INVARIANCE OF IMMERSED FLOER COHOMOLOGY UNDER LAGRANGIAN SURGERY

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Abstract. We show that cellular Floer cohomology of an immersed Lagrangian brane is invariant under smoothing of a self-intersection point if the quantum valuation of the weakly bounding cochain vanishes and the Lagrangian has dimension at least two. The chain-level map replaces the two orderings of the self-intersection point with meridional and longitudinal cells on the handle created by the surgery, and uses a bijection between holomorphic disks developed by Fukaya-Oh-Ohta-Ono [42, Chapter 10]. Our result generalizes invariance of potentials for certain Lagrangian surfaces in Dimitroglou-Rizell–Ekholm–Tonkonog [30, Theorem 1.2], and implies the invariance of Floer cohomology under mean curvature flow with this type of surgery, as conjectured by Joyce [53].

Contents

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1. Introduction

A Lagrangian immersion in a compact symplectic manifold with transverse self-intersection defines a homotopy-associative *Fukaya algebra* developed by Akaho-Joyce in [6]. The framework of Fukaya-Oh-Ohta-Ono [42] associates to this algebra a space of solutions to the projective Maurer-Cartan equation. For any solution, there is a *Lagrangian Floer cohomology group*, independent up to isomorphism of all choices. In Palmer-Woodward [66], we studied the behavior of Floer cohomology under variation of an immersion in the direction of the Maslov (relative first Chern) class, such as a coupled mean-curvature/Kähler-Ricci flow. The main result of [66] was that there exists a flow on the space of projective Maurer-Cartan solutions with the following property: The isomorphism class of the Lagrangian Floer cohomology is invariant as long as the valuation of the Maurer-Cartan solution with respect to the quantum parameter stays positive and the Lagrangian stays immersed. In particular, the Floer cohomology is invariant as the immersion passes through a self-tangency. Naturally a question arises whether one can continue the flow through a "wall" created by the vanishing valuation at a self-intersection point.

Via the mirror symmetry conjectures, this question is expected to be related to a question on deformation theory of vector bundles on a mirror complex manifold, or more precisely, matrix factorizations [55]. The mirror of the mean curvature flow is expected to be (a deformed version) of the Yang-Mills flow [51]. The isomorphism class of the bundle is constant under Yang-Mills flow and, in particular, the cohomology is invariant [8]. That is, there are no real-codimension-one "walls" on the mirror side, and so one does not expect such walls in the deformation spaces for Lagrangian branes either. For vector bundles on projective varieties there exist versal deformations [40] in the sense of Kuranishi; see for example [82] for coherent sheaves. The base of these versal deformations are complex-analytic spaces. The results of this paper can be viewed as giving a theory of versal deformations for immersed Lagrangians, in which solutions to the projective Maurer-Cartan equation with negative q-exponents parametrize the actual deformations of an immersed Lagrangian. As in the case of deformations of singular algebraic varieties [27, Chapter XI], in order to produce the expected space of deformations one must allow smoothings at the singularities.

A way of smoothing singularities of immersed self-transverse Lagrangians was introduced by Lalonde-Sikorav [58] and Polterovich [69]. Let $\phi_0 : L_0 \to X$ be a self-transverse Lagrangian immersion with compact domain L_0 with an self-intersection point $x \in \phi_0(L_0)$. For a sufficiently small *surgery parameter* $\epsilon \in \mathbb{R}$ denote by

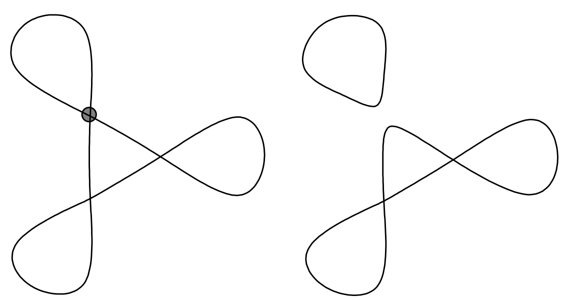

FIGURE 1. An immersion and its surgery

$\phi_\epsilon : L_\epsilon \to X$ the surgery obtained by removing small balls around the intersection and gluing in a cylinder. The surgery parameter ϵ is closely related to the difference $A(\epsilon)$ from (16) in the areas of disks bounding $\phi_0(L_0)$ and $\phi_\epsilon(L_\epsilon)$.

A long line of papers in symplectic geometry have studied the effect of Lagrangian surgery on Floer theory. Seidel's long exact triangle [76] is perhaps the first example, since a Dehn twist is a special case of a surgery. More generally, holomorphic[1]

[1]To save space, we refer to holomorphic disks with respect to some almost complex structure as holomorphic.

disks with boundary in the surgery were described in Fukaya-Oh-Ohta-Ono [42, Chapter 10]. Abouzaid [4], Mak-Wu [62], Tanaka [84], Chantraine-Dimitroglou-Rizell-Ghiggini-Golovko [20, Chapter 8], Fang [35], and Hong-Kim-Lau [50, Theorem B] proved various generalizations. The invariance of disk potentials was shown for certain Lagrangian surfaces by Pascaleff–Tonkonog [68, Theorem 1.2] and Dimitroglou-Rizell–Ekholm–Tonkonog [30, Theorem 1.2]. In dimension two, the Lagrangians related by the two different signs of surgery parameter are said to be related by *mutation*. Mutation-invariance of Lagrangian Floer homology was shown via Lagrangian cobordism techniques by Hicks [48]. The "wall-crossing" formula for the change in the local system given by the above formulas is discussed in Auroux [11], [12], Kontsevich-Soibelman [56], and Pascaleff-Tonkonog [68].

We construct a natural identification of solutions of the projective Maurer-Cartan equations for the surgered and unsurgered Lagrangian branes that preserves the disk potentials and Floer cohomology. The version of Floer cohomology used here is the cohomology of the twisted first composition map for a Fukaya algebra generated by cellular cochains and self-intersection points on the immersed Lagrangian, which counts *treed holomorphic disks* bounding the Lagrangian with cellular constraints. The treed holomorphic disks are natural generalizations of the treed holomorphic disks considered in the Morse model for Fukaya algebras considered in [75]. There the tree segments in the disk determine gradient trajectories while in this paper the parameter corresponds to an evaluation in some cellular degeneration of the diagonal embedding of the Lagrangian. The cellular model is essentially equivalent to the Morse model considered in the earlier paper [66] but we find that the bijection between Floer cochains is most naturally phrased as a correspondence involving cells, rather than Morse critical points.

Our results show that if the quantum valuation at a self-intersection point of a family of Maurer-Cartan solutions in a mean curvature flow of Palmer-Woodward [66] reaches zero then the solution may be continued by Lagrangian surgery so that the Floer cohomology of the surgery is invariant. Thus the flow may be continued after the singular time without changing the Floer cohomology.

The assumptions necessary for invariance of Floer cohomology to hold are encoded in the following definitions. Let

$$\Lambda = \mathbb{C}((q^{\mathbb{R}})) := \left\{ \sum_{i=0}^{\infty} a_i q^{d_i} \;\middle|\; \lim_{i\to\infty} d_i = \infty,\ \forall i,\ d_i \in \mathbb{R},\ a_i \in \mathbb{C} \right\} \tag{1}$$

denote the Novikov field with complex coefficients,[2] equipped with q-valuation

$$\mathrm{val}_q : \Lambda - \{0\} \to \mathbb{R}, \quad \sum_{i=0}^{\infty} a_i q^{d_i} \mapsto \min(d_i, a_i \neq 0).$$

[2]The Fukaya algebras in this paper are defined with rational coefficients, but allowing complex coefficients gives a possibly-larger Maurer-Cartan space.

Let Λ_0 denote the group of units in Λ with vanishing q-valuation

$$\Lambda_0 = \operatorname{val}_q^{-1}(0) = \left\{ a_0 + \sum_{i \geq 1} a_i q^{d_i} \in \Lambda \;\middle|\; a_0 \in \mathbb{C} - \{0\},\ \forall i,\ a_i \in \mathbb{C}, d_i > 0 \right\}.$$

Let $\phi_0 : L_0 \to X$ be a Lagrangian immersion. A *local system* on ϕ_0 is a flat Λ_0-line bundle y on $\phi_0(L_0)$, or equivalently, a flat line bundle on L_0 together with identifications of the fibers $y(x_-) \to y(x_+)$ at the self-intersection points

$$x = (x_-, x_+), \quad \phi_0(x_-) = \phi_0(x_+).$$

If $\phi_0(L_0)$ is connected with fundamental group $\pi_1(\phi_0(L_0))$ for some choice of base point then the space of isomorphism classes of local systems is isomorphic to the space of representations

$$\mathcal{R}(\phi_0) \cong \operatorname{Hom}(\pi_1(\phi_0(L_0)), \Lambda_0) \cong \operatorname{Hom}(H_1(\phi_0(L_0)), \Lambda_0).$$

For disconnected Lagrangians, $\mathcal{R}(\phi_0)$ is defined by replacing $\pi_1(\phi_0(L_0))$ with the product of the fundamental groups of the connected components of $\phi_0(L_0)$. Let $\phi_0 : L_0 \to X$ be equipped with a *brane structure* consisting of an orientation, relative spin structure, and Λ_0-valued local system $y \in \mathcal{R}(\phi_0)$. In Sections 5 and 6 we construct for any such datum a *Fukaya algebra* $CF(\phi_0)$, which is a strictly unital A_∞ algebra.

The Fukaya algebra has a canonical family of deformations parametrized by odd cochains, and the cohomology is defined for solutions to the *projective Maurer-Cartan equation.* By definition, any element $b \in CF(\phi_0)$ is given as a sum

$$b = \sum_{x \in \mathcal{I}(\phi_0)} b(x) x$$

over generators $x \in \mathcal{I}(\phi_0)$ corresponding to cells or self-intersection points. In particular, if b is odd then $b(x)$ vanishes for x even degree. Let $MC(\phi_0)$ denote the space of projective Maurer-Cartan solutions, as in (69). For $\delta > 0$ small let $MC_\delta(\phi_0)$ denote the subspace satisfying

$$\operatorname{val}_q(b(x)) \in (-\delta, \infty)$$

at the transverse self-intersection points x of ϕ_0. For sufficiently small δ, associated to any $b_0 \in MC_\delta(\phi_0)$, called a *weakly bounding cochain*, is a Floer cohomology group $HF(\phi_0, b_0)$, independent of all choices up to isomorphism. Given $x = (x_-, x_+)$, denote by $\overline{x} = (x_+, x_-) \in L_0^2$ the self-intersection point with the opposite ordering. The degree of x is even resp. odd if the natural map

$$T_{x_-} L \oplus T_{x_+} L \to T_{\phi(x_\pm)} X$$

is orientation preserving resp. reversing. If $\dim(L)$ is even then x is odd if and only if $\overline{x}$ is odd, while if $\dim(L)$ is odd then x is odd if and only if $\overline{x}$ even. By convention $b_0(\overline{x})$ vanishes if $\overline{x}$ is even.

Definition 1.1. Let $\phi_0 : L_0 \to X$ be a Lagrangian immersion and $b_0 \in MC_\delta(\phi_0)$ a Maurer-Cartan solution. An odd self-intersection point $x = (x_-, x_+) \in L_0^2$ is *admissible* for b_0 if and only if

(a) the q-valuation of the coefficient $b_0(x)$ is close to zero in the sense that

$$\operatorname{val}_q(b_0(x)) \in (-\delta, 0) \tag{2}$$

and

(b) either $\dim(L_0) = 2$ and the q-valuation of $b_0(\overline{x})$ is sufficiently large in the sense that [3]

$$\operatorname{val}_q(b_0(x)b_0(\overline{x})) > 0 \quad \text{or} \quad b_0(\overline{x}) = 0 \tag{3}$$

or $\dim(L_0) > 2$ and $b_0(\overline{x}) = 0$.

This ends the Definition.

We remark that for the bounding cochains arising in our previous study of invariance of Floer cohomology under the development of tangencies [66] in fact we have $b_0(\overline{x}) = 0$ since only one of the orderings was needed to cancel the obstruction arising from the additional contributions to the curvature of the immersed Fukaya algebra.

The invariance of Floer cohomology under surgery holds after the following change in the weakly bounding cochain. The surgered Lagrangian L_ϵ is obtained from L_0 by removing the self-intersection points $x_\pm \in L_0$ and gluing in a handle $H_\epsilon \cong S^{n-1} \times \mathbb{R}$, as in Section 2. We denote by

$$\mu \cong S^{n-1} \times \{0\}, \quad \lambda \cong \{\mathrm{pt}\} \times \mathbb{R}$$

the meridional and longitudinal cells on the handle H_ϵ, oriented so that the bijection of Proposition 7.2 is orientation preserving. These cells appear as generators of the space of Floer cochains in the cellular model.

Definition 1.2. Let $x = (x_-, x_+)$ with $\phi_0(x_-) = \phi_0(x_+)$. For $\epsilon > 0$ let

$$CF_\delta(\phi_0, \epsilon) \subset CF(\phi_0)$$

denote the space of elements $b_0 \in CF(\phi_0)$ satisfying

$$\operatorname{val}_q(b_0(x)q^{A(\epsilon)}) = 0$$

(see (16)) and (3). Let

$$MC_\delta(\phi, \epsilon) \subset CF_\delta(\phi, \epsilon)$$

[3]For the sake of discussing explicit examples, we also allow $\dim(L_0) = 1$ under the following assumptions (which do not typically hold): $b_0(\overline{x}) = 0$, every holomorphic disk $u : S \to X$ with boundary on ϕ meeting x has a branch change at every $z \in \partial S$ with $u(z) = x$, and there are no holomorphic disks $u : S \to X$ with exactly one corner at $\overline{x}$.

denote the space of Maurer-Cartan solutions b_0 that vanish on the closure of the cells containing $x_\pm$:

$$b_0(\sigma) = 0, \quad \forall \sigma \subset \overline{\tau}, \tau \ni x_\pm \tag{4}$$

and if $\dim(L_0) > 2$ then $b_0(\overline{x})$ vanishes. Define

$$\Psi : CF_\delta(\phi_0, \epsilon) \to CF(\phi_\epsilon), \quad b_0 \mapsto b_0 - b_0(x)x - b_0(\overline{x})\overline{x} + \begin{cases} \ln(b_0(x)q^{A(\epsilon)})\mu + \ln(b_0(x)b_0(\overline{x}) + 1)\lambda & \dim(L_0) = 2 \\ \ln(b_0(x)q^{A(\epsilon)})\mu + b_0(x)b_0(\overline{x})\lambda & \dim(L_0) > 2 \end{cases} \tag{5}$$

where the logarithms are defined by formal power series expansion at the leading order term for any choice of branch, well-defined by the assumption that $b_0(x)q^{A(\epsilon)}$ and $b_0(x)b_0(\overline{x}) + 1$ have vanishing q-valuation. This ends the Definition.

The vanishing condition (4) can always be achieved up to gauge equivalence by Lemma 5.11. In the case $\dim(L_0) = 2$, we assume that the surgered Lagrangian L_ϵ is equipped with a local system which has holonomy -1 around the meridian; note that this assumption constrains the topology of the surgery. The conditions in Definition 1.2 are satisfied in our application to mean curvature flow [66].

We may now state the main result. Let $MC_{\geq 0}(\phi_\epsilon)$ be the enlarged space of projective Maurer-Cartan solutions in (76) for ϕ_ϵ, in which one allows the coefficient of λ to have vanishing q-valuation.[4]

Theorem 1.3. *Let $\phi_0 : L_0 \to X$ be an immersed Lagrangian brane of dimension $\dim(L_0)$ at least two in a compact rational symplectic manifold X. There exists a constant $\delta > 0$ such that for any $b_0 \in MC_\delta(\phi_0)$ and any admissible transverse self-intersection point $x \in \mathcal{I}^{\text{si}}(\phi)$ as in Definition 1.1 there exist perturbation systems defining the Fukaya algebras $CF(\phi_0)$ and $CF(\phi_\epsilon)$ so that the following holds: The map Ψ of Definition 1.2 satisfies*

$$\Psi(MC_\delta(\phi_0, \epsilon)) \subset MC_{\geq 0}(\phi_\epsilon) \tag{6}$$

preserves the disk potentials

$$\Psi^* W_\epsilon = W_0, \quad W_0 : MC_\delta(\phi_0) \to \Lambda, \quad W_\epsilon : MC_{\geq 0}(\phi_\epsilon) \to \Lambda$$

and lifts to an isomorphism of Floer cohomologies

$$HF(\phi_0, b_0) \cong HF(\phi_\epsilon, b_\epsilon := \Psi(b_0)), \forall b_0 \in MC_\delta(\phi_0, \epsilon).$$

In other words, immersed Floer cohomology is invariant under surgery after a suitable change in the weakly bounding cochain.

[4] This enlargement is only relevant in the case $\dim(L) = 2$, and in this case we show that the Maurer-Cartan sum still converges.

Remark 1.4. J. Hicks [49, Section 2] has given examples of Lagrangian spheres that have surgeries that in the Fukaya category are non-isomorphic depending on the sign of the surgery parameter ϵ; the result above does not contradict these examples since we require the immersed Lagrangian $\phi_0 : L_0 \to X$ itself to have a non-zero weakly bounding cochain b_0.

Remark 1.5. Returning to the application to mean curvature flow, Theorem 1.3 suggests the possibility of mean curvature flow for Lagrangians with *preventive surgery*. Namely, similar to the set-up in the Thomas-Yau conjecture [85] suppose one performs coupled mean curvature/Kähler-Ricci flow on a Lagrangian immersion ϕ_t with unobstructed and non-trivial Floer theory $HF(\phi_t)$. The results of this paper and Palmer-Woodward [66] imply that the non-triviality of the Floer homology $HF(\phi_t)$ carries along with the flow ϕ_t, if a surgery *before* the time at which the geometric singularity forms is performed whenever the q-valuation $\operatorname{val}_q(b_t)$ of the Maurer-Cartan solution b_t crosses zero. This type of surgery is *preventive* rather than *emergency* in the sense that the Lagrangian immersion ϕ_t is not about to cease to exist. Non-triviality of the Floer cohomology affects the types of singularities that can occur as discussed by Joyce [53]. One naturally wonders what kind of singularities can occur *generically* (meaning allowing arbitrary Hamiltonian perturbations) in the case of non-trivial Floer cohomology.

We may upgrade the isomorphisms of Floer cohomology to a quasi-isomorphism in the Fukaya category as follows. A full construction of a Fukaya category containing all Lagrangians is beyond the techniques of this paper; we consider rather the following simplified Fukaya category with two objects. Let $\phi_0' : L_0' \to X$ be a Hamiltonian isotopy of $\phi_0 : L_0 \to X$ such that ϕ_0, ϕ_0' intersect transversally, as in Figure 15. We assume that ϵ is sufficiently small so that the surgered immersion $\phi_\epsilon : L_\epsilon \to X$ intersects ϕ_0' transversally as well. After a Hamiltonian perturbation we may assume that the Lagrangian

$$\tilde{\phi} : \phi_0 \cup \phi_0' : L_0 \cup L_0' \to X$$

is rational, immersed, and with transverse self-intersection, and similarly for

$$\tilde{\phi}_\epsilon : \phi_\epsilon \cup \phi_0' : L_\epsilon \cup L_0' \to X.$$

As such, we obtain a category $\operatorname{Fuk}_{\tilde{\phi}_0}(X)$ with two objects ϕ_0, ϕ_0' equipped with weakly bounding cochains, and with morphisms defined by the corresponding subspaces of $CF(\tilde{\phi}_0)$. For sufficiently small surgery parameter ϵ, the intersection $\phi_\epsilon \cap \phi_0'$ is still transverse and we obtain a category $\operatorname{Fuk}_{\tilde{\phi}_\epsilon}(X)$ with objects $(\phi_\epsilon, b_\epsilon), (\phi_0', b_0')$ with weakly bounding cochains. Invariance of Lagrangian Floer theory under Hamiltonian isotopy (specifically, the homotopy invariance of the A_∞ bimodule associated to a pair of Lagrangians) implies that that ϕ_0' admits a weakly bounding cochain ϕ_0' and so that the objects $(\phi_0, b_0), (\phi_0', b_0')$ are quasi-isomorphic in

$\mathrm{Fuk}_{\tilde{\phi}_0}(X)$. That is, there exist elements

$$\alpha_0 \in CF(\phi_0, \phi_0'), \quad \beta_0 \in CF(\phi_0', \phi_0), \quad \delta_0 \in CF(\phi_0, \phi_0), \quad \delta_0' \in CF(\phi_0', \phi_0') \tag{7}$$

so that

$$m_2^{b_0, b_0'}(\alpha_0, \beta_0) - 1_{\phi_0} = m_1^{b_0}(\delta_0), \quad m_2^{b_0', b_0}(\beta_0, \alpha_0) - 1_{\phi_0'} = m_1^{b_0'}(\delta_0').$$

It follows from the associativity of the composition law that

$$HF(\phi_0, b_0) \cong HF(\phi_0', b_0').$$

We prove a similar theorem for the surgered immersion:

Theorem 1.6. *Under the assumptions of Theorem 1.3, the objects $(\phi_\epsilon, b_\epsilon)$ and (ϕ_0', b_0') are quasi-isomorphic in $\mathrm{Fuk}_{\tilde{\phi}_\epsilon}(X)$, and in particular have their endomorphism algebras have isomorphic cohomology $HF(\phi_\epsilon, b_\epsilon) \cong HF(\phi_0, b_0)$.*

More generally, in any reasonable definition of the Fukaya category we expect that (ϕ_0, b_0) and $(\phi_\epsilon, b_\epsilon)$ are quasiisomorphic. This would be the mirror statement to invariance of the isomorphism class of the bundle under (deformed) Yang-Mills heat flow.

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2. Lagrangian surgery

Lagrangian surgery was introduced by Lalonde-Sikorav [58] in dimension two and Polterovich [69] for arbitrary dimension. Surgery smooths a self-intersection point by removing small balls around the preimages of the self-intersection point and gluing in a handle. Haug [47] introduced generalizations to handles of higher index, which we do not consider here.

2.1. **The local model.** The local model for the surgery is obtained by parallel transport of the vanishing cycle of the standard Lefschetz fibration along a line parallel to the real axis, as explained in Seidel [78, Section 2e]. The *standard Lefschetz fibration* is the map

$$\pi : \mathbb{C}^n \to \mathbb{C}, \quad (z_1, \ldots, z_n) \mapsto z_1^2 + \ldots + z_n^2. \tag{8}$$

Equip $\mathbb{C}^n$ with the standard symplectic form $\omega \in \Omega^2(\mathbb{C}^n)$. The space $\mathbb{C}^n - \{0\}$ has a natural connection given by a horizontal sub-bundle

$$T_z^h \subset T_z(\mathbb{C}^n - \{0\}), \quad T_z^h = (\mathrm{Ker}\, D_z\pi)^\omega$$

equal to the union of symplectic perpendiculars of the fibers $\pi^{-1}(z)$. For any path

$$\gamma : [0,1] \to \mathbb{C} - \{0\}$$

there is a symplectic parallel transport map

$$T_\gamma : \pi^{-1}(\gamma(0)) \to \pi^{-1}(\gamma(1))$$

by taking the endpoint of a horizontal lift of γ with any given initial condition. Let $\gamma : [0,1] \to \mathbb{C}$ be an embedded path with endpoint $\gamma(0) = 0$ at the critical point of the Lefschetz fibration. Each fiber of π over $\gamma([0,1])$ has a *vanishing cycle* $C_z \subset \pi^{-1}(z)$ defined as the set of elements $w \in \pi^{-1}(z)$ that limit to the origin $0 \in \mathbb{C}^n$ under symplectic parallel transport $C_z \to C_0$. If $S^{n-1} \subset \mathbb{R}^n \subset \mathbb{C}^n$ is the unit sphere, then explicitly

$$C_z := \sqrt{z} S^{n-1}, \quad z \in \mathbb{C}.$$

Now let $\gamma : \mathbb{R} \to \mathbb{C}$ be an embedded path so that except for t in some compact interval $[-c, c]$, we have

$$\gamma(t) = t + i2\epsilon, \quad \forall t \notin [-c, c]. \tag{9}$$

For example, one could assume that the path is the affine linear path $\gamma(t) = t + i2\epsilon$. The *handle Lagrangian* H_γ is the union of vanishing cycles over γ:

$$H_\gamma := \bigcup_{t \in \mathbb{R}} C_{\gamma(t)}.$$

As in [76, Discussion after (1.12)], H_γ may be equivalently defined as symplectic parallel transport of C_z along γ.

More generally, as pointed out by Seidel [78, Section 2e], one may define surgery by allowing more general paths in the base of the Lefschetz fibration. By bending the path somewhat below the real axis one can achieve a *zero-area surgery* for which the disks have the same area as for the original. However, we will only use the straight paths for the classification of disks in Section 7.

The handle has the following explicit description. Let $\mathbb{C}^n \cong \mathbb{R}^{2n}$ be equipped with Darboux coordinates

$$z = (z_1, \ldots, z_n), \quad z_k = q_k + ip_k, \quad k = 1, \ldots, n.$$

For a real number ϵ with $|\epsilon|$ small define a Lagrangian submanifold H_ϵ of $\mathbb{C}^n$, the *handle* of the surgery, by

$$H_\epsilon = \left\{ (q_1 + ip_1, \ldots, q_n + ip_n) \in \mathbb{C}^n \;\middle|\; q \neq 0,\ \forall k,\ p_k = \frac{\epsilon q_k}{|q|^2} \right\}. \tag{10}$$

Identify $\mathbb{C}^n = T^\vee \mathbb{R}^n$ in the standard way. Denote the standard symplectic form

$$\omega_0 = \sum_{k=1}^n \mathrm{d}q_k \wedge \mathrm{d}p_k \in \Omega^2(\mathbb{C}^n).$$

Define

$$f_\epsilon : \mathbb{R}^n - \{0\} \to \mathbb{R}, \quad q \mapsto \epsilon \ln(|q|).$$

The Lagrangian H_ϵ is the graph of the closed one-form $\mathrm{d}f_\epsilon$:

$$H_\epsilon = \mathrm{graph}(\mathrm{d}f_\epsilon) \subset \mathbb{R}^{2n}. \tag{11}$$

Also note that $H_\epsilon \subset \mathbb{C}^n$ of (11) is invariant under the anti-symplectic involution

$$\iota : \mathbb{C}^n \to \mathbb{C}^n, \quad (p,q) \mapsto (q,p).$$

For the purposes of symplectic field theory, it is convenient to replace the above Lagrangian with one that is cylindrical near infinity in the sense of Definition 6.7 rather than only asymptotically cylindrical. By the *flattened handle* we mean the Lagrangian defined by parallel transport of a sphere along a path γ with $\gamma(t) = t$ for t outside of a compact neighbourhood of 0, and passing slightly above the critical value $0 \in \mathbb{C}$. An equivalent definition can be given explicitly as follows. Define a Lagrangian submanifold $\check{H}_\epsilon \subset \mathbb{C}^n$ equal to H_ϵ in a compact neighborhood of 0 and equal to $\mathbb{R}^n \cup i\mathbb{R}^n$ outside a larger compact neighborhood of 0 as follows. Following Fukaya-Oh-Ohta-Ono [42, Chapter 10], let

$$\zeta > 0, \quad \epsilon \neq 0$$

be constants. The constant ϵ is the *surgery parameter* describing the "size" of the Lagrangian surgery, while the parameter ζ is a cutoff parameter describing the size of the ball on whose complement the surgery ϕ_ϵ agrees with the unsurgered immersion ϕ_0. These constants will be chosen later so that ζ is large and $\zeta|\epsilon|^{1/2}$ is small. Following Fukaya et al [42, 54.5,Chapter 10] consider a function

$$\rho \in C^\infty(\mathbb{R}_{>0}), \quad \rho(r) = \begin{cases} \ln(r) - |\epsilon| & r \leq |\epsilon|^{1/2}\zeta \\ \ln(|\epsilon|^{1/2}\zeta) & r \geq 2|\epsilon|^{1/2}\zeta \end{cases} \tag{12}$$

satisfying

$$\forall r \in \mathbb{R}_{>0}, \quad \rho''(r) \leq 0 \leq \rho'(r).$$

Define

$$\check{f}_\epsilon : \mathbb{R}^n \to \mathbb{R}, \qquad q \mapsto \epsilon\rho(|q|). \tag{13}$$

Consider the graph

$$\operatorname{graph}(\mathrm{d}\check{f}_\epsilon) \subset T^\vee\mathbb{R}^n \cong \mathbb{R}^{2n}.$$

Let $U \subset X$ be a Darboux chart near x so that the self-intersection of ϕ at x has the form (15). One realization of the *flattened handle* is the union of the graph of the differential of $\check{f}_\epsilon$ and its reflection:

$$\check{H}_\epsilon = \left(\operatorname{graph}(\mathrm{d}\check{f}_\epsilon) \cap (\mathbb{C}^n - B_{|\epsilon|^{1/2}\zeta}(0))\right) \cup \iota\left(\operatorname{graph}(\mathrm{d}\check{f}_\epsilon) \cap (\mathbb{C}^n - B_{|\epsilon|^{1/2}}\zeta(0))\right). \tag{14}$$

The inclusion

$$\check{\phi}_\epsilon : \check{H}_\epsilon \to U.$$

is then a Lagrangian embedding, with image equal to that of $\mathbb{R}^n \cup i\mathbb{R}^n$ outside of a compact set.

This explicit definition of the handle agrees with the previous one by parallel transport. Indeed, since H_ϵ projects to $\operatorname{Im}(z) = 2\epsilon$, the tangent space to TH_ϵ consists of a corank one sub-bundle $T^vH_\epsilon := TH_\epsilon \cap \operatorname{Ker}(D\pi)$ and the horizontal lift of $T(\{\operatorname{Im}(z) = 2\epsilon\})$. Hence H_ϵ is obtained by symplectic parallel transport of any

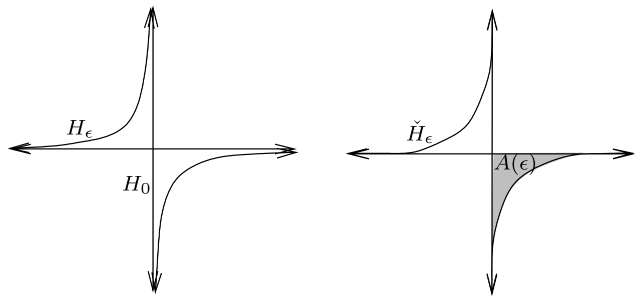

FIGURE 2. The local models

fiber. The Lagrangian $\hat{H}_\epsilon$ defined in this way is cylindrical near infinity and the argument of Proposition 2.2 (d) shows that after a change in surgery parameter the two definitions are Hamiltonian isotopic.

2.2. **Surgery and its properties.** The surgery of an immersed self-transverse Lagrangian is obtained by gluing in the local model of the previous section. Let X be a compact symplectic manifold. Let $\phi_0 : L_0 \to X$ be a self-transverse Lagrangian immersion with compact, connected domain L_0. Let

$$x = \phi_0(x_+) = \phi_0(x_-), \quad x_+ \neq x_- \in L_0$$

be an intersection point. The local model for transverse Lagrangian self-intersections (see for example Pozniak [70, Section 3.4] for the more general case of clean intersection) implies that there exist Darboux coordinates in an open ball $U \subset X$ of x

$$q_1, \ldots, q_n, p_1, \ldots, p_n \in C^\infty(U)$$

such that the two branches of ϕ_0 meeting at x are defined by

$$L_- = \{p_1 = \ldots = p_n = 0\}, \quad L_+ = \{q_1 = \ldots = q_n = 0\}. \tag{15}$$

Let $V \subset U$ be a subset so that H_0 agrees with $\check{H}_\epsilon$ outside of V.

Definition 2.1. The Lagrangian surgery of $\phi_0 : L_0 \to X$ is the immersion defined by replacing a neighborhood $U \cap L_0$ of the self-intersection points $x_-, x_+ \in L_0$ with an open subset $U \cap \check{H}_\epsilon$ of the cylindrical-near-infinity local model $\check{H}_\epsilon$:

$$L_\epsilon = \Big((L_0 - V) \cup (U \cap \check{H}_\epsilon)\Big) / \sim$$

where $\sim$ is the obvious identification of H_0 with $\check{H}_\epsilon$ on the complement of V. The *surgered immersion*

$$\phi_\epsilon : L_\epsilon \to X, \quad \phi_\epsilon = (\phi_0|_{L_0 - V}) \cup (\check{\phi}_\epsilon|_{\check{H}_\epsilon \cap U})$$

is defined by patching together the immersions $\check{\phi}_\epsilon$ of $\check{H}_\epsilon \cap U \to X$ and ϕ_0 on $L_\epsilon - V \cong L_0 - V \to X$. Let

$$A(\epsilon) = \int_S v^*\omega \tag{16}$$

be the area of a small holomorphic triangle $v : S \to X$ with boundary in $\phi_0(L_0) \cup \phi_\epsilon(L_\epsilon)$, as in Figure 2 and Equation (150) below. Equivalently, by Stokes' theorem, $A(\epsilon)$ is the difference of actions

$$A(\epsilon) = \int_{\mathbb{R}} \gamma_0^*\alpha - \gamma_\epsilon^*\alpha$$

given by the integral of the canonical one-form α over paths $\gamma_0, \gamma_\epsilon$ from ∞ in $\mathbb{R}^n$ to ∞ in $i\mathbb{R}^n$ along H_0 and $\check{H}_\epsilon$ in the local model; see the proof of Lemma 7.3. This ends the Definition.

We collect some basic properties of the surgery, most of which will be used later. See [69], [76], and [42, Chapter 10] for more details.

Proposition 2.2. *Let $\phi_0 : L_0 \to X$ be an immersed Lagrangian with transverse ordered self-intersection point $(x_-, x_+) \in L_0^2$.*

- *(a)* (Skew-symmetry) *The surgery ϕ_ϵ obtained from x with parameter $-\epsilon$ is equal to the surgery obtained from the conjugate $\overline{x}$ with parameter ϵ.*
- *(b)* (Orientation) *If L_0 is oriented and $\epsilon > 0$ then there exists an orientation on L_ϵ that agrees with that on L_0 in a complement of the handle $\check{H}_\epsilon$ if and only if the self-intersection $x \in L_0^2$ is odd.*
- *(c)* (Relative spin structure) *Any relative spin structure on $\phi_0 : L_0 \to X$ and an isomorphism* $\mathrm{Spin}(TL_0)_{x_-} \cong \mathrm{Spin}(TL_0)_{x_+}$ *defines a relative spin structure on the surgery $\phi_\epsilon : L_\epsilon \to X$.*
- *(d)* (Independence of choices) *The exact isotopy class of the surgery ϕ_ϵ is independent of all choices, up to a change in surgery parameter ϵ.*

Proof. Item (a) is immediate from the definition. Item (b) follows from the fact that the gluing maps on the ends of the handle are homotopic to $(t, v) \mapsto e^t v$ resp. $(t, v) \mapsto ie^{-t}v$. These maps are orientation preserving exactly if the intersection is odd[5]. For item (c), suppose a relative spin structure is given as a relative Čech cocycle as in [88]. Such a cocycle consists of charts $U_\alpha, \alpha \in A$ for X indexed by

[5] Recall that the self-intersection is odd if in the local model L_0 in a neighborhood of x_- resp. x_+ in L_0 is identified with $\mathbb{R}^n$ with the standard orientation induced by the volume form $\mathrm{d}q_1 \wedge \ldots \wedge \mathrm{d}q_n$ resp. $i\mathbb{R}^n$ with the opposite orientation $-\mathrm{d}p_1 \wedge \ldots \wedge \mathrm{d}p_n$. Reversing the sign of ϵ changes the order of the branches, and so changes the parity of the self-intersection if and only if $\dim(L_0)$ is odd. Thus in the case that $\dim(L_0)$ is odd, there is always some choice of sign ϵ for which the oriented surgery exists regardless of the parity of $x = (x_-, x_+)$. On the other hand, if $\dim(L_0)$ is even then either both surgeries exist as oriented surgeries or neither. The existence of orientations on the surgery is related to the fact that the monodromy of a Lefschetz fibration is orientation preserving exactly in odd dimensions.

some set A, corresponding charts $V_\alpha \subset \phi^{-1}(U_\alpha)$ for L_0, and transition functions defined as follows. For $\alpha, \beta \in A$ let

$$V_{\alpha\beta} = V_\alpha \cap V_\beta, \quad \text{resp. } g_{\alpha\beta} : V_{\alpha\beta} \to SO(n)$$

denote the intersections of the charts for L_0 resp. transition maps for the tangent bundle TL_0. A relative spin structure is a collection of lifts $\tilde{g}_{\alpha\beta}$ and signs $o_{\alpha\beta\gamma}$ given by maps

$$\tilde{g}_{\alpha\beta} : V_{\alpha\beta} \to \text{Spin}(n), \quad o_{\alpha\beta\gamma} : U_\alpha \cap U_\beta \cap U_\gamma \to \{\pm 1\}$$

such that the following relative cocycle condition holds:

$$\tilde{g}_{\alpha\beta}\tilde{g}_{\alpha\gamma}^{-1}\tilde{g}_{\beta\gamma} = \phi^* o_{\alpha\beta\gamma}, \quad \forall \alpha, \beta, \gamma \in A.$$

To obtain the relative spin structure on the surgery L_ϵ we take the cover on the surgery with a single additional open set on the handle $U_0 := H_\epsilon$ with no triple intersections. The relative spin structure is defined by transition maps near the handle $\tilde{g}_{0\alpha} = \tilde{g}_{0\beta} = \text{Id}$.

A more precise reformulation of the independence in item (d) is the following: Let $U^1, U^2 \subset X$ and $\check{H}^1_\epsilon, \check{H}^2_\epsilon \subset \mathbb{C}^n$ be two sets of such choices and $\phi^1_{\epsilon_1}, \phi^2_{\epsilon_2}$ the corresponding families of surgeries for parameters ϵ_1, ϵ_2. For any $\epsilon_1 \in \mathbb{R}$ with $|\epsilon_1|$ small there exists $\epsilon_2 \in \mathbb{R}$ so that $\phi^1_{\epsilon_1}$ is an exact deformation of $\phi^2_{\epsilon_2}$. In particular if both $\phi^1_{\epsilon_1}$ and $\phi^2_{\epsilon_2}$ are embeddings then $\phi^2_{\epsilon_2}(L_{\epsilon_2})$ is Hamiltonian isotopic to $\phi^1_{\epsilon_1}(L_{\epsilon_1})$. To prove this claim, recall that any Lagrangian $\phi'_\epsilon : L_\epsilon \to X$ nearby a given one is a graph of a one form $\phi'_\epsilon = \text{graph}(\alpha)$ for some $\alpha \in \Omega^1(L_\epsilon)$ and local model $T^\vee L_\epsilon \supset U \to X$. An exact deformation is one generated by exact one-forms, as in Weinstein [90]. Exact deformations are equivalent to deformation by Hamiltonian diffeomorphisms in the embedded case, but not in general. Any two Darboux charts are isotopic after shrinking, by Moser's argument. The approximations $\check{H}_\epsilon$ are also independent up to isotopy of the choice of cutoff function. Therefore, any two choices of surgery are isotopic through Lagrangian immersions $\phi^t_\epsilon : L_\epsilon \to X$. In particular, the infinitesimal deformation $\frac{d}{dt}\phi^t_\epsilon$ is given by a closed one-form $\alpha^t_\epsilon \in \Omega^1(L_\epsilon)$.

We distinguish the following two cases in item (d): Firstly, if the surgery connects different components of the Lagrangian L_0 then the positive-degree homology $H_{>0}(L_\epsilon)$ is isomorphic to $H_{>0}(L_0)$. On the other hand, if the surgery connects the same component of the Lagrangian, then by the Mayer-Vietoris Theorem $H_{>0}(L_\epsilon)$ has at most two additional generators. If $n = 2$ then the integral of α^t_ϵ on the additional generator corresponding to the meridian is, by Stokes' theorem, the evaluation of the relative symplectic class $[\omega] \in H^2(\mathbb{C}^2, \check{H}_\epsilon)$ on the generator in $H_2(\mathbb{C}^2, \check{H}\epsilon)$. Such a generator is given by a disk $u : S \to X, S = \{|z| \leq 1\}$ with boundary $u(\partial S)$ on the meridian $S^1 \times \{0\}$ of the handle. The disk u may be deformed to a disk $u_0 : S \to X$ taking values in $\mathbb{R}^2$, and so has vanishing area $A(u) = A(u_0) = 0$. Returning to the case of arbitrary $n \geq 2$, the action $\int_{\mathbb{R}} \gamma^*_\epsilon \alpha$

along a longitude $\gamma_\epsilon : \mathbb{R} \to \check{H}_\epsilon$ takes on all positive values near 0 as ϵ varies. It follows that for any such $\phi_\epsilon^t, t \in [0,1]$ there exists a family $\epsilon(t), \epsilon(0) = \epsilon$ such that the deformation is given by an exact form. Compare Sheridan-Smith [81, Section 2.6]. □

Remark 2.3. (Gradings) By Seidel [78], absolute gradings on Floer cohomology groups are provided by gradings of the Lagrangians: Given a positive integer N, an N-*grading* of a Lagrangian L is a lift of the natural map

$$L \to \operatorname{Lag}(TX), \quad x \mapsto T_x L$$

from L to the bundle of Lagrangian subspaces $\operatorname{Lag}(TX)$ to an N-fold Maslov cover

$$\operatorname{Lag}^N(TX) \to \operatorname{Lag}(TX).$$

If L_0 is graded by a map $\tilde{\phi}_0 : L_0 \to \operatorname{Lag}^N(X)$ and the self-intersection point has degree 1 then $\phi_\epsilon : L_\epsilon \to X$ is graded [78, Section 2e].

Remark 2.4. (Brane structures) A *brane structure* for ϕ_0 consists of an orientation, relative spin structure, and Λ_0-valued local system $y \in \mathcal{R}(\phi_0)$. For any holomorphic treed disk $u : S \to X$ the holonomy of the local system around the boundary is denoted by

$$y(\partial u) \in \Lambda_0, \quad y : \pi_1(\phi(L)) \to \Lambda_0.$$

Any local system on ϕ_0 induces a local system on ϕ_ϵ, trivial on the handle, by identifying the local system on the handle with the fiber of the local system over the self-intersection point. Remark 2.3 and Proposition 2.2 imply that any brane structure on ϕ_0 induces a brane structure on ϕ_ϵ, at least if the gradings are collapsed to $\mathbb{Z}_2$-gradings.

3. Treed holomorphic disks

We recall the construction of a strictly unital A_∞ algebra from Charest-Woodward [21]. We re-write the construction in terms of cellular cochains, rather than Morse cochains.

3.1. **Treed disks.** A *disk* will mean a 2-manifold-with-boundary $S_\circ$ equipped with a complex structure so that the surface $S_\circ$ is biholomorphic to the closed unit disk $\{z \in \mathbb{C} \mid |z| \le 1\}$. A *sphere* will mean a complex one-manifold $S_\bullet$ biholomorphic to the complex projective line $\mathbb{P}^1 = \{[\zeta_0 : \zeta_1] \mid \zeta_0, \zeta_1 \in \mathbb{C}\}$. A *nodal disk* S is a union

$$S = \left(\bigcup_{i=1}^{n_\circ} S_{\circ,i} \right) \cup \left(\bigcup_{i=1}^{n_\bullet} S_{\bullet,i} \right) \Big/ \sim$$

of a finite number of disks $S_{\circ,i}, i = 1, \ldots, n_\circ$ and spheres $S_{\bullet,i}, i = 1, \ldots, n_\bullet$ identified at pairs of distinct points called *nodes* $w_1, \ldots, w_m$. Each node

$$w_e = (w_e^-, w_e^+) \in S_{i_-(e)} \times S_{i_+(e)}$$

is a pair of distinct points (either both interior or both boundary points) where $S_{i_\pm(e)}$ are the (disk or sphere) components adjacent to the node; the resulting topological space S is required to be simply-connected and the boundary ∂S is required to be connected. The complex structures on the disks $S_{\circ,i}$ and spheres $S_{\bullet,i}$ induce a complex structure on the tangent bundle TS (which is a vector bundle except at the nodal points) denoted $j : TS \to TS$. A *boundary resp. interior marking* of a nodal disk S is an ordered collection of non-nodal points

$$\underline{z} = (z_0, \dots, z_d) \in \partial S^{d+1} \quad \text{resp.} \quad \underline{z}' = (z'_1, \dots, z'_c) \in \operatorname{int}(S)^c \tag{17}$$

on the boundary resp. interior, whose ordering is compatible with the orientation on the boundary ∂S. The *combinatorial type* $\Gamma(S)$ is the graph whose vertices, edges, and head and tail maps

$$(\operatorname{Vert}(\Gamma(S)), \operatorname{Edge}(\Gamma(S))), (h \times t) : \operatorname{Edge}(\Gamma(S)) \to \operatorname{Vert}(\Gamma(S)) \cup \{\infty\}$$

are obtained by setting $\operatorname{Vert}(\Gamma(S))$ to be the set of disk and sphere components and $\operatorname{Edge}(\Gamma(S))$ the set of nodes (each connected to the vertices corresponding to the disks or spheres they connect). The graph $\Gamma(S)$ is required to be a tree, which means that Γ is connected with no cycles among the combinatorially finite edges; each edge $e \in \operatorname{Edge}(\Gamma(S))$ is oriented so that it points towards the outgoing leaf $e_0 \in \operatorname{Edge}(\Gamma(S))$ corresponding to the marking z_0. An edge e is combinatorially finite neither of its ends are at infinity. The set of edges $\operatorname{Edge}(\Gamma(S))$ is equipped with a partition into subsets $\operatorname{Edge}_\bullet(\Gamma(S)) \cup \operatorname{Edge}_\circ(\Gamma(S))$ corresponding to interior resp. boundary markings respectively. The set of boundary edges $(h^{-1}(v) \cup t^{-1}(v)) \cap \operatorname{Edge}_\circ(\Gamma(S))$ meeting some vertex $v \in \operatorname{Vert}(\Gamma(S))$ is equipped with a cyclic ordering giving $\Gamma(S)$ the partial structure of a ribbon graph. Define

$$\begin{aligned} \operatorname{Edge}_\to(\Gamma(S)) &:= h^{-1}(\infty) \cup t^{-1}(\infty) \\ \operatorname{Edge}_{\circ,\to}(\Gamma(S)) &:= \operatorname{Edge}_\circ(\Gamma(S)) \cap \operatorname{Edge}_\to(\Gamma(S)) \\ \operatorname{Edge}_{\bullet,\to}(\Gamma(S)) &:= \operatorname{Edge}_\bullet(\Gamma(S)) \cap \operatorname{Edge}_\to(\Gamma(S)) \end{aligned}$$

The sets $\operatorname{Edge}_{\circ,\to}(\Gamma(S))$, $\operatorname{Edge}_{\bullet,\to}(\Gamma(S))$ of boundary and interior *semi-infinite edges* is each equipped with an ordering; these orderings will be omitted from the notation to save space. A marked disk $(S, \underline{z}, \underline{z}')$ is *stable* if it admits no non-trivial automorphisms $\varphi : S \to S$ preserving the markings $\underline{z}, \underline{z}'$. The moduli space of stable disks with fixed number $d \geq 0$ of boundary markings and no interior markings admits a natural structure of a cell complex which identifies the moduli space with Stasheff's associahedron.

Treed disks are defined by replacing nodes with broken segments as in the pearly trajectories of Biran-Cornea [14] and Seidel [80]. A *segment* will mean a closed one-manifold with boundary homeomorphic to a connected closed subset of the real line. Given two such subsets, one with a non-compact end at infinity and another with a non-compact end at infinity we may form a new closed manifold with boundary by gluing along the infinite ends, which we call a *broken segment.*

A *treed disk* is a topological space C obtained from a nodal disk S by replacing each boundary node or boundary marking corresponding to an edge $e \in \Gamma(S)$ with a (possibly broken) segment. Each such segment T_e is naturally equipped with a *length* $\ell(e) \in \mathbb{R}_{\geq 0} \cup \{\infty\}$, where the semi-infinite edges $e \in \Gamma(S)$ are automatically assigned infinite length. A treed disk C may be written as a union $C = S \cup T$ where the one-dimensional part T is joined to the two-dimensional part S at a finite set of points on the boundary of S, called the *nodes* $w \in C$ of the treed disk (as they correspond to the nodes in the underlying nodal disk.) The semi-infinite edges e in the one-dimensional part T are oriented by requiring that the root edge e_0 is outgoing while the remaining leaves $e_1, \ldots, e_d$ are incoming; the outgoing leaf e_0 is referred to as the *root* while the other semi-infinite edges $e_1, \ldots, e_d$ are *leaves.*

In particular, we have the following gluing construction which produces treed disks from a pair of treed disks. Given treed disks C_1, C_2 and and a leaf T_{e_2} of C_2 one may glue together C_1 and C_2 by identifying the point at infinity along the root edge T_{e_1} of C_1 with the point at infinity for an incoming leaf of C_2, creating a treed disk

$$C = C_1 \cup \{\infty\} \cup C_2, \quad T_e := T_{e_1} \cup \{\infty\} \cup T_{e_2} \tag{18}$$

with a broken edge $T_e \subset C$. We say that the treed disks C_1, C_2 are obtained from C by *cutting along* e. The combinatorial type

$$\Gamma(C) = (\mathrm{Vert}(\Gamma(C)), \mathrm{Edge}(\Gamma(C)))$$

of a treed disk C is defined similarly to that for disks with the following addition: The set of edges $\mathrm{Edge}(\Gamma(C))$ is equipped with a partition

$$\mathrm{Edge}(\Gamma(C)) = \mathrm{Edge}_0(\Gamma(C)) \cup \mathrm{Edge}_{(0,\infty)}(\Gamma(C)) \cup \mathrm{Edge}_\infty(\Gamma(C))$$

indicating whether the length is zero, finite and non-zero, or infinite.

The space of isomorphism classes of treed disks satisfying a stability condition is compact and Hausdorff with a universal curve. A treed disk $C = S \cup T$ is *stable* if the underlying nodal disk obtained by collapsing edges $T_e \subset T$ to points is stable. An example of a treed disk with one broken edge (indicated by a small hash through the edge) is shown in Figure 3. In the Figure, the interior leaves $e \in \mathrm{Edge}_\bullet(\Gamma)$ are not shown and only their attaching points $w_e \in S \cap T$ are depicted so as not to clutter the figure. For a given combinatorial type Γ, denote by $\mathcal{M}_\Gamma$ the moduli space of treed disks whose domains have combinatorial type Γ and

$$\overline{\mathcal{M}}_d = \cup_\Gamma \mathcal{M}_\Gamma$$

the union over stable types Γ with d leaves. The moduli space $\overline{\mathcal{M}}_d$ is compact. It admits a universal curve $\overline{\mathcal{U}}_d$, given as the space of isomorphism classes of pairs $[C, z]$ where C is a holomorphic treed disk and $z \in C$ is any point, either in S or T. The two cases correspond to a splitting

$$\overline{\mathcal{U}}_d = \overline{\mathcal{S}}_d \cup \overline{\mathcal{T}}_d \tag{19}$$

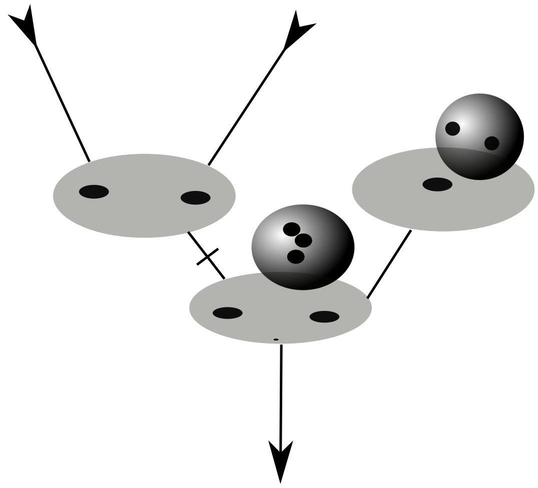

FIGURE 3. A treed disk with $d = 2$ incoming boundary edges

of the universal treed disk into one-dimensional and two-dimensional parts $\overline{\mathcal{T}}_d$ resp. $\overline{\mathcal{S}}_d$ where the fibers of $\overline{\mathcal{T}}_d \to \overline{\mathcal{M}}_d$ resp. $\overline{\mathcal{S}}_d \to \overline{\mathcal{M}}_d$ are one resp. two-dimensional. Denote by $\mathcal{S}_\Gamma$ resp. $\mathcal{T}_\Gamma$ the surface resp. tree parts of the universal treed disk living over $\mathcal{M}_\Gamma$, and similarly their closures $\overline{\mathcal{S}}_\Gamma, \overline{\mathcal{T}}_\Gamma$ over $\overline{\mathcal{M}}_\Gamma$. If for some types Γ', Γ the moduli space $\mathcal{M}_{\Gamma'}$ is contained in $\overline{\mathcal{M}}_\Gamma$ then we write $\Gamma' \preceq \Gamma$.

3.2. **Cell decompositions.** We introduce notation for cell decompositions. A *finite cell complex* of dimension d is a space L_d obtained from a finite cell complex L_{d-1} of dimension $d-1$ by attaching a collection of d-cells via *attaching maps*. The topology is induced from the quotient relation given by the attaching maps. The standard ball and sphere in dimension $d(i)$ are denoted

$$B^{d(i)} = \{v \in \mathbb{R}^{d(i)}, \|v\| \leq 1\}, \quad S^{d(i)-1} = \partial B^{d(i)}.$$

A *finite cellular decomposition* of an n-manifold L is a finite cell complex given by maps

$$\partial\sigma_i : S^{d(i)-1} \to L_{d(i)-1}, \quad i = 1, \ldots, k.$$

to $L_{d(i)-1}$ together with a homeomorphism of L_n with L. The induced maps from the cells $B^{d(i)}$ into L are denoted $\sigma_i : B^{d(i)} \to L$. For each d each point $x \in L$ is in the image of the interior of at most one of the d-cells $\sigma_i(\operatorname{int} B^{d(i)})$. For $d \geq 0$ the *d-skeleton* of a cellular structure on L consists of images of balls

$$L_d = \bigcup_{d(i) \leq d} \sigma_i(B^{d(i)}) \subset L.$$

of dimension $d(i) \leq d$. Our cell decompositions will be cell decompositions in the smooth sense. Thus the interiors $\sigma_i | \operatorname{int}(B^{d(i)})$ are diffeomorphisms onto their images in L.

The cellular chain complex is derived from the long exact sequence for pairs of skeleta, as in Hatcher [46, Section 2.2, Section 3.1, Section 4.1, Appendix A]. For $i \geq 0$ let $H_i(L_d, L_{d-1})$ denote the i-th relative singular homology group of the skeleta L_d relative to the lower-dimensional skeleta L_{d-1}. By excision,

$$H_d(L_d, L_{d-1}) \cong \bigoplus_{d(i)=d} H_d(B^i, \partial B^i) \cong \mathbb{Z}^{\#\{i \,|\, d(i)=d\}}, \quad \forall d = 0, \ldots, n.$$

A *cellular chain* of dimension d is a formal combination of the cells of dimension d, or equivalently an element of $H_d(L_d, L_{d-1})$. The space of cellular chains of dimension d is

$$C_d(L) = H_d(L_d, L_{d-1}).$$

The *cellular boundary operator* $\partial_d : C_d(L) \to C_{d-1}(L)$ is the connecting morphism in the long exact sequence for the inclusion $(L_{d-1}, L_{d-2}) \to (L_d, L_{d-2})$. For cells σ, τ of L denote

$$\partial(\sigma,\tau) \in \mathbb{Z}, \quad \partial\sigma = \sum_{\tau \in \mathcal{I}^c(\phi)} \partial(\sigma,\tau)\tau \tag{20}$$

the coefficient of τ in $\partial\sigma$.

In our applications, the cell decompositions will be induced by a gradient flow of a Morse function. In this case, the boundary operator in (20) has a simple description in terms of counts of gradient trajectories. Recall that a *Morse-Smale pair* on L is a pair (f, g) consisting of a Morse function and Riemannian metric

$$f : L \to \mathbb{R}, \quad g : TL \times_L TL \to \mathbb{R}$$

so that the stable and unstable manifolds of (f, g) meet transversally. Any Morse-Smale pair on L gives rise (somewhat non-canonically) to a cellular decomposition whose cellular chain complex is equal to the Morse chain complex of L (see for example [9, Theorem 4.9.3]). The images of the interiors of the cells σ are the stable manifolds of the critical points of f. In this description, the coefficient $\partial(\sigma,\tau)$ is the number of isolated Morse trajectories $\gamma : \mathbb{R} \to L$ connecting the unique critical points $x(\sigma), x(\tau) \in \mathrm{crit}(f)$ contained in σ, τ, counted with sign.

Cellular homology is functorial for cellular maps. A smooth map $\psi : L \to N$ between manifolds L, N equipped with cell decompositions is *cellular* if

$$\psi(L_d) \subseteq N_d, \quad \forall d = 0, \ldots, \dim(L).$$

Any cellular map $\psi : L \to N$ induces a chain homomorphism $\psi_* : C_\bullet(L) \to C_\bullet(N)$ independent of the cellular homotopy type of ψ. On the other hand, by the smooth cellular approximation theorem any map is homotopic to a cellular map, and any two homotopic cellular maps are cellularly homotopic. Let L_1, L_2 be smooth manifolds and $\psi : L_1 \to L_2$ a smooth map. A cellular approximation of ψ may be chosen inductively, starting with a map on the 0-skeleton $\psi_0 : L_{1,0} \to L_{2,0}$. Cellular approximations of maps naturally induce cellular approximations for their products: Let L_1', L_2', L_1'', L_2'' be smooth manifolds and $\psi' : L_1' \to L_2'$ and

$\psi'' : L_1'' \to L_2''$ be smooth maps. Any cellular approximations for ψ', ψ'' induce a cellular approximation for $\psi' \times \psi''$.

By a cellular approximation of the diagonal we mean the following. Choose a possibly different second cellular decomposition $\sigma^i : B^{d'(i)} \to L$ inducing skeleta L^d with cellular boundary operator

$$\partial^\vee : \bigoplus_{d=0}^{n} H_d(L^d, L^{d-1}) \to \bigoplus_{d=0}^{n} H_{d-1}(L^{d-1}, L^{d-2}).$$

Denote the cellular structure on the diagonal obtained by taking products by

$$\sigma_i \times \sigma^j : B^{d(i)} \times B^{d'(j)} \to L \times L, \quad \forall i \in \mathcal{I}^c(\phi), j \in \mathcal{I}^{c,\vee}(\phi).$$

The product $L \times L$ has d-skeleton the union of skeleta L_i, L^j of the factors whose dimension sum to the product:

$$(L \times L)_d = \bigcup_{i+j \leq d} (L_i \times L^j), \quad \forall d = 0, \ldots, 2n.$$

Choose a cellular approximation of the diagonal, that is, a homotopy

$$\delta_t : L \to L \times L, \quad t \in [0, 1]$$

where

$$\delta_0(\ell) = (\ell, \ell), \quad \forall \ell \in L$$

is the diagonal and $\delta_1 : L \to L \times L$ is a cellular map. The homology class of the diagonal has expansion in the cellular decomposition

$$[\delta_1(L)] = \sum_{i,j} c(\sigma_i, \sigma^j)[\sigma_i \times \sigma^j] \in H_n((L \times L)_n, (L \times L)_{n-1}) \tag{21}$$

for some coefficients

$$c(\sigma_i, \sigma^j) \in \mathbb{Z}, \quad i, j = 1, \ldots, k$$

and $n = \dim(L)$. Since $\delta_1 : L \to L \times L$ gives rise to a cycle in the cellular chain complex we have

$$(\partial \otimes 1 + 1 \otimes \partial^\vee)[\delta_1] = 0. \tag{22}$$

In terms of matrix coefficients (22) translates to the conditions

$$\sum_\alpha \partial(\alpha, \beta) c(\alpha, \gamma) = -\sum_\epsilon c(\beta, \epsilon) \partial^\vee(\epsilon, \gamma). \tag{23}$$

We require that our cell decompositions admit duals. A cell complex $\{\sigma\}$ is *dual* to a given cell decomposition $\{\tau\}$ if for each cell σ there is a unique dual cell τ of complementary dimension meeting σ transversally once, as in Seifert-Threlfall [77, p. 250]. Any smooth manifold admits a cell structure admitting a dual. In particular, for dual complexes the matrix $c(\alpha, \beta)$ is the identity matrix for a suitable indexing of the cells $\mathcal{I}^c(\phi), \mathcal{I}^{c,\vee}(\phi)$ and (23) says that the differentials are transposes. In the case of cellular decompositions defined by gradient flows of a Morse-Smale pair, a dual decomposition is that associated to the gradient flow of the additive inverse; see Abbondondalo-Majer [3] for the cell structure associated to a Morse function; the fact that the cell structure for the inverse is dual is immediate from the definitions. Cellular cochains are defined as sums of the relative cohomology groups of the skeleta:

$$C^\bullet(L) = \bigoplus_{d=0}^{\dim(L)} C^d(L), \quad C^d(L) = H^d(L_d, L_{d-1}).$$

A product on cellular cochains is obtained by pullback under a cellular approximation δ_1 of the diagonal, that is,

$$\delta_1^* : C^k(L) \otimes C^l(L) \to C^{k+l}(L), \quad k, l = 0, \ldots, \dim(L).$$

This induces an associative product in cohomology.

3.3. **Treed holomorphic disks.** Treed holomorphic disks for immersed Lagrangians are defined as in the embedded case, but requiring a double cover of the tree parts to obtain the boundary lift. To define holomorphic treed disks, choose an almost complex structure and cellular structure as follows. Let $J : TX \to TX$ be an almost complex structure compatible with the symplectic form $\omega \in \Omega^2(X)$. We will assume that J is adapted to the local intersections of $\phi : L \to X$ in the following sense: For any self-intersection there is a Darboux chart on $U \subset X$ as in (15) so that

$$L \cap U \cong \mathbb{R}^n \cup i\mathbb{R}^n$$

and J is given by the standard complex structure on $U \subset \mathbb{C}^n$. Choose a pair of cell decompositions

$$\sigma_i : B^{d(i)} \to L, i \in \mathcal{I}^c(\phi) \quad \sigma^j : B^{d(j)} \to L, j \in \mathcal{I}^{c,\vee}(\phi)$$

with index sets $\mathcal{I}^c(\phi), \mathcal{I}^{c,\vee}(\phi)$, where B^d is the closed unit ball of dimension d. The product $L \times L$ inherits the product cellular decomposition

$$\sigma_i \times \sigma^j : B^{d(i)+d(j)} \cong B^{d(i)} \times B^{d(j)} \to L \times L, \ (i,j) \in \mathcal{I}^c(\phi) \times \mathcal{I}^{c,\vee}(\phi). \tag{24}$$

Choose a cellular approximation of the diagonal

$$(\delta_t : L \to L \times L)_{t \in [0,1]}.$$

The reader may take the image $\delta_t(L)$ to be the image of the diagonal under the map $l \mapsto (l, \varphi_t(l))$ generated by the time t flow φ_t of a Morse-Smale pair, so that in particular $\delta_1(L)$ is the union (over critical points) of the products of stable and unstable manifolds.

A holomorphic treed disk consists of a map from the surface part of a treed disk, together with a lift of the boundary to a map to the Lagrangian. Since our Lagrangians are only immersed, the domain of the boundary map is a one-manifold with boundary as follows: Given a treed disk $C = S \cup T$, denote by

$$\widetilde{\partial S} = (\partial S - ((\partial S) \cap T)) \cup \{w_<, w_> \in (\partial S) \cap T\} \tag{25}$$

the compact one-manifold obtained by replacing each element w of $\partial S \cap T$ with a pair of points $w_<, w_>$ lying in the closure of the component of the boundary $\partial S - (\partial S) \cap T)$ which lies before resp. after the intersection point. Each component of the boundary $(\partial S)_i \subset \partial S - T$ has closure in $\widetilde{\partial S}$ that is homeomorphic to a closed interval. Let

$$\iota : \widetilde{\partial S} \to S$$

denote the canonical map that is generically $1-1$ except for the fibers over the intersection points $S \cap T$ which are $2-1$. Consider a pair of maps

$$u : S \to X, \quad \partial u : \widetilde{\partial S} \to L$$

so that

$$u \circ \iota = \phi \circ \partial u.$$

For each edge T_e adjacent to a component $u_v : S_v \to X$, evaluation at the end points $w_<, w_>$ defines an element

$$\operatorname{ev}_e(u_v) := (\operatorname{ev}_{e,<}(u_v), ev_{e,>}(u_v)) := (\partial u_v(w_<), \partial u_v(w_>)) \in L^2. \tag{26}$$

We introduce the following notation for vector fields. Let $\operatorname{Vect}(X)$ denote the space of vector fields on X, and let $\operatorname{Vect}_h(X) \subset \operatorname{Vect}(X)$ denote the subset of Hamiltonian vector fields. For any subset $U \subset X$, let

$$\operatorname{Vect}_h(X, U) \subset \operatorname{Vect}(X) \tag{27}$$

be the space of Hamiltonian vector fields vanishing on $U \subset X$. Let

$$H \in \Omega^1(S, \operatorname{Vect}_h(X))$$

be a one-form with values in Hamiltonian vector fields supported in the interior of S. Denote by

$$\mathrm{d}_H u = \mathrm{d}u - H \circ u \in \Omega^1(S, u^*TX)$$

the Hamiltonian-perturbed exterior derivative.

Definition 3.1. A (J,H)*-holomorphic treed disk* with boundary in $\phi : L \to X$ consists of a treed disk $C = S \cup T$ and continuous maps

$$u : S \to X, \quad \partial u : \widetilde{\partial S} \to L, \quad l : \operatorname{Edge}(\Gamma) \to L$$

so that $u \circ \iota = \phi \circ \partial u$ and

(a) the map u is (J,H)-holomorphic on $S - (S \cap T)$, that is,

$$J \mathrm{d}_H(u|_S) = \mathrm{d}_H(u|_S) j \tag{28}$$

(b) at each pair of components S_{v_-}, S_{v_+} joined by an edge $T_e \subset T$ either
  (i) ∂u has a branch change, in which case $\operatorname{ev}_e(u_{v_-}) = \operatorname{ev}_e(u_{v_+})$ with notation as in (26)

(ii) or ∂u has no branch change at $T_e \cap S$ in which case the matching condition

$$(\mathrm{ev}_e(u_{v_-}), \mathrm{ev}_e(u_{v_+}) = \delta_{\tilde{\ell}(e)}(l(e)) \tag{29}$$

is required where

$$\tilde{\ell}(e) = \frac{\ell(e)}{\ell(e)+1}$$

and $\ell(e)$ is the length of e.

We denote a treed disk by $(C, u : S \to X)$, omitting ∂u and v to save space. An *isomorphism* between treed disks $(C, u : S \to X)$ and $(C', u : S' \to X)$ is an isomorphism of treed disks $\psi : C \to C'$ so that $u' \circ (\psi_S) = u$.

A compactified moduli space for any type is obtained after imposing a stability condition.

Definition 3.2. A holomorphic treed disk $(C = S \cup T,\ u : S \to X)$ is *stable* if it has no non-trivial automorphisms $\psi : C \to C$, or equivalently

(a) each disk component $S_v \subset S, v \in \mathrm{Vert}_\circ(\Gamma)$ on which the map u is constant (that is, a ghost disk bubble) has at least one interior node $w_e \in \mathrm{int}(S_v)$ or has at least three boundary nodes $w_e \in \partial S_v$;
(b) each sphere component $S_v \subset S, v \in \mathrm{Vert}_\bullet(\Gamma)$ on which the map u is constant (that is, a ghost sphere bubble) has at least three nodes $w_e \in \partial S_v$.

The combinatorial data of a treed holomorphic disk is packaged into a labelled graph called the *combinatorial type*:

Definition 3.3. For a holomorphic treed disk $u : S \to X$ the combinatorial type $\mathbb{\Gamma}$ of u is the combinatorial type Γ of the underlying treed disk C together with

- the labelling of vertices $v \in \mathrm{Vert}(\Gamma)$ corresponding to sphere and disk components $S_v, v \in \mathrm{Vert}(\Gamma)$ by the (relative) homology classes $u_*[S_v] \in H_2(X) \cup H_2(X, \phi(L))$, and
- the labelling $t(e) \in \{1, 2\}$ of edges e by their branch type (whether they represent a branch change of the map $\partial u : \widetilde{\partial S} \to \phi(L)$ or not).

The total homology class of a type $\mathbb{\Gamma}$ of positive area is called *primitive* if it is not the sum of homology classes of types $\mathbb{\Gamma}_1, \mathbb{\Gamma}_2$ of smaller positive area. We write $\mathbb{\Gamma} \mapsto \Gamma$ if Γ is the domain type of a map type $\mathbb{\Gamma}$.

For any combinatorial type of map $\mathbb{\Gamma}$ denote by $\mathcal{M}_{\mathbb{\Gamma}}(\phi)$ the moduli space of stable treed holomorphic disks bounding ϕ of type $\mathbb{\Gamma}$. Denote by

$$\mathcal{M}_\Gamma(\phi) = \bigcup_{\mathbb{\Gamma} \mapsto \Gamma} \mathcal{M}_{\mathbb{\Gamma}}(\phi)$$

the union of strata of stable map type $\mathbb{\Gamma}$ with domain type Γ and

$$\overline{\mathcal{M}}_d(\phi) = \bigcup_{\Gamma} \mathcal{M}_{\Gamma}(\phi)$$

the union over combinatorial types with d incoming edges. The case that C consists of a single edge so that $S = \emptyset$ and $T = \mathbb{R}$ is allowed; in this case $\mathcal{M}_{\Gamma}(\phi)$ is defined to be the manifold L.

Each stratum is cut out by a Fredholm map of Banach spaces as follows. Let $u : S \to X$ be a map of type $\mathbb{\Gamma}$. Let

$$S^{\circ} = S - \{w \in S \cap T_e, t(e) = 2\}$$

(where $t(e)$ was the number of branches of the map u on the edge e defined in 3.3) denote the complement of the points $w \in S \cap T$ representing branch changes of the map $\partial u : \widetilde{\partial S} \to L$. The surface S° is naturally a surface with *strip-like ends*: For each $w \in S \cap T_e$ above there exists a proper embedding of manifolds with boundary

$$\epsilon_w : \mathbb{R}_{>0} \times [0,1] \to S^{\circ}, \quad \lim_{s \to \infty} \epsilon_w(s,t) = w, \quad \forall t \in [0,1]$$

such that the complex structure on S pulls back to the standard complex structure in the coordinates s, t. For a Sobolev exponent $p \geq 2$ and Sobolev differentiability constant $k \geq 1$ with $kp > 2$, let

$$\mathrm{Map}^{k,p}(S^{\circ}, X) = \{u = \exp_{u_0}(\xi), \quad \xi \in \Omega^0(S^{\circ}, TX)_{k,p}\}$$

denote the space of continuous maps $u : S^{\circ} \to X$ of the form $u = \exp_{u_0}(\xi)$ where u_0 is constant in a neighborhood of infinity along the strip-like ends and $\xi \in \Omega^0(S^{\circ}, TX)_{k,p}$ has finite $W^{k,p}$ norm. In particular, ξ has k covariant derivatives in L^p using a connection on X and a metric on S° that is of product form on the ends. For each branched edge e, let

$$w_{\pm}(e) \in \partial S \cap T$$

denote the points at the end of each finite edge $T_e \subset T$ (distinguished by requiring that with the given orientation of T_e, the segment T_e points from $w_-(e)$ to $w_+(e)$) and

$$\mathrm{ev}_{\pm,e}(u) = \lim_{s \to \infty} (u(s,0), u(s,1)) \in L^2$$

the limits along the boundary of the strip-like end approaching $w_{\pm}(e)$, while for e unbranched $\mathrm{ev}_{e,\pm}(u) = u(w_{\pm}(e))$ is simply the evaluation. We denote the similar limits at the end of any leaf by $\mathrm{ev}_e(u)$. Let T_1 resp. $T_2 \subset T$ be the locus on which u is unbranched resp. branched and $\mathrm{Edge}_1(\Gamma), \mathrm{Edge}_2(\Gamma)$ the corresponding subsets of edges. define

$$\mathcal{B}_\Gamma = \left\{ \begin{array}{l} (C,u,\partial u,l) \in \Big(\mathcal{M}_\Gamma(\phi) \times \mathrm{Map}^{k,p}(S^\circ, X) \\ \times \mathrm{Map}^{k-1/p,p}(\partial S^\circ, L) \times \mathrm{Map}(\mathrm{Edge}_1(\Gamma), L) \Big) \text{ so} \\ u \circ \iota = \phi \circ \partial u, \ (\mathrm{ev}_{e,-}(u), \mathrm{ev}_{e,+}(u)) = \delta_{\ell(e)}(l(e)), \ \forall e \in \mathrm{Edge}_1(\Gamma), \\ \mathrm{ev}_{e,-}(u) = \mathrm{ev}_{+,e}(u) \ \forall e \in \mathrm{Edge}_2(\Gamma) \end{array} \right\}. \tag{30}$$

Boundary values of $W^{k,p}$ maps lie in $W^{k-1/p,p}$ (see [60, (0.15)]). Maps close to any given pair $(u, \partial u)$ are exponentials $\exp_u(\xi), \exp_{\partial u}(\partial \xi)$ of sections

$$\xi \in \Omega^0(S^\circ, u^*TX)_{k,p}, \quad \partial\xi \in \Omega^0(\partial S^\circ, (\partial u)^* TL)_{k-1/p,p},$$

where the subscript denotes Sobolev class $W^{k,p}$, satisfying

$$\xi \circ \iota = D\phi \circ \partial \xi.$$

Here exponentiation means geodesic exponentiation using, for example, a metric on X for which each branch of $\phi(L)$ is totally geodesic. The fiber of the bundle $\mathcal{E}_\Gamma$ over some map u is the vector space of one-forms

$$\mathcal{E}_{\Gamma,u} := \Omega^{0,1}(S, u_S^* TX)_{k-1,p}. \tag{31}$$

Local charts are provided by almost complex parallel transport

$$\mathcal{T}_u^\xi : \Omega^{0,1}(S^\circ, \exp_u(\xi)^* TX)_{k-1,p} \to \Omega^{0,1}(S^\circ, u^*TX)_{k-1,p} \tag{32}$$

along $\exp_u(s\xi)$ for $s \in [0,1]$; note here that the connection used for parallel transport $\mathcal{T}_u^\xi$ need not be related to the metric used for geodesic exponentiation. In any local trivialization of the universal curve, one can obtain Banach bundles with arbitrarily high regularity. Let

$$\mathcal{U}_\Gamma^i \to \mathcal{M}_\Gamma^i \times C \tag{33}$$

be a collection of local trivializations of the universal curve. Let $\mathcal{B}_\Gamma^i$ denote the inverse image of $\mathcal{M}_\Gamma^i$ in $\mathcal{B}_\Gamma$ and $\mathcal{E}_\Gamma^i$ its preimage in $\mathcal{E}_\Gamma$. The Fredholm map cutting out the moduli space over $\mathcal{M}_\Gamma^i$ is

$$\mathcal{F}_\Gamma^i : \mathcal{B}_\Gamma^i \to \mathcal{E}_\Gamma^i, \quad u \mapsto \overline{\partial}_{J,H} u \tag{34}$$

The linearization of the map (34) cutting out the moduli space is a combination of the standard linearization of the Cauchy-Riemann operator with additional terms arising from the variation of conformal structure. With k, p integers determining the Sobolev class as above let

$$\begin{aligned} D_u : \Omega^0(S^\circ, u^*TX, (\partial u)^* TL)_{k,p} &\to \Omega^{0,1}(S^\circ, u^*TX)_{k-1,p} \\ \xi &\mapsto \nabla_H^{0,1} \xi - \frac{1}{2}(\nabla_\xi J) J \partial_J u \end{aligned} \tag{35}$$

denote the linearization of the Cauchy-Riemann operator, c.f. McDuff-Salamon [61, p. 258]; here

$$\partial_J u = \frac{1}{2}(\mathrm{d}_H u + J \mathrm{d}_H u j).$$

The complex structures on the fibers induce a family

$$\mathcal{M}^i_\Gamma \to \mathcal{J}(S), \quad m \mapsto j(m) \tag{36}$$

of complex structures on the two-dimensional locus $S \subset C$, and In particular, any tangent vector $\zeta \in T\mathcal{M}^i_\Gamma$ induces a variation $Dj : TS \to TS$ of complex structure on S: Let

$$\Omega^0(S^\circ, \partial S^\circ; u_S^* TX; (\partial u)^* TL)_{k,p} \subset \Omega^0(S^\circ; u_S^* TX)_{k,p}$$

denote the subspace of sections ξ whose boundary values lift to $W^{k-1/p,p}$-sections $\partial\xi$ with values in $(\partial u)^* TL$. The tangent space to $\mathcal{B}_\Gamma$ is the space of deformations (ζ_S, ζ_T, ξ) preserving the matching conditions given by

$$T_{(C,u)}\mathcal{B}_\Gamma = \left\{ (\zeta_S, \zeta_T, \xi, \xi') \;\middle|\; \begin{array}{l} (\mathrm{ev}_{e,-}(\xi), \mathrm{ev}_{e,+}(\xi)) = D\delta_{l,\ell(e)}(\xi', \zeta_T(e)) \\ \forall e \in \mathrm{Edge}_1(\Gamma) \end{array} \right\}$$

where $\delta(l,t) := \delta_t(l)$ is the cellular approximation. The *linearized operator* for the map u is given by the expression

$$\begin{aligned} \tilde{D}_u : T_{(C,u)}\mathcal{B}_\Gamma &\to \Omega^{0,1}(S^\circ, \partial S^\circ; u^* TX)_{k-1,p} \\ (\zeta_S, \zeta_T, \xi) &\mapsto \left(D_u \xi + \frac{1}{2} J \mathrm{d}u Dj(\zeta_S) \right). \end{aligned} \tag{37}$$

A holomorphic treed disk $u : S \to X$ with stable domain C is

regular if the linearized operator $\tilde{D}_u$ is surjective;

stratum-wise rigid if u is regular and $\tilde{D}_u$ is surjective and the kernel of $\tilde{D}_u$ is generated by the infinitesimal automorphism $\mathrm{aut}(S)$ of S; and

rigid if u is stratum-wise rigid and the domain C lies in a top-dimensional stratum $\mathcal{M}_\Gamma$ in the moduli space of domains $\overline{\mathcal{M}}_d$.

The moduli space of holomorphic treed disks admits a natural version of the Gromov topology which allows bubbling off spheres, disks, and cellular boundaries. For Hamiltonian-perturbed maps, the Hamiltonian-perturbed energy is

$$E(u) = \frac{1}{2} \int_S \|\mathrm{d}_H u\|^2 \mathrm{d}\,\mathrm{Vol}_S \tag{38}$$

where $\mathrm{d}\,\mathrm{Vol}_S$ is the area density on the surface S. The *area* of a map type $\mathbb{\Gamma}$ is the sum of the pairings of the homology classes of the disk and sphere components $u_*[S_v]$ with the symplectic class $[\omega]$. The energy $E(u)$ is equal to the area $A(u)$ up to a curvature term explained in [61, Chapter 8]. Consider a sequence

$$u_\nu : S_\nu \to X, \quad \partial u_\nu : \widetilde{\partial S_\nu} \to L, \quad \nu \in \mathbb{N}$$

of treed holomorphic disks with boundary in ϕ with bounded energy $E(u_\nu) > 0$. Gromov compactness with Lagrangian boundary conditions as in, for example, Frauenfelder-Zemisch [41] implies that there exists a subsequence with a stable limit

$$(C, u : S \to X) := \lim_{n\to\infty} (C_\nu, u_\nu : S_\nu \to X).$$

Standard arguments using local distance functions then show that for any fixed energy bound $E > 0$, the subset

$$\overline{\mathcal{M}}_d^{<E}(\phi) = \left\{ u \in \overline{\mathcal{M}}_d(\phi) \;\middle|\; E(u) < E \right\} \tag{39}$$

satisfying the given energy bound $E(u) < E$ is compact.

The moduli space further decomposes according to the limits at infinity and the expected dimension. Define

$$\mathcal{I}(\phi) = \mathcal{I}^c(\phi) \cup \mathcal{I}^{si}(\phi)$$

where

$$\mathcal{I}^c(\phi) := \{\sigma_i : B^{d(i)} \to L\}$$

is the set of cells. Thus each element of $\mathcal{I}^c(\phi)$ is a map σ_i from balls $B^{d(i)}$ of dimension $d(i)$ to L with boundary in the union of the images of j-cells for $j < i$; and

$$\mathcal{I}^{\mathrm{si}}(\phi) := (L \times_\phi L) - \Delta_L$$

is the set of ordered self-intersection points, where $L \times_\phi L$ is the fiber product and $\Delta_L \subset L^2$ the diagonal. Given

$$\underline{\sigma} = (\sigma_0, \ldots, \sigma_d) \in \mathcal{I}^\vee(\phi)$$

denote by

$$\overline{\mathcal{M}}(\phi, \underline{\sigma}) = \left\{ [C, u : S \to X] \in \overline{\mathcal{M}}_d(\phi) \;\middle|\; \mathrm{ev}_e(u) \in \sigma_e \ \forall e \in \mathrm{Edge}(\Gamma) \right\}$$

the locus of maps such that for each semi-infinite edge e the evaluation lands in σ_e. We consider the case of no disks in the configuration as a special case. Let

$$\sigma_- \in \mathcal{I}^c(\phi), \quad \sigma_+ \in \mathcal{I}^{c,\vee}(\phi), \quad \deg(\sigma_+) = \deg(\sigma_-) + 1.$$

Let $\mathcal{M}_\Gamma(\sigma_-, \sigma_+)$ be the oriented fiber $\sigma_-^{-1}(p)$ for generic p in the image $\mathrm{Im}(\sigma_+) \subseteq L$. The moduli space $\mathcal{M}_\Gamma(\sigma_-, \sigma_+)$ is independent of p up to oriented cobordism. Indeed, for any generic path γ from points p to p' in $\mathrm{Im}(\sigma_+)$ the inverse image $\sigma_-^{-1}(\gamma) \subset L$ is a one-manifold with boundary $\sigma_-^{-1}(p) \cup \sigma_-^{-1}(p')$. The moduli space $\mathcal{M}_\Gamma(\sigma_-, \sigma_+)$ would be the set of rigid Morse trajectories in the Morse model of the Fukaya algebra.

For any integer d denote by

$$\mathcal{M}_\Gamma(\phi, \underline{\sigma})_d = \left\{ [C, u : S \to X] \;\middle|\; \mathrm{Ind}(\tilde{D}_u) - \sum_{i=0}^{d} |\sigma_i| = d \right\} \tag{40}$$

the locus with *expected dimension* d, where $\tilde{D}_u$ is the operator of (37) and $|\sigma_i|$ is the codimension of the constraint σ_i for $i = 0, \ldots, d$. An element of $\mathcal{M}_\Gamma(\phi, \underline{\sigma})$ is *rigid* if it lies in the locus $\mathcal{M}_\Gamma(\phi, \underline{\sigma})_0$ of expected dimension zero and $\mathcal{M}_\Gamma$ is codimension zero in $\overline{\mathcal{M}}_d$. A *labelled type* of map is the map type $\mathbb{\Gamma}$ with a labelling $\underline{\sigma}$ of its edges. A labelled map type $\mathbb{\Gamma}$ is rigid if any (and so all) maps $(C, u : S \to X)$ of labelled type $\mathbb{\Gamma}$ are rigid.

4. Coherent perturbations

Regularization of the moduli spaces is achieved through domain-dependent perturbations, using a Donaldson hypersurface [32] to stabilize the domains as in Cieliebak-Mohnke [26].

4.1. **Donaldson hypersurfaces.**

Definition 4.1. A *Donaldson hypersurface* of a compact symplectic manifold X is a codimension two symplectic submanifold $D \subset X$ representing a multiple $k[\omega], k > 0$, of the symplectic class $[\omega] \in H^2(X)$. The integer k is the *degree* of D.

A *relative Donaldson hypersurface* for a Lagrangian immersion $\phi : L \to X$ is a codimension two symplectic submanifold $D \subset X$ disjoint from $\phi(L)$ representing a multiple $k[\omega], k > 0$, of the symplectic class $[\omega] \in H^2(\phi)$.

Donaldson's construction in [32] associates to any asymptotically holomorphic sequence of sections s_k of tensor powers $\hat{X}^k$ of a line bundle $\hat{X} \to X$ with first Chern class $c_1(\hat{X}) = [\omega]$ a sequence of hypersurfaces $D_k = s_k^{-1}(0)$; for k sufficiently large the submanifold D_k is a Donaldson hypersurface. A result of Auroux [10] provides a homotopy between any two such choices with the same degree. Results of Auroux-Gayet-Mohsen [13] show the existence of Donaldson hypersurfaces in the complement of an isotropic submanifold, and a result of Auroux included in Pascaleff–Tonkonog [68, Theorem 3.1] extends this to the case of cleanly-intersecting Lagrangians satisfying the Bohr-Sommerfeld condition that the pullback bundle $\phi^*(\hat{X}^k \to X)$ is trivial for some k. As in [68, Corollary 3.4] one may assume that the Lagrangian to be exact in the complement by choosing the approximately holomorphic section defining the Donaldson hypersurface to be flat on the Lagrangian. Then Stokes' theorem implies that k times the area $A(u)$ of any disk $u : S \to X$ bounding L is given by its intersection number $\langle [u], [D] \rangle$ with D. Equivalently, D represents $[\omega]$ in the relative cohomology group $H^2(\phi)$.

As explained in Cieliebak-Mohnke [26], the set of intersections of a holomorphic curve with a Donaldson hypersurface provides an additional set of marked points that stabilize the domain. Let $D \subset X$ be a Donaldson hypersurface. We say a compatible almost complex structure $J_D \in \mathcal{J}(X)$ is *stabilizing* if J preserves TD, a compatible almost complex structure preserving D so that D contains no non-constant holomorphic spheres as in Cieliebak-Mohnke [26, Section 8], and each non-constant J_D-holomorphic sphere $u : \mathbb{P}^1 \to X$ of energy at most E intersects D in finitely many but at least three points $u^{-1}(D)$. [6]

Lemma 4.2. [26, Section 8] *Suppose D has sufficiently large degree $k \gg 0$. Then any generic almost complex structure J_D preserving D is stabilizing, and for any energy bound $E > 0$ there exist an open neighborhood $\mathcal{J}(X, J_D, E)$ of J_D consisting of stabilizing almost complex structures.*

We use the additional markings provided by the Donaldson hypersurface to define domain-dependent perturbations. Choose an open neighborhood U of D. Recall from (27) that $\mathrm{Vect}_h(X, U)$ denotes the space of Hamiltonian vector fields $v : X \to TX$ vanishing on U.

Definition 4.3. For each combinatorial type of domain Γ,

(a) a *domain-dependent almost complex structure* for Γ is a map

$$J_\Gamma : \overline{\mathcal{S}}_\Gamma \to \mathcal{J}(X, J_D, E)$$

(notation from (19)) smooth as a map $\overline{\mathcal{S}}_\Gamma \times TX \to TX$.

(b) A *domain-dependent Hamiltonian perturbation* for Γ is a one-form

$$H_\Gamma \in \Omega^1(\overline{\mathcal{S}}_\Gamma, \mathrm{Vect}_h(X, U))$$

smooth as a map $T\overline{\mathcal{S}}_\Gamma \times X \to TX$.

(c) A *single-valued domain-dependent matching condition* for Γ is a map

$$M_\Gamma : (\overline{\mathcal{S}}_\Gamma \cap \overline{\mathcal{T}}_\Gamma) \times L \to L$$

such that $M_\Gamma(w_e, \cdot)$ is a diffeomorphism of L for each $w_e \in \overline{\mathcal{S}}_\Gamma \cap \overline{\mathcal{T}}_\Gamma$.

(d) A *perturbation datum* is a datum

$$P_\Gamma = (J_\Gamma, H_\Gamma, M_\Gamma)$$

such that J_Γ agrees with the given almost complex structure J_D on the hypersurface D and in a neighborhood of the nodes $w_e \in S$ and boundary ∂S for any fiber $S \subset \overline{\mathcal{S}}_\Gamma$, and takes values in $\mathcal{J}(X, J_D, \#\,\mathrm{Edge}_\bullet(\Gamma)/k)$. The space of P_Γ of perturbation data is denoted

$$\mathcal{P}_\Gamma = \{P_\Gamma\}.$$

To achieve certain symmetry properties of the Fukaya algebra, multi-valued perturbation data are required. For example, if expects divisor insertions to contribute exponentials to the disk potential then one expects the factorials to appear as an averaging factor as in Theorem 5.13.

[6] In order to prove independence from all choices, [26] also consider tamed almost complex structure in order to prove independence from all choices. However, in this paper we do not prove any independence results so compatible almost complex structures suffice.

Definition 4.4. (a) A *multivalued domain-dependent matching condition* for Γ is a formal sum

$$M_\Gamma = \sum_{i=1}^{k} c_i M_{\Gamma,i} \quad \sum_{i=1}^{k} c_i = 1 \quad c_i \in [0,1] \ \forall i \tag{41}$$

of single-valued matching conditions $M_{\Gamma,i}$.

(b) Similarly, a *multi-valued domain-dependent Hamiltonian* is a formal sum

$$H_\Gamma = \sum_{i=1}^{l} d_i H_{\Gamma,i} \quad \sum_{i=1}^{l} d_i = 1 \quad d_i \in [0,1] \ \forall i \tag{42}$$

of single-valued Hamiltonian perturbations.

For much of the paper, one could assume that M_Γ, H_Γ to be single-valued. However in order to deal with repeated inputs one must allow formal sums, that is, multivalued perturbations, as in Section 5.4.

Given perturbations, the perturbed moduli spaces are defined as follows.

Definition 4.5. For $P_\Gamma = (J_\Gamma, H_\Gamma, M_\Gamma)$, a P_Γ-perturbed treed holomorphic disk is a pair $(C, u : S \to X)$ where C is of type Γ and the equations (28) and (29) are replaced with the following conditions:

(a) The map u is perturbed holomorphic in the sense that

$$\overline{\partial}_{J_\Gamma, H_\Gamma} u(z) = \frac{1}{2} \left(\begin{array}{l} (\mathrm{d}u(z) - H_\Gamma(u(z)) \\ + J_\Gamma(z, u(z))(\mathrm{d}u(z) - H_\Gamma(u(z)))j(z)) \end{array} \right) = 0 \tag{43}$$

on the surface S;

(b) For each unbranched interior edge e the perturbed matching condition

$$(M_{\Gamma,i}(w_-(e), u(w_-(e))), M_{\Gamma,i}(w_+(e), u(w_+(e)))) \in \delta_{l(e)}(L) \tag{44}$$

holds for some i; and for each leaf e labelled by a cell σ_e for some i we have

$$M_{\Gamma,i}(w_e, u(w_e)) \in \sigma_e;$$

(c) the matching condition holds for each branched interior edge e

$$\mathrm{ev}_{e,-}(u) = \mathrm{ev}_{e,+}(u);$$

with notation as in (26).

The map is *adapted* if each connected component of $u^{-1}(D)$ contains an interior node $w_e \in S, e \in \mathrm{Edge}_\bullet(\Gamma)$ and each such w_e lies in $u^{-1}(D)$. This ends the Definition.

Remark 4.6. By Theorem 4.19 a generic adapted map $u : S \to X$ has the property that every holomorphic disk component $u|S_v$ meets D in finitely many points $u^{-1}(D)$, and positively many points if the disk is non-constant. The definition above, however, allows constant sphere components S_v mapping entirely to the divisor D, which would therefore have infinitely many intersections.

The construction above naturally produces a collection of moduli spaces satisfying an energy gap condition:

Lemma 4.7. *Let $\phi : L \to X$ be a self-transverse Lagrangian immersion. There exists an $\hbar > 0$ such that any treed holomorphic disk $u : S \to X$ with boundary on ϕ containing at least one non-constant holomorphic component $u_v : S_v \to X, \mathrm{d}u_v \neq 0, v \in \mathrm{Vert}(\Gamma)$ has area $A(u)$ at least $\hbar$.*

Proof. By Gromov compactness, for $E > 0$, the set of homotopy classes $[u] \in \pi_2(\phi)$ of stable holomorphic disks $u : S \to X, S = \{|z| \leq 1\}$ with energy bound $E(u) < E$ is finite. It follows that the set $\{A(u), \mathrm{d}u \neq 0\}$ of non-zero energies of disks $u : S \to X$ bounding ϕ has a non-zero minimum $\min\{A(u), \mathrm{d}u \neq 0\}$, which we may take to equal $\hbar$. □

Lemma 4.8. *For a regular Hamiltonian perturbation H_Γ that is sufficiently small in the C^∞ topology, the areas $A(u)$ of all rigid (J_Γ, H_Γ)-holomorphic treed disks $u : S \to X$ are non-negative.*

Proof. The areas of such configurations are topological quantities, that is, depend only on the homotopy type of the map. The set of homotopy types $[u]$ achieved by holomorphic maps $u : S \to X$ is unchanged by the introduction of a perturbation H_Γ, by a standard argument using Gromov compactness. Any (J_Γ, H_Γ)-holomorphic map may be written as a J'_Γ-holomorphic map for some almost complex structure J'_Γ obtained by pulling back J_Γ under a Hamiltonian flow as in [61, Chapter 8]. Suppose that $u_\nu : S_\nu \to X$ is a sequence of $(J_\Gamma, H_{\Gamma,\nu})$-holomorphic maps with $H_{\Gamma,\nu}$ converging to zero in C^∞. After passing to a subsequence, we may assume that the domain C_ν converges to a limit C. By the energy-area relation for Hamiltonian-perturbed maps, in particular the bound in [61, Remark 8.1.7], the energy of the sequence u_ν is bounded. By Gromov compactness (see for example [61, Chapter 4], although a modification is necessary to adapt for the varying domain) a subsequence of u_ν Gromov converges to a limiting J_Γ-holomorphic map $u : S \to X$ with the same area. Since the Hamiltonian perturbation $H_{\Gamma,\nu}$ vanishes in the limit, the area is necessarily non-negative. □

The combinatorial type of an adapted map is that of the map with the additional data of a labelling $i(e), e \in \mathrm{Edge}(\Gamma)$ of any interior node by intersection multiplicity $i(e)$ with the hypersurface D; let $i(e) = 0$ if the map $u : S \to X$ is constant with values in the hypersurface D near w_e. Denote the moduli space of D-adapted treed holomorphic disks bounding ϕ of type Γ with respect to the perturbation P_Γ by

$$\mathcal{M}_\Gamma(\phi, D) \subset \left\{ u : S \to X \,\middle|\, \overline{\partial}_{J_\Gamma, H_\Gamma} u = 0, \quad u(w_e) \in D, \quad \forall e \in \mathrm{Edge}_\bullet(\Gamma) \right\}.$$

Denote by

$$\overline{\mathcal{M}}(\phi, D) = \cup_\Gamma \mathcal{M}_\Gamma(\phi, D)$$

the union over combinatorial types Γ. As before, we may further refine to a union over map types

$$\overline{\mathcal{M}}(\phi, D) = \cup_{\mathbb{\Gamma}} \mathcal{M}_{\mathbb{\Gamma}}(\phi, D).$$

4.2. **Coherence.** In order to obtain good compactness properties of the moduli spaces of holomorphic curves, the following coherence properties of the perturbations are required.

Assumption 4.9. The perturbations $\underline{P} = (P_\Gamma)$ satisfy the following coherence axioms:

(Locality axiom) For the locality axiom we require the following notation. Given a type of map Γ, for each vertex $v \in \mathrm{Vert}(\Gamma)$, let $\Gamma(v)$ denote the subtree of Γ consisting of the vertex v and all edges e of Γ meeting v. Let $\Gamma_\circ$ denote the subgraph of Γ whose vertices are those of open type $v \in \mathrm{Vert}_\circ(\Gamma)$ and whose edges are $e \in \mathrm{Edge}_\circ(\Gamma)$. Let

$$\pi = \pi_\circ \times \pi_v : \mathcal{S}_\Gamma \to \mathcal{M}_{\Gamma_\circ} \times \mathcal{S}_{\Gamma(v)}$$

be the product of the maps where $\pi_\circ$ is given by projection followed by forgetful morphism and π_v is the map $S \mapsto S_v$ that collapses all components other than S_v onto the corresponding special points of S_v.

The locality property is the following: For each vertex v, the perturbation P_Γ restricts on S_v to the pull-back under π of some perturbation $P_{\Gamma,v}$ on $\mathcal{M}_{\Gamma_\circ} \times \mathcal{S}_{\Gamma(v)}$ to $\mathcal{S}_\Gamma$. [7]

(Cutting edges axiom) If Γ is obtained from types Γ_1, Γ_2 by gluing along semi-infinite edges e of Γ_1 and e' of Γ_2 as in (18) then let

$$\pi_1 : \mathcal{M}_\Gamma \to \mathcal{M}_{\Gamma_1}, \quad \pi_2 : \mathcal{M}_\Gamma \to \mathcal{M}_{\Gamma_2}$$

denote the projections obtained by mapping each curve $C = C_1 \cup_{e,e'} C_2$ to C_1 resp. C_2. For the coherence axioms P_Γ is the product of the perturbations $P_{\Gamma_1}, P_{\Gamma_2}$ under the isomorphism $\mathcal{S}_\Gamma \cong \pi_1^* \mathcal{S}_{\Gamma_1} \cup \pi_2^* \mathcal{S}_{\Gamma_2}$.[8]

(Collapsing edges axiom) If Γ' is obtained from Γ by setting a length equal to zero or infinity, or collapsing an edge, then the restriction of P_Γ to $\mathcal{S}_\Gamma | \mathcal{M}_{\Gamma'} \cong \mathcal{S}_{\Gamma'}$ is equal to $P_{\Gamma'}$.

[7] In other words, on each component S_v the perturbations only depend on the positions of the special points on that component and the boundary edge lengths. This locality principle is used later in Theorem 4.19 to rule out constant spheres with more than one marking, in the case of zero and one-dimensional moduli spaces.

[8] That is, on any configuration with a broken edge, the perturbations on the components separated by the broken edge depend only on the domains on that side of the edge, rather than the domain on the other side. This property is necessary for boundary description in Theorem 4.19, which in turn is used to prove the A_∞ axiom.

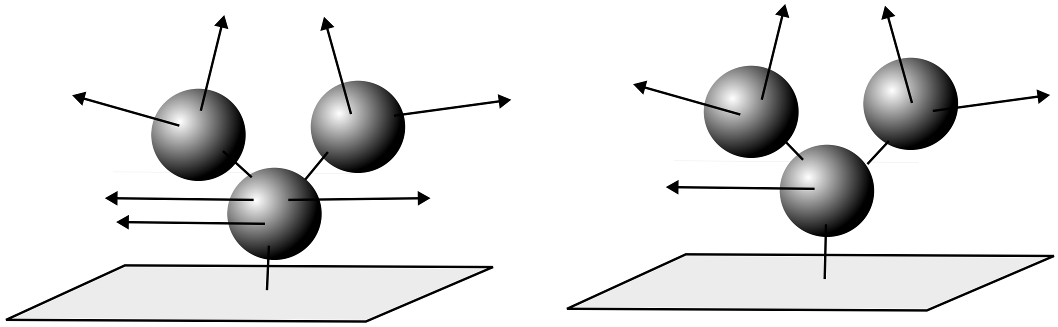

FIGURE 4. The types Γ and $f(\Gamma)$

Remark 4.10. (Forgetting markings on spheres) The locality axiom provides the following forgetful construction, which is a variation of the construction in Cieliebak-Mohnke [26]: Suppose that C is a curve of type Γ containing a sphere component S_v with more than one interior marking $w_e \in S_v$. Forgetting all but one marking, say w_{e_0} on S_v, and collapsing unstable components produces a marked curve $f(C)$ with type $f(\Gamma)$ possibly with a component $f(S_v)$ containing a single marking, as in Figure 4. Define a perturbation datum $f(P_\Gamma)$ for $f(\Gamma)$ by taking the almost complex structure $J_{f(\Gamma)}$ to equal the base almost complex structure J_D on $f(S_v)$, and the almost complex structures $P_\Gamma | f(C) - f(S_v)$ on the complement; while the Hamiltonian perturbation H_Γ and M_Γ remains the same. If $u : S \to X$ is a P_Γ-holomorphic map constant on S_v, then one obtains an $f(P_\Gamma)$-holomorphic map on $f(S_v)$ by forgetting all markings except w_{e_0} on S_v. Since each interior node $w_e, e \in \mathrm{Edge}_\bullet(\Gamma)$ is required to map to the divisor D, the resulting type $f(\Gamma)$ is the same expected dimension as that of maps of type Γ.

Remark 4.11. The (Cutting-edges) and (Collapsing-edges) axioms in particular imply that the part of the moduli space $\overline{\mathcal{M}}_{\mathbb{\Gamma}}(\phi, D)$ over the image of the inclusion $\mathcal{M}_{\Gamma_1} \times \mathcal{M}_{\Gamma_2} \to \overline{\mathcal{M}}_\Gamma$ is a union of products of moduli spaces $\mathcal{M}_{\mathbb{\Gamma}_1}(\phi, D)$ and $\mathcal{M}_{\mathbb{\Gamma}_2}(\phi, D)$ as $\mathbb{\Gamma}_1, \mathbb{\Gamma}_2$ range over map types with domain types Γ_1, Γ_2; this identification implies that the terms in the A_∞ axiom are associated to configurations obtained by gluing treed disks in the sense of (18).

Remark 4.12. The (Cutting edges) axiom implies the following relationship between moduli spaces.

(a) Suppose that the type Γ is obtained by gluing together types Γ_2 and Γ_1 along a boundary edge. An element of $\mathcal{M}_\Gamma(\phi, D)$ consists of
   (i) a pair $C_k = S_k \cup T_k,\ u_k : S_k \to X$ of treed holomorphic disks of combinatorial types Γ_k for $k \in \{1, 2\}$ and

(ii) an element $l \in L$ such that $\delta_1(l) = (l_1, l_2)$ where l_k is the evaluation of u_k at the node $w_e^\pm$ on the relevant side of the edge e of Γ being glued together.

Thus if i denotes the index of the incoming edge for Γ_2 glued at the outgoing edge of Γ_1 and $j+1$ the number of incoming edges of Γ_2 the map $u \mapsto (u_1, u_2)$ defines a map

$$\begin{aligned} \mathcal{M}_\Gamma(\phi, D, \sigma_0, \ldots, \sigma_d)_0 & \\ \to \cup_{\alpha,\beta} \mathcal{M}_{\Gamma_1}(\phi, D, \sigma_0, \sigma_1, \ldots, \sigma_{i-1}, \alpha, \sigma_{i+j+1}, \ldots, \sigma_d)_0 & \\ \times \mathcal{M}_{\Gamma_2}(\phi, D, \beta, \sigma_i, \ldots, \sigma_{i+j})_0 & \end{aligned} \tag{45}$$

with finite fiber over a pair (u_1, u_2) evaluating at

$$(l_-, l_+) := (u_1(w_e^-), u_2(w_e^+)) \in \sigma_i(B^{d(i)}) \times \sigma_j(B^{d(j)})$$

the set of points $l \in L$ with $\delta_1(l) = (l_-, l_+) \in L \times L$. The number of such points counted with sign is by definition $c(\alpha, \beta)$.

(b) The *true boundary components* of any one-dimensional component of the moduli space $\mathcal{M}(\phi, D)$ consist of these configurations as well as elements $C = S \cup T, u : S \to X$ of $\mathcal{M}_\Gamma(\phi, D)$ of unbroken type Γ whose evaluation at an end w_e of some leaf $e \subset T$ lies in the *boundary* of the cell σ_e of dimension $d(e)$. For generic maps, the boundary lies in the interior of a $(d(e)-1)$-cell σ_e' and the map

$$\begin{aligned} \pi_{\sigma_e \to \sigma_e'} : \{u \in \mathcal{M}_\Gamma(\phi, D, \sigma_0, \ldots, \sigma_e, \ldots, \sigma_d), u(w_e) \in \sigma_e'\} & \\ \to \mathcal{M}_\Gamma(\phi, D, \sigma_0, \ldots, \sigma_e', \ldots, \sigma_d) & \end{aligned} \tag{46}$$

is generically finite-to-one with finite fiber $\pi^{-1}_{\sigma_e, \sigma_e'}(p)$ over any generic element $p \in \sigma_e'$ having a signed count of $\partial(\sigma_e, \sigma_e')$ elements, by Definition (20).

Obtaining strict units requires the addition of *weightings* to the combinatorial types as in Ganatra [44, Section 10] and Charest-Woodward [21, Section 4]. When the weighting of an edge is infinite, we will assume that the perturbation data is pulled back under the forgetful map forgetting that edge and stabilizing. For this reason, the edges where the weightings are forced to be infinite are called *forgettable*.

Definition 4.13. A *weighting* of a treed disk $C = S \cup T$ of type Γ is

(a) a partition of the boundary semi-infinite edges

$$\mathrm{Edge}^{\mathrm{wt}}(\Gamma) \sqcup \mathrm{Edge}^{\mathrm{wt},\infty}(\Gamma) \sqcup \mathrm{Edge}^{\mathrm{wt},0}(\Gamma) = \mathrm{Edge}_{\circ,\to}(\Gamma)$$

into *weighted* resp. *forgettable* resp. *unforgettable* edges, and

(b) a map

$$\rho : \mathrm{Edge}_{\circ,\to}(\Gamma) \to [0, \infty]$$

satisfying the property: each of the semi-infinite e edges is assigned a *weight* $\rho(e)$ such that

$$\rho(e) \in \begin{cases} \{0\} & e \in \mathrm{Edge}^{\mathrm{wt},0}(\Gamma) \\ [0,\infty] & e \in \mathrm{Edge}^{\mathrm{wt}}(\Gamma) \\ \{\infty\} & e \in \mathrm{Edge}^{\mathrm{wt},\infty}(\Gamma) \end{cases}.$$

If the outgoing edge $e_0 \in \mathrm{Edge}_{\to}(\Gamma)$ is unweighted (forgettable or unforgettable) then an isomorphism $\psi : (C,\rho) \to (C',\rho')$ of weighted treed disks is an isomorphism of treed disks $C \to C'$ that preserves the types of semi-infinite edges $e \in \mathrm{Edge}_{\to}(\Gamma) \cong \mathrm{Edge}_{\to}(\Gamma')$ and weightings: $\rho(e) = \rho'(e')$ for all corresponding edges $e \in \mathrm{Edge}_{\circ,\to}(\Gamma),\ e' \in \mathrm{Edge}_{\circ,\to}(\Gamma')$. This ends the Definition.

There is an additional notion of equivalence in the case that the outgoing edge is weighted: If the outgoing edge e_0 is weighted then an isomorphism of weighted treed disks $C \to C'$ is an isomorphism of treed disks preserving the types of semi-infinite edges $e \in \mathrm{Edge}_{\circ,\to}(\Gamma)$ and the weights $\rho(e), e \in \mathrm{Edge}_{\circ,\to}(\Gamma)$ up to scalar multiples:

$$\exists \lambda \in (0,\infty),\ \forall e \in \mathrm{Edge}_{\circ,\to}(\Gamma), e' \in \mathrm{Edge}_{\circ,\to}(\Gamma'),\ \rho(e) = \lambda\rho'(e'). \tag{47}$$

In particular, any weighted tree T such that $\mathrm{Vert}(\Gamma) = \emptyset$ and a single edge $e \in \mathrm{Edge}_{\circ,\to}(\Gamma)$ that is weighted $\rho(e) \in (0,\infty)$ is isomorphic to any other such configuration T' with a different weight $\rho(e') \in (0,\infty), e \in \mathrm{Edge}(\Gamma')$.

The *combinatorial type* of any weighted treed disk is the tree associated to the underlying nodal disk with additional data recording which lengths resp. weights are zero or infinite. Namely if $C = S \cup T$ is a weighted treed disk then its combinatorial type is the tree $\Gamma = \Gamma(C)$ obtained by gluing together the combinatorial types $\Gamma(S_v)$ of the disks S_v along the edges corresponding to the edges of T; and equipped with the additional data of

(a) the subsets
$$\mathrm{Edge}^{\mathrm{wt}}(\Gamma) \text{ resp. } \mathrm{Edge}^{\mathrm{wt},\infty}(\Gamma) \text{ resp. } \mathrm{Edge}^{\mathrm{wt},0}(\Gamma) \subset \mathrm{Edge}_{\circ,\to}(\Gamma)$$
of weighted, resp. forgettable, resp. unforgettable semi-infinite edges;

(b) the subsets
$$\mathrm{Edge}_{-}^{\infty}(\Gamma) \text{ resp. } \mathrm{Edge}_{-}^{0}(\Gamma) \text{ resp. } \mathrm{Edge}_{-}^{(0,\infty)}(\Gamma) \subset \mathrm{Edge}_{-}(\Gamma)$$
of combinatorially finite edges of infinite resp. zero length resp. non-zero finite length;

A well-behaved moduli space of weighted treed disks is obtained after imposing a stability condition.

Definition 4.14. A weighted treed disk $C = S \cup T$ of type Γ is *stable* if either

(a) there is at least one disk component $S_v, v \in \mathrm{Vert}_{\circ}(\Gamma)$, and the following conditions hold:

(i) each disk component $S_v, v \in \mathrm{Vert}_\circ(\Gamma)$ has at least three edges $e \in \mathrm{Edge}(\Gamma)$ attached to the boundary ∂S_v or at least one edge attached to the boundary ∂S_v and one edge to the interior $\mathrm{int}(S_v)$;
(ii) each sphere component $S_v, v \in \mathrm{Vert}_\bullet(\Gamma)$ has at least three edges $e \in \mathrm{Edge}(\Gamma)$ attached;
(iii) each combinatorially-finite edge $e \in \mathrm{Edge}_-(\Gamma)$ is broken at most once, and each semi-infinite edge $e \in \mathrm{Edge}_\to(\Gamma)$ is unbroken;
(iv) if the outgoing edge is weighted $e_0 \in \mathrm{Edge}^{\mathrm{wt}}(\Gamma)$ then at least one leaf $e_i \in \mathrm{Edge}_{\circ,\to}(\Gamma), i > 0$ is also weighted, that is, $e_i \in \mathrm{Edge}^{\mathrm{wt}}(\Gamma)$.

(b) if there are no disks, so that $\mathrm{Vert}(\Gamma) = \emptyset$, there is a single weighted leaf $e_1 \in \mathrm{Edge}^{\mathrm{wt}}(\Gamma)$ and an unweighted (forgettable or unforgettable) root $e_0 \in \mathrm{Edge}^{\mathrm{wt},\infty}(\Gamma) \cup \mathrm{Edge}^{\mathrm{wt},0}(\Gamma)$.

Because a configuration with no disks is allowed (namely an infinite interval) the stability condition for weighted treed disks is not equivalent to the absence of non-trivial automorphisms. The moduli space of weighted treed disks of some type Γ is denoted $\mathcal{M}_\Gamma^{\mathrm{wt}}$. The natural map $\mathcal{M}_\Gamma^{\mathrm{wt}} \to \mathcal{M}_\Gamma$ forgetting the weightings is a fiber bundle with each fiber the product of intervals for each leaf, so that as long as there is one vertex,

$$\mathcal{M}_\Gamma^{\mathrm{wt}} \cong \mathcal{M}_\Gamma \times (0,\infty)^{\#\,\mathrm{Edge}_{\mathrm{wt}}(\Gamma)} \tag{48}$$

where $\#\,\mathrm{Edge}_{\mathrm{wt}}(\Gamma)$ is the number of weighted edges with weights in $(0,1)$; the case of trees with no vertex is exceptional, since in this case both the incoming edge may be weighted but the moduli space $\mathcal{M}_\Gamma^{\mathrm{wt}}$ is still dimension zero, and there are no stable strata $\mathcal{M}_\Gamma$.

A weighted treed holomorphic disk is a holomorphic treed disk with a weighting on the underlying treed disk and the following restriction on leaf labels. Given a non-constant pseudoholomorphic treed disk $u : S \to X$ with leaf e_i for which the weighting $\rho(e_i) = \infty$ resp. 0, we view u as obtained from gluing the pseudoholomorphic treed disk $u' : S \to X$ obtained by attaching to e_i a constant configuration u'' with weighted incoming e_i^- and forgettable resp. unforgettable outgoing edge e_i^+. See Figure 5. Also, any two configurations $u : S \to X, u' : S' \to X$ with an outgoing weighted edge e_0 with the same underlying tree Γ are considered equivalent. See Figure 6.

Remark 4.15. (Constant maps) If $x_1 = x^h$ and $x_0 = x^g$ resp. $x_0 = x^s$ then the moduli space $\mathcal{M}(L, x_0, x_1)$ contains a configuration with no disks and single edge on which u is constant, corresponding to a weighted leaf $e \in \mathrm{Edge}^{\mathrm{wt}}(\Gamma)$ and a root edge $e_0 \in \mathrm{Edge}(\Gamma)$ that is unforgettable resp. forgettable. These maps are pictured in Figure 7.

The set of generators of the space of Floer cochains $CF(\phi)$ is enlarged by adding two elements 1_ϕ^s (with superscript s denoting strict unit) resp. 1_ϕ^h (with superscript

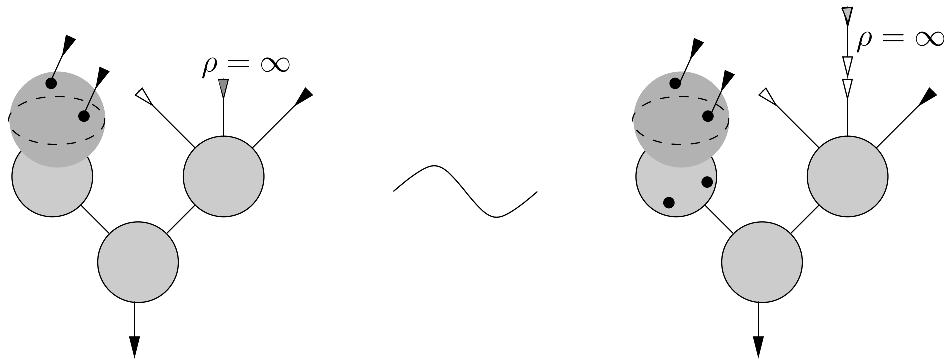

FIGURE 5. Equivalent weighted treed disks

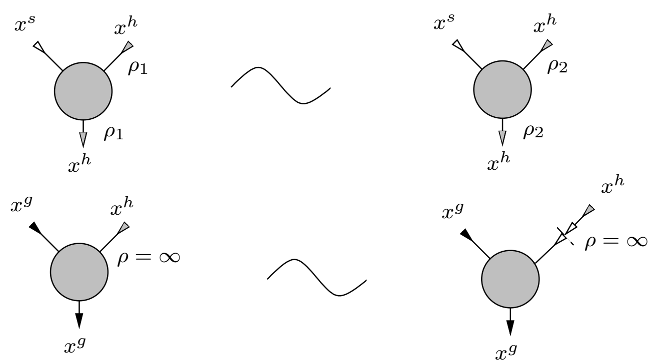

FIGURE 6. Equivalent weighted treed disks, ctd.

h denoting homotopy between strict and geometric unit) of degree 0 resp. -1 to $\mathcal{I}(\phi)$. Any edge e labelled 1_ϕ^s resp. 1_ϕ^g is required to have $\rho(e) = \infty$ resp. $\rho(e) = 0$ while an edge with label 1_ϕ^s may have weighting $\rho(e) \in [0, \infty]$. There is no constraint for the edges $e \in \mathrm{Edge}(\Gamma)$ labelled 1_ϕ^s, while any edge with label not equal to 1_ϕ^s or 1_ϕ^h must have zero weighting. The labels 1_ϕ^s and 1_ϕ^h are only allowed on the outgoing leaf only if the area of the treed disk is zero, the number of leaves is one or two and the expected dimension is zero. Thus either there are no disks and the incoming edge is labelled 1_ϕ^h and the outgoing leaf is labelled 1_ϕ^s or 1_ϕ^g or there is a single disk with no interior markings, one incoming leaf labelled 1_ϕ^s and the label of the other incoming leaf and outgoing leaf are the same. The outgoing weight $\rho(e_0)$ is required to be the product of the incoming weights $\prod \rho(e_i)$ and in the case of zero-area configurations all weightings are declared equivalent. By this definition, in each of these cases the moduli space $\mathcal{M}_\Gamma(\phi, D)$ is a point in each of these special configurations.

Definition 4.16. (Forgetful axiom) A perturbation datum P_Γ satisfies the forgetful axiom if for any leaf $e \in \mathrm{Edge}(\Gamma)$ with infinite weighting $\rho(e) = \infty$, the perturbation

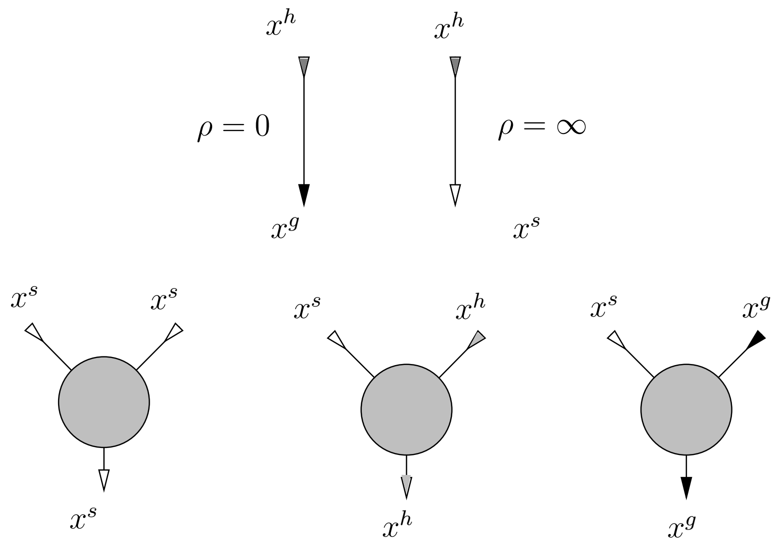

FIGURE 7. Unmarked treed disks

datum P_Γ is pulled back from the perturbation datum $P_{f(\Gamma)}$ for the type $f(\Gamma)$ obtained by forgetting the leaf e and stabilizing (that is, collapsing any unstable components) under the forgetful map $\mathcal{S}_\Gamma \to \mathcal{S}_{f(\Gamma)}$ of universal curves.

In particular, this axiom implies that the resulting moduli spaces admit forgetful morphisms $\mathcal{M}_\Gamma(\phi, D) \to \mathcal{M}_{\Gamma'}(\phi, D)$ whenever there is a leaf e with weighting $\rho(e) = \infty$. See [21, Section 4] for more details on the allowable weightings. We will show in Theorem 5.2 below that the resulting A_∞ algebra $CF(\phi)$ has 1_ϕ^s as a strict unit.

4.3. **Transversality and compactness.** Cieliebak-Mohnke perturbations [26] are not sufficient for achieving transversality if there are multiple interior nodes on ghost bubbles. Indeed, suppose there exists a sphere component $S_v \subset S, v \in \operatorname{Vert}_\bullet(\Gamma)$ on which the map $u|_{S_v}$ is constant and maps to the divisor so that $u(S_v) \subset D$. The domain S_v may meet any number of interior leaves $T_e \subset T$. Adding an interior leaf $T_{e'}$ to the tree meeting S_v increases the dimension of a stratum $\dim \mathcal{M}_{\mathbb{\Gamma}}(\phi, D)$, but leaves the expected dimension $\operatorname{Ind}(D_u), u \in \mathcal{M}_{\mathbb{\Gamma}}(\phi, D)$ unchanged. It follows that $\mathcal{M}_{\mathbb{\Gamma}}(\phi, D)$ is not of expected dimension for some types $\mathbb{\Gamma}$ that we call *crowded*:

Definition 4.17. A holomorphic treed disk $(C, u : S \to X)$ is *crowded* if each such ghost component $S_v \subset S$ meets at least two interior leaves T_e, so that $\#\{e, T_e \cap S_v \neq \emptyset\} \geq 2$, and *uncrowded* otherwise.

The construction of coherent perturbations for uncrowded types proceeds inductively. We summarize the properties that we wish our perturbations to satisfy in the following definition:

Definition 4.18. A perturbation datum $\underline{P} = \{P_\Gamma \in \mathcal{P}_\Gamma\}$ has *good properties* if the following hold for each uncrowded type of map $\mathbb{\Gamma}$ of expected dimension at most one:

(a) (Transversality) Every element of $\mathcal{M}_{\mathbb{\Gamma}}(\phi, D)$ is regular;
(b) (Compactness) the closure $\overline{\mathcal{M}}_{\mathbb{\Gamma}}(\phi, D)$ is a finite set, if expected dimension zero; or a compact one-manifold, if expected dimension one, with boundary contained in the adapted, uncrowded locus; and
(c) (Boundary description) the boundary of $\overline{\mathcal{M}}_{\mathbb{\Gamma}}(\phi, D)$ is a union of components $\mathcal{M}_{\mathbb{\Gamma}'}(\phi, D)$ where $\mathbb{\Gamma}'$ is a type with an edge e of length $\ell(e)$ zero, an infinite length edge $e, \ell(e) = \infty$ connecting two disk components, or a leaf $e \in \operatorname{Edge}(\mathbb{\Gamma})$ with ev_e mapping to the boundary $\sigma_i(\partial B^{d(i)})$ of a cell;

Suppose that perturbations $P_{\Gamma'}$ on the types $\Gamma' \prec \Gamma$ have been chosen in Definition 4.18 making the moduli space of type Γ' regular and all moduli spaces obtained by forgetting interior leaves in Remark 4.10 regular. Via a gluing construction the perturbations $P_{\Gamma'}$ induce regular perturbations in some neighborhood $\mathcal{S}^+_{\Gamma'}$ of $\mathcal{S}_{\Gamma'}$ in $\mathcal{S}_\Gamma$. Namely any curve C of type Γ near $\mathcal{M}_{\Gamma'}$ is obtained from a curve C' of type Γ' by some combination of removing small balls from the nodes and identifying the complements by gluing maps given in local coordinates $z \to \delta/z$; and varying the edge lengths. Since the perturbations by assumption vanish near the nodes, one obtains perturations on C from those on C'. Denote by $\overline{\mathcal{P}}_\Gamma \subset \mathcal{P}_\Gamma$ the subset of perturbations that agree with $P_{\Gamma'}$ on $\mathcal{S}^+_{\Gamma'}$ on the types $\Gamma' \prec \Gamma$.

Theorem 4.19. *There exists a comeager subset $\mathcal{P}^{\mathrm{reg}}_\Gamma$ of the subspace $\overline{\mathcal{P}}_\Gamma$ making the moduli space of type Γ regular and all moduli spaces obtained by forgetting interior leaves from domains of type Γ in Remark 4.10 regular. Furthermore, the perturbations chosen inductively in this way have the good properties in Definition 4.18.*

Sketch of proof. The statement of the Theorem is an application of the Sard-Smale theorem to the local universal moduli spaces. For $l \gg k$ consider the space of perturbation data of class C^l

$$\mathcal{P}^l_\Gamma = \{P_\Gamma = (J_\Gamma, H_\Gamma, M_\Gamma)\}$$

which have C^l-norm distance to the base almost complex structure resp. vanishing Hamiltonian perturbation resp. identity diffeomorphism smaller than some constant κ. The space of allowed perturbations is contractible, assuming that κ is sufficiently small. For almost complex structures J_Γ or Hamiltonian perturbations H_Γ, contractibility is standard. For the matching conditions M_Γ, convexity follows from the convexity of the space of solutions to the equation $\sum_{i=1}^k c_i = 1$ and the fact that any C^l diffeomorphism close to the identity is given by a flow of a C^{l-1} vector field. In particular, consider two matching conditions

$$M'_\Gamma = \sum c'_i M'_{\Gamma,i}, \quad M''_\Gamma = \sum c''_i M''_{\Gamma,i}.$$

The family

$$M_\Gamma = (1-t)M'_\Gamma + tM''_\Gamma \tag{49}$$

is a family of perturbed matching conditions for any $t \in [0,1]$. It follows that $\mathcal{P}^l_\Gamma$ is non-empty.

The moduli spaces $\mathcal{M}_\Gamma(\phi, D)$ are cut out locally by a section of a Banach vector bundle. Let $S^\circ = S - T_2$ denote the surface with strip-like ends obtained by removing the branched special points on the boundary. Recall from (33) the local trivializations $\mathcal{U}^i_\Gamma \to \mathcal{M}^i_\Gamma \times C$ of the universal bundle. Using the Sobolev mapping spaces from (30) define a universal space

$$\mathcal{B}^i_\Gamma \subset \mathcal{M}^i_\Gamma \times \mathrm{Map}^{k,p}(S^\circ, X) \times \mathrm{Map}^{k-1/p,p}(\partial S^\circ, X) \times \mathrm{Map}(\mathrm{Edge}(\Gamma), L) \times \mathcal{P}^l_\Gamma$$

to be the subset of tuples $(c, u, \partial u, l, P_\Gamma)$ satisfying the following conditions:

- the boundary condition $u \circ \iota = \phi \circ \partial u$ where ι is the canonical map $\partial S^\circ \to S^\circ$;
- the matching conditions

  $$(M_\Gamma(w_-, u(w_-(e))), M_\Gamma(w_+, u(w_+(e)))) = \delta_{l(e)}(l(e))$$

  at each pair of endpoints $(w_-(e), w_+(e))$ of combinatorially finite edges with no branching; otherwise if branched

  $$\mathrm{ev}_{e,-}(u) = \mathrm{ev}_{e,+}(u) \in \mathcal{I}^{\mathrm{si}}(\phi)$$

  in the sense of (26);
- the leaf conditions

  $$M_\Gamma(w_e, u(w_e)) \in \sigma_e(B^{d(e)}) \quad \text{resp.} \quad u(w_e) = \sigma_e$$

  if e is unbranched resp. branched;
- for each of the interior leaves $e \in \mathrm{Edge}_\bullet(\Gamma)$

  $$u(w_e) \in D$$

  with intersection multiplicities $d(e) \in \mathbb{Z}_{\geq 1}$.

As in the construction of the moduli space with fixed perturbation data, the universal moduli space is constructed locally over the trivializations given in (36). Consider the fiber bundle $\mathcal{E}^i = \mathcal{E}^i_\Gamma$ over $\mathcal{B}^i_\Gamma$ given by

$$(\mathcal{E}^i_\Gamma)_{c,u,P_\Gamma} \subset \Omega^{0,1}(S^\circ, u^*TX)_{k-1,p}$$

the space of $0,1$-forms with respect to $j(c), J_\Gamma$ on the surface with strip-like ends $S - T_2$ that vanish to order $d(e)-1$ at the node w_e. For $q < l-k$ the q-th derivatives of $J(u)$ with respect to $\xi \in W^{k,p}$ are in $W^{k-1,p}$ and the Cauchy-Riemann operator defines a C^q-section

$$\overline{\partial}_\Gamma : \mathcal{B}^i_\Gamma \to \mathcal{E}^i_\Gamma, \quad (c, u, P_\Gamma) \mapsto \overline{\partial}_\Gamma u \tag{50}$$

where

$$\overline{\partial}_\Gamma u(z) := \frac{1}{2}((\mathrm{d}u(z) - H_\Gamma(u(z)) + J_\Gamma(z, u(z))(\mathrm{d}u(z) - H_\Gamma(u(z)))j(m, z)), \quad \forall z \in S. \tag{51}$$

The *local universal moduli space* is

$$\mathcal{M}_\Gamma^{\mathrm{univ},i}(\phi, D) = \overline{\partial}_\Gamma^{-1} 0_{\mathcal{E}_\Gamma^i} \tag{52}$$

where $0_{\mathcal{E}_\Gamma^i} \subset \mathcal{E}_\Gamma^i$ is the zero section. As in [21], the local universal moduli spaces $\mathcal{M}_\Gamma^{\mathrm{univ},i}(\phi, D)$ are cut out transversally. Indeed, by unique continuation it suffices to show that any element η of the cokernel $\mathrm{coker}(\tilde{D}_u)$ of the linearized operator vanishes in an open neighborhood of a point z in each component of the domain S_v, T_e. In the case of constant components, the linearized operator $D_{u|S_v}$ is surjective for any choice of J and the matching conditions at the nodes are cut out transversally by an induction. Suppose that $\Gamma_0 \subset \Gamma$ is a connected subgraph consisting of vertices for which $u|S_v$ is constant and $S_0 \subset S$ the corresponding subset of S. Let $v \in \mathrm{Vert}(\Gamma_0)$ be a vertex with the property that v is adjacent to exactly one other vertex in Γ_0; such a vertex v exists since Γ_0 is a tree. The operator D_{u_v} is surjective, with kernel given by constants $\xi \in \Omega^0(u_v^* TL)$, if S_v is a disk, or $\Omega^0(u_v^* TX)$, if S_v is a sphere. Via evaluation $\ker(D_{u_v})$ surjects onto the fiber $T_{u_v(w)}TL$ resp. $T_{u_v(w)}TX$ connecting to the adjacent component in S_0, so the matching condition at $u_v(w)$ is transversally cut out. We may assume that the matching conditions at the remaining nodes of S_0 are transversally cut out by the inductive hypothesis, and the claim follows. The case of higher order tangencies with the divisor requires special treatment, as in [26, Section 6]. By the Sard-Smale theorem, for l sufficiently large the set of regular values $\mathcal{P}_\Gamma^{i,l,\mathrm{reg}}$ of the map

$$\varphi_i : \mathcal{M}_\Gamma^{\mathrm{univ},i}(\phi, D) \to \mathcal{P}_\Gamma^{i,l}, \quad (c, u, P_\Gamma) \mapsto P_\Gamma$$

is comeager. Let

$$\mathcal{P}_\Gamma^{l,\mathrm{reg}} = \cap_i \mathcal{P}_\Gamma^{i,l,\mathrm{reg}}.$$

The set of smooth domain-dependent perturbations $\mathcal{P}_\Gamma^{\mathrm{reg}}$ is open and dense and so also comeager. Fix $(J_\Gamma, H_\Gamma, M_\Gamma) \in \mathcal{P}_\Gamma^{\mathrm{reg}}$. By elliptic regularity, every element of $\mathcal{M}_\Gamma^i(\phi, D)$ is smooth. The transition maps for the local trivializations of the universal bundle define smooth maps

$$\mathcal{M}_\Gamma^i(\phi, D)|_{\mathcal{M}_\Gamma^i \cap \mathcal{M}_\Gamma^j} \to \mathcal{M}_\Gamma^j(\phi, D)|_{\mathcal{M}_\Gamma^i \cap \mathcal{M}_\Gamma^j}.$$

This construction equips the space $\mathcal{M}_\Gamma(\phi, D) = \cup_i \mathcal{M}_\Gamma^i(\phi, D)$ with a smooth atlas. Since $\mathcal{M}_\Gamma$ is Hausdorff and second-countable, so is $\mathcal{M}_\Gamma(\phi, D)$. It follows that $\mathcal{M}_\Gamma(\phi, D)$ has the structure of a smooth manifold.

Compactness is similar to the case of spheres in Cieliebak-Mohnke [26]. To see that compactness holds for the union of uncrowded types of expected dimension at most one, note that for any crowded stratum $\mathcal{M}_\Gamma(\phi)$ there exists a moduli space $\mathcal{M}_{f(\Gamma)}(\phi)$ of some other type $f(\Gamma)$ by forgetting all but one of the interior nodes on such ghost bubbles S_v as in Remark 4.10. Combinatorial types $f(\Gamma)$ of maps $u : S \to X$ with sphere bubbles $S_v \subset S,\ S_v \cong \mathbb{P}^1$ in the domain represent moduli spaces $\mathcal{M}_{f(\Gamma)}(\phi, D)$ of expected dimension two less than that of holomorphic disks $\mathcal{M}(\phi, D)$ of strata of top dimension. Thus, these moduli spaces are of negative expected dimension. If the sphere S_v is collapsed under stabilization, then the resulting marking $w_{e_0} \in f(C)$ represents a point whose intersection multiplicity with the divisor D is at least two, by topological invariance of intersection numbers, where (at least) two intersection points $w_{e_1}, w_{e_2} \in C$ have come together. Either way, such configurations lie in moduli spaces of negative expected dimension. Thus such configurations do not appear for generic choices of domain-dependent perturbations as above. The construction of tubular neighborhoods is similar to the case treated in [22]. □

4.4. **Orientations.** Orientations on the moduli spaces may be constructed following Fukaya-Oh-Ohta-Ono [42, Orientation chapter], [88], given a relative spin structure. For this purpose, we may ignore the constraints at the interior nodes $w_1, \ldots, w_m$ in $\operatorname{int}(S)$. The tangent spaces to these nodes and the linearized constraints $\mathrm{d}u(w_i) \in T_{u(w_i)}D$ are even dimensional and oriented by the given complex structures. Suppose the type $\mathbb{\Gamma}$ has at least one vertex $v \in \operatorname{Vert}(\mathbb{\Gamma})$. Consider a regular element

$$(C, u : S \to X) \in \mathcal{M}_{\mathbb{\Gamma}}(\phi, D, \underline{\sigma})$$

of type $\mathbb{\Gamma}$. The tangent space is the kernel of the linearized operator:

$$T_u\mathcal{M}_{\mathbb{\Gamma}}(\phi, D) \cong \ker(\tilde{D}_u)$$

where (abusing notation) $\tilde{D}_u$ is the restriction of the operator in (37) to the space of sections $(\zeta, \xi : S \to u^*TX)$ satisfying constraints

$$\xi(w_e) \in T\sigma_e, e \in \operatorname{Edge}_{\circ,\to}(\mathbb{\Gamma}), \quad \xi(w_e) \in TD, e \in \operatorname{Edge}_{\bullet,\to}(\mathbb{\Gamma}).$$

The operator $\tilde{D}_u$ admits a homotopy

$$\tilde{D}_u^t, t \in [0,1], \quad \tilde{D}_u^1 = \tilde{D}_u, \quad \tilde{D}_u^0 = 0 \oplus D_u$$

so that $\tilde{D}_u^0$ is a direct sum of the zero operator and the linearized Cauchy-Riemann operator D_u. For any vector spaces V, W, the determinant line of the direct sum admits an isomorphism $\det(V \oplus W) \cong \det(V) \otimes \det(W)$. The deformation $\tilde{D}_u^t, t \in [0,1]$ of operators induces a family of determinant lines $\det(\tilde{D}_u^t)$ over the interval $[0,1]$, necessarily trivial. One obtains by parallel transport of this family an identification of determinant lines

$$\det(T_u\mathcal{M}_{\mathbb{\Gamma}}(\phi, D)) \to \det(T_C\mathcal{M}_\Gamma) \otimes \det(D_u) \tag{53}$$

well-defined up to isomorphism. In the case of nodes of S mapping to self-intersection

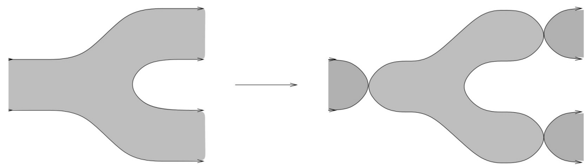

FIGURE 8. Bubbling off the strip-like ends

points $x \in \mathcal{I}^{\mathrm{si}}(\phi)$, the determinant line $\det(D_u)$ is oriented by "bubbling off one-pointed disks", as in [42, Theorem 44.1] or [88, Equation (36)]. For each self-intersection point

$$(x_- \neq x_+) \in L^2, \quad \phi(x_-) = \phi(x_+),$$

choose a path of Lagrangian subspaces

$$\gamma_x : [0,1] \to \mathrm{Lag}(T_{\phi(x_-)=\phi(x_+)}X) \tag{54}$$

$$\gamma_x(0) = D_{x_-}\phi(T_{x_-}L) \quad \gamma_x(1) = D_{x_+}\phi(T_{x_+}L).$$

Let S be the unit disk with a single boundary marking $1 \in \partial S$. The path γ_x defines a totally real boundary condition on S on the trivial bundle with fiber T_xX. Let $\det(D_x^+)$ denote the determinant line for the Cauchy-Riemann operator D_x^+ with boundary conditions γ_x as in [88]. Let D_x^- be the operator as in the previous discussion but with the direction of the path γ_x reversed and

$$\mathbb{D}_x^+ = \det(D_x^+), \quad \mathbb{D}_x^- = \det(D_x^-) \otimes \det(T_xL)$$

The once-marked disks with boundary conditions γ_x and $\gamma_{\overline{x}}$ glue together along the strip like end to a disk with no-strip like end whose boundary condition is the concatenation of γ_x and $\gamma_{\overline{x}}$. This boundary condition is isotopic to the constant boundary condition, and the determinant line extends over the isotopy giving a canonical isomorphism

$$\mathbb{D}_x^- \otimes \mathbb{D}_x^+ \to \mathbb{R}. \tag{55}$$

A choice of orientations $o_x \in \mathbb{D}_x^{\pm}$ for the self-intersection points x are *coherent* if the isomorphisms (55) are orientation preserving with respect to the standard orientation on $\mathbb{R}$. For each cell $\sigma_j \in \mathcal{I}^c(\phi)$ choose an orientation on the domain $B^{d(j)}$ and let $\mathbb{D}_{\sigma_j}^-$ denote the determinant line of the tangent space to $B^{d(j)}$ at any point. Choose orientations $\mathbb{D}_{\sigma_k^\vee}^+$ for cells in the dual decomposition. Given a relative spin structure for $\phi : L \to X$, the orientation at u is determined by an isomorphism

$$\det(D_u) \cong \mathbb{D}_{\sigma_0}^+ \otimes \mathbb{D}_{\sigma_1}^- \otimes \ldots \otimes \mathbb{D}_{\sigma_d}^-. \tag{56}$$

The isomorphism (56) is determined by degenerating the surface with strip-like ends to a nodal surface as in Figure 8. Thus each end $\epsilon_e, e \in \mathcal{E}(S_v)$ of a component S_v with a node w mapping to a self-intersection point is replaced by a disk $S_{v^\pm(e)}$

with one end attached to the rest of the surface by a node $w_e^{\pm}$. After combining the orientations o_e on the determinant lines on $S_{v^{\pm}(k)}$ with orientations o_σ on the tangent spaces to cells σ in the case of broken edges or semi-infinite edges $e \in \mathrm{Edge}(\Gamma), \ell(e) = \infty$, one obtains an orientation o_u on the determinant line of the parameterized linear operator $\det(\tilde{D}_u)$. The orientations on the determinant lines give orientations on the regularized moduli spaces $\mathcal{M}_\Gamma(\phi, D, \underline{\sigma})$.

There is a similar discussion for weighted moduli spaces. The moduli space of weighted trees $\mathcal{M}_\Gamma^{\mathrm{wt}}$ is oriented via the product description (48). in the case of labels 1_ϕ^s or 1_ϕ^g the orientations of the moduli spaces are defined by considering the normal bundles of the inclusions $\{\infty\} \to [0,\infty]$ resp. $\{0\} \to [0,\infty]$ to be positively resp. negatively weighted. As a corollary of this discussion we have:

Corollary 4.20. *The components $\mathcal{M}_\Gamma(\phi, D, \underline{\sigma})_{\le 1}$ of expected dimension at most one are equipped with orientations satisfying the standard gluing signs for inclusions of boundary components described in* [21].

In particular, for labelled map types $\mathbb{\Gamma}$ of expected dimension zero the strata $\mathcal{M}_{\mathbb{\Gamma}}(\phi, D)$ inherit orientation maps

$$o : \mathcal{M}_{\mathbb{\Gamma}}(\phi, D) \to \{+1, -1\} \tag{57}$$

comparing the constructed orientation to the canonical orientation of a point.

5. Fukaya algebras in the cellular model

Fukaya algebras of immersed Lagrangians with cell decompositions may be defined by adapting the Morse model definition in Palmer-Woodward [66] for cellular homology. The difference is mostly an aesthetical one; the cellular decompositions used in this paper are in fact associated to Morse-Smale pairs but conceptually the result is more easily understood in terms of cells rather than critical points.

5.1. Cellular Floer cochains.

The generators of the Floer complex in the cellular model consist of cells, self-intersection points, and additional generators for homotopy units. The set of generators is

$$\mathcal{I}(\phi) = \mathcal{I}^c(\phi) \cup \mathcal{I}^{\mathrm{si}}(\phi) \cup \mathcal{I}^{\mathrm{hu}}(\phi) \tag{58}$$

where

$$\mathcal{I}^c(\phi) := \{\sigma_i : B^{d(i)} \to L\}$$

is the set of cells, given as maps σ_i from balls $B^{d(i)}$ of dimension $d(i)$ to L with boundary in the union of the images of j-cells for $j < i$;

$$\mathcal{I}^{\mathrm{si}}(\phi) := (L \times_\phi L) - \Delta_L$$

is the set of ordered self-intersection points, where $L \times_\phi L$ is the fiber product and $\Delta_L \subset L^2$ the diagonal; and

$$\mathcal{I}^{\mathrm{hu}}(\phi) := \{1_\phi^s, 1_\phi^h\}$$

are two additional generators added as part of the homotopy unit construction. The sum

$$1_\phi^g := \sum_{\mathrm{codim}(\sigma_i)=0} \sigma_i$$

is the *geometric unit*. Thus $\mathcal{I}(\phi)$ consists of the cells in L together with two copies of each self-intersection point, plus two extra generators.

In order to obtain graded Floer cohomology groups, a grading on the set of generators is defined as follows. Given an orientation, there is a natural $\mathbb{Z}_2$-valued map

$$\mathcal{I}(\phi) \to \mathbb{Z}_2, \quad x \mapsto |x|$$

obtained by assigning to any cell $\sigma \in \mathcal{I}^c(\phi)$ the codimension mod 2 and to any self-intersection point $(x_-, x_+) \in \mathcal{I}^{\mathrm{si}}(\phi)$ the element $|x| = 0$ resp. $|x| = 1$ if the self-intersection is even resp. odd. The grading degrees of the cells are determined by the codimensions $\mathrm{codim}(\sigma_i) = \dim(L) - d(i)$ for the cells σ_i, and

$$|1_\phi^s| = 0, \quad |1_\phi^h| = -1$$

for the extra generators $1_\phi^s, 1_\phi^h$.[9] Denote by $\mathcal{I}^k(\phi)$ the subset of $\sigma \in \mathcal{I}(\phi)$ with $|\sigma| = k$ mod 2. Let $\mathcal{I}^\vee(\phi)$ denote the corresponding set for the dual cell decomposition.

The space of Floer cochains is freely generated by the above generators over the Novikov field. The space of Floer cochains is the $\mathbb{Z}_2$-graded vector space

$$CF(\phi) = \bigoplus_{k\in\mathbb{Z}_2} CF^k(\phi), \quad CF^k(\phi) = \bigoplus_{x\in\mathcal{I}^k(\phi)} \Lambda x.$$

The q-valuation on Λ extends naturally to $CF(\phi)$:

$$\mathrm{val}_q : CF(\phi) - \{0\} \to \mathbb{R}, \quad \sum_x c(x)x \mapsto \min_x(\mathrm{val}_q(c(x)), c(x) \neq 0).$$

5.2. **Composition maps.** The composition maps in the cellular Fukaya algebra are counts of rigid holomorphic treed disks weighted by areas and holonomies. For perturbations from the last section, define *higher composition maps*

$$m_d : CF(\phi)^{\otimes d} \to CF(\phi)[2-d], \quad d \geq 0$$

on generators as follows. Let $\sigma_1, \ldots, \sigma_d \in \mathcal{I}(\phi)$ and let

$$\mathcal{M}_\Gamma(\phi, D, \underline{\sigma})_0 \subset \mathcal{M}_\Gamma(\phi, D)$$

denote the subset of rigid maps with constraints given by generators $\underline{\sigma} = (\sigma_0, \ldots, \sigma_d)$ as defined in (40). Given cells α, β in the first resp. second cellular decomposition, $\mathcal{I}^c(\phi)$ resp. $\mathcal{I}^{c,\vee}(\phi)$ let $c(\alpha, \beta) = c^\vee(\beta, \alpha)$ denote the coefficient of $[\alpha] \times [\beta]$ in δ_1

[9] Here we work only with $\mathbb{Z}_2$ gradings, so the extra generators are simply even and odd respectively; see Remark 2.3.

as in (21). Extend $c(\cdot,\cdot)$ to $\mathcal{I}(\phi) \times \mathcal{I}^c(\phi)$ by defining the dual of x to be $\overline{x}$ and vice-versa, that is,

$$c(\sigma, x) = \begin{cases} 1 & \sigma = \overline{x} \\ 0 & \text{otherwise} \end{cases}. \tag{59}$$

Definition 5.1. (Composition maps) On generators $\sigma_1, \ldots, \sigma_d$ define

$$m_d(\sigma_1, \ldots, \sigma_d) = \sum_{\substack{\mathbb{\Gamma}, \ \sigma_0 \in \mathcal{I}^\vee(\phi), \gamma \in \mathcal{I}(\phi) \\ u \in \overline{\mathcal{M}}_{\mathbb{\Gamma}}(\phi, D, \sigma_0, \ldots, \sigma_d)}} \mathrm{wt}(u, \gamma)\gamma \tag{60}$$

where the weight $\mathrm{wt}(u,\gamma)$ is defined by

$$\mathrm{wt}(u, \gamma) = \frac{(-1)^{\heartsuit}}{\theta(u)!} y(\partial u) q^{A(u)} o(u) c^\vee(\sigma_0, \gamma)$$

with the notation

- $\theta(u) \in \mathbb{Z}_{>0}$ is the number of interior leaves $e \in \mathrm{Edge}_\bullet(\Gamma)$, corresponding to intersections $u(w_e) \in D$ with the Donaldson hypersurface D;
- $y(\partial u) \in \Lambda_0$ is the holonomy of the local system y around the boundary $u(\partial S) \subset \phi(L)$ as in 2.4;
- $A(u) \in \mathbb{R}_{\geq 0}$ is the sum of the areas $A(u_v)$ of the disks and spheres $u_v : S_v \to X$ for $v \in \mathrm{Vert}(\Gamma)$;
- $o(u) \in \{\pm 1\}$ is an orientation sign defined in (57) using the relative spin structure for $\phi : L \to X$;
- the exponent $\heartsuit \in \mathbb{Z}$ is given by

$$\heartsuit = \sum_{i=1}^{d} i|\sigma_i|; \tag{61}$$

- the sum is over all types of rigid maps $\mathbb{\Gamma}$;
- the sum $\sum_\gamma c^\vee(\sigma_0, \gamma)\gamma$ from (59) dualizes the output constraint σ_0, and

we have written tensor products as commas to save space. If a matching condition M_Γ is a formal sum (rather than a single diffeomorphism) the contributions are weighted by the coefficients c_i, d_j of the perturbations $M_{\Gamma,i}, H_{\Gamma,j}$ in (41), (42). This ends the Definition.

The composition maps involving one input of type $1_\phi^h, 1_\phi^s$ are also defined geometrically by the above sum, as in Lemma 5.4 below computing $m_1(1_\phi^h)$. In particular

$$m_2(1_\phi^s, 1_\phi^s) = 1_\phi^s, \quad -m_2(1_\phi^h, 1_\phi^s) = m_2(1_\phi^s, 1_\phi^h) = 1_\phi^h \tag{62}$$

since the corresponding moduli spaces are points. Recall that 1_ϕ^s is a strict unit if and only if

$$m_2(1_\phi^s, a) = a = (-1)^{|a|} m_2(a, 1_\phi^s), \quad m_d(\ldots, 1_\phi^s, \ldots) = 0, \forall d \neq 2. \tag{63}$$

Theorem 5.2. *For a perturbation system $\underline{P} = (P_\Gamma)$ with good properties as in Theorem 4.19 the maps $(m_d)_{d\geq 0}$ satisfy the axioms of a (possibly curved) A_∞ algebra $CF(\phi)$ with strict unit $1^s_\phi \in CF(\phi)$.*

Proof. We must show that the composition maps $(m_d)_{d\geq 0}$ satisfy the A_∞ *-associativity equations*

$$0 = \sum_{\substack{d_1,d_2\geq 0\\ d_1+d_2\leq d}} (-1)^{d_1+\sum_{i=1}^{d_1}\operatorname{codim}(\sigma_i)} m_{d-d_2+1}(\sigma_1,\ldots,\sigma_{d_1}, m_{d_2}(\sigma_{d_1+1},\ldots,\sigma_{d_1+d_2}),\sigma_{d_1+d_2+1},\ldots,\sigma_d) \tag{64}$$

for any $\sigma_1,\ldots,\sigma_d \in \mathcal{I}(\phi)$. Up to sign the relation (64) follows from the description of the boundary of the one-dimensional components in (c) of Definition 4.18, as we now explain. The condition (c) implies that any one-dimensional component $\mathcal{M}(\phi,D,\underline{\sigma})_1$ of $\mathcal{M}(\phi,D,\underline{\sigma})$ has true boundary points (that is, those 0-dimensional strata $\mathcal{M}_{\mathbb{F}}(\phi,D,\underline{\sigma})$ that form the topological boundary of $\overline{\mathcal{M}}(\phi,D,\underline{\sigma})_1$) that are given either by configurations $u: S \to X$ in which the length parameter $l(e)$ for an edge $e \in \mathrm{Edge}_\circ(\Gamma)$ is infinite or one of the semi-infinite edges $e \in \mathrm{Edge}_\infty(\Gamma)$ has node $w_e \in S$ mapping into the boundary of a cell $\sigma_i(\partial B^{d(i)})$.

In the first case of interior edge breaking, suppose that the combinatorial type Γ of the moduli space $\mathcal{M}_\Gamma(\phi,D)$ has all weights 0 and there is a single interior edge $e \in \mathrm{Edge}_-(\Gamma)$ of infinite length $\ell(e) = \infty$. The graph Γ is obtained by gluing together graphs Γ_1,Γ_2 with $d-d_2+1$ and d_2 leaves along leaves e_-,e_+, say with θ_1 resp. θ_2 interior leaves. There are $\theta!/\theta_1!\theta_2!$ orderings of the interior leaves $T_e, e \in \mathrm{Edge}_{\bullet,\to}(\Gamma)$ compatible with the fixed orderings on the two component graphs Γ_1,Γ_2. There is a natural map of moduli spaces with an infinite length edge

$$\bigcup_{\mathbb{F}} \mathcal{M}_{\mathbb{F}}(\phi,D,\sigma_0,\ldots,\sigma_d) \to \bigcup_{\alpha\in\mathcal{I}(\phi),\beta\in\mathcal{I}^\vee(\phi)} \mathcal{M}_{\mathbb{F}_1}(\phi,D,\sigma_0,\sigma_1,\ldots,\sigma_{i-1},\alpha,\sigma_{i+d_2},\ldots,\sigma_d) \times \mathcal{M}_{\mathbb{F}_2}(\phi,D,\beta,\sigma_i,\ldots,\sigma_{i+d_2-1}) \tag{65}$$

which has $c^\vee(\alpha,\beta)\theta!/\theta_1!\theta_2!$ points in the fiber. Combining(45) and (46) we have (up to signs to be determined) for any γ

$$
\begin{aligned}
(66)\quad 0 = &\sum_{\substack{i=1,\dots,d,\alpha\in\mathcal{I}(\phi)\\ u\in\mathcal{M}_\Gamma(\phi,D,\sigma_0,\dots,\sigma_{i-1},\alpha,\sigma_{i+1},\dots,\sigma_d)_0}} \pm\partial(\sigma_i,\alpha)\,\mathrm{wt}(u,\gamma)\\
&+\sum_{\alpha\in\mathcal{I}(\phi),u\in\mathcal{M}_\Gamma(\phi,D,\sigma_0,\sigma_1\dots,\sigma_d)_0} \pm\,\mathrm{wt}(u,\alpha)\partial(\alpha,\gamma)\\
&+\sum_{\substack{\alpha\in\mathcal{I}(\phi),\beta\in\mathcal{I}^\vee(\phi),d_1+d_2\le d\\ u_1\in\mathcal{M}_{\Gamma_1}(\phi,D,\sigma_0,\sigma_1,\dots,\sigma_{d_1},\alpha,\sigma_{d_1+d_2+1},\dots,\sigma_d)_0\\ u_2\in\mathcal{M}_{\Gamma_2}(\phi,D,\beta,\sigma_{d_1+1},\dots,\sigma_{d_1+d_2})_0}} \pm\,\mathrm{wt}(u_1,\gamma)\,\mathrm{wt}(u_2,\alpha)
\end{aligned}
$$

where the first sum consists of maps u with evaluation in the boundary of the given generator σ_i (corresponding to broken incoming edge in the Morse case); the second sum consists of maps u with an outgoing marking mapping to the boundary of σ_0; and the third type consists of maps u with an interior edge mapping to the degenerated diagonal $\delta_1(L)$. For a picture of maps of the third type, see Figure 3 which (after removing the spheres) shows a possible contribution to the A_∞ associative equation corresponding to the term $m_3(m_0(1),a_1,a_2))$. The sum over σ_0 gives the A_∞ axiom (64) up to sign. Following [63, Theorem 4.10] for the sign computation, we have the following: (As far as we can see one could also equally well use the treatment in Seidel [75, 12f], after redefining the generators of the Fukaya algebra to be orientations on the corresponding determinant lines.) The gluing map on determinant lines takes the form (omitting tensor products from the notation to save space)

$$
\begin{aligned}
(67)\quad &\det(\mathbb{R})\det(T\mathcal{M}_{d_2})\mathbb{D}^+_\alpha\mathbb{D}^-_{\sigma_{d_1+1}}\dots\mathbb{D}^-_{\sigma_{d_1+d_2}}\\
&\qquad\det(T\mathcal{M}_{d-d_2+1})\mathbb{D}^+_{\sigma_0}\mathbb{D}^-_{\sigma_1}\dots\mathbb{D}^-_{\beta}\dots\mathbb{D}^-_{\sigma_d}\\
&\qquad\qquad\to\det(T\mathcal{M}_d)\mathbb{D}^+_{\sigma_0}\mathbb{D}^-_{\sigma_1}\dots\mathbb{D}^-_{\sigma_d}
\end{aligned}
$$

where $\mathcal{M}_d$ is the moduli space of treed disks with d boundary leaves. The computation of the sign of this map is similar to that of [63, Theorem 4.10] and is congruent mod 2 to

$$
\sum_{k=1}^{d}(k+1)|\sigma_k|. \tag{68}
$$

Since (68) is independent of d_1,d_2, the A_∞ -associativity relation (64) follows for inputs not involving 1^s_ϕ or 1^h_ϕ.

The introduction of weighted semi-infinite edges produces boundary terms arising from a weight becoming 0 or 1, rather than an edge reaching infinity length. In the case of a weighted leaf e one has additional boundary components of the moduli space $\overline{\mathcal{M}}_d$ of treed disks $(C:u:S\to X)$ with weightings $\rho(e)$ either zero or infinity. Those configurations correspond to a weighted leaf $e \in \mathrm{Edge}^h_{\rightarrow}(\Gamma)$ and outgoing edge $e_0 \in \mathrm{Edge}^g_{\rightarrow}(\Gamma) \cup \mathrm{Edge}^s_{\rightarrow}(\Gamma)$. In the A_∞ maps, those configurations correspond to the terms involving 1^s and 1^g in $m_1(1^h)$.

The strict unitality follows from the existence of a forgetful map for perturbations with edges with an infinite weighting. Suppose that an incoming leaf is labelled 1^s_ϕ. The forgetful axiom implies the existence of a map

$$\mathcal{M}_{\mathbb{\Gamma}}(\phi, D, \underline{\sigma}) \to \mathcal{M}_{f(\mathbb{\Gamma})}(\phi, D, f(\underline{\sigma}))$$

where $f(\mathbb{\Gamma})$ is obtained by forgetting the edge and stabilizing, and $f(\underline{\sigma})$ is obtained by forgetting the input labelled 1^s_ϕ, assuming that $f(\mathbb{\Gamma})$ is non-empty. Since $\mathcal{M}_{\mathbb{\Gamma}}(\phi, D, \underline{\sigma})$ has expected dimension zero by assumption, $\mathcal{M}_{f(\mathbb{\Gamma})}(\phi, D, f(\underline{\sigma}))$ has expected dimension -1 and is therefore empty. This is a contradiction unless $f(\mathbb{\Gamma})$ is empty, which is to say that $\mathbb{\Gamma}$ is a type with no interior leaves and either one or two boundary leaves. In the case of one incoming leaf, the count of such configurations is zero since 1^s_ϕ is classically closed. For one of the generators $1^s_\phi, 1^h_\phi, 1^g_\phi$, the condition (63) follows as in Remark 4.15. □

Remark 5.3. The second A_∞ relation gives a condition for the existence of a coboundary operator. The element

$$m_0(1) \in CF(\phi)$$

is the *curvature* of the Fukaya algebra and has positive q-valuation $\mathrm{val}_q(m_0(1)) > 0$ by Remark 5.5. The Fukaya algebra $CF(\phi)$ is *flat* if $m_0(1)$ vanishes, and *projectively flat* if $m_0(1)$ is a multiple of the identity 1^s_ϕ. The first two A_∞ relations are the analogs of the Bianchi identity and definition of curvature respectively in differential geometry:

$$m_1(m_0(1)) = 0, \quad m_1^2(\sigma) = m_2(m_0(1), \sigma) - (-1)^{|\sigma|} m_2(\sigma, m_0(1)), \quad \forall \sigma \in \mathcal{I}(\phi).$$

Thus if $CF(\phi)$ is projectively flat then $m_1^2 = 0$ and the *undeformed Floer cohomology* $HF(\phi) = \ker(m_1)/\operatorname{im}(m_1)$ is defined.

Lemma 5.4. *For the composition maps m_d defined using (60), $m_1(1^h_\phi)$ is equal to $1^s_\phi - 1^g_\phi$ plus terms that are higher order in q.*

Proof. By definition, $m_1(1^h_\phi)$ counts configurations with a single input and output edge. By definition, constant configurations from a single edge T_e with input 1^h_ϕ and output 1^s_ϕ or 1^g_ϕ are stable. The moduli space of such configurations with weight $\rho(e) = \infty$ occur as the positive end of the moduli space $\mathcal{M}^{\mathrm{wt}}_\Gamma$ and by definition is positively oriented, while the locus with $\rho(e) = 0$ is negatively oriented. Configurations with no disks contribute $1^s_\phi - 1^g_\phi$, while configurations $(C, u : S \to X)$ with at least one disk $u_v : S_v \to X$ contribute terms with positive area $A(u) > 0$, since at least one of the disks u_v must be non-constant by the stability condition. □

Lemma 5.5. *For composition maps m_d defined using (60), the curvature $m_0(1)$ satisfies the gap condition $\mathrm{val}_q(m_0(1)) \geq \hbar$, where $\hbar > 0$ is the energy quantization constant of Lemma 4.7.*

Proof. Any configuration $(C, u : S \to X)$ with no leaves T_e must have at least one non-constant holomorphic disk $u_v|S_v : S_v \to X$, by the stability condition. Thus the area of any configuration $(C, u : S \to X)$ contributing to $m_0(1)$ must be at least $A(u_v) \geq \hbar$ by Lemma 4.7. □

More generally, the Fukaya algebra may admit projectively flat deformations even if it itself is not projectively flat. Consider the sub-space of $CF(\phi)$ consisting of elements with positive q-valuation

$$CF(\phi)_+ = \bigoplus_{\sigma \in \mathcal{I}(\phi)} \Lambda_{>0}\sigma.$$

where $\Lambda_{>0} = \{0\} \cup \mathrm{val}_q^{-1}(0, \infty)$.[10] Define the *Maurer-Cartan map*

$$m : CF(\phi)_+ \to CF(\phi), \quad b \mapsto m_0(1) + m_1(b) + m_2(b, b) + \dots .$$

Here $m_0(1)$ is the image of $1 \in \Lambda$ under

$$m_0 : CF(\phi)^{\otimes 0} \cong \Lambda \to CF(\phi).$$

Let $MC(\phi)$ denote the space of (weakly) *bounding cochains*:

$$MC(\phi) = \left\{ \begin{array}{c} b \in CF_+^{\mathrm{odd}}(\phi) \\ \mathrm{val}_q(b) > 0 \end{array} \middle| \; m(b) \in \mathrm{span}(1_\phi^s) \right\}. \tag{69}$$

The value $W(b)$ of $m(b)$ for $b \in MC(\phi)$ defines the *disk potential*

$$W : MC(\phi) \to \Lambda, \quad m(b) = W(b) 1_\phi^s.$$

For any $b \in MC(\phi)$ define a projectively flat *deformed Fukaya algebra* $CF(\phi, b)$ with the same underlying vector space but composition maps m_d^b defined by

$$m_d^b(a_1, \dots, a_d) = \sum_{i_1, \dots, i_{d+1}} m_{d+i_1+\dots+i_{d+1}}(\underbrace{b, \dots, b}_{i_1}, a_1, \underbrace{b, \dots, b}_{i_2}, a_2, b, \dots, b, a_d, \underbrace{b, \dots, b}_{i_{d+1}}); \tag{70}$$

note that these maps only satisfy the A_∞ axiom if b has odd degree because of additional signs that appear in the case b even. Occasionally we wish to emphasize the dependence of $MC(\phi)$ on the local system $y \in \mathcal{R}(\phi)$ and we write $MC(\phi, y)$ for

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[10] In fact one can only require positive valuations of the coefficients of the degree-one generators, and the self-intersection points. The requirement of positivity at the self-intersection points can be slightly weakened, see (76) below.

$MC(\phi)$. For $b \in MC(\phi)$, the maps $m_d^b, d \geq 1$ form a projectively flat A_∞ algebra. The resulting cohomology is denoted

$$HF(\phi, b) = \ker(m_1^b)/\operatorname{im}(m_1^b)$$

The union of $HF(\phi, b)$ for $b \in MC(\phi)$ mod gauge equivalence, see the following section, is a homotopy invariant of $CF(\phi)$ and independent of all choices up to isomorphism of groups and change of base point b.

In the case of self-intersection points, the condition that the Maurer-Cartan solutions have positive q-valuation may be relaxed using the following lemma, which is a sort of energy quantization for corners at self-intersections. The following is an analog of [30, Lemma 2.6].

Lemma 5.6. *Let* $\dim(L_0) > 2$ *and* $k > 2$. *There exists a constant* $\delta > 0$ *such that the following holds: Suppose that* $(C, u : S \to X)$ *is a rigid treed holomorphic disk with* $k+1$ *leaves. If* s *is the number of boundary nodes* $w_e \in S$ *mapping to transverse self-intersection points* $\sigma_e \in \mathcal{I}^{\mathrm{si}}(\phi)$, *then* $A(u) \geq s\delta$.

Proof. In each local chart near a self-intersection, we aim to show that the area of a holomorphic map as in the statement of the Lemma is controlled by the number of corners mapping to the self-intersection. Let $x \in \mathcal{I}^{\mathrm{si}}(\phi)$ be a self-intersection point. We may assume without loss of generality that the Darboux chart $X \supset U \to \mathbb{C}^n$ has image that contains the radius r ball $B_r(0) \subset \mathbb{C}^n$ for $r \in (0, \infty)$ small. Recall from Section 3.3 that the complex structure $J_\Gamma \in \mathcal{J}(X)$ near the self-intersection point is standard so that $J_\Gamma | U = J_0$, where $J_0 z = iz$ for any tangent vector $z \in T_x U \cong \mathbb{R}^{2n} \cong \mathbb{C}^n$. The symplectic form ω_0 on $\mathbb{C}^n$ is exact with

$$\omega_0 = \mathrm{d}\alpha_0, \quad \alpha_0 := \sum_{j=1}^n \frac{1}{2}(q_j \mathrm{d}p_j - p_j \mathrm{d}q_j) \in \Omega^1(\mathbb{C}^n).$$

By Stokes' theorem,

$$\int_{u^{-1}(U)} u^* \omega_0 = \int_{u^{-1}(\partial U)} u^* \alpha_0. \tag{71}$$

Here we have used that the restriction of α to the Lagrangian branches $\mathbb{R}^n, i\mathbb{R}^n$ vanishes.

We first deal with the case that the configuration is non-constant. On the locus $U^* = \{z \in S, u(z) \neq 0\}$ where u is non-zero in the local chart the map u descends to a map

$$[u] : U^* \to \mathbb{C}P^{n-1}, \quad z \mapsto \operatorname{span}(u(z)).$$

Consider the corresponding section $z \mapsto ([u(z)], u(z))$ of the pull-back $[u]^* T$ of the tautological bundle

$$T = \left\{ (\ell, z) \in \mathbb{C}P^{n-1} \times \mathbb{C}^n \;\middle|\; z \in \ell \right\} \to \mathbb{C}P^{n-1}.$$

The restriction of α_0 to the boundary of the ball $B_r(x)$, viewed as the unitary frame bundle of the tautological bundle T, is $-r^2$ times the standard connection one-form $\alpha_T \in \Omega^1(T_1)$ on the unit circle bundle T_1 in the tautological bundle $T \cong S^{2n-1}$ over $\mathbb{C}P^{n-1}$ with projection $\pi : T \to \mathbb{C}P^{n-1}$. Let

$$\operatorname{curv}(T) \in \Omega^2(\mathbb{C}P^{n-1}), \quad (\pi|_{T_1})^* \operatorname{curv}(T) = \mathrm{d}\alpha_T$$

denote the curvature two-form of α_T. One checks easily from, for example, a Taylor series expansion that removal of singularities holds in this case and the map $[u]$ extends to a holomorphic map $u^{-1}(U) \to \mathbb{C}P^{n-1}$. Since $[u]$ is also holomorphic, the pull-back of minus the curvature $-[u]^* \operatorname{curv}(T) \in \Omega^2(u^{-1}(U))$ is a positive two-form. On the other hand, on the locus $u \neq 0$ the map u determines a section of U whose normalization $v = u/\|u\|$ trivializes u^*T. The integral (71) is up to a scalar the parallel transport in the frame defined by the section u: Let $B_\epsilon(u^{-1}(0))$ denote a union of ϵ-balls around the finite set $u^{-1}(0)$, and denote the fractional winding number

$$d(u,z) := (2\pi)^{-1} \int_{\partial B_\epsilon(z) \cap S} v^* \alpha_T$$

of the phase of the section u along the path $\partial B_\epsilon(z) \cap S$; note that this integral is well-defined even if z is a boundary point. By Stokes' theorem

$$\int_{u^{-1}(\partial U)} u^* \alpha_0 = -r^2 \int_{u^{-1}(\partial U)} v^* \alpha_T \tag{72}$$

$$= \lim_{\epsilon \to 0} -r^2 \left(\int_{[u|U - B_\epsilon(u^{-1}(0))]} [v]^* \operatorname{curv}(T) \right. \tag{73}$$

$$\left. - \int_{\partial B_\epsilon(u^{-1}(0))} v^* \alpha_T \right) \tag{74}$$

$$= -r^2 \int_{[u|U]} [v]^* \operatorname{curv}(T) + 2\pi r^2 \sum_{z \in u^{-1}(0)} d(u,z). \tag{75}$$

The tautological bundle T has curvature $-\operatorname{curv}(T)$ that is a positive two-form, see for example Demailly [29, Section 15.B]. It follows that the first term on the right-hand side of (75) is non-negative. Let δ be the minimum of constants $r^2\pi/2$, as x varies over transverse self-intersection points. The angle change at any self-intersection point is a multiple of $\pi/2$, which proves the claim.

Finally, we deal with the case of constant disks mapping to self-intersections. Constant disks mapping $u : S \to X$ with image $\phi(x), x \in \mathcal{I}^{\mathrm{si}}(\phi)$ must have corners with alternating labels

$$\sigma_1 = x, \sigma_2 = \overline{x}, \sigma_3 = x, \sigma_4 = \overline{x}, \ldots, \sigma_{2k} = \overline{x}.$$

The sum of the degrees of these constraints is $k \dim(L)$, while the moduli space of $2k+1$-marked disks has dimension $2k-2$. The expected dimension of the moduli space of holomorphic treed disks is therefore

$$\begin{aligned}\dim \mathcal{M}_{\mathbb{\Gamma}}(\phi, D, \underline{\sigma}) &= (2k-2) - k(|x| + (\dim(L) - |x|) \\ &= (k-1)(2 - \dim(L)).\end{aligned}$$

Thus the labelled type $\mathbb{\Gamma}$ is rigid only if $\dim(L_0) = 2$ or $k = 1$. □

Corollary 5.7. *Let $\phi_0 : L_0 \to X$ be a self-transverse immersed Lagrangian brane of dimension $\dim(L_0) \geq 2$. The projective Maurer-Cartan equation*

$$\sum_{d \geq 0} m_d(b, \ldots, b) \in \text{span } 1_\phi^s \tag{76}$$

is well-defined for b of the form $b = b^{\text{si}} + b^c$ satisfying the condition in Definition 1.1 for the δ described in Lemma 5.6. Any such solution b has square-zero m_1^b and so a Floer cohomology group

$$HF(\phi, b) = \frac{\ker(m_1^b)}{\text{im}(m_1^b)}.$$

Proof. By Lemma 5.6, the infinite sum in the Maurer-Cartan equation (76) has q-valuations approaching infinity and is well-defined in $CF(\phi)$. A similar argument shows that the deformed Fukaya maps m_d^b from (70) are well-defined. □

Denote the set of solutions in Corollary 5.7 by

$$MC_\delta(\phi) = \{b \in CF(\phi) | (76)\}.$$

Remark 5.8. In the case $\phi = \phi_\epsilon$ is a surgery, we allow the coefficients $b_\epsilon(\mu), b_\epsilon(\lambda)$ of the meridian and longitude to have vanishing q-valuation. Theorem 5.13 implies that for the perturbation systems we use, the potential $W(b_\epsilon)$ and Floer cohomology $HF(\phi_\epsilon, b_\epsilon)$ are still well-defined for such elements.

Remark 5.9. We briefly describe the invariance properties of cellular Fukaya algebras. The argument using quilted disks with diagonal seam condition, see Charest-Woodward [21, Section 3] and Palmer-Woodward [66, Remark 6.3] extends to the cellular setting to define A_∞ morphisms between A_∞ algebras defined using different choices. Given two sets of choices $J_k, D_k, \underline{P}_k$ this argument gives an A_∞ morphism

$$CF(\phi, J_0, D_0, \underline{P}_0) \to CF(\phi, J_1, D_1, \underline{P}_1)$$

inducing in particular a morphism of Maurer-Cartan spaces

$$MC(\phi, J_0, D_0, \underline{P}_0) \to MC(\phi, J_1, D_1, \underline{P}_1)$$

preserving the Floer cohomologies. We expect that the homotopy type of the immersed Fukaya algebra $CF(\phi)$ is independent of the choices of almost complex structure, divisor, and perturbations. However, we prove no such invariance result here.

Example 5.10. The following example of an immersion of a circle in the plane shown in Figure 1 is an easily visualizable example of the invariance of the disk potential. In this case, the correspondence between holomorphic curves in X bounding ϕ_0 and ϕ_ϵ is an application of the Riemann mapping theorem. The Floer cohomology $HF(\phi_0)$ is trivial since the circle is displaceable by a compactly-supported Hamiltonian flow. The disk potential $W(\phi_0)$ is non-trivial and will be computed below. Let $\phi_0 : S^1 \to \mathbb{R}^2$ be the immersion with three self-intersection points

$$x, x', x'' \subset \phi_0(S^1).$$

The complement of the image $\phi_0(S^1) \subset X = \mathbb{R}^2$ has five connected components as in Figure 1.

We identify a particular weakly bounding cochain. Suppose that the area of the central region in $X - \phi_0(L)$ is $A_0 > 0$ while the area of each of the lobes is $A_1 > 0$. For simplicity, choose a cell structure on $L_0 \cong S^1$ with a single 0-cell σ_0 on the lobe containing x, and a single dual 1-cell σ_1; the actual cell structure used for the proof is somewhat more complicated but the difference in cell structures is irrelevant for the example. The coefficients of the cells σ_0, σ_1 in this cellular approximation are necessarily

$$c^\vee(\sigma_1, \sigma_0) = c(\sigma_0, \sigma_1) = 1$$

and all other coefficients vanish for degree reasons. Consider the cochain

$$b_0 = iq^{(-A_0+3A_1)/2} 1^h_{\phi_0} + iq^{(A_1-A_0)/2}(x + x' + x'') \in CF(\phi_0)$$

with coefficient $iq^{(A_1-A_0)/2}$ on the self-intersection points x, x', x'' and a multiple of the degree -1 element $iq^{(-A_0+3A_1)/2} 1^h_{\phi_0}$.

We compute the twisted curvature $m_0^{b_0}(1)$ as follows. The three outer lobes with no inputs contribute $q^{A_1}(\overline{x} + \overline{x'} + \overline{x''})$ to $m_0(1)$, and also to $m_0^{b_0}(1)$. The disk $u : S \to X$ whose interior $\mathrm{int}(S)$ maps to the central region of $X - \phi_0(L_0)$ contribute to $m_0^{b_0}(1)$ with outputs on x, x', x''. Since for each such output there are two inputs labelled b_0, the contribution of this region is

$$q^{A_0}(iq^{(A_1-A_0)/2})^2(\overline{x} + \overline{x'} + \overline{x''}) \in CF(\phi_0).$$

The holomorphic strip connecting x to the zero-dimensional cell contributes to $m_0^{b_0}(1)$ as well, with a single b_0 input and so a contribution of $iq^{(A_1-A_0)/2}q^{A_1}\sigma_1$. Finally, the constant disk with input $iq^{(-A_0+3A_1)/2}1^h_{\phi_0}$ contributes

$$iq^{(-A_0+3A_1)/2}(1^s_{\phi_0} - 1^g_{\phi_0}) \in CF(\phi_0)$$

to $m_1(b_0)$, hence $m_0^{b_0}(1)$. Thus

$$\begin{aligned} m_0^{b_0}(1) &= q^{A_1}(\overline{x} + \overline{x'} + \overline{x''}) + q^{A_0}(iq^{(A_1-A_0)/2})^2(\overline{x} + \overline{x'} + \overline{x''}) \\ &+ (iq^{(A_1-A_0)/2})q^{A_1}\sigma_1 + iq^{(-A_0+3A_1)/2}(1^s_{\phi_0} - 1^g_{\phi_0}) \\ &= iq^{(-A_0+3A_1)/2}1^s_{\phi_0} \end{aligned}$$

is a multiple of the unit $1^s_{\phi_0}$. Therefore, the element $b_0 \in MC(\phi_0)$ is a solution to the projective Maurer-Cartan equation.

The self-intersection points of ϕ_0 are admissible in the sense of Definition 1.1, which implies that the Floer cohomology is well-defined. Any disk $u : S \to X$ with boundary on ϕ_0 and meeting one of the self intersection points $x = (x_-, x_+) \in S^1$ without a branch change must contain in its image $u(S)$ the exterior non-compact region in X outside the curve $\phi_0(S^1)$. This is impossible since the image of a compact set must be compact.

A small Lagrangian surgery produces a Lagrangian immersion of a disjoint union of circles. Choose $\epsilon > 0$ sufficiently small so that the surgery is defined and

$$(A_1 - A_0)/2 = -A(\epsilon)$$

where $A(\epsilon) > 0$ is the area from Definition 2.1. Let σ_1', σ_1'' denote the top-dimensional cells on the two components near the self-intersection point x, as in Figure 1. As explained below in (187), the shift from b_0 to b_ϵ is equivalent to a shift in the local system. Define a local system on y_ϵ on ϕ_ϵ by

$$y_\epsilon([\sigma_1']) = y_\epsilon([\sigma_1'']) = iq^{(A_1 - A_0)/2} q^{A(\epsilon)} = i.$$

Define b_ϵ by removing the x-term so that

$$b_\epsilon = iq^{(-A_0 + 3A_1)/2} 1^h_{\phi_0} + iq^{(A_1 - A_0)/2}(x' + x'').$$

We have

$$\begin{aligned} m_0^{b_\epsilon}(1) &= iq^{A_1 - A(\epsilon)}\sigma_1' + (iq^{(A_1 - A_0)/2})^2 q^{A_0 - A(\epsilon)}\sigma_1'' \\ &\quad + q^{A_1}(\overline{x}' + \overline{x}'') + q^{A_0 - A(\epsilon)} i (iq^{(A_1 - A_0)/2})(\overline{x}' + \overline{x}'') \\ &\quad + iq^{(-A_0 + 3A_1)/2}(1^s_{\phi_0} - 1^g_{\phi_0}) \\ &= iq^{(-A_0 + 3A_1)/2} 1^s_{\phi_0}. \end{aligned}$$

It follows that $m_0^{b_\epsilon}(1)$ is a multiple of the strict unit $1^s_{\phi_0}$ on the right-hand-side with the same value of the potentials

$$W_0(b_0, y_0) = iq^{(3A_1 - A_0)/2} = W_\epsilon(b_\epsilon, y_\epsilon)$$

as the unsurgered immersion ϕ_0. This ends the Example.

5.3. **Gauge equivalence.** A notion of gauge equivalence relates solutions to the weak Maurer-Cartan equation so that cohomology is invariant under gauge equivalence. Let $b_0, \ldots, b_d \in CF(\phi)$ have odd degree and let $a_1, \ldots, a_d \in CF(\phi)$. Define

$$\begin{aligned} (77) \quad & m_d^{b_0, b_1, \ldots, b_d}(a_1, \ldots, a_d) = \\ & \sum_{i_0, \ldots, i_d} m_{d + i_0 + \ldots + i_d}(\underbrace{b_0, \ldots, b_0}_{i_0}, a_1, \underbrace{b_1, \ldots, b_1}_{i_2}, a_2, b_2, \\ & \qquad \ldots, b_2, \ldots, a_d, \underbrace{b_d, \ldots, b_d}_{i_d}). \end{aligned}$$

Two odd elements $b_0, b_1 \in CF(\phi)_+$ are *gauge equivalent* if and only if

$$\exists h \in CF(\phi)_+,\ b_1 - b_0 = m_1^{b_0,b_1}(h), \quad \deg(h) \text{ even}.$$

We then write $b_0 \sim_h b_1$. The discussion on [21, p. 75] shows that $\sim_h$ is an equivalence relation. The linearization of the above equation is $m_1(h) = b_1 - b_0$, in which case we say that b_0 and b_1 are *infinitesimally gauge equivalent.*

For notational convenience, we define a "shifted valuation"

$$\begin{array}{ll} \operatorname{val}_q^\delta(b^{\mathrm{si}}) = \operatorname{val}_q(b^{\mathrm{si}}) + \delta & b^{\mathrm{si}} \in \operatorname{span}(\mathcal{I}^{\mathrm{si}}(\phi)) - \{0\} \\ \operatorname{val}_q^\delta(b^c) = \operatorname{val}_q(b^c) & b^c \in \operatorname{span}(\mathcal{I}^c(\phi)) - \{0\} \\ \operatorname{val}_q^\delta(b^c + b^{\mathrm{si}}) = \min(\operatorname{val}_q^\delta(b^c), \operatorname{val}_q^\delta(b^{\mathrm{si}})), & b^c, b^{\mathrm{si}} \neq 0. \end{array}$$

Then $MC_\delta(\phi)$ is the space of solutions to the projective Maurer-Cartan equation with non-negative $\operatorname{val}_q^\delta$.

Lemma 5.11. *Let $\phi : L \to X$ be a self-transverse immersed Lagrangian brane and $b_0, b_1 \in CF(\phi)$.*

(a) (Preservation of the Maurer-Cartan space under gauge equivalence) *If $b_0 \sim_h b_1$ for some $h \in CF(\phi)_+$ and $b_0 \in MC_\delta(\phi)$ then $b_1 \in MC_\delta(\phi)$ as well.*

(b) (Integration of infinitesimal gauge equivalences into gauge equivalences) *Suppose that $h, b_0, b_1 \in CF(\phi)$ and $\zeta > 0$ are such that*

$$m_1^{b_0,b_1}(h) = b_1 - b_0, \quad \textit{mod } (\operatorname{val}_q^\delta)^{-1}((\zeta, \infty)), \quad \operatorname{val}_q^\delta(h) > \zeta. \tag{78}$$

Then there exists an element $b_\infty \in CF(\phi), \operatorname{val}_q^\delta(b_\infty) > 0$ with

$$m_1^{b_0,b_\infty}(h) = b_\infty - b_0, \quad \operatorname{val}_q^\delta(b_\infty - b_1) > \operatorname{val}_q^\delta(b_1 - b_0) + \zeta.$$

Proof. For item (a), define $W(b_1) \in \Lambda$ so that

$$m_0^{b_1}(1) = W(b_1) 1_\phi^s + c, \quad c := (m_0^{b_1}(1) - W(b_1)1_\phi^s). \tag{79}$$

The element $c \in CF(\phi)$ has coefficient of the strict unit 1_ϕ^s equal to zero. We have

$$\begin{aligned} m_0^{b_1}(1) - m_0^{b_0}(1) &= \sum_{d, i \leq d-1} m_d(\underbrace{b_0, \ldots, b_0}_{i}, b_1 - b_0, b_1, \ldots, b_1) \\ &= m_1^{b_0,b_1}(m_1^{b_0,b_1}(h)) \\ &= -m_2^{b_0,b_0,b_1}(m_0^{b_0}(1), h) + m_2^{b_0,b_1,b_1}(h, m_0^{b_1}(1)) \\ &= W(b_1)h - W(b_0)h + m_2^{b_0,b_1,b_1}(h, m_0^{b_1}(1) - W(b_1)1_\phi^s) \\ &= W(b_1)h - W(b_0)h + m_2^{b_0,b_1,b_1}(h, c) \end{aligned}$$

where the last inequality uses the definition of c in (79) and the strict unit identities (63). Rearranging terms we have

$$\begin{aligned} (W(b_1) - W(b_0))(1_\phi^s - h) &= ((m_0^{b_1}(1) - c) - W(b_1)h) - (m_0^{b_0}(1) - W(b_0)h) \\ &= m_2^{b_0,b_1,b_1}(h, c) - c. \end{aligned} \tag{80}$$

Since the two terms on the right have no coefficient of 1^s_ϕ by (62), we must have $W(b_0) = W(b_1)$.

We now apply an induction to show that the correction c vanishes. Suppose that there exists $\zeta > 0$ and $k \geq 1$ such that c is divisible by $q^{k\zeta}$ and $\mathrm{val}_q^\delta(h) > \zeta$; note that this holds for $k = 1$ and some $\zeta > 0$ sufficiently small by the previous paragraph. The equation (80) implies that

$$\begin{aligned} m_0^{b_1}(1) &= m_2^{b_0,b_1,b_1}(h,c) + W(b_1)1^s \\ &\in W(b_0)1^s_\phi + (\mathrm{val}_q^\delta)^{-1}(((k+1)\zeta,\infty)). \end{aligned}$$

Since this holds for every k, the claim (a) follows.

The second item (b) follows from a filtration argument. Suppose

$$b_k = m_1^{b_0,b_k}(h) + b_0 \mod (\mathrm{val}_q^\delta)^{-1}((k\zeta,\infty)).$$

Define a solution b_{k+1} to order $(k+1)\zeta$ by defining

$$b_{k+1} = m_1^{b_0,b_k}(h) + b_0.$$

The desired element is the limit of the elements b_k. □

The following gives a way of "gauging away" the weakly bounding cochain in a neighborhood of the self-intersection.

Proposition 5.12. *Let $\phi : L \to X$ be a Lagrangian immersion and $U \subset L$ an open set in L so that cellular differential on L is surjective onto the quotient of $C^{\mathrm{odd}}(L)$ by the span of odd cells not contained in U. Then any $b_0 \in MC_\delta(\phi)$ is gauge equivalent to some $b_\infty \in MC_\delta(\phi)$ that vanishes on cells contained in U.*

Proof. The leading order term in the Floer differential is the Morse differential, and by assumption the q^0 term in m_1^b is surjective as a map from $CF^0(\phi)$ to $CF^1(\phi)$ after modding out by cells not contained in U. Suppose that $b_k \in MC_\delta(\phi)$ vanishes on cells contained in U modulo terms of order $k\zeta$ for some $k \in \mathbb{Z}_+$. By Lemma 5.11 there exists $b_{k+1} \in MC_\delta(\phi)$ gauge equivalent to b_k such that

$$b_{k+1} - b_k = m_1^{b_k,b_{k+1}}(h), \quad \mathrm{val}_q^\delta(b_{k+1}(\sigma)) > (k+1)\zeta.$$

for any cell σ contained in U. The gauge transformation does not affect the lower order terms in b_k and so the limit

$$b_\infty := \lim_{k\to\infty} b_k$$

exists, lies in $MC_\delta(\phi)$, is gauge equivalent to b_0, and b_∞ vanishes on all cells contained in U. □

5.4. **Divisor insertions.** The divisor equation for Lagrangian Floer cohomology is a hoped-for relation for the insertion of a degree one cocycle into the composition maps. In this section, we prove a related result for the contribution of any configuration with a codimension one cell as input up to repetition of the input.[11] The divisor equation for Fukaya algebras is similar to the familiar divisor equation in Gromov-Witten theory. For $k \geq 0$ write

$$m_k = \sum_{\beta \in H_2(\phi)} m_{k,\beta} : CF(\phi)^{\otimes k} \to CF(\phi)$$

where $m_{k,\beta}$ is the contribution to m_k arising from holomorphic disks of class $\beta \in H_2(\phi)$. The divisor equation for a codimension one cycle c reads

$$\sum_{i=1}^{k+1} m_{k+1,\beta}(x_1, \ldots, x_{i-1}, c, x_i, \ldots, x_k) = \langle [c], [\partial \beta] \rangle m_{k,\beta}(x_1, \ldots, x_k) \tag{81}$$

see [23, Proposition 6.3]. In particular, the divisor equation implies that for x a degree one cocycle in $\phi(L)$

$$\sum_{k \geq 0} m_k(x, \ldots, x) = \sum_{k \geq 0} \sum_{\beta \in H_2(\phi)} \frac{\langle x, [\partial \beta] \rangle^k}{k!} m_{0,\beta}(1). \tag{82}$$

The right hand side of (82) is the contribution of $m_0(1)$ with local system y shifted by

$$\exp(x) \in \operatorname{Hom}(H_1(\phi(L), \mathbb{Z}), \Lambda_0) \cong \mathcal{R}(\phi).$$

In this sense, variations of the weakly bounding cochain $b \in MC(\phi)$ should be equivalent to variations of the local system $y \in \mathcal{R}(\phi)$. In general the truth of the divisor equation typically depends on the existence of regularized moduli spaces of holomorphic disks equipped with forgetful maps. The existence of such maps is rather difficult in the Morse or cellular settings.

We prove an identity for contributions to the composition maps with repeated cellular inputs related to the divisor equation (81); the terminology will be explained in the following discussion.

Theorem 5.13. *Suppose $\underline{P}^{\mathrm{red}}$ is a reduced-regular perturbation system. There exists a system of perturbations P_Γ satisfying the conditions in Definition 4.18 such that for each codimension one cell σ_i, rigid treed disk $u : S \to X$ whose boundary meets σ_i transversally, and collection of positive integers $\underline{d} = (d(z) \geq 0)$,*

[11]The results of this section are not necessary if $\dim(L_0) \geq 3$ and one uses the shift in local system (184) and

$$b_\epsilon = b_0 - b_0(x)x - b_0(\overline{x})\overline{x} + b_0(x)b_0(\overline{x})\lambda$$

instead of shifting the weakly bounding cochain in Definition 1.2, or in dimension $\dim(L_0) = 2$ with the local system formulas (184), (187).

there exists a unique configuration $u_{\underline{d}}$ obtained by repeating $d(z)$ inputs at each $z \in u^{-1}(\sigma_i)$ in the neighborhood U_u^{red} with weight given by the inverse factorial

$$\mathrm{wt}(u_{\underline{d}}, \gamma) = \prod_{z, \sigma_i} (d(z)!)^{-1} \, \mathrm{wt}(u, \gamma).$$

The terminology is explained by the following definitions.

Definition 5.14. (Inserting repeated inputs) Let $u : S \to X$ be a holomorphic disk bounding ϕ and let $z \in \partial S$ be a point where $u(z)$ intersects a codimension one cell σ transversally. Given such a disk and an integer $d \geq 1$, let

$$u_d^{\mathrm{red}} : C_d^{\mathrm{red}} \to X$$

be the configuration whose domain

$$C_d^{\mathrm{red}} := C \cup T_{e_0} \cup S_v \cup T_{e_1} \cup \ldots \cup T_{e_d}$$

consists of an additional disk S_v on which u_d^{red} is constant, attached at z via an edge T_{e_0} of some length $\ell(e_0)$ and d edges $T_{e_1}, \ldots, T_{e_d}$ attached to S_v mapping to σ, as in the middle drawing in Figure 9.

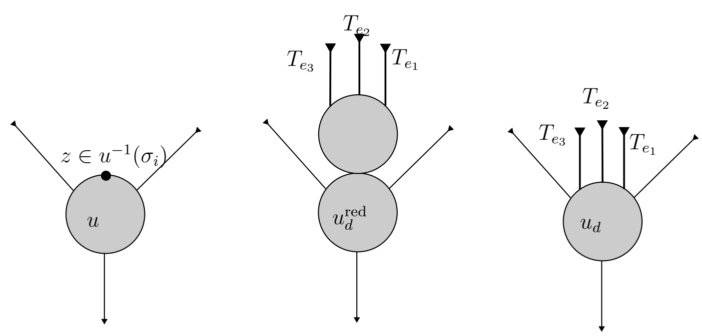

FIGURE 9. Configurations corresponding to an intersection with a codimension one cell: Left, a configuration meeting the cell transversally; Middle, adding a constant disk with multiple edges to the left configuration; Right, the center configuration after perturbation

Definition 5.15. A perturbation system $\underline{P}^{\mathrm{red}}$ is *reduced regular* for a cell σ if

(a) the matching conditions $\underline{M} = (M_\Gamma)$ are equal to the identity map in an open neighborhood of σ and
(b) all rigid configurations u not of the form $(u^{\mathrm{red}})_d$ for some u and d are regular.

In other words, rigid maps $u : S \to X$ not obtained by repeating inputs are all regular.

Suppose a reduced regular perturbation system $\underline{P}^{\mathrm{red}}$ has been chosen. Consider a new perturbation system $\underline{P}$ obtained from $\underline{P}^{\mathrm{red}}$ by perturbing the matching conditions for the semi-infinite edges $T_e, e \in \mathrm{Edge}_{\to}(\Gamma)$.

Lemma 5.16. *For a generic perturbation system $\underline{P}$ obtained from $\underline{P}^{\mathrm{red}}$ as above, each rigid treed disk u is regular and obtained from some u_d^{red} by removing the ghost component and changing the positions of the edges $T_{e_1}, \ldots, T_{e_d}$ so that the corresponding nodes map to the perturbed cell $M_\Gamma(w_{e_i}, \cdot)^{-1}(\sigma)$.*

Proof. Transversality is a standard consequence of Sard-Smale. To see that any rigid treed disk is of the form u_d^{red} for some u for sufficiently small perturbation, suppose that u_ν is a family of such maps for perturbations $M_{\Gamma,\nu}(w_{e_i}, \cdot)$ that converges to the identity in the C^∞ topology, but u_ν is not obtained as above. By Gromov compactness, there exists a subsequence of u_ν that converges to some u_d^{red}. But by transversality, there is a unique u_ν closed to u_d^{red} satisfying the perturbed matching conditions. This is a contradiction. □

That is, perturbed configurations are clustered around the configurations obtained by repeating inputs.

Example 5.17. We explain how repeating inputs appear in the product structure on the Floer cohomology of a Lagrangian given by the circle in the two-sphere. Let $X = S^2$ and $L = S^1$ a circle dividing X into two regions of areas A_+ and A_-. Equip L with its standard cell decomposition into two cells consisting of a 0-cell σ_0 and a 1-cell σ_1. Equip X with its standard complex structure. Since there are

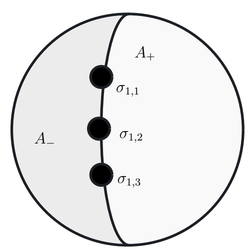

FIGURE 10. Disks with point constraints on the sphere

two Maslov-index-two disks corresponding to the two hemispheres, for the trivial bounding cochain we obtain

$$m_0(1) = (q^{A_+} + q^{A_-})\sigma_0$$

and the relation

$$m_1(\sigma_1) = (q^{A_+} - q^{A_-})\sigma_0.$$

Consider the family of maps u_d^{red} obtained from one of the two hemispheres as above with d by repeating incoming edges labelled by σ_1. The unperturbed complex structure and matching condition is reduced-regular for σ_1, since every configuration without constant disks labelled by multiple codimension-one constraints is regular. In particular, the map u_1 obtained from u by adding an edge T_{e_1} mapping to σ_0 is regular. A choice of perturbed matching conditions M_Γ at the incoming edges $T_{e_1}, \ldots, T_{e_d}$ amounts to a collection of perturbations $\sigma_{1,1}, \ldots, \sigma_{1,d}$, that is, points near σ_1. Suppose the perturbations $\sigma_{1,i}$ are in the same order around the boundary ∂S_v as the order given by the indices. Then there is a P_Γ-perturbed map u_d obtained from u by repeating inputs. Otherwise, if the order is different, then no such map exists. In particular, for $d = 2$ if the perturbations $\sigma_{0,1}, \sigma_{0,2}$ follow the cyclic order around the disk with area A_+ then we have

$$m_2(\sigma_1, \sigma_1) = q^{A_+}\sigma_0 \tag{83}$$

while if the perturbations are in the opposite order then

$$m_2(\sigma_1, \sigma_1) = q^{A_-}\sigma_0. \tag{84}$$

If $A_+ = A_1$, then the Floer cohomology is non-trivial and we obtain the relation on cohomology

$$m_2([\sigma_1], [\sigma_1]) = q^{A_+}[\sigma_0] = q^{A_-}[\sigma_0]$$

so the product on cohomology is independent of the choice of perturbation. The standard complex structure J on $X = S^2$ combined with the unperturbed matching condition $M_\Gamma(w, l) = l, \forall l \in L$ is reduced regular since the only maps of expected dimension zero are those containing a single Maslov index two disk constrained by degree one cells. Any such disk $u_v : S_v \to X$ is necessarily one of those above with area A_+ or A_- and is regular as long as there is a single incoming leaf. The only other configurations of expected dimension are those with multiple leaves labelled σ_1, and these are not required to be regular. The contributions to $m_d(\sigma_1, \ldots, \sigma_1)$ are clustered around the two maps arising from the two disks with areas A_+, A_-, and obtained by slightly perturbing the points $\sigma_1, \ldots, \sigma_1$ to points in general position $\sigma_{1,1}, \ldots, \sigma_{1,d}$.

We wish to choose perturbations so that the count of perturbed configurations with repeated inputs is controlled by the weight of the original configuration.

Definition 5.18. The matching condition M_Γ for a type Γ is *permutation-invariant* on an open subset $U \subset L$ for σ if for any two edges $e_1, e_2 \in \mathrm{Edge}(\Gamma)$ $M_\Gamma(w_{e_1}, l) = M_\Gamma(w_{e_2}, l)$ as multivalued perturbations for all $l \in U$.

Example 5.19. We continue the two-sphere example in 5.17 and compute the second structure map for a permutation-invariant matching condition. A permutation-invariant multi-valued perturbation is given by matching conditions assigning the nodes to map to perturbations of the 0-cells $\sigma_{1,1}, \sigma_{1,2}$ in clockwise order around the boundary of the disk with coefficient $1/2$, and $\sigma_{1,1}, \sigma_{1,2}$ in the counterclockwise order also with coefficient $1/2$. The resulting structure map is

$$m_2(\sigma_1, \sigma_1) = \frac{1}{2}(q^{A_+} + q^{A_-})\sigma_0.$$

In the case $A_=A_-$, this agrees with the formulas (83), (84) obtained without averaging, and so induces the same product on Floer cohomology.

Proof of Theorem 5.13. The statement of the Theorem follows by perturbing the perturbations so that they are permutation-invariant near the reduced configurations. Let $\underline{P}^{\mathrm{red}} = (P_\Gamma^{\mathrm{red}})$ be a reduced-regular perturbation system for $\sigma_1, \ldots, \sigma_k$. Inductively construct, as in the proof of Theorem 4.19, a nearby regular perturbation system $\underline{P}$ so that the perturbed matching conditions $\underline{M} = (M_\Gamma)$ are permutation-invariant in a neighborhood of each cell σ_i. Suppose that the reduced configurations of type Γ^{red} meet the codimension one cells σ_i in a finite set Z_Γ. It follows from the implicit function theorem that for sufficiently small perturbations P_Γ of P_Γ^{red} there is a bijection between P_Γ^{red}-perturbed configurations $u_{\underline{d}}^{\mathrm{red}}$ with $\ell(e_0) = 0$ and P_Γ-perturbed configurations. After a generic perturbation M_Γ° of M_Γ, the points $z_1, \ldots, z_d$ are distinct. Since the number of permutations is finite, the set of single-valued conditions $g^* M_\Gamma^\circ$ that are regular for all permutations g of the edges $T_{e_1}, \ldots, T_{e_d}$ is comeager. The average M_Γ is then regular. Assuming the perturbations M_Γ are invariant under permutations of the points on the constant disks, of the $d!$ possible orderings of the perturbations $M_{\Gamma,i}(\sigma)$ of σ induced by the matching conditions M_Γ exactly one ordering is achieved by a sequence of points $z_1, \ldots, z_d$ in cyclic order around the boundary of S. It follows that the weight of any point in the fiber is $(d!)^{-1}$ times the weight of the image configuration. In the recursive construction of the perturbation system $\underline{P} = (P_\Gamma)$, at each stage we are given P_Γ on the boundary of $\mathcal{U}_\Gamma$ and wish to extend it over the interior. Because the space of perturbations is contractible, the inductive procedure may be carried out as before. □

Remark 5.20. (Divisor edges attached to constant disks) In the moduli space of rigid treed disks there may also be configurations with boundary edges labelled by σ_i so that the adjacent disk S_v is constant. Suppose that the perturbations vanish, so that all treed disks are regular without perturbation. If the number of adjacent edges to S_v is at least four, then forgetting the boundary edge labelled σ_i produces a configuration of lower dimension which is impossible. On the other hand, if the number of adjacent edges is three, say with two incoming edges T_{e_1}, T_{e_2} then there is another configuration u' obtained by interchanging the order of the inputs T_{e_1}, T_{e_2} around the boundary of S_v. As long as the area of the configuration is positive, these two configurations contribute with opposite signs and cancel.

We will need a similar "repeating input" type formula for disks with repeating alternating inputs at the self-intersection points in the case of Lagrangians of dimension two. Suppose that $\dim(L_0) = 2$ and $x = (x_-, x_+), \overline{x} = (x_+, x_-)$ are ordered self-intersection points. In this case, there are additional constant disks $u|S_v : S_v \to \{\phi(x)\} \subset X$ of expected dimension zero with corners alternating

$$\sigma_1 = x, \sigma_2 = \overline{x}, \sigma_3 = x, \sigma_4 = \overline{x}, \ldots \in \mathcal{I}^{\mathrm{si}}(\phi)$$

where $\sigma_\pm \in \mathcal{I}^c(\phi)$ is the top-dimensional cell containing x_+ resp. x_-. Let Γ denote the corresponding combinatorial type of domain, with $2d+1$ boundary leaves and no interior leaves. The unperturbed relevant moduli space $\mathcal{M}_{\mathbb{F}}(\phi)$ is not of expected dimension. Indeed, any $2d+1$-marked constant disk $(C, u : S \to X)$ mapping to the self-intersection point $\phi(x)$ is holomorphic. The moduli space of such maps $\mathcal{M}_{\mathbb{F}}(\phi)$ is dimension $2d-2$, not the expected dimension zero.

Assumption 5.21. Let $\phi : L \to X$ be an immersed Lagrangian brane of dimension $\dim(L) = 2$ and $x \in \mathcal{I}^{\mathrm{si}}(\phi)$ a self-intersection point contained in a pair of cells $\sigma_\pm$ of top-dimension in the dual cell decomposition, each dual to 0-cell $\sigma_{\pm,0}$. The immersed Fukaya algebra $CF(\phi)$ is *rotation-invariant* at x if and only if for any $d \geq 1$ we have

$$\begin{aligned} m_{2d}(x, \overline{x}, x, \ldots, \overline{x}) &\in \frac{(-1)^{d-1}\sigma_{+,0}}{d} + \mathrm{val}_q^{-1}(0,\infty) \\ m_{2d}(\overline{x}, x, \ldots, \overline{x}, x) &\in -\frac{(-1)^{d-1}\sigma_{-,0}}{d} + \mathrm{val}_q^{-1}(0,\infty). \end{aligned}$$

Remark 5.22. It seems that rotation-invariant perturbations exist, essentially because perturbations for zero-energy disks are the first step in the inductive procedure for constructing perturbations. We take 5.21 as an assumption in the case $\dim(L) = 2$ and both $b_0(x)$ and $b_0(\overline{x})$ are non-vanishing. (Note that for the purposes of mean curvature flow we only need the case that one of $b_0(x)$ or $b_0(\overline{x})$ vanishes.)

6. Holomorphic disks and neck-stretching

In symplectic field theory, one studies the behavior of holomorphic curves as the almost complex structure on the target changes in a family corresponding to *neck-stretching*. Following Bourgeois-Eliashberg-Hofer-Wysocki-Zehnder [16] and Venugopalan-Woodward [86] we describe the limit of the Fukaya algebra of a Lagrangian under neck-stretching. The limit of a sequence of holomorphic disks with respect to such a neck-stretching is a *holomorphic building* in the language of Bourgeois-Eliashberg-Hofer-Wysocki-Zehnder [16].

The situation in this paper differes from the situation in the above papers in several ways. First of all, the Lagrangian is allowed to pass through the neck region, as it is in Fukaya-Oh-Ohta-Ono [42, Chapter 10]. Second, since our Fukaya algebras are defined using treed disks, it may be that the tree part, rather than surface part, breaks in the neck-stretching limit. Thus the levels of the buildings in this case have not only strip-like and cylindrical ends going to infinity, but also additional breakings of the segments. Finally, we wish to degenerate our moduli spaces to products of treed disks in the pieces, rather than fiber products. For this we take an an additional limit which degenerates the matching conditions at the separating hypersurface. The main result is Theorem 6.33 below which gives a homotopy-equivalent Fukaya algebra obtained by counting buildings with each level satisfying some constraints at infinity.

6.1. **Broken holomorphic disks.** Broken disks arise by the following neck-stretching limit studied by Bourgeois-Eliashberg-Hofer-Wysocki-Zehnder [16] in the context of symplectic field theory. Recall that if $Z \subset X$ is a coisotropic submanifold, then the *null foliation* of Z is the distribution defined by

$$\ker(\omega|_{TZ}) = \bigcup_{z \in Z} \{\xi \in T_z Z | \omega(\xi, \zeta) = 0, \quad \forall \zeta \in T_z Z\} \subset TZ.$$

The null foliation $\ker(\omega|_{TZ})$ is called *fibrating* if there exists a fiber bundle $p : Z \to Y$ so that the null foliation is $\ker(\omega|_{TZ}) = \ker(Dp)$. In this case, the bundle $p : Z \to Y$ is unique up to isomorphism and called the *null fibration*.

Definition 6.1. (Neck-stretching for almost complex structures on symplectic manifolds) Let $Z \subset X$ be a codimension one coisotropic submanifold admitting the structure of an S^1-null-fibration $p : Z \to Y$ over a symplectic manifold Y. Thus

$$\ker(Dp) = \ker(\omega|_{TZ}) \subset TZ$$

is the vertical subspace. Let

$$\omega_Z = p^* \omega_Y \in \Omega^2(Z)$$

denote the pullback of the symplectic form ω_Y to Z.

The neck-stretched manifold is obtained by cutting along the hypersurface and inserting a cylinder. Let X° denote the manifold with boundary obtained by cutting open X along Z. Let Z', Z'' denote the resulting copies of Z. For any $\tau > 0$ let

$$X^\tau = X^\circ \bigcup_{Z'' = \{-\tau\} \times Z, \{\tau\} \times Z = Z'} ([-\tau, \tau] \times Z) \tag{85}$$

be the manifold obtained by gluing together the ends Z', Z'' of X° using a neck $[-\tau, \tau] \times Z$ of length 2τ.

Define almost complex structures on the neck-stretched manifold as follows. The $\mathbb{R}$ action by translation on $\mathbb{R}$ and $U(1)$ action on Z combine to a smooth $\mathbb{C}^\times \cong \mathbb{R} \times U(1)$ action on $\mathbb{R} \times Z$ making $\mathbb{R} \times Z$ into a $\mathbb{C}^\times$-bundle. Consider the projections

$$p_\mathbb{R} : \mathbb{R} \times Z \to \mathbb{R}, \quad p_Z : \mathbb{R} \times Z \to Z, \quad p_Y : \mathbb{R} \times Z \to Y$$

onto factors $\mathbb{R}$ and Z resp. onto Y. An almost complex structure J on $\mathbb{R} \times Z$ is called *cylindrical*[12] if J is $\mathbb{C}^\times$-invariant, preserves the tangent spaces to the fibers of $p_Y : \mathbb{R} \times Z \to Y$ and J is equal to the standard almost complex structure on any fiber

$$p_Y^{-1}(y) = \mathbb{R} \times Z_y \cong \mathbb{R} \times U(1) \cong \mathbb{C}^\times.$$

[12]These conditions are stronger than the definition in Bourgeois-Eliashberg-Hofer-Wysocki-Zehnder [16], which deals with a more general situation.

In particular, each orbit of $\mathbb{C}^\times$ is holomorphic. Any cylindrical almost complex structure J on $\mathbb{R} \times Z$ induces an almost complex structure J_Y on Y by projection by the formula

$$Dp_Y(Jw) = J_Y Dp_Y w, \quad w \in T(\mathbb{R} \times Z).$$

We assume that J_Y is compatible with the symplectic form ω_Y on Y. [13] There are complementary *vertical* resp. *horizontal* rank resp. corank two sub-bundles

$$\begin{aligned} V &= \ker(Dp) \oplus \ker(Dp_Z) \subset T(\mathbb{R} \times Z) \\ H &= TZ \cap J(TZ) \subset p_Z^* TZ \subset T(\mathbb{R} \times Z). \end{aligned}$$

with the first bundle V being a a trivial bundle over $\mathbb{R} \times Z$. We have a splitting into complex vector bundles

$$T(\mathbb{R} \times Z) \cong H \oplus V. \tag{86}$$

Since Z is assumed to admit the structure of a principal S^1-bundle, there is unique connection one-form compatible with the splitting

$$\alpha \in \Omega^1(Z)^{S^1}, \quad \ker(\alpha) = H$$

(and in particular Z is a stable hypersurface in the terminology of symplectic field theory.) Conversely, given such a one-form, there is a unique almost complex structure J given by J_Y on H on the standard almost complex structure on V.

The neck-stretched submanifolds of (85) are all diffeomorphic, and the construction provides a family of almost complex structures on the original manifold. The neck-stretched manifold X^τ is diffeomorphic to X by a family of diffeomorphisms given on the neck region by a map

$$(-\tau, \tau) \times Z \to (-\tau_0, \tau_0) \times Z \tag{87}$$

equal to the identity on Z and a translation in a neighborhood of $\{\pm\tau\} \times Z$. Given an almost complex structure J on X that is of cylindrical form on $(-\tau_0, \tau_0) \times Z$, we obtain an almost complex structure J^τ on X^τ by using the same cylindrical almost complex structure on the neck region. Via the diffeomorphism $X^\tau \to X$ described in (87), we obtain an almost complex structure on X also denoted J^τ. This ends the Definition.

Compactness results in symplectic field theory [16] describe the limit of holomorphic curves as the length of the neck approaches infinity. The complement of Z in X divides X into regions $X_\subset$ and $X_\supset$, which we consider as symplectic manifolds with cylindrical ends. Similarly, suppose that $\phi : L \to X$ is a possibly immersed Lagrangian submanifold intersecting Z transversally in a submanifold $L_Z = Z \cap L$, so that in a neighborhood of $\{0\} \times Z$ in the tubular neighborhood $(-\epsilon, \epsilon) \times Z \to X$ L is the image of $(-\epsilon, \epsilon) \times L_Z \to X$. We denote by

$$L_\subset = \phi^{-1}(X_\subset), \quad L_\supset = \phi^{-1}(X_\supset)$$

[13] For many purposes, it suffices to assume that J_Y tames ω_Y, see for example [25].

the pieces of L in $X_\subset$ and $X_\supset$. In the neck-stretching limit, the symplectic field theory compactness results produce a configuration of holomorphic maps with Lagrangian boundary conditions called a *building*.

Definition 6.2. The *broken symplectic manifold* arising from the triple $(X_\subset, X_\supset, Y)$ above is the topological space

$$\mathbb{X} = X_\subset \cup Y \cup X_\supset$$

obtained by compactifying $X_\subset, X_\supset$ by adding a copy of Y and identifying the copies. Thus $\mathbb{X}$ is the singular space obtained by gluing the smooth manifolds

$$\overline{X}_\subset = X_\subset \cup Y, \quad \overline{X}_\supset = X_\supset \cup Y$$

along Y. The space $\mathbb{X}$ inherits a natural topology by viewing $\mathbb{X}$ as the quotient of X by the equivalence relation on Z given by the S^1-fibration. Thus $\mathbb{X}$ is a stratified space and the link of a point in Y in $\mathbb{X}$ is a disjoint union of two circles. The space $\mathbb{X}$ comes equipped with an isomorphism of normal bundles

$$N = \frac{(TX_\subset)_Y}{TY} \cong \left(\frac{(TX_\supset)_Y}{TY} \right)^{-1}. \tag{88}$$

The *infinite neck* is the product $\mathbb{R} \times Z$ and may be compactified by adding copies of Y at $\pm\infty$. For an integer $k \geq 1$, define the k*-broken symplectic manifold*

$$\mathbb{X}[k] = X_\subset \sqcup (\mathbb{R} \times Z) \sqcup \ldots \sqcup (\mathbb{R} \times Z) \sqcup X_\supset. \tag{89}$$

The $k-1$ copies of $\mathbb{R} \times Z$ are called the *neck pieces*. Define

$$\mathbb{X}[k]_0 = X_\subset, \quad \mathbb{X}[k]_1 = \mathbb{R} \times Z, \ldots, \mathbb{X}[k]_k = \mathbb{R} \times Z, \quad \mathbb{X}[k]_k = X_\supset. \tag{90}$$

For each piece we denote by $\overline{\mathbb{X}[k]_i}$ the compactified space obtained by adding one or two copies of Y at infinity. The complex torus $(\mathbb{C}^\times)^{k-1}$ acts $\mathbb{X}[k]$ via the action of $\mathbb{C}^\times$ on each neck piece,

$$\mathbb{C}^\times \times \mathbb{P}(N_\pm \oplus \underline{\mathbb{C}}) \to \mathbb{P}(N_\pm \oplus \underline{\mathbb{C}}), \quad (z, [n, w]) \mapsto z[n, w] := [zn, w].$$

Similarly, define the *broken Lagrangian*

$$\mathbb{L} = L_\subset \cup L_Y \cup L_\supset \tag{91}$$

where $L_Y = p(L_Z)$. Let

$$\mathbb{L}[k] = L_\subset \sqcup (\mathbb{R} \times L_Z) \sqcup \ldots \sqcup (\mathbb{R} \times L_Z) \sqcup L_\supset.$$

The group $(\mathbb{R}^\times)^{k-1}$ acts by real translations on the neck pieces.

Definition 6.3. A *holomorphic building* with $k+1$ levels consists of

(a) a collection of surfaces $S_i, i = 0, \ldots, k$ with strip-like and cylindrical ends and

(b) holomorphic maps
$$u_i : S_i \to \mathbb{X}[k]_i$$
called the *levels* of the building; satisfying the given boundary conditions $u_i(S_i) \subset \mathbb{L}[k]_i$

(c) a pairing of the outgoing ends of S_i with the incoming ends of S_{i+1} so that the limits along the ends satisfying matching conditions: For each such pair of ends, there exists a *multiplicity* $\mu \in \mathbb{R}$ (possibly non-integer in the case of strip-like ends) and coordinates (s,t) on the ends so that so that
$$\lim_{s \to \infty} \exp(-2\pi\mu(s+it))u_i(s,t) = \lim_{s \to -\infty} \exp(2\pi\mu(s+it))u_{i+1}(s,t). \tag{92}$$
In particular, the completions $\overline{u}_i$ to $\overline{u}_{i+1}$ have matching value at $Y = \mathbb{X}[k]_i \cap \mathbb{X}[k]_{i+1}$ and the same multiplicity of intersection with Y.

Automorphism of buildings $u : S \to \mathbb{X}[k]$ are pairs
$$\phi \in \operatorname{Aut}(S), \quad \psi \in \operatorname{Aut}(\mathbb{X}[k])$$
consisting of translations on the neck pieces so that
$$u \circ \phi = \psi \circ u.$$
A building $u : S \to \mathbb{X}$ is *stable* if it has finitely many automorphisms.

The limits of the levels in a building along the ends is a collection of Reeb chords and orbits.

Definition 6.4. (Reeb orbits and chords) Loops in a fiber of constant speed
$$\vartheta : S^1 \to Z_y, \quad \alpha\left(\frac{d}{dt}\vartheta(t)\right) \text{ constant}$$
are called *Reeb orbits*. Paths of constant speed beginning and ending at the Lagrangian
$$\vartheta : [0,1] \to Z_y, \quad \alpha\left(\frac{d}{dt}\vartheta(t)\right) \text{ constant}, \quad \vartheta(k) \in L_Z,\ k \in \{0,1\}$$
are called *Reeb chords*. This ends the definition.

Remark 6.5. Given a Reeb orbit or chord ϑ with $\alpha\left(\frac{d}{dt}\vartheta(t)\right) = \mu$ the *trivial strip* or *trivial cylinder* corresponding to ϑ is the map with domain $S = \mathbb{R} \times [0,1]$ resp. $S = \mathbb{R} \times S^1$
$$S \mapsto \mathbb{R} \times Z, \quad (s,t) \mapsto (s\mu, \vartheta(t))$$
Equivalently, a building u is stable if and only if each component
$$u_v : S_v \to \mathbb{X}[k]_0 \cup \mathbb{X}[k]_k$$
in the components $\mathbb{X}[k]_0, \mathbb{X}[k]_k$ without translation automorphisms is stable and each level
$$u_i : S_i \to \mathbb{X}[k]_i, i = 1, \dots, k-1$$

in the neck region has at least one component that is not a trivial cylinder or strip; the latter condition prevents the existence of automorphisms arising from the translation action on the neck pieces.

We now turn to treed holomorphic buildings. In the neck stretching limit, we assume that the cellular deformations of the diagonal are given by deformations on the neck in the translational direction only:

$$\delta_t(s,z) = ((s,z),(s+f(t),z))$$

on the neck region for some function $f(t)$ converging to infinity as t converges to 1. For each piece $L_\subset, L_\supset$ let

$$U_\subset \subset L_\subset, \quad U_\supset \subset L_\supset$$

be open sets describing the image of the neck region under cellular deformation. Let

$$\varphi_\subset : U_\subset \to L_Z, \quad \varphi_\supset : U_\supset \to L_Z \tag{93}$$

be maps so that a pair $(l_\subset, l_\supset)$ is in the image of the cellular deformation in the limit if and only if

$$\varphi_\subset(l_\subset) = \varphi_\supset(l_\supset).$$

Given a k-fold breaking $\mathbb{L}[k]$ define $\mathbb{U}[k]_i \subset \mathbb{L}[k]_i$ to be $U_\supset$ for the first level, $U_\supset$ for the last level, and $\mathbb{L}[k]_i$ for any intermediate levels. Thus we have maps

$$\varphi_i : \mathbb{U}[k]_i \to L_Z, i = 0,\ldots,k.$$

Definition 6.6. A *treed building* is a treed disk C equipped with a building structure $S = S_0 \cup \ldots \cup S_k$ on the surface part S, with each segment T_e connecting different levels only if those levels are adjacent S_i, S_{i+1} and the length $\ell(e)$ is infinite. Each component S_i together with the edges T_e attached to it will be called a *treed level*; see Figure 11.

A *holomorphic treed building* is a treed building $C = S \cup T$ equipped with a holomorphic map $u : S \to \mathbb{X}[k]$ for some k so that the intersections $T \cap S$ occur away from the nodes joining levels and the collection $u_i := u|S_i$ satisfies the conditions (92) as well as the matching conditions for the edges T_e connecting levels S_i, S_{i+1} with endpoints $w_\pm$

$$\varphi_i(w_-) = \varphi_{i+1}(w_+). \tag{94}$$

Stability for treed buildings is defined in the same way as for buildings, with the following special case: If a building C consists of a single edge T_e with no disks with an incoming label 1_ϕ^h and an outgoing label 1_ϕ^s or 1_ϕ^g then we declare the building stable; similarly a building consisting of a single edge with input labelled σ_{e_1} and output labelled σ_{e_0}, where σ_{e_0} appears in the boundary of σ_{e_1} is stable.

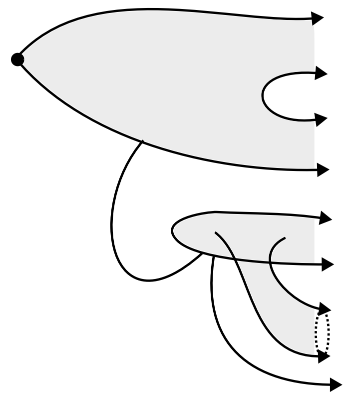

FIGURE 11. A treed level

For each treed level type $\mathbb{F}_i$ evaluation at the semi-infinite edges defines a map

$$\text{ev} : \mathcal{M}_{\mathbb{F}_i}(\mathbb{X}, \phi, \mathbb{D}) \to L_Z^{e_i(\circ)} \times Y^{e_i(\bullet)} \times L^{d_i(\circ)} \tag{95}$$

assigning to each map $u_i : S \to \mathbb{X}$ the beginning points $\vartheta_e(0) \in Z$ of the limiting $e_i(\circ)$ Reeb chords or $e_i(\bullet)$ orbits ϑ_e at infinity along each strip-like or cylindrical end of S, as well as the evaluations at the ends of the $d_i(\circ)$ semi-infinite edges. For any subset

$$\Sigma \subset L_Z^{e_i(\circ)} \times Y^{e_i(\bullet)} \times L^{d_i(\circ)} \tag{96}$$

denote by

$$\mathcal{M}_{\mathbb{F}_i}(\phi, \Sigma) = \text{ev}^{-1}(\Sigma) \tag{97}$$

the moduli space of maps with the given constraints.

We will regularize these moduli spaces by passing to maps adapted to a Donaldson hypersurface. A *broken divisor* $\mathbb{D} = (D_\subset, D_\supset)$ is a pair of divisors $D_\subset \subset X_\subset$ and $D_\supset \subset X_\supset$ with

$$D_\subset \cap Y = D_Y = D_\supset \cap Y$$

such that

$$\phi : L_Y \to Y, \quad \phi_\subset : L_\subset \to X_\subset, \quad \phi_\supset : L_\supset \to X_\supset$$

are exact in the complement of D_Y resp. $D_\subset$ resp. $D_\supset$. Any broken divisor $\mathbb{D} = (D_\subset, D_\supset)$ gives rise to a family of divisors D such that $\phi : L \to X$ is exact in the complement of D, since the section defining D is approximately holomorphic constant on $\phi(L)$. As in [21], one may first choose a Donaldson hypersurface D_Y for L_Y disjoint from the Lagrangian $L_Y \subset Y$. One may then extend to Donaldson hypersurfaces $D_\subset \subset \overline{X}_\subset$ and $D_\supset \subset \overline{X}_\supset$, by choosing extensions of the asymptotically holomorphic sequence of sections. The definition of adapted buildings is then similar to that of adapted maps: Each component of $u^{-1}(\mathbb{D})$ is required to contain an interior edge, and each such edge is required to map to $\mathbb{D}$.

6.2. **Fredholm theory and exponential decay.** In this section, we collect some technical results on holomorphic maps asymptotic to Reeb orbits or chords. In order to carry out the necessary classification of levels in the local model, we allow Lagrangian boundary conditions that are asymptotically cylindrical rather than cylindrical in a neighborhood of infinity.

Definition 6.7. (a) An almost complex manifold X has a *cylindrical end modelled on* Z if there exists an embedding

$$\kappa^X : \mathbb{R}_{>0} \times Z \to X$$

such that the image of κ^X has compact complement. A *cylindrical end almost complex structure* is an almost complex structure $J : TX \to TX$ for which the pull-back $J|_{\mathbb{R}_{>0}\times Z}$ to $\mathbb{R}_{>0} \times Z$ is of cylindrical form in the sense of Definition 6.1, that is, the restriction of a cylindrical almost complex structure $J_{\mathbb{R}\times Z}$ on $\mathbb{R} \times Z$.

(b) Let $\phi : L \to X$ be a Lagrangian immersion. Call ϕ *cylindrical near infinity* if there exists a smooth manifold L_Z of Z and $s \in \mathbb{R}$ so that L_Z is cylindrical in $(s, \infty) \times Z$ on the end: That is,

$$(\kappa^X)^{-1}(\phi(L)) \cap ((s, \infty) \times Z) = (s, \infty) \times L_Z.$$

Since we are considering only the circle-fibered case, our cylindrical end manifolds have natural compactifications at infinity. Given a manifold X with cylindrical almost complex structure J as above, the *compactification* of X is the almost complex manifold $\overline{X} = X \cup Y$ obtained by gluing in a copy of Y at infinity. In terms of charts, we have

$$\overline{X} = X \cup_{\mathbb{R}_{>0}\times Z} (Z \times_{\mathbb{C}^\times} \mathbb{C}) \tag{98}$$

where $Z \times_{\mathbb{C}^\times} \mathbb{C}$ is the line bundle associated to Z. The inclusion of $\mathbb{R}_{>0} \times Z$ in $Z \times_{\mathbb{C}^\times} \mathbb{C}$ is given by the isomorphism

$$\mathbb{R}_{>0} \times Z \cong Z \times_{S^1} \mathbb{C}^\times.$$

Proposition 6.8. *Suppose that $\phi : L \to X$ is cylindrical-near-infinity. Then the closure $\overline{\phi(L)} \subset \overline{X}$ is contained in the image of a Lagrangian immersion $\tilde{\phi} : \tilde{L} \to \overline{X}$ with (without boundary, but possibly non-compact) clean self-intersection.*

Proof. The subset $\mathbb{R}_{>0} \times Z$ glues into the chart $Z \times_{\mathbb{C}^\times} \mathbb{C}$ near infinity by the map $(s, z) \mapsto [z, e^{-s}]$. Let $\overline{L}$ denote the union

$$\overline{L} = \phi(L) \cup (\mathbb{R}_{>0} \times L_Z) \cup L_Y$$

in $\overline{X}$. In a neighborhood of Y the closure $\overline{L}$ is contained in the cleanly-self-intersecting submanifold $\tilde{L}$ given as the image of $(\mathbb{R}_{>0} \times (-L_Z \cup L_Z)) \cup L_Y$. $\square$

Definition 6.9. Let $\overline{X} = X \cup Y$ be as above equipped with a symplectic structure. A Lagrangian submanifold $L \subset X$ is *asymptotically cylindrical* to a cylindrical-near-infinity Lagrangian L_0 if the closure $\overline{L}$ is an immersed submanifold-with-boundary in $\overline{X}$ tangent to the closure of $\overline{L}_0$ at L_Y.

Proposition 6.10. *Let $L \subset X$ be an asymptotically cylindrical Lagrangian manifold asymptotic to a cylindrical-near-infinity Lagrangian submanifold L_0. The closure $\overline{L}$ is contained in a (possibly non-compact) cleanly-self-intersecting Lagrangian submanifold of $\overline{X}$.*

Proof. The closure of L is L_Y and $\overline{L}$ is tangent to $\overline{L}_0$, which is contained in a cleanly-self-intersecting Lagrangian by Proposition 6.8. We may write $\overline{L}$ near L_Y as the graph of an exact one-form $\mathrm{d}f$ on $\overline{L}_0$ where $f : \overline{L}_0 \to \mathbb{R}$ is smooth, using Weinstein neighborhoods of each branch of L' in $\overline{X}$ to write nearby Lagrangians as graphs of one-forms. By, for example, the Seeley extension theorem [74] extension theorem f extends to a function f' on L'. After possibly shrinking L', there are no self-intersection points of $\mathrm{graph}(\mathrm{d}f')$ other than L_Y, that is, the extensions of the branches do not intersect, and $\mathrm{graph}(\mathrm{d}f')$ provides the desired extension. □

Let S be a holomorphic curve with cylindrical and strip-like ends

$$\kappa_{e,\bullet} : \mathbb{R} \times S^1 \to S \quad e = 1, \dots, e(\bullet)$$
$$\kappa_{e,\circ} : \mathbb{R} \times [0,1] \to S \quad e = 1, \dots, e(\circ).$$

Definition 6.11. (Holomorphic maps asymptotic to Reeb chords) Given a cylindrical or asymptotically cylindrical Lagrangian $\phi : L \to X$, a *map from a surface S with strip and cylindrical ends to X with boundary in ϕ* is a map

$$u : S \to X, \quad u(\partial S) \subset \phi(L).$$

A map $u : S \to X$ is *asymptotic to a Reeb chord ϑ* on an end of S if there exist $s_0 \in \mathbb{R}$ and a *multiplicity* $\mu \in \mathbb{R}_+$ such that in cylindrical coordinates (s,t) on each end the distance in the cylindrical metric $\mathrm{d}_{\mathrm{cyl}}$ on $\mathbb{R} \times Z$, with coordinates given by $\kappa_{e,\bullet}$ or $\kappa_{e,\circ}$

$$\mathrm{d}_{\mathrm{cyl}}(u(s,t), (s_0 + \mu s, \vartheta(t))) < Ce^{-\theta s} \tag{99}$$

for some constant $\theta > 0$ and $s_0 \in \mathbb{R}$. The definition of an end asymptotic to a Reeb orbit is similar. This ends the Definition.

The exponential decay above is closely related to a finite-energy condition. Our case is a special case of a more general definition for stable Hamiltonian structures in [16]. For simplicity consider holomorphic maps to $U = \mathbb{R} \times Z$, where Z is equipped with closed two-form $\omega_Z \in \Omega^2(Z)$ with fibrating null-foliation $\ker(\omega_Z) \subset TZ$ and connection form $\alpha \in \Omega^1(Z)$. Let $J : TU \to TU$ be a cylindrical almost complex structure. The *horizontal energy* of a holomorphic map

$$u = (\psi, v) : (S, j) \to (\mathbb{R} \times Z, J)$$

is ([16, 5.3]) with $S^\circ \subset S$ denoting the complement of the corners (points where $u|\partial S$ has a branch change)

$$E^h(u) = \int_{S^\circ} v^* \omega_Z.$$

The *vertical energy* is ([16, 5.3])

$$E^v(u) = \sup_\zeta \int_{S^\circ} (\zeta \circ \psi) \mathrm{d}\psi \wedge v^* \alpha \tag{100}$$

where the supremum is taken over the set of all non-negative C^∞ functions

$$\zeta : \mathbb{R} \to \mathbb{R}, \quad \int_{\mathbb{R}} \zeta(s) \mathrm{d}s = 1$$

with compact support. The *Hofer energy* ([16, 5.3]) is the sum

$$E(u) = E^h(u) + E^v(u).$$

Let X° be a symplectic manifold with cylindrical end modelled on $\mathbb{R}_{>0} \times Z$. The vertical energy $E^v(u)$ on the end is defined as before in (100). The Hofer energy $E(u)$ of a map $u : S^\circ \to X^\circ$ from a surface S° with cylindrical ends to X° is defined by dividing X° into a compact piece X^{com} and a cylindrical end $\mathbb{R}_{>0} \times Z$, and defining

$$E(u) = E(u|_{X^{\mathrm{com}}}) + E(u|_{\mathbb{R}_{\geq 0} \times Z})$$

where $E(u)$ is the Hamiltonian-perturbed energy from (38).

Lemma 6.12. *Let $\phi : L \to X$ be an cylindrical-near-infinity Lagrangian immersion. Any J-holomorphic map $u : S \to X$ with boundary on $\phi(L)$ and finite Hofer energy extends to a $\overline{J}$-holomorphic map $\overline{u} : \overline{S} \to \overline{X}$, and the extension defines a bijection between maps to $\overline{X}$ and maps to X.*

Proof. Exponential convergence on strips with finite Hofer energy is proved in Cieliebak-Ekholm-Latschev [18, Proposition 3.2]. The exponential convergence implies that u is finite area. Removal of singularities for holomorphic maps with boundary on immersed Lagrangians with clean self-intersection Schmäshke [72] implies that the map u extends to a map $\overline{u} : \overline{S} \to \overline{X}$ with boundary on $\overline{L}$. Conversely, any map $\overline{u} : \overline{S} \to \overline{X}$ restricts to a map from S to X by removing the points mapping to $\overline{X} - X$. The finite Hofer energy condition $E(u) < \infty$ follows from the fact that by the constant rank embedding theorem, the symplectic form in a neighborhood of Y in $\overline{X} - X$ diffeomorphic to a neighborhood of the zero section in the normal bundle N_Y may be written $\mathrm{d}(\zeta v^* \alpha) + \pi_Y^* \omega_Y)$ where ζ is the norm-square function. This form is cohomologous to that one for which ζ has compact support, and the Stokes' formula computation in [16, Section 5.7] implies that bounded Hofer energy is equivalent to bounded area. See [16, Remark 5.9]. □

The condition that a holomorphic map has finite Hofer energy implies asymptotic convergence to Reeb chords at infinity for an exponential decay constant that is related to the minimum angle of intersection between the Lagrangians.

Lemma 6.13. (Removal of singularities for cylindrical maps) *Let $\phi : L \to X$ be an asymptotically cylindrical Lagrangian immersion. For any finite energy J-holomorphic map $u : S \to X$ either*

(a) there exist $x \in X$ such that $u(s,t)$ converges to x as $s \to \infty$, uniformly in t for cylindrical coordinates (s,t) along the end e (so that u has a removable singularity) or

(b) there exists a Reeb chord resp. orbit ϑ_e such that $u(s,t)$ converges exponentially fast to $\vartheta_e(s)$ as $s \to \infty$, for $s \to \infty$ with constant θ in the sense of (99) depending only on ϑ_e.

Proof. The desired convergence for strip-like ends $\kappa_{\circ,e}$ is a consequence of Schmäshke [72, Theorem 3.2]: There exist positive constants $\theta', c_0, c_1, c_2, \ldots$ and an eigenfunction

$$v : [0,1] \to T_x\overline{X}, \quad \partial_t v = \theta v, \quad v(0) \in T_x\overline{L}_{i_-},\ v(1) \in T_x\overline{L}_{i_+}$$

with eigenvalue θ so that for every integer $k \geq 0$

$$u(s,t) = \exp_{\overline{x}}\left(\frac{-1}{\theta} e^{-\theta s} v(t) + w(s,t)\right), \quad \|w\|_{C^k([s,\infty]\times[0,1])} \leq c_k e^{-(\theta+\theta')s}. \tag{101}$$

The eigenfunctions v of ∂_t on the vertical parts of $T_x\overline{L}_{k_-}, T_x\overline{L}_{k_+}$ correspond to Reeb chords (c.f. Robbin-Salamon [71, Appendix E]) ϑ, and in cylindrical coordinates on X the exponential of $e^{-\theta s}v(t)$ is equal to $(\theta s, \vartheta(t))$. The second estimate in (101) implies the desired exponential convergence. □

We develop Fredholm theory for holomorphic treed maps to cylindrical end manifolds. Given a holomorphic map $u : S \to X$ with finite Hofer energy, denote by Γ the type of the domain S and $\mathcal{M}_\Gamma(\phi, D)$ the space of maps $u : S \to X$ with domain type Γ.

Proposition 6.14. *For any domain type Γ, the space $\mathcal{M}_\Gamma(\phi, D)$ of finite-energy holomorphic maps $(C, u : S \to X)$ with domain type Γ is locally cut out by a Fredholm map of Banach spaces.*

Proof. There are two possible approaches to the Fredholm theory. By Lemma 6.12, the moduli space of finite-energy maps $\mathcal{M}_\Gamma(\phi, D)$ is in bijection with the space of maps $\overline{u} : \overline{S} \to \overline{X}$ to the compactification bounding $\overline{\phi}(\overline{L})$. The statement of the Proposition follows from the Fredholm theory for holomorphic maps with boundary on a clean intersection Lagrangian [72]. The second version of Fredholm theory treats the target as a cylindrical end manifold, and is required to prepare for the needed gluing result later in Section 6.5.

To carry out the second approach, we suppose for simplicity that the limits along the strip-like or cylindrical ends are Reeb chords or orbits, so that there are no self-intersection points. Since the intersection $L_k \cap Z_y$ with each branch L_k of L at infinity with each fiber $Z_y, y \in Y$ is by assumption finite, we may assume that the boundary of u on the strip-like ends maps to branches L_{k_-}, L_{k_+} of the Lagrangian on the boundary at infinity. The two branches differ by

$$L_{k_+} \cap Z_y \cong e^{i\theta}(L_{k_-} \cap Z_y) \tag{102}$$

for some angle $\theta \in [0, 2\pi)$. Choose a *Sobolev decay constant* $\lambda \in (0, 2\pi)$ smaller than the angles θ, if $\theta \neq 0$. Let β be a *good cutoff function*

$$\beta \in C^\infty(\mathbb{R}, [0,1]), \quad \begin{cases} \beta(s) = 0 & s \leq 0 \\ \beta(s) = 1 & s \geq 1 \end{cases}. \tag{103}$$

Define a *Sobolev weight function*

$$\aleph_\lambda : S^\circ \to [0, \infty), \quad (s,t) \mapsto \beta(s) p \lambda s \tag{104}$$

where $\beta(s)p\lambda$ is by definition zero on the complement of the cylindrical ends. Let

$$\Omega^0(S^\circ, u^*TX)'_{k,p,\lambda} = \left\{ \xi \in \Omega^0(S^\circ, u^*TX)_{W^{k,p}_{\mathrm{loc}}} \;\middle|\; \|\xi\|^p_{k,p,\lambda} < \infty \right\}$$

denote the *weighted Sobolev space* of exponent p, differentiability class k, and decay constant λ. By definition this space consists of sections with finite norm for sections $\xi : S^\circ \to u^*TX$ with limits

$$\lim_{s \to \infty} \xi \circ \kappa_{e,\circ}(s, \cdot) =: \xi(e) \in \Omega^0([0,1], \mathbb{R} \times \vartheta_e^* TZ)$$

at infinity defined by

$$\begin{aligned} \|\xi\|^p_{k,p,\lambda} := \sum_e \|(\xi(e))\|^p + \int_{S^\circ} \Bigg(\sum_{k>0} \|\nabla^k \xi\|^p \\ + \|\xi - \sum_e \beta(|s| - |\ln(\delta)|/2) \mathcal{T}^u(\xi(e))\|^p \Bigg) \exp(\aleph_\lambda) \mathrm{d}\,\mathrm{Vol}_{S^\circ} \end{aligned} \tag{105}$$

where $\mathcal{T}^u$ is parallel transport from $\vartheta_e(t)$ to $u(s,t)$ along $u(s', t)$. By definition, these Sobolev spaces have evaluation-at-infinity maps

$$\mathrm{ev}_\infty : \Omega^0(S^\circ, u^*TX)'_{k,p,\lambda} \to \bigoplus_{e \in \mathcal{E}(S^\circ)} T(\mathbb{R} \times Z), \quad \xi \mapsto (\xi(e))_{e \in \mathcal{E}(S^\circ)}. \tag{106}$$

Let

$$\mathrm{Map}(S^\circ, X)_{k,p,\lambda} = \{\exp_{u_0}(\xi), \quad \xi \in \Omega^0(S^\circ, u_0^*TX)_{k,p,\lambda}\}$$

denote the space of maps $u : S^\circ \to X$ equal to $\exp_{u_0}(\xi)$ for some $u_0 : S^\circ \to X$ constant near infinity on each strip like ends by an element of the weighted Sobolev space $\xi \in \Omega^0(S^\circ, u_0^*TX)_{k,p,\lambda}$. Let

$$\Omega^{0,1}(S^\circ, u^*TX)_{k-1,p,\lambda} = \{\eta \in \Omega^{0,1}(S^\circ, u^*TX)_{0,p,\lambda}, \quad \|\eta\|_{k-1,p,\lambda} < \infty\}$$

denote the space of $(0,1)$-forms with finite $(k-1, p, \lambda)$ norm, given in the case $k - 1 = 0$ by

$$\|\eta\|_{0,p,\lambda} = \left(\int_{S^\circ} \|\eta\|^p \exp(\aleph_\lambda) \mathrm{d}\,\mathrm{Vol}_{S^\circ} \right)^{1/p}.$$

Using the local trivializations (36) define a Banach manifold resp. Banach vector bundle

$$\begin{aligned}\mathcal{B}_\Gamma &= \mathcal{M}^i_\Gamma \times \operatorname{Map}\left(S^\circ, X, L\right)_{k,p,\lambda}\\ \mathcal{E}_\Gamma &= \cup_{u \in \mathcal{B}_\Gamma} \mathcal{E}^i_{\Gamma,u},\ \mathcal{E}^i_{\Gamma,u} = \Omega^{0,1}\left(S^\circ, u^*TX\right)_{k-1,p,\lambda}.\end{aligned}$$

Note that

$$T_{C,u}\mathcal{B}_\Gamma = T\mathcal{M}_\Gamma \oplus \Omega^0\left(S^\circ, u^*TX, (\partial u)^*TL\right)'_{k,p,\lambda}. \tag{107}$$

As usual we obtain a Cauchy-Riemann operator

$$\mathcal{F}^i_\Gamma : \mathcal{B}^i_\Gamma \to \mathcal{E}^i_\Gamma, \quad u \mapsto \overline{\partial}_{J,H} u \tag{108}$$

whose zeros cut out the space of holomorphic maps from S° to X locally. For cylindrical ends, the linearized operator $\tilde{D}_u$ is Fredholm by standard results on elliptic operators on cylindrical end manifolds in Lockart-McOwen [60], and in the case with Lagrangian boundary condition, results described in Schmäshke [72, Section 5]; note that these results require that the almost complex structure is compatible. □

We compare the linearized operators for the map to the cylindrical-end manifold and its compactification as follows. For holomorphic, finite energy $u : S \to X$, let $\overline{u} : \overline{S} \to \overline{X}$ denote its extension to the compactification described in Lemma 6.12 above. On the cylindrical end, we have a splitting

$$\kappa_X^* TX \cong \ker\left(Dp_Y\right) \oplus p_Y^* TY, \quad \ker\left(Dp_Y\right) \cong (\mathbb{R} \times Z) \times \mathbb{C} \tag{109}$$

which we call the splitting into the vertical and horizontal parts. Thus the restriction of u^*TX to $u^{-1}\left(\kappa_X\left(\mathbb{R}_{>0} \times Z\right)\right)$ splits into vertical and horizontal parts as well.

Definition 6.15. Define extensions of the pull-back bundles $u^*TX, (\partial u)^*TL$

$$\left(u^*TX\right)_c \to \overline{S}, \quad \left((\partial u)^*TL\right)_c \to \partial\overline{S}^\circ$$

(where $\partial\overline{S}^\circ$ denotes the complement of the corners in $\overline{S}$, that is, endpoints of strip-like ends limiting to self-intersection points) as follows. For each end asymptotic to some Reeb chord or orbit ϑ in a fiber over $\overline{u}(z) = y \in \overline{X}$, choose a trivialization of $Z_y \cong \mathbb{C}$. Consider the splitting

$$(u^*TX) \cong (p_Y \circ u)^*TY \oplus (X \times \mathbb{C}).$$

where the inclusion of the trivial summand $S \times \mathbb{C}$ is spanned by the partial derivatives $\partial_s u(s,t), \partial_t u(s,t)$ of $u(s,t)$; the fact that this gives a direct sum decomposition follows from the exponential convergence result (101). For the first summand $(p_Y \circ u)^*TY$ there is an obvious extension of $(p_Y \circ u)^*TY$ given by the extension of $p_Y \circ u$ using removal of singularities. The second summand has the trivial extension. Similarly, define $((\partial u)^*TL)_c$ by gluing in the trivial bundle $\partial S \times (T_y(L_Y) \oplus \mathbb{R})$ with local frame given by sections given by a chart for L_Y and the section $\partial_s u(s,t)$.

Proposition 6.16. *Suppose a finite-energy holomorphic map $u : S \to X$ bounding L extends to a holomorphic map $\overline{u} : S \to \overline{X}$ bounding $\overline{\phi(L)}$, and the almost complex structure preserves the splitting $T(\mathbb{R} \times Z) \cong \mathbb{C} \oplus (p \circ u)^* TY$ on the ends. Then the Cauchy-Riemann operator D_u extends to an operator $D_{\overline{u}}$ on $(u^*TX)_c$ and restriction defines an isomorphism of kernels and cokernels*

$$\ker(\tilde{D}_{\overline{u}}) \cong \ker(\tilde{D}_u), \quad \operatorname{coker}(\tilde{D}_{\overline{u}}) \cong \operatorname{coker}(\tilde{D}_u).$$

In particular these operators have the same index.

Proof. The statement of the Proposition is an application of removal of singularities. For sections of $(\pi \circ u)^* TY$ this result is found in Abouzaid [5, (4.19)], while in the cylindrical end case a version appears in Ekholm [33, Lemma 6.4]. The Cauchy-Riemann operator on the surface with ends is a restriction of a Cauchy-Riemann operator on the compactification. Indeed, since translation produces a family of solutions to the Cauchy-Riemann equation on each end, $D_u \partial_s u = D_u \partial_t u = 0$. Hence on each end D_u is equal to $D_{p \circ u}$ plus a trivial Cauchy-Riemann operator on the vertical part. Thus D_u extends to a Cauchy-Riemann operator $D_{\overline{u}}$, equal to $D_{\overline{p \circ u}}$ plus a trivial operator near infinity.

Restriction defines an isomorphism of kernels. Any section of $(u^*TX)_c$ bounding $(u^*TL)_c$ and in the kernel of $D_{\overline{u}}$ restricts to a section of (u^*TX). By definition, smooth sections of $(TX)_c$ may be written in cylindrical coordinates on each end

(110)
$$\xi(s,t) = \exp_{\overline{u}(s,t)} \left(v_0(t) + e^{-\theta s} v(t) + w(s,t) \right), \quad \|w\|_{C^k([s,\infty] \times [0,1])} \leq c_k e^{-(\theta + \theta')s}$$

where c is a real constant and $v_0(t)$ is a leading eigenvector the tangential part of D_u. Thus ξ has exponential convergence to a constant and lies in the kernel of D_u. On the other hand, c.f. Schmäshke [72, Appendix B], elements of the kernel of D_u have the same exponential convergence. Thus any $\xi \in \ker(D_u)$ extends to $\overline{\xi} \in \ker(D_{\underline{u}})$.

The identification of cokernels follows from a similar statement for the kernels of the adjoints: First note that the cokernel of D_u is identified with the subset of elements of the cokernel of the operator D_u acting on the space $\Omega^0(u^*TX, (\partial u)^*TL)_{k,p,\lambda}$ that are also orthogonal to the images of sections asymptotic to elements of the zero-eigenspace; these sections were thrown in by hand, so to speak, as in (105). As in Lockart-McOwen [60], this cokernel may be identified with the cokernel of the operator D_u acting on the space of sections $\Omega^0(u^*TX, (\partial u)^*TL)_{k,p,-\lambda}$ with small *negative* exponential decay constant $-\lambda$. The sign of the exponential decay constant allows sections asymptotic to the zero eigenspace in the cokernel. Thus, any such section extends to an element of the kernel of the adjoint $D^*_{\overline{u}}$, and this extension defines an isomorphism of adjoint kernels as before. ☐

6.3. **Compactness for buildings.** The relative form of the compactness theorem in Bourgeois-Eliashberg-Hofer-Wysocki-Zehnder [16, Section 11.3] and Abbas [1] in symplectic field theory describes the limits of subsequence of holomorphic maps with Lagrangian boundary conditions and Morse-Bott non-degeneracy conditions as in [16, Remark 5.9]. Compactness in symplectic field theory is also treated in related situations by Cieliebak-Mohnke [25] without Lagrangian boundary and Venugopalan-Woodward [86] in the case that the Lagrangian is disjoint from the stretching hypersurface. Chanda [19] gives further details in the Lagrangian case.

We will need an extension of these results to the case of treed holomorphic curves. The first result, Theorem 6.17 below, describes a compactness theorem in a neck-stretching limit. The second, Theorem 6.19, describes compactness for buildings under variation of the Lagrangian boundary condition. We assume that the cellular degeneration $\delta_s : L \to L \times L$ of the Lagrangian L near the cutting hypersurface Z is given by a flow $\varphi_s : Z \to Z$ that is a translation by s on the neck region in the sense that

$$\delta_s(x) = (\varphi_{-\tilde{s}}(x), \varphi_{\tilde{s}}(x)), \quad \forall x \in [-T, T] \times Z$$

for some function $\tilde{s}$ that tends to infinity as s tends to 1. In the Morse setting, this corresponds to assuming that the gradient flow on the edges is in the $\mathbb{R}$-direction on the neck region.

Theorem 6.17. *Given a sequence of adapted stable holomorphic treed disks $(C_\nu, u_\nu : S_\nu \to X^{\tau_\nu}), \tau_\nu \to \infty$ with Lagrangian boundary conditions in ϕ and bounded energy, there exists a subsequence of u_ν converging to an adapted stable holomorphic treed building $(C, u : S \to \mathbb{X})$ with boundary $(\partial u)(S)$ mapping to the broken Lagrangian $\mathbb{L}$. Furthermore, the limit of any Gromov convergent sequence is unique.*

Sketch of proof. Venugopalan-Woodward [86] prove sft compactness for the case of Lagrangians not meeting the neck, and Chanda [19] extends this to the case of Lagrangians passing through the neck. For completeness, we sketch the modifications of the argument in [86] necessary to handle the case in hand. We assume that we have chosen a broken divisor $\mathbb{D} = (D_\subset, D_\supset)$, and a family D_{τ_ν} of Donaldson hypersurfaces in X^{τ_ν} limiting to $\mathbb{D}$ in the sense that D_{τ_ν} is the pull-back of D_Y on the neck region, as in [86, Lemma 5.15]. We suppose furthermore that we have a collection of almost complex structures $\underline{J}(\mathbb{X}) = (J_\Gamma(\mathbb{X}))$ for $\mathbb{X}$ for which every adapted holomorphic building in $\mathbb{X}$ of expected dimension at most one is regular and for which there are no non-constant holomorphic spheres contained in $\mathbb{D}$. By [86, Lemma 5.29], there exists a collection $\underline{J}(X^{\tau_\nu}) = (J_\Gamma(X^{\tau_\nu}))$ of almost complex structures for X^{τ_ν} converging to $\underline{J}(\mathbb{X})$ so that every adapted stable map to X^{τ_ν} of expected dimension at most one is regular and for which there are no non-constant holomorphic spheres contained in D_{τ_ν}. Let

$$(C_\nu, u_\nu : S_\nu \to X^{\tau_\nu}), \tau_\nu \to \infty$$

be a sequence as in the statement of the Theorem. Since C_ν is stable, after passing to a subsequence the sequence C_ν Gromov-converges to a limiting treed disk C. The argument in [86, Step 2, Proof of Theorem 8.4] shows that the derivatives of u_ν

are bounded on the neck regions in C_ν, as otherwise one would obtain by rescaling a component of C mapping non-trivially into $\mathbb{X}$ but with no intersections with $\mathbb{D}$. This is impossible since $\mathbb{D}$ is a Donaldson hypersurface in each component. For each edge $e \in \mathrm{Edge}(\Gamma)$ the corresponding pairs of points $w_{\pm,\nu}(e) \in S_\nu \cap T_{e,\nu}$ have limits $w_\pm$ in different levels S_{i_-}, S_{i_+}, after passing to a subsequence. By assumption we have

$$(u_\nu(w_{-,\nu}), u_\nu(w_{+,\nu})) \in \delta_{\tilde{\ell}(e)}(L)$$

and by assumption $\delta_{\tilde{\ell}_\nu(e)}$ maps to the neck region to $L_\subset \times L_\supset$ and is given by a negative resp. positive translation in the first component on the neck region. Thus

$$(u_\nu(w_{-,\nu}), u_\nu(w_{+,\nu})) = \delta_{\tilde{\ell}_\nu(e)}(u_\nu(s_\nu, z_\nu))$$

for some sequence of points on the neck. This condition implies (94) in the limit $\nu \to \infty$. □

We will also need various compactness results for moduli spaces of buildings under variation of the Lagrangian boundary condition. Let $X_\subset$ be a manifold with a cylindrical end, and $\phi_\subset : L_\subset \to X_\subset$ an asymptotically cylindrical Lagrangian embedding. By Proposition 6.10, the closure $\overline{L}_\subset$ is contained in a cleanly-self-intersecting Lagrangian submanifold of $\overline{X}_\subset$.

Definition 6.18. Denote by $\mathbb{X}_\subset[k]$ the union of $X_\subset$ with $k-1$ neck pieces $\mathbb{P}(N_\pm \oplus \mathbb{C})$. A *treed holomorphic building* in $X_\subset$ is a collection of levels

$$(C_i, u_i : S_i \to \mathbb{X}_\subset[k]_i, i = 1, \ldots, k)$$

as in Definition 6.3, satisfying matching conditions for any collection of inter-level edges between u_i and u_{i+1}.

Any treed building in a broken manifold may be viewed as a pair of treed buildings in the corresponding cylindrical end manifolds, although not in a canonical way. Let $X = (X_\subset, X_\supset)$. A building in $\mathbb{X}[k]$ of type Γ consists of a building $u_\subset$ in $\mathbb{X}_\subset[k_\subset]$ and a building $u_\supset$ in $\mathbb{X}_\supset[k_\supset]$ for some $k_\subset, k_\supset$ with $k = k_\subset + k_\supset$ satisfying matching conditions at the leaves $e_\subset \in \mathrm{Edge}(\Gamma_\subset), e_\supset \in \mathrm{Edge}(\Gamma_\supset)$ corresponding to Reeb chords and orbits that are glued to form Γ.

The key point in the following Theorem, whose proof will occupy the rest of the section, is that the Lagrangians are not required to be cylinddrical-near-infinity, but only asymptotically cylindrical. As such, the Theorem is not a consequence of known results about compactness of buildings with Lagrangian boundary conditions.

Theorem 6.19. *Given a sequence of asymptotically stable Lagrangian boundary conditions $\phi_{\subset,\nu}$ converging to some limiting boundary condition $\phi_\subset$ (at least in the C^2 topology on submanifolds) and a sequence of stable holomorphic treed buildings $(C_\nu, u_\nu : S_\nu \to \mathbb{X}_\subset[k_\nu])$ bounding $\phi_{\subset,\nu}$, there exists a subsequence of u_ν converging to a stable holomorphic treed building $u : S \to \mathbb{X}_\subset[k]$ with boundary $(\partial u)(S)$ mapping* *to the broken Lagrangian* $\mathbb{L}_{\subset}[k]$. *Furthermore, the limit of any Gromov convergent sequence is unique.*

We will need the following generalization of Gromov compactness for Lagrangian boundary conditions with clean self-intersection in Schmäschke [72, Section 4].

Theorem 6.20. *Let* X *be a compact symplectic manifold and* $\phi : L \to X$ *a possibly non-compact Lagrangian immersion with clean self-intersection. Let* $L_0 \subset L$ *be a compact subset of* L *that is a submanifold with boundary. Suppose that* J_ν *is a sequence of tamed almost complex structures on* X *converging in* C^2 *to a limiting tamed almost complex structure* J. *Suppose that* $u_\nu : S_\nu \to X$ *is a sequence of* J_ν*-holomorphic maps with bounded area* $A(u_\nu)$ *bounding* L_0. *Then a subsequence of* u_ν *Gromov-converges to a* J*-holomorphic stable map* $u : S \to X$ *bounding* L.

Sketch of proof. With L compact, the statement is the standard Gromov compactness for clean intersection, as explained in [72, Section 4]. The extension is a kind of target-local Gromov-compactness theorem. One constructs the components $u_v : S_v \to X$ of the limit $u : S \to X$ by composing u with a sequence of embeddings $\phi_{v,\nu} : S_{v,\nu} \to S_\nu$, where $S_{v,\nu}$ is obtained from S_v by removing a sequence of small balls $B_{v,\nu}$ around the nodes $Z_v \subset S_v$. Consider a sequence $\phi_\nu : S_{v,\nu} \to S_\nu$ so that the maps $u_\nu \circ \phi_\nu$ have bounded first derivative on compact sets of $S_v - Z_v$. The compositions $u_\nu \circ \phi_\nu$ have boundary in L_0 and so converge, after passing to a subsequence, to a collection of components $u_v : S_v - B_{v,\nu} \to X$ bounding L_0, uniformly on compact sets. The exponential decay results on cylinders with small energy (used to show that bubbles connect) follow by considering u_ν as maps bounding L and do not require compactness of L. □

Sketch of proof of Theorem 6.19. We indicate the modifications necessary for sft compactness as presented in, for example, Venugopalan-Woodward [86] to go through. Consider a sequence of treed disks $(C_\nu, u_\nu : S_\nu \to X_{\subset})$ with bounded energy with boundary values in $L_{\subset}$. Because $\overline{L}_{\subset}$ has clean self-intersection, Gromov compactness for disks with clean self-intersection implies the existence of a subsequence converging to a limit $(C, u_\infty : S \to \overline{X}_{\subset})$ where C is a tree disk with surface component $S = \bigcup_v S_v$ mapping into the compactification $\overline{X}_{\subset}$.

By adding marked points, we may assume that the limiting stable map has stable domain. For example, choose a Donaldson hypersurface $\overline{D} \subset \overline{X}_{\subset}$ transverse to the limit u_∞ and add leaves to C_ν according to the intersections of u_ν with $\overline{D}$. We denote by S_v° the complement of the nodes in S_v. For each edge T_e meeting S_v choose ϵ_e small and denote by $B_v(\epsilon) = \cup_e B_{T_e \cap S_v}(\epsilon_e)$ the complement of the ϵ_e-balls around the intersection $T_e \cap S_v$. The surface S_ν is obtained by gluing together the surfaces $S_v - B_v(\epsilon_\nu)$ for suitable choices of $\epsilon_{e,\nu}$ converging to 0 as $\nu \to \infty$. We denote by $u_{\nu,v}$ the restriction of u_ν to $S_v - B_v(\epsilon_\nu)$.

Construct the levels of the limiting building by rescaling the target locally as follows. By assumption, an open neighborhood of Y in $\overline{X}_{\subset}$ is isomorphic to the normal bundle N_- of Y. Identify the complement $N_-^\times$ of the zero section with $\mathbb{R} \times Z$ as above, and consider the action $e^s : N_-^\times \to N_-^\times$ of scalar multiplication of e^s for a real number $s \in \mathbb{R}$, equivalent to translation in the $\mathbb{R}$-factor by s. Suppose u_v has image in Y. Fix a point $z \in S_v^\circ$ and choose a sequence $s_{\nu,v} \in \mathbb{R}$ so that the translations $e^{s_{\nu,v}} u_{\nu,v}(z)$ converge to a point in $N_-^\times$. The argument in [86, Section 10.4] shows that the derivatives of $u_{\nu,v}$ are bounded with respect to the cylindrical-end metric on S_v°, so that after passing to a subsequence we may assume that $u_{\nu,v}$ converges to a level in $\mathbb{P}(N_\pm \oplus \mathbb{C})$.

It remains to show that the matching conditions between levels are satisfied. Suppose that u_{v_1} and u_{v_2} are adjacent components of the limit. Denote by $u_{\nu,e}$ the restriction of u_ν to the neck region $[-\zeta_e, \zeta_e] \times S^1$ resp. strip $[-\zeta_e, \zeta_e] \times S^1$ connecting the two components of the limit. For z lying in some such strip, choose a sequence s_ν so that the maps $e^{s_\nu} u_{\nu,e}(z)$ converge. Since the derivative of $u_{\nu,e}$ is bounded and the Lagrangians $e^{s_\nu} L_{\subset,\nu}$ converge in C^∞ to a boundary condition $\mathbb{R} \times L_Z$, after passing to a subsequence the maps $e^{s_\nu} u_{\nu,e}(z)$ converge in C^∞ on compact sets to a limit u_e which is contained in a fiber of $\mathbb{P}(N_- \oplus \mathbb{C})$, necessarily with a single intersection with the divisors at zero and infinity corresponding to the two ends of $S_{e,\nu}$. In the case of a strip, the boundary conditions on $e^{s_\nu} u_{e,\nu}$ converge to the cylindrical boundary condition $\mathbb{R} \times L_Z$ as $\nu \to \infty$. It follows that $u_{e,\nu}$ converges to a trivial cylinders resp. strip of the form $u_e(s,t) = (\mu s, \vartheta_\mu(t))$ in some fiber $N_{-,y}^\times \cong \mathbb{R} \times S^1$ for some $\mu \in \mathbb{R}$ and Reeb chord or orbit ϑ_μ with total angle change μ.

The Reeb orbit appearing in the limit on the thin parts of the surface is independent of the choice of rescaling. Indeed, suppose by way of contradiction that there exist two rescaling sequences $e^{s_\nu} u_{e,\nu}$ and $e^{s'_\nu} u_{e,\nu}$ converging to different to trivial cylinders corresponding to different Reeb chords or orbits $\vartheta_\mu, \gamma_{\mu'}$ with different angle changes μ, μ'. Since the angle change of $u_{e,\nu}(s, \cdot)$ is a continuous function of s and the set of angle changes

$$\int \gamma^* \alpha, \quad \gamma : [0,1] \to Z \text{ resp. } S^1 \to Z \tag{111}$$

of Reeb chords and orbits is discrete, the intermediate value theorem implies the existence of a rescaling sequence s''_ν for which the limit of $e^{s'_\nu} u_{e,\nu}(0, \cdot)$ has angle change $\mu'' \in (\mu, \mu')$ which is not the angle change of any Reeb chord or orbit. This is a contradiction.

The limiting building is constructed as follows. Assign each component u_v to a level S_i by comparing the translation sequences $s_{v,\nu}$ necessary to construct the limit. By the discussion from the previous paragraph, if two components u_{v_-}, u_{v_+} are in different levels and are joined by an edge then for one of the components, say u_{v_-}, the image of $S_{v_-} \cap S_{v_+}$ maps to the divisor at infinity in $\mathbb{P}(N_- \oplus \mathbb{C})$ and the other maps to the divisor at zero. After possibly adding trivial strip or cylinders, we obtain a building $(C, u = (u_v, v \in \mathrm{Vert}(\Gamma)))$ with the matching conditions that the Reeb chords at either side of the node match, the projections to Y match on either side of the node, and the matching occurs at the same copy of Y in $\mathbb{X}_{\subset}[k]$. Since the limit in $\overline{X}_{\subset}$ was unique and the rescaling sequences $s_{\nu,v}$ are unique up to addition of constants, the limiting building in $\mathbb{X}_{\subset}[k]$ is unique up to translation in the neck pieces. The statement of the Theorem follows. $\square$

6.4. **Transversality for buildings.** Regularization of the moduli spaces of buildings may be carried out using Donaldson hypersurfaces following Charest-Woodward [21] and Venugopalan-Woodward [86]. We modify the construction to allow boundary in asymptotically-cylindrical broken Lagrangians

$$\mathbb{L} = (L_{\subset}, L_{\supset}).$$

The broken analog of Theorem 4.19 gives an inductive construction of regular perturbation data. For any type $\mathbb{\Gamma}$ we denote by $\mathcal{M}_{\mathbb{\Gamma}}(\mathbb{X}, \phi, \mathbb{D})$ the regularized moduli space of buildings with type $\mathbb{\Gamma}$. As in Proposition 6.14, the moduli space of buildings $\mathcal{M}_{\mathbb{\Gamma}}(\mathbb{X}, \phi, \mathbb{D})$ is locally cut out by a Fredholm map. Let $S_i \subset S$ denote the subset of the domain mapping to $\mathbb{X}[l]_{i(v_\pm)}$, and u_i the restriction of u to S_i. Using the Sobolev norms from (105), let

$$\mathcal{B}_\Gamma \subset \mathcal{M}_\Gamma \times \Pi_{i=0}^{l} \mathrm{Map}(S_i^\circ, \mathbb{X}[l]_i)_{k,p,\lambda} \times \mathrm{Map}(\partial S_i^\circ, \mathbb{L}[l]_i)_{k-1/p,p,\lambda} \times \mathrm{Map}(\mathrm{Edge}_{<\infty}(\Gamma_i), \mathbb{L})$$

be the space of maps of class k, p from each S_i° to the spaces $\mathbb{X}[l]_i$, lifting to a map to $\mathbb{L}[l]_i$ on the boundary, and satisfying the following conditions: The matching conditions along cylindrical and strip-like ends via the evaluation maps including (106) and the deformed matching conditions of (44). The linearized operator for such buildings is defined as in the discussion after (108).

Definition 6.21. The *linearized operator for a holomorphic building* (C, u) is

$$\tilde{D}_u : T_{(C,u)}\mathcal{B}_\Gamma \to \Omega^{0,1}(S, u^*T\mathbb{X})_{k-1,p,\lambda} \tag{112}$$

$$(\zeta_S, \zeta_T, \xi, \partial\xi) \mapsto D_u\xi - \frac{1}{2} J \mathrm{d}u Dj(\zeta_S) \tag{113}$$

where $T_{(C,u)}\mathcal{B}_\Gamma = \{(\zeta_S, \zeta_T, \xi, \partial\xi)\}$ restricts to deformations ξ satisfying in addition to the conditions in (37) the matching conditions at infinity

$$\mathrm{ev}_e(\xi_{S_i}) = \mathrm{ev}_e(\xi_{S_{i+1}})$$

for all edges e of Γ connecting different levels S_i, S_{i+1}. A holomorphic building $u : S \to \mathbb{X}$ is called *regular* if the operator $\tilde{D}_u$ is surjective.

Theorem 6.22. *For any broken type Γ, given perturbations $P_{\Gamma'}$ for strata $\mathcal{M}_{\Gamma'}(\mathbb{X}, \phi, \mathbb{D})$ with $\Gamma' \prec \Gamma$, there exists a perturbation P_Γ so that for each uncrowded type $\mathbb{\Gamma}$ with expected dimension at most one, the closure of $\mathcal{M}_{\mathbb{\Gamma}}(\mathbb{X}, \phi, \mathbb{D})$ is contained in the uncrowded locus and, if non-empty, is a finite set or a compact one-manifold with boundary.*

Sketch of proof. The proof of this theorem is similar to that in Charest-Woodward [21]. At any point $(C, u : S \to \mathbb{X}, P_\Gamma)$ in the universal moduli space $\mathcal{M}_\Gamma^{\mathrm{univ},i}(\mathbb{X}, \phi, \mathbb{D})$ one must show that any element in the cokernel of $\tilde{D}_u$ vanishes. The elements of the cokernel $\eta = (\eta_v)$ have vanishing restriction $\eta_v = 0$ to any component S_v on which u has non-trivial horizontal derivative. The restriction η_v of η must satisfy $D_{u_v}^* \eta_v = 0$ and be perpendicular to domain-dependent variations of the cylindrical almost-complex structure J_Γ. The last condition in particular implies that η_v vanishes in a neighborhood of any point at which $\mathrm{d}(p \circ u)$ is non-zero. The claim follows by unique continuation.

Multiple covers of trivial cylinders, meaning maps whose image is contained in a fiber of the projection $\mathbb{R} \times Z \to Y$, are transversally cut out. Indeed, if u maps to $\mathbb{R} \times Z_y$ for some $y \in Y$ then the Cauchy-Riemann operator D_u splits

$$D_u \cong \overline{\partial}_{T_y Y} \oplus D_u^v$$

as the standard Cauchy-Riemann operator $\overline{\partial}_{T_y Y}$ on maps to $T_y Y$ with boundary $T_y L_Y$ plus the linearized operator D_u^v for a map of a genus zero surface S_v° into the fiber $\mathbb{C} - \{0\}$ with boundary conditions $(\mathbb{R} \cup i\mathbb{R}) - \{0\}$. Such operators are surjective by any number of arguments; for example, by Proposition 6.16, D_u^v compactifies to a rank one Cauchy-Riemann operator $D_{\overline{u}}^v$ with a non-trivial kernel

$$\ker(D_{\overline{u}}^v) \cong \mathbb{R}$$

given by dilation. In rank one, any Cauchy-Riemann operator cannot have both non-trivial kernel and cokernel by Oh [64], so the cokernel of D_u^v must vanish. Similarly, $\overline{\partial}_{T_y Y}$ has non-trivial kernel and vanishing cokernel as well, so D_u is surjective. In particular, the usual problem in symplectic field theory of multiple covers of trivial cylinders or strips lacking regularity does not occur. Thus, the operator $\tilde{D}_{u_v}$ on any component u_v that covers a trivial cylinder is surjective. In particular, $\eta_v = \tilde{D}_{u_v} \xi_v$ for some ξ_v possibly with non-trivial evaluations on the ends of S_v.

An induction shows that the restriction of $\tilde{D}_u$ to any union of components S_v that are covers of trivial components, or on which the map is constant, is also surjective: The kernel of D_u on any disk with strip like ends consists of constant sections and is identified with $\mathbb{R} \oplus T_y Y$ via the splitting of the symplectization. As such, the matching conditions are cut out transversally as in the proof of Theorem 4.19. Thus there are no non-trivial elements of the cokernel. Compare also Ekholm [33, Lemma 6.4] and especially Venugopalan-Woodward [86, Corollary 6.35] where a similar proof is given for the context of multi-directional symplectic field theory. □

6.5. **Gluing with Lagrangian boundary conditions.** The gluing argument produces from any holomorphic building a limiting family of holomorphic maps. The proof is probably standard and similar to that in Charest-Woodward [21].

First recall the gluing construction on domains and targets. Given gluing parameters $\delta_1, \ldots, \delta_k > 0$, the glued domain $S^{\delta_1, \ldots, \delta_k}$ is obtained from S by gluing necks

$[-|\ln(\delta_i)|/2, |\ln(\delta_i)|/2] \times S^1$ of length $|\ln(\delta_i)|$ at each node of S separating two levels. There is a similar construction of the glued target X^δ obtained by gluing in a neck of length $|\ln(\delta)|$ in $\mathbb{X}$. In the case that the Lagrangian meets the neck region, we assume the Lagrangian is asymptotically cylindrical on the neck so that

$$L \cap ([-|\ln(\delta)|/2, |\ln(\delta)|/2] \times Z) = [-|\ln(\delta)|/2, |\ln(\delta)|/2] \times (L \cap Z). \tag{114}$$

If $(L_\subset, L_\supset)$ is only asymptotically cylindrical, then we may achieve the condition (114) by a diffeomorphism on the neck region which makes the almost complex structure J cylindrical only up to exponentially decreasing terms on the cylindrical ends of $X_\subset, X_\supset$.

For simplicity we state the gluing result for the case of two levels only:

Theorem 6.23. *Let $\mathbb{X} = \overline{X}_\subset \cup_Y \overline{X}_\supset$ and $\mathbb{L} = L_\subset \cup_{L_Y} L_\supset$ be a broken rational symplectic manifold and rational self-transverse immersed asymptotically-cylindrical broken Lagrangian in the sense of (91). Suppose that $(C, u : S \to \mathbb{X})$ is a regular treed building with limiting eigenvalues $\mu_1, \ldots, \mu_k$ of Reeb chords or orbits at the separating hypersurface $Y \subset \mathbb{X}$ with boundary in ϕ_ϵ for some $\epsilon < 0$. Then there exists $\delta_0 > 0$ such that for each gluing parameter $\delta \in (0, \delta_0)$ there exists a treed building*

$$(C^{\delta/\mu_1,\ldots,\delta/\mu_k}, u_\delta : S^{\delta/\mu_1,\ldots,\delta/\mu_k} \to X^\delta)$$

with the property that u_δ depends smoothly on δ. Furthermore the Gromov limit recovers the original map:

$$\lim_{\delta \to 0} u_\delta = u.$$

We construct from any holomorphic building a holomorphic map to the manifold with long neck, using Floer's version of the Picard Lemma. Afterwards we show that any such map for sufficiently long neck length is obtained by such a construction. Recall Floer's version of the Picard Lemma, [36, Proposition 24]).

Lemma 6.24. *Let $f : V_1 \to V_2$ be a smooth map between Banach spaces that admits a Taylor expansion*

$$f(v) = f(0) + df(0)v + N(v)$$

satisfying the following condition: There exists a constant $C > 0$ such that $df(0) : V_1 \to V_2$ has a right inverse $G : V_2 \to V_1$ satisfying the uniform bound

$$\|GN(u) - GN(v)\| \le C(\|u\| + \|v\|)\|u - v\|, \quad \forall u, v \in V_1.$$

Let $B_\epsilon(0)$ denote the open ϵ-ball centered at $0 \in V_1$ and assume that

$$\|Gf(0)\| \le 1/(8C).$$

For $\epsilon < 1/(4C)$, the zero-set of $f^{-1}(0) \cap B_\epsilon(0)$ is a tranversally-cut-out (hence smooth) submanifold of dimension $\dim(\mathrm{Ker}(df(0)))$ *diffeomorphic to the ϵ-ball in* $\mathrm{Ker}(df(0))$.

Proof of Theorem. To simplify notation, we consider only the case that the building consists of a pair of maps joined by strip-like ends; the general case is left to the reader. To construct the approximate solution, we begin by recalling the construction of the deformation of a complex curve at a node. Let S be a broken curve with two sublevels S_+, S_-. Let $\delta > 0$ be a small gluing parameter. Variations of the domain may be represented as variations of the conformal structure on a fixed curve together with variations of the edge lengths. Let

$$\begin{aligned} u_- : S_- &\to X_- := X_{\subset} \\ u_+ : S_+ &\to X_+ := X_{\supset} \end{aligned}$$

be maps from components $S_\mp$ containing points $w_\pm \in S_\pm$ corresponding to the ends satisfying (92) so that $u = (u_-, u_+)$ form a building in $\mathbb{X}$. Let $\Gamma_\pm$ denote the combinatorial types of the domains of $u_\pm$ and let

$$\mathcal{S}^i_{\Gamma_\pm} \to \mathcal{M}^i_{\Gamma_\pm} \times S_\pm, , i = 1, \ldots, l \tag{115}$$

be local trivializations of the universal treed disk. These local trivializations identify each nearby fiber with $(S_\pm, \underline{z}, \underline{w})$ such that each point in the universal treed disk is contained in one of the local trivializations (115). We may assume that $\mathcal{M}^i_{\Gamma_\pm}$ is identified with an open ball in Euclidean space so that nodal fiber containing $S_- \cup S_+$ lies over 0. Similarly, we assume we have a local trivialization of the universal bundle near the glued curve as a smooth fiber bundle. The local trivialization gives rise to a family of complex structures

$$\mathcal{M}^i_\Gamma \to \mathcal{J}(S^\delta) \tag{116}$$

that are constant on the neck region. We consider metrics on the punctured curves $S^\circ_\pm$ that are cylindrical on the neck region. That is, on the images of the maps

$$\kappa^C_\pm : \pm[0, \infty) \times [0, 1] \to S_\pm$$

the metrics are the product of the standard metrics on the two factors. By assumption we have cylindrical ends so that the images of

$$\kappa^X_\pm : \ \mp[0, \infty) \times Z \to X_\pm$$

are isometric. Both the glued target $X^{\delta\mu}$ and glued domain S^δ are defined by removing the part of the end with $|s| > |\ln(\delta)|$ and identifying

$$\begin{aligned} (s, t) \sim (s - |\ln(\delta)|, t) \quad (s, t) \in (0, |\ln(\delta)|) \times S^1 \\ (s, t) \sim (s - |\mu \ln(\delta)|, t) \quad (s, t) \in (0, |\ln(\delta)|) \times Z. \end{aligned}$$

The prerequisite for Floer's version of the Picard lemma is an approximate solution to the Cauchy-Riemann equation on the glued curve. Choose a cutoff function

$$\beta \in C^\infty(\mathbb{R}, [0, 1]), \quad \begin{cases} \beta(s) = 0 & s \le 0 \\ \beta(s) = 1 & s \ge 1 \end{cases}. \tag{117}$$

We denote by $\exp_x : T_x X^\delta \to X^\delta$ geodesic exponentiation, using the given cylindrical metric on the neck region. We write using geodesic exponentiation in cylindrical coordinates

$$u_\pm(s,t) = \exp_{(\mp\mu s, t^\mu z)}(\zeta_\pm(s,t)).$$

Define u_δ^{pre} to be equal to $u_\pm$ away from the neck region, while on the neck region of S^δ with coordinates s,t define

$$\begin{aligned}(118)\quad u_\delta^{\mathrm{pre}}(s,t) &= \exp_{(\mu s, t^\mu z)}(\zeta^\delta(s,t)),\\ \zeta^\delta(s,t) &= \beta(-s)\zeta_-\left(-s + \frac{|\ln(\delta)|}{2}, t\right) + \beta(s)\zeta_+\left(s - \frac{|\ln(\delta)|}{2}, t\right).\end{aligned}$$

In other words, one translates u_+, u_- by some amount $|\ln(\delta)|$, and then patches them together using the cutoff function and geodesic exponentiation.

To obtain the estimates necessary for the application of the Picard lemma, we work in Sobolev spaces with weighting functions close to those needed for the Fredholm property on cylindrical and strip-like ends in (104). The surface part S^δ satisfies a uniform cone condition and the metrics on $X^{\delta\mu}$ are uniformly bounded. These uniform estimates imply uniform Sobolev embedding estimates and multiplication estimates. Denote by

$$(s,t) \in \left[-\frac{|\ln(\delta)|}{2}, \frac{|\ln(\delta)|}{2}\right] \times S^1$$

the coordinates on the neck region in S^δ created by the gluing. For $\lambda > 0$ small, define a *Sobolev weight function*

$$\aleph_\lambda^\delta : S^\delta \to [0,\infty), \quad (s,t) \mapsto \beta\left(\frac{|\ln(\delta)|}{2} - |s|\right) p\lambda \left(\frac{|\ln(\delta)|}{2} - |s|\right).$$

By definition $\aleph_\lambda^\delta$ is zero on the complement of the neck region. We will also use similar weight functions on the punctured curves

$$\aleph_\lambda^\pm : S_\pm^\circ \to [0,\infty), \quad (s,t) \mapsto \beta(|s|)p\lambda|s|.$$

Holomorphic maps near the pre-glued solution are cut out locally by a smooth map of Banach spaces. Given an element $m \in \mathcal{M}_\Gamma^i$ and a section $\xi : S^\delta \to u^* TX^\delta$ define as in Abouzaid [5, 5.38] a norm based on the decomposition of the section into a part constant on the neck and the difference:

$$\begin{aligned}(119)\quad \|(m,\xi)\|_{1,p,\lambda}^p &:= \|m\|^p + \|\xi\|_{1,p,\lambda}^p\\ \|\xi\|_{1,p,\lambda}^p &:= \|(\xi(0,0))\|^p + \int_{S^\delta} (\|\nabla \xi\|^p\\ &\quad + \|\xi - \beta(|\ln(\delta)|/2 - |s|)\mathcal{T}^u(\xi(0,0))\|^p) \exp(\aleph_\lambda^\delta) \mathrm{d}\,\mathrm{Vol}_{S^\delta}\end{aligned}$$

where $\mathcal{T}^u$ is parallel transport from $u^{\mathrm{pre}}(0,t)$ to $u^{\mathrm{pre}}(s,t)$ along $u^{\mathrm{pre}}(s',t)$. Pointwise geodesic exponentiation defines a map (using Sobolev multiplication estimates)

$$\exp_{u_\delta^{\mathrm{pre}}} : \Omega^0(S^\delta, (u_\delta^{\mathrm{pre}})^* TX^{\delta\mu})_{1,p,\lambda} \to \mathrm{Map}^{1,p}(S^\delta, X^{\delta\mu}) \tag{120}$$

and $\mathrm{Map}^{1,p}(S^\delta, X^{\delta\mu})$ denotes maps of class $W^{\mathrm{loc}}_{1,p}$ from S^δ to $X^{\delta\mu}$. In the case of Lagrangian boundary conditions, we have a similar map assuming that the exponential map sends tangent vectors to the Lagrangian to points in the Lagrangian boundary condition; we omit the Lagrangian boundary condition from the notation. Similarly, for the punctured surfaces we have Sobolev norms

$$\|(m,\xi)\|_{1,p,\lambda} := \left(\|m\|^p + \|\xi\|^p_{1,p,\lambda}\right)^{1/p}, \tag{121}$$

$$\|\xi\|_{1,p,\lambda} := \begin{pmatrix} \|\xi(0,0)\|^p + \int_{S^\delta}(\|\nabla\xi\|^p + \\ \|\xi - \beta(|s|)\mathcal{T}^u\xi(0,0)\|^p)\exp(\aleph^\pm_\lambda)\mathrm{d}\,\mathrm{Vol}_{S^\circ_\pm} \end{pmatrix}^{1/p}.$$

Geodesic exponentiation defines maps

$$\exp_{u_\delta^{\mathrm{pre}}} : \Omega^0(S^\circ_\pm, (u_\delta^{\mathrm{pre}})^* TX)'_{1,p,\lambda} \to \mathrm{Map}^{1,p,\lambda}(S^\circ_\pm, X^\circ_\pm) \tag{122}$$

where, by definition, $\mathrm{Map}^{1,p,\lambda}(S^\circ_\pm, X^\circ_\pm)$ is the space of $W^{\mathrm{loc}}_{1,p}$ maps from $S^\circ_\pm$ to $X_\pm$ that differ from a Reeb chord at infinity by an element of $\Omega^0(S^\circ_\pm, (u_\delta^{\mathrm{pre}})^* TX^\circ_\pm)'_{1,p,\lambda}$ (which may vary at infinity because of the inclusion of constant maps on the end in the Banach space). In the case of the cylindrical end manifolds , the assumption λ small on the Sobolev decay constant implies that the linearized operators

$$D_{u_\pm} : \Omega^0(S_\pm, u_\pm^* TX_\pm)'_{1,p,\lambda} \to \Omega^{0,1}(S_\pm, u_\pm^* TX_\pm)_{0,p,\lambda}$$

are Fredholm. The kernel contains any infinitesimal variation of the map by Lemma 6.13. By the regularity assumption, the fiber products

$$\ker(\tilde{D}_{u_-}) \times_{\mathrm{ev}_{\infty,-},\mathrm{ev}_{\infty,+}} \ker(\tilde{D}_{u_+}) \tag{123}$$

are transversally cut out, where $\mathrm{ev}_{\infty,\pm}$ are the maps of (106).

The space of holomorphic maps near the pre-glued solution is cut out locally by a smooth map of Banach spaces. For a $0,1$-form $\eta \in \Omega^{0,1}(S^\delta, u^*TX)$ define

$$\|\eta\|_{0,p,\lambda} = \left(\int_{S^\delta} \|\eta\|^p \exp(\aleph^\delta_\lambda)\mathrm{d}\,\mathrm{Vol}_{S^\delta}\right)^{1/p}.$$

Parallel transport using an almost-complex connection defines a map

$$\mathcal{T}_{u_\delta^{\mathrm{pre}}}(\xi) : \ \Omega^{0,1}(S^\delta, (u_\delta^{\mathrm{pre}})^* TX)_{0,p,\lambda} \to \Omega^{0,1}(S^\delta, (\exp_{u_\delta^{\mathrm{pre}}}(\xi))^* TX)_{0,p,\lambda}.$$

Because we are working in the adapted setting, our curves S^δ are attached to a collection of interior leaves $T_{e_1}, \ldots, T_{e_n}$. We require

$$(\exp_{u_\delta^{\mathrm{pre}}}(\xi))(T_{e_i}) \in D, \quad i = 1, \ldots, n. \tag{124}$$

By choosing local coordinates near the attaching points $w_e = T_e \cap S$, the constraints (124) may be incorporated into the map $\mathcal{F}_\delta$ to produce a map

(125)
$$\mathcal{F}_\delta : \mathcal{M}^i_\Gamma \times \Omega^0(S^\delta, (u^{\mathrm{pre}}_\delta)^* TX, (\partial u^{\mathrm{pre}}_\delta)^* TL)'_{1,p,\lambda} \to \Omega^{0,1}(S^\delta, (u^{\mathrm{pre}}_\delta)^* TX)_{0,p,\lambda} \times V$$

where the space V is the direct sum of additional factors enforcing the matching and divisor conditions; namely those in (92) together with the sum $\bigoplus_{e=1}^n T_{u(w_e)} X / T_{u(w_e)} D$ enforcing the conditions that the interior markings map to the Donaldson hypersurface. The first component of this map is

$$\mathcal{F}_\delta(m,\xi) = \left(\mathcal{T}_{u^{\mathrm{pre}}_\delta}(\xi)^{-1} \overline{\partial}_{J_\Gamma, H_\Gamma, j(m)} \exp_{u^{\mathrm{pre}}_\delta}(\xi), \ldots \right).$$

Zeroes of $\mathcal{F}_\delta$ correspond to *adapted* holomorphic maps near the preglued map u^{pre}_δ. The expression $\mathcal{F}_\delta(0)$ has contributions created by the cutoff function and difference in the maps:

$$\begin{aligned}
\|\mathcal{F}_\delta(0)\|_{0,p,\lambda} &= \Bigg\| \overline{\partial}_{J_\Gamma, H_\Gamma} \exp_{(\mu s, t^\mu z)} \left(\beta(-s)\zeta_- \left(-s + \frac{|\ln(\delta)|}{2}, t \right) \right. \\
&\qquad \left. + \beta(s)\zeta_+ \left(s - \frac{|\ln(\delta)|}{2}, t \right) \right) \Bigg\|_{0,p,\lambda} \\
&= \Bigg\| \left(D \exp_{(\mu s, t^\mu z)} \left(\mathrm{d}\beta(-s)\zeta_- \left(-s + \frac{|\ln(\delta)|}{2}, t \right) \right. \right. \\
&\qquad \left. + \mathrm{d}\beta(s)\zeta_+ \left(s - \frac{|\ln(\delta)|}{2}, t \right) \right) + \\
&\qquad \left(\beta(-s)\mathrm{d}\zeta_- \left(-s + \frac{|\ln(\delta)|}{2}, t \right) \right. \\
&\qquad \left. \left. + \ \beta(s)\mathrm{d}\zeta_+ \left(s - \frac{|\ln(\delta)|}{2}, t \right) \right) \right)^{0,1} \Bigg\|_{0,p,\lambda}.
\end{aligned}$$

Holomorphicity of $u_\pm$ implies an estimate

$$\begin{aligned}
&\left\| \left(\left(\beta(-s)\mathrm{d}\zeta_- \left(-s + \frac{|\ln(\delta)|}{2}, t \right) + \beta(s)\mathrm{d}\zeta_+ \left(s - \frac{|\ln(\delta)|}{2}, t \right) \right) \right)^{0,1} \right\|_{0,p,\lambda} \\
&\qquad \le c e^{-|\ln(\delta)|(1-\lambda)} = c\delta^{1-\lambda},
\end{aligned} \tag{126}$$

c.f. Abouzaid [5, 5.10]. Similarly, from the terms involving the derivatives of the cutoff function and exponential convergence of $\zeta_\pm$ to 0 we obtain an estimate

$$\|\mathcal{F}_\delta(0)\|_{0,p,\lambda} < c \exp(-|\ln(\delta)|(1-\lambda)) = c\delta^{1-\lambda} \tag{127}$$

with c independent of δ.

To perform the iteration, we apply a uniformly bounded right inverse to the failure of the approximate solution to solve the Cauchy-Riemann equation. Given

$$\eta \in \Omega^{0,1}(S^\delta, (u^{\mathrm{pre}})^* TX)_{0,p}$$

one obtains elements

$$\underline{\eta} = (\eta_-, \eta_+) \in \Omega^{0,1}(S_\pm, u_\pm^* TX_\pm)$$

by multiplication with the cutoff function β and parallel transport $\mathcal{T}^{u_\pm}$ to $u_\pm$ along the path

$$\exp_{(\mu s, t^\mu z)}(\rho(\zeta^\delta(s,t) + (1-\rho)\zeta_\pm(s,t))), \quad \rho \in [0,1].$$

Define

$$\eta_+ = \mathcal{T}^{u_+}\beta(s - 1/2)\eta, \quad \eta_- = \mathcal{T}^{u_-}\beta(1/2 - s)\eta.$$

Since the fiber product (123) is transversally cut out, there exists

$$(\xi_+, \xi_-) \in \Omega^0(S_\pm, u^* TX_\pm)_{1,p,\lambda}, \quad D_{u_\pm}\xi_\pm = \eta_\pm, \quad \mathrm{ev}_\infty(\xi_+) = \mathrm{ev}_\infty(\xi_-)$$

where ev_∞ are the evaluation-at-infinity maps of (106). Denote

$$\xi_\infty = \mathrm{ev}_\infty(\xi_\pm) \in \mathbb{R} \times T_{\mathrm{ev}_\infty(u_\pm)} Z.$$

Define $Q^\delta\eta$ equal to (ξ_-, ξ_+) away from $[-\frac{|\ln(\delta)|}{2}, \frac{|\ln(\delta)|}{2}] \times Z$ and on the neck region by patching the solutions (ξ_-, ξ_+) together using a cutoff function that vanishes three-quarters of the way along the neck:

$$\begin{aligned}
Q^\delta\eta := {} & \beta\left(-s + \frac{1}{4}|\ln(\delta)|\right)((\mathcal{T}^{u_-})^{-1}\xi_- - \mathcal{T}^u\xi_\infty) \\
& + \beta\left(s + \frac{1}{4}|\ln(\delta)|\right)((\mathcal{T}^{u_+})^{-1}\xi_+ - \mathcal{T}^u\xi_\infty) \\
& + \mathcal{T}^u\xi_\infty \in \Omega^{0,1}(S^\delta, (u_\delta^{\mathrm{pre}})^* TX)_{1,p,\lambda}
\end{aligned} \tag{128}$$

where $\mathcal{T}^{u_\pm}$ denotes parallel transport to $u_\pm$ from u_δ^{pre} along the path

$$\exp_{(\mu s, t^\mu z)}(\rho(\zeta^\delta(s,t) + (1-\rho)\zeta_\pm(s,t))), \rho \in [0,1].$$

Since

$$\eta = (\mathcal{T}^{u_-})^{-1}\eta_- + (\mathcal{T}^{u_+})^{-1}\eta_+$$

we have

$$\begin{aligned}
\|D_{u^\delta_{\mathrm{pre}}} Q^\delta\eta - \eta\|_{1,p,\lambda} &= \|D_{u_\delta^{\mathrm{pre}}} Q^\delta\eta - (\mathcal{T}^{u_-})^{-1} D_{u_-^\delta}\xi_- - (\mathcal{T}^{u_+})^{-1} D_{u_+^\delta}\xi_+\|_{1,p,\lambda} \\
&\leq c\exp((1-\lambda)|\ln(\delta)/4|)\|\eta\|_{0,p,\lambda} \\
&\quad + c\|\mathrm{d}\beta(s - |\ln(\delta)|/4) Q_-^\delta\underline{\eta}\|_{0,p,\lambda} \\
&\quad + c\|\mathrm{d}\beta(-s + |\ln(\delta)|/4) Q_+^\delta\underline{\eta}\|_{0,p,\lambda}
\end{aligned}$$

where the first term arises from the difference between $D_{u_\delta^{\mathrm{pre}}}$ and $(\mathcal{T}^{u_\pm})^{-1} D_{u_\pm} \mathcal{T}^{u_\pm}$ and the second from the derivative $\mathrm{d}\beta$ of the cutoff function β. The difference in the exponential factors

$$\aleph_\lambda^\pm = \aleph_\lambda^\delta \exp(\pm 2 s\lambda), \quad \mp s \geq \frac{|\ln(\delta)|}{2}$$

in the definition of the Sobolev weight functions implies that possibly after changing the constant c, we have since $|\ln(\delta)| = -\ln(\delta)$

$$\| \mathrm{d}\beta(s - |\ln(\delta)|/4) Q_\pm^\delta \eta \|_{1,p,\lambda} < c e^{-\lambda \frac{|\ln(\delta)|}{2}} = c\delta^{\lambda/2}.$$

Hence one obtains an estimate as in Fukaya-Oh-Ohta-Ono [42, 7.1.32], Abouzaid [5, Lemma 5.13]: for some constant $c > 0$, for any $\delta > 0$,

$$\| D_{u_\delta^{\mathrm{pre}}} Q^\delta - \mathrm{Id} \| < c \min(\delta^{\lambda/2}, \delta^{(1-\lambda)/4}). \tag{129}$$

It follows that for δ sufficiently large an actual inverse may be obtained from the Taylor series formula

$$D_{u_\delta^{\mathrm{pre}}}^{-1} = Q^\delta (D_{u_\delta^{\mathrm{pre}}} Q^\delta)^{-1} = \sum_{k \geq 0} Q^\delta (I - Q^\delta D_{u_\delta^{\mathrm{pre}}})^k.$$

The variation in the linearized operators can be estimated as follows. After redefining $c > 0$ we have for all ξ_1, ξ sufficiently small

$$\| D_\xi \mathcal{F}_\delta(0, \xi_1) - D_{u_\delta^{\mathrm{pre}}} \xi_1 \| \leq C \|\xi_1\|_{1,p,\lambda} \|\xi\|_{1,p,\lambda}. \tag{130}$$

To prove this we require some estimates on parallel transport. Let

$$\mathcal{T}_z^{\delta,x}(m,\xi) : \Lambda^{0,1} T_z^* S_\delta \otimes T_x X \to \Lambda^{0,1}_{j^\delta(m)} T_z^* S_\delta \otimes T_{\exp_x(\xi)} X$$

denote pointwise parallel transport. Consider its derivative

$$D\mathcal{T}_z^{\delta,x}(m,\xi,m_1,\xi_1;\eta) = \nabla_t|_{t=0} \mathcal{T}_{u_\delta^{\mathrm{pre}}}(m + t m_1, \xi + t\xi_1)\eta.$$

For a map $u : S \to X$ we denote by $D\mathcal{T}_u$ the corresponding map on sections. By Sobolev multiplication (for which the constants are uniform because of the uniform cone condition on the metric on S^δ and uniform bounds on the metric on X^{δ^μ}) there exists a constant c such that

$$\| D\mathcal{T}_u^{\delta,x}(m,\xi,m_1,\xi_1;\eta) \|_{0,p,\lambda} \leq c \|(m,\xi)\|_{1,p,\lambda} \|(m_1,\xi_1)\|_{1,p,\lambda} \|\eta\|_{0,p,\lambda}. \tag{131}$$

Differentiate the equation

$$\mathcal{T}_u^{\delta,x}(m,\xi) \mathcal{F}_\delta(m,\xi) = \overline{\partial}_{J_\Gamma, H_\Gamma, j^\delta(m)}(\exp_{u^\delta_{\mathrm{pre}}}(\xi)))$$

with respect to (m_1, ξ_1) to obtain

$$\begin{aligned} D\mathcal{T}_{u^\delta_{\mathrm{pre}}}(m,\xi,m_1,\xi_1,\mathcal{F}_\delta(m,\xi)) + \mathcal{T}_u^\delta(m,\xi)(D\mathcal{F}_\delta(m,\xi,m_1,\xi_1)) = \\ (D\overline{\partial})_{j^\delta(m), \exp_{u_\delta^{\mathrm{pre}}}(\xi)} (Dj^\delta(m,m_1), D\exp_{u^\delta}(\xi,\xi_1)). \end{aligned} \tag{132}$$

Using the pointwise inequality

$$|\mathcal{F}_\delta(m,\xi)| < c|\mathrm{dexp}_{u_\delta^{\mathrm{pre}}(z)}(\xi)| < c(|\mathrm{d}u_\delta^{\mathrm{pre}}| + |\nabla\xi|)$$

for m, ξ sufficiently small, the estimate (131) yields a pointwise estimate

$$|\mathcal{T}_{u_\delta^{\mathrm{pre}}}(\xi)^{-1} D\mathcal{T}_{u_{\mathrm{pre}}^\delta}(m,\xi,m_1,\xi_1,\mathcal{F}_\delta(m,\xi))| \leq c(|\mathrm{d}u_{\mathrm{pre}}^\delta| + |\nabla\xi|)\,|(m,\xi)|\,|(\xi_1,m_1)|.$$

Hence

$$\begin{aligned}(133)\quad &\|\mathcal{T}_{u_\delta^{\mathrm{pre}}}(\xi)^{-1} D\mathcal{T}_{u_{\mathrm{pre}}^\delta}(m,\xi,m_1,\xi_1,\mathcal{F}_\delta(m,\xi))\|_{0,p,\lambda} \\ &\qquad\leq c(1 + \|\mathrm{d}u^\delta\|_{0,p,\lambda} + \|\nabla\xi\|_{0,p,\lambda})\|(m,\xi)\|_{L^\infty}\|(\xi_1,m_1)\|_{L^\infty}.\end{aligned}$$

It follows that

$$(134)\quad \|\mathcal{T}_{u_\delta^{\mathrm{pre}}}(\xi)^{-1} D\mathcal{T}_{u_{\mathrm{pre}}^\delta}(m,\xi,m_1,\xi_1,\mathcal{F}_\delta(m,\xi))\|_{0,p,\lambda} \leq c\|(m,\xi)\|_{1,p,\lambda}\|(m_1,\xi_1)\|_{1,p,\lambda}$$

since the $W^{1,p}$ norm controls the L^∞ norm by the uniform Sobolev estimates. As in McDuff-Salamon [61, Chapter 10], Abouzaid [5] there exists a constant $c > 0$ such that for all δ sufficiently small,

$$\begin{aligned}(135)\quad &\|\mathcal{T}_{u_{\mathrm{pre}}^\delta}(\xi)^{-1} D_{\exp_{u_\delta^{\mathrm{pre}}(\xi)}}(D_m j^\delta(m_1), D_{\exp_{u_{\mathrm{pre}}^\delta}(\xi)}\xi_1)) - D_{u_\delta^{\mathrm{pre}}}(m_1,\xi_1)\|_{0,p,\lambda} \\ &\qquad\leq c\|m,\xi\|_{1,p,\lambda}\|m_1,\xi_1\|_{1,p,\lambda}.\end{aligned}$$

Combining the estimates (134) and (135) and integrating completes the proof of claim (130). Applying the estimates (127), (129), (130) produces a unique solution $m(\delta), \xi(\delta)$ to the equation

$$\mathcal{F}_\delta(m(\delta),\xi(\delta)) = 0$$

for each δ, such that the maps

$$u_\delta := \exp_{u_\delta^{\mathrm{pre}}}(\xi(\delta))$$

depend smoothly on δ. Note that the implicit function theorem by itself does not give that the maps u_δ are distinct, since each u_δ is the result of applying the contraction mapping principle in a different Sobolev space. □

We now state the main result on the behavior of the moduli spaces under the neck-stretching limit. Let $\mathcal{M}^{<E}(X,\phi_\gamma,D)$ denotes the locus in $\mathcal{M}(X,\phi_\gamma,D)$ with area less than E. Similarly, let $\mathcal{M}^{<E}(\mathbb{X},\phi_\gamma,\mathbb{D})$ denote the locus with area less than E in $\mathcal{M}(\mathbb{X},\phi_\gamma,\mathbb{D})$.

Theorem 6.25. *Let $\mathbb{X} = \overline{X}_\subset \cup_Y \overline{X}_\supset$ be a broken rational symplectic manifold and $\phi : \mathbb{L} \to \mathbb{X}$ a broken self-transverse Lagrangian immersion in the sense of (91). Suppose perturbations $\underline{P} = (P_\Gamma)$ have been chosen so that every rigid labelled map in $\mathcal{M}^{<E}(\mathbb{X},\phi_\gamma,\mathcal{D},\underline{\sigma})$ is regular. There exists δ_0 such that for $\delta < \delta_0$, the assignment $[u] \mapsto [u_\delta]$ from Theorem 6.23 defines a bijection between the rigid moduli spaces $\mathcal{M}^{<E}(\mathbb{X},\phi_\gamma,\mathbb{D},\underline{\sigma})_0$ and $\mathcal{M}^{<E}(X^\delta,\phi_\gamma,D,\underline{\sigma})_0$.*

Proof. To prove injectivity, suppose that $u_{\delta_\nu} = v_{\delta_\nu}$ for some pseudoholomorphic buildings $u \neq v$ and gluing parameters $\delta_\nu \to 0$. Then the sequence u_{δ_ν} has two stable Gromov limits, which is a contradiction to Theorem 6.17. To prove surjectivity of gluing, it suffices to prove the following: Given a converging family u'_δ with parameter δ converging to u, the map u'_δ is close to u_δ in the norms used in the gluing formula. Indeed, this closeness implies that $u'_\delta = u_\delta$ by the uniqueness part of the implicit function theorem. By definition of Gromov convergence, the surface S_δ is obtained from S using a gluing parameter δ_C. The parameter δ_C is a function of the gluing parameter δ for the breaking of target to $\mathbb{X}$ and converges to zero as $\delta \to 0$. The map on the neck region may be decomposed into horizontal and vertical component. First consider the horizontal part of the map $p_Y \circ u'_\delta : S_\delta \to Y$. Denote by $R(l)$ the rectangle

$$R(l) = [-l/2, l/2| \times [0,1].$$

Since there is no area loss in the limit $\delta \to 0$, for any $C > 0$ there exists $\delta' > \delta_C$ such that the restriction of $p_Y \circ u'_\delta$ to the annulus $R(|\ln(\delta')|/2)$ satisfies the energy estimate of [41, Lemma 3.1]. Thus

(136)
$$p_Y u'_\delta(s,t) = \exp_{p_Y u_\delta^{\mathrm{pre}}(s,t)} \xi^h(s,t), \quad \|\xi^h(s,t)\| \leq C(e^{\pi(s-|\ln(\delta')|/2)} + e^{\pi(|\ln(\delta')|/2-s)})$$
$$s \in [-|\ln(\delta')|/2, |\ln(\delta')|/2].$$

A similar estimate holds for the higher derivatives $D^k\xi^h(s,t)$ by elliptic regularity, for any $k \geq 0$. The necessary exponential decay result on the neck region is proved in the case of cylinders in Venugopalan-Woodward [86, (8.13)]; the case of strips is similar. □

Corollary 6.26. *If $\mathcal{M}^{<E}(\mathbb{X}, \phi_\gamma, \mathbb{D})_0$ is regular, then there exists δ_0 such that for $\delta > \delta_0$, $\mathcal{M}^{<E}(X^\delta, \phi_\gamma, D)_0$ is regular.*

Proof. We may assume that every $(C^{\delta/\mu_1,\ldots,\delta/\mu_k}, u_\delta)$ in $\mathcal{M}^{<E}(X^\delta, \phi_\gamma, D)$ for δ sufficiently small is obtained by an application of Lemma 6.24 to the approximate solution in the proof of Theorem 6.23. By the transversality statement in the Picard Lemma 6.24, the nearby solution u_δ produced from u_0 by the implicit function theorem also has surjective linearized operator $\tilde{D}_{u_\delta}$. (In fact, the operator is surjective even restricting to variations of conformal structure on $C^{\delta/\mu_1,\ldots,\delta/\mu_k}$ arising from variations on S.) □

We will need a similar bijection for the case of buildings in $X_\subset$ consisting of a map $u_\subset : S_\subset \to X_\subset$ and a neck piece $u_0 : S_0 \to \mathbb{P}(N_- \oplus \mathbb{C})$, where the Lagrangian $L_\subset$ in $X_\subset$ is only asymptotically cylindrical.

Theorem 6.27. *Let $\mathbb{\Gamma}$ be a type of building in $\mathbb{X}_\subset$ with two components as above. Suppose perturbations $\underline{P} = (P_\Gamma)$ have been chosen so that every rigid labelled map in*

$\mathcal{M}_{\mathbb{F}}(\mathbb{X}, \phi_\gamma, \mathbb{D}, \underline{\sigma})$ is regular, and let $\mathbb{F}'$ be the type with a single level in $X_\subset$ obtained by gluing. Then each $u \in \mathcal{M}_{\mathbb{F}}(\mathbb{X}, \phi_\gamma, \mathbb{D}, \underline{\sigma})$ is in the closure of a unique component of $\mathcal{M}_{\mathbb{F}'}(\mathbb{X}, \phi_\gamma, \mathbb{D}, \underline{\sigma})$.

Proof. The statement of the Theorem is a type of result known as "surjectivity of gluing" in the literature, in which one must show that the sequence on the long cylinders is close to the approximate solution in the chosen Sobolev norm. Let u be as in the statement of the Theorem. By the gluing construction, there exists a one-parameter family u_δ of buildings of type Γ' Gromov-converging to u in the limit $\delta \to 0$. To see that u_δ is the unique such limit, it suffice to check that if u'_ν converges to $u = (u_\subset, u_0)$ then u'_ν is close to the approximate solution u_δ^{pre} of (118).

To prove this, we examine the vertical and horizontal parts of the map. Denote by $\overline{u}$ the map to $\overline{X}_\subset$ obtained by projecting u_0 to the base Y of the neck piece $\mathbb{P}(N_- \oplus \mathbb{C})$. Similarly, let $\overline{u}_\nu$ denote the map to $\overline{X}_\subset$ induced by u_ν. Then $\overline{u}_\nu$ Gromov converges to $\overline{u}$, and in particular the domain C_ν of $\overline{u}_\nu$ converges to the domain C of $\overline{u}$. Hence C_ν is obtained from C by removing small balls around the node and gluing in cylinders of length $|\ln(\delta_\nu)|$ for some sequence of gluing parameters δ_ν. Let $y \in Y$ be the image of the node in $\overline{u}$. We trivialize the bundle $\mathbb{R} \times Z \to Y$ in a neighborhood $B_\epsilon(y)$ of y. Denote by

$$e^{-\tau_\nu} : \mathbb{R} \times Z \to \mathbb{R} \times Z, \quad (\sigma, z) \mapsto (\sigma - \tau_\nu, z)$$

translation by $-\tau_\nu$. We may pass to a subsequence so that

$$\lim_{\nu \to \infty} e^{-\tau_\nu} u_\nu(0,0) = (0, z)$$

for some point $(0, z)$ over y. By the annulus lemma for maps to $\overline{X}_\subset$, we have on the long strips of length $|\ln(\delta_\nu)|$ connecting the components

$$\overline{u}_\nu(s,t) = \exp_y \xi(s,t), \quad \|\xi(s,t)\| \le C(e^{\pi(s - |\ln(\delta_\nu)|/2)\mu_0} + e^{\pi(|\ln(\delta_\nu)|/2 - s)\mu_0}) \qquad s \in [-|\ln(\delta_\nu)|/2, |\ln(\delta_\nu)|/2] \tag{137}$$

where the exponential decay constant μ_0 is determined by the angles at the clean intersection; see Abouzaid [5, 10.12].

To control the Sobolev norms of the vertical part of the map on the neck pieces, we compare the given almost complex structure and boundary conditions to a model problem in which the almost complex structure and boundary conditions are constant. Let

$$y(s,t) = p(e^{-\tau_\nu} u_\nu(s,t)) \in Y, \quad s \in [-|\ln(\delta_\nu)|/2, |\ln(\delta_\nu)|/2]$$

be the projection of the given map to Y. Choose local coordinates on $B_\epsilon(y)$ so that L_Y is linear. Denote by $L_\subset^\pm$ the branches of $L_\subset$ containing the images of $u_\nu(s,t)$ for $t = 0$ resp. $t = 1$ and s sufficiently large and $\theta_\pm \in S^1$ the angles of the branches at y. Define affine-linear model boundary conditions in $\mathbb{R} \times S^1 \times B_\epsilon(y)$ by

$$L_\nu^{\mathrm{model},\pm} = \mathbb{R} \times \{\theta_\pm\} \times L_Y.$$

Let σ_ν be the $\mathbb{R}$-coordinate of the evaluations $u_\nu(0,0)$. Define $T_\nu^\pm$ by

$$u([-|\ln(\delta_\nu)|/2, |\ln(\delta_\nu)|/2] \times 0) = [T_\nu^-, T_\nu^+].$$

Since $L_\subset$ has smooth cleanly-self-intersecting compactification $\overline{L}_\subset$ in $\overline{X}_\subset$, the translations

$$e^{-\tau_\nu}(L_\subset^\pm \cap [T_\nu^- + \tau_\nu, T_\nu^+ + \tau_\nu] \times Z) \subset \mathbb{R} \times S^1 \times B_\epsilon(y)$$

differ from $L_\nu^{\mathrm{model},\pm}$ near $u(s,t)$ by a map

$$\beta_\pm : L_\subset^\pm \cap \Big([T_\nu^- + \tau_\nu, T_\nu^+ + \tau_\nu] \times Z\Big) \to N_{L_\nu^{\mathrm{model},\pm}}$$

satisfying an exponential decay estimate

$$\|e^{-\tau\nu}\beta_\pm(\sigma, z)\| \le C(e^{-\tau_\nu - \sigma} + \mathrm{dist}(z, z')) \quad \sigma \in [T_\nu^-, T_\nu^+], z \in L_\nu^{\mathrm{model}} \cap Z. \tag{138}$$

Choose a diffeomorphism identifying the boundary condition with its model

$$\psi_\nu \in \mathrm{Diff}(\mathbb{R} \times S^1 \times B_\epsilon), \quad \psi_\nu(e^{-\tau_\nu} L_\subset^\pm) = L_\nu^{\mathrm{model},\pm}.$$

Because of (138), the diffeomorphism $D\psi_\nu$ may be taken to satisfy an estimate similar to that for the boundary conditions:

$$\|D\psi_\nu(\sigma, z') - \mathrm{Id}\,\| \le C(e^{-\tau_\nu - \sigma} + \mathrm{dist}(z, z')) \quad \sigma \in [T_\nu^-, \infty). \tag{139}$$

The composition $\psi_\nu e^{-\tau_\nu} u_\nu$ satisfies the model boundary conditions and the Cauchy-Riemann equation up to an error term arising from the failure of the map ψ_ν to be J-holomorphic: Let $J_0(t)$ denotes the almost complex structure obtained by evaluating at $\psi_\nu e^{-\tau_\nu} u_\nu(0,t)$. By assumption, the $\mathbb{R}$-derivative of $u_\nu(s,t)$ on the neck so that if $\sigma_\nu(s,t)$ denotes the $\mathbb{R}$ coordinate of $e^{-\tau\nu} u_\nu(s,t)$ then for any $\epsilon > 0$ we have for ν sufficiently large

$$\partial_s \sigma_\nu(s,t) \in \left(\frac{\mu}{2}, \frac{3\mu}{2}\right).$$

This implies

$$|\sigma_\nu(s,t) - \sigma_\nu(0,t)| \ge \frac{\mu}{2}|s|.$$

Let

$$u^{\mathrm{model}}(s,t) = (\mu s, t^\mu z)$$

as in (118). As in the discussion around (111), we have

$$p_Z(\psi_\nu e^{-\tau_\nu} u_\nu(s_\nu, t)) \to t^\mu z \quad \text{in} \quad C^\infty([0,1])$$

for any sequence s_ν with $|s_\nu|/|\ln(\delta_\nu)| \to 0$. Define

$$\zeta_\nu(s,t) = (\psi_\nu e^{-\tau_\nu} u_\nu)(s,t) - u^{\mathrm{model}}(s,t).$$

Then ζ_ν satisfies linear totally-real boundary conditions and is approximately holomorphic: Since the difference between J and J_0 depends only on the projection to Y, after re-defining μ_0 we have

$$\begin{aligned}\|\overline{\partial}_{J_0}\zeta_\nu(s,t)\| &\leq \|\overline{\partial}_J\zeta_\nu(s,t)\| + \|(\overline{\partial}_J - \overline{\partial}_{J_0})\zeta_\nu(s,t)\| \\ &\leq C(e^{-\tau_v - \mu(s-|\ln(\delta_\nu)|/2)/2} + e^{(s-|\ln(\delta_\nu)|/2)\mu_0} + e^{(|\ln(\delta_\nu)|/2-s)\mu_0}) \\ &\quad s \in [-|\ln(\delta_\nu)|/2, |\ln(\delta_\nu)|/2]\end{aligned}$$

where the first term arises from exponential convergence of the vertical part of the boundary conditions, and the second from the annulus lemma for the horizontal map. Write

$$\eta_\nu := \overline{\partial}_{J_0}\zeta_\nu \in \Omega^{0,1}([-|\ln(\delta_\nu)|/2, |\ln(\delta_\nu)|/2] \times [0,1], \mathbb{C}^n).$$

Denote by

$$f_i \in C^\infty([0,1], \mathbb{R}^{2n}), i \in I$$

the eigenfunctions of $J_0\partial_t$ with boundary conditions the linear subspaces $TL_\nu^{\mathrm{model},\pm}$ and the eigenvalues $\lambda_i \in \mathbb{R}$, the operator $J_0\partial_t$ being self-adjoint. Consider the decomposition of the maps ζ_ν, η_ν into eigenfunctions of $J\partial_t$ with boundary conditions $TL_\nu^{\mathrm{model},\pm}$ with coefficients $c_{\nu,i}, d_{\nu,i} \in \mathbb{R}$:

$$\zeta_\nu(s,t) = \sum_{i\in I} c_{\nu,i}(s) f_i(t), \quad \eta_\nu(s,t) = \sum_{i \in I} d_{\nu,i}(s) f_i(t).$$

We obtain a solution for the coefficients by integration

$$\begin{aligned}(140)\quad c_{\nu,i}(s) &= c_{\nu,i}(|\ln(\delta_\nu)|/2)\exp\left(\lambda_i(s-|\ln(\delta_\nu)|/2) + \int_{|\ln(\delta_\nu)|/2}^s d_{\nu,i}(s')\mathrm{d}s'\right) \\ (141)\quad &= c_{\nu,i}(-|\ln(\delta_\nu)|/2)\exp\left(\lambda_i(s+|\ln(\delta_\nu)|/2) + \int_{-|\ln(\delta_\nu)|/2}^s d_{\nu,i}(s')\mathrm{d}s'\right).\end{aligned}$$

We may assume, by redefining μ_0, that the exponential decay constant μ_0 is smaller than the minimum of the non-zero eigenvalues $|\lambda_i|$. For any $i \in I$, (140) gives

(142)
$$\|c_{\nu,i}(s)\| \leq C\left(\|c_{\nu,i}(|\ln(\delta_\nu)|/2)\|\ e^{(s-|\ln(\delta_\nu)|/2)\mu_0} + \|c_{\nu,i}(-|\ln(\delta_\nu)|/2)\|\ e^{(|\ln(\delta_\nu)|/2-s)\mu_0}\right)$$
$$s \in [-|\ln(\delta_\nu)|/2, |\ln(\delta_\nu)|/2].$$

Using elliptic estimates, one obtains similar estimates for the derivatives of ζ_ν. We obtain by (142) and elliptic regularity that for any constant C, for ν sufficiently large the estimate

$$(143)\qquad \|\psi_\nu e^{-\tau_\nu} u_\nu(s,t) - u^{\mathrm{model}}(s,t)\|_{k,2,\lambda} < C(1 + \delta_\nu^{\mu_0-\lambda}/(\mu_0 - \lambda))$$

holds for any $k \geq 0$. Note that in the norm defined in (121), the zero modes $c_i(s,t), \lambda_i = 0$ have norm determined by evaluation, and are not required to have small k, p norm on the neck. The Sobolev embedding theorem implies the same estimate for the k, p, λ norm for any kp with $kp > 1$. Write

$$\psi_\nu e^{-\tau_\nu} u_\nu(s,t) = \exp_{e^{-\tau_\nu} u_{\delta_\nu^{\mathrm{pre}}(s,t)}}(\xi_\nu(s,t))$$

with notation as around (121). The difference between geodesic exponentiation and addition vanishes uniformly in the limit. Hence, the estimate (143) holds for ξ_ν by comparability of geodesic exponentiation with addition in the local model. Away from the neck u_ν, converges to $u_{\delta_\nu^{\mathrm{pre}}}$ uniformly in all derivatives, so the k, p, λ norm of ξ_ν tends to zero on S^{δ_ν} as $\nu \to \infty$. Thus the map u'_ν is the unique solution appearing in the implicit function theorem. $\square$

6.6. **Deformation to split form.** As in Charest-Woodward [21] we consider a deformation of the matching conditions between levels to split form. In this limit, the moduli spaces of treed buildings become products, rather than fiber products, of moduli spaces of their treed levels. This deformation is similar to the theory introduced by Bourgeois [16]. The resulting Fukaya algebra is homotopy equivalent to the original.

The deformation replaces the matching conditions at the Reeb orbits and chords with deformed matching conditions using deformations of the diagonal. Choose cellular deformations of the diagonal on $Y = Z/S^1$ and $L_Z = L \cap Z$:

$$\delta_t^Y : Y \to Y \times Y, \quad \delta_t^{L_Z} : L_Z \to L_Z \times L_Z.$$

In the examples considered in this paper, the submanifold L_Z is always a sphere $L_Z \cong S^{n-1}$. A cellular deformation of the diagonal in $L_Z \times L_Z \cong (S^{n-1})^2$ is given by the standard degeneration of the diagonal $S^{n-1} \to S^{n-1} \times S^{n-1}$ so that the diagonal is degenerated to the split form corresponding to its Künneth decomposition

$$\delta_1(S^{n-1}) = (\{(1,\ldots,0)\} \times S^{n-1}) \cup (S^{n-1} \times \{(-1,0,\ldots,0)\}).$$

Similarly, $Y \cong \mathbb{C}P^{n-1}$ will be a complex projective space in our application and δ_t^Y deforms the diagonal embedding δ_0^Y to the map δ_1^Y whose image is the union of products of projective space $\mathbb{C}P^k \times \mathbb{C}P^{n-k-1}$.

Definition 6.28. A *treed $\wp$-building* C is a treed building C^{pre} together with *splitting parameters* $\wp_1, \ldots, \wp_k \in [0, \infty]$ so that the sum

$$\sum_{i=1}^k \wp_i = \wp$$

is equal to $\wp$.

A *holomorphic treed $\wp$-building* is a pair $(C, u : S \to \mathbb{X}[k])$ where C is a treed $\wp$-building and a holomorphic map u satisfying the deformed matching conditions

$$(\vartheta_{e,-}(0), \vartheta_{e,+}(0)) \in \mathrm{Im}(\delta_{\tilde{\wp}_i}), \quad \tilde{\wp}_i = \frac{\wp_i}{\wp_i + 1}. \tag{144}$$

and instead of the condition (94) the deformed condition

$$\left(\lim_{s\to\infty}\varphi_s(w_{e,-}),\lim_{s\to-\infty}\varphi_{-s}(w_{e,+})\right)\in \operatorname{Im}(\delta_{\tilde{\wp}_i})$$

for the original edges T_e connecting levels S_i and S_{i+1} with endpoints $u(w_{e,-}), u(w_{e,+})$ in L.

We now repeat the construction of the broken Fukaya category using $\wp$-buildings rather than treed buildings. The moduli space of holomorphic $\wp$-treed buildings is denoted $\mathcal{M}^{\wp}(\mathbb{X},\phi)$. The same regularization procedure as in the previous section leads to regularized moduli spaces $\mathcal{M}^{\wp}(\mathbb{X},\phi,\mathbb{D})$ with good compactness properties for the components of expected dimension at most one.

Remark 6.29. Treed buildings with the deformation above have the following boundary strata. Boundary types $\mathbb{\Gamma}$ of codimension one occur when one of the components S_v develops a boundary node in the interior of one of the pieces $\mathbb{X}[k]_i$. Such a boundary type $\mathbb{\Gamma}$ has stratum $\mathcal{M}^{\wp}_{\mathbb{\Gamma}}(\mathbb{X},\phi,\mathbb{D})$ that is a fake boundary component of the one-dimensional component of $\mathcal{M}^{\wp}(\mathbb{X},\phi,\mathbb{D})$. Indeed, given a configuration (C,u) of type $\mathbb{\Gamma}$ the length $\ell(e)$ of the corresponding edge may also be deformed to a non-zero value. This fact implies that $\mathcal{M}^{\wp}_{\mathbb{\Gamma}}(\mathbb{X},\phi,\mathbb{D})$ lies in the closure of two one-dimensional strata. Another codimension one type occurs when a level $u_i : S_i \to \mathbb{X}[k]_i$ splits into two levels, so that u_i is replaced by maps u_i' and u_{i+1}' joined by inter-level edges $\ell(e)$ of length zero. This locus is again a fake boundary component since the length of those edges $\ell(e)$ may be made non-zero. Note that for $\wp < \infty$, an increase in the length of one of the lengths $\wp_i$ must be compensated by a decrease in one of the other lengths $\wp_j$. In any one-parameter family of buildings with non-zero $\wp_i$, eventually one of the lengths $\wp_i$ becomes zero or one of the lengths $\ell(e)$ of the edges connecting to boundary points of S becomes infinite. In particular, the topological boundary of any one-dimensional component of $\mathcal{M}^{\wp}(\mathbb{X},\phi,\mathbb{D})$ consists of buildings with an infinite length boundary edge $e, \ell(e) = \infty$ so that e is not an inter-level edge. This ends the Remark.

Using moduli spaces of buildings, we may define a broken analog of the Fukaya algebra. The underlying vector space $CF(\mathbb{X},\phi)$ is defined in the same way as $CF(X,\phi)$, but in equation (60) the count of elements of $\mathcal{M}_{\mathbb{\Gamma}}(X,\phi,D)$ is replaced by a count of elements of $\mathcal{M}_{\mathbb{\Gamma}}(\mathbb{X},\phi,\mathbb{D})$. Counts of $\wp$-treed buildings lead to a family of broken Fukaya algebras $CF^{\wp}(\mathbb{X},\phi)$.

6.7. **Homotopy equivalences.** We consider various kinds of homotopy equivalences of A_∞ algebras involving broken Fukaya algebras in this section. The first Theorem 6.30 is an A_∞ equivalence which arises in the limit of infinite neck length under the neck stretching limit; this is essentially the same as considered in Charest-Woodward [21]. For this theorem, the Lagrangian boundary condition is required to be cylindrical on the neck. The second Theorem 6.32 describes a homotopy equivalence of Fukaya algebras where the boundary conditions are allowed to vary in an asymptotically cylindrical (rather than cylindrical-near-infinity) way, which in particular allow a local model for the surgery in which the Lagrangians are only asymptotically cylindrical.

First, suppose that $\phi : L \to X$ is a Lagrangian boundary condition that is cylindrical in a neighborhood of a hypersurface $Z \subset X$. We denote by m_d^τ the composition maps on $CF(X, \phi)$ associated to the neck-stretched almost complex structure $J^\tau \in \mathcal{J}(X)$.

Theorem 6.30. *The maps m_d^τ have a limit m_d^∞ as $\tau \to \infty$ equal to the composition map m_d for the algebra $CF(\mathbb{X}, \phi)$. The broken Fukaya algebra $CF(\mathbb{X}, \phi)$ is homotopy-equivalent to $CF(X, \phi)$.*

The Fukaya algebras associated to different choices of almost complex structure are homotopy equivalent. In particular, denote by J^τ the almost complex structure stretched by gluing in a neck of length τ. As in [21] counts of quilted treed disks (C, Q, u) with diagonal seam condition define homotopy-equivalences

$$CF(X^\tau, \phi) \underset{\psi_\tau}{\overset{\zeta_\tau}{\rightleftarrows}} CF(X^{\tau+1}, \phi).$$

We refer to [21] for the detailed definition of quilted disks; in particular there is a collection of disk components $S' \subset S$ equipped with *quiltings*, meaning circles in S' intersecting a boundary component exactly once. These components are called *quilted components* and, in the treed context, the lengths $\ell(e)$ of edges connecting these components satisfy a system of equalities, if the number of quilted components is greater than one. We claim that the composition of these homotopy equivalences converges to a homotopy equivalence with the broken Fukaya algebra.

Lemma 6.31. *For any energy bound E, the terms in the homotopy-equivalence ζ_τ with coefficient $q^{A(u)}, A(u) < E$ vanish for sufficiently large τ except for constant disks.*

Proof. Suppose, by way of contradiction, that there exists a sequence $(C_\nu, u_\nu : S_\nu \to X)$ of non-constant treed quilted disks with arbitrarily large τ in a component of the moduli space with expected dimension zero and bounded energy $E(u_\nu)$. By forgetting the quilting and stabilizing, we obtain a sequence v_ν of treed disks with domain dependent almost complex structures and bounded area. By sft compactness, the limit of a subsequence of v_ν is a stable broken disk $(C, u : S \to \mathbb{X})$ with boundary in ϕ. The limiting cylindrical-end manifolds are independent of the position of the quilting, and the map (C, u) lies in a component of the moduli space of expected dimension -1, which is a contradiction. □

Proof of Theorem 6.30. Lemma 6.31 implies that there exist limits of the successive compositions of the homotopy equivalences. For $N, \tau \in \mathbb{Z}_{>0}$ consider the composition

$$\zeta_{N,\tau} := \zeta_{N+\tau} \circ \zeta_{N+\tau-1} \circ \ldots \circ \zeta_N : CF(X^N, \phi) \to CF(X^{N+\tau}, \phi).$$

Because of the bijection in Theorem 6.25, the limit

$$\zeta_N = \lim_{\tau\to\infty} \zeta_{N,\tau} : CF(X^N, \phi) \to \lim_{\tau\to\infty} CF(X^{N+\tau}, \phi)$$

exists. Similarly, the limit

$$\psi_N = \lim_{\tau\to\infty} \psi_{N,\tau}, \quad \psi_{N,\tau} := \psi_{N+\tau} \circ \psi_{N+\tau-1} \circ \ldots \circ \psi_N$$

exists. The composition of strictly unital morphisms is strictly unital, so the composition ψ is strictly unital mod terms divisible by q^E for any E. So ψ is strictly unital.

The limiting morphisms are also homotopy-equivalences. Let h_τ, g_τ denote the homotopies satisfying

$$\zeta_\tau \circ \psi_\tau - \mathrm{id} = m_1(h_\tau), \quad \psi_\tau \circ \zeta_\tau - \mathrm{id} = m_1(g_\tau),$$

from the homotopies relating $\zeta_\tau \circ \psi_\tau$ and $\psi_\tau \circ \zeta_\tau$ to the identities in [75, Section 1e]. In particular, $h_{\tau+1}, g_{\tau+1}$ differ from h_τ, g_τ by expressions counting *twice-quilted* disks. For any $E > 0$ and τ sufficiently large, all terms in $h_{\tau+1} - h_\tau$ are divisible by q^E. It follows that the infinite composition

$$h_N = \lim_{\tau\to\infty} h_{N,\tau}, \quad g_N = \lim_{\tau\to\infty} g_{N,\tau}$$

exists and gives a homotopy-equivalence between $\zeta_N \circ \psi_N$ resp. $\psi_N \circ \zeta_N$ and the identities on $CF(\mathbb{X}, \phi)$ and $CF(X, \phi)$. □

In the second homotopy equivalence result, we study the broken Fukaya category as the Lagrangian boundary condition is varied in a family that is only asymptotically cylindrical. The regularization procedure for the moduli spaces of buildings is the same as before, while the necessary compactness result is Theorem 6.19. One obtains by counting such buildings broken Fukaya algebra $CF(\mathbb{X}, \phi)$ for any Lagrangian boundary condition that is asymptotically cylindrical. Given a family ϕ_ρ of such boundary conditions, one may consider moduli spaces of quilted buildings where the boundary condition ϕ_ρ depends on the distance from the quilted component (similar to the proof of independence of homotopy type of the Fukaya algebra on the choice of almost complex structure.) Since the moduli spaces on each disk are defined with a fixed ϕ_ρ, Theorem 6.19 implies that these moduli spaces are also compact. One obtains by counting quilted buildings:

Theorem 6.32. *Let $\mathbb{L}_\rho = (L_{\subset,\rho}, L_{\supset,\rho})$ be a path of asymptotically-cylindrical broken immersed Lagrangians in $\mathbb{X} = (X_\subset, X_\supset)$ with immmersions ϕ_ρ so that $L_{\subset,\rho}$ and $L_{\supset,\rho}$ are Hamiltonian isotopic for all values of ρ. Then the corresponding broken Fukaya algebras are homotopy equivalent:*

$$CF(\mathbb{X}, \phi_{\rho_1}) \simeq CF(\mathbb{X}, \phi_{\rho_2})$$

if the area parameters in (150) defined by the two paths are equal.

Theorem 6.33. *Let $\phi : \mathbb{L} \to \mathbb{X}$ be an asymptotically-cylindrical Lagrangian boundary condition. For any $\wp \in [0, \infty]$ the broken Fukaya algebra $CF^{\wp}(\mathbb{X}, \phi)$ is homotopy-equivalent to $CF(\mathbb{X}, \phi)$.*

Sketch of proof. The necessary homotopy-equivalences

$$\psi_{\wp_1,\wp_2} : CF^{\wp_1}(\mathbb{X}, \phi) \to CF^{\wp_2}(\mathbb{X}, \phi)$$

are given by a count of quilted disks $(C, u : S \to \mathbb{X})$ of the following type: The buildings before the quilted components and at infinite distance from the quilted components (that is, the sum of the lengths of boundary edges $\ell(e)$ connecting S_v to a quilted component is infinite) are $\wp_1$-buildings and the buildings after the quilted components at infinite distance from them are $\wp_2$-buildings; see [21, Section 8.1] for the corresponding construction with the Morse model; for components $S_v \subset S$ at finite distance from the quilted components one takes $\wp$ a function interpolating between $\wp_1$ and $\wp_2$. Any two such choices of interpolation give homotopy-equivalent morphisms. This fact implies

$$\psi_{\wp_2,\wp_3} \circ \psi_{\wp_1,\wp_2} = \psi_{\wp_1,\wp_3}$$

for any $\wp_1, \wp_2, \wp_3$. For $\wp_1 = \wp_3$ one may take $\wp = \wp_1 = \wp_3$ for all buildings in the construction and (taking perturbations that are independent of the quiltings) one obtains that $\psi_{\wp_1,\wp_3}$ is the identity functor, so that $\psi_{\wp_1,\wp_2}$ and $\psi_{\wp_2,\wp_1}$ are homotopy equivalences. □

Remark 6.34. In the case of infinite breaking parameter, each moduli space of buildings is a product of the moduli spaces of levels in the following sense. Each holomorphic building $(C, u : S \to \mathbb{X}[k])$ breaks up into a collection of pairs

$$(C_i, u_i : S_i \to \mathbb{X}[k]_{l_-(i), l_+(i)})$$

where $\mathbb{X}[k]_{l_-(i), l_+(i)} \subset \mathbb{X}[k]$ is the union of components in the decomposition (89) between levels $l_-(i), l_+(i)$. We denote in particular $\mathbb{X}_{\subset}[k] \subset \mathbb{X}[k]$ the union of the components except the last $X_{\supset}$, and similarly for $\mathbb{X}_{\supset}[k]$. Thus, in particular an ∞-level may be a pair $(C, u : S \to \mathbb{X}_{\subset}[k])$ which consists of a map to $X_{\subset}$ and some collection of maps to $\mathbb{R} \times Z$ with matching conditions. For a type of building $\mathbb{\Gamma}$ and collection of constraints Σ the moduli space of treed buildings $\mathcal{M}^{\infty}_{\mathbb{\Gamma}}(\mathbb{X}, \phi, \mathbb{D})$ of labelled type $\mathbb{\Gamma}$ is a product of the moduli space of its labelled treed levels

$$\mathcal{M}^{\infty}_{\mathbb{\Gamma}}(\mathbb{X}, \phi, \mathbb{D}) = \bigcup_{(\Sigma_i)} \prod_{i=0}^{k} \mathcal{M}_{\mathbb{\Gamma}_i}(\mathbb{X}, \phi, \mathbb{D}, \Sigma_i).$$

Here the union is over possible labellings Σ_i, representing the collection of constraints given by the (un)stable manifolds of the Morse functions chosen on Π and Y for the inter-level edges and the cellular constraints for each boundary leaf. We adjust our terminology and call the elements u_i of $\mathcal{M}_{\mathbb{\Gamma}_i}(\mathbb{X}, \phi, \mathbb{D}, \Sigma_i)$ *levels.* Each

u_i further decomposes into *sublevels* $u_{i,j}$ with domain $S_{i,j} \subset S_i$ mapping into some $\mathbb{X}[k]_{l(i,j)}$.

7. Holomorphic disks bounding the handle

In this section, we review some results of Fukaya-Oh-Ohta-Ono [42, Chapter 10] on the moduli spaces of holomorphic disks with boundary in the local model. The main results are Proposition 7.2 and Lemma 7.18, which combined give a correspondence, up to repetition of codimension one inputs, between rigid maps in the local model with surgered and unsurgered boundary condition (after adding a longitudinal constraint, in the case of wrong-way corners.)

7.1. **Classifying disks with a single end.** We first classify disks with a single end. Let $\gamma(t) = t + i2\epsilon$ be the standard path and $\mathcal{M}_{\mathbb{\Gamma}}(\phi_\gamma)$ denote the space of holomorphic maps $u : S \to X = \mathbb{C}^n$ with boundary condition in $\phi_\gamma : H_\gamma \to X$ of some type of map $\mathbb{\Gamma}$. The target $X = \mathbb{C}^n$ is naturally a cylindrical-end manifold with cylindrical end modelled on a cylinder $\mathbb{R} \times Z$ on the unit sphere $Z = S^{2n-1}$ defined using coordinates $q_j + ip_j, j = 1, \ldots, n$ on $\mathbb{C}^n$ by

$$Z = \{q_1^2 + p_1^2 + \ldots + q_n^2 + p_n^2 = 1\}.$$

The Reeb flow on Z is periodic with period 2π and the quotient Z/S^1 is a complex projective space

$$Y = Z/S^1 \cong \mathbb{C}P^{n-1}.$$

The handle Lagrangian H_γ defines a Lagrangian in the projective space $\mathbb{C}P^n$, whose intersection with the divisor at infinity is $\mathbb{R}P^{n-1}$. The Reeb chords from $\mathbb{R}^n$ to $i\mathbb{R}^n$ (or vice-versa) through $0 \neq (a_1, \ldots, a_n) \in \mathbb{R}^n$ are classified by half-integer $m \in \mathbb{Z}/2$ and are of the form

$$\vartheta_{m,\underline{a}}(t) = e^{m\pi it/2}(a_1, \ldots, a_n).$$

Consider the case that Γ is a type of configuration consisting of a disk S attached to single leaf T at a node $w \in S$. The following classification of curves with *right-way* and *wrong-way* corners is a modification of Fukaya-Oh-Ohta-Ono [42, Theorem 60.26].

Definition 7.1. Let Γ be a type of domain S with a single strip-like end $e \in \mathcal{E}(S)$. Let $\mathbb{\Gamma}_+$ resp. $\mathbb{\Gamma}_-$ be a type of finite-energy map given by sections of the Lefschetz fibration $\pi : \mathbb{C}^n \to \mathbb{C}$ over a half space bounding ϕ_γ asymptotic to a Reeb chord of angle change $\pi/2$ from $\mathbb{R}^n$ to $i\mathbb{R}^n$ resp. $i\mathbb{R}^n$ to $\mathbb{R}^n$. We say that the map types $\mathbb{\Gamma}_\pm$ are *minimal types* and all other map types are *non-minimal*. Let $\hat{\mathbb{\Gamma}}_\pm$ be the type obtained from the minimal types $\mathbb{\Gamma}_\pm$ by adding a boundary leaf $e \in \operatorname{Edge}(\hat{\mathbb{\Gamma}}_\pm)$. Let

$$\operatorname{ev} : \mathcal{M}_{\hat{\mathbb{\Gamma}}_-}(\phi_\gamma) \to H_\gamma \times S^{n-1}$$

denote the combined evaluation map for the leaf and end.

Proposition 7.2. *For γ be the standard path $t \mapsto t + i2\epsilon, \epsilon > 0$ and $J_\Gamma = J_0$ the standard complex structure, the maps of type $\mathbb{\Gamma}_\pm$ and $\hat{\mathbb{\Gamma}}_\pm$ are regular and the following hold:*

(a) (Right-way corners) *Evaluation at infinity* (95) *defines a diffeomorphism*

$$\mathcal{M}_{\mathbb{\Gamma}_+}(\phi_\gamma) \to S^{n-1}, \quad u \mapsto \vartheta_e(0).$$

(b) (Wrong-way corners) *Evaluation at infinity* (95) *defines a map*

$$\mathcal{M}_{\mathbb{\Gamma}_-}(\phi_\gamma) \to S^{n-1}, \quad u \mapsto \vartheta_e(0) \tag{145}$$

giving $\mathcal{M}_{\mathbb{\Gamma}_-}(\phi_\gamma)$ the structure of an S^{n-2} bundle over S^{n-1} diffeomorphic to the unit sphere bundle T_1S^{n-1} in TS^{n-1}. For generic $a, c \in S^{n-1}$, the inverse image $\mathrm{ev}^{-1}(\mathbb{R} \times \{c\} \times \{a\})$ *is a single transverse point.*

Furthermore, the homology classes of maps of type $\mathbb{\Gamma}_\pm$ are primitive. In the dimension two case $\dim(H_0) = 2$*, the orientations of the two points in any fiber of* (145) *agree for the trivial relative spin structure.*

Proof. We adopt a proof similar to Seidel's computation in [76], which studied a boundary value problem for sections of a Lefschetz fibration with Lagrangian boundary condition obtained by parallel transport of the vanishing cycle around a circle, rather than a line considered here.

We compare the indices of the map with its projection to the base of the standard Lefschetz fibration. Let $u : S \to X = \mathbb{C}^n$ be a map with boundary in H_γ. The composition $\pi \circ u$ of u with the Lefschetz fibration $\pi : \mathbb{C}^n \to \mathbb{C}$ of (8) produces a map $\pi \circ u$ from $\mathbb{H}$ to $\mathbb{C}$ with boundary condition $(\pi \circ u)(\partial S) \subset \mathbb{R} + i2\epsilon$. The map $\pi \circ u$ is an isomorphism from $\mathbb{H}$ to $\mathbb{H} + i2\epsilon$ by assumption. After composing on the right with the shift $z \mapsto z + i2\epsilon$ and an automorphism of $\mathbb{H}$, the map u becomes a section of the Lefschetz fibration:

$$\pi \circ u(z) = z, \quad \forall z \in \mathbb{H} + i2\epsilon.$$

Thus the components $u_j, j = 1, \ldots, n$ of the map

$$u : (\mathbb{H}, \partial\mathbb{H}) \to (\mathbb{C}^n, H_\gamma)$$

satisfy equations

$$u_j(z + i2\epsilon) \in (z + i2\epsilon)^{1/2}\mathbb{R}, \quad z \in \mathbb{R}.$$

The rank one problems in the previous paragraph are easily solvable. A change in sign of ϵ is equivalent to an interchange of the types $\mathbb{\Gamma}_\pm$, so it suffices to consider the case of maps whose image is half-plane above the line $\mathrm{Im}(z) = \epsilon$ and consider the cases $\epsilon > 0$ and $\epsilon < 0$ respectively. The components u_j are solutions to a rank one boundary value problem of index zero resp. one in the case $\epsilon > 0$ resp. $\epsilon < 0$. Each component u_j of u must be of the form for $z \in \mathbb{H}$

$$u_j(z) = \begin{cases} a_j(z + i2\epsilon)^{1/2} & \epsilon > 0 \\ (a_j z + b_j)(z - i2\epsilon)^{-1/2} & \epsilon < 0 \end{cases} \tag{146}$$

for some $a_j \in \mathbb{R}_{>0}$ resp. $a_j \in \mathbb{R}_{>0}, b_j \in \mathbb{R}$. One can check explicitly that each such u is a solution to the given boundary value problem: In the first case $\epsilon > 0$ the map has the required boundary values by inspection while in the second case we have for $x \in \mathbb{R}$,

$$\begin{aligned} u_j(x)(x+i2\epsilon)^{-1/2} &= (a_j x + b_j)(x+i2\epsilon)^{-1/2}(x-i2\epsilon)^{-1/2} \\ &= (a_j x + b_j)(x^2+4\epsilon^2)^{-1/2} \in \mathbb{R}. \end{aligned}$$

Rank one Cauchy-Riemann operators on a disk with non-negative index are always regular [64, Section 5]. It follows that (146) gives all the solutions.

The constants are fixed by requiring that the given map is a section of the Lefschetz fibration over its projection to the base. Solving for the condition $\pi u(z) = z$, that is, $u(z)$ is a section of the Lefschetz fibration, we obtain

$$\begin{cases} a^2 = 1 & \epsilon > 0 \\ a^2 = 1, \quad a \cdot b = 0, \quad b^2 = \epsilon^2 & \epsilon < 0. \end{cases} \tag{147}$$

Indeed, if $\epsilon < 0$ then

$$\pi u(z) = z \iff (a(z - i2\epsilon) + b)^2 = (z - i4\epsilon)z \tag{148}$$

$$\iff \left(\begin{array}{rcl} a^2 &=& 1 \\ 2a \cdot b - 4a^2\epsilon i &=& -2\epsilon i \\ -4\epsilon^2 a^2 - 4\epsilon i a \cdot b + b^2 &=& 0 \end{array} \right). \tag{149}$$

The equations (149) are equivalent to the equations

$$a^2 = 1, \quad a \cdot b = 0, \quad b^2 = 4\epsilon^2.$$

A similar computation computes the kernel of the linearization. In the first case $\epsilon > 0$, the kernel at a is the set of solutions a' to $a'a = 0$, and so has dimension $n-1$. In the second case $\epsilon < 0$, the kernel of the linearization at (a, b) is the set of solutions (a', b') to

$$a' \cdot a = 0, \quad a' \cdot b + a \cdot b' = 0, \quad b \cdot b' = 0$$

and so dimension $2n - 3$.

An index computation implies that the cokernel is trivial. Indeed, for $\epsilon > 0$ the bundle $u^*(\mathbb{C}^n \to \mathbb{C})$ is a trivial symplectic fibration. It follows the vertical part of the index is equal to the dimension of the boundary condition, that is, $n - 1$. On the other hand, if $\epsilon < 0$ then the index problem is related to that obtained by a connect sum with the index problem over the disk, which has index $2n - 3$ by Seidel [76, Proof of Lemma 2.16]. Since the horizontal index is the same as the dimension of the space of automorphisms of the domain, the total index is $2n - 3$ as well. Triviality of the cokernel implies that the moduli spaces are transversally cut out by the equations (147). The equations give a sphere of dimension $n - 1$ in the first case, and fibration in the second case with spherical fibers of dimension $n - 2$.

It remains to prove the claim on the intersection with a generic line on the handle. Given $c \in S^{n-1}$ and $a \in S^{n-1}$ such that $c \neq a, -a$, there exist unique $x \in \mathbb{R}$ and $b \in S^{n-1}$ with $a \cdot b = 0$ such that

$$\frac{u(x+i2\epsilon)}{|u(x+i2\epsilon)|} = \frac{ax+b}{((ax)^2+b^2)^{1/2}} = c.$$

Indeed, the set of points

$$\Big\{(ax+b)/\|ax+b\|, \quad x \in \mathbb{R}, b \in \operatorname{span}(a)^\perp\Big\}$$

is the complement of the two poles $a, -a$ in S^{n-1}. The claim follows.

To prove the claim about primitivity, we classify the possible homology classes. The second relative homology group $H_2(\mathbb{C}P^n, \overline{H}_\gamma) \cong H_2(\mathbb{C}P^n - \{0\}, \overline{H}_\gamma)$ can be computed by Mayer-Vietoris. Write $\mathbb{C}P^n = \mathbb{C}^n \cup \mathbb{C}P^{n-1}$ and consider the cover of $\mathbb{C}P^n$ by the open sets $(\mathbb{C}^n - \{0\})$ and $\mathbb{C}P^n - B_R(0))$ where $B_R(0)$ is a ball around $0 \in \mathbb{C}^n$ of radius R. The classes corresponding to the first homology of the intersection are generated by disks in the line $\mathbb{C}$ with boundary in $H_\gamma \cap \mathbb{C}$ that have area $\pi/2 \pm A(\epsilon)$. The remaining classes arise from the classes of disks and spheres in $\mathbb{C}P^{n-1}$ with boundary in $\mathbb{R}P^{n-1}$, which have areas equal to multiples of π. It follows that there are no decompositions of the classes with areas $\pi/2 \pm A(\epsilon)$ into classes with positive smaller areas, so that the homology classes of the maps in the Proposition are primitive.

To prove the claim on orientations in the dimension two case, we must compare the contributions from the two points u, u' in each fiber of the fibration of (145). The orientations $o(u), o(u')$ may be compared by deforming the Lefschetz fibration by bubbling off a disk containing the critical value of the Lefschetz fibration as in Seidel [76]. By the computation in [87, Proof of Corollary 4.31] the orientations of the two different elements u, u' in a single fiber agree. □

The areas of the disks on the handle and on the self-transverse Lagrangian are related by the area correction in Definition 2.1 and indicated (conceptually; the graph does not exactly match the definition) in Figure 2. As in the introduction we denote by ϕ_ϵ resp. ϕ_0 the surgered resp. unsurgered immersion.

Lemma 7.3. *Suppose that $u_0, u_\epsilon : S \to X$ are maps with boundary in ϕ_0 resp. ϕ_ϵ that are equal except in a neighborhood of a self-intersection point $x \in \mathcal{I}^{\mathrm{si}}(\phi_0)$ as in Figure 2; and suppose that in a neighborhood of the surgery, the map u_ϵ is obtained by by replacing a right-way resp. wrong-way corner in u_0 by its smoothing above. Then the areas of u_ϵ and u_0 are related by*

$$A(u_\epsilon) = A(u_0) - (\kappa - \overline{\kappa})A(\epsilon)$$

where $\kappa \in \mathbb{Z}_{\geq 0}$ resp. $\overline{\kappa} \in \mathbb{Z}_{\geq 0}$ is the number of times u_0 passes through x resp. $\overline{x}$.

Proof. We compute the difference in areas using Stokes' theorem. The symplectic form ω_0 on $\mathbb{C}^n$ is exact with bounding cochain

$$\alpha_0 = \sum_{j=1}^{n} \frac{1}{2}(q_j \mathrm{d}p_j - p_j \mathrm{d}q_j), \quad \mathrm{d}\alpha_0 = \omega_0.$$

The restriction of α_0 to the Lagrangian branches $\mathbb{R}^n, i\mathbb{R}^n$ vanishes. The maps u_0, u_ϵ agree away from the corner, and the difference of u_0, u_ϵ in the relative homology with respect to $\phi_0 \cup \phi_\epsilon$ is the class of a map

$$v : \mathbb{R} \times [0,1] \to \mathbb{C}^n$$

bounding H_0, H_ϵ and constant outside a compact set $[-T,T] \times [0,1]$ with boundary

$$\gamma_0 : [-T,T] \to \phi_0(H_0), \quad \gamma_\epsilon : [-T,T] \to \phi_\epsilon(H_\epsilon).$$

The first path γ_0 travels from the negative to positive branches of $\phi_0(H_0)$, while the second γ_ϵ travels along $\phi_\epsilon(H_\epsilon)$ in the same direction, as in Figure 2. By Stokes' theorem, the area of v is

$$A(\epsilon) = \int_{\mathbb{R}\times[0,1]} v^*\omega = \int_{[-T,T]} \gamma_\epsilon^*\alpha_0 - \gamma_0^*\alpha_0. \tag{150}$$

independent of the homotopy class of the map v. □

7.2. **Ruling out disks with large angle in the unsurgered handle.** In this section we show that the only rigid curves for the unsurgered handle are those appearing in Proposition 7.2, that is, those curves with a single strip-like end asymptotic to a Reeb chord of minimal length. Given a map type of punctured surface $\mathbb{\Gamma}$, denote by $\mathcal{M}_{\mathbb{\Gamma}}(\phi_0)$ the moduli space of holomorphic treed disks bounding $H_0 = \mathbb{R}^n \cup i\mathbb{R}^n$ of type $\mathbb{\Gamma}$. Denote by $e(\bullet)$ resp. $e(\circ)$ the number of Reeb orbits resp. chords at infinity, and $d(\circ)$ the number of boundary leaves in total, so that $d_c(\circ) = d(\circ) - e(\circ)$ represents the number of boundary leaves not corresponding to Reeb chords. We have a natural evaluation map

$$\mathrm{ev} : \mathcal{M}_{\mathbb{\Gamma}}(\phi_0) \to H_0^{d_c(\circ)} \times (S^{n-1})^{e(\circ)} \times \mathbb{C}P^{e(\bullet)}$$

which assigns to any configuration $(C, u : S \to X)$ the projection of the limiting Reeb orbits and chords, and evaluates the map at the intersection of the remaining leaves T_e with S. As in (96) let

$$\Sigma \subset H_0^{d_c(\circ)} \times (S^{n-1})^{e(\circ)} \times \mathbb{C}P^{e(\bullet)}$$

be a submanifold (later, Σ will be the image under the evaluation map of the "outside pieces" in the symplectic field theory decomposition) intersecting the evaluation map for $\mathcal{M}_{\mathbb{\Gamma}}(\phi_0)$ transversally. As in (97) define

$$\mathcal{M}_{\mathbb{\Gamma}}(\phi_0, \Sigma) = \mathrm{ev}^{-1}(\Sigma)$$

denote the moduli space of maps with the given constraints. We wish to know in what conditions $\mathcal{M}_{\mathbb{\Gamma}}(\phi_0, \Sigma)$ may be rigid, that is, of expected dimension zero.

We recall the classification of holomorphic maps of disks to the complex projective line. Let $L \subset S^2$ be an embedded circle. For the case $\gamma(t) = t + i2\epsilon$, the Blaschke classification in [24] implies that after a change of coordinates any holomorphic disk u bounding L is of the form

$$u(z) = \left[c_- \prod_{i=1}^{d_-} \frac{z - a_{i,-}}{1 - z\overline{a}_{i,-}}, c_+ \prod_{i=1}^{d_+} \frac{z - a_{i,+}}{1 - z\overline{a}_{i,+}} \right] \tag{151}$$

for some $c_\pm$ with

$$|c_\pm| = 1, \quad a_{i,\pm} \in \mathbb{C}, \quad d_\pm \in \mathbb{Z}_{\geq 0}.$$

Returning to the case of arbitrary paths, we introduce the following notation for topological type. Identify $H_2(\mathbb{C}P^1, \mathbb{R}P^1) \cong \mathbb{Z}^2$. Let

$$d = (d_-, d_+) \in \mathbb{Z}^2$$

denote the *degree* of the composition $\pi \circ \overline{u}$ given by the image $u_*([S, \partial S])$ in the relative homology group $H_2(\mathbb{C}P^1, \mathbb{R}P^1) \cong \mathbb{Z}^2$. For each point $z \in \overline{S}$ mapping to infinity ∞, let $m_\circ(z), m_\bullet(z)$ denote the "multiplicity" of the corresponding Reeb chord or orbit, indicating how many Taylor coefficients vanish at z. Thus $m_\bullet(z)$ are the intersection numbers with the divisor at infinity Y at z, while $m_\circ(z) = k$ if the Reeb chord represents an angle change of $k\pi$ for some $k \in \mathbb{Z}_{\geq 1}$.

Lemma 7.4. *For a holomorphic map $u : S \to \mathbb{C}$ bounding $\mathbb{R}$ asymptotic to Reeb chords and orbits, the Fredholm index of the linearized operator D_u is*

$$\operatorname{Ind}(D_u) = 1+2(d_-+d_+) - \sum_{z \in \overline{u}^{-1}(\infty) \cap \partial S} (m_\circ(z)-1) - \sum_{z \in \overline{u}^{-1}(\infty) \cap \operatorname{int}(S)} 2(m_\bullet(z)-1) \tag{152}$$

Proof. The Maslov index of the map $\overline{u}$ is $I(\overline{u}) = 2(d_- + d_+)$. The Fredholm index is $\operatorname{Ind}(D_{\overline{u}}) = 1 + 2(d_- + d_+)$ by Riemann-Roch. The tangency requirements lower the index by $m_\circ(z) - 1$ resp. $2(m_\bullet(z) - 1)$ at each Reeb chord or orbit; indeed, the spectrum of the tangential operator at infinity may be identified with the integers and the change in index following from change in Sobolev decay constant λ follows as in Lockart-McOwen [60]. □

Lemma 7.5. *Let $n \geq 2$. Assume that the standard complex structure makes every punctured holomorphic disk or sphere $u : S \to \mathbb{C}^n$ bounding $\mathbb{R}^n \cup i\mathbb{R}^n$ regular. Then any rigid such map is an isomorphism from a disk onto a quadrant $(\mathbb{R}_{\geq 0} + i\mathbb{R}_{\geq 0})v$ in* $\operatorname{span}(z) \cong \mathbb{C} \subset \mathbb{C}^n$ *for some $v \in \mathbb{R}^n \subset \mathbb{C}^n$.*

Proof. We use the fact that the boundary condition is invariant under dilation. The action of $\lambda \in \mathbb{R}^\times$ on $\mathbb{C}^n$ induces a one-parameter family of punctured curves λu with the same evaluation at infinity. Suppose u is rigid. The dilation λu must equal the composition $u \circ \phi_\lambda$ for some automorphism $\phi_\lambda : S \to S$. Thus, in particular S has a non-trivial automorphism, at most two strip-like or cylindrical ends, and image contained in a line in $\mathbb{C}^n$, and we are in the one-dimensional case considered in (152). For holomorphic spheres, the degree equals the signed count of points in a fiber $u^{-1}(\zeta)$ over a point ζ near ∞, so each puncture with multiplicity $m(z)$ contributes $m(z)$ to the degree. Hence, for spheres u we have

$$\mathrm{Ind}(D_u) \geq 2 + (d_- + d_+).$$

Since the automorphism group is at most dimension four, rigidity forces $(d_-, d_+) = (1,1)$. In this case, the map $u(z) = v_1 z$ is linear with first derivative given by some $v_1 \in \mathbb{C}^n$. The map u admits a deformation $u_t(z) = v_1 z + t v_0$ with v_0 not in the span of v_1, and so is not rigid.

Thus the domain is a disk, and we claim that the map is linear. Since the image of the boundary is $\mathbb{R}$-invariant, $u(\partial S)$ must intersect infinity and so u has at least one strip-like end. If u has only one strip-like end (mapping either to 0 or to ∞) then u is a disk in $\mathbb{C}P^1 \cong \mathrm{span}(c) \cup \{\infty\}$ bounding $\mathbb{R}P^1$ or $i\mathbb{R}P^1$. By (152), u cannot be rigid. Thus S has two strip-like ends, and by homogeneity one must map to the corner $\{0\} = \mathbb{R}^n \cap i\mathbb{R}^n$. The map $u \cdot u$ has real boundary conditions, so u is of the form

$$u(z) = \sqrt{a(z)}v, \quad \mathrm{Im}(z) \geq 0$$

for some $v \in \mathbb{R}^n$ and polynomial $a(z)$. Such maps are rigid only if $a(z)$ has degree one, since otherwise the variation in the sub-leading coefficients in $a(z)$ preserving the condition $a(0) = 0$ produces a non-trivial variation of the map. □

In order to prove a similar result for treed disks, we assume the cellular deformation of the diagonal is of a specific form given by a family of dilations.

Lemma 7.6. *Assuming the standard complex structure on $\mathbb{C}^n$ is regular, for any rigid treed level $(C, u : S \to \mathbb{C}^n[k])$ bounding $\mathbb{R}^n \cup i\mathbb{R}^n$ with at least one component in $\mathbb{C}^n$, the domain S is connected with image $u(S)$ equal to a quadrant in a one-dimensional subspace of $\mathbb{C}^n$.*

Proof. The general linear group acts on the set of disks with the given boundary condition. Let $u : S \to \mathbb{C}^n$ be a rigid treed disk for the given constraint Σ with components $u_v : S_v \to \mathbb{C}^n, v \in \mathrm{Vert}(\Gamma)$. Any deformation (ξ_v) of the maps u_v by, for example, a dilation by $\lambda \in \mathbb{R} - \{0\}$ extends to a dilation of u, by translating the remaining components $u_{v'}, v' \neq v$ by elements of $GL(n, \mathbb{R})$ so that the matching conditions hold. It follows that each component u_v must be rigid separately. Each u_v is therefore an isomorphism onto a quadrant in some line $\mathrm{span}(c) \subset \mathbb{C}^n$. The attaching points w_e of the edges T_e to the surface part S are necessarily invariant under the action of $\mathbb{R}^\times$. Each such w_e must be a point on the boundary ∂S with $u(w_e) = 0 \in \mathbb{C}^n$. That is, the non-constant components of u must be quadrants, with adjacent components joined by nodes mapping to $0 \in \mathbb{C}^n$. (For example, the domain S of u could consist of two components S_{v_1}, S_{v_2}, with the first component

S_{v_1} mapping onto the first quadrant, and the second S_{v_2} mapping onto onto the second quadrant.) The edge length $\ell(e)$ of the edge T_e connecting two such quadrants is free to vary, so the configuration u cannot be rigid. $\square$

It remains to justify the regularity assumption.

Lemma 7.7. *The standard complex structure on $\mathbb{C}^n$ makes all treed levels $(C, u : S \to \mathbb{C}^n[k])$ bounding the unsurgered handle $\mathbb{R} \cup i\mathbb{R}^n$ regular.*

Proof. We use the fact that the boundary value problem for the unsurgered problem splits into one-dimensional problems. Let $u : S \to \mathbb{C}^n$ be a holomorphic map from a connected surface S to $\mathbb{C}^n$ asymptotic to some collection $\underline{\vartheta}$ of Reeb chords and orbits. The pair $(\mathbb{C}^n, \mathbb{R}^n \cup i\mathbb{R}^n)$ splits into one-dimensional problems $(\mathbb{C}, \mathbb{R} \cup i\mathbb{R})$, each invariant under the action of dilation. It follows that each summand has a non-trivial element of the kernel and vanishing cokernel. Thus any such holomorphic map u is regular.

The case of disconnected domain requires an induction. Let $u : S \to \mathbb{C}^n[k]$ be a treed level with surface components $S_v, v \in \operatorname{Vert}(\Gamma)$ and line segments $T_e, e \in \operatorname{Edge}(\Gamma)$; recall that the notation $\mathbb{C}^n[k]$ means that u has various components some of which map to neck pieces $\mathbb{C}^n - \{0\}$. The moduli space of configurations of type $\mathbb{\Gamma}$ with any given set of finite edge lengths $\ell(e)$ is transversally cut out. Let $\mathcal{M}'_{\mathbb{\Gamma}}(\phi_0)$ denote the moduli space of configurations with no matching conditions at the nodes. Denote by Δ_t the image of δ_t. It suffices to show that the evaluation map $\mathcal{M}'_{\mathbb{\Gamma}}(\phi_0) \to (S^{n-1})^{\#(T \cap S)}$ is transverse to the diagonal $\Delta_t^{\#(T \cap S)/2}$. This follows from an inductive argument: If e is the only finite edge adjacent to a vertex v and $\Gamma(v) \subset \Gamma$ is the subgraph of edges containing v then one sees from the previous paragraph that the map $\mathcal{M}'_{\mathbb{\Gamma}(v)}(\phi_0) \to L$ is a submersion at $T_e \cap S_v$. Removing v and e one obtains a tree Γ_1 with fewer vertices and edges, and the claim follows from the inductive assumption of transversality for such trees. $\square$

Remark 7.8. Alternatively, one may prove regularity using a long exact sequence. Let $u : S \to \mathbb{C}^n$ be a punctured holomorphic disk or sphere avoiding $0 \in \mathbb{C}^n$. Let

$$p : \mathbb{C}^n - \{0\} \to \mathbb{C}P^{n-1}$$

denote the projection. Consider the short exact sequence of vector bundles

$$u^* T^v \mathbb{C}^n \to u^* T\mathbb{C}^n \to u^* p^* T\mathbb{C}P^{n-1}$$

where the vertical sub-bundle $T^v\mathbb{C}^n$ is the kernel of Dp. Denote by D_u^v, D_u^h the vertical and horizontal parts of the linearized operator D_u. The short exact sequence of complexes induces a long exact sequence of kernels and cokernels denoted

$$\begin{tikzcd}
\operatorname{coker}(D_u^v) \arrow[r] & \operatorname{coker}(D_u) \arrow[r] & \operatorname{coker}(D_u^h) \\
\ker(D_u^v) \arrow[r] & \ker(D_u) \arrow[r] & \ker(D_u^h) \arrow[ull]
\end{tikzcd} . \tag{153}$$

As in Proposition 6.16, the kernel and cokernel of $D_{p\circ u}$ may be identified with those of its extension $D_{\overline{p\circ u}}$ obtained by adding in the points at infinity along the cylindrical and strip-like ends. It is a standard consequence of homogeneity of $\mathbb{C}P^{n-1}$ that the cokernel of such operators vanish: Since $\overline{p\circ u}^* T\mathbb{C}P^{n-1}$ is generated by the image of $\mathfrak{o}(n)$ in $\ker(D_{\overline{p\circ u}})$ at any point, $\overline{p\circ u}^* T\mathbb{C}P^{n-1}$ must be a sum of a line bundles with boundary conditions with positive Maslov index as in Oh [64]. Since the cokernel of a Cauchy-Riemann operator on any one-dimensional problem with positive Maslov index vanishes [64], the cokernel of $D_{p\circ u}$ vanishes. Similarly, the vertical bundle $u^*T^v(\mathbb{C}^n - \{0\})$ has a section given by the action of $\mathbb{R}^\times$ on $\mathbb{C}^n - \{0\}$ which preserves the boundary conditions. It follows that $\operatorname{coker}(D_u^v)$ also vanishes. By the long exact sequence (153), $\operatorname{coker}(D_u)$ vanishes as well.

7.3. **Ruling out disks with large angle in the surgered handle.** We now wish to show a similar statement for curves bounding the surgered handle, namely that rigidity forces strip-like ends of minimal length. We first introduce some notation. Consider the natural projections

$$\pi : \mathbb{C}^n \to \mathbb{C}, \quad (z_1, \ldots, z_n) \mapsto z_1^2 + \ldots z_n^2$$

and

$$p : \mathbb{C}^n - \{0\} \to \mathbb{C}P^{n-1}, \quad z \mapsto \operatorname{span}(z).$$

The null-cone in $\mathbb{C}P^{n-1}$ is

$$N := p(\pi^{-1}(0)) = \{[z_1, \ldots, z_n], z_1^2 + \ldots + z_n^2 = 0\}.$$

Given a holomorphic map bounding the handle, we obtain maps by composition with either projection. Namely let $u : S \to \mathbb{C}^n$ be a smooth map with boundary on the handle H_γ and avoiding 0. The composition $\pi \circ u$ maps to $\mathbb{C}$ with boundary in γ, while the composition $p \circ u$ maps to $\mathbb{C}P^{n-1}$ with boundary in $\mathbb{R}P^{n-1}$. We wish to compare the indices of the linearized operator of the map u with that of its projections. Because the map u may limit to the null-cone along the strip-like ends, the composition $\pi \circ u$ may have finite limits along any particular end.

We introduce convenient terminology corresponding to the "multiplicity" of the map at infinity. For each point $z \in S$ mapping to infinity in $\overline{X}$, let $m_\circ(z), m_\bullet(z)$ denote the “multiplicity” of the corresponding Reeb chord or orbit, indicating which eigenvector appears in (101). Thus $m_\bullet(z)$ are the intersection numbers with the divisor at infinity Y at z, while $m_\circ(z) = k$ if the Reeb chord represents an angle change of $k\pi$ for some $k \in \mathbb{Z}_{\geq 1}/2$. Define $D^m_{\pi\circ u}$ to be the same differential operator as $D_{\pi\circ u}$, but using a Sobolev weight on the ends which is slightly smaller than $-2m_\bullet(z_i')$ so that sections with poles of order $2m_\bullet(z_i')$ have finite Sobolev norm. Recall that $D_{\pi\circ u}$ denotes the operator acting on the weighted Sobolev space with a negative weight, which disallows poles.

Lemma 7.9. *Let $u : S \to \mathbb{C}^n - \{0\}$ be a holomorphic map bounding H_γ as above. The map $D\pi$ induces an isomorphism $\ker(D_u^h) \cong \ker(D^m_{\pi\circ u})$. In particular, $\ker(D_{\pi\circ u})$ injects into $\ker(D_u^h)$ with equality if there are no cylindrical ends.*

Proof. Let $\xi \in \ker(D_u)$. Identifying $u^*T\mathbb{C}^n$ with $S\times\mathbb{C}^n$, the maps ξ and u have at worst poles of order $m_\bullet(z_i')$, in standard coordinates on $\mathbb{C}^n$. So $D_u\pi\xi = u\cdot\xi$ has, at worst, a pole of order $2m_\bullet(z_i')$. Conversely, any section ζ of $(\pi\circ u)^*T\mathbb{C}$ with these growth conditions pulls back to a section ξ of $u^*T^h\mathbb{C}^n \cong S\times\mathbb{C}^n$ which has a pole of order $m_\bullet(z_i')$. Adjusting for the trivialization over the cylindrical end $\mathbb{R}_{>0}\times S^{2n-1}\to\mathbb{C}^n$ which is geodesic exponentiation from u in the product metric, the section ξ is asymptotic to a constant at infinity and so lies in the domain of D_u^h. □

To compare indices we introduce a long exact sequence. We have a short exact sequence of bundles over $(S,\partial S)$

$$0\to(T^v\mathbb{C}^n,T^vH_\gamma)\to(T\mathbb{C}^n,TH_\gamma)\to(T^h\mathbb{C}^n,T^hH_\gamma)\to 0. \tag{154}$$

The short exact sequence of bundles (154) induces a short exact sequence of complexes of 0 and 0,1-forms. We denote by D_u^v, D_u^h the vertical and horizontal parts of the linearized operator D_u, and $\tilde{D}_u^h$ the parametrized linear operator for the horizontal part. Consider the map

$$D\pi : u^*TX \to (\pi\circ u)^*T\mathbb{C}.$$

The short exact sequence of complexes induces a long exact sequence of kernels and cokernels given by

$$\begin{tikzcd} \operatorname{coker}(D_u^v) \arrow[r] & \cdots & \\ \ker(D_u^v) \arrow[r] & \ker(D_u) \arrow[r] & \ker(D_u^h) \arrow[ull] \end{tikzcd} \tag{155}$$

The long exact sequence equally holds with the parametrized linear operator $\tilde{D}_u$ replaced with the unparametrized linear operator D_u whose domain does not allow variations of the conformal structure, by the same argument. Define

$$m^{\mathrm{ex}}(z) = \begin{cases} 2m_\circ(z)-1 & \text{if } z \text{ represents a strip-like end, or} \\ 4m_\bullet(z)-2 & \text{if } z \text{ represents a cylindrical end.}\end{cases}$$

Proposition 7.10. *Let $u : S^\circ \to \mathbb{C}^n$ be a finite energy holomorphic map bounding H_γ whose evaluations $\mathrm{ev}_e(u)$ at cylindrical ends avoid the null-cone N. The Fredholm index of the horizontal part D_u^h of the linearized operator D_u is by (152)*

$$\mathrm{Ind}(D_u^h) = 1 + 2(d_+ + d_-) - \sum_{z \in \overline{u}^{-1}(Y)} m^{\mathrm{ex}}(z)$$

while the index of the vertical part D_u^v is

$$\mathrm{Ind}(D_u^v) = (n-1) + d_-(n-2).$$

Proof. Denote the extensions of Lemma 6.16 by $((\pi \circ u)^* T\mathbb{C}, (\pi \circ u)^* T\gamma(\mathbb{R}))_c$ over $(\mathbb{C}P^1, \overline{\gamma(\mathbb{R})})$. The Maslov index of the horizontal part is

$$I((\pi \circ u)^* T\mathbb{C}, (\pi \circ u)^* T\gamma(\mathbb{R}))_c = 2(d_+ + d_-) - \sum_{z \in \overline{u}^{-1}(Y)} m^{\mathrm{ex}}(z).$$

To compute the vertical part of the index, note that the map u may be viewed as a section of the pull-back of the Lefschetz fibration $\pi : \mathbb{C}^n \to \mathbb{C}$ under $(\pi \circ u)$. The "bubbling off singularities" computation in Seidel [75, p. 253] implies that each bubble containing a singularity contributes the Maslov index of the corresponding holomorphic map to the disk, computed in Seidel [76, Proof of Lemma 2.16] to equal $n-2$. □

Lemma 7.11. *Let $\gamma : \mathbb{R} \to \mathbb{C}$ be an asymptotically cylindrical path and $u : S \to \mathbb{C}$ a level bounding ϕ_γ. Then the parametrized linearized operator $\tilde{D}_u$ is surjective. For the standard path γ, the unparametrized linear operator D_u is surjective.*

Proof. We first prove the last claim for the standard path. The kernel and cokernel are identified via Lemma 6.16 with a one-dimensional real boundary value problem. Note that $(\mathbb{C}P^1, \mathbb{R}P^1)$ has a family of automorphisms ϕ_s preserving ∞. The derivative $\frac{d}{ds}|_{s=0} \phi_s \circ u$ gives a non-trivial element of the kernel $\ker(D_u)$. Hence $\mathrm{coker}(D_u)$ vanishes, since in rank one either the kernel or cokernel of any Cauchy-Riemann operator vanishes by Oh [64].

For arbitrary paths that are asymptotically cylindrical, the statement of the Lemma regarding the parametrized linear operator is an instance of automatic regularity in dimension one as in Seidel [75, Lemma 13.2]. The cokernel of $\tilde{D}_u$ may be identified with the kernel of the adjoint operator. Necessarily, any $\eta \in \mathrm{coker}(\tilde{D}_u)$ is perpendicular to variations of the form $(J du \alpha)^{0,1}$ produced by variations

$$T_j \mathcal{J}(S) = \{\alpha \in \Omega^0(S, \mathrm{End}(TS)) \mid j\alpha + \alpha j = 0\}$$

of the conformal structure on $(S, \underline{z}, \underline{z}')$, with notation from (17). By definition the moduli space of curves is a slice for the action of the diffeomorphism group on the space of conformal variations

$$T_j \mathcal{J}(S) = T_S \mathcal{M} \oplus \mathrm{Vect}(S, \underline{z}, \underline{z}')$$

with

$$\mathrm{Vect}(S, \underline{z}, \underline{z}') := \{v \in \mathrm{Vect}(S) | v(z) = 0, \forall z \in \underline{z} \cup \underline{z}'\}$$

where if S unstable then $T_S\mathcal{M}$ is defined to be trivial. The image of the operator $\tilde{D}_u$ is unchanged if one extends the domain to allow all deformations: Let $v \in \mathrm{Vect}(S)$ vanish at the points $\underline{z}, \underline{z}' \in u^{-1}(\infty)$. The variation of the complex structure and map with respect to v is given by elements

$$\alpha(v) \in \Omega^0(S, \mathrm{End}(TS)), \quad L_v u \in \Omega^0(S, u^* T\mathbb{P}^1)$$

with

$$(J\mathrm{d}u\alpha(v))^{0,1} - D_u L_v u = 0.$$

Since u has derivatives vanishing up to order $m(z_j)$ at each $z_j \in \underline{z}, \underline{z}'$, the derivative $L_\alpha u$ has derivatives vanishing up to order $m(z_j) - 1$. Similarly, for the points $z \in u^{-1}(\infty)$, if u has derivatives up to order $m(z)$ vanishing at z then $\mathrm{d}u$ has derivatives up to order $m-1$ vanishing at z. Since v vanishes at z, the derivative $L_v u$ has derivatives up to order m vanishing at z as well. Hence $L_v u$ defines an element in the domain of D_u. Thus, the term $(J\mathrm{d}u\alpha(v))^{0,1}$ lies in the image of D_u. Since $\mathrm{d}u$ is an isomorphism away from the finitely many critical points of u, there are no $0,1$-forms perpendicular to such variations $(J\mathrm{d}u\alpha)^{0,1}$ for all α. The claim follows. $\square$

Lemma 7.12. *For the standard complex structure on $\mathbb{C}^n$ and a boundary condition given by an asymptotically cylindrical path $\gamma : \mathbb{R} \to \mathbb{C}$, every holomorphic treed disk $u : S \to \mathbb{C}^n[k]$ with boundary on H_γ not meeting $0 \in \mathbb{C}^n$ is regular and the evaluation map at any point on the boundary $z \in \partial S$ has surjective linearization in $T_{u(z)}H_\gamma$.*

Proof. We will show that the vertical and horizontal parts of the linearized operator are both surjective. First let $\mathbb{\Gamma}$ be a combinatorial type of map with a single vertex v and strip-like and cylindrical ends S. Consider the moduli space $\mathcal{M}_{\mathbb{\Gamma}}(\phi_\gamma)$ of holomorphic maps $u : S \to X$ with boundary in H_γ. By the long exact sequence, to show regularity it suffices to show that the cokernels of D_u^v and $\tilde{D}_u^h$ vanish. The homogeneity argument implies that the higher cohomology of the vertical part vanishes: The action of $SO(n)$ on $\mathbb{C}^n$ preserves the complex structure and Lagrangians H_γ. Given a holomorphic disk $u : S \to \mathbb{C}^n$ with boundary on H_γ, one obtains an inclusion

$$\mathfrak{so}(n) \to \ker(D_{u^*T^v\mathbb{C}^n}), \quad \xi \mapsto \frac{d}{dt}|_{t=0} \exp(t\xi)u$$

by mapping each Lie algebra element $\xi \in \mathfrak{so}(n)$ to the corresponding infinitesimal deformation of the map. In particular, the evaluation map

$$\ker(D_{u^*T^v\mathbb{C}^n}) \to T^v H_\gamma, \quad \xi \mapsto \xi(z)$$

at any point $z \in \partial S^\circ$ is a submersion. By Lemma 7.11, $\mathrm{coker}(\tilde{D}_u^h)$ vanishes as well. The vertical part $(u^*T^v\mathbb{C}^n, (\partial u)^*T^v H_\gamma)$ has spanning sections at any point given by the action of $SO(n)$. This fact implies that the linearization of the evaluation map is surjective.

Similar arguments apply to the case of sublevels mapping to the neck piece, that is, components u_v mapping to $\mathbb{C}^n - \{0\}$ with boundary on $(\mathbb{R}^n \cup i\mathbb{R}^n) - \{0\}$. Given such a map, the projection $\pi \circ u_v$ is a map to $\mathbb{C}$ with boundary values on $\mathbb{R}$ with corners of u_v mapping to zeros of $\pi \circ u_v$. Such maps have surjective linearization by Lemma 7.7.

The previous paragraphs show that the moduli space $\mathcal{M}_{\mathbb{\Gamma}(v)}(\phi_\gamma)$ is cut out transversally for each vertex $v \in \mathrm{Vert}(\Gamma)$, and the evaluation map at any point on the boundary is a submersion. As in the proof of Lemma 7.7, an induction shows that the evaluation maps at the nodes from $H^0(u^*T\mathbb{C}^n)$ are transverse to the diagonal and so the moduli space $\mathcal{M}_{\mathbb{\Gamma}}(\phi_\gamma)$ is transversally cut out. □

Denote by $\mathcal{M}^f_{\mathbb{\Gamma}}(\phi_\gamma) \subset \mathcal{M}_{\mathbb{\Gamma}}(\phi_\gamma)$ the locus of levels u with $\pi \circ u$ having finite limit $\mathrm{ev}_e(\pi \circ u) \neq \infty$ along some cylindrical end $e \in \mathrm{Edge}_{\to,\bullet}(\Gamma)$. Let $\mathcal{M}^n_{\mathbb{\Gamma}}(\phi_\gamma) \subset \mathcal{M}_{\mathbb{\Gamma}}(\phi_\gamma)$ the locus of maps u with $\mathrm{ev}_e(u) \in N$ at a cylindrical end e.

Lemma 7.13. *For each type $\mathbb{\Gamma}$, the loci $\mathcal{M}^f_{\mathbb{\Gamma}}(\phi_\gamma), \mathcal{M}^n_{\mathbb{\Gamma}}(\phi_\gamma)$ are transversally cut out and codimension at least two. In particular, for rigid map types $\mathbb{\Gamma}$ the locus of maps having limit in the null-cone or with $\pi \circ u$ having a finite limit along some cylindrical end is empty.*

Proof. Let $u : S \to \mathbb{C}^n$ be a map so that $\pi \circ u : S \to \mathbb{C}$ has finite evaluation at a cylindrical end e. The locus of such maps has formal tangent space given by sections ξ of $u^*T\mathbb{C}^n$ in $\ker(\tilde{D}_u)$ such that $D\pi\xi$ is finite at the end. For simplicity, suppose there is a single such end with multiplicity $m_\bullet(z_1')$. Evaluating the coefficients of $z^{-1}, \dots, z^{-2m_\bullet(z_1')}$ of any section at the end induces a long exact sequence

$$\ker(\tilde{D}_{\pi\circ u}) \to \ker(\tilde{D}^h_u) \to (T_{u(z_1')}\mathbb{C})^{2m_\bullet(z_1)} \to \operatorname{coker}(\tilde{D}_{\pi\circ u}) \to \dots .$$

The connecting homomorphism $(T_{u(z_1')}\mathbb{C})^{2m_\bullet(z_1)} \to \operatorname{coker}(\tilde{D}_{\pi\circ u})$ maps the Taylor coefficients $(c_1, \dots, c_{2m_\bullet(z_1')})$ to the image of $\tilde{D}^h_u\xi$ in $\operatorname{coker}(\tilde{D}_{\pi\circ u})$ where ξ has the given coefficients in the expansion at z_1'. Since $\tilde{D}_{\pi\circ u}$ is surjective, the long exact sequence implies that the evaluation map ev_e at the cylindrical end from $\ker(\tilde{D}_{\pi\circ u})$ to $(u^*T\mathbb{C})_c$ has surjective linearization $D\,\mathrm{ev}_e$. It follows that the locus of maps u with $\mathrm{ev}_e(\pi \circ u) \neq \infty$ is transversally cut out. The first claim follows. The second claim follows from the last statement in Lemma 7.11. □

By Lemma 7.13 rigid maps have evaluations only in the complement of the null-cone, assuming transversality holds.

Remark 7.14. To give the reader an idea of what kind of maps occur, we classify such maps for the standard path $\gamma(t) = t + i2\epsilon$. We claim that any map u from the complement S of a finite set in $\mathbb{H}$ whose evaluations $\mathrm{ev}_e(u)$ along the cylindrical ends $e \in \mathcal{E}_\bullet(S)$ do not lie in the null-cone N is of the form

$$u(z) = b(z)^{-1/2} \prod_{\operatorname{Im}(\alpha_i)<0} (z-\alpha_i)^{1/2} \prod_{\operatorname{Im}(\alpha_i)>0} (z-\overline{\alpha_i})^{-1/2} (c_{d_-} z^{d_-} + \ldots + c_1 z + c_0).$$

for some constants $c_0, , \ldots, c_{d_-}$, polynomial $b(z)$ and complex numbers α_i. Indeed, suppose the domain of u is the punctured upper half-plane

$$S = \{\operatorname{Im}(z) \geq 0\} - \{z_1, \ldots, z_{e(\circ)-1}\} - \{z_1', \ldots, z_{e(\bullet)}'\}.$$

The difference $(\pi \circ u)(z) - i\epsilon$ is a rational function $a(z)/b(z)$ of z by the reflection principle. Since $(\pi \circ u)(z) - i\epsilon$ has real boundary values, there exist real polynomials $a(z), b(z)$ so that

$$(\pi \circ u)(z) = \frac{a(z)}{b(z)} + i2\epsilon.$$

By assumption $u(z)$ goes to infinity as $z \to \infty$ and the limit in $\mathbb{C}P^{n-1}$ is not in the null-cone. So

$$\deg(a) \geq \deg(b), \quad \deg((\pi \circ u) b) = \deg(a).$$

The composition $\pi \circ u$ admits a factorization

$$a(z) + i2\epsilon b(z) = c f_+(z) f_-(z)$$

where

$$f_\pm(z) := \prod_{\pm, i=1}^{d_\pm} (z - \alpha_{i,\pm})$$

has $d_\pm$ roots $\alpha_{i,\pm}$ in the lower resp. upper halfplane, that is, with $\mp \operatorname{Im}(\alpha_{i,\pm}) > 0$. Define

$$f_\pm^*(z) := \prod_{i=1}^{d_\pm} (z - \overline{\alpha}_{i,\pm})$$

and

$$\tilde{u} : \mathbb{H} \to \mathbb{C}^n, \quad z \mapsto u(z) f_+(z)^{-1/2} f_-^*(z)^{1/2} b(z)^{1/2}.$$

Since by assumption the limits of u along the punctures do not lie in the null-cone, the poles of $\pi \circ u$ are twice the order of those of u. Since these are the order of vanishing of $b(z)$, the map $\tilde{u}$ is a polynomial. The degree of $\tilde{u}$ is

$$\begin{aligned} \deg(\tilde{u}) &= \frac{1}{2}(\deg(\pi \circ u) + \deg(f_-) - \deg(f_+) + \deg(b)) \\ &= \frac{1}{2}(\deg(a) + \deg(f_-) - \deg(f_+)) = d_- \end{aligned}$$

and so of the claimed form. Conversely, u may be constructed from $\tilde{u}$ as follows. Assume that for z real

$$\tilde{u}(z) \cdot \tilde{u}(z) = f_-(z) f_-^*(z) = \prod_{\operatorname{Im}(\alpha_i)>0} |z - \alpha_i|^2.$$

This condition imposes $2d_- + 1$ real constraints on the coefficients of $\tilde{u}$. The map

$$
\begin{aligned}
(156) \qquad u(z) &:= f_+(z)^{1/2} f_-^*(z)^{-1/2} b(z)^{-1/2} \tilde{u}(z) \\
&= b(z)^{-1/2} \prod_{\mathrm{Im}(\alpha_i)<0} (z-\alpha_i)^{1/2} \prod_{\mathrm{Im}(\alpha_i)>0} (z-\overline{\alpha_i})^{-1/2} \\
&\quad (c_{d_-} z^{d_-} + \ldots + c_1 z + c_0)
\end{aligned}
$$

has the required boundary values. This ends the Remark.

Lemma 7.15. *For any type $\mathbb{\Gamma}$ of level in $\mathbb{C}^n$ bounding ϕ_γ, there exists a dense open subset of group elements*

$$g = (g_e)_{e \in \mathrm{Edge}_\to(\Gamma)} \subset U(n)^{\# \mathrm{Edge}_{\to,\bullet}(\Gamma)} \times O(n)^{\# \mathrm{Edge}_{\to,\circ}(\Gamma)}$$

for which the moduli spaces $\mathcal{M}_{\mathbb{\Gamma}}(\mathbb{C}^n - \{0\}, \phi_\gamma, g\Sigma)$ with translated constraints $g\Sigma$ are cut out transversally.

Proof. Let $\mathcal{M}_{\mathbb{\Gamma}}^{\mathrm{univ}}(\mathbb{C}^n - \{0\}, \phi_\gamma, \Sigma)$ denote the universal space of tuples (C, u, g) where (C, u) is a treed disk with contraints Σ and

$$g = \begin{pmatrix} g_e \in U(n), e \in \mathrm{Edge}_{\to,\bullet}(\Gamma) \\ g_e \in O(n), e \in \mathrm{Edge}_{\to,\circ}(\Gamma) \end{pmatrix}$$

for the interior resp. boundary semi-infinite edges. Transitivity implies that the universal space $\mathcal{M}_{\mathbb{\Gamma}}^{\mathrm{univ}}(\mathbb{C}^n - \{0\}, \phi_\gamma, \Sigma)$ is transversally cut out. The claim follows from Sard's theorem for the projection to (g_e). $\square$

Remark 7.16. In the above construction, we have shown that the moduli spaces of levels in $\mathbb{C}^n$ are already regular without using a domain-dependent almost complex structure. By an argument using Sard's theorem, there exists a complex hyperusrface $D_\subset$ in $\mathbb{C}P^n$ so that the moduli spaces $\mathcal{M}_{\mathbb{\Gamma}}(\phi_\gamma, D_\subset, \Sigma)$ are cut out transversally whenever $\mathbb{\Gamma}$ is an uncrowded type of expected dimension at most one. Let $f(\mathbb{\Gamma})$ be the type of map without interior edges obtained by forgetting the interior edges of $\mathbb{\Gamma}$. There is a forgetful map

$$\mathcal{M}_{\mathbb{\Gamma}}(\mathbb{C}^n - \{0\}, \phi_\gamma, D_\subset, \Sigma) \to \mathcal{M}_{f(\mathbb{\Gamma})}(\mathbb{C}^n - \{0\}, \phi_\gamma, \Sigma)$$

forgetting the interior edges which produces a bijection between the moduli spaces with and without the Donaldson hypersurface. For this reason, in this section we use treed disks rather than the adapted treed disks in the earlier chapters.

We introduce further notation on the combinatorial type. Assume that $\mathbb{\Gamma}$ has $e(\circ)$ boundary leaves representing strip-like ends asymptotic to Reeb chords, $d(\circ)$ boundary leaves representing possibly constraints and $e(\bullet)$ interior leaves mapping to infinity representing cylindrical ends asymptotic to Reeb orbits.

Lemma 7.17. *For any rigid map type $\mathbb{\Gamma}$ of treed disks $(C < u)$ bounding H_γ of bidegree* $\deg(\pi \circ u) = (d_-, d_+)$, *we have $d_+ \leq d_- - 1$.*

Proof. First consider the case without cylindrical ends. By the index formula 7.10, in the case without cellular constraints the moduli space $\mathcal{M}_{\mathbb{\Gamma}}(\phi_\gamma)$ has dimension

$$\dim \mathcal{M}_{\mathbb{\Gamma}}(\mathbb{C}^n - \{0\}, \phi_\gamma) = 2d_+ + n(1+d_-) - 3 - \sum_{i=1}^{e(\circ)} (2m_\circ(z_i) - 1) \tag{157}$$

$$= 2d_+ + n(1+d_-) - 3 + d(\circ) - (d_- + d_+) \tag{158}$$

$$= d_+ + n + (n-1)d_- + d(\circ) - 3 \tag{159}$$

where (d_-, d_+) is the bidegree of $\pi \circ u$. Denote by $p(\mathbb{\Gamma})$ the corresponding type of map bounding $\mathbb{R}P^{n-1}$ obtained by replacing the homology classes in $\mathbb{C}^n$ with homology classes in $\mathbb{C}P^{n-1}$, and forgetting the intersection multiplicities at the ends. We have an induced map

$$p_{\mathbb{\Gamma}} : \mathcal{M}_{\mathbb{\Gamma}}(\mathbb{C}^n - \{0\}, \phi_\gamma) \to \mathcal{M}_{p(\mathbb{\Gamma})}(\mathbb{C}P^{n-1}, \mathbb{R}P^{n-1}), \quad u \mapsto p \circ u. \tag{160}$$

Denote by $\mathcal{M}_{\mathbb{\Gamma}}(\mathbb{C}^n - \{0\}, \phi_\gamma)$ the locus of maps disjoint from 0, that is, the critical locus of π. The dimension of the moduli space of maps to $\mathbb{C}P^{n-1}$ (dropping the constraints at the Donaldson hypersurface) is

$$\mathcal{M}_{p(\mathbb{\Gamma})}(\mathbb{C}P^{n-1}, \mathbb{R}P^{n-1}) = (n-1) + nd_- + d(\circ) - 3.$$

Comparing dimensions we see that the map from (160) satisfies

$$\ker(Dp_{\mathbb{\Gamma}}) \geq 1 + 2d_+ - \sum_{i=1}^{e(\circ)} 2m_\circ(z_i) \tag{161}$$

where $e(\circ)$ is the number of strip-like ends. We have $Dp_{\mathbb{\Gamma}} \subset \ker(\tilde{D}_u)$ and such deformations do not change the constraints. Thus rigidity forces $\ker(Dp_{\mathbb{\Gamma}}) = \{0\}$ and so

$$1 + 2d_+ \leq \sum_{i=1}^{e(\circ)} 2m_\circ(z_i) = d_+ + d_-. \tag{162}$$

Cylindrical ends contribute an additional term of $2e(\bullet)$ to the dimension of both moduli spaces and do not affect the computation. □

We now wish to classify which configurations in the local model may be rigid. Let $d_1(\mathbb{\Gamma})$ denote the number of cellular constraints labelled σ_1 and $e_m(\mathbb{\Gamma})$ the number of strip-like ends labelled with minimal length Reeb chords and with a non-trivial constraint.

Lemma 7.18. *Let $\gamma(t) = t + i2\epsilon$ be the standard path. Let $\mathbb{\Gamma}$ be a labelled type of treed holomorphic building $(C, u : S \to \mathbb{C}^n)$ bounding ϕ_γ. Assume that either $n = 2$ or $d_1(\mathbb{\Gamma}) \leq 2$, so that there are at most two boundary constraints labelled by the cell σ_1. If $\mathbb{\Gamma}$ is a rigid type then one of the following holds:*

(a) *The number of strip-like ends is $e(\circ) = 1$ and the Reeb chord at the puncture is minimal length, or*

(b) *the number of strip-like ends is* $e(\circ) \geq 2$ *and the numbers* $d_(\mathbb{\Gamma})$ *resp.* $e_m(\mathbb{\Gamma})$ *of contraints labelled* σ_1 *resp. strip-like ends asymptotic to minimal-length Reeb chords satisfies*

$$d_1(\mathbb{\Gamma}) + e_m(\mathbb{\Gamma}) \geq 3.$$

If $n = 2$ *then only the first possibility* (a) *occurs.*

Proof. We first rule out the case that the domain is a treed sphere, that is, the domain has empty boundary. In this case, the dimension of the moduli space of maps $\mathcal{M}_{\mathbb{\Gamma}}(\phi_\gamma, D)$ of any type $\mathbb{\Gamma}$ with Γ representing a top-dimensional stratum and map with degree d to $\mathbb{C}P^n$ is by Riemann-Roch

$$\begin{aligned}
\dim(\mathcal{M}_{\mathbb{\Gamma}}(\phi_\gamma, D)) &= 2d(n+1) + 2n - 6 - \sum_{i=1}^{e(\bullet)} 2(m_\bullet(z_i) - 1) \\
&= (2d+2)n - 6 + 2e(\bullet) \\
&\geq 2e(\bullet)(n+1) + 2n - 6.
\end{aligned}$$

Each cylindrical end has a constraint which cut down the dimension by at most $2(n-1)$. It follows that for $n \geq 2$ such types cannot be rigid.

Therefore, it suffices to consider the case that the domain has non-empty boundary, that is, the domain is a treed disk. Suppose first $n = 2$. The dimension of the moduli space from (157) is

$$d_+ + 2 + d_- + d(\circ) - 3 \geq 2d(\circ) - 1.$$

Since the constraints at the boundary leaves cut down the dimension by at most 1, the constrained moduli space is expected dimension at least $d(\circ) - 1$, and we must have $d(\circ) = 1$ for rigidity to hold.

In the case of arbitrary dimension, we first deal with the case that there are only boundary leaves, that is, no cylindrical ends. Each end e contributes some pair $(d_+(e), d_-(e))$ to the bidegree of the composition $\pi \circ u$, measured as the number of points in a fiber of $\pi \circ u$ near infinity, with $|d_+(e) - d_-(e)| \leq 1$.

Case 1: There is a single strip-like end. The dimension of the unconstrained moduli space from (157) is $d_+ + n + (n-1)d_- - 2$. In the case of at most one cellular constraint and one constraint at the Reeb chord, this dimension must be at most $2(n-1)$. Thus either

$$(d_-(e), d_+(e)) = (0,1) \quad \text{or} \quad (d_-(e), d_+(e)) = (1,0);$$

in the latter case by rigidity there is a constraint at the end and a cellular constraint labelled σ_1. Consider the case of two cellular constraints and one constraint at the end. Since the expected dimension is non-negative,

$$d_+ + n + (n-1)d_- - 2(n-1) \geq 0$$

which implies $d_+ + d_- \geq 2$. Lemma 7.17 implies $d_- \geq 2$. Since $d_-(e) - 1 \leq d_+(e)$, d_+ is at least one and the dimension with constaints is

$$d_+ + n + (n-1)d_- - 2(n-1) - n \geq 1 + n + 2(n-1) - 3(n-1) \geq 2.$$

Thus such configurations cannot be rigid.

Case 2: There there are two strip-like ends. Lemma 7.17 implies that $d_- \geq 1$, and if $d_- = 1$ then $d_+ = 0$. This is impossible if there are two ends since both have non-zero contributions $(d_-(e), d_+(e))$ to the bidegree. Suppose $d_- \geq 2$ and there is at most one cellular constraint. The dimension from (157) is then at least $3n-3$. There are at most two constraints each of which cuts down the dimension by at most $n-1$, and one cellular constraint which cuts down the dimension by $n-2$. Thus such configurations cannot be rigid.

Suppose there are two cellular constraints and no minimal length Reeb chords. Then one of the ends has $d_-(e) \geq 2$, so that $d_- \geq 3$, and both ends have $d_+(e) \geq 1$ so $d_+ \geq 2$. The unconstrained moduli space has dimension

$$d_+ + n + (n-1)d_- - 1 \geq 2 + n + 3(n-1) - 1 = 4n - 2.$$

Again, the cellular constraints and constraints at the chords cut down the dimension by at most $4(n-1)$ and cannot make the configuration rigid. If there is a minimal length Reeb chord, then $d_- \geq 2$ and $d_+ \geq 1$ and the dimension is at least

$$d_+ + n + (n-1)d_- - 1 \geq 1 + n + 2(n-1) - 1 = 3n - 1.$$

Two cellular constraints and one constraint at a strip-like end are not enough to produce rigidity, so both ends must have constraints.

Case 3a: There are three or more strip-like ends and there is no boundary constraint labelled σ_1. Suppose there are at most two ends e_1, e_2 labelled by minimal length Reeb chords ϑ_1, ϑ_2. If one of the other ends, say e_3 has $d_-(e_3) \geq 2$ then the dimension of the moduli space is at least

$$d_+ + n + (n-1)d_- + d(\circ) - 3 \geq d(\circ)n - 2$$

and so cannot be made rigid by adding constraints, each of which lowers the dimension by at most $n-1$. Lemma 7.17 implies that if there two minimal length ends then both have type $(1,0)$, so again the moduli space is at least $d(\circ)n - 2$, which cannot be made rigid. Hence at least three ends are labelled by minimal length Reeb chords.

To see that at least three of the minimal-length ends have a non-trivial constraint, suppose otherwise. Let $e_{\min}(\circ)$ be the number of ends asymptotic to a minimal-length Reeb chord. If one of the non-minimal-length ends has $d_-(e) > d_+(e)$, then the dimension of the constrained moduli space from (157) is at least

$$n + (n-1)(e(\circ) - e_{\min}(\circ) + 1) - (n-1)(e(\circ) - e_{\min}(\circ)) - 2(n-1) > 0$$

and cannot be rigid. Otherwise, one of the minimal-length-ends must have $d_-(e) > 0$ and the dimension of the constrained moduli space satisfies the same inequality.

Case 3b: There are at least three strip-like ends and there is a boundary constraint labelled σ_1. Assume there is one end e_1 labelled by a minimal length Reeb chord. If one of the other ends e_2 has $d_-(e_2) \geq 2$, or all the ends are labelled by non-minimal-length Reeb chords, then the dimension of the moduli space is at least $(e(\circ)+1)n-2$. The dimension with constraints is at least

$$(e(\circ)+1)n-2-(e(\circ)+1)(n-1) > 0$$

and so rigidity is impossible. So all of the non-minimal ends have type $(1,1)$ or $(1,2)$. Lemma 7.17 implies that the type of the minimal-length end cannot be $(0,1)$. On the other hand, if the minimal-length end has type $(1,0)$ then the dimension of the moduli space is at least $(e(\circ)+1)n-2$, which cannot be made rigid as before.

To see that at least two of the minimal-length ends have non-trivial constraint, suppose otherwise. If one of the non-minimal-length ends has $d_-(e) > d_+(e)$, then the dimension of the constrained moduli space is at least

$$n+(n-1)(e(\circ)-e_{\min}(\circ)+1)-(n-1)(e(\circ)-e_{\min}(\circ))-2(n-1) > 0$$

and cannot be rigid. Otherwise, one of the minimal-length-ends must have $d_-(e) > 0$ and the dimension of the constrained moduli space satisfies the same inequality.

Case 3c: There are at least three strip-like ends and there are two boundary constraints labelled σ_1. If none of the Reeb chords are minimal length then each end has $d_+(e) \geq 1$. Since $d_+ \leq d_- - 1$ we must have $d_- \geq e(\circ)+1$ and $d_+ \geq e(\circ)$. The dimension of the moduli space is at least

$$\begin{aligned} d_+ + n + (n-1)d_- + d(\circ) - 3 &\geq e(\circ) + n + (e(\circ)+1)(n-1) + d(\circ) - 3 \\ &= e(\circ)(n+2) + 2n - 4. \end{aligned}$$

The moduli space cannot be made rigid by constraints at the Reeb chords and two cellular constraints, since each reduces the dimension by at most $n-1$. The argument that at least one of the minimal-length ends has non-trivial constraint is similar to the previous cases.

Case 4: There are cylindrical ends. Each cylindrical end with multiplicity $m_\bullet(z_i')$ contributes $(2n+4)m_\bullet(z_i')$ to the degree terms $2d_+ + nd_-$ in the dimension formula (157). On the other hand, any constraint at the Reeb orbit cuts down the dimension by at most $\dim(\mathbb{C}P^{n-1}) = 2n-2$. Since

$$(2n+4)m_\bullet(z_i') - (2n-2) > 0,$$

the arguments of the previous three cases still apply. □

We require similar results for ruling out holomorphic disks mapping to $\mathbb{C}^n - \{0\}$ with boundary in $(\mathbb{R}^n \cup i\mathbb{R}^n) - \{0\}$. Consider a map u from S to $X_0 := \mathbb{C}^n - \{0\}$ with boundary on $\phi_0 : (\mathbb{R}^n \cup i\mathbb{R}^n) - \{0\} \to \mathbb{C}^n - \{0\}$ with multiple ends or non-minimal Reeb orbits. We assume that the deformation of the diagonal δ_t is the product of a deformation of the diagonal $\delta_t^{\mathbb{R}}$ of $\mathbb{R}$ and $\delta_t^{S^{n-1}}$ of S^{n-1}. In the Morse case, this corresponds to taking the Morse function and metric on the boundary condition of product form.

Lemma 7.19. *Suppose $(C, u : S \to \mathbb{X}_0[k_-, k_+])$ is a treed map bounding ϕ_0 and Σ is a collection of constraints so that all cellular constraints in ϕ_0 are equal to σ_1. Then (C, u) consists of a single component with image in a fiber of $\mathbb{C}^n - \{0\} \to \mathbb{C}P^{n-1}$.*

Proof. The proof is similar to that of Lemma 7.5 and uses the action of the group $\mathbb{R}_{>0}$ on $\mathbb{C}^n - \{0\}$ by dilations preserving the boundary condition $(\mathbb{R}^n \cup i\mathbb{R}^n) - \{0\}$ preserving the limits at infinity. Suppose the surface part S of the domain has components S_v connected by edges T_e. Suppose some component u_v is a component. of u. By dilation by a constant $c \in \mathbb{R}_{>0}$ one obtains a new map cu_v with the same boundary condition not isomorphic to the original map. Given two strips $S_v, S_{v'}$ joined by an edge T_e joined at points $z \in S_v, z' \in S_{v'}$, and given c the point $u(cz), u(c'z')$ lie in the image of the deformation of the diagonal for some c' with parameter given by the length of T_e. Indeed the points $u(cz), u(cz')$ lies in the deformation of the diagonal of S^{n-1} and $u(c'z')$ ranges over all such possible lifts of $u(cz')$, as c' varies. It follows that S has a single component that is a trivial strip. □

7.4. **Ruling out maps intersecting the critical locus.** In the previous section, we ruled out rigid levels with more than one end or non-minimal Reeb length assuming that the maps avoid the critical locus. To deal with the case that the map intersects the critical locus, we introduce a blow-up which allows us to extend the splitting of the tangent bundle into horizontal and vertical parts over the critical locus. Let

$$\mathrm{Bl}(\mathbb{C}^n) = \{(z, l) \in \mathbb{C}^n \times \mathbb{C}P^{n-1}, z \in l\}$$

denote the blow-up of $\mathbb{C}^n$ at 0, and let

$$p : \mathrm{Bl}(\mathbb{C}^n) \to \mathbb{C}P^{n-1}, \quad (z, l) \mapsto z$$

denote the natural projection. For any path γ avoiding 0, the boundary condition ϕ_γ naturally lifts to $\mathrm{Bl}(\mathbb{C}^n)$ and we denote the lift with the same notation. Removal of singularities defines a bijective correspondence between maps to the blow-up and to its projection. For some type $\mathbb{\Gamma}$ of level, let $\widetilde{\mathcal{M}}_{\mathbb{\Gamma}}(\mathrm{Bl}(\mathbb{C}^n), \phi_\gamma)$ denote the moduli space of tuples $(C, u, \underline{\zeta})$ of maps $u : S \to \mathrm{Bl}(\mathbb{C}^n)$ where (C, u) is a treed disk and with additional markings $\underline{\zeta} \subset S$ is a finite set describing the intersection set with the exceptional divisor in the sense that

$$u^{-1}(\mathbb{C}P^{n-1}) = \underline{\zeta}.$$

Similarly, let $\widetilde{\mathcal{M}}_{p(\mathbb{\Gamma})}(\mathbb{C}^n, \phi_\gamma)$ denote the moduli space of pairs $(u, \underline{\zeta})$ of maps $u : S \to \mathbb{C}^n$ with additional markings $\underline{\zeta}$ describing the intersection set with the exceptional divisor $u^{-1}(0) = \underline{\zeta}$; here $p(\mathbb{\Gamma})$ is the obvious type of level in $\mathbb{C}^n$ obtained by projecting the homology classes of the components. Composition with the projection

$$\widetilde{\mathcal{M}}_{\mathbb{\Gamma}}(\mathrm{Bl}(\mathbb{C}^n), \phi_\gamma) \to \widetilde{\mathcal{M}}_{p(\mathbb{\Gamma})}(\mathbb{C}^n, \phi_\gamma), \quad u \mapsto p \circ u$$

is a bijection, by removal of singularities.

Lemma 7.20. *For the standard complex structure on* $\mathrm{Bl}(\mathbb{C}^n)$*, every holomorphic treed disk* $u : S \to \mathrm{Bl}(\mathbb{C}^n)$ *bounding* ϕ_γ *asymptotic to some collection of Reeb orbits and chords at infinity is regular.*

Proof. The proof is essentially the same as that of Lemma 7.12. The splitting of $u^*T\,\mathrm{Bl}(\mathbb{C}^n)$ into vertical and horizontal parts, corresponding to the tangent space to $\mathbb{P}^{n-1}$ and $\mathcal{O}(-1)$, induces a long exact sequence involving the kernels and cokernels of D_u^v, D_u and $D_{p\circ u}$. Homogeneity implies vanishing of the vertical cokernel $\mathrm{coker}(D_u^v)$, while existence of a section implies vanishing of the horizontal cokernel $D_{p\circ u}$. The claim now follows from the long exact sequence similar to (153). □

Proposition 7.21. *For generic constraints* Σ*, there are no rigid treed levels* $u : S \to X$ *bounding* H_γ *passing through the critical locus* $0 \in \mathbb{C}^n$*.*

Proof. By Lemma 7.20 and a Sard-Smale argument, the moduli space of configurations in the blow-up $\widetilde{\mathcal{M}}_{\mathbb{\Gamma}}(\mathrm{Bl}(\mathbb{C}^n), \phi_\gamma, \Sigma)$ is transversally cut out and of expected dimension for generic translates of the constraints Σ as in Lemma 7.15. The relative Chern class of $\mathrm{Bl}(\mathbb{C}^n) \to \mathbb{C}^n$ is dual to $(1-n)[\mathbb{C}P^{n-1}]$, and so the Maslov index of u differs from that of $p \circ u$ by $2(1-n)$ times the intersection number of u with $\mathbb{C}P^{n-1}$. For $\underline{\zeta}$ non-empty, if $\mathcal{M}_{\mathbb{\Gamma}}(\phi_\gamma, \Sigma)$ is expected dimension zero then $\widetilde{\mathcal{M}}_{\mathbb{\Gamma}}(\phi_\gamma, \Sigma)$ is empty, and so the elements of $\mathcal{M}_{\mathbb{\Gamma}}(\phi_\gamma, \Sigma)$ have images disjoint from $0 \in \mathbb{C}^n$. □

Lemma 7.22. *For generic constraints* Σ*, for any type* $\mathbb{\Gamma}$ *of treed level the moduli space* $\mathcal{M}_{\mathbb{\Gamma}}(\phi_\gamma, \Sigma)$ *is cut out transversally and any rigid treed level* $u \in \mathcal{M}_{\mathbb{\Gamma}}(\phi_\gamma, \Sigma)$ *has image disjoint from* $0 \in \mathbb{C}^n$*.*

Proof. By Lemma 7.20, the moduli spaces $\widetilde{\mathcal{M}}_{\mathbb{\Gamma}}(\phi_\gamma, \Sigma)$ are regular. A standard codimension argument shows that $\widetilde{\mathcal{M}}_{\mathbb{\Gamma}}(\phi_\gamma, \Sigma)$ has dimension $2(n-1)$ less than that of $\mathcal{M}_{\mathbb{\Gamma}}(\phi_\gamma, \Sigma)$, with the obvious identification of map types. Indeed, the relative Chern class of $\mathrm{Bl}(\mathbb{C}^n) \to \mathbb{C}^n$ is dual to $(1-n)[\mathbb{C}P^{n-1}]$. Thus, the Maslov index of u differs from that of $p \circ u$ by $2(1-n)$ times the intersection number of u with $\mathbb{C}P^{n-1}$. For $\underline{\zeta}$ non-empty, if $\mathcal{M}_{\mathbb{\Gamma}}(\phi_\gamma, \Sigma)$ is expected dimension zero then $\widetilde{\mathcal{M}}_{\mathbb{\Gamma}}(\phi_\gamma, \Sigma)$ is empty, and the elements of $\mathcal{M}_{\mathbb{\Gamma}}(\phi_\gamma, \Sigma)$ have images disjoint from $0 \in \mathbb{C}^n$. □

7.5. **Comparing disks on the flattened and unflattened handles.** We wish to show that the unflattened handle is asymptotically cylindrical so that we may use Theorem 6.32.

Lemma 7.23. *The handle $H_\gamma \subset \mathbb{C}^n$ for the standard path $\gamma(t) = t + 2i\epsilon$ is asymptotically cylindrical.*

Proof. We first consider the case of dimension one targets. The image of the line $\mathbb{R} + i2\epsilon$ under the coordinate change $z \mapsto 1/z$ is a circle of diameter $1/2\epsilon$ with center at $i/4\epsilon$, described in coordinates $z = x + iy$ as the solution set to

$$(\mathbb{R} + i2\epsilon)^{-1} = \{x^2 + (y - (4\epsilon)^{-1})^2 = (4\epsilon)^{-2}\}.$$

It suffices to consider the case $\epsilon = 1/4$. The handle H_γ is the pre-image of $\mathbb{R} + i/2$ under the square map

$$(x, y) \mapsto (x^2 - y^2, 2xy).$$

Thus H_γ is given in coordinates near infinity by

$$\begin{aligned} \{(x^2 - y^2)^2 + ((2xy) - 1)^2 = 1\} &= \{y^4 - 2x^2y^2 + x^4 + 4x^2y^2 - 4xy + 1 = 1\} \\ &= \{y^4 + 2x^2y^2 + x^4 - 4xy = 0\}. \end{aligned}$$

Write $y = xz$. The equation

$$xz = (x^2z^2 + x^2)^2/4x = x^3(1 + z^2)^2/4$$

has a smooth solution of z in terms of x, since $z \mapsto z/(1 + z^2)^2$ is a diffeomorphism near $z = 0$. By symmetry, each branch is a smooth manifold with boundary at $x = y = 0$.

For higher dimension, the handle is the flow-out of the dimension one handle under the action of the special orthogonal group. Let $\overline{H}_\gamma^n \subset \mathbb{C}P^n$ denote the handle in dimension n. Consider the action of $O(n)$ on $\mathbb{C}P^n$ on the first n coordinates. The locus

$$\overline{H}_\gamma^1 = \{[z, 0, \ldots, 0, 1]\} \cap \overline{H}_\gamma^n$$

has stabilizer groups contained in $O(1) \times O(n-1)$. Thus we have a homeomorphism

$$\overline{H}_\gamma^n = O(n)\overline{H}_\gamma^1 \cong O(n) \times_{O(1)\times O(n-1)} \overline{H}_\gamma^1$$

which is a diffeomorphism away from the boundary. Since each branch of $\overline{H}_\gamma^1$ is a smooth submanifold with boundary, $\overline{H}_\gamma^n$ has a smooth structure for which the inclusion in $\overline{X}$ is an immersion. □

Proposition 7.24. *Let $\mathbb{\Gamma}$ be a primitive type. There exists an oriented cobordism from $\mathcal{M}_{\mathbb{\Gamma}}(\check{\phi}_\gamma, \Sigma)$ to $\mathcal{M}_{\mathbb{\Gamma}}(\phi_\gamma, \Sigma)$, where $\check{\phi}_\gamma : \check{H}_\gamma \to \mathbb{C}^n$ is the flattened embedding of* (14).

Proof. We will construct a cobordism between the two moduli spaces by viewing them as moduli spaces of curves bounding cleanly-intersecting Lagrangians. The closures of $\check{H}_\gamma$ and H_γ in $\mathbb{C}P^n$ are contained in cleanly-intersecting Lagrangians by Proposition 6.10. Furthermore, this extension of $\check{H}_\gamma$ is an exact deformation of H_γ, by the description from (13). Choose a family of Lagrangians H_γ^t interpolating between H_γ and $\check{H}_\gamma$, for example, by using $\rho_t = (1-t)\rho + t(\ln(r) - |\epsilon|)$ in the definition (13). Let $\widetilde{\mathcal{M}}_{\mathbb{F}}(\phi_\gamma, \Sigma)$ denote the parametrized moduli space of J_0-holomorphic maps for the family consisting of triples $(t, C, \overline{u})$ where $t \in [0,1]$ and $(C, \overline{u})$ is a J_0-holomorphic curve bounding $\overline{H}^t_\gamma$ above.

There is no bubbling in the parametrized moduli space by the primitivity assumption. By definition any element $(t, C, \overline{u})$ in $\widetilde{\mathcal{M}}_{\mathbb{F}}(\phi_\gamma, \Sigma)$, consists of a collection of disks or spheres meeting the interior and disks or spheres contained in the boundary $\mathbb{C}P^{n-1}$ with homology class summing to the homology class of the sections described above. Since the homology class is primitive, no bubbling is possible and the domain of $\overline{u}$ consists of a single disk, with at least one corner mapping to $\mathbb{C}P^{n-1}$. In particular, for every $\overline{u} \in \widetilde{\mathcal{M}}_{\mathbb{F}}(\phi_\gamma, \Sigma)$, the set of intersections $\overline{u}^{-1}(\mathbb{C}P^n)$ with the divisor at infinity consists only of a single corner, corresponding to a single strip-like end asymptotic to a Reeb chord ϑ_e of minimal length. □

Remark 7.25. In fact in the case of type $\mathbb{F} = \mathbb{F}_+$ the cobordism provided by Lemma 7.24 is trivial. By Lemma 7.12, the linearized operators $\tilde{D}_u$ are surjective and the evaluation map at the end is surjective on the kernel of $\tilde{D}_u$. It follows that $\widetilde{\mathcal{M}}_{\mathbb{F}}(\phi_\gamma, \Sigma)$ is smooth and compact. Furthermore, the natural map $\widetilde{\mathcal{M}}_{\mathbb{F}}(\phi_\gamma, \Sigma) \to [0,1]$ is a submersion (since the map is index one and the kernel of the linearization is exactly the kernel of $\tilde{D}_u$) and the moduli space $\widetilde{\mathcal{M}}_{\mathbb{F}}(\phi_\gamma, \Sigma)$ is a trivial cobordism between the moduli spaces $\mathcal{M}_{\mathbb{F}}(\phi_\gamma, \Sigma)$ and $\mathcal{M}_{\mathbb{F}}(\check{\phi}_\gamma, \Sigma)$.

8. Fukaya algebras under surgery

In this section, we prove the main result Theorem 1.3 by combining the homotopy equivalences in the previous section with broken Fukaya algebras with the local computation in Section 7. We work with the unflattened boundary condition on the handle. By Theorem 6.30, the resulting Fukaya algebra is homotopy equivalent to that of the surgered Lagrangian immersion ϕ_ϵ. Theorem 8.5 globalizes the results of the previous section to a bijection between admissible buildings. Given this, some algebra identifies the potentials up to the chain in projective Maurer-Cartan solution in Theorem 1.3. Proposition 8.11 shows that the count of inadmissible buildings vanishes. After this, a stabilization argument identifies the Floer cohomologies.

8.1. **The cell structure on the handle.**

The isomorphism of Floer cohomologies is induced by a map of Floer cochains that maps the ordered self-intersection points of the original Lagrangian to the longitudinal and meridian cells in the surgered Lagrangian. Topologically, the surgered Lagrangian L_ϵ is obtained from the unsurgered Lagrangian L_0 by attaching the handle

$$H_\epsilon \cong (-1,1) \times S^{n-1}.$$

The boundary $\partial H_\epsilon \cong \{-1,+1\} \times S^{n-1}$ is glued in along small spheres around the preimages $x_\pm \in L_0$ of the self-intersection point $\phi(x_+) = \phi(x_-) \in X$. Choose a cell structure on L_0 that includes cells consisting of small balls $\sigma_{n,\pm}$ and spheres $\sigma_{n-1,\pm}$ around the self-intersection points $x_\pm$. Let $\sigma_{0,\pm}$ be the zero cells in the boundary of $\sigma_{n-1,\pm}$. A cell structure L_ϵ is derived from that on L_0 by removing a ball around each $x_\pm$ and gluing in a single 1-cell and single n-cell

$$\sigma_1 : B^1 := [-1,1] \to L_\epsilon, \quad \sigma_n : B^n \cong B^{n-1} \times [-1,1] \to L_\epsilon$$

along the boundary of the 0-cells resp. $n-1$-cells

$$\sigma_{0,\pm} : \{0\} \to L_\epsilon, \quad \sigma_{n-1,\pm} : B^{n-1} \to L_\epsilon$$

as shown in Figure 13. A dual cell structure is obtained locally by dualizing the cell structures on the interval and the sphere.

Lemma 8.1. *The cell structures on the handles H_0 and H_ϵ given above extend to dualizable cell structures on L_0 and L_ϵ agreeing in the complement of the handle.*

Proof. In the case of H_0, view the cell structure as that associated to a Morse function f given locally as the product of Morse functions $f_{\mathbb{R}}, f_{S^{n-1}}$ on the factors $\mathbb{R}$ and S^{n-1}, where $f_{S^{n-1}}$ is the standard height function and $f_{\mathbb{R}}$ is a function with three local maxima on the handle. The Morse function on the handle extends to a Morse function f on all of L_0. The gradient flow $\operatorname{grad}(f)$ and its additive inverse $-\operatorname{grad}(f)$ define dual cell decompositions, as in Section 3.2. Similarly, in the case of H_ϵ, assume that f is the product of $f_{\mathbb{R}}, f_{S^{n-1}}$ on the complement of the top-dimensional cells, and on each top-dimensional cell f has a single critical point of index 0. □

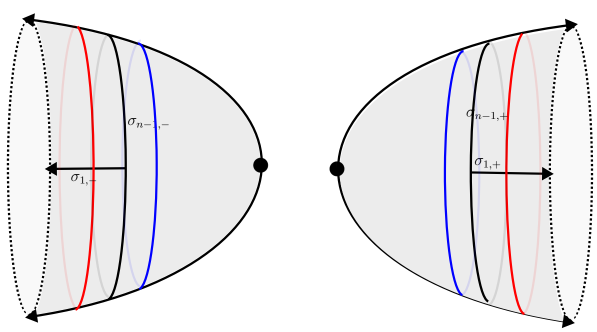

FIGURE 12. Cell structure on the unsurgered handle

8.2. **The surgered-unsurgered bijection.** With the cellular structures on the Lagrangian and its surgery defined, we now define the chain-level map which replaces the ordered self-intersection points to be surgered with the longitude and meridian on the handle. The neck-stretching argument in Theorem 6.25 produces

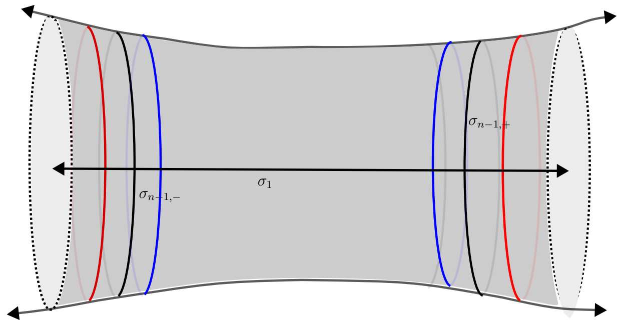

FIGURE 13. Cell structure on the surgered handle

a cobordism between rigid holomorphic maps with rigid broken holomorphic maps. Let $\mathbb{X}$ denote the broken manifold obtained by quotienting the spheres S^{2n-1} on either side of the $n-1$-cells $\sigma_{n-1,\pm}$ by the S^1-action. This cell structure is shown in Figure 12 and 13 as the collection of blue and red spheres. The pieces of $\mathbb{X}$ are

$$\mathbb{X} = X_\subset \cup X_0 \cup X_\supset \tag{163}$$

where

$$\overline{X}_\subset \cong \mathbb{C}P^n, \quad X_0 \cong \mathrm{Bl}(\mathbb{C}P^n) \quad \overline{X}_\supset = \mathrm{Bl}(X) \tag{164}$$

is a projective space resp. the blow-up $\mathrm{Bl}(\mathbb{C}P^n)$ of projective space at a point resp. the blow-up $\mathrm{Bl}(X)$ of X at the self-intersection point $\phi_0(x_-) = \phi_0(x_+)$. The handle H_γ admits the standard cellular deformation of the diagonal induced by positive translation on $\sigma_1 \cong [-1,1]$ and the standard Morse flow on $\sigma_{n-1} \cong S^{n-1}$. The Fukaya algebra $CF(X, \phi_\gamma)$ is then homotopy equivalent to the broken Fukaya algebra $CF^\infty(\mathbb{X}, \phi_\gamma)$ as in Theorem 6.30 whose structure maps have levels with cellular constraints.

Lemma 8.2. *There exists a regular perturbation datum $\underline{P} = (P_\Gamma)$ for holomorphic buildings in $\mathbb{X}$ with boundary in ϕ_γ such that J_Γ is the standard complex structure on $X_\subset = \mathbb{C}^n$ and $X_0 = \mathbb{C}^n - \{0\}$*

Proof. Lemma 7.7 shows that rigid holomorphic maps to $X_\subset = \mathbb{C}^n$ are automatically regular, while Lemma 7.20 shows that rigid holomorphic maps to $X_0 = \mathbb{C}^n - \{0\}$ are regular for the standard complex structure. □

Definition 8.3. A component $u_v : S_v \to X_\subset$ of a building $u : S \to \mathbb{X}$ is *inadmissible* if it has more than one strip-like end. A building $u : S \to \mathbb{X}$ is *admissible* if it has no inadmissible components.

Lemma 7.18 classifies the possible rigid configurations with inadmissible components. We apply the classification of rigid maps bounding the handle in the previous section to classify rigid holomorphic buildings. Denote by ϕ_γ the Lagrangian boundary condition in $\mathbb{X}$ defined using the Lagrangian H_γ using a path γ in the local model corresponding to the surgered or unsurgered Lagrangian.

Lemma 8.4. *Suppose the perturbations on $X_\subset$ and X_0 vanish. There are no rigid ∞-buildings in $\mathbb{X}$ with inter-level edges between levels of non-zero areas in $X_\subset$ and X_0.*

Proof. Let u_{v_-}, u_{v_+} be components of a rigid configuration u with u_{v_-} mapping to $X_\subset$ and u_{v_+} mapping to X_0, connected by an inter-level edge T_e at points $w_\pm \in S_{v_\pm}$. Necessarily either $\varphi_-(u_{v_-}(w_-))$ is required to be a point and there is no constraint on $\varphi_+(u_{v_+}(w_+))$, or vice versa, In the first resp. second case, forgetting the edge T_e and stabilizing produces a rigid configuration u'_{v_+} resp. u'_{v_-} in a moduli space of expected dimension one lower. The assumption on areas guarantees that these configurations are non-empty, and one obtains a contradiction. □

Define the map between generators as follows. Denote by

$$\mu = \sigma_{n-1,\pm}, \quad \lambda = \sigma_1$$

the meridienal and longitudinal cells; the choice of which cell $\sigma_{n-1,\pm}$ is immaterial as both choices give the same counts. Define a map

$$\mathcal{I}(\phi_0) \to \mathcal{I}(\phi_\epsilon), \quad \sigma_0 \mapsto \sigma_\epsilon$$

by mapping the surgered self-intersection points

$$\begin{aligned} x = (x_-, x_+) \in \mathcal{I}^{\mathrm{si}}(\phi_0) &\mapsto \mu \in \mathcal{I}^c(\phi_\epsilon) \\ \overline{x} = (x_+, x_-) \in \mathcal{I}^{\mathrm{si}}(\phi_0) &\mapsto \lambda \in \mathcal{I}^c(\phi_\epsilon) \end{aligned}$$

to the meridianal resp. longitudinal cells and leaving the remaining generators unchanged. Given $\underline{\sigma}_0 \in \mathcal{I}(\phi_0)^{d+1}$ define $\underline{\sigma}_\epsilon \in \mathcal{I}(\phi_\epsilon)^{d+1}$ by applying this map to each generator. We view this as a bijection up to the homology equivalences between $\sigma_{n-1,+}$ and $\sigma_{n-1,-}$ and similar which will be explained as a stabilization.

The bijection between maps with boundary in the unsurgered handle ϕ_0 and those in the surgered handle ϕ_ϵ in the local model in Proposition 7.2 and Lemma 7.18 and Lemma 7.5 produces a correspondence between admissible rigid types with only contraints λ on each level in the neck region. More rigid types are obtained by repeating inputs labelled by μ (or in the case $n = 2$ by the constraint σ_1.)

Theorem 8.5. (c.f. Fukaya-Oh-Ohta-Ono [42, 55.11, Chapter 10]) *For any labelled rigid type $\mathbb{\Gamma}_0$ with positive energy, there exists a type $\mathbb{\Gamma}_\epsilon$ of building obtained by replacing components in $X_\subset$ bounding ϕ_0 with those bounding ϕ_ϵ, possibly after adding edges labelled σ_1, so that there is a bijection between rigid moduli spaces of admissible buildings of positive area*

$$\mathcal{M}^\infty_{\mathbb{\Gamma}_0}(\mathbb{X}, \phi_0, \mathbb{D}) \to \mathcal{M}^\infty_{\mathbb{\Gamma}_\epsilon}(\mathbb{X}, \phi_\epsilon, \mathbb{D}), \quad u_0 \mapsto u_\epsilon \tag{165}$$

preserving orientations. If u_ϵ, u_0 are related by this bijection then the symplectic areas are related as in Lemma 7.3:

$$A(u_\epsilon) = A(u_0) + (\kappa - \overline{\kappa})A(\epsilon)$$

where κ resp. $\overline{\kappa}$ is the number of corners with boundary in ϕ_0 which map to x resp. $\overline{x}$ and $A(\epsilon)$ is the area of (150).

Proof. Replacing the levels in $X_\subset$ bounding ϕ_0 with the corresponding levels in $X_\subset$ bounding ϕ_ϵ (with constraint on σ_1, in the wrong-way case) gives the desired bijection. We compare the numerical invariants of the corresponding buildings. Lemma 7.3 then implies that the areas differ by $(\kappa - \overline{\kappa})A(\epsilon)$. The bijection in (165) is sign-preserving if and only if the bijection between moduli spaces on the broken piece $X_\subset$ of (164) containing the self-intersection point $x \in \phi(L)$ is orientation preserving. This can always be achieved by changing the orientation on the determinant line $\mathbb{D}_x^+$. The bijection preserves the cellular constraints at the evaluation maps by construction. □

Remark 8.6. We discuss constant disks on the handle region; these will be needed later to prove the invariance of the potential. On the pre-surgered side, there are two constant disks

$$u_\pm : S \to X, \quad u_\pm(S) = \phi(x_-) = \phi(x_+)$$

in the case $\dim(L_0) > 2$. The first constant disk u_- has inputs $x, \overline{x}$ and output $\sigma_{n,-}^\vee$, while the second u_+ has inputs $\overline{x}, x$ and output $\sigma_{n,+}^\vee$. We have $c^\vee(\sigma_{n,+}^\vee, \sigma_{0,+}) = 1$ and all other pairing vanish, since $\sigma_{0,+}$ is the unique 0-cell in the closure. Thus the outgoing labels of these two disks u_+ resp. u_- are $\sigma_{0,+}$ resp. $\sigma_{0,-}$. In the case $\dim(L_0) = 2$, there are arbitrary numbers of inputs, as in (5.21). For the surgered Lagrangian, we have two constant configurations $u_\pm : S \to X$ corresponding to the classical boundary $\sigma_{0,+} - \sigma_{0,-}$ of σ_1.

Remark 8.7. The bijections between moduli spaces of holomorphic treed disks bounding the immersion and its surgery extend to repeated inputs. Suppose $\mathbb{\Gamma}_0$ is a type of building bounding ϕ_0 with label x appearing l times, and

$$\underline{r} = (r_1, \ldots, r_l)$$

is a collection of integers. Let $\mathbb{\Gamma}_\epsilon^{\underline{r}}$ denote the type obtained by repeating the edge labelled $\sigma_{n-1,\pm}$ at the i-th place r_i times. Combining Theorem 8.5 with Theorem 5.13 (or rather, it's extension to buildings, whose proof is the same) gives for permutation-invariant matching conditions bijections between moduli spaces

$$\mathcal{M}^\infty_{\mathbb{\Gamma}_0}(\mathbb{X}, \phi_0, D) \to \mathcal{M}^\infty_{\mathbb{\Gamma}_\epsilon^{\underline{r}}}(\mathbb{X}, \phi_\epsilon, D), \quad u_0 \mapsto u_\epsilon.$$

Each disk passing once through the handle in the positive direction meets each generic translate of the meridian $\sigma_{n-1,\pm}$ exactly once. If $\dim(L_0) = 2$, then the longitudinal cell σ_1 is also codimension one. In this case, let

$$\underline{r} = (r_1^+, \ldots, r_l^+, r_1^-, \ldots, r_s^-)$$

be a tuple of integers represented a pattern of repetitions. If $\underline{\sigma}_\epsilon^{\underline{r}}$ is obtained by replacing the i-th occurrence x resp. $\overline{x}$ with r_i^+ resp. r_i^- copies of $\sigma_{n-1,\pm}$ resp σ_1, then there is a bijection as above for exactly one of the $r_i^+!$ resp. $r_i^-!$-factorial of the perturbations of the cycles $\sigma_{n-1,\pm}$ resp. σ_1. Indeed, each curve hitting λ hits each generic translate of σ_1 exactly once. This ends the Remark.

Lemma 8.8. *The rigid moduli spaces $\mathcal{M}^\infty(\mathbb{X}, \phi_\epsilon, D)_0$ are invariant under replacement of a constraint $\sigma_{n-1,+}$ with constraint $\sigma_{n-1,-}$ and vice-versa.*

Proof. By Proposition 7.2, for each rigid building (C, u) the boundary $\partial u : \partial S \to L$ meets each meridian $\sigma_{n-1,\pm}$ the same number of times that ∂u passes through the handle $H_\gamma \subset L$ (counted with sign), and the claim follows. □

8.3. **Equivalence of potentials.** We may now prove the first part of main result Theorem 1.3 using the bijection between curves contributing to the potentials. First we relate the curvatures of the immersion and its surgery. We work with the broken Fukaya algebras

$$CF(\phi_0) = CF^\infty(\mathbb{X}, \phi_0), \quad CF(\phi_\epsilon) = CF^\infty(\mathbb{X}, \phi_\epsilon)$$

where ϕ_ϵ is the asymptotically-cylindrical Lagrangian defined in the local model by the standard path $\gamma(t) = t + 2i\epsilon$. These broken Fukaya algebras are homotopy equivalent to the unbroken Fukaya algebras by Theorem 6.30. Define $\Psi : b_0 \mapsto b_\epsilon$ as in (5). We assume that b_0 vanishes in the neighborhood of the attaching spheres in L_0 by Lemma 5.11. The following is straight-forward from Definition 1.2:

Proposition 8.9. *The derivative*

$$D_{b_0}\Psi : CF(\phi_0) \to CF(\phi_\epsilon)$$

is given by the identity on all generators in $\mathcal{I}(\phi_0)$ except $x, \overline{x}$. On these generators we have

$$x \mapsto (b_0(x) q^{A(\epsilon)})^{-1}\mu + b_0(\overline{x})\lambda \tag{166}$$

$$\overline{x} \mapsto b_0(x)\lambda \tag{167}$$

for $\dim(L_0) > 2$*; while*

$$x \mapsto (b_0(x) q^{A(\epsilon)})^{-1}\mu + b_0(\overline{x})(b_0(x)b_0(\overline{x}) + 1)^{-1}\lambda \tag{168}$$

$$\overline{x} \mapsto b_0(x)(b_0(x)b_0(\overline{x}) + 1)^{-1}\lambda \tag{169}$$

for $\dim(L_0) = 2$.

We may write the higher composition maps in terms of correlators as follows. For $\bullet = 0, \epsilon$ define *correlators*

$$p_{d+1}^\bullet(\sigma_0, \ldots, \sigma_n) \in \Lambda, \quad m_d^\bullet(\sigma_1, \ldots, \sigma_n) = \sum_{\sigma_0, \alpha} p_{d+1}^\bullet(\sigma_0, \ldots, \sigma_n) c^\vee(\sigma_0, \alpha)\alpha.$$

Theorem 8.10. *Assume that if* $\dim(L_0) = 2$, *then the condition in Definition 5.21 holds. Then*

$$\sum_{r \geq 0} p^{\epsilon}_{r+1}(D_{b_0}\Psi(\sigma), b_\epsilon, \ldots, b_\epsilon) = \sum_{r \geq 0} p^0_{r+1}(\sigma, b_0, \ldots, b_0) \tag{170}$$

for each generator $\sigma \in \mathcal{I}(\phi_0)$. *Similarly, for any generator* $\tau \in \mathcal{I}(\phi_0)$

$$\sum_{r \geq 0} p^{\epsilon}_{r+1}(D_{b_0}\Psi(\sigma), b_\epsilon, \ldots, D_{b_0}\Psi(c), b_\epsilon, \ldots, b_\epsilon) = \sum_{r \geq 0} p^0_{r+1}(\sigma, b_0, \ldots, b_0, c, b_0, \ldots b_0) \tag{171}$$

We first show that the inadmissible configurations, in the sense of Definition 8.3, do not contribute to the sum on the left-hand-side.

Proposition 8.11. *The weighted count of configurations* $(C, u : S \to \mathbb{X})$ *bounding* ϕ_ϵ *with inadmissible surface components* $u_v : S_v \to X_\subset$ *in* (170) *and* (171) *vanishes.*

Proof. We will show that the weighted count of configurations is a multiple of a coefficient of a self-intersection point in the curvature of the weakly bounding cochain for the unsurgered Lagrangian, which vanishes. First, suppose there is a single inadmissible component S_v. Lemma 7.18 implies that such configurations cannot be rigid unless one of the minimal length Reeb chords connects to a configuration not containing the output or input labelled $D_{b_0}\Psi(c)$.

Choose such a minimal length chord ϑ leading to a component in X_0 which does not contain the output or constraint σ_1. Splitting at ϑ divides the configuration into two pieces, which we denote by u_+ and u_- as in Figure 14. Since there is a single inadmissible component, the piece u_+ has no inadmissible components. We obtain a building bounding the unsurgered Lagrangian as follows. Let $\hat{u} : \hat{S} \to \mathbb{X}$ denote the configuration obtained by replacing u_- with the map to a single quadrant bounding ϕ_0, and all levels in u_+ mapping to $X_\subset$ with similar quadrants, as in Figure 14. The resulting building $\hat{u} = (\hat{u}_-, \hat{u}_+)$ has inputs labelled b_0 and output labelled $\overline{x}$, in the component adjacent to ϑ. Let $\hat{\Gamma}$ denote the type of $\hat{u}$. We have a bijection

$$\bigcup_{\Gamma \supset \Gamma_-} \mathcal{M}_\Gamma(\phi_\epsilon) \to \mathcal{M}_{\Gamma_-}(\phi_\epsilon) \times \bigcup_{\hat{\Gamma}} \mathcal{M}_{\hat{\Gamma}}(\phi_0)$$

which maps u to $(u_-, \hat{u}_+)$, and whose inverse is given by replacinng the quadrants in $\hat{u}_+$ with the corresponding maps in Proposition 7.2. Here $\hat{\Gamma}$ ranges over types with an output labelled x or $\overline{x}$, (depending on which minimal length chord ϑ is). The weighted sum over such configurations $\hat{u}$ is the coefficient of x or $\overline{x}$ in $m_0^{b_0}(1)$ which by assumption vanishes.

Suppose now that the configuration has an arbitrary number of inadmissible components. Removing the *inadmissible vertices* $\mathrm{Vert}^{in}(\Gamma) \subset \mathrm{Vert}(\Gamma)$ corresponding to inadmissible components S_v of Definition 8.3 creates a union of trees $\Gamma_1, \ldots, \Gamma_k$. At least one of these trees $\Gamma_+ := \Gamma_i$ must be adjacent to a single inadmissible

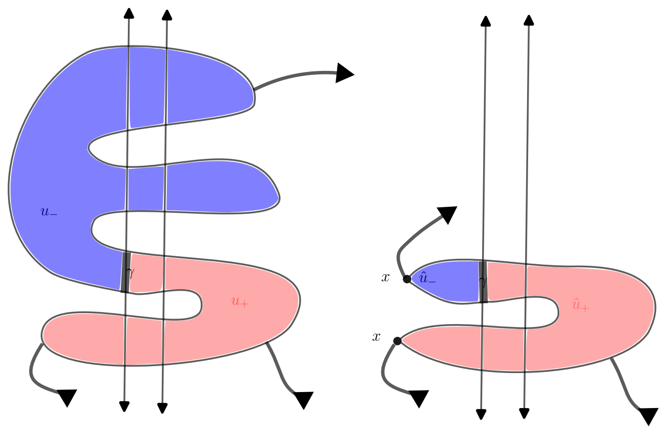

FIGURE 14. Eliminating levels with multiple ends

vertex joined at a minimal-length orbit ϑ, since by Lemma 7.18 any inadmissible component has

$$e_m(\mathbb{\Gamma}) \geq 3 - d_1(\mathbb{\Gamma}) \geq 1.$$

That is, one starts from any inadmissible vertex and moves outwards along the tree away from the output, choosing an edge in $\mathbb{\Gamma}$ that corresponds to a minimal length Reeb chord at each inadmissible vertex. If there are several such graphs, we may choose $\mathbb{\Gamma}_i$ to be the graph containing the last interior edge in the given ordering, so that the decomposition of $\mathbb{\Gamma}$ into

$$\mathbb{\Gamma}_- := \mathbb{\Gamma} - \mathbb{\Gamma}_i, \quad \mathbb{\Gamma}_+ := \mathbb{\Gamma}_i$$

is unique.

We now sum over the moduli spaces corresponding to the subgraphs separately. Let u_i denote the part of u corresponding to $\mathbb{\Gamma}_i$. Replacing each component of u_i with the corresponding quadrant, and adding a single quadrant at ϑ, produces a configuration $\overline{u}$ bounding ϕ_0. Let $\hat{\mathbb{\Gamma}}$ denote the type of $\hat{u}$. The weighted count of configurations $\hat{u}$ over types $\hat{\mathbb{\Gamma}}$ vanishes, being the coefficient of x or $\overline{x}$ in $m_0^{b_0}(1)$ depending on whether ϑ is a minimal length chord from $\mathbb{R}^n$ to $i\mathbb{R}^n$ or vice-versa. $\square$

Proof of Theorem 8.10. Each correlator is a sum over contributions from disks that pass k_- resp. k_+ times through the neck region in the negative resp. positive direction:

$$p^{\bullet}_{d+1}(\sigma_0, \ldots, \sigma_d) = \sum_{k_-,k_+} p^{\bullet,k_-,k_+}_{d+1}(\sigma_0, \ldots, \sigma_d).$$

Each non-zero contribution to $p^{\epsilon,k_-,k_+}_{d+1}$ has up to k_+ groups of inputs labelled μ and up to k_- groups of inputs labelled μ or λ. Let

$$b_\cap = b_0 - b_0(x)x - b_0(\overline{x})$$

which is the collection of terms of b_0 and b_ϵ that both share. Choose a generator $\sigma \in \mathcal{I}(\phi_0)$ not equal to $x, \overline{x}$ so that $D_{b_0}\Psi(\sigma) = \sigma$. Let r_j resp. s_j be the number of repetitions of μ resp. λ in the j-th group. Set

$$c(\mu) = \ln(b_0(x)q^{A(\epsilon)}), \quad c(\lambda) = \ln(b_0(x)b_0(\overline{x}) + 1).$$

Suppose first that $\dim(L_0) = 2$. By Remark 8.7, the j-th group of repetitions may be removed at the cost of changing the correlator by a factorial $r_j!$, where r_j is the length of the group, so that

$$\begin{aligned}\sum_{r \geq 1} p_r^\epsilon(\sigma, b_\epsilon, \ldots, b_\epsilon) \quad = \quad & \sum_{r \geq 1, k_-, k_+} \left(\prod_{j=1}^{k_+} \sum_{r_j \geq 0} c(\mu)^{r_j} (r_j!)^{-1} \right) \\ & \left(\prod_{j=1}^{k_-} \sum_{r_j \geq 0} (-c(\mu))^{r_j} (r_j!)^{-1} \left(-1 + \sum_{s_j \geq 0} c(\lambda)^{s_j} (s_j!)^{-1} \right) \right) \qquad (172) \\ & p_r^{\epsilon, k_-, k_+}(\sigma, b_\cap, \ldots, b_\cap). \qquad (173)\end{aligned}$$

Here the terms in the sum

$$-1 + \sum_{s_j \geq 0} c(\lambda)^{s_j} (s_j!)^{-1}$$

come from the two "wrong-way" curves in the handle, corresponding to the points in $S^{n-2} = S^0 = \{1, -1\}$. We assume that the local system is chosen so that the boundary of the holomorphic disk in the handle bounding ϕ_ϵ not passing through λ has parallel transport -1. The other curve crosses the longitude once, possibly with repetitions, hence the sum over s_j in the second term. Denote by

$$\underline{i}_+ \text{ resp. } \underline{i}_- \ \subset \{1, \ldots, r + k_- + k_+\}$$

the positions of these groups of label μ resp. λ. Conversely, define

$$j_0(\underline{i}_-, \underline{i}_+) : \mathcal{I}(\phi_0)^{r - k_- - k_+} \to \mathcal{I}(\phi_0)^{k + k_- + k_+} \qquad (174)$$

the map defined by inserting $k_\pm$ labels x resp. $\overline{x}$ at the positions $\underline{i}_-, \underline{i}_+$. Continuing we have

$$\begin{aligned}
(173) \quad &= \sum_{k_-,k_+} (\exp(\ln(b_0(x)q^{A(\epsilon)}))^{k_+-k_-}(-1+\exp(\ln(b_0(x)b_0(\overline{x})+1))^{k_-} \\
&\quad p_r^{\epsilon,k_-,k_+}(\sigma, b_\cap, \dots, b_\cap) \\
&= \sum_{r,k_-,k_+} (b_0(x)q^{A(\epsilon)})^{k_+}(b_0(x)^{-1}q^{-A(\epsilon)}(-1+(b_0(x)b_0(\overline{x})+1)))^{k_-} \\
&\quad p_r^{0,k_-,k_+}(\sigma, b_\cap, \dots, b_\cap) \\
&= \sum_{r,k_-,k_+} q^{(k_+-k_-)A(\epsilon)} b_0(x)^{k_+} b_0(\overline{x})^{k_-} p_r^{\epsilon,k_-,k_+}(\sigma, b_\cap, \dots, b_\cap)) \\
&= \sum_{r,\underline{i}_-,\underline{i}_+} b_0(x)^{k_+} b_0(\overline{x})^{k_-} p_r^{0,k_-,k_+}(\sigma, j_0(\underline{i}_-, \underline{i}_+)(b_\cap, \dots, b_\cap)) \\
(175) \quad &= \sum_{r\geq 1} p_r^0(\sigma, b_0, \dots, b_0).
\end{aligned}$$

where j_0 is defined in (174). For the contributions from non-constant disks, the first equality above is an application of Remark 8.7, the second is by the power series of the exponential function, the third and fourth equalities are algebraic simplifications, the fifth is by Theorem 8.5 and Proposition 8.11, and the last is the expansion $b_0 = b_\cap + b_0(x)x + b_0(\overline{x})\overline{x}$.

The contributions of the constant disks in the above computation match by the following argument. The contribution of the two constant disks with inputs $x, \overline{x}$ in Remark 8.6 to $m_2(x, \overline{x})$ is

$$m_2(x, \overline{x}) = b_0(x)b_0(\overline{x})(\sigma_{0,+} - \sigma_{0,-}) + \dots .$$

We also have contributions from alternating inputs $x, \overline{x}, \dots, \overline{x}$ to $\sigma_{0,\pm}$ with coefficient $(-1)^{d-1}/d$ by assumption, see Definition 5.21. The sum of these contributions is

$$\sum_{d\geq 1} \frac{(-1)^{d-1}}{d}(b_0(x)b_0(\overline{x}))^d(\sigma_{0,+} - \sigma_{0,-}) = \ln(b_0(x)b_0(\overline{x})+1)\ (\sigma_{0,+} - \sigma_{0,-}).$$

Since the classical boundary of σ_1 is $\sigma_{0,+} - \sigma_{0,-}$, this sum matches the classical terms in $p_\epsilon(\sigma_{n,\pm}, c(\lambda)\lambda)$.

It remains to deal with the cases that the constraint on the output is one of the cells on the neck. In the case $\sigma = \mu$, the contributions to $p_d(\mu, \dots)$ arise from configurations$(C, u : S \to \mathbb{X})$ passing either positively or negatively through the neck region at the outgoing node. Write

$$\delta(\rho) = \ln((b_0(x)+\rho)q^{A(\epsilon)})\mu + \ln(b_0(x)b_0(\overline{x})+1)\lambda.$$

Let p_{r,r_+,r_-} denote contributions to p_r from disks where the first $r_+ + 1$ labels and last r_- labels lie on the handle. Then

$$\begin{aligned}
\sum_{r\geq 1} p_r^\epsilon(\mu, b_\epsilon, \ldots, b_\epsilon) &= b_0(x) \sum_{r \geq 1} p_r^\epsilon(b_0(x)^{-1}\mu, b_\epsilon, \ldots, b_\epsilon) \\
&= b_0(x) \sum_{r\geq 1, r_\pm \geq 0} p^\epsilon_{r,r_-,r_+}(b_0(x)^{-1}\mu, \delta(0), \ldots, \delta(0), \\
&\quad b_\epsilon, b_\epsilon, \ldots, b_\epsilon, \delta(0), \ldots, \delta(0)) \\
&= b_0(x) \sum_{r\geq 1, r_\pm \geq 0} \frac{\partial}{\partial\rho}|_{\rho=0} p^\epsilon_{r,r_-,r_+}(\delta(\rho), \ldots, \delta(\rho), \\
&\quad b_\epsilon, b_\epsilon, \ldots, b_\epsilon, \delta(\rho), \ldots, \delta(\rho)) \\
&= b_0(x) \sum_{r\geq 1} \partial\rho|_{\rho=0}(b_0(x)+\rho) p_r^0(x, b_0, \ldots, b_0) \\
&\quad + \partial\rho|_{\rho=0}(b_0(x)+\rho)^{-1}(-1+\exp(\ln(b_0(x)b_0(\overline{x})+1)) \\
&\quad p_r^0\,(\overline{x}, b_0, \ldots, b_0) \\
&= \sum_{r\geq 1} p_r^0(b_0(x)x, b_0, \ldots, b_0) \\
&\quad - \sum_{r\geq 1} q^{A(\epsilon)} p_r^0 \left(\frac{-1 + (b_0(x)b_0(\overline{x})+1)}{b_0(x)} \overline{x}, b_0, \ldots, b_0 \right) \\
&= \sum_{r\geq 1} p_r^0(b_0(x)x - b_0(\overline{x})\overline{x}, b_0, \ldots, b_0)
\end{aligned}$$

where the terms involving $p_r^0(x, b_0, \ldots, b_0)$ in the sum arises from configurations passing through the handle positively and terms involving $p_r^0(\overline{x}, b_0, \ldots, b_0)$ arise configurations passing through the handle negatively. The presence of a label μ in the first entry forces the first node to map to the handle. There are contributions from any number r_- entries $\delta(\rho)$ at the end of the string $\underline{\sigma}$ and r_+ entries $\delta(\rho)$ where those labels appear on the same level of the building in the configuration. These contribute by Remark 8.7 with a factorial entry $l = (1 + r_- + r_+)!^{-1}$. Since there are $1 + r_- + r_+$ such entries for each l (depending on where the 0-th entry appears in the string), we obtain a contribution of $(r_+ + r_+)!^{-1}$ after summing over these positions. Similarly for $\sigma = \lambda$ we have

$$\begin{aligned}
\sum_{d\geq 1} p_d^\epsilon(\lambda, b_\epsilon, \ldots, b_\epsilon) &= \sum_{r\geq 1} q^{A(\epsilon)} p_r^0(b_0(x)^{-1} q^{-A(\epsilon)}(b_0(x)b_0(\overline{x})+1))\overline{x}, b_0, \ldots, b_0) \\
&= \sum_{r\geq 1} p_r^0(b_0(x)^{-1}(b_0(\overline{x})b_0(x)+1)\overline{x}, b_0, \ldots, b_0).
\end{aligned}$$

Hence

$$\begin{aligned}
\sum_{r\geq 1} p_r^\epsilon(D_{b_0}\Psi(x), b_\epsilon, \ldots, b_\epsilon) &= \sum_{r\geq 1} p_r^\epsilon\left(b_0(x)^{-1}\mu + \frac{b_0(\overline{x})}{(b_0(x)b_0(\overline{x})+1)}\lambda, b_\epsilon, \ldots, b_\epsilon\right) \\
&= \sum_{r\geq 1} p_r^0(b_0(x)^{-1}(b_0(x)x - b_0(\overline{x})\overline{x}) \qquad (176)\\
&\quad + \frac{b_0(\overline{x})(b_0(\overline{x})b_0(x)+1)}{(b_0(x)b_0(\overline{x})+1)b_0(x)}\overline{x}, b_0, \ldots, b_0) \qquad (177)\\
&= \sum_{r\geq 1} p_r^0(x, b_0, \ldots, b_0). \qquad (178)
\end{aligned}$$

Finally

$$\begin{aligned}
\sum_{r\geq 1} p_r^\epsilon(D_{b_0}\Psi(\overline{x}), b_\epsilon, \ldots, b_\epsilon) &= \sum_{r\geq 1} p_r^\epsilon(b_0(x)(b_0(x)b_0(\overline{x})+1)^{-1}\lambda, b_\epsilon, \ldots, b_\epsilon) \\
&= \sum_{r\geq 1} p_r^0(\overline{x}, b_0, \ldots, b_0). \qquad (179)
\end{aligned}$$

If σ has no $x, \overline{x}$ terms then $D_{b_0}\Psi(\sigma) = \sigma$. Together (175), (178), (179) imply the result for $\dim(L_0) = 2$.

The case $\dim(L_0) > 2$ is easier and details are left to the reader. By assumption on $b_0(\overline{x}) = 0$, any disk meeting λ has a single (not repeated) label λ on the boundary. Furthermore, each disk passing through the handle in the negative direction must have one λ label to be rigid. The computation is then the same as in the case $\dim(L_0) = 2$, but without the repeated λ inputs and defining $c(\lambda) = b_0(x)b_0(\overline{x})$. □

8.4. **Equivalence of Floer cohomologies.** To prove the isomorphisms of Floer cohomology, we introduce a quotient $CF^{\text{ess}}(\phi_0)$ of $CF(\phi_0)$ that captures the cohomology $HF(\phi_0, b_0)$, and a quotient $CF^{\text{ess}}(\phi_\epsilon)$ of $CF(\phi_\epsilon)$ capturing the cohomology $HF(\phi_\epsilon, b_\epsilon)$. Recall that the generators for ϕ_ϵ are obtained by removing two top-dimensional cells and two ordered self-intersections and gluing in cells of codimension $0, n-1, 1, n$. Let

$$CF^{\text{loc}}(\phi_0) = \text{span}(\{\sigma_{n-1,\pm}, \sigma_{n,\pm}\}) \subset CF(\phi_0).$$

Lemma 8.12. *$CF^{\text{loc}}(\phi_0)$ is a sub-complex of $CF(\phi_0)$.*

Proof. By assumption, the almost complex structure J_Γ near the self-intersection points $x, \overline{x}$ is the standard one. For index reasons, there are no rigid buildings $(C, u : S \to \mathbb{X})$ with positive area $A(u)$ having input σ_n: Forgetting the constraint would produce a building in a moduli space of negative expected dimension. Thus

$$m_1^{b_0}\sigma_{n,\pm} = \partial\sigma_{n,\pm} = \sigma_{n-1,\pm}.$$

Since $b_0 \in MC(\phi_0)$, we have $(m_1^{b_0})^2 = 0$ and so $m_1^{b_0}\sigma_{n-1,\pm} = 0$ which proves the claim. □

A long exact sequence argument implies that the quotient complex has isomorphic cohomology: The quotient

$$CF^{\mathrm{ess}}(\phi_0) = CF(\phi_0)/CF^{\mathrm{loc}}(\phi_0)$$

fits into a short exact sequence

$$0 \to CF^{\mathrm{loc}}(\phi_0) \to CF(\phi_0) \to CF^{\mathrm{ess}}(\phi_0) \to 0$$

inducing a long exact sequence in cohomology. Since $CF^{\mathrm{loc}}(\phi_0)$ is acyclic,

$$H(CF^{\mathrm{ess}}(\phi_0), m_1^{b_0})) = H(CF(\phi_0), m_1^{b_0}). \tag{180}$$

Similar arguments apply to the cohomology of the surgered Lagrangian. Define a subspace $CF^{\mathrm{loc}}(\phi_\epsilon)$ generated by the cells $\sigma_n, \sigma_{n,\pm}, \sigma_{1,\pm}$ and their classical boundaries:

$$CF^{\mathrm{loc}}(\phi_\epsilon) = \mathrm{span}(\{\sigma_n, \sigma_{n-1,+} - \sigma_{n-1,-}\}) \subset CF(\phi_\epsilon)$$

Lemma 8.13. *$CF^{\mathrm{loc}}(\phi_\epsilon)$ is a sub-complex of $CF(\phi_\epsilon)$*

Proof. Since there are no rigid treed disks with positive area and a constraint σ_n on the neck, there are no quantum corrections in the formula

$$m_1^{b_\epsilon}(\sigma_n) = \partial\sigma_n = \sigma_{n-1,+} - \sigma_{n-1,-}.$$

Since $(m_1^{b_\epsilon})^2 = 0$, we have $m_1^{b_\epsilon}(\sigma_{n-1,+} - \sigma_{n-1,-}) = 0$. □

The quotient complex

$$CF^{\mathrm{ess}}(\phi_\epsilon) = CF(\phi_\epsilon)/CF^{\mathrm{loc}}(\phi_\epsilon)$$

fits into a short exact sequence

$$0 \to CF^{\mathrm{loc}}(\phi_\epsilon) \to CF(\phi_\epsilon) \to CF^{\mathrm{ess}}(\phi_\epsilon) \to 0.$$

Since $CF^{\mathrm{loc}}(\phi_\epsilon)$ is acyclic,

$$H(CF^{\mathrm{ess}}(\phi_\epsilon), m_1^{b_\epsilon}) \cong H(CF(\phi_\epsilon,), m_1^{b_\epsilon}). \tag{181}$$

Lemma 8.14. *The complexes $CF^{\mathrm{ess}}(\phi_\epsilon)$ and $CF^{\mathrm{ess}}(\phi_0)$ have the same dimension.*

Proof. The quotient $CF^{\mathrm{ess}}(\phi_\epsilon)$ has two new generators compared to $CF^{\mathrm{ess}}(\phi_0)$ corresponding to the longitudinal cell in dimension 1 and the meridional cell in dimension $n-1$ compared to $CF^{\mathrm{ess}}(\phi_0)$, but two fewer generators corresponding to ordered self-intersection points $(x_+, x_-), (x_-, x_+) \in \mathcal{I}^{\mathrm{si}}(\phi_0)$. □

Proof of Theorem 1.3. First we prove the equivalence of potentials. Let c_0 resp. c_ϵ denote the coefficients of the approximation for the diagonal for L_0 resp. L_ϵ from (59). For the standard approximation of the diagonal we have with superscript $^\vee$ denoting the corresponding cells in the dual decomposition

$$c_0(x, \overline{x}) = c_0(\overline{x}, x) = c_\epsilon(\sigma_{n-1,\pm}, \sigma_{1,\pm}^\vee) = c_\epsilon(\sigma_{1,\pm}, \sigma_{n-1,\pm}^\vee) = 1 \tag{182}$$

and all other coefficients involving cells on the neck vanish. For the cohomology, we denote by $c_0^{\mathrm{ess}}, c_\epsilon^{\mathrm{ess}}$ the induced coefficients for the quotients $CF^{\mathrm{ess}}(\phi_0), CF^{\mathrm{ess}}(\phi_\epsilon)$. Let $\sigma_{n-1,\pm}$ denote the image of $\sigma_{n,-}, \sigma_{n,+}$ in $CF^{\mathrm{ess}}(\phi_\epsilon)$. The pairings between cells and dual cells are given by

$$c_0^{\mathrm{ess}}(x, \overline{x}) = c_0^{\mathrm{ess}}(\overline{x}, x) = c_\epsilon^{\mathrm{ess}}(\sigma_{n-1,\pm}, \sigma_1^\vee) = c_\epsilon^{\mathrm{ess}}(\sigma_1, \sigma_{n-1,\pm}^\vee) = 1. \tag{183}$$

We may assume that the diagonal approximations agree away from the cells $\sigma_{n,\pm}$ in L_0 and σ_n in L_ϵ. The derivative $D_{b_0}\Psi$ induces a map on quotient complexes by Lemma 8.8, for which we use the same notation. The complex $CF^{\mathrm{ess}}(\phi_0)$ is generated by the images of

$$\mathcal{I}^{\mathrm{ess}}(\phi_0) = \mathcal{I}(\phi_0) - \{\sigma_{n-1,\pm}, \sigma_{n,\pm}\}$$

and similarly for $\mathcal{I}^{\mathrm{ess}}(\phi_\epsilon)$. By Theorem 8.10, the coefficients of cells not on the neck in $m_0^{b_0}(1)$ and $m_0^{b_\epsilon}(1)$ agree. Furthermore, for β in the surgery region

$$\begin{aligned}
(m_0^{b_0}(1), \beta) &:= \sum_{\sigma,\gamma,r} p_r^0(\sigma, b_0, \dots, b_0) c_0^{\mathrm{ess},\vee}(\sigma, \gamma) c_0^{\mathrm{ess}}(\gamma, \beta) \\
&= \sum_r p_r^0(\beta, b_0, \dots, b_0) \\
&= \sum_r p_r^\epsilon(D_{b_0}\Psi(\beta), b_\epsilon, \dots, b_\epsilon) \\
&= \sum_{\sigma,r} p_r^\epsilon(D_{b_0}\Psi(\sigma), b_\epsilon, \dots, b_\epsilon) c_\epsilon^{\mathrm{ess},\vee}(\sigma, \gamma) c_\epsilon^{\mathrm{ess}}(\gamma, \beta) \\
&=: (m_0^{b_\epsilon}(1), (D_{b_0}\Psi)\beta).
\end{aligned}$$

Since $D_{b_0}\Psi$ preserves the identity $1_{\phi_0} \mapsto 1_{\phi_\epsilon}$, the potential is preserved by surgery:

$$W_0(b_0) = W_\epsilon(\Psi(b_0)).$$

The derivative of this identity with respect to a cochain γ is (171) and gives the identity

$$(m_1^{b_0}(\gamma), \alpha) = (m_1^{b_\epsilon}(D_{b_0}\Psi\gamma), (D_{b_0}\Psi)\alpha), \quad \forall\alpha, \quad \text{so } m_1^{b_0} = (D_{b_0}\Psi)^t m_1^{b_\epsilon}(D_{b_0}\Psi).$$

Since $D_{b_0}\Psi$ is invertible, the kernels and cokernels are related by

$$(D_{b_0}\Psi) \ker m_1^{b_0} = \ker m_1^{b_\epsilon}, \quad \operatorname{im} m_1^{b_0} = (D_{b_0}\Psi)^t \operatorname{im} m_1^{b_\epsilon}.$$

Hence, as claimed

$$HF(\phi_\epsilon, b_\epsilon) \cong HF^{\mathrm{ess}}(\phi_\epsilon, b_\epsilon) \cong HF^{\mathrm{ess}}(\phi_0, b_0) \cong HF(\phi_0, b_0).$$

□

Remark 8.15. By Proposition 5.12 the map of Maurer-Cartan spaces in Theorem 1.3 is surjective up to gauge transformation. We expect that there is an identification of surgered and unsurgered Lagrangian branes equipped with Maurer-Cartan solutions as objects in the Fukaya category, given the construction of a Fukaya category in the cellular model.

Remark 8.16. Recall that a deformation of a complex space X_0 over a pointed base (S, s_0) is a pair

$$(\pi : X \to S, \phi : \pi^{-1}(s_0) \to X_0)$$

consisting of germ π of a flat map together with an identification of the central fiber ϕ. A deformation is *versal* if it is complete, that is, if every deformation is obtained by pullback by some map; note that this is the weakest notion of versality in the literature [17]. There are natural notions of deformation of morphisms, coherent sheaves, and so on [82]. A naive notion of deformation of an immersed Lagrangian brane $\phi \to L$ is given by a family of pairs

$$(\phi_s : L \to X, b_s \in MC(\phi_s))$$

parametrized by a point s in a space S. Depending on the structure of ϕ_s, b_s, one could speak of analytic, smooth, continuous deformations and so on. Clearly, this notion is inadequate as the deformation does not include the surgered branes near L, and one seems to have codimension walls at $\mathrm{val}_q(b) = 0$. The results of this paper imply that those walls vanish by adjoining the Maurer-Cartan spaces of the surgeries. In this somewhat vague sense, we have shown the existence of versal deformations of Lagrangian branes including the surgered Lagrangians. It would be interesting to know whether there is a more precise definition of deformation of a Lagrangian brane similar to that of coherent sheaf in algebraic geometry.

Remark 8.17. In the proof of Theorem 1.3, we assume that the Fukaya algebras $CF(\phi_0)$ and $CF(\phi_\epsilon)$ have been defined using perturbation data satisfying good invariance properties in Definition 5.18 and, for Lagrangian surfaces, Definition 5.21, explained in Section 5.4. We were left feeling that we only partially understood Definition 5.21, and future work will hopefully clarify the situation. Note that in dimension two, one can also assume (186) and shift the local system rather than the Maurer-Cartan solution to prove invariance which avoids the assumption in 5.21.

Remark 8.18. The almost complex structures admit an sft-style limit in Section 6, in which the self-intersection point is isolated by a neck-stretching. For arbitrary choices of perturbation data, the conclusion of the Theorem holds without the explicit formula in Definition 1.2 for the change in the weakly bounding cochains b_0, b_ϵ.

8.5. **Variations of local system.** The formulas in Definition 1.2 are equivalent to slightly different formulas using changes in the local system rather than the weakly bounding cochain. Suppose that the parallel longitudinal transport $\mathcal{L}_\epsilon$ from one side of the handle $\{-\infty\} \times S^{n-1}$ to the other $\{\infty\} \times S^{n-1}$ using y_ϵ is given by

$$\mathcal{L}_\epsilon = b_0(x) q^{A(\epsilon)} \in \Lambda_0. \tag{184}$$

Theorem 8.19. *Given the local system $\mathcal{L}_\epsilon$ above, the conclusions of Theorem 1.3 hold for the surgered bounding cochain*

$$\mathfrak{b}_\epsilon = b_0(x)x - b_0(\overline{x})\overline{x} + \begin{cases} \ln(b_0(x)b_0(\overline{x})+1)\lambda & \dim(L_0)=2 \\ b_0(x)b_0(\overline{x})\lambda & \dim(L_0)>2 \end{cases} \tag{185}$$

The proof is essentially the same as that of the main result Theorem 1.3. In the dimension two case one may sometimes completely replace the change in weakly bounding cochain with a change in local system. Suppose that $\dim(L_0)=2$, L_0 is connected, and the weakly bounding cochain b_0 vanishes except on a single one-chain $\kappa : [-1,1] \to L_0$ connecting x_+ with x_- which has only classical boundary

$$m_1(\kappa) = x_+ - x_-. \tag{186}$$

Define $b_\epsilon = 0$ and set the parallel transport $\mathcal{M}_\epsilon$ around the meridian of the local system y_ϵ to be

$$\mathcal{M}_\epsilon = b_0(x)b_0(\overline{x}) - 1 \in \Lambda_0. \tag{187}$$

Indeed, variation of a weakly bounding cochain b by a degree one element $b' \in CF^1(\phi)$ is equivalent to a variation of the local system y by the corresponding representation $\exp(b')$ by the divisor equation (81) in Section 5.4.

9. Quasi-isomorphisms

In this section we show Theorem 1.6, namely that the objects defined by the surgered and unsurgered Lagrangian are quasi-isomorphic in a simplified version of the Fukaya category.

9.1. **Quasi-isomorphisms induced by Hamiltonian perturbation.** Let ϕ_0' be a Hamiltonian perturbation of ϕ_0 as in the statement of the Theorem and $MC(\phi_0'), MC(\phi_0)$ the corresponding Maurer-Cartan spaces. Symplectomorphisms induce A_∞ isomorphisms, since one can use the pull-back almost complex structure for which the holomorphic disk counts are the same. For each $b_0 \in MC(\phi_0)$ there exists a $b_0' \in MC(\phi_0')$ with the same value of the potential:

$$w(b_0) = w(b_0') \in \Lambda.$$

Fix such a b_0 and let $\mathrm{Fuk}_{\tilde{\phi}_0}(X)$ denote the category with objects (ϕ_0, b_0) and (ϕ_0', b_0') with higher compositions for $d \geq 1$

$$m_d^{b_0,\ldots,b_d}(a_1,\ldots,a_d) = \sum_{k_0,\ldots,k_d \geq 0} m_{d+k_0+\ldots+k_d}(\underbrace{b_0,\ldots,b_0}_{k_0}, a_1, \ldots, \underbrace{b_{d-1},\ldots,b_{d-1}}_{k_{d-1}}, a_d, \underbrace{b_d,\ldots,b_d}_{k_d}) \tag{188}$$

and define $m_0(1) = 0$. The A_∞ associativity relation for the algebra $CF(\tilde{\phi})$ implies that $\mathrm{Fuk}_{\tilde{\phi}_0}(X)$ is a flat A_∞ category.

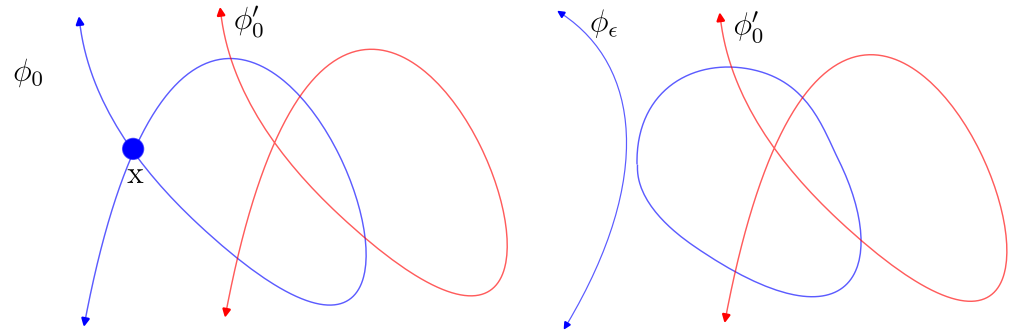

FIGURE 15. The unsurgered immersion and its perturbation; the surgered immersion and its perturbation

Lemma 9.1. *(ϕ_0', b_0') is quasi-isomorphic to (ϕ_0, b_0) in* $\mathrm{Fuk}_{\tilde{\phi}_0}(X)$.

Sketch of proof. The space $CF(\phi_0, \phi_0')$ is naturally a $(CF(\phi_0), CF(\phi_0'))$-bimodule as explained in Charest-Woodward [21, Chapter 6]. The structure maps are defined by a count of (J, H)-holomorphic strips bounding ϕ_0, ϕ_0', where $H : [0,1] \times X \to \mathbb{R}$ is the Hamiltonian whose flow defines ϕ_0'. Let

$$\tilde{H} \in C^\infty(\mathbb{R} \times [0,1] \times X))$$

be a function limiting to 0 as $s \to -\infty$ and H as $s \to +\infty$. A count of treed $(J, \tilde{H})$-holomorphic strips implies that the bimodule $(CF(\phi_0), CF(\phi_0'))$ is homotopy equivalent to $CF(\phi_0)$. The image of the unit 1_{ϕ_0} under the homotopy equivalence with $CF(\phi_0, \phi_0')$, and the image of the unit 1_{ϕ_0} under the homotopy equivalence with $CF(\phi_0', \phi_0)$ provides the necessary elements

$$\alpha_0 \in CF(\phi_0, \phi_0'), \quad \beta_0 \in CF(\phi_0', \phi_0), \quad \delta_0 \in CF(\phi_0, \phi_0), \quad \delta_0' \in CF(\phi_0', \phi_0') \tag{189}$$

as in (7). The fact that the composition of these two homotopy equivalences is homotopic to the identity implies the necessary composition relation for $m_2(\alpha_0, \beta_0)$ and $m_2(\beta_0, \alpha_0)$. $\square$

We also have a Fukaya category with the same two objects, but where the structure maps are defined by pseudoholomorphic buildings. As a special case of Theorem 6.33 we have an A_∞ homotopy equivalence

$$CF(X, \tilde{\phi}_0) \cong CF^\infty(\mathbb{X}, \tilde{\phi}_0).$$

Let $\mathrm{Fuk}^\infty_{\tilde{\phi}_0}(\mathbb{X})$ be the category whose objects are ϕ_0 and ϕ_0', and whose morphisms are the sub-spaces of $CF(X, \tilde{\phi}_0)$ in the obvious way. The homotopy equivalence of A_∞ algebras induces a homotopy equivalence of A_∞ categories

$$\mathrm{Fuk}_{\tilde{\phi}_0}(X) \to \mathrm{Fuk}^\infty_{\tilde{\phi}_0}(\mathbb{X}).$$

The existence of quasi-isomorphisms with the broken limit $CF^\infty(\mathbb{X}, \underline{\phi}_0)$ implies that (ϕ_0, b_0) and (ϕ_0', b_0') define quasi-isomorphic objects in $\mathrm{Fuk}^\infty_{\tilde{\phi}_0}(\mathbb{X})$.

9.2. **Quasi-isomorphism with the surgery.** We now use the quasi-isomorphisms above to show that the surgery is also quasi-isomorphic. Since the intersections of ϕ_0 and ϕ_0' are disjoint from the surgery regions, we have equalities

$$(\phi_0 \times \phi_0')^{-1}(\Delta) = (\phi_\epsilon \times \phi_0')^{-1}(\Delta)$$

and identifications

$$CF(\phi_0, \phi_0') \cong CF(\phi_\epsilon, \phi_0'), \quad CF(\phi_0', \phi_0) \cong CF(\phi_0', \phi_\epsilon). \tag{190}$$

We denote by

$$\alpha_\epsilon \in CF(\phi_\epsilon, \phi_0'), \quad \beta_\epsilon \in CF(\phi_0', \phi_\epsilon)$$

the images of α_0 and β_0 under the isomorphisms (190). Furthermore, let

$$\delta_\epsilon \in CF(\phi_\epsilon, \phi_\epsilon)$$

the image of δ_0 under the map $D\Psi$.

We claim that the bijection between disks bounding the surgered and unsurgered Lagrangians in Theorem 6.23 applies; in fact the corresponding moduli space have no more than one edge labelled by the cell σ_1 so the argument is somewhat easier than the first proof given. Since δ_ϵ is an odd element, the output $m_1^{b_\epsilon}(\delta_\epsilon)$ is even and does not contain σ_{n-1}. Thus the computation of $m_1^{b_\epsilon}$ involves at most one constraint labelled σ_1. Similarly, the computation of $m_2^{b_\epsilon, b_0'}(\beta_\epsilon, \alpha_\epsilon)$ has inputs labelled by intersection points of ϕ_ϵ and ϕ_0' and inputs labelled odd cells in b_ϵ or b_0', so there are no inputs labelled by σ_1. Therefore the only possibly constraint labelled σ_1 might be at the outgoing edge, and so the number of such constraints is at most one. The correspondence in Theorem 6.23 therefore gives a bijection between the moduli spaces computing $m_1^{b_\epsilon}(\delta_\epsilon)$ and $m_1^{b_0}(\delta_0)$, the moduli spaces computing $m_2^{b_\epsilon, b_0'}(\alpha_\epsilon, \beta_\epsilon)$ and $m_2^{b_0, b_0'}(\alpha_0, \beta_0)$, and the moduli spaces computing $m_2^{b_0', b_\epsilon}(\beta_\epsilon, \alpha_\epsilon)$ and $m_2^{b_0', b_0}(\beta_0, \alpha_0)$, with the count of inadmissible configurations vanishing by Proposition 8.11. It follows that $m_1^{b_\epsilon}(\delta_\epsilon)$ agrees with the image of $m_1^{b_0}(\delta_0)$ under the identification of $CF^{\text{ess}}(\phi_\epsilon)$ with $CF^{\text{ess}}(\phi_0)$, the element $m_2^{b_\epsilon, b_0'}(\alpha_\epsilon, \beta_\epsilon)$ agrees with the image of $m_2^{b_0, b_0'}(\alpha_0, \beta_0)$, and the element $m_2^{b_0', b_\epsilon}(\beta_\epsilon, \alpha_\epsilon)$ agrees with the image of $m_2^{b_0', b_0}(\beta_0, \alpha_0)$. Hence $\alpha_\epsilon, \beta_\epsilon$ are quasi-isomorphisms as claimed. This completes the proof of Theorem 1.6.

9.3. **Mapping cones.** In the case of a single intersection point of a pair of embedded Lagrangians, the main result of this paper reproduces the identification of the surgery with the mapping cone, which was the original intent of Fukaya-Oh-Ohta-Ono [42, Chapter 10], see also Abouzaid [4], Mak-Wu [62], Tanaka [84], and Chantraine-Dimitroglou-Rizell-Ghiggini-Golovko [20, Chapter 8]. The special case that one of the Lagrangians is a Lagrangian sphere was treated earlier by Seidel

[76] in his paper on symplectic Dehn twists. Pascaleff-Tonkonog [68] have developed a generalization to clean intersections, related to higher-dimensional analogs of Lagrangian mutation.

We put ourselves in the following simple version of the Fukaya category, generated by two branes. Suppose that the immersion $\phi_0 : L_0 \to X$ is the disjoint union of immersions $\phi_\pm : L_\pm \to X$ intersecting transversally equipped with weakly bounding cochains $b_\pm \in MC(\phi_\pm)$. Denote the combined immersion by

$$\phi_0 = \phi_- \sqcup \phi_+ : L_- \sqcup L_+ \to X.$$

Recall that $CF(\phi_-, \phi_+)$ is the subspace of $CF(\phi_0)$ generated by the intersection points of ϕ_- and ϕ_+. As vector spaces

$$CF(\phi_0) \cong CF(\phi_-) \oplus CF(\phi_+) \oplus CF(\phi_-, \phi_+) \oplus CF(\phi_+, \phi_-)$$

and $CF(\phi_\pm)$ are A_∞ sub-algebras. The space $CF(\phi_-, \phi_+)$ is naturally an A_∞ bimodule over the A_∞ algebras $CF(\phi_-)$ and $CF(\phi_+)$. Let

$$c \in CF(\phi_-, \phi_+), \quad m_1^{b_-, b_+}(c) = 0$$

be a cocycle. Let $\psi : K \to X$ be another immersed Lagrangian brane in X meeting ϕ_+, ϕ_- transversally and disjoint from $\phi(L_+) \cap \phi(L_-)$. Suppose that K is equipped with a bounding cochain $k \in MC(\psi)$ with

$$W(k) = W(b_-) = W(b_+).$$

The complex $\operatorname{Hom}(\operatorname{Cone}(c), K)$ is by definition

$$\operatorname{Hom}(\operatorname{Cone}(c), K) = CF(L_-, K)[1] \oplus CF(L_+, K)$$

with differential $m_1^{b_- + b_+ + c, k}$ induced by the differentials on $CF(L_\pm, K)$ and composition with c, see for example Seidel [79, 2.10].

Theorem 9.2. (c.f. [42, Remark 54.9, Chapter 10]) *Suppose $L_\pm, K$ are as above and $\dim(L_\pm) > 2$. Suppose that $x \in \phi_-(L_-) \cap \phi_+(L_+)$ is an odd self-intersection point and*

$$c = q^{-A(\epsilon)} x \in CF(L_-, L_+), \quad m_1^{b_-, b_+} c = 0$$

is a cocycle. Let ϕ_ϵ denote the ϵ-surgery at x with cochain $b_\epsilon = b_+ + b_-$ with $b_\pm$ vanishing in an open neighborhood of x. Then the complex $CF(\phi_\epsilon, K)$ with differential $m_1^{b_\epsilon, k}$ is homotopy equivalent to the mapping cone $\operatorname{Hom}(\operatorname{Cone}(c), K)$.

Remark 9.3. The special case that one of the Lagrangians is a Lagrangian sphere was treated earlier by Seidel [76]. In this case, say L_- is a sphere, the surgery $\phi_\epsilon : L_\epsilon \to X$ is embedded and Hamiltonian isotopic to the *Dehn twist* $\tau_{L_-} L_+$ of L_+ around L_-. Here the Dehn twist $\tau_{L_-} \in \operatorname{Aut}(X, \omega)$ is a symplectomorphism on X that restricts to minus the identity on L_- and is supported on a neighborhood of L_-. Surgering all self-intersections simultaneously gives an exact triangle in the derived Fukaya category

$$\mathrm{Hom}(L_-, L_+)L_- \to L_+ \to \tau_{L_-}(L_+) \to \mathrm{Hom}(L_-, L_+)L_-[1],$$

see Seidel [79, Proposition 9.1]. This ends the Remark.

Proof of Theorem 9.2. Let $\phi_\pm : L_\pm \to X$ be embeddings as in the statement of the theorem and $b_\pm \in MC_\delta(L_\pm)$ projective Maurer-Cartan solutions. As in Theorem 6.30, the complexes $CF(\phi_\epsilon, K)$, $\mathrm{Hom}(\mathrm{Cone}(c), K)$ are homotopy equivalent to those defined by curve counts in the broken limit $\mathbb{X}$ in which the almost complex structure is stretched along a sphere enclosing the given intersection point $x \in L_- \cap L_+$. Any configuration $(C, u_0 : S \to \mathbb{X})$ contributing to a structure map of $\mathrm{Hom}(\mathrm{Cone}(c), K)$ corresponds under the map Theorem 8.5 with a curve $(C, u_\epsilon : S \to \mathbb{X})$ with boundary on (ϕ_ϵ, K). The number of corners of u_0 on x is equal to the number of times that u_ϵ passes through the handle H_ϵ positively. Since $\dim(L_0) > 2$ by assumption, any rigid curve u_ϵ passes in the positive direction on the handle by Theorem 8.5. That is, there are no "wrong way" corners to deal with in the bijection between holomorphic disks. On the surgered side, only one end represents a change of boundary condition from L_0 to L_1. The rest must have angle change θ_e a multiple of π, and so $d_-(e) \geq 1$ and $d_+(e) \geq 1$ for all but one end. As in the analysis in Lemma 7.18, the dimension of the unconstrained moduli space is at least $e(\circ)n - 1$ and such configurations cannot be rigid unless there is a single end asymptotic to a minimal length Reeb chord. The area of $A(u_\epsilon)$ is $A(u_0) - \kappa A(\epsilon)$ as in Lemma 7.3. Counting rigid curves u_ϵ defines the differential on $CF(\phi_\epsilon, K)$ using the bounding cochain $b_- + b_+$. One obtains an identification of complexes as before. $\square$

Remark 9.4. Fukaya-Oh-Ohta-Ono [42, Theorem 56.14, Chapter 10] use this identification with the mapping cone to show that there exists a Lagrangian in the six-dimensional symplectic torus whose Fukaya algebra (defined using their foundational system, presumably equivalent to ours) has no projective Maurer-Cartan solutions.

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