Quilted disks, A_∞ functors, and Lagrangian correspondences

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joint with Katrin Wehrheim (MIT), and Sikimeti Mau (Rutgers Ph.D. candidate)

Part I: Stable disks, A_∞ categories, and symplectic manifolds

Summary: The associahedron is homeomorphic to the moduli space of stable marked disks.

The facets correspond to terms in the A_∞ associativity relations.

The associahedron

Let $d > 2$ be an integer. The d-th associahedron K_d is a CW-complex of dimension $d - 2$ whose vertices correspond to the possible ways of parenthesizing d variables $x_1, \ldots, x_d$.

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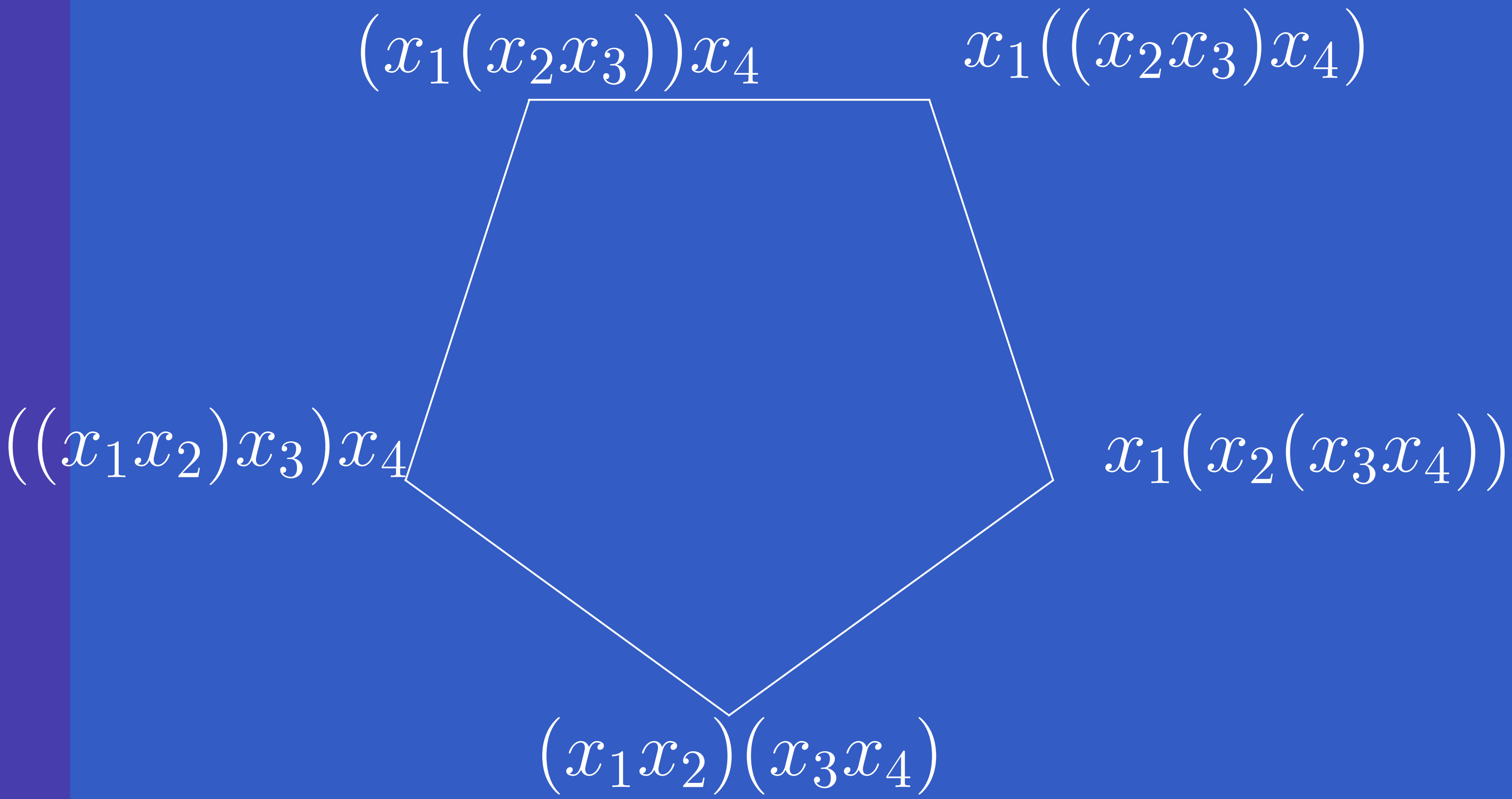
The associahedron K_4 is the pentagon shown below.



Inductive definition of associahedra

Each facet of K_d is the image of an embedding $K_{j+1} \times K_{d-j} \longrightarrow K_d$ corresponding to the expression $x_1 \ldots x_{i-1}(x_i \ldots x_{i+j})x_{i+j+1} \ldots x_d$.

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The associahedra are defined by induction

$$\partial K_d := \bigcup_{i,j} (K_{j+1} \times K_{d-j})/\sim$$

$$K_d := \mathrm{Cone}(\partial K_d).$$

Moduli space of nodal disks

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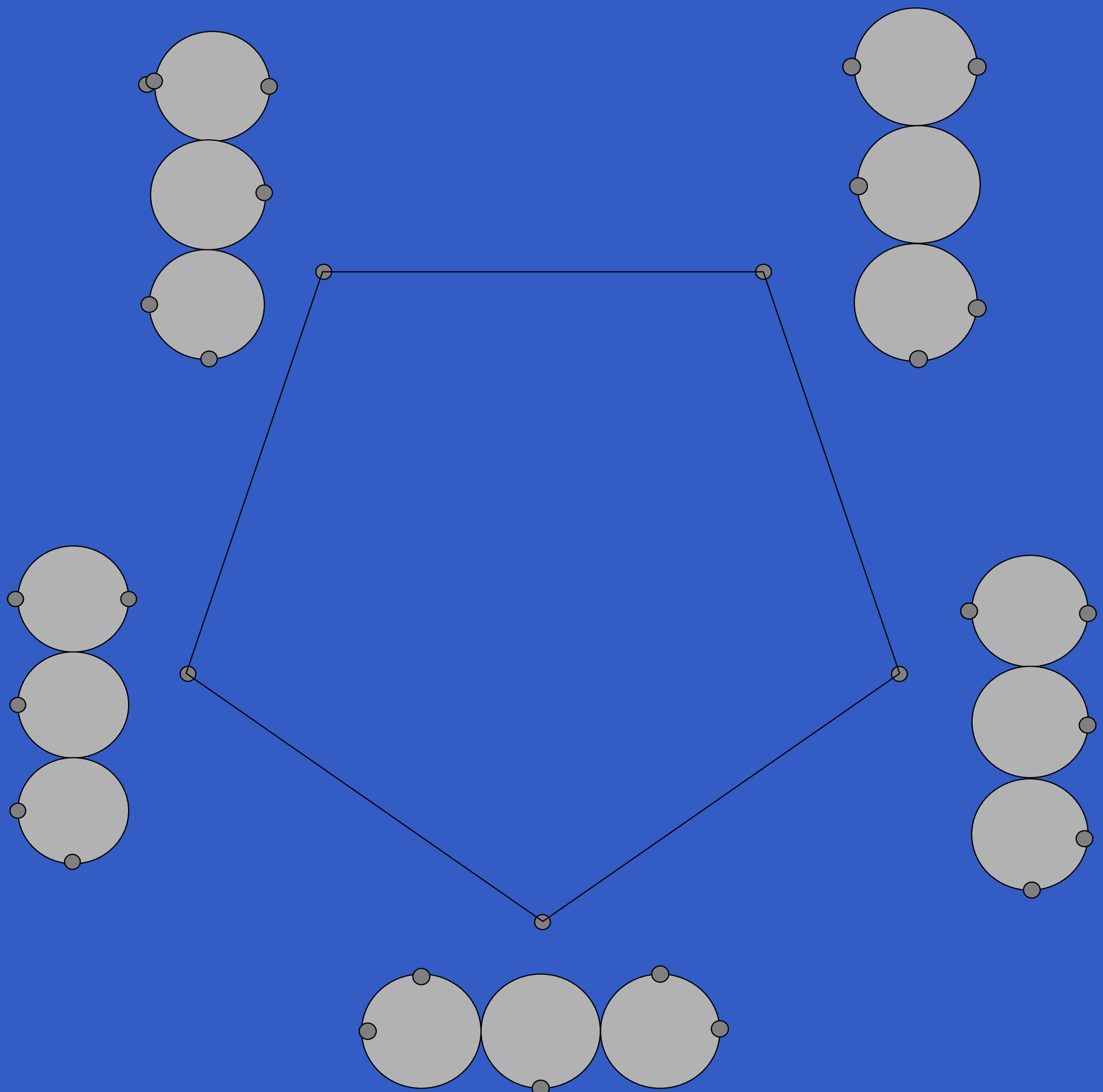
A set of markings is a set $\{z_0, \ldots, z_d\}$ of the boundary ∂D in clockwise order, distinct from the singularities.

A nodal disk with markings is *stable* if each disk component contains at least three singularities or markings.

Picture of $\overline{M}_4$

The moduli space $\overline{M}_4$ is



Topology of $\overline{M}_d$

The topology on $\overline{M}_d$ is the smallest topology so that the cross-ratios

$$\rho_{ijkl}(z_i, z_j, z_k, z_l) = \frac{(z_i - z_j)(z_k - z_l)}{(z_i - z_l)(z_j - z_k)}$$

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(Haiman etc.) $K_d, \overline{M}_d$ are isomorphic (as CW-complexes) to a convex polytope.

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$$\mu^d_{\mathcal{C}} : \mathrm{Hom}_{\mathcal{C}}(C_{d-1}, C_d) \times \ldots \times \mathrm{Hom}_{\mathcal{C}}(C_0, C_1) \longrightarrow \mathrm{Hom}_{\mathcal{C}}(C_0, C_d)$$

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satisfying the A_∞ *-associativity equations*

$$0 = \sum_{i,j} \pm \mu^{d-j}_{\mathcal{C}}(a_d, \ldots, a_{i+j+1}, \mu^j_{\mathcal{C}}(a_{i+j}, \ldots, a_i), a_{i-1}, \ldots, a_1).$$

Each term corresponds to a facet of the associahedron.

Part II: Stable quilted disks and A_∞ functors

Summary: The multiplihedron is homeomorphic to the moduli space of stable marked quilted disks.

The facets correspond to terms in the definition of A_∞ functor.

Associated to any monotone symplectic manifold M is an A_∞ category, called the Fukaya category of M.

Multiplihedra

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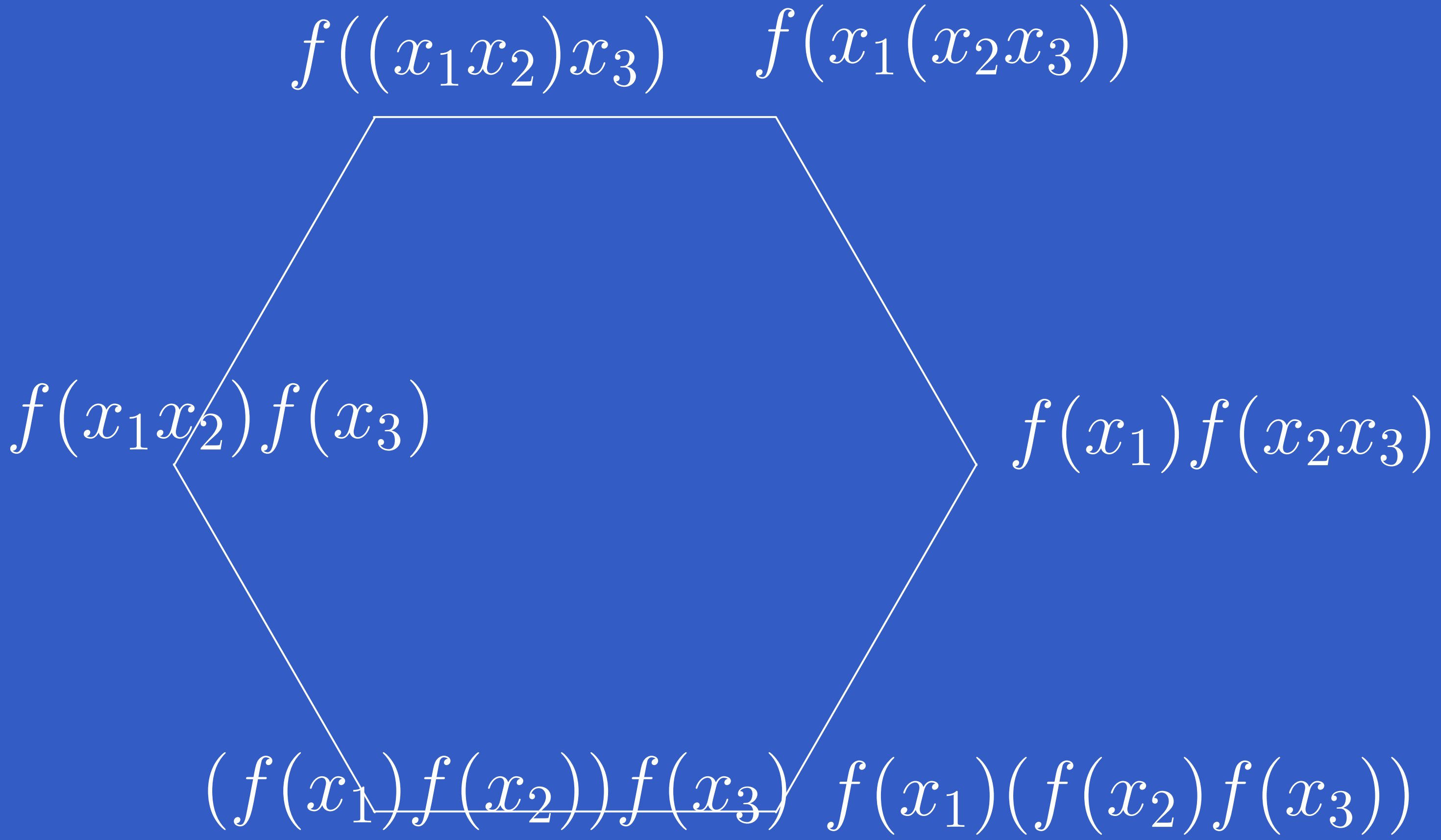
The multiplihedron $K_{3,1}$ is



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$K_{d,1}$ is NOT a manifold with corners.

NOT KNOWN whether $K_{d,1}$ is CW-isomorphic to a convex polytope.

Moduli spaces of quilted disks

In this section, a *quilted disk* refers to a disk D together with a circle $C \subset D$ (the *seam* of the quilt) passing through a unique *root point* in the boundary. Thus C divides the interior of D into two components.

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Each non-quilted disk component contains at least 3 singular or marked points.

$\overline{M}_{3,1}$

The set $\overline{M}_{3,1}$ consists of nodal quilted disks of three types, shown below. There is unique isomorphism class for the left, right pictures and a one-dimensional family for the middle picture. Hence the moduli space is an interval.

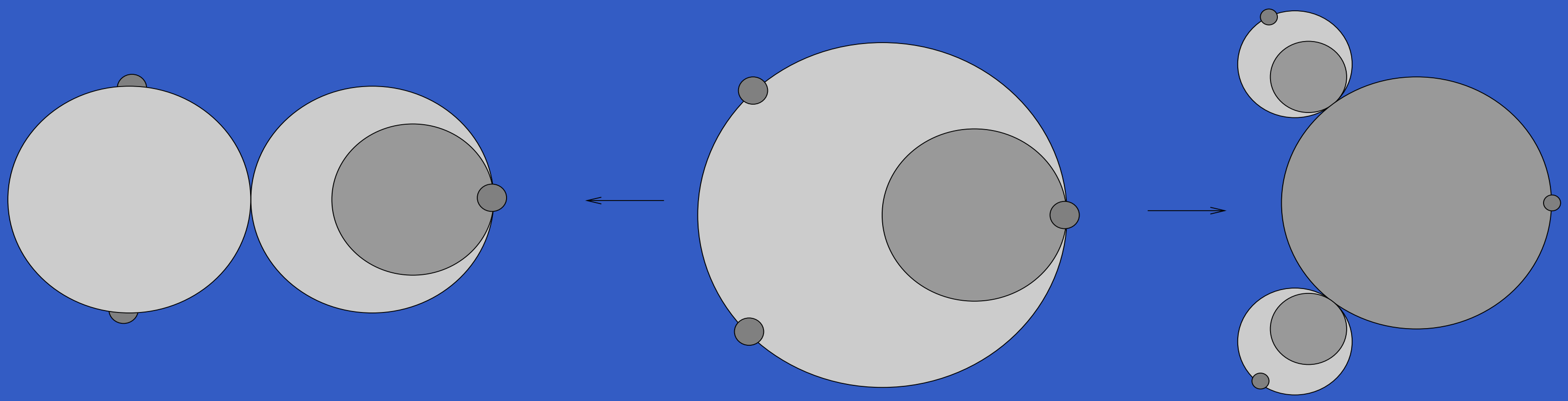
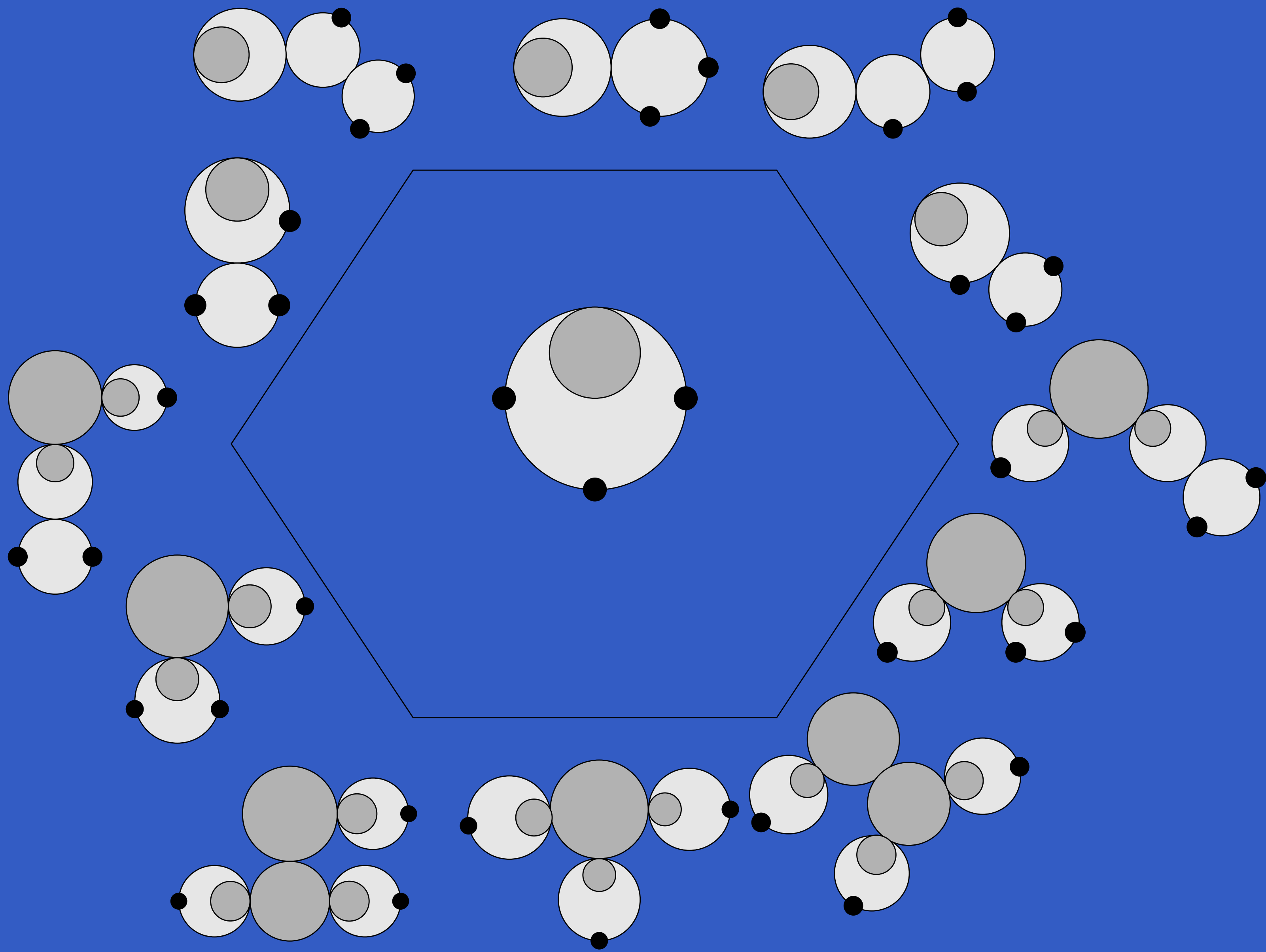


Figure 2: $\overline{M}_{3,1}$

$\overline{M}_{4,1}$



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Theorem (Mau-W) $\overline{M}_{d,1}$ is homeomorphic to $K_{d,1}$.

Local structure on $\overline{M}_{d,1}$

Explicit charts in terms of gluing parameters and relations on them.

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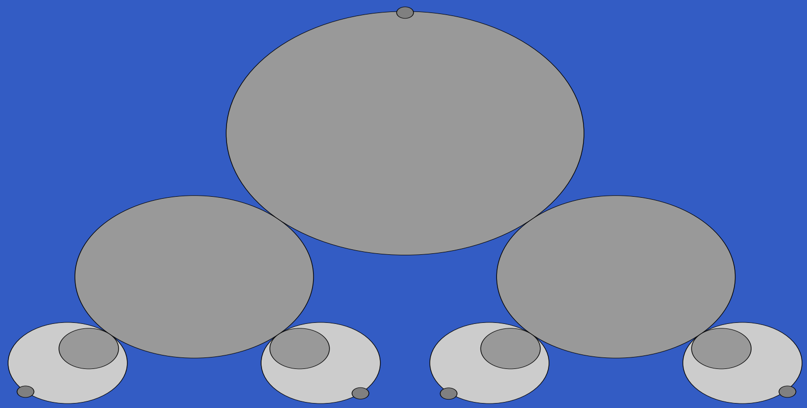
For example, consider the following singular point in $\overline{M}_{4,1}$. 

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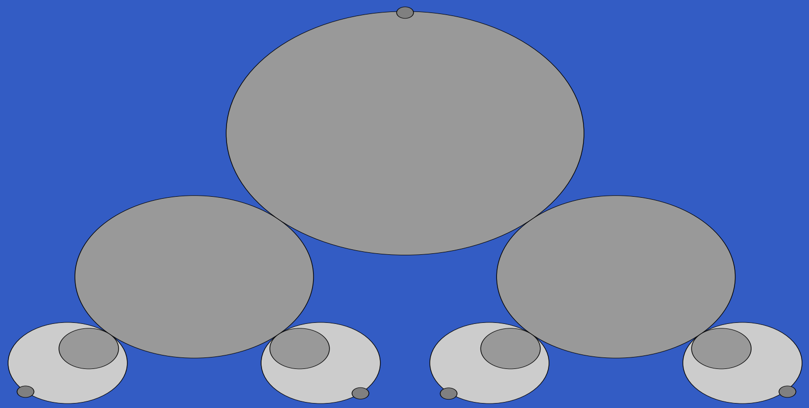
$\overline{M}_{4,1}$. 

Six gluing parameters $\delta_1, \delta_2, \delta_3, \delta_4, \delta_5, \delta_6$.

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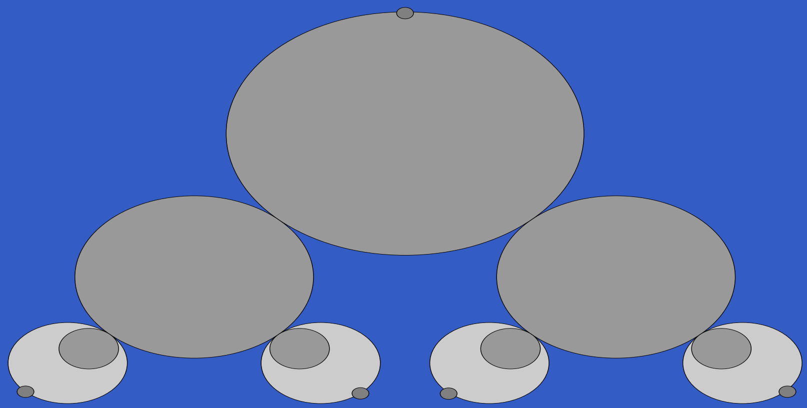
For example, consider the following singular point in $\overline{M}_{4,1}$. 

Six gluing parameters $\delta_1, \delta_2, \delta_3, \delta_4, \delta_5, \delta_6$.

Locally the moduli space is the set of solutions to $\delta_1\delta_2 = \delta_1\delta_3 = \delta_4\delta_5 = \delta_5\delta_6$.

Local structure continued

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$\overline{M}_{d,1}$ is in particular a topological manifold with boundary.

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Proof follows by induction on d.

Part III: Functoriality for Lagrangian correspondences in Floer-Fukaya theory

Summary: A notion of morphism in symplectic geometry is a Lagrangian correspondence.

Associated to any Lagrangian correspondence is an A_∞ functor between Fukaya categories.

There is a good calculus of these functors, similar to Hörmander's calculus of Fourier integral operators from the 1960's.

Fukaya categories

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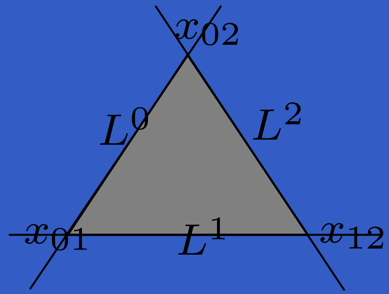
The composition map μ^d counts pseudoholomorphic $d+1$-gons with boundary in $L^0, \ldots, L^d$.

The composition map μ^2

$$\mu^2 : Hom(L^1, L^2) \times Hom(L^0, L^1) \to Hom(L^0, L^2)$$

$$\mu(<x_{12}> \otimes <x_{01}>) = \sum_{x_{02}} \# \Bigg\{$$



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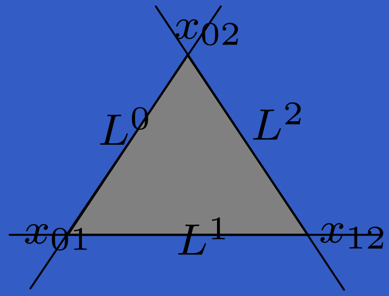

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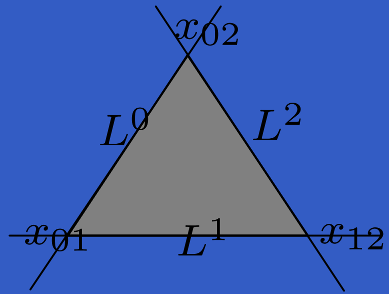

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μ^2 descends to an associative composition on the corresponding homological category.

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This is like only allowing invertible linear transformations in linear algebra.

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$\Phi^d_{L_{01}}(<x_1>, \ldots, <x_d>) = \sum_{u \in M_{d,1}(x_0,\ldots,x_d)} \pm\epsilon(u)\ <x_0>$ where $M_{d,1}(x_0, \ldots, x_d)$ is the number of isolated perturbed J-holomorphic quilted disks with boundary in $L^0, \ldots, L^d$,

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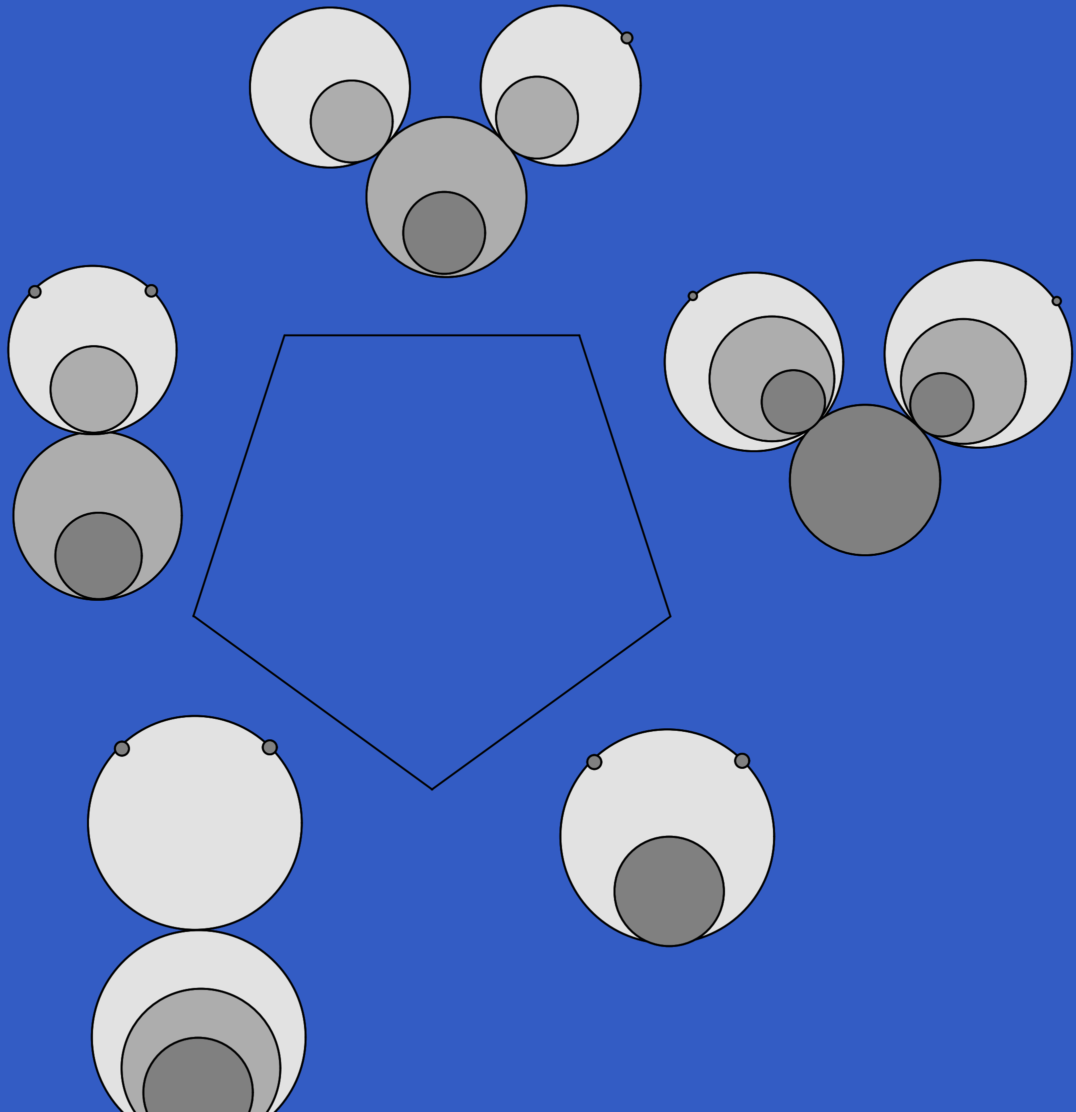
Because the boundary of $\overline{M}_{d,1}$ is a union of facets corresponding to terms in A_∞ functor axiom, $\Phi(L_{01})$ is an A_∞ functor.

Composition of A_∞ functors for Lagrangian correspondences

Main Theorem: (M-W-W) $\Phi_{L_{12}} \circ \Phi_{L_{01}}$ is homotopic to $\Phi_{L_{01} \circ L_{12}}$ if $L_{12} \circ L_{01}$ is a smooth embedded Lagrangian correspondence. Proof uses *biquilted disks*



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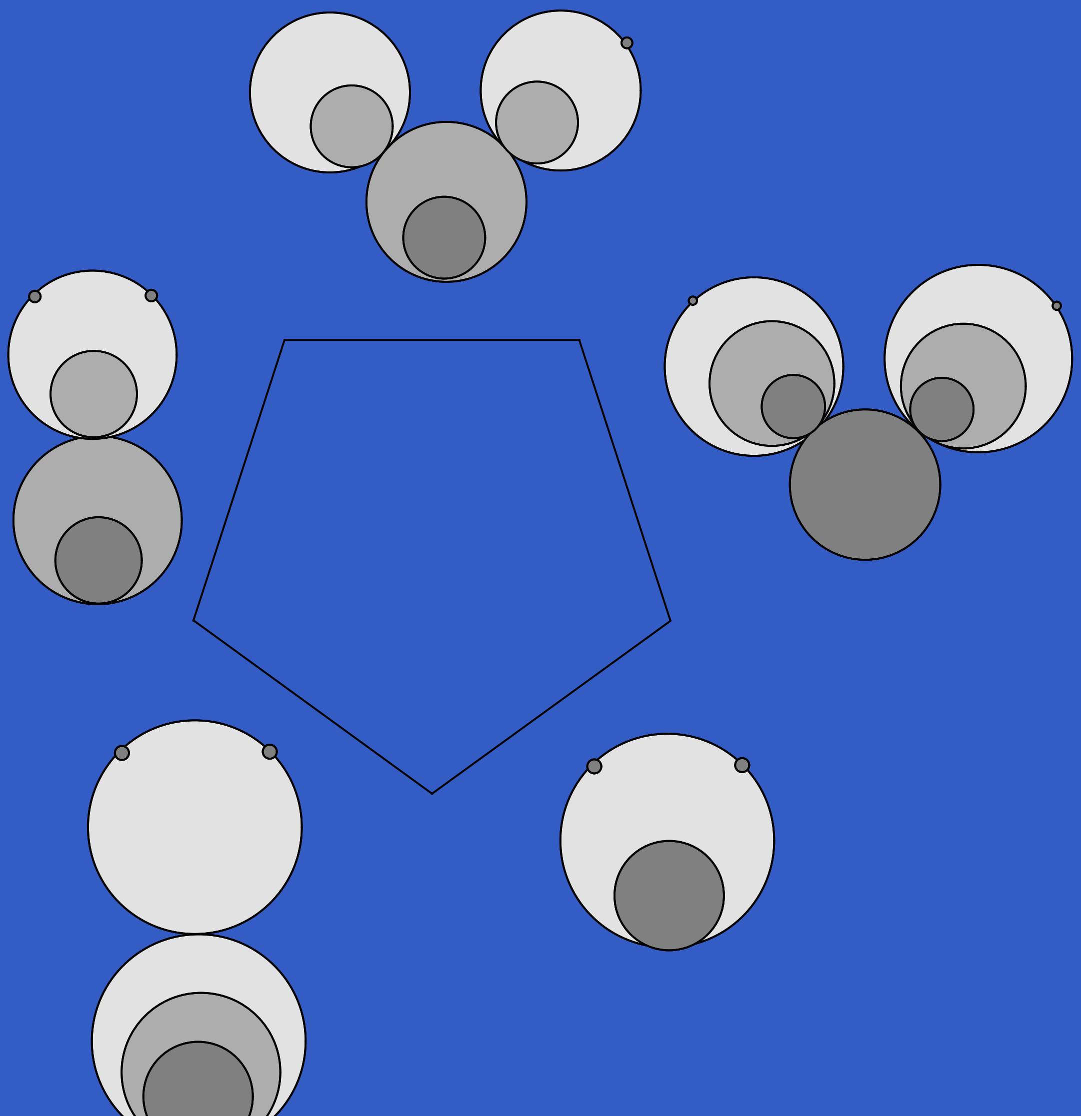

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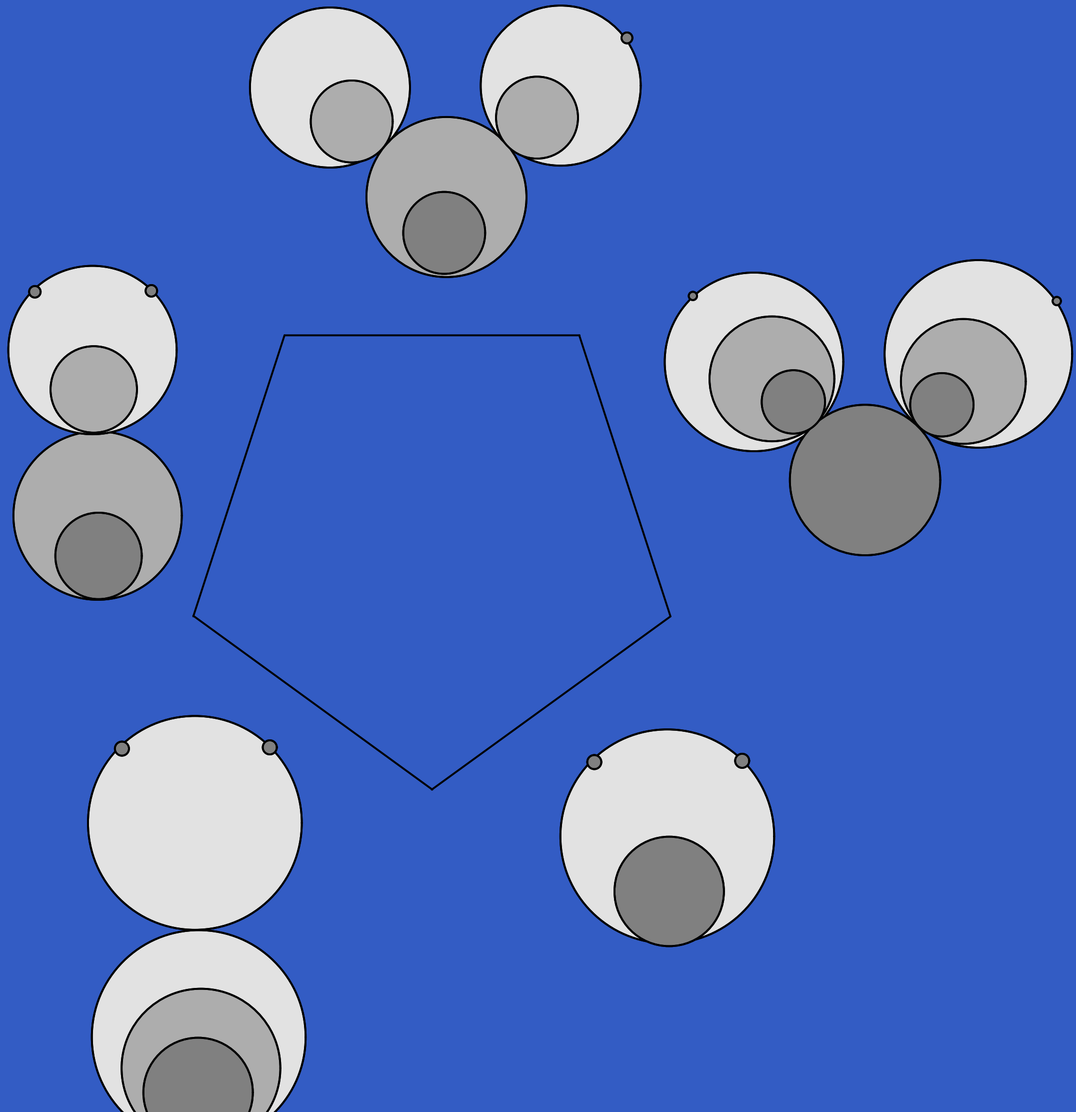

Conjectures on A_∞ 2-categories

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Conjectures on A_∞ 2-categories

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Conjectural Categorification Functor: There is an A_∞ 2-functor from Example 1 to Example 2. Papers on my web-page with W proves this for $H(Fuk)$.

Part IV: Fukaya categorification of moduli space of flat bundles

Donaldons and Floer partially constructed a functor from $4d$ cobordisms to vector spaces

Using the main result we can partially extend this to a 2-functor from 2-manifolds, $2+1$-dimensional cobordisms, $2+1+1$-dimensional cobordisms of cobordisms to A_∞ categories.

Resulting theory for knots is related by a spectral sequence to Khovanov homology

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$X \mapsto M(X), Y \mapsto M(Y)$ is a functor from $Cob_{2+1} \to Wein$.

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One can get A_∞ -natural transformations for 4-dimensional cobordisms of cobordisms.

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- Consequence: there are geometrically interesting differentials on Khovanov-Rozansky homology. Which we knew anyway.