SPHERICAL TETRAHEDRA AND INVARIANTS OF 3-MANIFOLDS

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1. Introduction

Let Y be an oriented closed three-manifold and r a positive integer. The Reshetikhin-Turaev invariant $Z(Y,r)$ and Turaev-Viro invariant $TV(Y,r)$ are three-manifold invariants that attempt to make rigorous the Hamiltonian formulation of quantum Chern-Simons theory. $Z(Y,r)$ is constructed using the R-matrix of the quantized enveloping algebra $U_q(\mathfrak{sl}_2)$ and Kirby moves, while $TV(Y,r)$ is based on the $6j$ symbols for $U_q(\mathfrak{sl}_2)$ and a choice of triangulation. Turaev [18] and Roberts [15] independently showed that the $TV(Y,r)$ is the square of the modulus of $Z(Y,r)$.

On the other hand, the Lagrangian (path integral) formulation of quantum Chern-Simons theory leads to perturbative invariants developed in [1, 2, 10]. The leading term for the invariant is conjectured in [5] to involve the torsion, the Chern-Simons invariant, and the spectral flow for flat $SU(2)$ bundles on Y. An interesting mathematical problem is whether the two formulations can be shown to agree. A proof that the leading term is the same for lens spaces and torus bundles was given in Jeffrey [6]. Yoshida [19] recently announced a proof of equality of the leading term for a rational homology sphere, using a different definition of $Z(Y,r)$.

In this paper we apply our previous work on asymptotics of the quantum $6j$ symbols [16] to the asymptotics of $TV(Y,r)$ as $r \to \infty$. Substituting the asymptotic formula and applying stationary phase yield a finite dimensional integral involving Gram matrices of spherical tetrahedra which turns out to be a spherical version of an integral considered by Ponzano-Regge [14] and Korepanov [9]; see also Mizoguchi and Tada [12]. Unfortunately, we have nothing rigorous to say about the asymptotics because of various problems involving convergence of the integral and error estimates for the asymptotics of the $6j$ symbols. The modest results of this paper are a proof that the integral is invariant under the Pachner moves, as one would expect from the connection with Turaev-Viro, and of convergence for the sphere S^3.

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2. $6j$ Symbol for $U_q(\mathfrak{sl}_2)$ and its asymptotic formula

Let $U_q(\mathfrak{sl}_2)$ denote the quantized enveloping algebra at a primitive $2r$-th root of unity $q = \exp(\pi i/r)$. Let $[n]_q$ be the *quantum integer* n defined by

$$[n]_q = \frac{q^n - q^{-n}}{q - q^{-1}} = \frac{\sin(n\pi/r)}{\sin(\pi/r)}$$

for $n \in \mathbb{Z}$. We say that a half-integer j is a *color* at level r if

$$0 \leq j \leq \frac{r-2}{2}.$$

For any color j, define

$$\Delta_j = (-1)^{2j}[2j+1].$$

A triple of colors $j_1, j_2, j_3 \in \mathbb{Z}/2$ is called *admissible* if

$$\max(j_1 - j_2, j_2 - j_1) \leq j_3 \leq \min(j_1 + j_2, r - 2 - j_1 - j_2)$$

and

$$j_1 + j_2 + j_3 \in \mathbb{Z}.$$

The quantity

$$\Delta = \Delta_a^{-1} \sum_{b,c,(a,b,c) \text{ admissible}} \Delta_b \Delta_c = r \sin(\pi/r)^{-2}$$

[17] and in particular is independent of a.

For any 6-tuple of colors $j_{ab}, 1 \leq a < b \leq 4$, the *quantum* $6j$ *symbol*

$$\left\{ \begin{array}{ccc} j_{12} & j_{23} & j_{13} \\ j_{34} & j_{14} & j_{24} \end{array} \right\}$$

is a rational number obtained from associativity of the tensor product for representations of $U_q(\mathfrak{sl}_2)$. There are two standard conventions for $U_q(\mathfrak{sl}_2)$ with tetrahedral symmetry, which are related by a sign

$$(-1)^{\sum_{a<b} 2j_{ab}}.$$

The reader should note that the Turaev-Viro convention is different from the convention in our earlier paper [16], which was chosen because it agrees with the accepted conventions for $q = 1$. The $6j$ symbol satisfies the *orthogonality relations* [4]

$$\sum_{j_{14}} \Delta_{j_{14}} \Delta_m \left\{ \begin{array}{ccc} j_{12} & j_{13} & n \\ j_{34} & j_{24} & j_{14} \end{array} \right\} \left\{ \begin{array}{ccc} j_{12} & j_{13} & m \\ j_{34} & j_{24} & j_{14} \end{array} \right\} = \delta_{m,n}, \tag{1}$$

and the *pentagon* or Biedenharn-Elliot relation

$$\tau(1234)\}\{\tau(2345)\} = \sum_{j_{15}} (-1)^z [2j_{15}+1]\{\tau(1235)\}\{\tau(1345)\}\{\tau(1245)\} \tag{2}$$

where z is the sum of all j_{ab}, $a, b \in \{1,2,3,4,5\}$ and (j_{23}, j_{34}, j_{24}) is q-admissible, and $\{\tau(abcd)\}$ is short for

$$\{\tau(abcd)\} = \left\{ \begin{array}{ccc} j_{ab} & j_{bc} & j_{ac} \\ j_{cd} & j_{ad} & j_{bd} \end{array} \right\}.$$

In our previous paper [16] we obtained the following result on the asymptotics of the quantum $6j$ symbols as the labels and level are simultaneously rescaled. Set

$$r(k) \equiv k(r-2) + 2.$$

Let τ denote the tetrahedron in the sphere S^3 with edge lengths

$$l_{ab} = 2\pi \left(\frac{kj_{ab} + \frac{1}{2}}{r(k)} \right), \tag{3}$$

if it exists, and let θ_{ab} denote the exterior dihedral angles. Define

$$\phi = \frac{r(k)}{2\pi} \left(\sum_{a<b} l_{ab}\theta_{ab} - 2\,\mathrm{vol}(\tau) \right)$$

and

$$G(\tau) = \det(\cos(l_{ab}))$$

where $(\cos(l_{ab}))$ is the spherical 4×4 Gram matrix. Then,

$$\left\{ \begin{array}{ccc} kj_{12} & kj_{13} & kj_{23} \\ kj_{34} & kj_{24} & kj_{14} \end{array} \right\}_q \sim \frac{2\pi \cos(\frac{\pi}{4} + \phi)}{(r(k))^{\frac{3}{2}} G^{\frac{1}{4}}(\tau(l_{ab}))}, \tag{4}$$

if τ exists and is non-degenerate.

3. The Turaev-Viro invariant

Let Y be a compact triangulated 3-manifold with tetrahedra $\mathrm{Tet}(Y)$, triangles $\mathrm{Tri}(Y)$, edges $\mathrm{Edge}(Y)$, and vertices $\mathrm{Vert}(Y)$. A *coloring* of Y at an integer $r \geq 2$ is a map

$$j : \mathrm{Edge}(Y) \to \left\{ 0, \frac{1}{2}, \dots, \frac{r-2}{2} \right\}.$$

For each such coloring, define

$$TV(Y, r, j) = \Delta^{-v(Y)} \prod_{e \in \mathrm{Edge}(Y)} \Delta_{j(e)} \prod_{\tau \in \mathrm{Tet}(Y)} \{j(\tau)\}_q$$

$j(\tau)$ denotes the vector of values of j on the 6 edges of a tetrahedron τ, $\{j(\tau)\}_q$ is the $6j$-symbol for $U_q(\mathfrak{sl}_2), q = \exp(\pi i/r)$ for the colors associated to the edges of the tetrahedron τ. The Turaev-Viro invariant of Y is

$$TV(Y, r) = \sum_j TV(Y, r, j). \tag{5}$$

The pentagon and orthogonality identities imply that $TV(Y,r)$ is invariant under the Pachner 2-3 and 1-4 moves and hence independent of the triangulation, that is, a topological invariant of Y.

4. Non-Euclidean Tetrahedra

This section provides various elementary facts about non-Euclidean tetrahedra relevant to this paper. For the proof, we refer to [16]. Let E^n, S^n denote n-dimensional Euclidean, spherical space respectively. Let S_n denote an n-dimensional simplex and l_{ab} edge lengths in S_n. The *Cayley-Menger determinant* for a Euclidean simplex S_n, denoted by $G_0(l_{ab})$, is defined by

$$G_0(l_{ab}) = \det \begin{pmatrix} 0 & 1 & 1 & 1 & \dots & 1 \\ 1 & 0 & -\frac{1}{2}l_{12}^2 & -\frac{1}{2}l_{13}^2 & \dots & -\frac{1}{2}l_{1n}^2 \\ 1 & -\frac{1}{2}l_{21}^2 & 0 & -\frac{1}{2}l_{23}^2 & \dots & -\frac{1}{2}{l_{2n}}^2 \\ \vdots & \vdots & \vdots & & \dots & \vdots \\ 1 & -\frac{1}{2}l_{n1}^2 & -\frac{1}{2}l_{n2}^2 & \dots & \dots & 0 \end{pmatrix}. \tag{6}$$

For a spherical simplex, we define $n \times n$ *Gram matrix*

$$G(l_{ab}) = \det(\cos(l_{ab})).$$

Note that this is the volume of a Euclidean $n+1$-simplex with n vertices on the unit sphere and one at 0. We will need later the following facts on Euclidean and spherical simplices; the hyperbolic versions are discussed in [16].

Theorem 4.0.1. (a) *(Cayley formula,* [3, p. 98]*) If a simplex S with edge lengths l_{ab} exists in E^n, then*

$$(n!\,\mathrm{Vol}(S))^2 = G_0(l_{ab}).$$

(b) *(Schläfli formula,* [11, p. 281]*) For an n-simplex S in E^n or S^n,*

$$(n-1)\kappa\, d\mathrm{Vol}_n(S) = \sum \mathrm{Vol}_{n-2}(F)d\theta_F$$

qwhere the sum is over $(n-2)$-dimensional faces F of the simplex S, θ_F is the exterior dihedral angle around F and $\kappa = 0, 1$ is the curvature.

(c) *For the case of a triangle, one has the factorizations:*

$$G_0 = \frac{1}{4}(l_{12}+l_{23}+l_{13})(l_{12}+l_{23}-l_{13})(l_{12}-l_{23}+l_{13})(-l_{12}+l_{23}+l_{13})$$
$$G = 4\sin(\frac{1}{2}(l_{12}+l_{23}+l_{13}))\sin(\frac{1}{2}(l_{12}+l_{23}-l_{13}))\sin(\frac{1}{2}(l_{12}-l_{23}+l_{13}))$$
$$\sin(\frac{1}{2}(-l_{12}+l_{23}+l_{13}))$$

(d) *A Euclidean triangle exists if and only if*

$$l_{12} \le l_{13} + l_{23},\ \ l_{13} \le l_{12} + l_{23}\ l_{23} \le l_{12} + l_{13}. \tag{7}$$

A spherical triangle exists if and only if (7) *and*

$$l_{12}+l_{13}+l_{23}\leq 2\pi.$$

(e) *A non-degenerate tetrahedron with edge lengths l_{ab} exists in E^3, S^3 respectively if and only if l_{ab} satisfy* (7) *for faces and $I>0$, $G>0$ respectively.*

(f) *The derivative of an edge length l_{ab} in a Euclidean resp. spherical tetrahedron τ with respect to an opposite dihedral angle θ_{cd} is given by*

$$\frac{\partial l_{ab}}{\partial\theta_{cd}}=\pm\frac{G_0^{1/2}(l_{ij})}{l_{ab}l_{cd}},\qquad \frac{\partial l_{ab}}{\partial\theta_{cd}}=\pm\frac{G^{1/2}(l_{ij})}{\sin(l_{ab})\sin(l_{cd})},$$

5. Asymptotic Pentagon and Normalization Identities

In this section, we prove several geometric identities which may be viewed as semi-classical analogs of the identities (2), (9) for $6j$ symbols. We will use them when we discuss 3-manifold in section 6. The Euclidean versions are due to Ponzano and Regge [14]. Starting from this section, we fix an integer $r\geq 3$ and $q=\exp(\frac{\pi i}{r})$.

A simplex spanned by vertices $v_0,\ldots,v_n$ is denoted by $S_{0\ldots n}$. If the simplex is two-dimensional, then we sometime denote S_{kl} by e_{kl}. Also, a vector from v_i to v_j is denoted by v_{ij}. A simplex spanned by vertices $v_0,\ldots,\hat{v}_i,\ldots,v_n$, in which the vertex v_i is omitted, is denoted by S_i. A volume of a simplex is written as $\mathrm{vol}(S_{0\ldots n})$, or $\mathrm{vol}(S_{h_1,\ldots,h_n})$,where h_i are spanning vectors of the simplex. If a simplex is two-dimensional, we sometimes write l_{kl} for the length $\mathrm{vol}(S_{kl})=\mathrm{vol}(e_{kl})$.

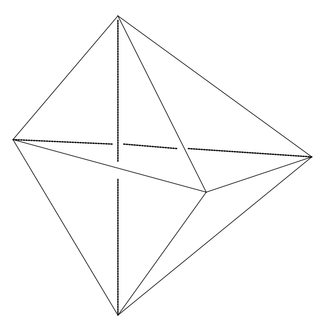

Figure 1. 2-3 move

Consider a complex with vertices $v_0,\ldots,v_4$ and five tetrahedra S_0, S_4, S_1, S_2, S_3. Suppose the complex described above is embedded in S^3. For the spherical Gram matrix associated with S_i, let G_i denote its determinant. For $j=1,\ldots,5$ define $s_j=1$ resp. -1 if the embedding is orientation preserving resp. reversing. Around the edge e_{04}, we have three exterior dihedral angles θ_{04}^1, θ_{04}^2, θ_{04}^3. Define the *defect angle* around the edge e_{04} by

$$\omega_{04}=\sum_{j=1}^{3}s_j(\pi-\theta_{04}^j).$$

Theorem 5.0.2. *The determinants G_i of spherical Gram matrices for tetrahedra S_i satisfy the identities*

(a) *(Asymptotic pentagon identity) In the same situation as in* (2)

$$\frac{\partial \omega_{04}}{\partial l_{04}} = s_1 s_2 s_3 \sin^2(l_{04}) \sqrt{\frac{G_0 G_4}{G_1 G_2 G_3}}, \tag{8}$$

(b) *(Asymptotic normalization identity) In the same situation described as in* (1)

$$\sin(l_{cd}) \int \frac{\sin(l_{ab})}{\sqrt{G(\tau(l_{ij}))}}\, dl_{ab} = \pi. \tag{9}$$

Here, l_{ab} and l_{cd} are the lengths of opposite edges and $G(\tau(l_{ab}))$ is the determinant of the spherical Gram matrix associated with the tetrahedron with edge lengths l_{ij}.

The equation (8) can be obtained heuristically via stationary phase applied to (2). To prove (9), note that

$$\int \frac{\partial \theta_{cd}}{\partial l_{ab}} \mathrm{d} l_{ab} = \int_0^\pi \mathrm{d}\theta_{cd} = \pi.$$

by Theorem 4.0.1 (f). The proof of (8) is by a series of lemmas. Suppose $v_0, \ldots, v_n$ in $\mathbb{R}^n$ form a $n-$simplex $S_{0,\ldots,n}$. Consider the $(n-3)-$simplex $S_{0,4,5,\ldots,n}$. Let h_i be the vector starting at the vertex v_i perpendicular to the simplex $S_{0,4,5,\ldots,n}$ for each $i = 1, 2, 3$.

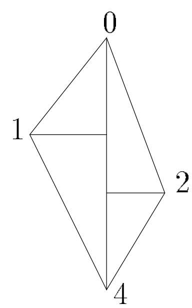

FIGURE 2. The vectors h_1 and h_2

The length $\|h_i\|$ of the vector is the distance from v_i to $S_{0,4,5,\ldots,n}$. Also, the dihedral angle between $3-$dimensional simplices $S_{0,i,j,4}$ and $S_{0,j,k,4}$ for $i \neq j \neq k \in \{1,2,3\}$ around $S_{0,4,\ldots,n}$ is the same as the angle ϕ_{ik} between vectors h_i and h_k. In particular, the exterior dihedral angle between $S_{0,i,j,4}$ and $S_{0,j,k,4}$ is $\pi - \phi_{ik}$, which we denote by θ_{ik}. The volume of the $(n-1)$-simplex $S_i = (v_0, \ldots, \hat{v}_i, \ldots, v_n)$, denoted by V_i, is

$$V_i = \frac{1}{(n-1)(n-2)} \mathrm{Vol}(0, h_j, h_k)\, \mathrm{Vol}(S_{0,4,\ldots,n}) \tag{10}$$

$$= \frac{1}{(n-1)(n-2)} \|h_j\| \|h_k\| \,\mathrm{Vol}(S_{0,4,\ldots,n}) \sin\theta_{jk} \tag{11}$$

where $i,j,k \in \{1,2,3\}$, $i \neq j \neq k$. The volume of the n-simplex $S_{0,\ldots,n}$ is

$$\binom{n}{3}\operatorname{Vol}(S_{0,\ldots,n}) = \operatorname{Vol}(h_1,h_2,h_3)\operatorname{Vol}(S_{0,4,\ldots,n}). \tag{12}$$

Indeed,

$$\begin{aligned} n!\operatorname{Vol}(S(0,\ldots,n)) &= \det(e_{01},e_{02},e_{03},e_{04},\ldots,e_{0n}) \\ &= \det(h_1,h_2,h_3)\det(e_{04},\ldots,e_{0n}) \\ &= 3!(n-3)!\operatorname{Vol}(h_1,h_2,h_3)\operatorname{Vol}(S_{0,4,\ldots,n}). \end{aligned}$$

Lemma 5.0.3.

$$\frac{\partial(\operatorname{Vol}(h_1,h_2,h_3))^2}{\partial\omega_{04}}|_{\omega_{04}=0} = \frac{2(n-1)^3(n-2)^3}{(3!)^2}\frac{s_1s_2s_3V_1V_2V_3}{(\operatorname{Vol}(S_{0,4,\ldots,n}))^3} \tag{13}$$

Proof. We know that

$$\operatorname{Vol}(h_1,h_2,h_3) = \frac{1}{3!}\det(h_i\cdot h_j)^{1/2} = \frac{1}{3!}\|h_1\|\|h_2\|\|h_3\|\det(\cos\phi_{ij})^{\frac{1}{2}}.$$

Substituting $\phi_{ij} = \pi - \theta_{ij}$ and expanding the determinant yield

$$(\operatorname{Vol}(h_1,h_2,h_3))^2 = (\frac{1}{3!})^2(\|h_1\|\|h_2\|\|h_3\|)^2(\cos^2\theta_{12}+\cos^2\theta_{13}+\cos^2\theta_{23}+2\cos\theta_{12}\cos\theta_{13}\cos\theta_{23}).$$

The differential with respect to $\theta_{12},\theta_{13},\theta_{23}$ is

$$\begin{aligned} \mathrm{d}(\operatorname{Vol}(h_1,h_2,h_3))^2 = (\frac{1}{3!})^2(\|h_1\|\|h_2\|\|h_3\|)^2\{&2\sin\theta_{12}(\cos\theta_{12}+\cos\theta_{13}\cos\theta_{23})\mathrm{d}\theta_{12}+ \\ &2\sin\theta_{13}(\cos\theta_{13}+\cos\theta_{12}\cos\theta_{23})\mathrm{d}\theta_{13}+ \\ &2\sin\theta_{23}(\cos\theta_{23}+\cos\theta_{12}\cos\theta_{13})\mathrm{d}\theta_{23}\}. \end{aligned}$$

The double angle formula, together with $\omega_{04}=0$ gives

$$\cos\theta_{12} = \cos(\pi-(s_2\theta_{13}+s_1\theta_{23})) = -\cos\theta_{13}\cos\theta_{23}+\sin s_2\theta_{13}\sin s_1\theta_{23}.$$

Therefore, $\mathrm{d}(\operatorname{Vol}(h_1,h_2,h_3))^2\ |_{\omega_{04}=0}$ is equal to

$$\frac{2}{(3!)^2}(\|h_1\|\|h_2\|\|h_3\|)^2\sin s_3\theta_{12}\sin s_2\theta_{13}\sin s_1\theta_{23}(s_3d\theta_{12}+s_2d\theta_{13}+s_1d\theta_{23}).$$

By (10),

$$V_1V_2V_3 = (\frac{1}{(n-1)(n-2)})^3(\|h_1\|\|h_2\|\|h_3\|)^2(\operatorname{Vol}(S_{0,4,\ldots,n}))^3\sin\theta_{12}\sin\theta_{23}\sin\theta_{13}.$$

The lemma follows since $\mathrm{d}\omega_{04} = \sum_{k\neq i\neq j} s_k\mathrm{d}\theta_{ij}$. □

Let x be the length of the edge e_{ij} from the vertex v_i to the vertex v_j and $x_\pm$ be the roots of the Cayley-Menger determinant associated with the n-simplex.

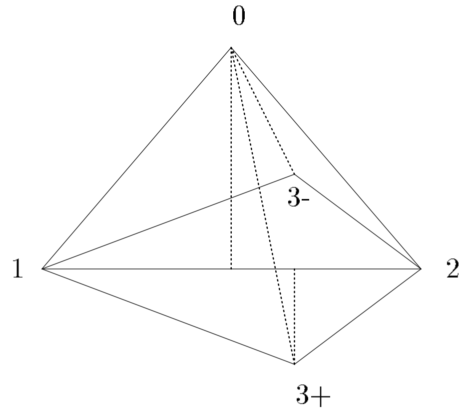

FIGURE 3. n=3

Lemma 5.0.4.

$$\frac{\partial \operatorname{Vol}(S_{0,\ldots,n})^2}{\partial x^2}|_{x^2=x_\pm^2} = \pm\frac{1}{n^2}V_iV_j, \tag{14}$$

Proof. Without loss of generality, assume x is the length of e_{0n}. Using the Cayley-Menger determinant,

$$\operatorname{Vol}^2(S_{0,\ldots,n}) = \frac{-1}{4(n(n-1))^2}\operatorname{Vol}^2(S_{1,\ldots,n-1})(x^2-x_-^2)(x^2-x_+^2)$$

and so

$$\frac{\partial \operatorname{Vol}^2(S_{0,\ldots,n})}{\partial x^2}|_{x^2=x_\pm^2} = \frac{\pm 1}{4(n(n-1))^2}\operatorname{Vol}^2(S_{1,\ldots,n-1})(x_+^2-x_-^2). \tag{15}$$

The roots $x_\pm$ correspond to values of the length for which the simplex embeds into $\mathbb{R}^n$. We can choose the embeddings so that only the image v_{n+}, v_{n-} of the vertex v_n varies; see Figure 3. Let h_0, resp. h_n, denote distance of v_0, resp. $v_{n,\pm}$, to $S(1,\ldots,n-1)$, so that

$$\operatorname{Vol}(S_n) = \frac{1}{n-1}\operatorname{Vol}(S_{1,\ldots,n-1})h_0, \qquad \operatorname{Vol}(S_0) = \frac{1}{n-1}\operatorname{Vol}(S_{1,\ldots,n-1})h_n.$$

Let w denote the distance from the projection of v_n, to the projection of v_0 in $S(1,\ldots,n-1)$. By the Pythagorean theorem,

$$x_+^2 = (h_0+h_n)^2 + w^2, \quad x_-^2 = (h_0-h_n)^2+w^2.$$

Hence $x_+^2 - x_-^2 = 4h_0h_n$, so the lemma follows from (15). ☐

Finally we prove the asymptotic pentagon identity. We use the above lemmas for $n=5$. Suppose that the vertices v_0, v_1, v_2, v_3, v_4 lie in S^3 and $v_5 = 0$. Let

$$I = \operatorname{Vol}^2(S_{0,\ldots,4}).$$

It suffices to compute

$$\frac{\partial \omega_{04}}{\partial y} = \frac{\partial \omega_{04}}{\partial I}\frac{\partial I}{\partial x^2}\frac{\partial x^2}{\partial y},$$

where x is the Euclidean length between v_0 and v_4 and y the spherical geodesic distance. By (12),

$$I = \frac{\mathrm{Vol}^2(S_{0,4,5})\,\mathrm{Vol}^2(h_1,h_2,h_3)}{5^2 2^2}.$$

Because $\mathrm{Vol}(S_{0,4,5})$ is independent of ω_{04},

$$\frac{\partial I}{\partial \omega_{04}} = \frac{\mathrm{Vol}^2(S_{0,4,5})}{5^2 2^2}\frac{\partial(\mathrm{Vol}^2(h_1,h_2,h_3))}{\partial \omega_{04}}.$$

By (13),

$$\frac{\partial \omega_{04}}{\partial I} = \frac{5^2 2^2}{96}\frac{\mathrm{Vol}(S_{0,4,5}) s_1 s_2 s_3}{V_1 V_2 V_3}.$$

By (14),

$$s_1 s_2 s_3 \frac{\partial \omega_{04}}{\partial x^2} = \frac{5^2 2^2}{96}\frac{\mathrm{Vol}(S_{0,4,5})}{V_1 V_2 V_3}\frac{V_0 V_4}{5^2} = \frac{1}{24}\,\mathrm{Vol}(S_{0,4,5})\frac{V_0 V_4}{V_1 V_2 V_3}.$$

Note that $\mathrm{Vol}(S_{0,4,5}) = \frac{1}{2}\sin(y)$ and $x = 2\sin(\frac{y}{2})$, where x is the length of the straight line from v_0 and v_4 and $v_5 = 0$. Hence,

$$\frac{\mathrm{d}x^2}{\mathrm{d}y} = 4\sin(\frac{y}{2})\cos(\frac{y}{2}) = 2\sin(y).$$

Thus,

$$\frac{\partial \omega_{04}}{\partial y} = \frac{\partial \omega_{04}}{\partial x^2}\frac{\partial x^2}{\partial y} = \frac{s_1 s_2 s_3}{24}\frac{\sin(y)}{2} 2\sin(y)\frac{V_0 V_4}{V_1 V_2 V_3} = s_1 s_2 s_3 \sin^2(y)\sqrt{\frac{G_0 G_4}{G_1 G_2 G_3}}.$$

6. A semiclassical three-manifold Invariant

In this section, we explain how to use (8),(9) to define a formal three-manifold invariant which is a spherical version of the formal invariant introduced by Korepanov in [9] and [8]. By formal we mean that the existence of the invariant depends on the convergence of certain finite dimensional integrals, which we can only prove in the case of S^3.

6.1. **Definition of the Invariant.** Let Y be a triangulated, closed, and oriented three-manifold with vertices $\mathrm{Vert}(Y)$, edges $\mathrm{Edge}(Y)$, triangles $\mathrm{Tri}(Y)$, and tetrahedra $\mathrm{Tet}(Y)$. Let $\mathcal{L}$ denote the space of the edge-labellings

$$\mathcal{L} = \{l:\ \mathrm{Edge}(Y) \to [0,\pi],\quad G(l(\tau)) > 0\ \forall \tau \in \mathrm{Tet}(Y)\}.$$

Here, $l(\tau)$ denotes the 6-tuple which is a restriction of a labelling l on the edges in τ and $G(l(\tau))$ the determinant of the spherical 4×4 Gram matrix associated with $l(\tau)$, and the edge length l_{ab} is as defined in (3).

By Theorem 4.0.1, if $G(l_{ab}) > 0$, there is a non-degenerate spherical tetrahedron with edge length l_{ab}. So, given an $l \in \mathcal{L}$ and $\tau \in \mathrm{Tet}(Y)$, there is an embedding $\varphi : \tau \to S^3$

such that for any edge $e \subset \tau$, the length of the edges of the tetrahedron $\varphi(e)$ is $l(e)$. For any coloring l and any edge $l(e) := l_e$ in the spherical tetrahedron $\varphi(\tau)$, let $\phi_{l_e,t}$ resp. $\theta_{l_e,t}$ denote the interior resp. exterior dihedral angle at l_e in $\varphi(\tau)$. Let

$$s : \mathrm{Tet}(Y) \to \{\pm 1\}$$

be a sign assignment to each tetrahedron in Y. For each $e \in \mathrm{Edge}(Y)$ and labelling l, define the *defect angle around the edge* e to be

$$\omega_{l_e,s} = 2\pi - \sum_{\tau \supset e} s(\tau)\phi_{l_e,\tau}. \tag{16}$$

We say that a labelling l is *flat* with respect to the sign choice s if

$$\omega_{l_e,s} = 0 \bmod 2\pi \quad \forall e \in \mathrm{Edge}(Y).$$

Definition 6.1.1. $\mathcal{L}_{\flat,s}$ denotes the set of flat labellings with a fixed sign assignment s. That is, $\mathcal{L}_{\flat,s} = \{l \in \mathcal{L} : \omega_{l_e,s} = 0 \bmod 2\pi\}$.

Proposition 6.1.2. *Suppose that Y is simply connected. For a given flat labelling l and a fixed sign assignment there exists a map $\varphi : \ Y \to S^3$ such that $\varphi|_\tau$ is an embedding of τ with length l_τ, for all tetrahedra $\tau \in \mathrm{Tet}(Y)$. Any other map $\varphi' : \ Y \to S^3$ whose restriction to a tetrahedron is an embedding is obtained by composing $\varphi : \ Y \to S^3$ with an element of $SO(4)$.*

The proof is similar to the construction of developing maps for hyperbolic or spherical manifolds and is left to the reader.

Suppose that Y is not necessarily simply connected. Let $\tilde{Y} \to Y$ be the universal cover of Y. Each flat labelling l with a fixed sign assignment s defines $\varphi_l : \ \tilde{Y} \to S^3$. Let $|\tau|$ denote the spherical tetrahedron $\varphi_l(\tau)$ realized from $l(\tau)$. For any $\gamma \in \pi_1(Y)$, $\gamma|\tau|$ is a spherical tetrahedron, related to $|\tau|$ by an element $\rho(\gamma)$ in the isometry group $SO(4)$ of S^3. By construction

$$\varphi_l(\gamma_1\gamma_2|\tau|) = \rho(\gamma_1)\rho(\gamma_2)\varphi_l(|\tau|).$$

It follows that ρ is a homomorphism

$$\rho : \ \pi_1(Y) \to SO(4) = (SU(2) \times SU(2))/\{\pm 1\}.$$

Let $[\rho]$ denote the conjugacy class of ρ in the representation variety

$$R(Y, SO(4)) := \mathrm{Hom}(\pi_1(Y), SO(4))/SO(4).$$

Because of the last statement in proposition 6.1.2, $[\rho]$ is independent of the choice of the base tetrahedron τ or an embedding $\tau \to S^3$. Let $\mathcal{L}_{\flat,[\rho]} = \cup_s \mathcal{L}_{\flat,[\rho],s}$ denote the set of flat labellings l which give rise to the class $[\rho]$.

Given $l \in \mathcal{L}_{\flat,[\rho],s}$, recall that the defect angle ω_{l_e} around an edge e is defined by (16). Let H denote the matrix

$$H = (\frac{\mathrm{d}\omega_i}{\mathrm{d}l_j})_{i,j \in \mathrm{Edge}(Y)}.$$

By Schläfli's formula b, H is the Hessian of the function

$$\sum_{e\in\mathrm{Edge}(Y)} \omega_{l_e,s} l_e - \sum_{\tau\in\mathrm{Tet}(Y)} s(\tau) 2\,\mathrm{vol}(|\tau|);$$

in particular, H is symmetric.

For any matrix $M=(m_{ij}), i,j\in\mathrm{Edge}(Y)$, and subsets $I,J\subset\mathrm{Edge}(Y)$, we denote by M_{IJ} the sub-matrix of M obtained by restricting the index set for rows, resp. columns, to I, resp. to J. Let $\mathcal{C}\subset\mathrm{Edge}(Y)$ be a maximal subset of edges such that the sub-matrix $H_{\mathcal{C}\mathcal{C}}\subset A$ is positive definite. Let $\overline{\mathcal{C}}$ denote its complement $\mathrm{Edge}(Y)\smallsetminus\mathcal{C}$. Define

$$I(Y,[\rho]) := \left(\frac{1}{2\pi}\right)^{\#\,\mathrm{Vert}} \sum_s \int_{l\in\mathcal{L}_{\flat,[\rho],s}} \prod_{\tau\in\mathrm{Tet}} G(l(\tau))^{-1/4} \prod_{e\in\mathrm{Edge}} \sin(l_e) \frac{\bigwedge_{e\in\overline{\mathcal{C}}} \mathrm{d}l_e}{\sqrt{\det(H_{\mathcal{C}\mathcal{C}})}}. \tag{17}$$

If $R(Y,SO(4))$ is finite, then we define

$$I(Y) := \sum_{[\rho]\in R(Y,SO(4))} I(Y,[\rho]).$$

This is not exactly the expression predicted by stationary phase applied to $TV(Y,r)$; that expression is (even) more complicated due to the inclusion of phases and certain powers of 2 which we have ignored. These omissions are partly discussed in the last section of the paper.

6.2. **Formal topological invariance.** By Pachner's theorem [13], any two triangulations of a given 3-manifold are related by a sequence of 1-4 and 2-3 moves. The 1-4 move replaces a tetrahedron with four tetrahedra by adding a vertex or vice versa. The Pachner 2-3 move replaces two tetrahedra sharing a face with three tetrahedra by adding an edge or vice versa.

Theorem 6.2.1. *$I(Y,[\rho])$ is a formal topological invariant, i.e., independent of the choice of $\mathcal{C}$ and invariant under the Pachner moves assuming convergence.*

First we show invariance of the integral under a 2-3 move. In the triangulation of Y, find a complex of two tetrahedra with vertices v_0,v_1,v_2,v_3,v_4. Denote it by X. For $\mathrm{Edge}(Y)$, we have the set of labellings $\mathcal{L}_\flat=\bigcup_s\mathcal{L}_{\flat,s}$, where s is a sign assignment $\mathrm{Tet}(Y)\to\{\pm1\}$. Consider a new triangulation T' of Y, obtained by adding an edge e_{04} to the complex X. We denote the new complex by X'. The set of data for the new triangulation is

$$\mathrm{Tet}'(Y)=\mathrm{Tet}(Y)-\{S(0123),S(1234)\}\cup\{S(0234),S(0134),S(0124)\},$$

$$\mathrm{Edge}'(Y)=\mathrm{Edge}(Y)\cup\{e_{04}\},\quad \mathrm{Vert}'(Y)=\mathrm{Vert}(Y).$$

Any flat labelling l of $\mathrm{Edge}(T)$ induces a flat labelling l' of $\mathrm{Edge}(T')$. Since any loop in Y can be deformed so as not to intersect $S(0123)\cup S(1234)$, $[\rho]$ is the same for l and l'. Let $l^{(0)}_{\mathrm{new}}$ denote the function of the lengths $l_1,\dots,l_N$ given by the implicit function theorem so that if $l_{\mathrm{new}}^{(0)}$ is the length of the edge (v_0v_4), and l_j are other lengths, then $\omega_{\mathrm{new}} = 0$. Let l_{new} denote the length of the edge (v_0v_4), and

$$\tilde{l}_{\mathrm{new}} = l_{\mathrm{new}} - l_{\mathrm{new}}^{(0)}. \tag{18}$$

Since

$$0 = \frac{\partial \omega_{\mathrm{new}}(l_{\mathrm{new}}^{(0)})}{\partial l_j} = \frac{\partial \omega_{\mathrm{new}}}{\partial l_{\mathrm{new}}} \frac{\partial l_{\mathrm{new}}^{(0)}}{\partial l_j} + \frac{\partial \omega_{\mathrm{new}}}{\partial l_j}$$

we have

$$d\tilde{l}_{\mathrm{new}} = dl_{\mathrm{new}} + \sum_j \frac{\partial \omega_{\mathrm{new}}/\partial l_j}{\partial \omega_{\mathrm{new}}/\partial l_{\mathrm{new}}} \mathrm{d}l_j.$$

It follows that

$$\begin{pmatrix} d\omega_{\mathrm{new}} \\ d\omega_1 \\ \vdots \\ d\omega_N \end{pmatrix} = \begin{pmatrix} \partial\omega_{\mathrm{new}}/\partial l_{\mathrm{new}} & 0 \cdots 0 \\ \partial\omega_1/\partial l_{\mathrm{new}} & \\ \vdots & H \\ \partial\omega_N/\partial l_{\mathrm{new}} & \end{pmatrix} \begin{pmatrix} d\tilde{l}_{\mathrm{new}} \\ dl_1 \\ \vdots \\ dl_N \end{pmatrix}. \tag{19}$$

Hence

$$H_{\mathrm{new}} = \begin{pmatrix} \partial\omega_{\mathrm{new}}/\partial l_{\mathrm{new}} & 0 \cdots 0 \\ \partial\omega_1/\partial l_{\mathrm{new}} & \\ \vdots & H \\ \partial\omega_N/\partial l_{\mathrm{new}} & \end{pmatrix} \begin{pmatrix} 1 & \frac{\partial\omega_{\mathrm{new}}/\partial l_1}{\partial\omega_{\mathrm{new}}/\partial l_{\mathrm{new}}} & \cdots & \frac{\partial\omega_{\mathrm{new}}/\partial l_N}{\partial\omega_{\mathrm{new}}/\partial l_{\mathrm{new}}} \\ & 1 & & \\ & & \ddots & 0 \\ & 0 & & 1 \end{pmatrix}. \tag{20}$$

Since both matrices are block triangular,

$$\det(H_{\mathrm{new}}) = \frac{\partial \omega_{\mathrm{new}}}{\partial l_{\mathrm{new}}} \det(H).$$

The new triangulation has $\mathcal{C}' = \mathcal{C} \cup \{(v_0v_4)\}$, so that $\overline{\mathcal{C}'} = \mathrm{Edge}' - \mathcal{C}' = \overline{\mathcal{C}}$. Invariance now follows from (8).

Next we show that $I(Y, [\rho])$ is independent of the choice of $\mathcal{C}$. We write

$$l = (l', l''), \quad \omega = (\omega', \omega'')$$

where l' is the vector of edge lengths in $\mathcal{C}$, and l'' the the remaining edge lengths, and similarly for ω. Generically the length l'' may be written as a function of l', by requiring that the defect angles $\omega = 0$. With respect to this decomposition, the matrix H may be written in block diagonal form as follows. Let

$$D = \frac{\partial l''_i}{\partial l'_j} \tag{21}$$

denote the matrix of partial derivatives. Define

$$\mathrm{d}\tilde{l}' = \mathrm{d}l' + \frac{\partial l''}{\partial l'} \mathrm{d}l''$$

similar to (18). It follows from the definition that

$$\begin{pmatrix} \mathrm{d}\omega' \\ \mathrm{d}\omega'' \end{pmatrix} = \begin{pmatrix} B & 0 \\ C & 0 \end{pmatrix} \begin{pmatrix} \mathrm{d}\tilde{l}' \\ \mathrm{d}l'' \end{pmatrix}$$

for some matrices B, C. We have an equation similar to (19)

$$H = \begin{pmatrix} B & 0 \\ C & 0 \end{pmatrix} \begin{pmatrix} I & D \\ 0 & I \end{pmatrix} = \begin{pmatrix} B & BD \\ C & CD \end{pmatrix}.$$

It follows from the fact that H is symmetric that

$$H = \begin{pmatrix} H_{\mathcal{C}\mathcal{C}} & H_{\mathcal{C}\mathcal{C}}D \\ D^T H_{\mathcal{C}\mathcal{C}} & D^T H_{\mathcal{C}\mathcal{C}} D \end{pmatrix}. \tag{22}$$

Let $\mathcal{C}'$ be a different maximal subset of edges, such that $H_{\mathcal{C}'\mathcal{C}'}$ is non-degenerate. Take $X \subset \mathcal{C}$, $Y \subset \overline{\mathcal{C}}$ such that $|X| = |Y|$. Set $\mathcal{C}' = (\mathcal{C} - X) \cup Y$. From (22) we see that

$$H_{\mathcal{C}'\mathcal{C}'} = \begin{pmatrix} H_{\mathcal{C}-X\mathcal{C}-X} & H_{\mathcal{C}-XX} D_{XY} \\ D_{XY}^T H_{X\mathcal{C}-X} & D_{XY}^T H_{XX} D_{XY} \end{pmatrix}, \tag{23}$$

since $\frac{\partial \omega_i}{\partial l_j} = \frac{\partial \omega_j}{\partial l_i}$. Thus,

$$H_{\mathcal{C}'\mathcal{C}'} = F^T (H_{\mathcal{C}\mathcal{C}}) F$$

where F is the matrix in block diagonal form with respect to the decomposition $\mathcal{C} = (\mathcal{C} \setminus X) \bigcup X$ for columns and $\mathcal{C}' = (\mathcal{C}' \setminus Y) \bigcup Y$ for rows

$$F = \begin{pmatrix} I & 0 \\ 0 & D_{XY} \end{pmatrix}.$$

It follows that

$$\det(H_{\mathcal{C}'\mathcal{C}'}) = \det(H_{\mathcal{C}\mathcal{C}}) \det(F)^2 = \det(H_{\mathcal{C}\mathcal{C}}) \det(D_{XY})^2.$$

Together with (21) this implies that the differential form

$$\frac{\bigwedge_{e \in \overline{\mathcal{C}}} dl_e}{\sqrt{\det(H_{\mathcal{C}\mathcal{C}})}}$$

in (17) is the same for $\mathcal{C}$ and $\mathcal{C}'$.

To prove invariance under a 1-4 move we will use the following lemma, whose proof is left to the reader:

Lemma 6.2.2. *The integral over the region*

$$\{l_b, l_c : l_a \le l_b + l_c, l_b \le l_a + l_c, l_c \le l_a + l_b, l_a + l_b + l_c \le 2\pi\}$$

$$\frac{1}{\sin(l_a)} \int \int \sin(l_b) \sin(l_c) dl_b dl_c = 2 \tag{24}$$

for any $l_a \in [0, \pi]$.

Let S_{0123} be a tetrahedron with vertices $v_0, \ldots, v_3$ in Y. We consider the effect of adding an extra vertex v_4 in the interior and replacing the tetrahedron S_{0123} with the four tetrahedra S_{1234}, S_{0234}, S_{0134}, S_{0124}. We use the notation τ_i for $S_{0\ldots\hat{i}\ldots 4}$. We have

$$\mathrm{Vert}' = \mathrm{Vert} \cup \{v_4\}, \quad \mathrm{Tet}' = (\mathrm{Tet} - \{S(0123)\}) \cup \{S(1234), S(0234), S(0134), S(0124)\}$$

$$\mathrm{Edge}' = \mathrm{Edge} \cup \{e_{04}, e_{14}, e_{24}, e_{34}\}.$$

Also $\mathcal{C}' = \mathcal{C} \cup e_{34}$ since adding any other edge would allow a deformation of the new vertex changing only the lengths of edges in $\mathcal{C}'$. Hence

$$\overline{\mathcal{C}'} = \overline{\mathcal{C}} \cup \{e_{04}, e_{14}, e_{24}\}.$$

Exactly the same argument as in the 2-3 case shows that

$$\det(H_{\mathcal{C}'\mathcal{C}'}) = \frac{\partial \omega_{34}}{\partial l_{34}} \det(H_{\mathcal{C}\mathcal{C}}).$$

Hence

$$I(Y') = (\frac{1}{2\pi})^{(\#\,\mathrm{Vert}\,+1)} \int_{\mathcal{L}'} \frac{\prod_{\tau \in \mathrm{Tet} - \{S(0123)\}} (G(l(\tau))^{-1/4}) \prod_{e \in \mathrm{Edge}'} \sin(l_e) \bigwedge_{e \in \overline{\mathcal{C}'}} dl_e}{(G_0 G_1 G_2 G_3)^{1/4} \sqrt{(\det H_{\mathcal{C}\mathcal{C}})(\frac{\partial \omega_{34}}{\partial l_{34}})}}.$$

After substituting the Jacobian

$$\frac{\partial \omega_{34}}{\partial l_{34}} = \sin^2(l_{34}) \sqrt{\frac{G_3 G_4}{G_0 G_1 G_2}}$$

we need to compute the integral

$$\int \frac{\sin(l_{04}) \sin(l_{14}) \sin(l_{24}) dl_{04} dl_{14} dl_{24}}{\sqrt{G_3}}.$$

The equations (9) and (24) give

$$\frac{\pi}{\sin(l_{12})} \int \sin(l_{14}) \sin(l_{24}) dl_{14} dl_{24} = 2\pi.$$

This cancels with the extra factor of 2π in the coefficient, and completes the proof that $I(Y, [\rho])$ is invariant under the Pachner moves, assuming it converges.

6.3. **An acyclic complex and its torsion.** In this section we relate the determinant appearing in $I(Y, [\rho])$ to the torsion of an acyclic complex, following Korepanov [7]. Recall the infinitesimal action of the group of gauge transformations $\mathrm{Map}(Y, SO(4))$ on the space of connections $\Omega^1(Y, \mathfrak{so}(4))$ at a connection A is given by

$$\Omega^0(Y, \mathfrak{so}(4)) \to \Omega^1(Y, \mathfrak{so}(4)), \quad \xi \mapsto -d_A \xi \tag{25}$$

where d_A is the associated covariant derivative. Hence the infinitesimal stabilizer of A is

$$\Omega^0(Y, \mathfrak{so}(4))_A = H_0(d_A).$$

Let $\mathrm{SO}(4)_\rho$ denote the stabilizer of $\rho : \pi_1(Y) \to SO(4)$, and $\mathfrak{so}(4)_\rho$ its Lie algebra. If A is a flat connection defining the holonomy representation ρ, then evaluation at the identity induces an isomorphism

$$\mathrm{Map}(Y, SO(4))_A \to SO(4)_\rho.$$

Hence $H^0(d_A)$ is isomorphic to $\mathfrak{so}(4)_\rho$. Let

$$h^0(d_A) = \dim(\mathfrak{so}(4)_\rho) = \dim(H^0(d_A)).$$

The cohomology group $H^1(d_A)$ parameterizes first-order deformations of ρ; in particular, if $H^1(d_A) = 0$ then $[\rho]$ is isolated in $R(Y, SO(4))$. Suppose that $H^1(d_A) = 0$. Let

$$V = \mathrm{Map}(\mathrm{Vert}(\tilde{Y}), S^3)^{\pi_1(Y)}$$

denote the space of maps invariant under $\pi_1(Y)$, acting on $\mathrm{Vert}(\tilde{Y})$ by deck transformations and S^3 via the representation ρ. Let

$$E = \mathrm{Map}(\mathrm{Edge}(\tilde{Y}), [0, \pi])^{\pi_1(Y)} = \mathrm{Map}(\mathrm{Edge}(Y), [0, \pi])$$

and $\delta : V \to E$ the map taking edge lengths of edges. Let $\omega : E \to E$ be the map which assigns to a set of edge lengths the set of defect angles. The action of $SO(4)_\rho$ on V induces a map

$$\lambda : \ \mathfrak{so}(4)_\rho \to \mathrm{Vect}(V).$$

Evaluating the vector field at $p \in V$ gives

$$\lambda_p : \ \mathfrak{so}(4)_\rho \to T_p V.$$

For any $p \in V$, let $l = \delta(p)$, $\hat{l} = \omega(l)$ and $\hat{p}$ any point in $\delta^{-1}(\hat{l})$. Consider the sequence

$$0 \to \mathfrak{so}(4)_\rho \to T_p V \to T_l E \to T_{\hat{l}} E \to T_{\hat{p}} V \to \mathfrak{so}(4)_\rho \to 0 \tag{26}$$

with maps $\lambda_p, D_p\delta, H, D_p\delta^T, \lambda_{\hat{p}}^T$. It follows from the fact that H is symmetric and a straight-forward calculation that the sequence (26) is exact, that is, (26) is an acyclic complex. Let $\tau(l, s)$ denote the torsion, which is defined as follows. Let $\mathrm{Vert}'(Y)$ denote a maximal subset of the space of vertices so that δ is injective on the corresponding subspace of $T_p V$. Let δ' denote the restriction of δ to $\mathrm{Vert}'(Y)$, followed by projection onto the subspace of $T_l E$ corresponding to the complement of $\mathcal{C}$. Then

$$\tau(l, s) = \det(\lambda)^{-2} \det(\delta')^2 \det(H_{\mathcal{C}\mathcal{C}})^{-1}.$$

7. Computations of the Invariant for the sphere S^3

A triangulation of S^3 consists of the following data:

$$\mathrm{Vert} = \{0, 1, 2, 3, 4\},$$

$$\mathrm{Edge} = \{01, 02, 03, 04, 12, 13, 14, 23, 24, 34\},$$

$$\mathrm{Face} = \{012, 023, 013, 124, 123, 134, 234, 014, 024, 034\},$$

$$\mathrm{Tet} = \{0123, 1234, 0124, 0234, 0134\}.$$

Since S^3 is simply-connected, the representation variety $R(S^3, SO(4))$ is trivial. So, $I(S^3) = I(S^3, [1])$. Using the acyclic complex in the previous section, we find that the rank of $\mathcal{C}$ is 1. Since there is no distinction among the edges, we choose $\mathcal{C} = \{04\}$. Thus, $\overline{\mathcal{C}} = \mathrm{Edge} - \{04\}$. Denote by G_i the determinant of the Gram matrix associated with the tetrahedron $(0 \ldots \hat{i} \ldots 4)$. We must compute

$$(\frac{1}{2\pi})^5 \int_{\mathcal{L}_{b,\mathrm{I}}} \frac{\prod_{e\in \mathrm{Edge}} \sin(l_e) \bigwedge_{e\in\overline{\mathcal{C}}} dl_e}{(G_0G_1G_2G_3G_4)^{1/4}\sqrt{\det H_{\mathcal{C}\mathcal{C}}}}. \tag{27}$$

Note that around the edge (04), there are three tetrahedra (0234), (0124), (0134). When these tetrahedra match with each other in S^3 under the curvature zero condition around the edge (04), we have two tetrahedra (0123) and (1234) as well. In other words, we are in the situation where the spherical Jacobian (8) is equal to $H_{\mathcal{C}\mathcal{C}}$. So, the integral reduces to

$$(\frac{1}{2\pi})^5 \int \frac{\prod_{e\in\overline{\mathcal{C}}} \sin(l_e) \bigwedge_{e\in\overline{\mathcal{C}}} dl_e}{\sqrt{G_0G_4}}.$$

Apply the orthogonality identity (9) to the tetrahedra (1234) and (0123) respectively and integrate the rest from 0 to π in each variable. Then, the integral (27) is computed to be $\frac{1}{\pi^3}$. There are 2^5 ways of assigning signs to each tetrahedron in the triangulation, but the above argument is applied to each assignment of the sign. Therefore,

$$I(S^3) = \frac{2^5}{\pi^3}.$$

8. Remarks on the semiclassical limit of Turaev-Viro

Throughout this section we assume that Y is a rational homology sphere. The stationary phase approximation to the Chern-Simons path integral predicts [5]

$$Z(Y,r) \sim \frac{1}{2} r^{-\frac{1}{2}h^0(d_A)} e^{-3\pi i/4} \sum_{[A]\in R(Y,SU(2))} \sqrt{\tau(A)} e^{-2\pi i I_A/4} e^{2\pi i CS(A,r)}$$

where $\tau(A)$ is the torsion of A, I_A is the spectral flow, and $CS(A,r)$ the Chern-Simons invariant at level r

$$CS(A,r) = \frac{r}{8\pi^2}\int_Y \mathrm{Tr}(A\wedge \mathrm{d}A + \frac{2}{3}A\wedge A\wedge A\}.$$

We write any $SO(4)$ connection as a pair of $SU(2)$-connections. The norm-square of the asymptotic formula for $Z(Y,r)$ is

$$TV(Y,r) \sim \frac{1}{4} \sum_{[A]\in R(Y,SO(4))} r^{-\frac{1}{2}h^0(d_A)} \sqrt{\tau(A_1)\tau(A_2)} e^{-2\pi(I_{A_1}-I_{A_2})/4} e^{2\pi i(CS(A_1,r)-CS(A_2,r))} \tag{28}$$

where $A = (A_1, A_2)$.

8.1. **The leading power of r.** It follows from $\Delta(r) = r\sin(\pi/r)^{-2}$ that $\Delta \sim \frac{r^3}{\pi^2}$ as $r \to \infty$. Let t, e, v denote the size of the sets $\mathrm{Tet}(Y)$, $\mathrm{Edge}(Y)$, $\mathrm{Vert}(Y)$. Collecting together the powers of r in the asymptotic $6j$ formula (4), the definition of the Turaev-Viro invariant (5), and the acyclicity of (26) we obtain the prediction for leading power of r in the Turaev-Viro invariant

$$-\frac{3}{2}v + \frac{3}{2}e - \frac{3}{2}t - \frac{1}{2}h^0(d_A) = -\frac{1}{2}h^0(d_A).$$

This agrees with the prediction in (28).

8.2. **The Volumes/Chern-Simons invariant.** The terms $\exp(\pm i\phi)$ appearing in the stationary phase approximation to Turaev-Viro lead to a factor

$$\exp\left(\frac{i}{\pi}\sum_{\tau\in\mathrm{Tet}(Y)} \pm \mathrm{Vol}(\tau)\right).$$

Let $\phi : \tilde{Y} \to S^3$ denote the developing map as in Proposition 6.1.2. Let $\mathrm{d}\,\mathrm{Vol}(S^3)$ denote the volume form on S^3 so that $\int_{S^3} \mathrm{d}\,\mathrm{Vol}(S^3) = 2\pi^2$.

Let $\pi :\ SO(4) \to S^3$ denote the map given by action on $(1,0,0)$. We have $\pi^*\mathrm{d}\,\mathrm{Vol}(S^3) = 2\pi^2\chi$ where $\chi = (\alpha, [\alpha,\alpha]) \in \Omega^3(SO(4))$ is the Chern-Simons three-form on $SO(4)$ with $\alpha \in \Omega^1(SO(4), \mathfrak{so}(4))$ the left Maurer-Cartan form and $(\ ,\ )$ the inner product equal to the basic inner product on one $\mathfrak{su}(2)$-factor and minus the basic inner product on the other. Let $A = (A_1, A_2)$ be an $SU(2)^2$ connection on Y with holonomy representation ρ and $g : \tilde{Y} \to SU(2)^2$ a gauge transformation trivializing the lift $\tilde{A}$ of A to $\tilde{Y}$. For any $\gamma \in \pi_1(Y)$, we have $\gamma^* g = \rho(\gamma) g$. This implies that $g^{-1}\cdot\phi$ is π_1-invariant, and hence descends to a map $Y \to S^3$. Hence

$$\begin{aligned}
\frac{1}{\pi}\sum_{\tau\in\mathrm{Tet}(Y)} \pm \mathrm{Vol}(\tau) &= \frac{1}{\pi}(\#\pi_1(Y))^{-1}\int_{\tilde{Y}} \phi^*\mathrm{d}\,\mathrm{Vol}(S^3) \\
&= \frac{1}{\pi}(\#\pi_1(Y))^{-1}\int_{\tilde{Y}} g^*\pi^*\mathrm{d}\,\mathrm{Vol}(S^3) \mod 2\pi\mathbb{Z} \\
&= 2\pi(\#\pi_1(Y))^{-1}\int_{\tilde{Y}} g^*\chi \mod 2\pi\mathbb{Z} \\
&= 2\pi(\#\pi_1(Y))^{-1}(CS(\tilde{A}_1) - CS(\tilde{A}_2)) \mod 2\pi\mathbb{Z} \\
&= 2\pi(CS(A_1) - CS(A_2)) \mod 2\pi\mathbb{Z}
\end{aligned}$$

which also matches (28).

8.3. **The Maslov indices and torsion.** Each tetrahedron contributes $\exp(\pm\pi i/4)$ from the formula (4). Stationary phase leads to a factor $\exp(\pi i\,\mathrm{sign}(H_{\mathcal{C}\mathcal{C}})/4)$. It seems natural to conjecture that these combine to the spectral flow factor $\exp(2\pi i I_A/4)$ in the Freed-Gompf formula. One expects the torsion to correspond to our three-manifold invariant. However, it is not clear to us how to perform the integral over flat labellings.

References

[1] Scott Axelrod and I. M. Singer. Chern-Simons perturbation theory. In *Proceedings of the XXth International Conference on Differential Geometric Methods in Theoretical Physics, Vol. 1, 2 (New York, 1991)*, pages 3–45, River Edge, NJ, 1992. World Sci. Publishing.

[2] Scott Axelrod and I. M. Singer. Chern-Simons perturbation theory. II. *J. Differential Geom.*, 39(1):173–213, 1994.

[3] Leonard M. Blumenthal. *Theory and applications of distance geometry.* Second edition. Chelsea Publishing Co., New York, 1970.

[4] J. Scott Carter, Daniel E. Flath, and Masahico Saito. *The classical and quantum 6j-symbols*, volume 43 of *Mathematical Notes.* Princeton University Press, Princeton, NJ, 1995.

[5] Daniel S. Freed and Robert E. Gompf. Computer calculation of Witten's 3-manifold invariant. *Comm. Math. Phys.*, 141(1):79–117, 1991.

[6] Lisa C. Jeffrey. Chern-Simons-Witten invariants of lens spaces and torus bundles, and the semi-classical approximation. *Comm. Math. Phys.*, 147(3):563–604, 1992.

[7] I. G. Korepanov. Euclidean 4-simplices and invariants of four-dimensional manifolds. I. Surgeries $3 \to 3$. *Teoret. Mat. Fiz.*, 131(3):377–388, 2002.

[8] I. G. Korepanov and E. V. Martyushev. Distinguishing three-dimensional lens spaces $L(7,1)$ and $L(7,2)$ by means of classical pentagon equation. *J. Nonlinear Math. Phys.*, 9(1):86–98, 2002.

[9] I. G. Korepanov. Invariants of PL manifolds from metrized simplicial complexes. Three-dimensional case. *J. Nonlinear Math. Phys.*, 8(2):196–210, 2001.

[10] Thang T. Q. Le, Jun Murakami, and Tomotada Ohtsuki. On a universal perturbative invariant of 3-manifolds. *Topology*, 37(3):539–574, 1998.

[11] John Milnor. *Collected papers. Vol. 1.* Publish or Perish Inc., Houston, TX, 1994. Geometry.

[12] S. Mizoguchi and T. Tada. 3-dimensional gravity and the Turaev-Viro invariant. *Progr. Theoret. Phys. Suppl.*, (110):207–227, 1992. Recent developments in string and field theory (Kyoto, 1991).

[13] Udo Pachner. P.L. homeomorphic manifolds are equivalent by elementary shellings. *European J. Combin.*, 12(2):129–145, 1991.

[14] G. Ponzano and T. Regge. Semiclassical limit of Racah coefficients. In *Spectroscopic and Group Theoretical Methods in Physics (Amsterdam, 1968)*, pages 1–58. North-Holland Publ. Co., 1989.

[15] Justin Roberts. Skein theory and Turaev-Viro invariants. *Topology*, 34(4):771–787, 1995.

[16] Y. Taylor and C. Woodward. Non-Euclidean tetrahedra and $6j$ symbols for $U_q(\mathfrak{sl}_2)$. preprint, available as math.QA/0305113.

[17] V. G. Turaev and O. Ya. Viro. State sum invariants of 3-manifolds and quantum $6j$-symbols. *Topology*, 31(4):865–902, 1992.

[18] V. Turaev. Quantum invariants of 3-manifold and a glimpse of shadow topology. In *Quantum groups (Leningrad, 1990)*, volume 1510 of *Lecture Notes in Math.*, pages 363–366. Springer, Berlin, 1992.

[19] T. Yoshida. An abelianization of $SU(2)$ conformal field theory and Witten invariant. Undated preprint.

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