^{\vee} \otimes_{\mathbb{Z}} \mathbb{Q}$$

the *weight lattice* of derivatives of characters of G and the set of *rational weights*, respectively. More generally if G is non-abelian then we still denote by $\mathfrak{g}_{\mathbb{Q}}$ the set of derivatives of rational one-parameter subgroups.

The targets of our maps are quotient stacks defined as follows. Let X be a smooth projective G-variety. Let X/G denote the quotient stack, that is, the category fibered in groupoids whose fiber over a scheme S has objects pairs $v = (P, u)$ consisting of a principal G-bundle $P \to S$ and a section $u : S \to P \times_G X$; and whose morphisms are given by diagrams

$$\begin{array}{ccc} P_1 & \xrightarrow{\phi} & P_2 \\ \downarrow & & \downarrow \\ S_1 & \xrightarrow[\psi]{} & S_2 \end{array}, \quad \phi(X) \circ u_1 = u_2 \circ \psi$$

where $\phi(X) : P_1(X) \to P_2(X)$ denotes the map of associated fiber bundles [20], Tag 04UV [19].

Mumford's geometric invariant theory quotient [54] is traditionally defined as the projective variety associated to the graded ring of invariant sections of a linearization of the action in the previous paragraph. Let $\tilde{X} \to X$ be a linearization, that is, ample G-line bundle. Then

$$X /\!\!/ G := \operatorname{Proj}\left(\oplus_{k \geq 0} H^0(\tilde{X}^k)^G\right).$$

Mumford [54] realizes this projective variety as the quotient of a *semistable locus* by an equivalence relation. The semistable locus consists of points $x \in X$ such that some tensor power $\tilde{X}^k, k > 0$ of $\tilde{X}$ has an invariant section non-vanishing at x, while the unstable locus is the complement of the semistable locus:

$$X^{\mathrm{ss}} = \{x \in X \mid \exists k > 0, \sigma \in H^0(\tilde{X}^k)^G, \quad \sigma(x) \neq 0\}, \quad X^{\mathrm{us}} := X - X^{\mathrm{ss}}.$$

A point $x \in X$ is *polystable* if its orbit is closed in the semistable locus $\overline{Gx \cap X^{\mathrm{ss}}} = Gx \cap X^{\mathrm{ss}}$. A point $x \in X$ is *stable* if it is polystable and the stabilizer G_x of x is finite. In Mumford's definition the git quotient is the quotient of the semistable locus by the *orbit equivalence relation*

$$(x_1 \sim x_2) \iff \overline{Gx_1} \cap \overline{Gx_2} \cap X^{\mathrm{ss}} \neq \emptyset.$$

Each semistable point is then orbit-equivalent to a unique polystable point. However, here we define the git quotient as the stack-theoretic quotient

$$X /\!\!/ G := X^{\mathrm{ss}}/G.$$

We shall always assume that X^{ss}/G is a Deligne-Mumford stack (that is, the stabilizers G_x are finite) in which case the coarse moduli space of X^{ss}/G is the git quotient in Mumford's sense. The Luna slice theorem [49] implies that X^{ss}/G is étale-locally the quotient of a smooth variety by a finite group, and so has finite diagonal. By the Keel-Mori theorem [43], explicitly stated in [18, Theorem 1.1], the morphism from X^{ss}/G to its coarse moduli space is proper. Since the coarse moduli space of X^{ss}/G is projective by Mumford's construction, it is proper, hence X^{ss}/G is proper as well.

Later we will need the following observation about the unstable locus. As the quotient $X/\!\!/G$ is non-empty, there exists an ample divisor D containing the unstable locus: take D to be the vanishing locus of any non-zero invariant section of $\tilde{X}^k$ for some $k > 0$:

$$D = \sigma^{-1}(0), \quad \sigma \in H^0(\tilde{X}^k)^G - \{0\}. \tag{9}$$

The Hilbert-Mumford numerical criterion [54, Chapter 2] provides a computational tool to determine the semistable locus: A point $x \in X$ is G-semistable if and only if it is $\mathbb{C}^\times$-semistable for all one-parameter subgroups $\mathbb{C}^\times \to G$. Given an element $\lambda \in \mathfrak{g}_{\mathbb{Z}}$ denote the corresponding one-parameter subgroup $\mathbb{C}^\times \to G,\ z \mapsto z^\lambda$. Denote by

$$x_\lambda := \lim_{z \to 0} z^\lambda x$$

the limit under the one-parameter subgroup. Let $\mu(x, \lambda) \in \mathbb{Z}$ be the weight of the linearization $\tilde{X}$ at x_λ defined by

$$z\tilde{x} = z^{\mu(x,\lambda)}\tilde{x}, \quad \forall z \in \mathbb{C}^\times, \tilde{x} \in \tilde{X}_{x_\lambda}.$$

By restricting to the case of a projective line one sees that the point $x \in X$ is semistable if and only if $\mu(x, \lambda) \leq 0$ for all $\lambda \in \mathfrak{g}_{\mathbb{Z}}$. Polystability is equivalent to semistability and the additional condition $\mu(x, \lambda) = 0 \iff \mu(x, -\lambda) = 0$. Stability is the condition that $\mu(x, \lambda) < 0$ for all $\lambda \in \mathfrak{g}_{\mathbb{Z}} - \{0\}$.

The Hilbert-Mumford numerical criterion can be applied explicitly to actions on projective spaces as follows. Suppose that G is a torus and $X = \mathbb{P}(V)$ the projectivization of a vector space V. Let $\tilde{X} = \mathcal{O}_X(1) \otimes \mathbb{C}_\theta$ be the G-equivariant line bundle given by tensoring the hyperplane bundle $\mathcal{O}_X(1)$ and the one-dimensional representation $\mathbb{C}_\theta$ corresponding to some weight $\theta \in \mathfrak{g}_{\mathbb{Z}}^\vee$. Recall if $p \in X$ is represented by a line $l \subset V$ then the fiber of $\mathcal{O}_X(1) \otimes \mathbb{C}_\theta$ at p is $l^\vee \otimes \mathbb{C}_\theta$. In particular if z^λ fixes p then z^λ scales l by some $z^{\mu(\lambda)}$ so that $z^\lambda \tilde{x} = z^{-\mu(\lambda)+\theta(\lambda)}\tilde{x}$, for $\tilde{x} \in l^\vee \otimes \mathbb{C}_\theta$. Let $k = \dim(V)$ and decompose V into weight spaces $V_1, \ldots, V_k$ with weights $\mu_1, \ldots, \mu_k \in \mathfrak{g}_{\mathbb{Z}}^\vee$. Identify

$$H^2_G(X) \cong H^2_{\mathbb{C}^\times \times G}(V) \cong \mathbb{Z} \oplus \mathfrak{g}_{\mathbb{Z}}^\vee$$

Under this splitting the first Chern class $c_1^G(\tilde{X})$ becomes identified up to positive scalar multiple with the pair

$$c_1^G(\tilde{X}) \mapsto (1, \theta) \in \mathbb{Z} \oplus \mathfrak{g}_{\mathbb{Z}}^\vee. \tag{10}$$

The following is essentially [54, Proposition 2.3].

Lemma 3.1. *The semistable locus for the action of a torus G on the projective space $X = P(V)$ with weights $\mu_1, \ldots, \mu_k$ and linearization shifted by θ is $X^{\mathrm{ss}} = \mathbb{P}(V)^{\mathrm{ss}} = \{[x_1, \ldots, x_k] \in \mathbb{P}(V) \mid \mathrm{hull}(\{\mu_i | x_i \neq 0\}) \ni \theta\}$. A point x is polystable iff θ lies in the interior of the hull above, and stable if in addition the hull is of maximal dimension.*

Proof. The Hilbert-Mumford weights are computed as follows. For any non-zero $\lambda \in \mathfrak{g}_{\mathbb{Z}}$, let

$$\nu(x, \lambda) := \min_i \{-\mu_i(\lambda), x_i \neq 0\}.$$

Then

$$\begin{aligned} z^\lambda[x_1,\ldots,x_k] &= [z^{\mu_1(\lambda)}x_1,\ldots,z^{\mu_k(\lambda)}x_k] \\ &= [z^{\mu_1(\lambda)+\nu(x,\lambda)}x_1,\ldots,z^{\mu_k(\lambda)+\nu(x,\lambda)}x_k] \end{aligned}$$

and

$$(-\mu_i(\lambda)\neq\nu(x,\lambda)) \implies \left(\lim_{z\to 0} z^{\mu_i(\lambda)+\nu(x,\lambda)}=0\right).$$

Let

$$x_\lambda := \lim_{z\to 0} z^\lambda x = \lim_{z\to 0}[z^{\mu_i(\lambda)}x_i]_{i=1}^k \in X$$

Then

$$x_\lambda = [x_{\lambda,1},\ldots,x_{\lambda,k}], \quad x_{\lambda,i} = \begin{cases} x_i & -\mu_i(\lambda)=\nu(x,\lambda) \\ 0 & \text{otherwise} \end{cases}.$$

The Hilbert-Mumford weight is therefore

$$\mu(x,\lambda)=\nu(x,\lambda)+(\theta,\lambda). \tag{11}$$

By the Hilbert-Mumford criterion, the point x is semistable if and only if

$$\nu(x,\lambda) := \min\{-\mu_i(\lambda)\mid x_i\neq 0\} \leq (-\theta,\lambda), \quad \forall\lambda\in\mathfrak{g}_\mathbb{Z}-\{0\}.$$

That is,

$$(x\in X^{\rm ss}) \iff (\theta\in\mathrm{hull}\{\mu_i\mid x_i\neq 0\}).$$

This proves the claim about the semistable locus. To prove the claim about polystability, note that $\mu(x,\lambda)=0=\mu(x,-\lambda)$ implies that the minimum $\nu(x,\lambda)$ is also the maximum. Thus the only affine linear functions $\xi:\mathfrak{g}^\vee\to\mathbb{R}$ which vanish at θ are those ξ that are constant on the hull of μ_i with x_i nonzero. This implies that the span of μ_i with x_i non-zero contains θ in its relative interior. The stabilizer G_x of x has Lie algebra $\mathfrak{g}_x$ the annihilator of the span of the hull of the μ_i with $x_i\neq 0$. So the stabilizer G_x is finite if and only if the span of μ_i with $x_i\neq 0$ is of maximal dimension $\dim(G)$. This implies the claim on stability. □

We introduce the following notation for weight and coweight lattices. As above G is a connected complex reductive group with maximal torus T and $\mathfrak{g},\mathfrak{t}$ are the Lie algebras of G,T respectively. Fix an invariant inner product $(\ ,\ ):\mathfrak{g}\times\mathfrak{g}\to\mathbb{C}$ on $\mathfrak{g}$ inducing an identification $\mathfrak{g}\to\mathfrak{g}^\vee$. By taking a multiple of the basic inner product on each factor we may assume that the inner product induces an identification $\mathfrak{t}_\mathbb{Q}\to\mathfrak{t}_\mathbb{Q}^\vee$. Denote by

$$\|\cdot\|:\mathfrak{q}_\mathbb{Q}\to\mathbb{R}_{\geq 0}, \quad \|\xi\|:=(\xi,\xi)^{1/2}$$

the norm with respect to the induced metric.

Next recall the theory of Levi decompositions of parabolic subgroups from Borel [10, Section 11]. A parabolic subgroup Q of G is one for which G/Q is complete, or equivalently, containing a maximal solvable subgroup $B\subset G$. Any parabolic Q admits a Levi decomposition $Q=L(Q)U(Q)$ where $L(Q)$ denote a maximal reductive subgroup of Q and $U(Q)$ is the maximal unipotent subgroup. Let $\mathfrak{l}(Q),\mathfrak{u}(Q)$ denote the Lie algebras of $L(Q),U(Q)$. Let $\mathfrak{g}=\mathfrak{t}\oplus\bigoplus_{\alpha\in R(G)}\mathfrak{g}_\alpha$ denote the root space decomposition of $\mathfrak{g}$, where $R(G)$ is the set of roots. The Lie algebras $\mathfrak{l}(Q),\mathfrak{u}(Q)$ decompose into root spaces as

$$\mathfrak{q}=\mathfrak{t}\oplus\bigoplus_{\alpha\in R(Q)}\mathfrak{g}_\alpha, \quad \mathfrak{l}(Q)=\mathfrak{t}\oplus\bigoplus_{\alpha\in R(Q)\cap -R(Q)}\mathfrak{g}_\alpha, \quad \mathfrak{u}(Q)=\mathfrak{q}/\mathfrak{l}(Q)$$

where $R(Q) \subset R(G)$ is the set of roots for $\mathfrak{l}(Q)$. Let $\mathfrak{z}(Q)$ denote the center of $\mathfrak{l}(Q)$ and

$$\mathfrak{z}_+(Q) = \{\xi \in \mathfrak{z}(Q) \mid \alpha(\xi) \geq 0, \ \forall \alpha \in R(Q)\}$$

the *positive chamber* on which the roots of Q are non-negative. The Levi decomposition induces a homomorphism

$$\pi_Q : Q \to Q/U(Q) \cong L(Q). \tag{12}$$

This homomorphism has the following alternative description as a limit. Let $\lambda \in \mathfrak{z}_+(Q) \cap \mathfrak{g}_{\mathbb{Q}}$ be a positive coweight and

$$\phi_\lambda : \mathbb{C}^\times \to L(Q), \quad z \mapsto z^\lambda$$

the corresponding central one-parameter subgroup. Then

$$\pi_Q(g) = \lim_{z \to 0} \mathrm{Ad}(z^\lambda) g.$$

In the case of the general linear group in which the parabolic consists of block-upper-triangular matrices, this limit projects out the off-block-diagonal terms.

The unstable locus admits a stratification by maximally destabilizing subgroups, as in Hesselink [42], Kirwan [44], and Ness [56]. The stratification reads

$$X = \bigcup_{\lambda \in \mathcal{C}(X)} X_\lambda, \quad X_\lambda = G \times_{Q_\lambda} Y_\lambda, \quad Y_\lambda \mapsto Z_\lambda \text{ affine fibers} \tag{13}$$

where $Y_\lambda, Z_\lambda, Q_\lambda, \mathcal{C}(X)$ are defined as follows. For each fixed point component $\overline{Z}_\lambda$ of z^λ there exist a weight $\mu(\lambda)$ so z^λ acts on $\tilde{X}|\overline{Z}_\lambda$ with weight $\mu(\lambda)$:

$$z^\lambda \tilde{x} = z^{\mu(\lambda)} \tilde{x}, \quad \forall \tilde{x} \in \tilde{X}|\overline{Z}_\lambda.$$

The quotient $G_\lambda/\mathbb{C}^\times_\lambda$ acts on $\overline{Z}_\lambda$ and we denote by Z_λ the semistable locus. Define

$$\mathcal{C}(X) = \{\lambda \in \mathfrak{t}_+ \mid \exists Z_\lambda, \ \mu(\lambda) = (\lambda, \lambda)\} \tag{14}$$

using the metric, where $\mathfrak{t}_+$ is the closed positive Weyl chamber. The variety Y_λ is the set of points that flow to Z_λ under $z^\lambda, z \to 0$:

$$Y_\lambda = \left\{ x \in X \mid \lim_{z \to 0} z^\lambda x \in Z_\lambda \right\}$$

The group Q_λ is the parabolic of group elements that have a limit under $\mathrm{Ad}(z^\lambda)$ as $z \to 0$:

$$Q_\lambda = \left\{ g \in G \mid \exists \lim_{z \to 0} \mathrm{Ad}(z^\lambda) g \in G \right\}.$$

Then Y_λ is a Q_λ-variety; and X_λ is the flow-out of Y_λ under G. By taking quotients we obtain a stratification of the quotient stack by locally-closed substacks

$$X/G = \bigcup_{\lambda \in \mathcal{C}(X)} X_\lambda/G.$$

This stratification was used in Teleman [64] to give a formula for the sheaf cohomology of bundles on the quotient stack.

4. Kontsevich stability

In this section we recall the definition of Kontsevich's moduli stacks of stable maps [45] as generalized to orbifold quotients by Chen-Ruan [12] and in the algebraic setting by Abramovich-Graber-Vistoli [2]. Let X be a smooth projective variety. Recall that a *prestable map* with target X consists of a prestable curve $C \to S$, a morphism $u : C \to X$, and a collection $z_1, \ldots, z_n : S \to C$ of distinct non-singular points called *markings*. An automorphism of a prestable map $(C, u, \underline{z})$ is an automorphism

$$\varphi : C \to C, \quad \varphi \circ u = u, \quad \varphi(z_i) = z_i, \quad i = 1, \ldots, n.$$

A prestable map $(C, u, \underline{z})$ is *stable* if the number $\# \operatorname{Aut}(C, u, \underline{z})$ of automorphisms is finite. For $d \in H_2(X, \mathbb{Z})$ we denote by $\overline{\mathcal{M}}_{g,n}(X, d)$ the moduli stack of stable maps $(C, u, \underline{z})$ of genus $g = \operatorname{genus}(C)$ and class $d = v_*[C]$ with n markings.

The notion of stable map generalizes to orbifolds [12], [2] as follows. These definitions are needed for the construction of the moduli stack of affine gauged maps in the case that the git quotient is an orbifold, but not if the quotient is free. First we recall the notion of twisted curve:

Definition 4.1. (Twisted curves) *Let S be a scheme. An n-*marked twisted curve *over S is a collection of data $(f : \mathcal{C} \to S, \{\ddagger_i \subset \mathcal{C}\}_{i=1}^n)$ such that*

(a) (Coarse moduli space) *$\mathcal{C}$ is a proper stack over S whose geometric fibers are connected of dimension 1, and such that the coarse moduli space of $\mathcal{C}$ is a nodal curve C over S.*
(b) (Markings) *The $\ddagger_i \subset \mathcal{C}$ are closed substacks that are gerbes over S, and whose images in C are contained in the smooth locus of the morphism $C \to S$.*
(c) (Automorphisms only at markings and nodes) *If $\mathcal{C}^{ns} \subset \mathcal{C}$ denotes the* non-special locus *given as the complement of the $\ddagger_i$ and the singular locus of $\mathcal{C} \to S$, then $\mathcal{C}^{ns} \to C$ is an open immersion.*
(d) (Local form at smooth points) *If $p \to C$ is a geometric point mapping to a smooth point of C, then there exists an integer r, equal to 1 unless p is in the image of some $\ddagger_i$, an étale neighborhood* $\operatorname{Spec}(R) \to C$ *of p and an étale morphism* $\operatorname{Spec}(R) \to \operatorname{Spec}_S(\mathcal{O}_S[x])$ *such that the pull-back* $\mathcal{C} \times_C \operatorname{Spec}(R)$ *is isomorphic to* $\operatorname{Spec}(R[z]/z^r = x)/\mu_r$.
(e) (Local form at nodal points) *If $p \to C$ is a geometric point mapping to a node of C, then there exists an integer r, an étale neighborhood* $\operatorname{Spec}(R) \to C$ *of p and an étale morphism* $\operatorname{Spec}(R) \to \operatorname{Spec}_S(\mathcal{O}_S[x, y]/(xy - t))$ *for some $t \in \mathcal{O}_S$ such that the pull-back $\mathcal{C} \times_C \operatorname{Spec}(R)$ is isomorphic to* $\operatorname{Spec}(R[z, w]/zw - t', z^r - x, w^r - y)/\mu_r$ *for some $t' \in \mathcal{O}_S$.*

Next we recall the notion of twisted stable maps. Let $\mathcal{X}$ be a proper Deligne-Mumford stack with projective coarse moduli space X. Algebraic definitions of twisted curve and twisted stable map to a $\mathcal{X}$ are given in Abramovich-Graber-Vistoli [2], Abramovich-Olsson-Vistoli [3], and Olsson [57].

Definition 4.2. *A* twisted stable map *from an n-marked twisted curve $(\pi : \mathcal{C} \to S, (\ddagger_i \subset \mathcal{C})_{i=1}^n)$ over S to $\mathcal{X}$ is a representable morphism of S-stacks $u : \mathcal{C} \to \mathcal{X}$ such that the induced morphism on coarse moduli spaces $u_c : C \to X$ is a stable map in the sense of Kontsevich from the n-pointed curve $(C, \underline{z} = (z_1, \ldots, z_n))$ to*

X, where z_i is the image of $\ddagger_i$. The homology class *of a twisted stable curve is the homology class $u_*[\mathcal{C}_s] \in H_2(X, \mathbb{Q})$ of any fiber $\mathcal{C}_s$.*

Twisted stable maps naturally form a 2-category. Every 2-morphism is unique and invertible if it exists, and so this 2-category is naturally equivalent to a 1-category which forms a stack over schemes [2].

Theorem 4.3. *([2, 4.2]) The stack $\overline{\mathcal{M}}_{g,n}(\mathcal{X})$ of twisted stable maps from n-pointed genus g curves into $\mathcal{X}$ is a Deligne-Mumford stack. If $\mathcal{X}$ is proper, then for any $c > 0$ the union of substacks $\overline{\mathcal{M}}_{g,n}(\mathcal{X}, d)$ with homology class $d \in H_2(\mathcal{X}, \mathbb{Q})$ satisfying $(d, c_1(\tilde{X})) < c$ is proper.*

The Gromov-Witten invariants takes values in the cohomology of the *inertia stack*

$$\mathcal{I}_{\mathcal{X}} := \mathcal{X} \times_{\mathcal{X} \times \mathcal{X}} \mathcal{X}$$

where both maps are the diagonal. The objects of $\mathcal{I}_{\mathcal{X}}$ may be identified with pairs (x, g) where $x \in \mathcal{X}$ and $g \in \mathrm{Aut}_{\mathcal{X}}(x)$. For example, if $\mathcal{X} = X/G$ is a global quotient by a finite group then

$$\mathcal{I}_{\mathcal{X}} = \bigcup_{[g] \in G/\operatorname{Ad}(G)} X^g/Z_g$$

where $G/\operatorname{Ad}(G)$ denotes the set of conjugacy classes in X and Z_g is the centralizer of g. Let $\mu_r = \mathbb{Z}/r\mathbb{Z}$ denote the group of r-th roots of unity. The inertia stack may also be written as a hom stack [2, Section 3]

$$\mathcal{I}_{\mathcal{X}} = \cup_{r>0} \mathcal{I}_{\mathcal{X},r}, \quad \mathcal{I}_{\mathcal{X},r} := \mathrm{Hom}^{\mathrm{rep}}(B\mu_r, \mathcal{X}).$$

The classifying stack $B\mu_r$ is a Deligne-Mumford stack and if $\mathcal{X}$ is a Deligne-Mumford stack then

$$\overline{\mathcal{I}}_{\mathcal{X}} := \cup_{r>0} \overline{\mathcal{I}}_{\mathcal{X},r}, \quad \overline{\mathcal{I}}_{\mathcal{X},r} := \mathcal{I}_{\mathcal{X}/r}/B\mu_r.$$

is the *rigidified inertia stack* of representable morphisms from $B\mu_r$ to $\mathcal{X}$, see [2, Section 3]. There is a canonical quotient cover $\pi : \mathcal{I}_{\mathcal{X}} \to \overline{\mathcal{I}}_{\mathcal{X}}$ which is r-fold over $\overline{\mathcal{I}}_{\mathcal{X},r}$. Since the covering is finite, from the point of rational cohomology there is no difference between $\overline{\mathcal{I}}_{\mathcal{X}}$ and $\mathcal{I}_{\mathcal{X}}$; that is, pullback induces an isomorphism

$$\pi^* : H^*(\overline{\mathcal{I}}_{\mathcal{X}}, \mathbb{Q}) \to H^*(\mathcal{I}_{\mathcal{X}}, \mathbb{Q}).$$

For the purposes of defining orbifold Gromov-Witten invariants, $\overline{\mathcal{I}}_{\mathcal{X}}$ can be replaced by $\mathcal{I}_{\mathcal{X}}$ at the cost of additional factors of r on the r-twisted sectors. If $\mathcal{X} = X/G$ is a global quotient of a scheme X by a finite group G then

$$\overline{\mathcal{I}}_{X/G} = \coprod_{(g)} X^g/(Z_g/\langle g \rangle)$$

where $\langle g \rangle \subset Z_g$ is the cyclic subgroup generated by g. For example, suppose that X is a polarized linearized projective G-variety such that $X/\!\!/G$ is locally free. Then

$$\mathcal{I}_{X/\!\!/G} = \coprod_{(g)} X^{\mathrm{ss},g}/Z_g$$

where $X^{\mathrm{ss},g}$ is the fixed point set of $g \in G$ on X^{ss}, Z_g is its centralizer, and the union is over all conjugacy classes,

$$\overline{\mathcal{I}}_{X/\!\!/G} = \coprod_{(g)} X^{\mathrm{ss},g}/(Z_g/\langle g \rangle)$$

where $\langle g \rangle$ is the (finite) group generated by g. The moduli stack of twisted stable maps admits evaluation maps to the rigidified inertia stack

$$\mathrm{ev} : \overline{\mathcal{M}}_{g,n}(\mathcal{X}) \to \overline{\mathcal{I}}^n_{\mathcal{X}}, \quad \overline{\mathrm{ev}} : \overline{\mathcal{M}}_{g,n}(\mathcal{X}) \to \overline{\mathcal{I}}^n_{\mathcal{X}},$$

where the second is obtained by composing with the involution $\overline{\mathcal{I}}_{\mathcal{X}} \to \overline{\mathcal{I}}_{\mathcal{X}}$ induced by the map $\mu_r \to \mu_r, \zeta \mapsto \zeta^{-1}$.

Constructions of Behrend-Fantechi [6] provide the stack of stable maps with virtual fundamental classes. The virtual fundamental classes

$$[\overline{\mathcal{M}}_{g,n,\Gamma}(\mathcal{X}, d)] \in H(\overline{\mathcal{M}}_{g,n}(\mathcal{X}), \mathbb{Q})$$

(where the right-hand-side denotes the singular homology of the coarse moduli space) satisfy the splitting axioms for morphisms of modular graphs similar to those in the case that X is a variety. Orbifold Gromov-Witten invariants are defined by virtual integration of pull-back classes using the evaluation maps above. The orbifold Gromov-Witten invariants satisfy properties similar to those for usual Gromov-Witten invariants, after replacing rescaling the inner product on the cohomology of the inertia stack by the order of the stabilizer. The definition of orbifold Gromov-Witten invariants leads to the definition of orbifold quantum cohomology as follows.

Definition 4.4. (Orbifold quantum cohomology) *To each component $\mathcal{X}_k$ of $\mathcal{I}_{\mathcal{X}}$ is assigned a rational number* age$(\mathcal{X}_k)$ *as follows. Let (x, g) be an object in $\mathcal{X}_k$. The element g acts on $T_x\mathcal{X}$ with eigenvalues $\alpha_1, \ldots, \alpha_n$ with $n = \dim(\mathcal{X})$. Let r be the order of g and define $s_j \in \{0, \ldots, r-1\}$ by $\alpha_j = \exp(2\pi i s_j / r)$. The* age *is defined by* $\mathrm{age}(\mathcal{X}_k) = (1/r) \sum_{j=1}^n s_j$. *Let*

$$\Lambda_{\mathcal{X}} = \left\{ \sum b_i q^{d_i} \mid b_i \in \mathbb{Q}, d_i \in H_2(\mathcal{X}, \mathbb{Q}), \ \forall c > 0, \#\{d_i \mid (d_i, c_1(\tilde{X})) < c\} < \infty \right\}$$

denote the Novikov field of sums of formal symbols $q^{d_i}, d_i \in H_2(\mathcal{X}, \mathbb{Q})$ where for each $c > 0$, only finitely many q^{d_i} with $(d_i, c_1(\tilde{X})) < c$ have non-zero coefficient. Denote the quantum cohomology

$$QH(\mathcal{X}) = \bigoplus QH^\bullet(\mathcal{X}), \quad QH^\bullet(\mathcal{X}) = \bigoplus_{\mathcal{X}_k \subset \mathcal{I}_{\mathcal{X}}} H^{\bullet + 2\,\mathrm{age}(\mathcal{X}_k)}(\mathcal{X}_k) \otimes \Lambda_{\mathcal{X}}.$$

The genus zero Gromov-Witten invariants define on $QH(\mathcal{X})$ the structure of a Frobenius manifold [12], [2].

5. Mundet stability

In this section we explain the Ramanathan condition for semistability of principal bundles [60] and its generalization to maps to quotients stacks by Mundet [55], and the quot-scheme and stable-map compactification of the moduli stacks.

5.1. **Ramanathan stability.** Morphisms from a curve to a quotient of a point by a reductive group are by definition principal bundles over the curve. Bundles have a natural semistability condition introduced half a century ago by Mumford, Narasimhan-Seshadri, Ramanathan and others in terms of *parabolic reductions* [60]. First we explain stability for vector bundles. A vector bundle $E \to C$ of degree zero over a smooth projective curve C is semistable if there are no sub-bundles of positive degree:

$$(E \text{ semistable}) \quad \text{iff} \quad (\deg(F) \leq 0, \quad \forall F \subset E \text{ sub-bundles}).$$

A generalization of the notion of semistability to principal bundles is given by Ramanathan [60] in terms of *parabolic reductions*. A parabolic reduction of P consists of a pair

$$Q \subset G, \quad \sigma : C \to P/Q$$

of a parabolic subgroup of G, that is and a section $\sigma : C \to P/Q$. Denote by $\sigma^* P \subset P$ the pull-back of the Q-bundle $P \to P/Q$, that is, the reduction of structure group of P to Q corresponding to σ. Associated to the homomorphism π_Q of (12) is an *associated graded* bundle $\mathrm{Gr}(P) := \sigma^* P \times_Q L(Q) \to C$ with structure group $L(Q)$. In the case that P is the frame bundle of a vector bundle $E \to C$ of rank r, that is,

$$P = \cup_z P_z, \quad P_z = \{(e_1, \ldots, e_r) \in E_z^r \mid e_1 \wedge \ldots \wedge e_r \neq 0\}$$

a parabolic reduction of P is equivalent to a flag of sub-vector-bundles of E

$$\{0\} \subset E_{i_1} \subset E_{i_2} \subset \ldots \subset E_{i_l} \subset E.$$

Explicitly the parabolic reduction $\sigma^* P$ given by frames adapted to the flag:

$$\sigma(z) = \{(e_1, \ldots, e_r) \in E_z^r \mid e_j \in E_{i_k,z}, \ \forall j \leq i_k, k = 1, \ldots, l\}.$$

Conversely, given a parabolic reduction the associated vector bundle has a canonical filtration.

An analog of the degree of a sub-bundle for parabolic reductions is the degree of a line bundle defined as follows. Given $\lambda \in \mathfrak{g}_{\mathbb{Z}} - \{0\}$ we obtain from the identification $\mathfrak{g} \to \mathfrak{g}^\vee$ a rational weight $\lambda^\vee$. Denote the corresponding characters $\chi_\lambda : L(Q) \to \mathbb{C}^\times$ and $\chi_\lambda \circ \pi_Q : Q \to \mathbb{C}^\times$. Consider the associated line bundle over C defined by $P(\mathbb{C}_{\lambda^\vee}) := \sigma^* P \times_Q \mathbb{C}_{\lambda^\vee}$. The *Ramanathan weight* [60] is the degree of the line bundle $P(\mathbb{C}_{\lambda^\vee})$, that is,

$$\mu_{BG}(\sigma, \lambda) := \langle [C], c_1(P(\mathbb{C}_{\lambda^\vee})) \rangle \in \mathbb{Z}.$$

The bundle $P \to C$ is *Ramanathan semistable* if for all (σ, λ) with λ dominant,

$$\mu_{BG}(\sigma, \lambda) \leq 0, \quad \forall (\sigma, \lambda).$$

As in the case of vector bundles, it suffices to check semistability for all reductions to *maximal parabolic* subgroups. In fact, any dominant weight may be used in the definition of $\mu_{BG}(\sigma, \lambda)$, which shows that Ramanathan semistability is independent of the choice of invariant inner product on the Lie algebra and one obtains the definition given in Ramanathan [60].

5.2. **Mundet semistability.** The Mundet semistability condition generalizes Ramanathan's condition to morphisms from a curve to the quotient stack [55], [61]. Let

$$(p : P \to C, u : C \to P(X)) \in \mathrm{Obj}(\mathrm{Hom}(C, X/G))$$

be a gauged map. Let (σ, λ) consist of a parabolic reduction $\sigma : C \to P/Q$ and a positive coweight $\lambda \in \mathfrak{z}_+(Q)$. Consider the family of bundles $P^\lambda \to S := \mathbb{C}^\times$ obtained by conjugating by z^λ. That is, if P is given as a cocycle in nonabelian cohomology with respect to a covering $\{U_i \to X\}$

$$[P] = [\psi_{ij} : (U_i \cap U_j) \to G] \ \in \ H^1(C, G)$$

then the twisted bundle is given by

$$[P^\lambda] = [z^\lambda \psi_{ij} z^{-\lambda} : (U_i \cap U_j) \to G] \ \in \ H^1(C \times S, G).$$

Define a family of sections

$$u^\lambda : S \times C \to P^\lambda(X)$$

by multiplying u by $z^\lambda, z \in \mathbb{C}^\times$. This family has an extension over $s = \infty$ called the *associated graded* bundle and stable section

$$\operatorname{Gr}(P) \to C, \quad \operatorname{Gr}(u) : \hat{C} \to \operatorname{Gr}(P)(X) \tag{15}$$

whose bundle $\operatorname{Gr}(P)$ agrees with the definition of associated graded above. Note that the associated graded section $\operatorname{Gr}(u)$ exists by properness of the moduli space of stable maps to $\operatorname{Gr}(P)(X)$. The composition of $\operatorname{Gr}(u)$ with projection $\operatorname{Gr}(P)(X) \to C$ is a map of degree one; hence there is a unique component $\hat{C}_0$ of $\hat{C}$ that maps isomorphically onto C. The construction above is $\mathbb{C}^\times$-equivariant and in particular over the central fiber $z = 0$ the group element z^λ acts by an automorphism of $\operatorname{Gr}(P)$ fixing $\operatorname{Gr}(u)$ up to automorphism of the domain.

For each pair of a parabolic reduction and one-parameter subgroup as above, the Mundet weight is a sum of *Ramanathan* and *Hilbert-Mumford* weights. To define the Mundet weight, consider the action of the automorphism z^λ on the associated graded $\operatorname{Gr}(P)$. The automorphism of X by z^λ is $L(Q)$-invariant and so defines an automorphism of the associated line bundle $\operatorname{Gr}(u)^* P(\tilde{X}) \to \operatorname{Gr}(C)$. The weight of the action of z^λ on the fiber of $\operatorname{Gr}(u)^* P(\tilde{X})$ over the root component $\hat{C}_0$ is the *Hilbert-Mumford weight*

$$\mu_X(\sigma, \lambda) \in \mathbb{Z}, \quad z^\lambda \tilde{x} = z^{\mu_X(\sigma,\lambda)} \tilde{x}, \quad \forall \tilde{x} \in (\operatorname{Gr}(u)|_{\hat{C}_0})^* \operatorname{Gr}(P) \times_G \tilde{X}.$$

Definition 5.1. (Mundet stability) *Let (P, u) be a gauged map from a smooth projective curve C to the quotient stack X/G. The* Mundet weight *of a parabolic reduction σ and dominant coweight λ is*

$$\mu(\sigma, \lambda) = \mu_{BG}(\sigma, \lambda) + \mu_X(\sigma, \lambda) \in \mathbb{Z}.$$

The gauged map (P, u) is Mundet semistable *resp.* stable *if and only if*

$$\mu(\sigma, \lambda) \leq 0, \ \textit{resp.}\ < 0, \quad \forall (\sigma, \lambda).$$

A pair (σ, λ) such that $\mu(\sigma, \lambda) \geq 0$ is a destabilizing pair*. A pair (P, u) is* polystable *iff*

$$\mu(\sigma, \lambda) = 0 \iff \mu(\sigma, -\lambda) = 0, \quad \forall (\sigma, \lambda). \tag{16}$$

That is, a pair (P, u) is polystable if for any destabilizing pair the opposite pair is also destabilizing.

More conceptually the semistability condition above is the Hilbert-Mumford stability condition adapted to one-parameter subgroups of the complexified gauge group, as explained in [55]. Semistability is independent of the choice of invariant inner product as follows for example from the presentation of the semistable locus in Schmitt [62, Section 2.3].

We introduce notation for various moduli stacks. Let $\mathcal{M}^G(C, X)$ denote the moduli space of Mundet semistable pairs; in general, $\mathcal{M}^G(C, X)$ is an Artin stack as follows from the git construction given in Schmitt [61, 62] or the more general construction of hom stacks in Lieblich [48, 2.3.4]. For any $d \in H_2^G(X, \mathbb{Z})$, denote by $\mathcal{M}^G(C, X, d)$ the moduli stack of pairs $v = (P, u)$ with

$$v_*[C] := (\phi \times_G \operatorname{id}_X)_* u_*[C] = d \in H_2^G(X, \mathbb{Z})$$

where $\phi : P \to EG$ is the classifying map.

5.3. **Compactification.** Schmitt [61] constructs a Grothendieck-style compactification of the moduli space of Mundet-semistable obtained as follows. Suppose X is projectively embedded in a projectivization of a representation V, that is $X \subset \mathbb{P}(V)$. Any section $u : C \to P(X)$ gives rise to a line sub-bundle $L := u^*(\mathcal{O}_{\mathbb{P}(V)}(-1) \to \mathbb{P}(V))$ of the associated vector bundle $P \times_G V$. From the inclusion $\iota : L \to P(V)$ we obtain by dualizing a surjective map

$$j : P(V^\vee) := P \times_G V^\vee \to L^\vee.$$

A *bundle with generalized map* in the sense of Schmitt [62] is a pair (P, j) such that j has base points in the sense that

$$\zeta \in C \text{ basepoint } \iff ((\operatorname{rank}(j_\zeta) : P(V)^\vee_\zeta \to L^\vee_\zeta) = 0).$$

Schmitt [62] shows that the Mundet semistability condition extends naturally to the moduli stack of bundles with generalized map. Furthermore, the compactified moduli space $\overline{\mathcal{M}}^{\mathrm{quot},G}(C, X)$ is projective, in particular proper.

Schmitt's construction of the moduli space of bundles with generalized maps uses geometric invariant theory. After twisting by a sufficiently positive bundle we may assume that $P(V^\vee)$ is generated by global sections. A collection of sections $s_1, \ldots, s_l$ generating $P(V^\vee)$ is called a *level structure*. Equivalently, an l-level structure is a surjective morphism $q : \mathcal{O}_C^{\oplus l} \to P(V^\vee)$. Denote by $\mathcal{M}^{G,\mathrm{lev}}(C, \mathbb{P}(V))$ the stack of gauged maps to $\mathbb{P}(V)$ with l-level structure. The group $GL(l)$ acts on the stack of l-level structures, with quotient

$$\mathcal{M}^{G,\mathrm{lev}}(C, \mathbb{P}(V))/GL(l) = \mathcal{M}^G(C, \mathbb{P}(V)). \tag{17}$$

Denote by $\mathcal{M}^{G,\mathrm{lev}}(C, X) \subset \mathcal{M}^{G,\mathrm{lev}}(C, \mathbb{P}(V))$ the substack whose sections $u : C \to \mathbb{P}(V)$ have image in $P(X) \subset P(\mathbb{P}(V))$. Then by restriction we obtain a quotient presentation

$$\mathcal{M}^{G,\mathrm{lev}}(C, X)/GL(l) = \mathcal{M}^G(C, X).$$

Allowing the associated quotient $P \times_G V^\vee \to P \times_G L^\vee$ to develop base points gives a compactified moduli stack of gauged maps with level structure $\overline{\mathcal{M}}^{G,\mathrm{quot},\mathrm{lev}}(C, X)$. Schmitt [61, 62] shows that the stack $\overline{\mathcal{M}}^{G,\mathrm{quot},\mathrm{lev}}(C, X)$ has a canonical linearization and the git quotient $\overline{\mathcal{M}}^{G,\mathrm{quot},\mathrm{lev}}(C, X) /\!\!/ GL(l)$ defines a compactification $\overline{\mathcal{M}}^{G,\mathrm{quot}}(C, X)$ of $\mathcal{M}^G(C, X)$ independent of the choice of l as long as l is sufficiently large. A version of the quot-scheme compactification with markings is obtained by adding tuples of points to the data. That is,

$$\overline{\mathcal{M}}_n^{G,\mathrm{quot}}(C, X) := \overline{\mathcal{M}}^{G,\mathrm{quot}}(C, X) \times \overline{\mathcal{M}}_n(C)$$

where we recall that $\overline{\mathcal{M}}_n(C)$ is the moduli stack of stable maps $p : \hat{C} \to C$ of class $[C]$ with n markings and genus that of C. The orbit-equivalence relation can be described more naturally in terms of *S-equivalence*: Given a family (P_S, u_S) of semistable gauged maps over a scheme S, such that the generic fiber is isomorphic to some fixed (P, u), then we declare (P, u) to be S-equivalent to (P_s, u_s) for any $s \in S$. Any equivalence class of semistable gauged maps has a unique representative that is polystable, by the git construction in Schmitt [61, Remark 2.3.5.18]. From the construction evaluation at the markings defines maps to the quotient stack

$$\overline{\mathcal{M}}_n^{G,\mathrm{quot}}(C, X, d) \to (V/\mathbb{C}^\times)^n, \quad ((p \circ z_i)^* L, j \circ p \circ z_i)$$

rather than to the git quotient $X^n \subset \mathbb{P}(V)^n$.[2]

Example 5.2. (Mundet semistable maps in the toric case) If G is a torus and $X = \mathbb{P}(V)$ then we can given an explicit description of Schmitt's quot-scheme compactification $\overline{\mathcal{M}}^{G,\mathrm{quot}}(C, X, d)$ of Mundet semistable maps [61].

Specifically let $X = \mathbb{P}(V)$ where V is a k-dimensional vector space and

$$V = \bigoplus_{i=1}^{k} V_i \tag{18}$$

is the decomposition of V into weight spaces V_i with weight $\mu_i \in \mathfrak{g}_{\mathbb{Z}}^{\vee}$.

A point of $\mathcal{M}^G(C, X, d)$ is a pair (P, u):

$$P \to C \qquad u \colon C \to P(X),$$

where P is a G-bundle and u is a section. We consider u as a morphism $\widetilde{u} \colon L \to P(V)$ with $L \to C$ a line bundle [40, Theorem 7.1]. Via the decomposition of V, we can write $\widetilde{u}$ as a k-tuple:

$$(\widetilde{u}_1, \ldots, \widetilde{u}_k) \in \bigoplus_{i=1}^{k} H^0(P(V_i) \otimes L^{\vee}).$$

The compactification $\overline{\mathcal{M}}^{G,\mathrm{quot}}(C, X, d)$ is obtained by allowing the $\widetilde{u}_i$ to have simultaneous zeros:

$$\widetilde{u}_1^{-1}(0) \cap \cdots \cap \widetilde{u}_k^{-1}(0) \neq \emptyset.$$

We make use of this example later on so we collect a few results about $\overline{\mathcal{M}}^{G,\mathrm{quot}}(C, X, d)$ below.

Recall (10) there is a projection $H_G^2(X) \to H^2(BG) = \mathfrak{g}_{\mathbb{Z}}^{\vee}$ and similarly we have $H_2^G(X) \to H_2(BG) = \mathfrak{g}_{\mathbb{Z}}$. Associated to $v = (P, u)$ is the discrete data:

- $v_*[C] = d \in H_2^G(X, \mathbb{Z})$ and its image $d(P) \in H_2(BG)$
- $c_1^G(\widetilde{X}) \in H_G^2(X)$ and its image $\theta \in H^2(BG)$
- $d(u) := -c_1(L) \in H^2(C, \mathbb{Z}) \cong \mathbb{Z}$.

Note that $d(P)$ is the degree of P; that is, $d(P) = c_1(P) \in \mathfrak{g}_{\mathbb{Z}}$ under the identification $H^2(C, \mathfrak{g}_{\mathbb{Z}}) \cong \mathfrak{g}_{\mathbb{Z}}$. We can now state the following:

Lemma 5.3. *Let G be a torus acting on a vector space V. Let $V = \bigoplus_{i=1}^k V_i$ be its decomposition into weight spaces with weights $\mu_1, \ldots, \mu_k$. The Mundet semistable locus consists of pairs (P, u) such that*

$$\mathrm{hull}(\{-d(P)^{\vee} + \mu_i | \tilde{u}_i \neq 0\}) \ni \theta. \tag{19}$$

Furthermore let $W = \bigoplus_{i=1}^k H^0(P(V_i) \otimes L^{\vee})$ and let W^{ss} consist of $(\widetilde{u}_1, \ldots, \widetilde{u}_k)$ such that (19) holds. Then $\overline{\mathcal{M}}^{G,\mathrm{quot}}(C, X, d) \cong W^{ss}/G$.

Proof. Since G is abelian, $\mathrm{Gr}(P) = P$ for any pair (λ, σ). It follows that for any $\lambda \in \mathfrak{g}_{\mathbb{Q}}$, the Mundet weight is

$$\mu(\sigma, \lambda) := \min_i \{(d(P), \lambda) - \mu_i(\lambda) + \theta(\lambda), \tilde{u}_i \neq 0\}.$$

[2]The Ciocan-Fontanine-Kim-Maulik [15] moduli space of *stable quotients* remedies this defect by imposing a stability condition at the marked points $z_1, \ldots, z_n \in C$. The moduli stack then admits a morphism to $\overline{\mathcal{I}}^n_{X/\!\!/G}$ by evaluation at the markings.

Hence the semistable locus is the space of pairs (P, u) where

$$\operatorname{hull}(\{-d(P)^\vee + \mu_i | \tilde{u}_i \neq 0\}) \ni \theta.$$

This proves the first claim. The second claim is an immediate consequence.

□

Example 5.4. Consider $G = \mathbb{C}^\times$ and $V = \mathbb{C}^k$. Then

$$\deg(P(V_i) \otimes L^\vee) = \deg(P(V_i)) - \deg(L) = d(P) + d(u), \quad i = 1, \ldots, k.$$

It follows that the moduli stack admits an isomorphism

$$\overline{\mathcal{M}}^{G,\mathrm{quot}}(C, X, d) \cong \mathbb{C}^{k(d(P)+d(u)+1),\times}/\mathbb{C}^\times \cong \mathbb{P}^{k(d(P)+d(u)+1)-1}.$$

This moduli stack is substantially simpler in topology than the moduli space of stable maps to $C \times X/\!\!/G$, despite the dramatically more complicated stability condition. This ends the example.

A *Kontsevich-style compactification* of the stack of Mundet-semistable gauged maps which admits evaluation maps as well as a Behrend-Fantechi virtual fundamental class [35] is defined as follows. The objects in this compactification allow *stable sections*, that is, stable maps $u : \hat{C} \to P(X)$ whose composition with $P(X) \to C$ has class $[C]$. Thus objects of $\overline{\mathcal{M}}_n^G(C, X)$ are triples $(P, \hat{C}, u, \underline{z})$ consisting of a G-bundle $P \to C$, a projective nodal curve $(\hat{C}, \underline{z})$, and a stable map $u : \hat{C} \to P \times_G X$ whose class projects to $[C] \in H_2(C, \mathbb{Z})$. Morphisms are the obvious diagrams. To see that this category forms an Artin stack, note that the moduli stack of bundles $\operatorname{Hom}(C, BG)$ has a universal bundle

$$U \to C \times \operatorname{Hom}(C, BG).$$

Consider the associated X-bundle

$$U \times_G X \to C \times \operatorname{Hom}(C, BG).$$

The stack $\overline{\mathcal{M}}_n^G(C, X)$ is a substack of the stack of stable maps to $U \times_G X$, and is an Artin stack by e.g. Lieblich [48, 2.3.4], see [69] for more details. Note that hom-stacks are not in general algebraic [8].

Properness of the Kontsevich-style compactification follows from a combination of Schmitt's construction and the Givental map. A proper relative Givental map is described in Popa-Roth [59], and in this case gives a morphism

$$\overline{\mathcal{M}}^G(C, X, d) \to \overline{\mathcal{M}}^{G,\mathrm{quot}}(C, X, d). \tag{20}$$

For each fixed bundle this map collapses bubbles of the section u and replaces them with base points with multiplicity given by the degree of the bubble tree. Since the Givental morphism (20), the quot-scheme compactification $\overline{\mathcal{M}}_n^{G,\mathrm{quot}}(C, X, d)$ and the forgetful morphism $\overline{\mathcal{M}}_n^G(C, X, d) \to \overline{\mathcal{M}}^G(C, X, d)$ are proper, which implies that $\overline{\mathcal{M}}_n^G(C, X, d)$ is proper.

6. Applications

6.1. **Presentations of quantum cohomology and quantum K-theory.** The first group of applications use the that the linearized quantum Kirwan map is a ring homomorphism. In good cases, such as the toric case, one can prove that the (non-circle-equivariant) linearized quantum Kirwan map is a surjection

$$D_\alpha \kappa_X^G : T_\alpha QH_G(X) \to T_{\kappa_X^G(\alpha)} QH(X /\!\!/ G)$$

and so obtain a presentation for the quantum product in quantum cohomology [34] or quantum K-theory at $\kappa_X^G(\alpha)$. A simple example is projective space itself, which is the git quotient of a vector space by scalar multiplication of $X = \mathbb{C}^k$ by $G = \mathbb{C}^\times$:

$$\mathbb{P}^{k-1} = \mathbb{C}^k /\!\!/ \mathbb{C}^\times$$

We show how to derive the relation in quantum cohomology or quantum K-theory. The moduli space of affine gauged maps of class $d \in H_2^G(X) \cong \mathbb{Z}$ is the space of k-tuples of degree d polynomials

$$u(z) = (u_1(z), \ldots, u_k(z))$$

with non-zero leading order term given by the top derivative $u^{(d)}$. Thus its evaluation at infinity lies on the semistable locus.

$$\mathcal{M}_{1,1}^G(\mathbb{A}, X) \cong (\mathbb{C}^{k(d+1)} - (\mathbb{C}^{kd} \times \{0\}))/\mathbb{C}^\times.$$

Let $\alpha = 0$ and let $\beta \in H_2^G(X)$ be the Euler class of the trivial vector bundle $X \times \mathbb{C} \to X$ where G acts on $\mathbb{C}$ with weight one. The pull-back

$$\mathrm{ev}_1^*((X \times \mathbb{C} \to X)^{\oplus d}) \to \overline{\mathcal{M}}_{1,1}^G(\mathbb{A}, X, d)$$

has a canonical section given by the first d derivatives,

$$\sigma : \overline{\mathcal{M}}_{1,1}^G(\mathbb{A}, X, d) \to \mathrm{ev}_1^*((X \times \mathbb{C} \to X)^{\oplus d}), \quad [u] \mapsto [u^{(i)}(z_1)]_{i<d}.$$

One can check that this section extends to a section over the compactification $\overline{\mathcal{M}}_{1,1}^G(\mathbb{A}, X, d)$ which is non-vanishing on the boundary. The map $\mathrm{ev}_{\infty,d} : \sigma^{-1}(0) \to X /\!\!/ G$ is an isomorphism, since the evaluation takes the terms of top order. This implies

$$(\mathrm{ev}_{\infty,d,*}\, \mathrm{ev}_1^*)\beta^{dk} = \mathrm{ev}_{\infty,d,*}|_{\sigma^{-1}(0)} 1 = 1.$$

In particular for $d = 1$ the image of β^k under the linearized quantum Kirwan map is

$$D_0 \kappa_X^G(\beta^k) = q.$$

In quantum cohomology one obtains

$$QH(\mathbb{P}^{k-1}) = \mathbb{Q}[\beta, q]/(\beta^k - q).$$

This was one of the first computations in quantum cohomology but the advantage of this approach is that it works equally well in quantum K-theory. In this case β is the exterior algebra on the dual of L, $\beta = 1 - L^{-1} \in K(\mathbb{P}^{k-1})$, where L is the class of the hyperplane line bundle. One obtains

$$QK(\mathbb{P}^{k-1}) = \mathbb{Q}[L, L^{-1}, q]/((1 - L^{-1})^k - q).$$

More generally, one gets Batyrev-type presentations of the quantum cohomology and quantum K-theory of toric smooth Deligne-Mumford stacks with projective coarse moduli spaces.

6.2. **Solutions to quantum differential equations.** A second group of applications concerns explicit formulas for solutions to quantum differential equations. Recall that a fundamental solution in quantum cohomology is given by one-point descendent potentials (J-function)

$$\tau^{\pm}_{X/\!\!/G} : QH(X/\!\!/G) \to QH(X/\!\!/G)[[\zeta]]$$

which appear when one factorizes the genus zero graph potential $\tau_{X/\!\!/G}$ for $C = \mathbb{P}^1$ into contributions from $\mathbb{C}^\times$-fixed points at $0, \infty$; there is a parallel discussion in quantum K-theory. One has a similar factorization of the gauged potential into contributions from $0, \infty$ in terms of *localized gauged potentials*

$$\tau^{\pm}_{X,G} : QH_G(X) \to QH(X/\!\!/G)[[\zeta]].$$

Using the fact that the cobordism given by scaled gauged maps is equivariant, one obtains the following "localized" version of the adiabatic limit theorem:

$$\tau^{\pm}_{X,G} = \tau^{\pm}_{X/\!\!/G} \circ \kappa^G_X.$$

Often, the localized gauged potentials are easier to compute than the localized graph potentials. For example, for the case of a vector space X with a representation of a torus G one obtains at $\alpha = 0$

$$\tau^{\pm}_{X,G}(0) = \sum_{d \in H_2^G(X)} q^d \frac{\prod_{j=1}^k \prod_{m=-\infty}^{0} (1 - X_j^{-1}\zeta^m)}{\prod_{j=1}^k \prod_{m=-\infty}^{\mu_j(d)} (1 - X_j^{-1}\zeta^m)}$$

where X_j is the class of the line bundle given by the j-th weight space of X. In this case the relationship between the two potentials has essentially been established in a series of papers by Givental [32], [31], [30], [29], [28], [27]. This gives yet another way of producing relations in the quantum cohomology or quantum K-theory.

For actions of non-abelian groups, one may also obtain formulas for fundamental solutions to the quantum differential equation by an *abelianization* procedure suggested by Martin [50] in the classical case and by Bertram-Ciocan-Fontanine-Kim [14] in the quantum case. In the case of gauged maps, a proof in K-theory can be reduced to Martin's case by taking the *small linearization* chamber in which the bundle part of a gauged genus zero map is automatically trivial. As a result, one also gets formulas for fundamental solutions to quantum differential equations for such spaces as Grassmannians and moduli of framed rank r sheaves, charge k (in the Atiyah-Hitchin-Drinfeld-Manin description). For example for framed sheaves one obtains the formula for the fundamental solution (with equivariant parameters ξ_1, ξ_2 corresponding to the action of the torus on the plane):

$$\tau_{X/\!\!/G,-}(0) = \sum_{d \geq 1} q^d \sum_{\underline{d}: d_1 + \ldots + d_k = d} \prod_{i \neq j} \frac{\Delta_{\underline{d}}(\theta_i - \theta_j, \xi_1 + \xi_2) \Delta_{\underline{d}}(\theta_i - \theta_j, 0)}{\Delta_{\underline{d}}(\theta_i - \theta_j, \xi_1) \Delta_{\underline{d}}(\theta_i - \theta_j, \xi_2)} \prod_{i=1}^k \Delta_{\underline{d}}(\theta_i, 0)^{-r} \Delta_{\underline{d}}(-\theta_i, \xi_1 + \xi_2)^{-r}. \tag{21}$$

where $\Delta_{\underline{d}}(\theta, w) := (\prod_{l=-\infty}^{\theta \cdot \underline{d}} (\theta + w + l\zeta)) / (\prod_{l=-\infty}^{0} (\theta + w + l\zeta))$ and $\theta_1, \ldots, \theta_k$ are the Chern roots of the tautological bundle, $\theta = \sum k_i \theta_i$ any integral linear combination and $\theta \cdot \underline{d} = \sum d_i k_i$. A version of this formula for $r = 1$ (with a "mirror map" correction) is proved by Ciocan-Fontanine-Kim-Maulik. The $r > 1$ formula seems to be new. Furthermore, we claim that same approach works in quantum K-theory. For the Grassmannian, one obtains a formula for the descendent potential in both quantum cohomology and quantum K-theory. From this one may deduce the standard presentation of the quantum cohomology or quantum K-theory of the Grassmannian as a quotient of the ring of symmetric functions as in Gorbounov-Korff [38] and Buch-Chaput-Mihalcea-Perrin [11].

6.3. **Wall-crossing for Gromov-Witten invariants.** A final application is a conceptual result on the relationship between Gromov-Witten invariants of symplectic quotients. For simplicity let $G = \mathbb{C}^\times$ and let X be a smooth projective G-variety equipped with a family of linearizations $\tilde{X}_t$. Associated to the family of linearizations we have a family of git quotients $X /\!\!/_t G$; we suppose that stable $\neq$ semistable at $t = 0$ in which case $X /\!\!/_t G$ undergoes a flip. For example, if $X = \mathbb{P}^4$ with weights $0, 1, 1, -1, -1$ then $X /\!\!/_t G$ undergoes an Atiyah flop as t passes through zero. Using $\mathbb{C}^\times$-equivariant versions of the gauged Gromov-Witten invariants we obtain an explicit formula for the difference

$$\tau_{X /\!\!/_+ G} \kappa^G_{X,+} - \tau_{X /\!\!/_- G} \kappa^G_{X,-} \tag{22}$$

in either cohomology or K-theory. We say that the variation is *crepant* if the weights at the G-fixed points at $t = 0$ sum to zero; in this case the birational equivalence between $X /\!\!/_- G$ and $X /\!\!/_+ G$ is a combination of flops. In this case one can show that the difference in (22) is, as a distribution in q, vanishing almost everywhere:

$$\tau_{X /\!\!/_+ G} \kappa^G_{X,+} =_{\text{a.e. in q}} \tau_{X /\!\!/_- G} \kappa^G_{X,-}$$

in both quantum cohomology and K-theory as well. In the later this holds after shifting by the square root of the canonical line bundle. This is a version of the so called *crepant transformation* conjecture in [16], with the added benefit that the proof is essentially the same for both cohomology (covered in [36]) and K-theory.

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