Geometry of $6j$ symbols

Chris Woodward (Rutgers New Brunswick), based on joint work with Yuka U. Taylor (George Washington U.)

Outline:

I Kähler quantization and the Ponzano-Regge-Roberts formula.

II Asympotics of quantum 6j symbols and spherical tetrahedra

III Puzzles on the asymptotics of the Turaev-Viro invariant

I Kähler quantization and the Ponzano-Regge-Roberts formula.

Abstract: I will derive the Ponzano-Regge-Roberts formula from functorial properties of Kähler quantization.

Background on symplectic geometry:

A symplectic manifold is a smooth manifold M with a *symplectic form* $\omega \in \Omega^2(M)$:

$$d\omega = 0, \quad \omega(m)^{\dim(M)/2} \neq 0 \quad \forall m \in M.$$

For any function, we have a Hamiltonian vector field:

$$C^\infty(M) \to \mathrm{Vect(M)}, \quad \mathrm{f} \mapsto \mathrm{v_f}, \quad \iota(\mathrm{v_f})\omega = \mathrm{df}.$$

A Lagrangian submanifold is an immersed submanifold $i : L \to M$ with

$$\dim(L) = \dim(M)/2, \qquad i^*\omega = 0.$$

Example: $M = \mathbb{R}^{2n} = \{(p,q)\}$, $\omega = \sum_{i=1}^n \mathrm{d}p_i \wedge \mathrm{d}q_i$. $L = \{(p,\cdot)\}$ or $\{(\cdot,q)\}$ are Lagrangians.

Example: $M = S^2$, $\omega =$ volume form, L any one-manifold.

Operations on symplectic manifolds:

Duals: $\overline{(M,\omega)} = (M,-\omega)$.

Sums: $(M_1,\omega_1)\cup(M_2,\omega_2) = (M_1\cup M_2,\omega_1\cup\omega_2)$.

Products: $(M_1,\omega_1)\times(M_2,\omega_2) = (M_1\times M_2,\pi_1^*\omega_1 + \pi_2^*\omega_2)$.

The symplectic category is the "category" with

Obj(Sp) = Symplectic manifolds

$\mathrm{Hom}(\mathrm{M}_1,\mathrm{M}_2)$ = Lagrangian submanifolds of $\overline{M_1}\times M_2$.

The identity is the diagonal in $\mathsf{Hom}(\mathsf{M},\mathsf{M})$.

Composition is $\mathsf{Hom}(\mathsf{M}_1,\mathsf{M}_2)\times\mathsf{Hom}(\mathsf{M}_2,\mathsf{M}_3) \to \mathsf{Hom}(\mathsf{M}_1,\mathsf{M}_3)$, $L_{12} \circ L_{23} = \{(l_1, l_3),\ \exists l_2 s.t. (l_1, l_2) \in L_{12},\ (l_2, l_3) \in L_{23}\}$.

Warning: composition is defined only if these conditions are transversal.

Addition $\mathsf{Hom}(\mathsf{M}_1,\mathsf{M}_2)^2 \to \mathsf{Hom}(\mathsf{M}_1,\mathsf{M}_2)$ is disjoint union.

Operations make this a "tensor category".

Equivariant version:

Let G be a group, here either $SO(3)$ or $SU(2)$, acting on M with generating vector fields

$$\mathsf{Lie(G)} \to \mathsf{Vect(M)}, \quad \xi \mapsto \xi_{\mathsf{M}}.$$

The action is Hamiltonian if there exists a moment map

$$\Phi: \ M \to \mathsf{Lie(G)}^*, \quad \mathsf{v}_{(\Phi,\xi)} = -\xi_{\mathsf{M}}$$

for all $\xi \in \mathsf{Lie(G)}$. A Hamiltonian G-manifold is a triple (M, ω, Φ). (Might also include the action.)

Sums, Duals, Products as before.

Example: $G = SO(3)$ acts on $M = S^2$ with moment map the inclusion $S^2 \to \mathbb{R}^3 \cong \mathsf{Lie(SO(3))}$.

For any $a \in R$, let $S^2_a = (S^2, a\omega_{\mathsf{vol}})$ sphere of radius a. G acts on $S^2_{l_1} \times \ldots \times S^2_{l_n}$ with moment map $(v_1, \ldots, v_n) \mapsto v_1 + \ldots + v_n$.

The symplectic quotient is $M_0 = \Phi^{-1}(0)/G$. If G acts freely on $\Phi^{-1}(0)$, this is a symplectic manifold.

Canonical morphism $L_\Phi \in \mathrm{Hom}(\mathrm{M_0}, \mathrm{M})$,

$$L_\Phi = \{(m_0, m) \in M_0 \times \Phi^{-1}(0), Gm = m_0\}.$$

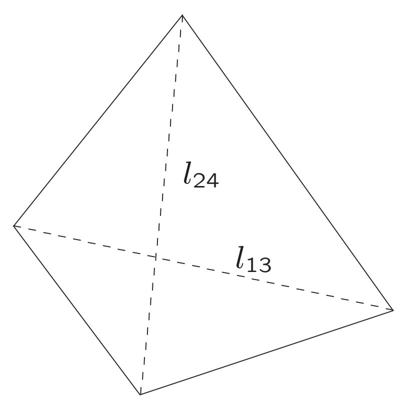
For $L \subset M$ invariant, lagrangian we have

$$L_\Phi \circ L = L_0 := (L \cap \Phi^{-1}(0))/G.$$

Example: $(S^2_{l_1} \times \ldots \times S^2_{l_n})_0$ is the moduli space of closed n-gons with lengths $l_1, \ldots, l_n$, and has dimension $2n-6$. For $n=4$, M_0 is diffeomorphic to S^2, a point, or empty.

Let $l_{13} > 0$ and $L = \{(v_{12}, v_{23}, v_{34}, v_{14}), \;\; v_{12} + v_{23} = v_{34} + v_{14}, \; \|v_{12} + v_{23}\| = l_{13}\}$.

Let $\lambda_{ij} : M_0 \to \mathbb{R}$ be the length between the i-th and j-th vertices. Then $L_0 = \lambda_{13}^{-1}(l_{13})$.



Geometric Quantization:

Given a symplectic manifold (M, ω) ("phase space") we would like a vector space $\mathbb{H}_M$ ("quantum phase space"). For each $L \in \mathrm{Hom}(\mathrm{M}_1, \mathrm{M}_2)$ should be a subset of $\mathrm{Hom}(\mathbb{H}_{\mathrm{M}_1}, \mathbb{H}_{\mathrm{M}_2})$.

In other words, we want a quantum category Q together with a symbol functor to the symplectic category which keeps only the semiclassical data and a forgetful functor from Q to the category of vector spaces, which forgets the geometric data.

(With apologies to E. Nelson) First quantization is a category, second quantization is a functor. (Which category is still a mystery.)

Two situations in which this “works”:

Homogeneous Case. $M = T^*X$, for some manifold X and $\mathrm{Hom}(\mathsf{M}_1, \mathsf{M}_2)$ = homogeneous Lagrangians. The quantum Hilbert space is $L^2(X)$. Attached to $L_{12} \in \mathrm{Hom}(\mathsf{M}_1, \mathsf{M}_2)$ is the set of Fourier integral operators (FIO) of Hörmander. (In order to act on L^2, these have to be of non-negative order.)

Kähler case. M is a compact Kähler manifold. What is the analog of FIO's in this case? I will describe a (not very deep) candidate, Bohr-Sommerfeld sequences.

Prequantization with half-forms: (Hörmander, analysis of ψdo's III: In order to get good functorial properties [for FIO's], one needs half-forms.)

A pre-quantum line bundle is a Hermitian line bunde with connection $(\mathcal{L}, \nabla)$ such that the curvature is $(2\pi/i)\omega$.

A pre-quantized symplectic manifold is a symplectic manifold with pre-quantum line bundle.

A Lagrangian L is Bohr-Sommerfeld if $(\mathcal{L}, \nabla)|L$ is trivial.

Let $\mathcal{K} \to M$ denote the canonical bundle of M, the top complex exterior power of the tangent bundle. Let $\omega_{\mathcal{K}} \in \Omega^2(M)$ denote its curvature, and for any $k > 0$ let $\omega_k = k\omega - \frac{1}{2}\omega_{\mathcal{K}}$ the shifted two-form. For k sufficiently large, this is symplectic.

A Bohr-Sommerfeld Lagrangian with half form at level k is a Bohr-Sommerfeld Lagrangian for (M, ω_k) together with a non-vanishing section σ_L of $\mathcal{K}^{1/2}|L$ and a trivialization τ_L of $\mathcal{L} \otimes \mathcal{K}^{-1/2}|L$.

Pre-quantized symplectic manifolds, with Bohr-Sommerfeld Lagrangians equipped with half-forms form a category in a natural way.

Composition involves a pairing of the dual of the symplectic form, with the half-forms attached to the Lagrangians.

If M is a point, then a Bohr-Sommerfeld Lagrangian with half-form is a point equipped with a complex number.

If L_1, L_2 are Bohr-Sommerfeld Lagrangians with half-forms in $\overline{M}, M$, $\dim(M) = n$, then their composition (pairing) $L_1 \circ L_2$ is

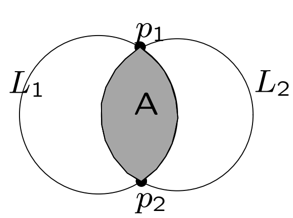
$$L_1 \circ L_2 = \sum_{p \in L_1 \cap L_2} \frac{\tau_{L_1}(p)\overline{\tau_{L_2}(p)}\exp(\pi i \mu(p)/4)}{\sqrt{\omega^n(\sigma^2_{L_1}(p) \wedge \sigma^2_{L_2}(p))}}$$

where $\mu(p)$ is a Maslov index.

If $\dim(M) = 2$ and L_1, L_2 are circles invariant under an anti-symplectic involution, then

$$|L_1 \circ L_2| = \frac{\cos(A/2 + \pi/4)}{\sqrt{\omega(\frac{\partial}{\partial\theta_1}\frac{\partial}{\partial\theta_2})}}$$

where $\frac{\partial}{\partial\theta_j}$ are the vector fields on L_j induced from the half-forms.



Special case M is the moduli space of closed four-gons with lengths $l_{12}, l_{23}, l_{34}, l_{14} \in \mathbb{Z}/2$.

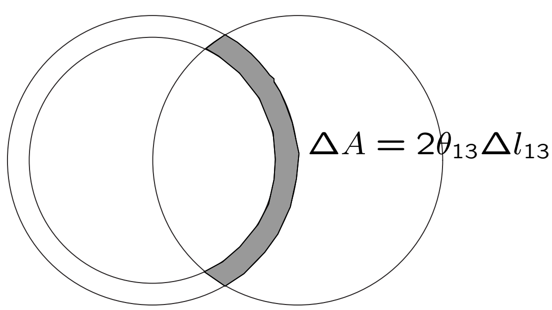
Let $L_{13} = \lambda_{13}^{-1}(l_{13})$ and $L_{24} = \lambda_{24}^{-1}(l_{24})$.

Let T be the tetrahedron with edge lengths l_{ab} and θ_{ab} its exterior dihedral angles. Take any half-forms whose square correspond to the vector fields $l_{13}\frac{\partial}{\partial\theta_{13}}$, $l_{24}\frac{\partial}{\partial\theta_{24}}$ on L_{13}, L_{24}.

Lemma: l_{ab}, θ_{ab} for $ab = 13$ or 24 are action-angle coordinates on M:

$$\omega = \mathrm{d}\lambda_{13} \wedge \mathrm{d}\theta_{13} = \mathrm{d}\lambda_{24} \wedge \mathrm{d}\theta_{24}.$$

This implies



Using Schlaffli's formula $0 = \sum_{a<b} \mathrm{d}\theta_{ab} l_{ab}$ we get

$$A = \sum_{a<b} \theta_{ab} l_{ab} \ \mathrm{mod} \pi\mathbb{Z}.$$

Wigner shows

$$\omega(\frac{\partial}{\partial\theta_{13}}, \frac{\partial}{\partial\theta_{24}}) = \frac{\partial l_{24}}{\partial\theta_{13}} = \pm \frac{12\mathrm{Vol(T)}}{l_{13} l_{24}}.$$

This gives

$$|L_{13} \circ L_{24}| = \frac{|\cos(\sum_{a<b} \theta_{ab} l_{ab} + \pi/4)|}{\sqrt{12\mathrm{Vol(T)}}}.$$

Kähler Quantization:

Obj(Q): pre-quantized compact Kähler manifolds with square root $\mathcal{K}^{1/2}$ of $\mathcal{K}$.

Let L_k be a sequence of Bohr-Sommerfeld Lagrangians with half-forms for $(M_1 \times M_2, -\omega_{1,k} + \omega_{2,k})$ converging to a limiting Lagrangian L_∞.

$\mathrm{Hom}(\mathrm{M}_1, \mathrm{M}_2)$: sequence $s_k \in H^0(L_k)$ of holomorphic sections such that s_k is asymptotically Guassian near L_k as $k \to \infty$.

In the trivialization given by parallel transport normal to L_k require

$$s_k \cong \exp(-2\pi g_k(\xi, \xi))$$

where g_k is the metric associated to ω_k,

Require the error $\epsilon_k = LHS/RHS - 1$ to satisfy $\epsilon_k(l, \xi k^{-1/2}) = O(k^{-1/2})$.

$1 \in \mathrm{Hom}(\mathrm{M}, \mathrm{M})$: the sequence $\mathrm{Id} \in \mathrm{H}^0(\mathrm{L}_\mathrm{k}^{-1} \otimes \mathrm{L}_\mathrm{k}) \cong \mathrm{Hom}(\mathrm{H}^0(\mathrm{L}_\mathrm{k}), \mathrm{H}^0(\mathrm{L}_\mathrm{k}))$.

Theorem: Q is a "tensor category" with symbol functor to sequences of Bohr-Sommerfeld Lagrangians with half-forms.

Warning: I can only prove the composition axiom in a weak sense, that is, I cannot prove that if $s_1 \in \mathrm{Hom}(\mathrm{M}_1, \mathrm{M}_2)$ and $s_2 \in \mathrm{Hom}(\mathrm{M}_2, \mathrm{M}_3)$ then $s_1 \circ s_2 \in \mathrm{Hom}(\mathrm{M}_1, \mathrm{M}_3)$. Instead, I can prove that $(s_1 \circ s_2, s_3)$ has the expected asymptotics.

This is a horrible situation: the Kähler case should be easier than the homogeneous case considered by Hörmander. Yet fifty years later, we still do not seem to have comparable results.

Reduction axiom: under the $H^0(\mathcal{L}_k)^G \cong H^0(\mathcal{L}_{k,0})$, a Bohr-Sommerfeld sequence concentrating near L_k maps to a Bohr-Sommerfeld sequence concentrating near $L_{k,0}$, and vice versa.

Application to 6j symbols:

$G = SU(2), V_j \cong V_j^*, j \in (\mathbb{Z}/2)_{\geq 0}$ the irreducible representations. Fix $j_{ab} \in (\mathbb{Z}/2)_{\geq 0}$ for $1 \leq a, b \leq 4$.

For a, b, c, d distinct, there is a (canonical) tri-linear invariant

$$\epsilon_a \in (V_{j_{bc}} \otimes V_{j_{cd}} \otimes V_{j_{bd}})^G$$

if and only if the Euclidean triangle inequalities hold:

$$j_{bc} \leq j_{cd} + j_{bd}, \;\; j_{cd} \leq j_{bc} + j_{bd}, \;\; j_{bd} \leq j_{bc} + j_{cd}.$$

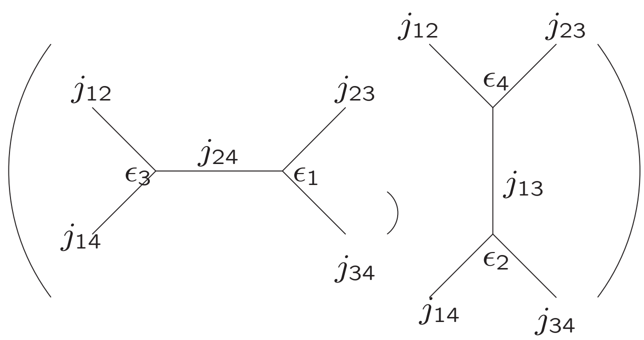
Note

$$\epsilon_1 \circ \epsilon_3, \epsilon_2 \circ \epsilon_4 \in (V_{12} \otimes V_{23} \otimes V_{34} \otimes V_{14})^G.$$

Define the $6j$ symbol

$$\left\{ \begin{array}{ccc} j_{12} & j_{23} & j_{13} \\ j_{34} & j_{14} & j_{24} \end{array} \right\} = (\epsilon_1 \circ \epsilon_3, \epsilon_2 \circ \epsilon_4).$$

Graphically $(\epsilon_1 \circ \epsilon_3, \epsilon_2 \circ \epsilon_4) =$



Proof of Ponzano-Regge:

Set $j_{ab}(k) = k(j_{ab} - \frac{1}{2}) + \frac{1}{2}$. Reduction Property: $\epsilon_{d,k} \in \mathrm{Hom}(\mathrm{pt}, \mathsf{S}^2_{\mathsf{j_{ab}(k)}} \times \mathsf{S}^2_{\mathsf{j_{bc}(k)}} \times \mathsf{S}^2_{\mathsf{j_{ac}(k)}})$ (Ignoring factor $\mathrm{Vol}(\mathrm{SO}(3))^{1/2}\mathrm{k}^{3/4}$). Concentrates to $\{v_{ab} + v_{bc} + v_{ac} = 0\}$.

Composition Property: $\epsilon_{1,k} \circ \epsilon_{3,k} \in \mathrm{Hom}(\mathrm{pt}, \mathsf{S}_{\mathsf{j_{12}(k)}} \times \mathsf{S}_{\mathsf{j_{23}(k)}} \times \mathsf{S}_{\mathsf{j_{34}(k)}} \times \mathsf{S}_{\mathsf{j_{14}(k)}})$. Concentrates to $\{v_{23} + v_{24} = v_{12} + v_{14}, |v_{23} + v_{24}| = l_{24}\}$. Similar for $\epsilon_{2,k} \circ \epsilon_{4,k}$.

Reduction Property: $\epsilon_{1,k} \circ \epsilon_{3,k}, \epsilon_{2,k} \circ \epsilon_{4,k} \in \mathrm{Hom}(\mathrm{pt}, (\mathsf{S}_{\mathsf{j_{12}(k)}} \times \mathsf{S}_{\mathsf{j_{23}(k)}} \times \mathsf{S}_{\mathsf{j_{34}(k)}} \times \mathsf{S}_{\mathsf{j_{14}(k)}})_0)$. Concentrate to $\lambda_{13}^{-1}(l_{13})$ and $\lambda_{24}^{-1}(l_{24})$ respectively.

Pairing formula (special case of Composition): Ponzano-Regge-Roberts formula.

Questions I would like to see answered:

(1) Is there a composition property for Kähler quantization. Or at least, show that $\epsilon_{1,k} \circ \epsilon_{3,k}$ and $\epsilon_{2,k} \circ \epsilon_{4,k}$ really do concentrate to the Lagrangians I described.

(2) What does the moduli space of closed 4-gons look like as a Kähler manifold? It is homeomorphic to a two-sphere, but not isometric.

(3) Extend the category above to the case where the Lagrangians do not meet transversally, e.g., the caustic case should involve the Airy function.

(4) What if the sequence of holomorphic sections concentrates to an isotropic submanifold. (E.g. coherent states?)

Warning: this subject is not very fashionable among symplectic geometers right now.

II Semiclassical asymptotics of quantum $6j$ symbols

Abstract: joint work with Yuka Taylor on the asymptotics of $6j$ symbols for $U_q(\mathfrak{sl}_2)$ for $|q| = 1$ (spherical case) or q real (hyperbolic case). There are eight different cases; we solved seven of them. The last, classically forbidden case, is expected to show exponential decay, only if q is a primitive root of unity.

Warning: the word quantum here is not the same quantum as in the first part.

Fix $q = \exp(\pi i/r)$, primitive $2r$-th root of unity.

Kulish and Reshetikhin: quantized enveloping algebra is the algebra $U_q(\mathfrak{sl}_2)$ with generators E, F, K, K^{-1} and relations

$$[E, F] = \frac{K^2 - K^{-2}}{q - q^{-1}}, \;\; KE = qEK, \quad KF = q^{-1}FK.$$

A $U_q(\mathfrak{sl}_2)$-module V is traceless if

$$\mathrm{Tr_q}(\cdot) = \mathrm{Tr}(\mathrm{K}^2 \cdot)$$

vanishes on $\mathrm{End}_{\mathrm{U_q}(\mathfrak{sl}_2)}(\mathrm{V})$.

Consider the tensor category of finite dimensional $U_q(\mathfrak{sl}_2)$-modules/traceless representations.

Irreducible repns $V_j, j = 0, \frac{1}{2}, 1 \ldots, (r-2)/2$.

The multiplicity space $\mathrm{Hom}_{\mathrm{U_q}(\mathfrak{sl}_2)}(\mathrm{V_j}, \mathrm{V_{j_1}} \otimes \mathrm{V_{j_2}})$ is non-zero if and only if the spherical triangle inequalities hold:

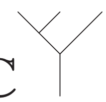
$$|j_1 - j_2| \leq j \leq \min(j_1 + j_2, r - 2 - j_1 - j_2).$$

This is the condition for a triangle with side lengths j, j_1, j_2 to exist in a sphere of radius $(r-2)/2\pi$.

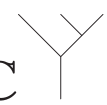
Question: How does this generalize to spherical tetrahedra (or more generally, any simplex)?

If $\mathrm{Hom}_{U_q(\mathfrak{sl}_2)}(V_j, V_{j_1} \otimes V_{j_2})$ is non-trivial then there is a canonical choice of generator $\curlyvee(j; j_1, j_2)$

By associativity of the tensor product we have a basis for $\mathrm{Hom}_{U_q(\mathfrak{sl}_2)}(V_{j_{14}}, V_{j_{12}} \otimes V_{j_{23}} \otimes V_{j_{34}})$ for each way of parenthesizing:

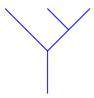
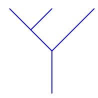
$$\mathrm{Hom}_{U_q(\mathfrak{sl}_2)}(V_{j_{14}}, (V_{j_{12}} \otimes V_{j_{23}}) \otimes V_{j_{34}}) = \bigoplus_{j_{13}} \mathbb{C}$$

$$(j_{13})$$

and

$$\mathrm{Hom}_{U_q(\mathfrak{sl}_2)}(V_{j_14}, V_{j_{12}} \otimes (V_{j_{23}} \otimes V_{j_{34}})) = \bigoplus_{j_{24}} \mathbb{C}$$

$$(j_{24}).$$

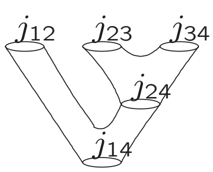
Convention: Vertical stacking = composition.

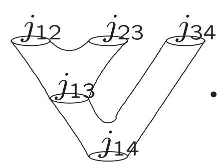
The $6j$ symbol $\left\{ \begin{matrix} j_{12} & j_{23} & j_{13} \\ j_{34} & j_{14} & j_{24} \end{matrix} \right\}$ is defined by

 $(j_{24}) = \sum_{j_{13}} \left\{ \begin{matrix} j_{12} & j_{23} & j_{13} \\ j_{34} & j_{14} & j_{24} \end{matrix} \right\}$  (j_{13}). (Wigner for $q = 1$, An. Kirillov-Reshetikhin for $q \neq 1$.)

One can modify the definition slightly so that the symbols have tetrahedral symmetry.

By applying the standard string theory idea of replacing the lines with tubes etc. one obtains the more suggestive equation (“WZW picture”)



$= \sum_{j_{13}} \left\{ \begin{matrix} j_{12} & j_{23} & j_{13} \\ j_{34} & j_{14} & j_{24} \end{matrix} \right\}$


.

Racah hypergeometric formula (easily implemented on a computer)

$$\left\{ \begin{array}{ccc} j_{12} & j_{23} & j_{13} \\ j_{34} & j_{14} & j_{24} \end{array} \right\} = \Delta(j_{12}j_{23}j_{31})\Delta(j_{13}j_{34}j_{41})\Delta(j_{23}j_{34}j_{24})\Delta(j_{12}j_{24}j_{14})$$

$\sum(-1)^z[z+1]!/([z-j_{12}-j_{23}-j_{13}]![z-j_{13}-j_{34}-j_{14}]![z-j_{23}-j_{34}-j_{24}]![z-j_{12}-j_{24}-j_{14}]![j_{12}+j_{23}+j_{34}+j_{14}-z]![j_{12}+j_{13}+j_{34}+j_{24}-z]![j_{23}+j_{13}+j_{14}+j_{24}-z]!)$

where

$$\Delta(j_{ab}, j_{bc}, j_{ac}) = \left(\frac{[j_{ab}+j_{bc}-j_{ac}]![j_{ab}-j_{bc}+j_{ac}]![-j_{ab}+j_{bc}+j_{ac}]!}{[j_{ab}+j_{bc}+j_{ac}+1]!}\right)^{1/2}.$$

$[n] = \sin(n\frac{\pi}{r})/\sin(\frac{\pi}{r})$ is the quantum integer and $[n]! = [n][n-1]\ldots[1]$ is the quantum factorial.

However, it is impossible to read off the asymptotics from this formula.

Question: What are the asymptotics of the $6j$ symbols?

For $q = 1$, Wigner,Ponzano-Regge,Schulten-Gordon, Roberts.

Let $\tau(k)$ be the Euclidean tetrahedron with lengths $l_{ab}(k) = kj_{ab} + \frac{1}{2}$, if it exists, and $\theta_{ab}(k)$ denote the exterior dihedral angles.

$$\phi(k) = \sum_{a<b} l_{ab}(k)\theta_{ab}(k).$$

Theorem: [conjectured by Ponzano-Regge, proved by J. Roberts]

$$\begin{Bmatrix} kj_{12} & kj_{23} & kj_{13} \\ kj_{34} & kj_{14} & kj_{24} \end{Bmatrix} \sim \frac{\cos(\phi(k)+\pi/4)}{\sqrt{12\pi \mathrm{Vol}(\tau(\mathrm{k}))}}$$

if τ, the tetrahedron with edge lengths j_{ab}, exists and is non-degenerate.

Remarks:

The classically forbidden formula conjectured by Ponzano-Regge and Schulten-Gordon is still open. Its meaning is not clear to me.

For $q = \exp(\pi i/r)$, studied by Mizoguchi-Tada (quantum gravity with cosmological constant). Their formula seems to be different.

Re-scale

$$j_{ab} \mapsto kj_{ab}, \quad r \mapsto r(k) := k(r-2)+2.$$

If it exists, let $\tau(k)$ denote the spherical tetrahedron with edge lengths

$$l_{ab}(k) = \frac{2\pi(kj_{ab}+\frac{1}{2})}{r(k)}$$

and exterior dihedral angles $\theta_{ab}(k)$. Define

$$\phi(k) = \frac{r(k)}{2\pi}\left(\sum_{a<b}\theta_{ab}(k)l_{ab}(k) - 2\mathrm{Vol}(\tau(\mathrm{k}))\right).$$

Let τ denote the spherical tetrahedron with edge lengths

$$l_{ab} = \frac{2\pi j_{ab}}{r-2}$$

and exterior dihedral angles θ_{ab}.

Theorem [Taylor-Woodward] If τ exists and is non-degenerate, then

$$\left\{\begin{array}{ccc} kj_{12} & kj_{23} & kj_{13} \\ kj_{34} & kj_{14} & kj_{24} \end{array}\right\} \sim \frac{2\pi\cos(\phi(k)+\pi/4)}{r(k)^{3/2}\det(\cos(l_{ab}))^{1/4}}.$$

Meanings of $\sum \theta_{ab} l_{ab} - 2\text{Vol}(\tau)$:

(a) Integral of $\sum \theta_{ab} \mathrm{d} l_{ab}$, by Schläfli,

(b) Legendre transform of $2\text{Vol}(\tau)$.

(c) $2\text{Vol}(\tau')$, where τ' is the dual spherical tetrahedron. (Interchange edge lengths and dihedral angles.)

Meanings of $\det(\cos(l_{ab}))$:

(a) Euclidean volume of the 4-simplex obtained by adding a vertex at 0.

(b) Poisson bracket of l_{13}, l_{24} on the moduli space of spherical 4-gons.

(c) $G(l) = (\cos(l_{ab}))$ is the spherical generalization of the Cayley-Menger matrix

$$G_0(l) = \begin{bmatrix} 0 & 1 & 1 & 1 & \dots & 1 \\ 1 & 0 & -\frac{1}{2}l_{12}^2 & -\frac{1}{2}l_{13}^2 & \dots & -\frac{1}{2}l_{1n}^2 \\ 1 & -\frac{1}{2}l_{21}^2 & 0 & -\frac{1}{2}l_{23}^2 & \dots & -\frac{1}{2}l_{2n}^2 \\ \vdots & \vdots & \vdots & & & \vdots \\ 1 & -\frac{1}{2}l_{n1}^2 & -\frac{1}{2}l_{n2}^2 & \dots & & 0 \end{bmatrix}. \tag{1}$$

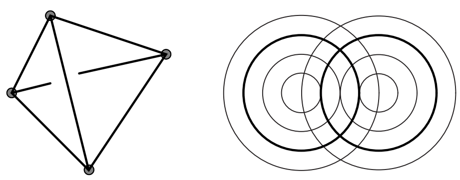
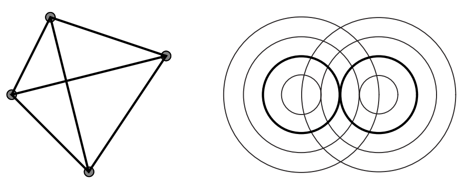
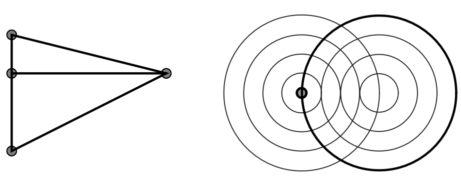
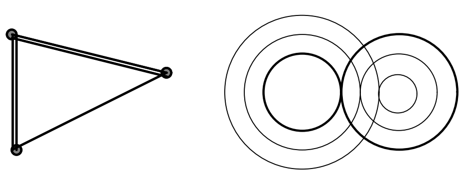
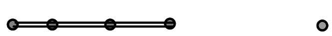
That is, a Euclidean (resp. spherical) n-simplex exists if and only if all principal minors of $G_0(l)$ resp. $G(l)$ are non-negative.

A similar result holds for $-\cosh(l_{ab})$ in the hyperbolic case.

(d) If $n = 2$, then $\det(\cos(l_{ab}))$ is the numerator in the spherical version of Heron formula for the area of a triangle.

Question: Relation of $G(l)$ to spherical volume, for $n > 2$?

Types of degeneracy of the tetrahedron τ and the corresponding asymptotics of the quantum $6j$ symbols.:

Tetrahedron Diagram	Asymptotic Formula
	$\frac{2\pi\cos(\phi+\pi/4)}{r(k)^{3/2}\det(\cos(l_{ab}))^{1/4}}$ $\phi(k) := \frac{r(k)}{2\pi}(\sum\theta_{ab}(k)l_{ab}(k) - 2\mathrm{Vol}(\tau(k)))$ $l_{ab}(k) := 2\pi\frac{kj_{ab}+\frac{1}{2}}{r(k)}$
	$r(k)^{-\frac{4}{3}}2^{\frac{2}{3}}3^{-\frac{1}{3}}\pi^{4/3}\Gamma(\frac{2}{3})^{-1}\frac{\cos(k\sum\theta_{ab}j_{ab})}{(A_1A_2A_3A_4)^{1/6}}$ $A_a := \det(\cos(l_{bc})_{b,c\neq a})^{1/2}$
	$(-1)^{k(j_{13}+j_{34}+j_{14})}2^{-\frac{1}{4}}r(k)^{-\frac{5}{4}}\pi\left\{\frac{I(l_{13},l_{34},l_{41})}{I(l_{12},l_{24},l_{41})I(l_{23},l_{34},l_{24})}\right\}^{\frac{1}{2}}$ $I(l_{ab},l_{bc},l_{ac}) := \frac{H(l_{ab},l_{bc},l_{ac})^{1/2}\sin((l_{ab}+l_{bc}+l_{ac})/2)}{\sin((l_{bc}+l_{ac}-l_{ab})/2)}$ $H(l_{ab},l_{bc},l_{ac}) := \frac{\sin(\frac{l_{ab}+l_{bc}+l_{ac}}{2})\sin(\frac{l_{ab}+l_{bc}-l_{ac}}{2})}{\sin(\frac{l_{ab}-l_{bc}+l_{ac}}{2})\sin(\frac{-l_{ab}+l_{bc}+l_{ac}}{2})}$
	$(-1)^{k(j_{13}+j_{34}+j_{14})}\pi r(k)^{-1}(\sin(l_{12})\sin(l_{34}))^{-\frac{1}{2}}$
	$(-1)^{2kj_{14}}\pi r(k)^{-1}(\sin(l_{13})\sin(l_{24}))^{-1/2}$
	$(-1)^{2kj_{14}}\sqrt{\frac{\pi}{r(k)}}\sin(l_{14})^{-\frac{1}{2}}$
	1

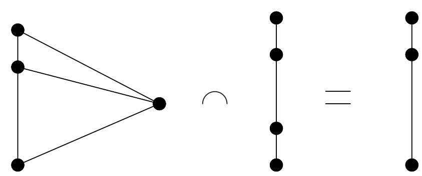
Remarks:

The powers of k in the asymptotics are

$$-3/2 \to -4/3 \to -5/4 \to -1 \to -1/2 \to 0$$

respectively.

There is an overlap between cases (d) and (e), when the vertices are colinear but $l_{23} = 0$; perhaps cases (d) and (e) should not really be considered separate:



We tested some of the formulas numerically. For instance, here is the sequence of $6j$ symbols $\left\{ \begin{array}{ccc} 25 & 25 & 25 \\ 25 & 15 & j \end{array} \right\}$ as j varies from 0 to 50, at $q = \exp(\pi i/202)$.

The amplitude $2\pi r(k)^{-\frac{3}{2}} \det(\cos(l_{ab}))^{-\frac{1}{4}}$ is also shown. The volume was computed by numerical integration.

We guessed the formulas geometrically, then proved them algebraically. First I will explain the geometry.

Let X be a compact, oriented, connected surface and $x_1, \ldots, x_n$ distinct points. Let $l_1, \ldots, l_n \in [0, \pi]$. The subspace $M(X, l_1, \ldots, l_n)$ of

$$\mathrm{Hom}(\pi_1(\mathrm{X} - \{\mathrm{x}_1, \ldots, \mathrm{x}_n\}, \mathrm{SU}(2)))/\mathrm{SU}(2)$$

such that a small loop around x_i is conjugate to $\mathrm{diag}(\exp(\mathrm{il_j}), \exp(-\mathrm{il_j}))$ is the moduli space of flat connections on $X - \{x_1, \ldots, x_n\}$ with fixed holonomy around the punctures.

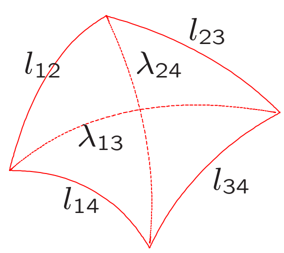
Narasimhan-Seshadri,Mehta-Seshadri,Steer-Furuta: $\mathcal{M}(X, l_1, \ldots, l_n)$ is a normal projective variety, smooth for $n \geq 3$ and $l_1, \ldots, l_n$ generic.

The complex structure on $M(X, l_1, \ldots, l_n)$ depends on the choice of location of $x_1, \ldots, x_n$.

Case $X = S^2, n = 4$.

Since $SU(2) = S^3$, and $2\pi j_{a(a+1)}/(r-2))$ is the distance to the identity, $\mathcal{M}$ is the moduli space of closed 4-gons in S^3 with edge lengths

$$l_{a(a+1)} = 2\pi j_{a(a+1)}/(r-2)).$$



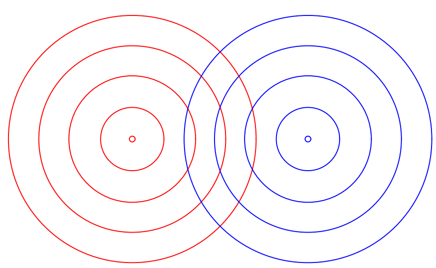
$\mathcal{M}$ is either empty, a point, or a symplectic sphere.

As a symplectic manifold, it has two sets of action-angle coordinates: The length λ_{13} and dihedral angle θ_{13}, or the length λ_{24} and dihedral angle θ_{24}.

The symplectic form on $\mathcal{M}$ is

$$\mathrm{d}\lambda_{13} \wedge \mathrm{d}\theta_{13} = \mathrm{d}\lambda_{24} \wedge \mathrm{d}\theta_{24}.$$

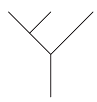
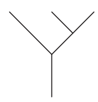
We can picture the two coordinate systems as overlapping families of circles:



The equation $l_{ab} = 0$ defines not a point but a circle.

Kohno, Drinfeld, Kazhdan, Luszdig, Finkelberg, Narasimhan, Seshadri, Mehta, Segal, Beauville, Laszlo, Pauly ....:

The space $\mathrm{Hom}_{U_q(\mathfrak{sl}_2)}(V_{j_{14}}, V_{j_{12}} \otimes V_{j_{23}} \otimes V_{j_{34}})$ is isomorphic to the space of conformal blocks: Sections of a holomorphic line bundle $\mathcal{L}(X) \to M(X)$.

The elements  (j_{13}),  (j_{24}) are represented by holomorphic sections of a holomorphic line bundle over $\mathcal{M}(X, l_{12}, l_{23}, l_{34}, l_{14})$.

According to the standard semiclassics discussed in Part I, as $k \to \infty$ the sections should concentrate near the Lagrangian manifolds defined by

$$\lambda_{13} = j_{13}/r, \quad \lambda_{24} = j_{24}/r.$$

Concentration to Lagrangians is rigorously provable using

(1) a result of Daskalopolous-Wentworth (personal communication, partly published) which says that as $z \to 0$ or 1 the complex structure becomes invariant under the 13 or 24 edge

(2) the leading term in KZ is a first-order differential operator vanishing at the lagrangian

(3) results that do not appear in the literature but everyone thinks are true!

At this point the geometry fails.

However, the same pairing computation from Part I predicts a formula.

Checks on the formula:

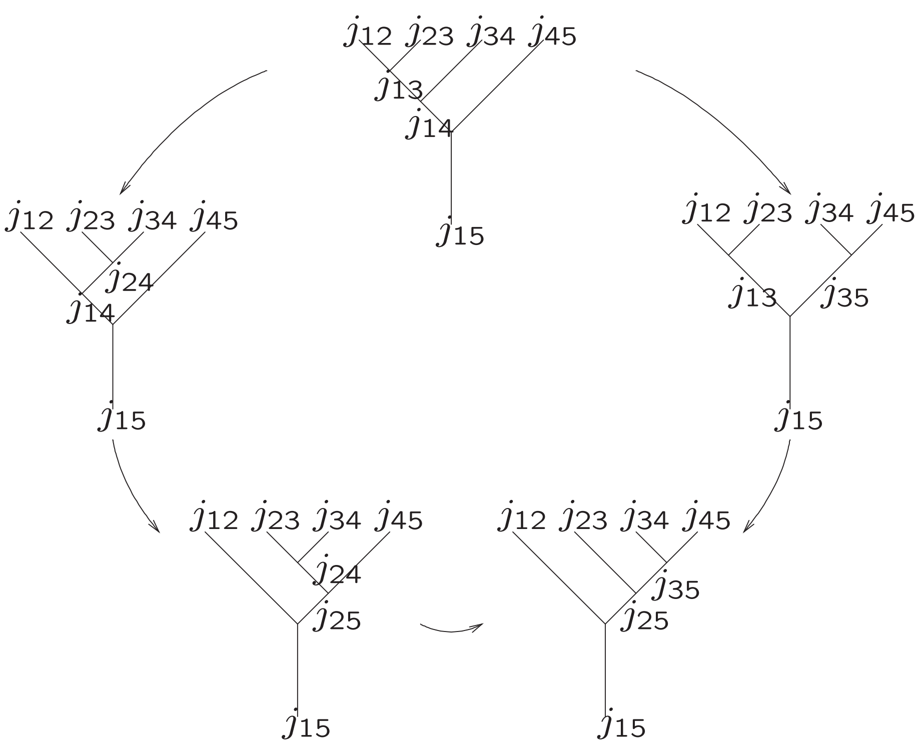
Pentagon or Biedenharn-Elliot relation

$$\left\{ \begin{matrix} j_{23} & j_{34} & j_{24} \\ j_{14} & j_{12} & j_{13} \end{matrix} \right\} \left\{ \begin{matrix} j_{23} & j_{34} & j_{24} \\ j_{45} & j_{25} & j_{35} \end{matrix} \right\} = \sum_{j_{15}} (-1)^z [2j_{15}+1]$$

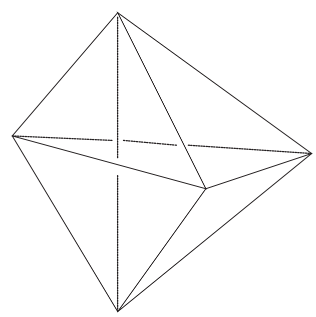
$$\left\{ \begin{matrix} j_{13} & j_{34} & j_{14} \\ j_{45} & j_{15} & j_{35} \end{matrix} \right\} \left\{ \begin{matrix} j_{12} & j_{24} & j_{14} \\ j_{45} & j_{15} & j_{25} \end{matrix} \right\} \left\{ \begin{matrix} j_{12} & j_{23} & j_{13} \\ j_{35} & j_{15} & j_{25} \end{matrix} \right\}$$

where

$$z = j_{12} + j_{13} + j_{14} + j_{15} + j_{23} + j_{24} + j_{25} + j_{34} + j_{35} + j_{45}.$$



Semiclassical version: Consider two tetrahedra T_0, T_4 intersecting in a face. The union $T_0 \cup T_4$ can be divided into three tetrahedra intersecting at the edge 04.



Proposition: $\frac{\partial \omega_{04}}{\partial l_{04}} = \sin^2(l_{04}) \sqrt{\frac{G_0 G_4}{G_1 G_2 G_3}}$.

Euclidean case by Ponzano-Regge.

Orthogonality relation:

$$\sum_{j_{14}} \Delta_{j_{14}} \Delta_m \left\{ \begin{array}{ccc} j_{12} & j_{13} & n \\ j_{34} & j_{24} & j_{14} \end{array} \right\} \left\{ \begin{array}{ccc} j_{12} & j_{13} & m \\ j_{34} & j_{24} & j_{14} \end{array} \right\} = \delta_{m,n},$$

Semiclassical version of the case $m = n$:

$$\sin(l_{cd}) \int \frac{\sin(l_{ab})}{\sqrt{G(\tau(l_{ij}))}}\, dl_{ab} = \pi$$

Rigorous proof of the asymptotic formulas:

Combining various pentagon relations gives a second order difference equation for the $6j$ symbols as one label is varied by ± 1: Set

$$f(j_{ab}) = \left(\frac{|\det(\cos(l_{ab}))|^{\frac{1}{2}}}{\sin(\theta_{ab})} \right)^{\frac{1}{2}} \left\{ \begin{matrix} j_{12} & j_{23} & j_{13} \\ j_{34} & j_{14} & j_{24} \end{matrix} \right\}.$$

Then

$$(\Delta_{ab} + 2 - 2\cos\theta_{ab} + O(r^{-2}))f(j_{ab}) = 0.$$

Here $O(r^{-2})$ denotes a second order difference operator of norm less than Cr^{-2}, for some constant C, and Δ_{ab} is the discrete Laplacian in the variable j_{ab}

$$(\Delta_{ab} f)(j_{ab}) = f(j_{ab} + 1) - 2f(j_{ab}) + f(j_{ab} - 1).$$

One can show that the asymptotic expression satisfies the same equation.

One can use PR formula (Roberts' theorem) as initial condition: scale the level $r(k) = k^N(r-2)+2$ for some $N >> 0$ and labels kj_{ab}. In this case, the quantum $6j$ symbols are asymptotic to the classical $6j$ symbols.

Take a sequence of lattice paths between the labels kj_{ab} and $k^N j_{ab}$ such that every label in the path is a non-degenerate tetrahedron, and the length of the path grows at most linearly (lemma). Using the difference equation write

$$\begin{Bmatrix} k^N j_{12} & k^N j_{23} & k^N j_{13} \\ k^N j_{34} & k^N j_{14} & k^N j_{24} \end{Bmatrix}$$

in terms of

$$\begin{Bmatrix} kj_{12} & kj_{23} & kj_{13} \\ kj_{34} & kj_{14} & kj_{24} \end{Bmatrix}$$

and then apply Roberts theorem.

The degenerate cases

Case (b) is obtained from (a) by degeneration. ("uniform WKB method").

The remaining cases are obtained from Racah's formula and "q-Stirling's formula" describing the asymptotics of quantum factorials:

Then as $k \to \infty$,

$$[kn]! \sim \sqrt{2\pi}[kn]^{kn+\frac{1}{2}}[e]^{-kn}$$

where

$$[e] := \sin(n\tfrac{\pi}{r}) \exp\left(-\tfrac{r}{n\pi}\int_0^{\frac{\pi n}{r}} \ln(\sin(x))\mathrm{d}x\right).$$

Presumably, one can derive this and the other terms in the asymptotic expansion from Kashaev's work, but I have not done so.

Remark:

Cases (a) and (b) fit into a uniform formula, involving the Airy function.

Case (c) - (f) do not, in part because we treated them by a different method.

In other words, except for case (b) we have no error estimates which describe what happens close to a degenerate case.

This will come back to haunt us later.

III Puzzles on the asymptotics of Turaev-Viro.

The Turaev-Viro invariant for a triangulated compact three-manifold Y is a the sum over labellings

$$l : \mathrm{Edge(Y)} \to \{0, \frac{1}{2}, \ldots, (\mathrm{r}-2)/2\}$$

of the product

$$\Delta^{-\#Vert(Y)} \prod_{e \in \mathrm{Edge(Y)}} \Delta_{l(e)} \prod_{t \in Tetra(Y)} \{l(t)\}_q$$

Here

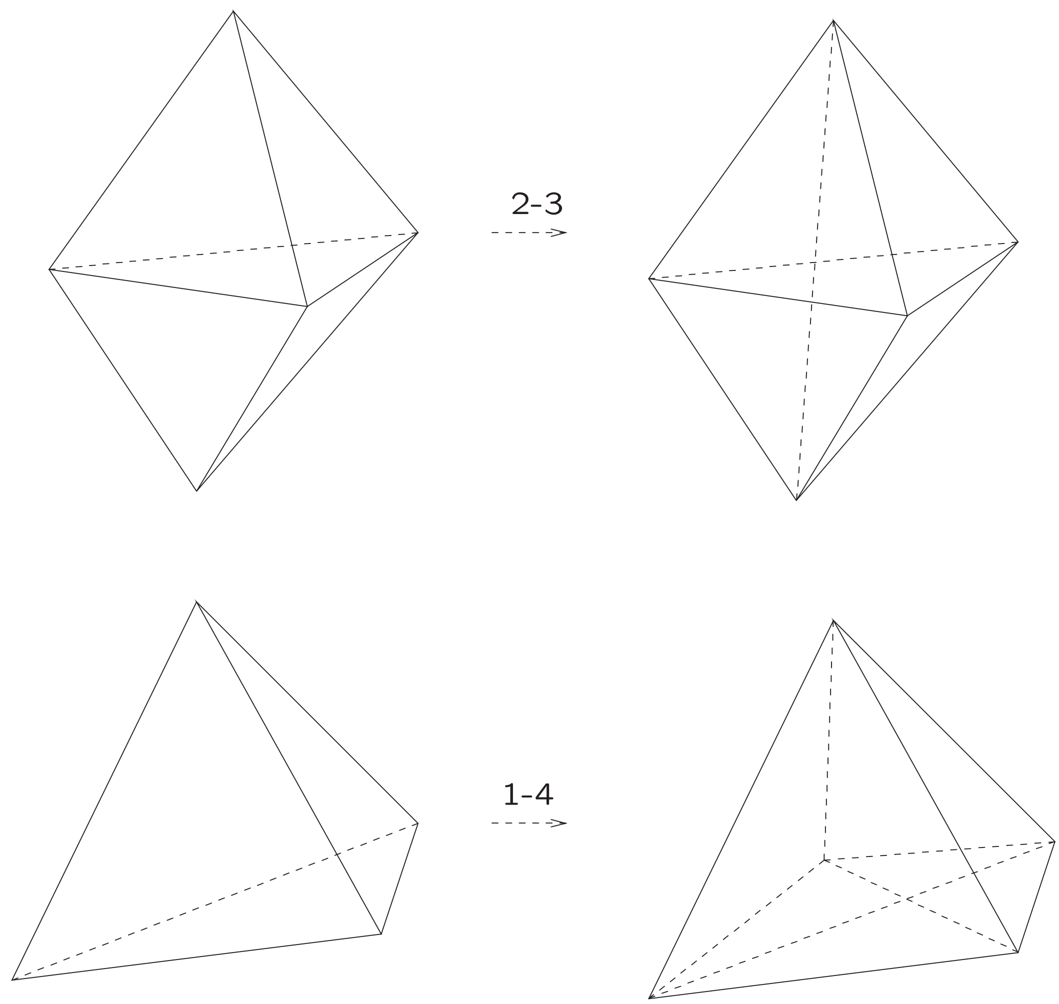
$$\Delta_l = (-1)^{2l}[2l+1].$$

and

$$\Delta = \Delta_a^{-1} \sum_{b,c} \Delta_b \Delta_c$$

and $\{l(t)\}_q$ is the quantum $6j$ symbol for the labellings of the edges of t.

Topological invariance follows from invariance under the Pachner $2-3$ and $1-4$ moves



Invariance under the $2-3$ move is exactly the pentagon identity. Invariance under the $1-4$ move is the pentagon plus orthogonality identities.

Turaev, Roberts: $TV(Y,r)$ is the norm-square of the Reshetikhin-Turaev invariant $Z(Y,r)$.

Conjectures for the asymptotics of $Z(Y,r), TV(Y,r)$:

Witten, Freed-Gompf: Say Y is a rational homology sphere. Then

$$Z(Y,r) \sim \frac{1}{2} r^{-\frac{1}{2}h^0(d_A)} e^{-3\pi i/4} \sum_{[A]\in R(Y,SU(2))} \sqrt{\tau(A)} e^{-2\pi i I_A/4} e^{2\pi i CS(A,r)}$$

where $\tau(A)$ is the torsion of A, I_A is the spectral flow, and $CS(A,r)$ the Chern-Simons invariant at level r

$$CS(A,r) = \frac{r}{8\pi^2} \int_Y \mathrm{Tr}(\mathrm{A} \wedge \mathrm{dA} + \frac{2}{3} \mathrm{A} \wedge \mathrm{A} \wedge \mathrm{A}\}.$$

(Proved by Yoshida, using a different definition of $Z(Y,r)$. Is it the same as Reshetikhin-Turaev?)

Since $Lie(SU(2)^2) = \mathrm{Lie(SO(4))}$, the prediction for Turaev-Viro is $TV(Y,r) \sim$

$$\frac{1}{4} \sum_{[(A_1,A_2)]\in R(Y,SO(4))} r^{-\frac{1}{2}h^0(d_A)} \sqrt{\tau(A_1)\tau(A_2)} e^{-2\pi(I_{A_1}-I_{A_2})/4} e^{2\pi i (CS(A_1,r)-CS(A_2,r))}.$$

Question: Can one see this using asymptotics of the quantum $6j$ symbols? Beyonds this, can one show the Reshetikhin-Turaev invariants equal the perturbative invariants of Axelrod-Singer etc.?

Answer: No because of missing error estimates and convergence results. But there is clearly more to be understood in this direction and I will go over the approach.

Each $\cos(\phi(k) + \pi/4)$ can be expanded

$$\frac{1}{2}\exp(i(\phi(k) + \pi/4)) + \exp(-i(\phi(k) + \pi/4)).$$

In the limit, the Turaev-Viro sum becomes an integral over all labellings

$$l : Edge(M) \to [0, \frac{1}{2}]$$

of a sum over sign assignments (“orientations of the tetrahedra”)

$$s : Tetra(M) \to \{\pm 1\}.$$

For each edge e, there is a factor

$$\exp(i \sum_{e \subset t} s(t)(\phi(k) + \pi/4)).$$

Applying stationary phase formally gives an integral over flat labellings, that is, labellings l such that

$$\exp(i \sum \theta_i(t) s(t)) = 1$$

where θ_i are the interior dihedral angles. If $s(t) = 1$ for all t, these correspond to spherical metrics on M. (But fixing the value of s does not define a topological invariant!)

Any flat labelling defines a developing map of the universal cover $\tilde{Y}$ of Y to S^3, and hence a representation

$$\rho : \ \pi_1(Y) \to Isom(S^3) = SO(4).$$

Contribution from the Gram matrices and the stationary phase determinant:

Let $H = (\frac{\mathsf{d}\omega_i}{\mathsf{d}l_j})_{i,j\in\mathsf{Edge(Y)}}$. By Schläfli, H is the Hessian of

$$\sum_{e\in\mathsf{Edge(Y)}} \omega_{l_e,s} l_e - \sum_{\tau\in\mathsf{Tet(Y)}} s(\tau)2\mathsf{vol}(|\tau|)$$

so H is symmetric.

Let $\mathcal{E} \subset$ Edge(Y) be a maximal subset of edges such that the sub-matrix $H_{\mathcal{E}\mathcal{E}} \subset A$ is positive definite. The contribution is

$$I(Y,[\rho]) := (\frac{1}{2\pi})^{\#\mathsf{Vert}} \sum_s \int_{l\in\mathcal{L}_{\flat,[\rho],s}} \prod_{\tau\in\mathsf{Tet}} G(l(\tau))^{-1/4} \prod_{e\in\mathsf{Edge}} \sin(l_e) \frac{\bigwedge_{e\in\overline{\mathcal{E}}} \mathsf{d}l_e}{\sqrt{\det(H_{\mathcal{E}\mathcal{E}})}}.$$

Theorem: $I(Y,[\rho])$, if convergent, is invariant under the Pachner moves. It converges for S^3.

Warning: The answer differs from the predicted asymptotics of $TV(S^3,r)$ by factors of 2 and π.

Leading power of k:

Let t, e, v denote the size of the sets Tet(Y), Edge(Y), Vert(Y).

The product over vertices contributes k^{-3v}.

The sum in Turaev-Viro contributes k^{e}.

The product over edges contributes k^{e}.

The stationary phase contributes $k^{\#\mathcal{E}/2}$.

The product over tetrahedra contributes $k^{-3t/2}$.

Collecting these together and using that the Euler characteristic $v - e + t$ is zero, we obtain

$$k^{-\frac{1}{2}h^0(d_A)}$$

as the leading power of k, as expected.

The volumes/Chern-Simons invariant:

The volumes in the numerator contribute

$$\exp\left(\frac{i}{\pi}\sum_{\tau\in\mathsf{Tet(Y)}}\pm\mathsf{Vol}(\tau)\right).$$

Let $A = (A_1, A_2)$ be an $SU(2)^2$ connection, with holonomy representation ρ. A computation shows this is $2\pi(CS(A_1) - CS(A_2))$, as expected.

Factors of $\pi i/4$:

Each tetrahedron contributes $\exp(\pm\pi i/4)$. Stationary phase contributes $\exp(\pi i \mathrm{sign}(\mathrm{H}_{\mathcal{E}\mathcal{E}})/4)$.

Question: Do these combine to the spectral flow? Is there a combinatorial formula for the spectral flow?

Question: Is the integral I equal to the square root of the torsion of ρ?

The determinant of $H_{\mathcal{E}\mathcal{E}}$ does have an interpretation in terms of torsion (Korepanov for Euclidean case) Let

$$V = \mathsf{Map}(\mathsf{Vert}(\tilde{\mathsf{Y}}), \mathsf{S}^3)^{\pi_1(\mathsf{Y})}$$

$$E = \mathsf{Map}(\mathsf{Edge}(\mathsf{Y}), [0, \pi])$$

and $SO(4)_\rho$ the centralizer of the holonomy of ρ.

Then there is an acyclic complex

$$0 \to \mathfrak{so}(4)_\rho \to T_p V \to T_l E \to T_{\hat{l}} E \to T_{\hat{p}} V \to \mathfrak{so}(4)_\rho \to 0$$

The third map takes lengths of edges, the fourth map takes defect angles. How does its torsion relate to the torsion of ρ?

Fundamental problem for this approach:

Recall that the asymptotic decay of the degenerate cases

$$k^{-4/3}, k^{-1}, k^{-1/2}, 1$$

is slower than the asymptotic decay $k^{-3/2}$ of the non-degenerate case.

Show that the contribution from degenerate configurations to the Turaev-Viro invariant vanishes. (Or does it?)