PROPERNESS FOR SCALED GAUGED MAPS

EDUARDO GONZÁLEZ, PABLO SOLIS, AND CHRIS T. WOODWARD

Abstract. We prove properness of moduli stacks of gauged maps satisfying a stability condition introduced by Mundet [40], Schmitt [46] and Ziltener [57]. The proof combines a git construction of Schmitt [46], properness for twisted stable maps by Abramovich-Vistoli [1], a variation of a boundedness argument due to Ciocan-Fontanine-Kim-Maulik [13], and a removal of singularities for bundles on surfaces in Colliot-Thélène-Sansuc [14].

Contents

1. Introduction

The moduli stack of maps from a curve to the stack quotient of a smooth projective variety by the action of a complex reductive group has a natural stability condition introduced by Mundet in [40] and investigated further in Schmitt [46, 47]; the condition generalizes stability for bundles over a curve introduced by Mumford, Narasimhan-Seshadri and Ramanathan [45]. In an earlier paper [24] the first and third authors used the moduli of Mundet-stable maps to give a formula that relates the genus zero gauged Gromov-Witten invariants and Gromov-Witten invariants of the git quotient of a smooth projective variety with reductive group action, termed a quantum

---

Partially supported by NSF grants DMS-1207194 and DMS-1510518.

analog of Witten's localization theorem. The proof of the formula depended on the properness of the stack. This properness was proved via symplectic geometry and results of Ziltener [57] and Ott [43]. In this paper we give a purely algebraic proof of properness via the valuative criterion for stacks [37, Chapter 7].

The stability condition for maps to quotient stacks combines several stability conditions already present in the literature, and leads to a notion of gauged Gromov-Witten invariant. Let X be a smooth projective G-variety such that the semi-stable locus is equal to the stable locus, and X/G the quotient stack. By definition a map from a curve C to X/G is a pair that consists of a bundle $P \to C$ and a section u of the associated bundle $P \times_G X \to C$. We denote by $\pi : X/G \to \mathrm{pt}/G =: BG$ the projection to the classifying space. In case X is a point, a stability condition for $\mathrm{Hom}(C, X/G)$, bundles on C, was introduced by Ramanathan [45]. A stability condition that combines bundle and target stability was introduced by Mundet [40]. There is a compactified moduli stack $\overline{\mathcal{M}}_n^G(C, X, d)$ whose open locus consists of Mundet semistable maps of class $d \in H_2^G(X, \mathbb{Z})$ with markings:

$$C \to S, \quad v : C \to X/G, \quad (z_1, \ldots, z_n) : S \to C^n \text{ distinct.}$$

The compactification uses the notion of Kontsevich stability for maps [53], [54], [55]. The stack admits evaluation maps to the quotient stack

$$\mathrm{ev} : \overline{\mathcal{M}}_n^G(C, X, d) \to (X/G)^n, \quad (\hat{C}, P, u, \underline{z}) \mapsto (z_i^* P, u \circ z_i).$$

In addition, assuming stable=semistable there is a virtual fundamental class constructed via the machinery of Behrend-Fantechi [6].

Let $\widehat{QH}_G(X)$ denote the formal completion of $QH_G(X)$ at 0. The *gauged Gromov-Witten trace* is the map

$$\tau_X^G : \widehat{QH}_G(X) \to \Lambda_X^G, \quad \alpha \mapsto \sum_{n,d} \frac{q^d}{n!} \int_{\overline{\mathcal{M}}_n^G(C,X,d)} \mathrm{ev}^*(\alpha, \ldots, \alpha). \tag{1}$$

The derivatives of the potential will be called gauged Gromov-Witten invariants. For toric varieties, the potential τ_X^G already appears in Givental [23] and Lian-Liu-Yau [32] under the name of *quasimap potential*.[1] In those papers (following earlier work of Morrison-Plesser [38]) the gauged potential is explicitly computed in the toric case, and questions about Gromov-Witten invariants of toric varieties or complete intersections therein reduced to a computation of quasimap invariants. We wish re-prove and extend the results of those papers in a uniform and geometric way that extends to quantum K-theory and non-abelian quotients and does not use any assumption such as the existence of a torus action with isolated fixed points. The splitting axiom for the gauged invariants is somewhat different than the usual splitting axiom in Gromov-Witten theory: the potential τ_X^G is a non-linear

[1] We are simplifying things a bit for the sake of exposition; actually the quasimap potentials in those papers involve an additional determinant line bundle in the integrals.

version of a *trace* on the Frobenius manifold $QH_G(X)$. Note that there are several other notions of gauged Gromov-Witten invariants, for example, Ciocan-Fontanine-Kim-Maulik [13], Frenkel-Teleman-Tolland [20], as well as a growing body of work on gauged Gromov-Witten theory with potential [51], [19].

The gauged Gromov-Witten invariants so defined are closely related to, but different from in general, the Gromov-Witten invariants of the stack-theoretic geometric invariant theory quotient. The stack of marked maps to the git quotient

$$v : C \to X /\!\!/ G, \quad (z_1, \dots, z_n) \in C^n \text{ distinct}$$

is compactified by the *graph space*

$$\overline{\mathcal{M}}_n(C, X /\!\!/ G, d) := \overline{\mathcal{M}}_{g,n}(C \times X /\!\!/ G, (1, d))$$

the moduli stack of stable maps to $C \times X /\!\!/ G$ of class $(1, d)$; in case $X /\!\!/ G$ is an orbifold the domain is allowed to have orbifold structures at the nodes and markings as in [11], [2]. The stack admits evaluation maps

$$\mathrm{ev} : \overline{\mathcal{M}}_n(C, X /\!\!/ G, d) \to (\overline{\mathcal{I}}_{X /\!\!/ G})^n$$

where $\overline{\mathcal{I}}_{X /\!\!/ G}$ is the *rigidified inertia stack* of $X /\!\!/ G$. The *graph trace* is the map

$$\tau_{X /\!\!/ G} : \widehat{QH}_{\mathbb{C}^\times}(X /\!\!/ G) \to \Lambda_X^G, \quad \alpha \mapsto \sum_{n,d} \frac{q^d}{n!} \int_{\overline{\mathcal{M}}_n(C, X /\!\!/ G, d)} \mathrm{ev}^*(\alpha, \dots, \alpha)$$

and where the equivariant parameter for the $\mathbb{C}^\times$-action is interpreted as a ψ-class at the corresponding marking. The relationship between the graph Gromov-Witten invariants of $X /\!\!/ G$ and Gromov-Witten invariants arising from stable maps to $X /\!\!/ G$ in the toric case is studied in [23], [32], and other papers.

The goal of this paper is to construct, using only algebraic geometry, a proper algebraic cobordism between the moduli stack of Mundet semistable maps and the moduli stack of stable maps to the git quotient with corrections coming from "affine gauged maps". Affine gauged maps are maps

$$v : \mathbb{P}^1 \to X/G, \quad u(\infty) \in X^{\mathrm{ss}}/G, \quad z_1, \dots, z_n \in \mathbb{P}^1 - \{\infty\} \text{ distinct}$$

where $\infty = [0, 1] \in \mathbb{P}^1$ is the point "at infinity", modulo *affine* automorphisms, that is, automorphisms of $\mathbb{P}^1$ which preserve the standard affine structure on $\mathbb{P}^1 - \{0\}$. Denote by $\overline{\mathcal{M}}_{n,1}^G(\mathbb{A}, X)$ the compactified moduli stack of such affine gauged maps to X; we use the notation $\mathbb{A}$ to emphasize that the equivalence only uses affine automorphisms of the domains. A table with the different kinds of stable maps to quotients stacks is presented in Section 12. Evaluation at the markings defines a morphism

$$\mathrm{ev} \times \mathrm{ev}_\infty : \overline{\mathcal{M}}_{n,1}^G(\mathbb{A}, X, d) \to (X/G)^n \times \overline{\mathcal{I}}_{X /\!\!/ G}.$$

In the case $d = 0$, the moduli stack $\overline{\mathcal{M}}^G_{0,1}(\mathbb{A}, X, d)$ is isomorphic to $\overline{\mathcal{I}}_{X/\!\!/G}$ via evaluation at infinity. The *quantum Kirwan map* is the map

$$\kappa^G_X : \widehat{QH}_G(X) \to QH_{\mathbb{C}^\times}(X/\!\!/G)$$

defined as follows. Let $\mathrm{ev}_{\infty,d} : \overline{\mathcal{M}}^G_n(\mathbb{A}, X, d) \to \overline{\mathcal{I}}_{X/\!\!/G}$ be evaluation at infinity restricted to affine gauged maps, and

$$\mathrm{ev}_{\infty,d,*} : H(\overline{\mathcal{M}}^G_n(\mathbb{A}, X, d)) \otimes_{\mathbb{Q}} \Lambda^G_X \to H_G(\overline{\mathcal{I}}_{X/\!\!/G}) \otimes_{\mathbb{Q}} \Lambda^G_X$$

push-forward using the virtual fundamental class. The quantum Kirwan map is

$$\kappa^G_X : \widehat{QH}_G(X) \to QH_{\mathbb{C}^\times}(X/\!\!/G), \quad \alpha \mapsto \sum_{n,d} \frac{q^d}{n!} \mathrm{ev}_{\infty,d,*}\, \mathrm{ev}^*(\alpha, \ldots, \alpha).$$

As a formal map each term in the Taylor series of κ^G_X is well-defined on $QH_G(X)$, but in general the sum of terms may have convergence issues. The $q = 0$ specialization of κ^G_X is the Kirwan map to the cohomology of a git quotient studied in [29].

The cobordism relating stable maps to the quotient with Mundet semistable maps is itself a moduli stack of gauged maps with scaling defined by allowing the linearization to tend towards infinity. In order to determine which stability condition to use, the source curves must be equipped with additional data of a *scaling*: a section

$$\delta : \hat{C} \to \mathbb{P}\left(\omega_{\hat{C}/(C\times S)} \oplus \mathcal{O}_{\hat{C}}\right)$$

of the projectivized relative dualizing sheaf. If the section is finite, one uses the Mundet semistability condition, while if infinite one uses the stability condition on the target. The possibility of constructing a cobordism in this way was suggested by a symplectic argument of Gaio-Salamon [22]. A *scaled gauged map* is a map to the quotient stack whose domain is a curve equipped with a section of the projectivized dualizing sheaf and a collection of distinct markings: A datum

$$\hat{C} \to S, \quad v : \hat{C} \to C \times X/G, \quad \delta : \hat{C} \to \mathbb{P}\left(\omega_{\hat{C}/(C\times S)} \oplus \mathcal{O}_{\hat{C}}\right), \quad z_1, \ldots, z_n \in \hat{C}$$

where $\hat{C} \to S$ is a nodal curve of genus $g = \mathrm{genus}\, C$, $v = (P, u)$ is a morphism to the quotient stack X/G that consists of a principal G-bundle $P \to \hat{C}$ and a map $u : \hat{C} \to P \times_G X$ of whose class projects to $[C] \in H_2(C)$, and δ is a section of the projectivization of the relative dualizing sheaf $\omega_{\hat{C}/(C\times S)}$ satisfying certain properties. In the case that $X/\!\!/G$ is an orbifold, the domain $\hat{C}$ is allowed to have orbifold singularities at the nodes and markings and the morphism is required to be representable. The moduli stack of stable scaled gauged maps $\overline{\mathcal{M}}^G_{n,1}(C, X, d)$ with n markings and class $d \in H^G_2(X, \mathbb{Q})$ is equipped with a forgetful map

$$\rho : \overline{\mathcal{M}}^G_{n,1}(C, X, d) \to \overline{\mathcal{M}}_{0,1} \cong \mathbb{P}^1, \quad [\hat{C}, u, \delta, \underline{z}] \mapsto \delta.$$

The fibers of ρ over zero $0, \infty \in \mathbb{P}^1$ consist of either Mundet semistable gauged maps, in the case $\delta = 0$, or stable maps to the git quotient together with affine gauged maps, in the case $\delta = \infty$: In notation,

$$(2)\quad \rho^{-1}(0) = \overline{\mathcal{M}}^G_n(C, X, d), \quad \rho^{-1}(\infty) = \bigcup_{d_0+\ldots+d_r=d} \bigcup_{I_1\cup\ldots\cup I_r=\{1,\ldots,n\}} \left(\overline{\mathcal{M}}^{\mathrm{fr}}_{g,r}(C \times X/\!\!/G, (1, d_0)) \times_{(\overline{\mathcal{I}}_{X/\!\!/G})^r} \prod_{j=1}^{r} \overline{\mathcal{M}}^G_{|I_j|,1}(\mathbb{A}, X, d_j) \right) /(\mathbb{C}^\times)^r$$

where the superscipt fr indicates the inclusion of framings at the tangent spaces to the markings, $(\mathbb{C}^\times)^r$ acts diagonally on the framings and on the scalings, and we identify $H_2(X/\!\!/G)$ as a subspace of $H_2^G(X)$ via the inclusion $X/\!\!/G \subset X/G$. The properness of these moduli stacks was argued via symplectic geometry in [54]. The advantage of the symplectic proof is that the compactness is somewhat more natural; it follows by a combination of Gromov and Uhlenbeck compactness theorems as in Ott [43] and also applies in the presence of Lagrangian boundary conditions as in Xu [56], for an arbitrary symplectic manifold. However, in the setting of virtual fundamental classes constructed algebraically, one prefers to stay in the framework of algebraic geometry. This is especially true in a subsequent paper of the first and third authors in which we extend the results to quantum K-theory, and in particular give presentations of the quantum K-theory ring of toric stacks; the definition of quantum K-theory is at the moment heavily algebraic, and there is no known definition purely in terms of symplectic geometry. Also, it is good to have several proofs. In this paper we give an algebraic proof of the following:

Theorem 1.1. *For any real $E > 0$, the union of components $\overline{\mathcal{M}}^G_{n,1}(C, X, d)$, $\overline{\mathcal{M}}^G_n(C, X, d)$, and $\overline{\mathcal{M}}^G_{n,1}(\mathbb{A}, X, d)$ with $(d, c_1^G(\tilde{X})) < E$ is proper.*

The proof is a combination of boundedness arguments and valuative criteria. By integration over the moduli stack of stable scaled gauged maps one obtains the following identity: Let $\tau_X^{G,k}$ denote the gauged potential of (1) defined using the polarization $\tilde{X}^k$ for k a positive integer.

$$(3)\qquad \lim_{k\to\infty} \tau_X^{G,k} = \tau_{X/\!\!/G} \circ \kappa_X^G.$$

This is called in [22] and [53] the *adiabatic limit theorem*.

2. Scaled curves

Scaled curves are curves with a section of the projectivized dualizing sheaf incorporated, intended to give complex analogs of spaces introduced by Stasheff [48] such as the multiplihedron, cyclohedron etc. Recall from Deligne-Mumford [17] and Behrend-Manin [7, Definition 2.1] the definition of stable and prestable curves. A *prestable curve* over the scheme S is a flat proper morphism $\pi : C \to S$ of schemes such that the geometric fibers of π are reduced, connected, one-dimensional and have at most ordinary double points (nodes) as singularities. A *marked prestable curve* over S is a prestable curve $\pi : C \to S$ equipped with a tuple $\underline{z} = (z_1, \ldots, z_n) : S \to C^n$ of distinct non-singular sections. A *morphism* $p : C \to D$ of prestable curves over S is an S-morphism of schemes, such that for every geometric point s of S we have (a) if η is the generic point of an irreducible component of D_s, then the fiber of p_s over η is a finite η-scheme of degree at most one, (b) if C' is the normalization of an irreducible component of C_s, then $p_s(C')$ is a single point only if C' is rational. A prestable curve is *stable* if it has finitely many automorphisms. Denote by $\overline{\mathcal{M}}_{g,n}$ the proper Deligne-Mumford stack of stable curves of genus g with n markings [17]. The stack $\overline{\mathfrak{M}}_{g,n}$ of prestable curves of genus g with n markings is an Artin stack locally of finite type [5, Proposition 2].

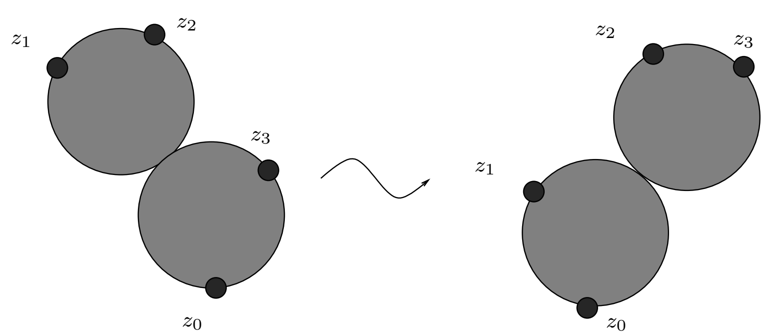

Figure 1. Associativity divisor relation

The following constructions give complex analogs of the spaces constructed in Stasheff [48]. For any family of possibly nodal curves $C \to S$ we denote by ω_C the relative dualizing sheaf defined for example in Arbarello-Cornalba-Griffiths [4, p. 97]. Similarly for any morphism $\hat{C} \to C$ we denote by $\omega_{\hat{C}/C}$ the relative dualizing sheaf and $\mathbb{P}(\omega_{\hat{C}/C} \oplus \mathcal{O}_{\hat{C}}) \to \hat{C}$ the projectivization. A *scaling* is a section

$$\delta : \hat{C} \to \mathbb{P}(\omega_{\hat{C}/C} \oplus \mathcal{O}_{\hat{C}}), \quad \mathbb{P}(\omega_{\hat{C}/C} \oplus \mathcal{O}_{\hat{C}}) = (\omega_{\hat{C}/C} \oplus \mathcal{O}_{\hat{C}})^\times / \mathbb{C}^\times.$$

If $\hat{C} \to C$ is an isomorphism then $\omega_{\hat{C}/C}$ is trivial:

$$(\hat{C} \cong C) \implies (\mathbb{P}(\omega_{\hat{C}/C} \oplus \mathcal{O}_{\hat{C}}) \cong C \times \mathbb{P}^1).$$

In this case a scaling δ is a section $C \to \mathbb{P}^1$, and δ is required to be constant. Thus the space of scalings on an unmarked, irreducible curve is $\mathbb{P}^1$.

Scalings on nodal curves with markings are required to satisfy the following properties. First, δ should satisfy the *affinization* property that on any component $\hat{C}_i$ of $\hat{C}$ on which δ is finite and non-zero, δ has no zeroes and a single double pole. In particular, this implies that in the case $\hat{C} \cong C$, then

δ is a constant section as in the last paragraph, while on any component $\hat{C}_i$ of $\hat{C}$ with finite non-zero scaling which maps to a point in C, δ defines an affine structure on the complement of the pole. To define the second property, note that any morphism $\hat{C} \to C$ of class $[C]$ defines a *rooted tree* whose vertices are components $\hat{C}_i$ of $\hat{C}$, whose edges are nodes $w_j \in \hat{C}$, and whose root vertex is the vertex corresponding to the component $\hat{C}_0$ that maps isomorphically to C. Let $\mathcal{T}$ denote the set of indices of *terminal* components $\hat{C}_i$ that meet only one other component of $\hat{C}$:

$$\mathcal{T} = \{i \mid \#\{j \neq i | \hat{C}_j \cap \hat{C}_i \neq \emptyset\} = 1\}$$

as in Figure 2. The *bubble components* are the components of $\hat{C}$ mapping to a point in C. For each terminal component $\hat{C}_i, i \in \mathcal{T}$ there is a canonical

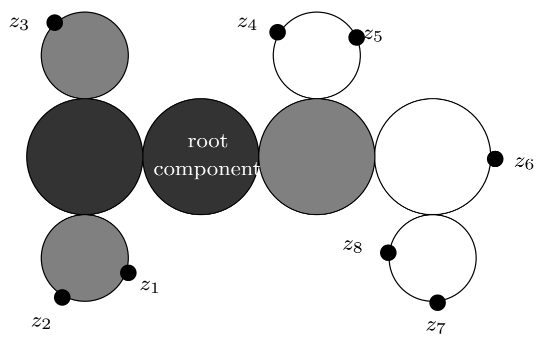

FIGURE 2. A scaled marked curve

non-self-crossing path of components $\hat{C}_{i,0} = \hat{C}_0, \ldots, \hat{C}_{i,k(i)} = \hat{C}_i$. Define a partial order on components by $\hat{C}_{i,j} \preceq \hat{C}_{i,k}$ for $j \leq k$. The *monotonicity property* requires that δ is finite and non-zero on at most one of these (gray shaded) components, say $\hat{C}_{i,f(i)}$, and

$$\delta | \hat{C}_{i,j} = \begin{cases} \infty & j < f(i) \\ 0 & j > f(i) \end{cases}. \tag{4}$$

We call $\hat{C}_{i,f(i)}$ a *transition component.* That is, the scaling δ is infinite on the components before the transition components and zero on the components after the transition components, in the ordering $\preceq$. See Figure 2. In addition the *marking condition* requires that the scaling is finite at the markings:

$$\delta(z_i) < \infty, \quad \forall i = 1, \ldots, n.$$

Definition 2.1. *A* prestable scaled curve *with target a smooth projective curve C is a morphism from a prestable map $\hat{C}$ to C of class $[C]$ equipped with section δ and n markings $\underline{z} = (z_1, \ldots, z_n)$ satisfying the affinization,* *monotonicity and marking properties. Isomorphisms of prestable scaled curves are diagrams*

$$\begin{array}{ccc} \hat{C}_1 & \xrightarrow{\varphi} & \hat{C}_2 \\ \downarrow & & \downarrow \\ S_1 & \longrightarrow & S_2 \end{array}, \quad (D\varphi^*)\varphi^*(\delta_2) = \delta_1, \quad \varphi(z_{i,1}) = z_{i,2}, \quad \forall i = 1, \dots, n$$

where the top arrow is an isomorphism of prestable curves and

$$D\varphi^* : \varphi^* \mathbb{P}(\omega_{\hat{C}_2/C} \oplus \mathcal{O}_{\hat{C}_2}) \to \mathbb{P}(\omega_{\hat{C}_1/C} \oplus \mathcal{O}_{\hat{C}_1})$$

is the associated morphism of projectivized relative dualizing sheaves. A scaled curve is stable *if on each bubble component $\hat{C}_i \subset \hat{C}$ (that is, component mapping to a point in C) there are at least three special points (markings or nodes),*

$$(\delta | \hat{C}_i \in \{0, \infty\}) \implies \#((\{z_i\} \cup \{w_j\}) \cap \hat{C}_i) \geq 3$$

or the scaling is finite and non-zero and there are least two special points

$$(\delta | \hat{C}_i \notin \{0, \infty\}) \implies \#((\{z_i\} \cup \{w_j\}) \cap \hat{C}_i) \geq 2.$$

Introduce the following notation for moduli spaces. Let $\overline{\mathfrak{M}}_{n,1}(C)$ denote the category of prestable n-marked scaled curves and $\overline{\mathcal{M}}_{n,1}(C)$ the subcategory of stable n-marked scaled curves.

The *combinatorial type* of a prestable marked scaled curve is defined as follows. Given such $(\hat{C}, u : \hat{C} \to C, \underline{z}, \delta)$ Let Γ be the graph whose vertex set $\mathrm{Vert}(\Gamma)$ is the set of irreducible components of C, finite edges $\mathrm{Edge}_{<\infty}(\Gamma)$ correspond to nodes, semi-infinite edges $\mathrm{Edge}_{\infty}(\Gamma)$ correspond to markings, and equipped with the labelling of semi-infinite edges by $\{1, \dots, n\}$ a distinguished *root vertex* $v_0 \in \mathrm{Vert}(\Gamma)$ corresponding to the root component and a set of *transition vertices* $\mathrm{Vert}^t(\Gamma) \subset \mathrm{Vert}(\Gamma)$ corresponding to the transition components. Graphically we represent a combinatorial type as a graph with transition vertices shaded by grey, and the vertices lying on three levels depending on whether they occur before or after the transition vertices. See Figure 3. Note that the combinatorial type is functorial; in particular any automorphism of prestable marked scaled curves induces an automorphism of the corresponding type, that is, an automorphism of the graph preserving the additional data.

We note that the graphical representation of the combinatorial type of a curve can be viewed as the graph of a Morse/height function on the curve. In general this gives a spider like figure with the root component being the body of the spider. From this perspective the paths used in the monotonicity property of scalings are the legs of the spider.

Example 2.2. (a) For $n = 0$, no bubbling is possible and $\overline{\mathcal{M}}_{0,1}(C)$ is the projective line, $\overline{\mathcal{M}}_{0,1}(C) \cong \mathbb{P}^1$.

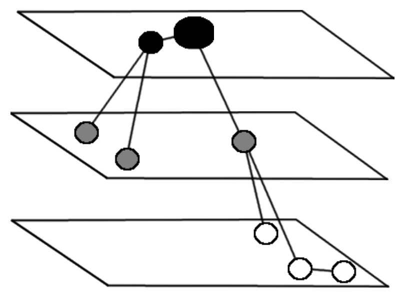

FIGURE 3. Combinatorial type of a scaled marked curve

(b) For $n = 1$, $\overline{\mathcal{M}}_{1,1}(C)$ consists of configurations $\mathcal{M}_{1,1}(C) \cong C \times \mathbb{C}$ with irreducible domain and finite scaling; a configurations $\overline{\mathcal{M}}_{1,1} - \mathcal{M}_{1,1}$ with one component $\hat{C}_0 \cong C$ with infinite scaling $\delta|\hat{C}_0$, and another component $\hat{C}_1$ mapping trivially to C, equipped with a one-form $\delta|\hat{C}_1$ with a double pole at the node and a marking $z_1 \in \hat{C}_1$. Thus $\overline{\mathcal{M}}_{1,1}(C) \cong C \times \mathbb{P}^1$.

(c) For $n = 2$, $\overline{\mathcal{M}}_{2,1}(C)$ consists of configurations $\mathcal{M}_{2,1}(C)$ with two distinct points $z_1, z_2 \in C$ and a scaling $\delta \in \mathbb{P}^1$; configurations $\mathcal{M}_{2,1,\Gamma_1}$ where the two points z_1, z_2 have come together and bubbled off onto a curve $z_1, z_2 \in \hat{C}_1$ with zero scaling $\delta|\hat{C}_1$, so that $\mathcal{M}_{2,1,\Gamma_1} \cong C \times \mathbb{P}^1$; configurations $\mathcal{M}_{2,1,\Gamma_2}$ with a root component $\hat{C}_0$ with infinite scaling $\delta|\hat{C}_0$, and two components $\hat{C}_1, \hat{C}_2$ with non-trivial scalings $\delta|\hat{C}_1, \delta|\hat{C}_2$ containing markings $z_1 \in \hat{C}_1, z_2 \in \hat{C}_2$; a stratum $\mathcal{M}_{2,1,\Gamma_2}$ of configurations with a component $\hat{C}_1$ containing two markings $z_1, z_2 \in \hat{C}_1$ and $\delta|\hat{C}_1$ non-zero; a stratum $\mathcal{M}_{2,1,\Gamma_3}$ containing with three components, one $\hat{C}_0$ mapping isomorphically to C; one $\hat{C}_1$ with two nodes and a one form $\delta|\hat{C}_1$ with a double pole at the node attaching to $\hat{C}_0$; and a component $\hat{C}_2$ with two markings $z_1, z_2 \in \hat{C}_2$, a node, and vanishing scaling $\delta|\hat{C}_2$; and a stratum a stratum $\mathcal{M}_{2,1,\Gamma_4}$ containing the root component $\hat{C}_0$, a component $\hat{C}_1$ with infinite scaling with three nodes, and two components $\hat{C}_2, \hat{C}_3$ with finite, non-zero scaling, each containing a node and a marking. The two evaluation maps at the markings, together with the forgetful map to $\overline{\mathcal{M}}_{0,1}(C)$, define an isomorphism $\overline{\mathcal{M}}_{2,1}(C) \to C \times C \times \mathbb{P}^1$.

Remark 2.3. The extension of the one-form in a family of scaled curves may be explicitly described as follows. On each component of the limit, the one-form is determined by the limiting behavior of the product of deformation parameters for the nodes connecting that component to the root component of the limit: Let

$$\hat{C} \to S, \delta : \hat{C} \to \mathbb{P}(\omega_{\hat{C}/C\times S} \oplus \mathcal{O}_{\hat{C}}), \underline{z} : S \to \hat{C}^n$$

be a family of scaled curves over a punctured curve $S = \overline{S} - \{\infty\}$ and $\hat{C}_\infty$ a curve over ∞ extending the family $\hat{C}$. Let $\operatorname{Def}(\hat{C}_\infty)/\operatorname{Def}_\Gamma(\hat{C}_\infty)$ denote the deformation space of the curve $\hat{C}_\infty$ normal to the stratum of curves of the same combinatorial type Γ as $\hat{C}_\infty$. This normal deformation space is canonically identified with the sum of products of cotangent lines at the nodes

$$\operatorname{Def}(\hat{C}_\infty)/\operatorname{Def}_\Gamma(\hat{C}_\infty) = \sum_w T_w^\vee \hat{C}_{i_-(w)} \otimes T_w^\vee \hat{C}_{i_+(w)}$$

where $\hat{C}_{i_\pm(w)}$ are components of $\hat{C}_\infty$ adjacent to w, see [4, p. 176]. Over the deformation space $\operatorname{Def}(\hat{C}_\infty)$ lives a semiversal family, universal if the curve is stable. Given family of curves $\hat{C} \to S$ as above the curve $\hat{C}$ is obtained by pull-back of the semiversal family by a map

$$S \to \sum_w T_w^\vee \hat{C}_{i_-(w)} \otimes T_w^\vee \hat{C}_{i_+(w)}, \quad z \mapsto (\delta_w(z))$$

describing the curves as local deformations (non-uniquely, since the curves themselves may be only prestable.) Let

$$\hat{C}_0 = \hat{C}_{i,0}, \ldots, \hat{C}_{i,l(i)} := \hat{C}_i$$

denote the path of components from the root component, and

$$w_{i,0}, \ldots, w_{i,l(i)-1} \in \hat{C}_\infty$$

the corresponding sequence of nodes. The nodes $w_{i,j}, w_{i,j+1}$ lie in the same component $C_{i,j+1}$ and we have a canonical isomorphism

$$T^\vee_{w_{i,j}} C_{i,j+1} \cong T_{w_{i,j+1}} C_{i,j+1}$$

corresponding to the relation of local coordinates $z_+ = 1/z_-$ near $w_{i,j}$. Deformation parameters for this chain lie in the space

$$\begin{aligned}(5) \quad \operatorname{Hom}(T^\vee_{w_{i,0}}\hat{C}_{i,0}, T^\vee_{w_{i,1}}\hat{C}_{i,1}) \oplus \operatorname{Hom}(T^\vee_{w_{i,1}}\hat{C}_{i,1}, T^\vee_{w_{i,2}}\hat{C}_{i,2}) \ldots \\ \oplus \operatorname{Hom}(T^\vee_{w_{i,l(i)-2}}\hat{C}_{i,l(i)-2}, T^\vee_{w_{i,l(i)-1}}\hat{C}_{i,l(i)-1}).\end{aligned}$$

In particular, the product of deformation parameters

$$(6) \qquad \gamma_{w_{i,0}}(z) \cdot \cdots \cdot \gamma_{w_{i,l(i)-1}}(z) \in \operatorname{Hom}(T^\vee_{w_{i,0}}\hat{C}_{i,0}, T^\vee_{w_{i,l(i)-1}}\hat{C}_{i,l(i)-1})$$

is well-defined. The product represents the *scale* at which the bubble component $\hat{C}_i$ forms in comparison with $\hat{C}_0 = \hat{C}_{i,0}$, that is, the ratio between the derivatives of local coordinates on $\hat{C}_i$ and $\hat{C}_0$. If z is a point in $\hat{C}_i$ then we also have a canonical isomorphism $T^\vee_z\hat{C}_i \to T_{w_{i,0}}\hat{C}_0$. The product (6) gives an isomorphism $T^\vee_z\hat{C}_i \to T^\vee_{w_0}\hat{C}_0$.

$$(7) \qquad \delta|\hat{C}_i = \lim_{z\to 0} \delta(z)(\gamma_{w_{i,0}}(z) \cdot \cdots \cdot \gamma_{w_{i,l(i)-1}}(z))$$

the ratio of the scale of the bubble component with the parameter $\delta(z)^{-1}$. This ends the Remark.

One may view a scaled curve with infinite scaling on the root component as a nodal curve formed from the root component and a collection of bubble trees as follows.

Definition 2.4. *An* affine prestable scaled curve *consists of a tuple* $(C, \delta, \underline{z})$ *where* C *is a connected projective nodal curve,* $\delta : C \to \mathbb{P}(\omega_C \oplus \mathcal{O}_C)$ *a section of the projectivized dualizing sheaf, and* $\underline{z} = (z_0, \ldots, z_n)$ *non-singular, distinct points, such that*

(a) δ *is monotone in the following sense: For each terminal component* $\hat{C}_i, i \in \mathcal{T}$ *there is a canonical non-self-crossing path of components*

$$\hat{C}_{l(i),0} = \hat{C}_0, \ldots, \hat{C}_{i,k(i)} = \hat{C}_i.$$

The monotonicity condition is for any such non-self-crossing path of components starting with a root component, that δ *is finite and non-zero on at most one of these* transition components, *say* $\hat{C}_{i,f(i)}$, *and the scaling is infinite for all components before the transition component and zero for components after the transition component:*

$$\delta|\hat{C}_{i,j} = \begin{cases} \infty & j < f(i) \\ 0 & j > f(i) \end{cases}.$$

(b) δ *is infinite at* z_0, *and finite at* $z_1, \ldots, z_n$.

A prestable affine scaled curve is stable *if it has finitely many automorphisms, or equivalently, if each component* $C_i \subset C$ *has at least three special points (markings or nodes),*

$$(\delta|C_i \in \{0, \infty\}) \implies \#((\{z_i\} \cup \{w_j\}) \cap C_i) \geq 3$$

or the scaling is finite and non-zero and there are least two special points

$$(\delta|C_i \notin \{0, \infty\}) \implies \#((\{z_i\} \cup \{w_j\}) \cap C_i) \geq 2.$$

We will see below in Theorem 2.5 that scaled marked curves have no automorphisms. Examples of stable affine scaled curves are shown in Figure 4. Denote the moduli stack of prestable affine scaled curves resp. stable affine n-marked scaled curves by $\overline{\mathfrak{M}}_{n,1}(\mathbb{A})$ resp. $\overline{\mathcal{M}}_{n,1}(\mathbb{A})$.

Theorem 2.5. *For each* $n \geq 0$ *and smooth projective curve* C *the moduli stack* $\overline{\mathcal{M}}_{n,1}(C)$ *resp.* $\overline{\mathcal{M}}_{n,1}(\mathbb{A})$ *of stable scaled affine curves is a proper scheme locally isomorphic to a product of a number of copies of* C *with a toric variety. The stack* $\overline{\mathfrak{M}}_{n,1}(C)$ *resp.* $\overline{\mathfrak{M}}_{n,1}(\mathbb{A})$ *of prestable scaled curves is an Artin stack of locally finite type.*

Proof. Standard arguments on imply that $\overline{\mathcal{M}}_{n,1}(C)$ and $\overline{\mathfrak{M}}_{n,1}(C)$ are stacks, that is, categories fibered in groupoids satisfying effective descent for objects and for which morphisms form a sheaf. An object $(\hat{C}, \underline{z}, \delta)$ of $\overline{\mathcal{M}}_{n,1}(C)$ over

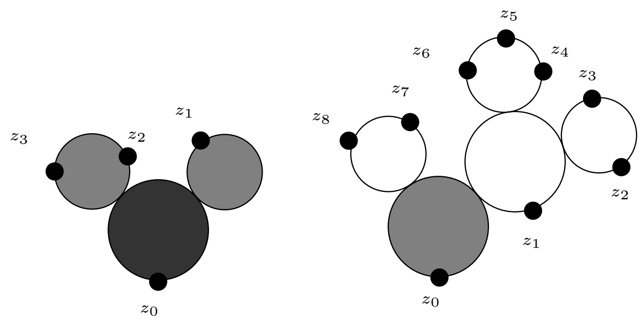

FIGURE 4. Examples of stable affine scaled curves

a scheme S is a family of curves with sections. Families of curves with markings and sections satisfy the gluing axioms for objects; similarly morphisms are determined uniquely by their pull-back under a covering. Standard results on hom-schemes imply that the diagonal for $\overline{\mathfrak{M}}_{n,1}(C)$, hence also $\overline{\mathcal{M}}_{n,1}(C)$, is representable, see for example [17, 1.11] for similar arguments, hence the stacks $\overline{\mathfrak{M}}_{n,1}(C)$ and $\overline{\mathcal{M}}_{n,1}(C)$ are algebraic.

In preparation for showing that $\overline{\mathcal{M}}_{n,1}(C)$ is a variety we claim that for any object $(\hat{C}, \underline{z}, \delta)$ of the moduli stack $\overline{\mathcal{M}}_{n,1}(C)$ the automorphism group is trivial. Let Γ be the combinatorial type. The association of Γ to $(\hat{C}, \underline{z}, \delta)$ is functorial and any automorphism of $(\hat{C}, \underline{z}, \delta)$ induces an automorphism of Γ. The graph Γ is a tree with labelled semi-infinite edges, each vertex is determined uniquely by the partition of semi-infinite edges given by removing the vertex; hence the automorphism acts trivially on the vertices of Γ. Each component has at least three special points, or two special points and a non-trivial scaling and so has trivial automorphism group fixing the special points. Thus the automorphism is trivial on each component of $\hat{C}$. The claim follows.

The moduli space of stable scaled curves has a canonical covering by varieties corresponding to the versal deformations of prestable curves constructed by gluing. Suppose that $(u : \hat{C} \to C, \underline{z}, \delta)$ is an object of $\overline{\mathcal{M}}_{n,1}(C)$ of combinatorial type Γ. Let $\rho : \overline{\mathcal{M}}_{n,1}(C) \to \overline{\mathcal{M}}_{0,1}(C) \cong \mathbb{P}^1$ denote the forgetful morphism. The locus $\rho^{-1}(\mathbb{C}) \subset \overline{\mathcal{M}}_{n,1}(C)$ of curves with finite scaling is isomorphic to $\overline{\mathcal{M}}_n(C) \times \mathbb{C}$, where the last factor denotes the scaling. In the case that the root component has infinite scaling, let $\Gamma_1, \ldots, \Gamma_k$ denote the (possibly empty) combinatorial types of the bubble trees attached at the special points. The stratum $\mathcal{M}_{n,1,\Gamma}(C)$ is the product of C^k with moduli stacks of scaled affine curves $\mathcal{M}_{n_i,1,\Gamma_i}(\mathbb{A})$ for $i = 1, \ldots, k$, each isomorphic to an affine space given by the number of markings and scalings minus the dimension of the automorphism group $(n_i + 1) + 1 - \dim(\mathrm{Aut}(\mathbb{P}^1)) = n_i - 1$ [35]. Let

$$\gamma_e \in T^{\vee}_{w(e)} \hat{C}_{i_-(e)} \otimes T^{\vee}_{w(e)} \hat{C}_{i_+(e)}, \quad e \in \mathrm{Edge}_{<\infty}(\Gamma)$$

be the deformation parameters for the nodes. A collection of deformation parameters $\underline{\gamma} = (\gamma_e)_{e \in \mathrm{Edge}(\Gamma)}$ is *balanced* if the signed product

$$\prod_{e \in P} \gamma_e^{\pm 1} \tag{8}$$

of parameters corresponding to any non-self-crossing path P between transition components is equal to 1, where the sign is positive for edges pointing towards the root vertex and equal to -1 if the edge is oriented away from it. Let Z_Γ denote the set of deformation parameters satisfying the condition (8). Then there is a morphism

$$\mathcal{M}_{n,1,\Gamma}(C) \times Z_\Gamma \to \overline{\mathcal{M}}_{n,1}(C)$$

described as follows. Choose local étale coordinates $z_e^{\pm}$ on the adjacent components to each node $w_e, \in \mathrm{Edge}_{<\infty}(\Gamma)$ and glue together the components using the identifications $z_e^+ \mapsto \gamma_e / z_e^-$, see for example [4, p. 176], [42, 2.2]. Set the scaling on the root component

$$\delta = \prod_{e \in P} \gamma_e$$

where P is a path of nodes from the root component to the transition component, independent of the choice of component by (8). This gives a family $(\hat{C}, u, \delta, \underline{z})$ of stable scaled curves over $\mathcal{M}_{n,1,\Gamma}(C) \times Z_\Gamma$ and hence a morphism to $\overline{\mathcal{M}}_{n,1}(C)$. The family $(\hat{C}, \underline{z}, u, \delta)$ defines a universal deformation of any curve of type Γ. Indeed, $(\hat{C}, \underline{z})$ is a versal deformation of any of its prestable fibers by [4], and it follows that the family $(\hat{C}, \underline{z}, u)$ is a versal deformation of any of its fibers since there is a unique extension of the stable map on the central fiber, up to automorphism. The equation (6) implies that any family of stable scaled curves satisfies the balanced relation (8) between the deformation parameters for any family of marked curves with scalings. This provides a cover of $\overline{\mathcal{M}}_{n,1}(C)$ by varieties. It follows that $\overline{\mathcal{M}}_{n,1}(C)$ is a variety.

The stack of prestable scaled curves $\overline{\mathfrak{M}}_{n,1}(C)$ is an Artin stack of locally finite type. Charts for the stack $\overline{\mathfrak{M}}_{n,1}(C)$, as in the case of prestable curves in [5], are given by using forgetful morphisms $\overline{\mathcal{M}}_{n+k,1}(C) \to \overline{\mathfrak{M}}_{n,1}(C)$. Since these morphisms admit sections locally, they provide a smooth covering of $\overline{\mathfrak{M}}_{n,1}(C)$ by varieties.

We check the valuative criterion for properness for $\overline{\mathcal{M}}_{n,1}(C)$. Given a family of stable scaled marked curves over a punctured curve S with finite scaling δ

$$(\hat{C}, u : \hat{C} \to C, \underline{z}, \delta) \to S = \overline{S} - \{\infty\}$$

we wish to construct there exists an extension over $\overline{S}$. We consider only the case $\hat{C} \cong C \times S$; the general case is similar. After forgetting the scaling δ and stabilizing we obtain a family of stable maps to C of degree $[C]$,

$$(\hat{C}^{\mathrm{st}}, u : \hat{C}^{\mathrm{st}} \to C, \underline{z}^{\mathrm{st}}) \to \overline{S} - \{\infty\}.$$

By properness of the stack $\overline{\mathcal{M}}_n(C)$ of stable maps to C, this family extends over the central fiber ∞ to give a family over $\overline{S}$. The section δ of $\omega_{\hat{C}^{\rm st}/C}$ defines an extension over $\overline{S}$ except possibly at the nodes. Here there are possible irremovable singularities corresponding to the following situation: suppose that $\hat{C}_0, \ldots \hat{C}_i$ is a chain of components in the curve at the central fiber, with $\hat{C}_0 \cong C$ the root component. Suppose that $\hat{C}_i, \hat{C}_{i+1}$ are adjacent component with δ infinite on $\hat{C}_i$ and zero on $\hat{C}_{i+1}$. Taking the closure of the graph of δ gives a family $\hat{C}$ of curves over C given by replacing some of the nodes of $\hat{C}^{\rm st}$ with fibers of $\mathbb{P}(\omega_{\hat{C}^{\rm st}/C} \oplus \mathcal{O}_{\hat{C}^{\rm st}})$ over the node. The relative cotangent bundle of $\hat{C}$ is related to that of $\hat{C}^{\rm st}$ by a twist at D_0, D_∞: If $\pi : \hat{C} \to \hat{C}^{\rm st}$ denotes the projection onto $\hat{C}$ then on the components of $\hat{C}$ collapsed by π we have

$$\omega_{\hat{C}/C} = \pi^* \omega_{\hat{C}^{\rm st}/C}(-D_0 - D_\infty)$$

where D_0, D_∞ are the inverse images of the sections at zero and infinity in $\mathbb{P}(\omega_{\hat{C}^{\rm st}/C} \oplus \mathcal{O}_{\hat{C}^{\rm st}})$. Abusing notation $\omega_{\hat{C}_i^{\rm st}/C}(-D_0) = \omega_{\hat{C}_i^{\rm st}/C}$ resp. $\omega_{\hat{C}_i^{\rm st}/C}(-D_\infty) = \omega_{\hat{C}_i^{\rm st}/C}$ on components $\hat{C}_i^{\rm st}$ contained in D_0 resp. D_∞. The extension of δ to a rational section of $\pi^*\omega_{\hat{C}^{\rm st}/C}$ has, by definition a zero at $\delta^{-1}(D_0)$ and a pole at $\delta^{-1}(D_\infty)$. Hence the extension of δ to a section of $\pi^*\omega_{\hat{C}^{\rm st}/C}(-D_0 - D_\infty)$ has no zeroes at D_0 and a double pole at D_∞. This implies that δ extends uniquely as a section of $\mathbb{P}(\omega_{\hat{C}/C} \oplus \mathcal{O}_{\hat{C}})$ to all of $\overline{S}$.

By the construction (7), the extension of δ satisfies the monotonicity condition (4). Indeed suppose that a component $\hat{C}_i$ is further away from a component $\hat{C}_j$ in the path of components from the root component $\hat{C}_0$. Since all deformation parameters $\gamma_{w_{i,k}}(z)$ are approaching zero, from (7), at most one of the limits $\delta|\hat{C}_i, \delta|\hat{C}_j$ can be finite, and

$$\begin{cases} \delta|\hat{C}_i \text{ finite} \implies \delta|\hat{C}_j & \text{zero} \\ \delta|\hat{C}_j \text{ finite} \implies \delta|\hat{C}_i & \text{infinite.} \end{cases} .$$

The condition (4) follows. □

3. Mumford stability

In this section we review the relationship between the stack-theoretic quotient and Mumford's geometric invariant theory quotient [39]. First we introduce various Lie-theoretic notation. Let G be a connected complex reductive group with Lie algebra $\mathfrak{g}$. When G is abelian (so a complex torus) we denote by

$$\mathfrak{g}_{\mathbb{Z}} = \{ D\phi(1) \in \mathfrak{g} \mid \phi \in \operatorname{Hom}(\mathbb{C}^\times, G) \}, \quad \mathfrak{g}_{\mathbb{Q}} = \mathfrak{g}_{\mathbb{Z}} \otimes_{\mathbb{Z}} \mathbb{Q}$$

the *coweight lattice* of derivatives of one-parameter subgroups resp. rational one-parameter subgroups. Dually denote by

$$\mathfrak{g}_{\mathbb{Z}}^\vee = \{ D\chi \in \mathfrak{g}^\vee \mid \chi \in \operatorname{Hom}(G, \mathbb{C}^\times) \}, \quad \mathfrak{g}_{\mathbb{Q}}^\vee = \mathfrak{g}_{\mathbb{Z}}^\vee \otimes_{\mathbb{Z}} \mathbb{Q}$$

the *weight lattice* of derivatives of characters of G and the set of *rational weights*, respectively. If G is non-abelian then we still denote by $\mathfrak{g}_{\mathbb{Q}}$ the set of derivatives of rational one-parameter subgroups.

The targets of our maps are quotient stacks defined as follows. Let X be a smooth projective G-variety. Let X/G denote the quotient stack, that is, the category fibered in groupoids whose fiber over a scheme S has objects pairs $v = (P, u)$ consisting of a principal G-bundle $P \to S$ and a section $u : S \to P \times_G X$; and whose morphisms are given by diagrams

$$\begin{tikzcd} P_1 \arrow[r, "\phi"] \arrow[d] & P_2 \arrow[d] \\ S_1 \arrow[r, "\psi"'] & S_2 \end{tikzcd}, \quad \phi(X) \circ u_1 = u_2 \circ \psi$$

where $\phi(X) : P_1(X) \to P_2(X)$ denotes the map of associated fiber bundles [17], Tag 04UV [16].

Mumford's geometric invariant theory quotient [39] is traditionally defined as the projective variety associated to the graded ring of invariant sections of a linearization of the action in the previous paragraph. Let $\tilde{X} \to X$ be a linearization, that is, ample G-line bundle. Then

$$X/\!\!/G := \operatorname{Proj}\left(\oplus_{k \geq 0} H^0(\tilde{X}^k)^G\right).$$

Mumford [39] realizes this projective variety as the quotient of a *semistable locus* by an equivalence relation. The semistable locus consists of points $x \in X$ such that some tensor power $\tilde{X}^k, k > 0$ of $\tilde{X}$ has an invariant section non-vanishing at x, while the unstable locus is the complement of the semistable locus:

$$X^{\mathrm{ss}} = \{x \in X \mid \exists k > 0, \sigma \in H^0(\tilde{X}^k)^G, \quad \sigma(x) \neq 0\}, \quad X^{\mathrm{us}} := X - X^{\mathrm{ss}}.$$

A point $x \in X$ is *polystable* if its orbit is closed in the semistable locus $\overline{Gx \cap X^{\mathrm{ss}}} = Gx \cap X^{\mathrm{ss}}$. A point $x \in X$ is *stable* if it is polystable and the stabilizer G_x of x is finite. In Mumford's definition the git quotient is the quotient of the semistable locus by the *orbit equivalence relation*

$$(x_1 \sim x_2) \iff \overline{Gx_1} \cap \overline{Gx_2} \cap X^{\mathrm{ss}} \neq \emptyset.$$

Each semistable point is then orbit-equivalent to a unique polystable point. However, here we define the git quotient as the stack-theoretic quotient

$$X/\!\!/G := X^{\mathrm{ss}}/G.$$

We shall always assume that X^{ss}/G is a Deligne-Mumford stack (that is, the stabilizers G_x are finite) in which case the coarse moduli space of X^{ss}/G is the git quotient in Mumford's sense. The Luna slice theorem [34] implies that X^{ss}/G is étale-locally the quotient of a smooth variety by a finite group, and so has finite diagonal. By the Keel-Mori theorem [28], explicitly stated in [15, Theorem 1.1], the morphism from X^{ss}/G to its coarse moduli space is proper. Since the coarse moduli space of X^{ss}/G is projective by Mumford's construction, it is proper, hence X^{ss}/G is proper as well.

Later we will need the following observation about the unstable locus. As the quotient $X/\!\!/G$ is non-empty, there exists an ample divisor D containing the unstable locus: take D to be the vanishing locus of any non-zero invariant section of $\tilde{X}^k$ for some $k > 0$:

$$D = \sigma^{-1}(0), \quad \sigma \in H^0(\tilde{X}^k)^G - \{0\}. \tag{9}$$

The Hilbert-Mumford numerical criterion [39, Chapter 2] provides a computational tool to determine the semistable locus: A point $x \in X$ is G-semistable if and only if it is $\mathbb{C}^\times$-semistable for all one-parameter subgroups $\mathbb{C}^\times \to G$. Given a rational element $\lambda \in \mathfrak{g}_{\mathbb{Z}}$ denote the corresponding one-parameter subgroup $\mathbb{C}^\times \to G, \ z \mapsto z^\lambda$. Denote by

$$x_\lambda := \lim_{z \to 0} z^\lambda x$$

the limit under the one-parameter subgroup. Let $\mu(x, \lambda) \in \mathbb{Z}$ be the weight of the linearization $\tilde{X}$ at x_λ defined by

$$z\tilde{x} = z^{\mu(x,\lambda)}\tilde{x}, \quad \forall z \in \mathbb{C}^\times, \tilde{x} \in \tilde{X}_{x_\lambda}.$$

By restricting to the case of a projective line one sees that the point $x \in X$ is semistable if and only if $\mu(x, \lambda) \le 0$ for all $\lambda \in \mathfrak{g}_{\mathbb{Z}}$. Polystability is equivalent to semistability and the additional condition $\mu(x, \lambda) = 0 \iff \mu(x, -\lambda) = 0$. Stability is the condition that $\mu(x, \lambda) < 0$ for all $\lambda \in \mathfrak{g}_{\mathbb{Z}} - \{0\}$.

The Hilbert-Mumford numerical criterion [39, Chapter 2] can be applied explicitly to actions on projective spaces as follows. Suppose that G is a torus and $X = \mathbb{P}(V)$ the projectivization of a vector space V. Let $\tilde{X} = \mathcal{O}_X(1) \otimes \mathbb{C}_\theta$ be the G-equivariant line bundle given by tensoring the hyperplane bundle $\mathcal{O}_X(1)$ and the one-dimensional representation $\mathbb{C}_\theta$ corresponding to some weight $\theta \in \mathfrak{g}_{\mathbb{Z}}^\vee$. Recall if $p \in X$ is represented by a line $l \subset V$ then the fiber of $\mathcal{O}_X(1) \otimes \mathbb{C}_\theta$ at p is $l^\vee \otimes \mathbb{C}_\theta$. In particular if z^λ fixes p then z^λ scales l by some $z^{\mu(\lambda)}$, so that $z^\lambda \tilde{x} = z^{-\mu(\lambda)+\theta(\lambda)}\tilde{x}$. Let $k = \dim(V)$ and decompose V into weight spaces $V_1, \ldots, V_k$ with weights $\mu_1, \ldots, \mu_k \in \mathfrak{g}_{\mathbb{Z}}^\vee$. Identify

$$H_G^2(X) \cong H^2_{\mathbb{C}^\times \times G}(V) \cong \mathbb{Z} \oplus \mathfrak{g}_{\mathbb{Z}}^\vee$$

Under this splitting the first Chern class $c_1^G(\tilde{X})$ becomes identified up to positive scalar multiple with the pair

$$c_1^G(\tilde{X}) \mapsto (1, \theta) \in \mathbb{Z} \oplus \mathfrak{g}_{\mathbb{Z}}^\vee. \tag{10}$$

The following is essentially [39, Proposition 2.3].

Lemma 3.1. *The semistable locus for the action of a torus G on the projective space $X = P(V)$ with weights $\mu_1, \ldots, \mu_k$ and linearization shifted by θ is $X^{\mathrm{ss}} = \mathbb{P}(V)^{\mathrm{ss}} = \{[x_1, \ldots, x_k] \in \mathbb{P}(V) \mid \operatorname{hull}(\{\mu_i | x_i \neq 0\}) \ni \theta\}$. A point x is polystable iff θ lies in the interior of the hull above, and stable if in addition the hull is of maximal dimension.*

Proof. The Hilbert-Mumford weights are computed as follows. For any non-zero $\lambda \in \mathfrak{g}_{\mathbb{Z}}$, let

$$\nu(x,\lambda) := \min_i \{-\mu_i(\lambda), x_i \neq 0\}.$$

Then

$$\begin{aligned} z^\lambda[x_1,\ldots,x_k] &= [z^{\mu_1(\lambda)}x_1,\ldots,z^{\mu_k(\lambda)}x_k] \\ &= [z^{\mu_1(\lambda)+\nu(x,\lambda)}x_1,\ldots,z^{\mu_k(\lambda)+\nu(x,\lambda)}x_k] \end{aligned}$$

and

$$(-\mu_i(\lambda) \neq \nu(x,\lambda)) \implies \left(\lim_{z\to 0} z^{\mu_i(\lambda)+\nu(x,\lambda)} = 0\right).$$

Let

$$x_\lambda := \lim_{z\to 0} z^\lambda x = \lim_{z\to 0}[z^{\mu_i(\lambda)}x_i]_{i=1}^k \in X$$

Then

$$x_\lambda = [x_{\lambda,1},\ldots,x_{\lambda,k}], \quad x_{\lambda,i} = \begin{cases} x_i & -\mu_i(\lambda) = \nu(x,\lambda) \\ 0 & \text{otherwise} \end{cases}.$$

The Hilbert-Mumford weight is therefore

$$\mu(x,\lambda) = \nu(x,\lambda) + (\theta,\lambda). \tag{11}$$

By the Hilbert-Mumford criterion, the point x is semistable if and only if

$$\nu(x,\lambda) := \min\{-\mu_i(\lambda) \mid x_i \neq 0\} \leq (-\theta,\lambda), \quad \forall \lambda \in \mathfrak{g}_{\mathbb{Z}} - \{0\}.$$

That is,

$$(x \in X^{\mathrm{ss}}) \iff (\theta \in \mathrm{hull}\{\mu_i \mid x_i \neq 0\}).$$

This proves the claim about the semistable locus. To prove the claim about polystability, note that $\mu(x,\lambda) = 0 = \mu(x,-\lambda)$ implies that the minimum $\nu(x,\lambda)$ is also the maximum. Thus the only affine linear functions $\xi : \mathfrak{g}^\vee \to \mathbb{R}$ which vanish at θ are those ξ that are constant on the hull of μ_i with x_i nonzero. This implies that the span of μ_i with x_i non-zero contains θ in its relative interior. The stabilizer G_x of x has Lie algebra $\mathfrak{g}_x$ the annihilator of the span of the hull of the μ_i with $x_i \neq 0$. So the stabilizer G_x is finite if and only if the span of μ_i with $x_i \neq 0$ is of maximal dimension $\dim(G)$. This implies the claim on stability. □

We introduce the following notation. As above G is a connected complex reductive group with maximal torus T and $\mathfrak{g},\mathfrak{t}$ are the Lie algebras of G,T respectively. Fix an invariant inner product $(\ ,\ ) : \mathfrak{g}\times\mathfrak{g} \to \mathbb{C}$ on $\mathfrak{g}$ inducing an identification $\mathfrak{g} \to \mathfrak{g}^\vee$. By taking a multiple of the basic inner product on each factor we may assume that the inner product induces an identification $\mathfrak{t}_{\mathbb{Q}} \to \mathfrak{t}_{\mathbb{Q}}^\vee$. Denote by

$$\|\cdot\| : \mathfrak{q}_{\mathbb{Q}} \to \mathbb{R}_{\geq 0}, \quad \|\xi\| := (\xi,\xi)^{1/2}$$

the norm with respect to the induced metric.

Next recall the theory of Levi decompositions of parabolic subgroups from Borel [10, Section 11]. A parabolic subgroup Q of G is one for which G/Q is complete, or equivalently, containing a maximal solvable subgroup $B \subset$

G. Any parabolic Q admits a Levi decomposition $Q = L(Q)U(Q)$ where $L(Q)$ denote a maximal reductive subgroup of Q and $U(Q)$ is a maximal unipotent subgroup. Let $\mathfrak{l}(Q), \mathfrak{u}(Q)$ denote the Lie algebras of $L(Q), U(Q)$. Let $\mathfrak{g} = \mathfrak{t} \oplus \bigoplus_{\alpha \in R(G)} \mathfrak{g}_\alpha$ denote the root space decomposition of $\mathfrak{g}$, where $R(G)$ is the set of roots. The Lie algebras $\mathfrak{l}(Q), \mathfrak{u}(Q)$ decompose into root spaces as

$$\mathfrak{q} = \mathfrak{t} \oplus \bigoplus_{\alpha \in R(Q)} \mathfrak{g}_\alpha, \quad \mathfrak{l}(Q) = \mathfrak{t} \oplus \bigoplus_{\alpha \in R(Q) \cap -R(Q)} \mathfrak{g}_\alpha, \quad \mathfrak{u}(Q) = \mathfrak{q}/\mathfrak{l}(Q)$$

where $R(Q) \subset R(G)$ is the set of roots for $\mathfrak{l}(Q)$. Let $\mathfrak{z}(Q)$ denote the center of $\mathfrak{l}(Q)$ and

$$\mathfrak{z}_+(Q) = \{\xi \in \mathfrak{z}(Q) \mid \alpha(\xi) \geq 0, \ \forall \alpha \in R(Q)\}$$

the *positive chamber* on which the roots of Q are non-negative. The Levi decomposition induces a homomorphism

$$\pi_Q : Q \to Q/U(Q) \cong L(Q). \tag{12}$$

This homomorphism has the following alternative description as a limit. Let $\lambda \in \mathfrak{z}_+(Q) \cap \mathfrak{g}_\mathbb{Q}$ be a positive coweight and

$$\phi_\lambda : \mathbb{C}^\times \to L(Q), \quad z \mapsto z^\lambda$$

the corresponding central one-parameter subgroup. Then

$$\pi_Q(g) = \lim_{z \to 0} \mathrm{Ad}(z^\lambda) g.$$

In the case of the general linear group in which the parabolic consists of block-upper-triangular matrices, this limit projects out the off-block-diagonal terms.

The unstable locus admits a stratification by maximally destabilizing subgroups, as in Hesselink [27], Kirwan [29], and Ness [41]. The stratification reads

$$X = \bigcup_{\lambda \in \mathcal{C}(X)} X_\lambda, \quad X_\lambda = G \times_{Q_\lambda} Y_\lambda, \quad Y_\lambda \mapsto Z_\lambda \text{ affine fibers} \tag{13}$$

where $Y_\lambda, Z_\lambda, Q_\lambda, \mathcal{C}(X)$ are defined as follows. For each fixed point component $\overline{Z}_\lambda$ of z^λ there exist a weight $\mu(\lambda)$ so z^λ acts on $\tilde{X}|Z_\lambda$ with weight $\mu(\lambda)$:

$$z^\lambda \tilde{x} = z^{\mu(\lambda)} \tilde{x}, \quad \forall \tilde{x} \in \tilde{X}|Z_\lambda.$$

The group $G_\lambda/\mathbb{C}^\times_\lambda$ acts on $\overline{Z}_\lambda$ and we denote by $Z_\lambda \subset \overline{Z}_\lambda$ the semistable locus. Define

$$\mathcal{C}(X) = \{\lambda \in \mathfrak{t}_+ \mid \exists Z_\lambda, \ \mu(\lambda) = (\lambda, \lambda)\} \tag{14}$$

using the metric, where $\mathfrak{t}_+$ is the closed positive Weyl chamber. The variety Y_λ is the set of points that flow to Z_λ under $z^\lambda, z \to 0$:

$$Y_\lambda = \left\{ x \in X \mid \lim_{z \to 0} z^\lambda x \in Z_\lambda \right\}$$

The group Q_λ is the parabolic of group elements that have a limit under $\mathrm{Ad}(z^\lambda)$ as $z \to 0$:

$$Q_\lambda = \left\{ g \in G \mid \exists \lim_{z \to 0} \mathrm{Ad}(z^\lambda) g \in G \right\}.$$

Then Y_λ is a Q_λ-variety; and X_λ is the flow-out of Y_λ under G. By taking quotients we obtain a stratification of the quotient stack by locally-closed substacks

$$X/G = \bigcup_{\lambda \in \mathcal{C}(X)} X_\lambda / G.$$

This stratification was used in Teleman [49] to give a formula for the sheaf cohomology of bundles on the quotient stack.

4. Kontsevich stability

In this section we recall the definition of Kontsevich's moduli stacks of stable maps [30] as generalized to orbifold quotients by Chen-Ruan [11] and in the algebraic setting by Abramovich-Graber-Vistoli [2]. Let X be a smooth projective variety. Recall that a *prestable map* with target X consists of a prestable curve $C \to S$, a morphism $u : C \to X$, and a collection $z_1, \dots, z_n : S \to C$ of distinct non-singular points called *markings*. An automorphism of a prestable map $(C, u, \underline{z})$ is an automorphism

$$\varphi : C \to C, \quad \varphi \circ u = u, \quad \varphi(z_i) = z_i, \quad i = 1, \dots, n.$$

A prestable map $(C, u, \underline{z})$ is *stable* if the number $\# \mathrm{Aut}(C, u, \underline{z})$ of automorphisms is finite. For $d \in H_2(X, \mathbb{Z})$ we denote by $\overline{\mathcal{M}}_{g,n}(X, d)$ the moduli stack of stable maps $(C, u, \underline{z})$ of genus $g = \mathrm{genus}(C)$ and class $d = v_*[C]$ with n markings.

The notion of stable map generalizes to orbifolds [11], [2] as follows. These definitions are needed for the construction of the moduli stack of affine gauged maps in the case that the git quotient is an orbifold, but not if the quotient is free. First we recall the notion of twisted curve:

Definition 4.1. (Twisted curves) *Let S be a scheme. An n-*marked twisted curve* over S is a collection of data $(f : \mathcal{C} \to S, \{\ddagger_i \subset \mathcal{C}\}_{i=1}^n)$ such that*

(a) (Coarse moduli space) *$\mathcal{C}$ is a proper stack over S whose geometric fibers are connected of dimension 1, and such that the coarse moduli space of $\mathcal{C}$ is a nodal curve C over S.*
(b) (Markings) *The $\ddagger_i \subset \mathcal{C}$ are closed substacks that are gerbes over S, and whose images in C are contained in the smooth locus of the morphism $C \to S$.*
(c) (Automorphisms only at markings and nodes) *If $\mathcal{C}^{ns} \subset \mathcal{C}$ denotes the* non-special locus *given as the complement of the $\ddagger_i$ and the singular locus of $\mathcal{C} \to S$, then $\mathcal{C}^{ns} \to C$ is an open immersion.*
(d) (Local form at smooth points) *If $p \to C$ is a geometric point mapping to a smooth point of C, then there exists an integer r, equal to 1 unless p is in the image of some $\ddagger_i$, an étale neighborhood $\mathrm{Spec}(R) \to C$ of*

p and an étale morphism $\operatorname{Spec}(R) \to \operatorname{Spec}_S(\mathcal{O}_S[x])$ *such that the pull-back* $\mathcal{C} \times_C \operatorname{Spec}(R)$ *is isomorphic to* $\operatorname{Spec}(R[z]/z^r = x)/\mu_r$.

(e) (Local form at nodal points) *If $p \to C$ is a geometric point mapping to a node of C, then there exists an integer r, an étale neighborhood* $\operatorname{Spec}(R) \to C$ *of p and an étale morphism* $\operatorname{Spec}(R) \to \operatorname{Spec}_S(\mathcal{O}_S[x,y]/(xy-t))$ *for some $t \in \mathcal{O}_S$ such that the pull-back* $\mathcal{C} \times_C \operatorname{Spec}(R)$ *is isomorphic to* $\operatorname{Spec}(R[z,w]/zw-t', z^r - x, w^r - y)/\mu_r$ *for some $t' \in \mathcal{O}_S$.*

Next we recall the notion of twisted stable maps. Let $\mathcal{X}$ be a proper Deligne-Mumford stack with projective coarse moduli space X. Algebraic definitions of twisted curve and twisted stable map to a $\mathcal{X}$ are given in Abramovich-Graber-Vistoli [2], Abramovich-Olsson-Vistoli [3], and Olsson [42].

Definition 4.2. *A* twisted stable map *from an n-marked twisted curve* $(\pi : \mathcal{C} \to S, (\ddagger_i \subset \mathcal{C})_{i=1}^n)$ *over S to $\mathcal{X}$ is a representable morphism of S-stacks $u : \mathcal{C} \to \mathcal{X}$ such that the induced morphism on coarse moduli spaces $u_c : C \to X$ is a stable map in the sense of Kontsevich from the n-pointed curve $(C, \underline{z} = (z_1, \ldots, z_n))$ to X, where z_i is the image of $\ddagger_i$. The* homology class *of a twisted stable curve is the homology class $u_*[\mathcal{C}_s] \in H_2(X, \mathbb{Q})$ of any fiber $\mathcal{C}_s$.*

Twisted stable maps naturally form a 2-category. Every 2-morphism is unique and invertible if it exists, and so this 2-category is naturally equivalent to a 1-category which forms a stack over schemes [2].

Theorem 4.3. *([2, 4.2]) The stack $\overline{\mathcal{M}}_{g,n}(\mathcal{X})$ of twisted stable maps from n-pointed genus g curves into $\mathcal{X}$ is a Deligne-Mumford stack. If $\mathcal{X}$ is proper, then for any $c > 0$ the union of substacks $\overline{\mathcal{M}}_{g,n}(\mathcal{X}, d)$ with homology class $d \in H_2(\mathcal{X}, \mathbb{Q})$ satisfying $(d, c_1(\tilde{X})) < c$ is proper.*

The Gromov-Witten invariants takes values in the cohomology of the *inertia stack*

$$\mathcal{I}_{\mathcal{X}} := \mathcal{X} \times_{\mathcal{X} \times \mathcal{X}} \mathcal{X}$$

where both maps are the diagonal. The objects of $\mathcal{I}_{\mathcal{X}}$ may be identified with pairs (x, g) where $x \in \mathcal{X}$ and $g \in \operatorname{Aut}_{\mathcal{X}}(x)$. For example, if $\mathcal{X} = X/G$ is a global quotient by a finite group then

$$\mathcal{I}_{\mathcal{X}} = \bigcup_{[g] \in G/\operatorname{Ad}(G)} X^g / Z_g$$

where $G/\operatorname{Ad}(G)$ denotes the set of conjugacy classes in X and Z_g is the centralizer of g. Let $\mu_r = \mathbb{Z}/r\mathbb{Z}$ denote the group of r-th roots of unity. The inertia stack may also be written as a hom stack [2, Section 3]

$$\mathcal{I}_{\mathcal{X}} = \cup_{r > 0} \mathcal{I}_{\mathcal{X},r}, \quad \mathcal{I}_{\mathcal{X},r} := \operatorname{Hom}^{\mathrm{rep}}(B\mu_r, \mathcal{X}).$$

The classifying stack $B\mu_r$ is a Deligne-Mumford stack and if $\mathcal{X}$ is a Deligne-Mumford stack then

$$\overline{\mathcal{I}}_{\mathcal{X}} := \cup_{r>0} \overline{\mathcal{I}}_{\mathcal{X},r}, \quad \overline{\mathcal{I}}_{\mathcal{X},r} := \mathcal{I}_{\mathcal{X}/r}/B\mu_r.$$

is the *rigidified inertia stack* of representable morphisms from $B\mu_r$ to $\mathcal{X}$, see [2, Section 3]. There is a canonical quotient cover $\pi : \mathcal{I}_{\mathcal{X}} \to \overline{\mathcal{I}}_{\mathcal{X}}$ which is r-fold over $\overline{\mathcal{I}}_{\mathcal{X},r}$. Pullback acts on cohomology by an isomorphism

$$\pi^* H^*(\overline{\mathcal{I}}_{\mathcal{X}}, \mathbb{Q}) \to H^*(\mathcal{I}_{\mathcal{X}}, \mathbb{Q}).$$

For the purposes of defining orbifold Gromov-Witten invariants, $\overline{\mathcal{I}}_{\mathcal{X}}$ can be replaced by $\mathcal{I}_{\mathcal{X}}$ at the cost of additional factors of r on the r-twisted sectors. If $\mathcal{X} = X/G$ is a global quotient of a scheme X by a finite group G then

$$\overline{\mathcal{I}}_{X/G} = \coprod_{(g)} X^g/(Z_g/\langle g\rangle)$$

where $\langle g\rangle \subset Z_g$ is the cyclic subgroup generated by g. For example, suppose that X is a polarized linearized projective G-variety such that $X/\!\!/G$ is locally free. Then

$$\mathcal{I}_{X/\!\!/G} = \coprod_{(g)} X^{\mathrm{ss},g}/Z_g$$

where $X^{\mathrm{ss},g}$ is the fixed point set of $g \in G$ on X^{ss}, Z_g is its centralizer, and the union is over all conjugacy classes,

$$\overline{\mathcal{I}}_{X/\!\!/G} = \coprod_{(g)} X^{\mathrm{ss},g}/(Z_g/\langle g\rangle)$$

where $\langle g\rangle$ is the (finite) group generated by g. The moduli stack of twisted stable maps admits evaluation maps to the rigidified inertia stack

$$\mathrm{ev} : \overline{\mathcal{M}}_{g,n}(\mathcal{X}) \to \overline{\mathcal{I}}_{\mathcal{X}}^n, \quad \overline{\mathrm{ev}} : \overline{\mathcal{M}}_{g,n}(\mathcal{X}) \to \overline{\mathcal{I}}_{\mathcal{X}}^n,$$

where the second is obtained by composing with the involution $\overline{\mathcal{I}}_{\mathcal{X}} \to \overline{\mathcal{I}}_{\mathcal{X}}$ induced by the map $\mu_r \to \mu_r, \zeta \mapsto \zeta^{-1}$.

Constructions of Behrend-Fantechi [6] provide the stack of stable maps with virtual fundamental classes. The virtual fundamental classes

$$[\overline{\mathcal{M}}_{g,n,\Gamma}(\mathcal{X}, d)] \in H(\overline{\mathcal{M}}_{g,n}(\mathcal{X}), \mathbb{Q})$$

(where the right-hand-side denotes the singular homology of the coarse moduli space) satisfy the splitting axioms for morphisms of modular graphs similar to those in the case that X is a variety. Orbifold Gromov-Witten invariants are defined by virtual integration of pull-back classes using the evaluation maps above. The orbifold Gromov-Witten invariants satisfy properties similar to those for usual Gromov-Witten invariants, after replacing rescaling the inner product on the cohomology of the inertia stack by the order of the stabilizer. The definition of orbifold Gromov-Witten invariants leads to the definition of orbifold quantum cohomology as follows.

Definition 4.4. (Orbifold quantum cohomology) *To each component* $\mathcal{X}_k$ *of* $\mathcal{I}_{\mathcal{X}}$ *is assigned a rational number* $\operatorname{age}(\mathcal{X}_k)$ *as follows. Let* (x, g) *be an object in* $\mathcal{X}_k$*. The element* g *acts on* $T_x\mathcal{X}$ *with eigenvalues* $\alpha_1, \ldots, \alpha_n$ *with* $n = \dim(\mathcal{X})$*. Let* r *be the order of* g *and define* $s_j \in \{0, \ldots, r-1\}$ *by* $\alpha_j = \exp(2\pi i s_j/r)$*. The* age *is defined by* $\operatorname{age}(\mathcal{X}_k) = (1/r)\sum_{j=1}^n s_j$*. Let* $\Lambda_{\mathcal{X}} \subset \operatorname{Hom}(H_2(\mathcal{X}, \mathbb{Q}), \mathbb{Q})$ *denote the Novikov field of linear combinations of formal symbols* $q^d, d \in H_2(\mathcal{X}, \mathbb{Q})$ *where for each* $c > 0$*, only finitely many* q^d *with* $(d, c_1(\tilde{X})) < c$ *have non-zero coefficient. Denote the quantum cohomology*

$$QH(\mathcal{X}) = \bigoplus QH^\bullet(\mathcal{X}), \quad QH^\bullet(\mathcal{X}) = \bigoplus_{\mathcal{X}_k \subset \mathcal{I}_{\mathcal{X}}} H^{\bullet + 2\operatorname{age}(\mathcal{X}_k)}(\mathcal{X}_k) \otimes \Lambda_{\mathcal{X}}.$$

The genus zero Gromov-Witten invariants define on $QH(\mathcal{X})$ the structure of a Frobenius manifold [11], [2].

5. Mundet stability

In this section we explain the Ramanathan condition for semistability of principal bundles [45] and its generalization to maps to quotients stacks by Mundet [40], and the quot-scheme and stable-map compactification of the moduli stacks.

5.1. **Ramanathan stability.** Morphisms from a curve to a quotient of a point by a reductive group are by definition principal bundles over the curve. Bundles have a natural semistability condition introduced half a century ago by Mumford, Narasimhan-Seshadri, Ramanathan and others in terms of *parabolic reductions* [45]. First we explain stability for vector bundles. A vector bundle $E \to C$ of degree zero over a smooth projective curve C is semistable if there are no sub-bundles of positive degree:

$$(E \text{ semistable}) \quad \text{iff} \quad (\deg(F) \leq 0, \quad \forall F \subset E \text{ sub-bundles}).$$

A generalization of the notion of semistability to principal bundles is given by Ramanathan [45] in terms of *parabolic reductions*. A parabolic reduction of P consists of a pair

$$Q \subset G, \quad \sigma : C \to P/Q$$

of a parabolic subgroup of G, that is and a section $\sigma : C \to P/Q$. Denote by $\sigma^* P \subset P$ the pull-back of the Q-bundle $P \to P/Q$, that is, the reduction of structure group of P to Q corresponding to σ. Associated to the homomorphism π_Q of (12) is an *associated graded* bundle $\operatorname{Gr}(P) := \sigma^* P \times_Q L(Q) \to C$ with structure group $L(Q)$. In the case that P is the frame bundle of a vector bundle $E \to C$ of rank r, that is,

$$P = \cup_z P_z, \quad P_z = \{(e_1, \ldots, e_r) \in E_z^r \mid e_1 \wedge \ldots \wedge e_r \neq 0\}$$

a parabolic reduction of P is equivalent to a flag of sub-vector-bundles of E

$$\{0\} \subset E_{i_1} \subset E_{i_2} \subset \ldots \subset E_{i_l} \subset E.$$

Explicitly the parabolic reduction $\sigma^* P$ given by frames adapted to the flag:

$$\sigma(z) = \{(e_1, \ldots, e_r) \in E_z^r \mid e_j \in E_{i_k,z}, \ \forall j \le i_k, k = 1, \ldots, l\}.$$

Conversely, given a parabolic reduction the associated vector bundle has a canonical filtration.

An analog of the degree of a sub-bundle for parabolic reductions is the degree of a line bundle defined as follows. Given $\lambda \in \mathfrak{g}_{\mathbb{Z}} - \{0\}$ we obtain from the identification $\mathfrak{g} \to \mathfrak{g}^\vee$ a rational weight $\lambda^\vee$. Denote the corresponding characters $\chi_\lambda : L(Q) \to \mathbb{C}^\times$ and $\chi_\lambda \circ \pi_Q : Q \to \mathbb{C}^\times$. Consider the associated line bundle over C defined by $P(\mathbb{C}_{\lambda^\vee}) := \sigma^* P \times_Q \mathbb{C}_{\lambda^\vee}$. The *Ramanathan weight* [45] is the degree of the line bundle $P(\mathbb{C}_{\lambda^\vee})$, that is,

$$\mu_{BG}(\sigma, \lambda) := ([C], (c_1(P(\mathbb{C}_{\lambda^\vee})) \in \mathbb{Z}.$$

The bundle $P \to C$ is *Ramanathan semistable* if

$$\mu_{BG}(\sigma, \lambda) \le 0, \quad \forall (\sigma, \lambda).$$

As in the case of vector bundle, it suffices to check semistability for all reductions to *maximal parabolic* subgroups. In fact, any dominant weight may be used in the definition of $\mu_{BG}(\sigma, \lambda)$, which shows that Ramanathan semistability is independent of the choice of invariant inner product on the Lie algebra and one obtains the definition given in Ramanathan [45].

5.2. **Mundet semistability.** The Mundet semistability condition generalizes Ramanathan's condition to morphisms from a curve to the quotient stack [40], [46]. Let

$$(p : P \to C, u : C \to P(X)) \in \mathrm{Obj}(\mathrm{Hom}(C, X/G))$$

be a gauged map. Let (σ, λ) consist of a parabolic reduction $\sigma : C \to P/Q$ and a positive coweight $\lambda \in \mathfrak{z}_+(Q)$. Consider the family of bundles $P^\lambda \to S := \mathbb{C}^\times$ obtained by conjugating by z^λ. That is, if P is given as a cocycle in nonabelian cohomology with respect to a covering $\{U_i \to X\}$

$$[P] = [\psi_{ij} : (U_i \cap U_j) \to G] \ \in \ H^1(C, G)$$

then the twisted bundle is given by

$$[P^\lambda] = [z^\lambda \psi_{ij} z^{-\lambda} : (U_i \cap U_j) \to G] \ \in \ H^1(C \times S, G).$$

Define a family of sections

$$u^\lambda : S \times C \to P^\lambda(X)$$

by multiplying u by $z^\lambda, z \in \mathbb{C}^\times$. This family has an extension over $s = \infty$ called the *associated graded* bundle and stable section

$$\mathrm{Gr}(P) \to C, \quad \mathrm{Gr}(u) : \hat{C} \to \mathrm{Gr}(P)(X) \tag{15}$$

whose bundle $\mathrm{Gr}(P)$ agrees with the definition of associated graded above. Note that the associated graded section $\mathrm{Gr}(u)$ exists by compactness of the moduli space of stable maps to $\mathrm{Gr}(P)(X)$. The composition of $\mathrm{Gr}(u)$ with projection $\mathrm{Gr}(P)(X) \to C$ is a map of degree one; hence there is a unique component $\hat{C}_0$ of $\hat{C}$ that maps isomorphically onto C. The construction above is $\mathbb{C}^\times$-equivariant and in particular over the central fiber $z = 0$ the group element z^λ acts by an automorphism of $\mathrm{Gr}(P)$ fixing $\mathrm{Gr}(u)$ up to automorphism of the domain.

For each pair of a parabolic reduction and one-parameter subgroup as above, the Mundet weight is a sum of *Ramanathan* and *Hilbert-Mumford* weights. To define the Mundet weight, consider the action of the automorphism z^λ on the associated graded $\mathrm{Gr}(P)$. The automorphism of X by z^λ is $L(Q)$-invariant and so defines an automorphism of the associated line bundle $\mathrm{Gr}(u)^* P(\tilde{X}) \to \mathrm{Gr}(C)$. The weight of the action of z^λ on the fiber of $\mathrm{Gr}(u)^* P(\tilde{X})$ over the root component $\hat{C}_0$ is the *Hilbert-Mumford weight*

$$\mu_X(\sigma, \lambda) \in \mathbb{Z}, \quad z^\lambda \tilde{x} = z^{\mu_X(\sigma,\lambda)} \tilde{x}, \quad \forall \tilde{x} \in (\mathrm{Gr}(u)|_{\hat{C}_0})^* \,\mathrm{Gr}(P) \times_G \tilde{X}.$$

Definition 5.1. (Mundet stability) *Let (P, u) be a gauged map from a smooth projective curve C to the quotient stack X/G. The* Mundet weight *of a parabolic reduction σ and dominant coweight λ is*

$$\mu(\sigma, \lambda) = \mu_{BG}(\sigma, \lambda) + \mu_X(\sigma, \lambda) \in \mathbb{Z}.$$

The gauged map (P, u) is Mundet semistable *resp.* stable *if and only if*

$$\mu(\sigma, \lambda) \leq 0, \ \textit{resp.}\ < 0, \quad \forall(\sigma, \lambda).$$

A pair (σ, λ) such that $\mu(\sigma, \lambda) \geq 0$ is a destabilizing pair. *A pair (P, u) is* polystable *iff*

$$\mu(\sigma, \lambda) = 0 \iff \mu(\sigma, -\lambda) = 0, \quad \forall(\sigma, \lambda). \tag{16}$$

That is, a pair (P, u) is polystable if for any destabilizing pair the opposite pair is also destabilizing.

More conceptually the semistability condition above is the Hilbert-Mumford stability condition adapted to one-parameter subgroups of the complexified gauge group, as explained in [40]. Semistability is independent of the choice of invariant inner product as follows for example from the presentation of the semistable locus in Schmitt [47, Section 2.3].

We introduce notation for various moduli stacks. Let $\mathcal{M}^G(C, X)$ denote the moduli space of Mundet semistable pairs; in general, $\mathcal{M}^G(C, X)$ is an Artin stack as follows from the git construction given in Schmitt [46, 47] or the more general construction of hom stacks in Lieblich [33, 2.3.4]. For any $d \in H_2^G(X, \mathbb{Z})$, denote by $\mathcal{M}^G(C, X, d)$ the moduli stack of pairs $v = (P, u)$ with

$$v_*[C] := (\phi \times_G \mathrm{id}_X)_* u_*[C] = d \in H_2^G(X, \mathbb{Z})$$

where $\phi : P \to EG$ is the classifying map.

5.3. **Compactification.** Schmitt [46] constructs a Grothendieck-style compactification of the moduli space of Mundet-semistable obtained as follows. Suppose X is projectively embedded in a projectivization of a representation V, that is $X \subset \mathbb{P}(V)$. Any section $u : C \to P(X)$ gives rise to a line subbundle $L := u^*(\mathcal{O}_{\mathbb{P}(V)}(-1) \to \mathbb{P}(V))$ of the associated vector bundle $P \times_G V$. From the inclusion $\iota : L \to P(V)$ we obtain by dualizing a surjective map

$$j : P(V^\vee) := P \times_G V^\vee \to L^\vee.$$

A *bundle with generalized map* in the sense of Schmitt [47] is a pair (P, j) as above where j is allowed to have base points in the sense that

$$\zeta \in C \text{ basepoint} \iff ((\operatorname{rank}(j_\zeta) : P(V)^\vee_\zeta \to L^\vee_\zeta) = 0).$$

Schmitt [47] shows that the Mundet semistability condition extends naturally to the moduli stack of bundles with generalized map. Furthermore, the compactified moduli space $\overline{\mathcal{M}}^{\mathrm{quot},G}(C, X, d)$ is projective, in particular proper.

Schmitt's construction of the moduli space of bundles with generalized maps uses geometric invariant theory. After twisting by a sufficiently positive bundle we may assume that $P(V^\vee)$ is generated by global sections. A collection of sections $s_1, \ldots, s_l$ generating $P(V^\vee)$ is called an *l-level structure*. Equivalently, an l-level structure is a surjective morphism $q : \mathcal{O}_C^{\oplus l} \to P(V^\vee)$. Denote by $\mathcal{M}^{G,\mathrm{lev}}(C, \mathbb{P}(V))$ the stack of gauged maps to $\mathbb{P}(V)$ with level structure. The group $GL(l)$ acts on the stack of l-level structures, with quotient

$$\mathcal{M}^{G,\mathrm{lev}}(C, \mathbb{P}(V))/GL(l) = \mathcal{M}^G(C, \mathbb{P}(V)). \tag{17}$$

Denote by $\mathcal{M}^{G,\mathrm{lev}}(C, X) \subset \mathcal{M}^{G,\mathrm{lev}}(C, \mathbb{P}(V))$ the substack whose sections $u : C \to \mathbb{P}(V)$ have image in $P(X) \subset P(\mathbb{P}(V))$. Then by restriction we obtain a quotient presentation

$$\mathcal{M}^{G,\mathrm{lev}}(C, X)/GL(l) = \mathcal{M}^G(C, X).$$

Allowing the associated quotient $P \times_G V^\vee \to P \times_G L^\vee$ to develop base points gives a compactified moduli stack of gauged maps with level structure $\overline{\mathcal{M}}^{G,\mathrm{quot},\mathrm{lev}}(C, X)$. Schmitt [46, 47] shows that the stack $\overline{\mathcal{M}}^{G,\mathrm{quot},\mathrm{lev}}(C, X)$ has a canonical linearization and the git quotient $\overline{\mathcal{M}}^{G,\mathrm{quot},\mathrm{lev}}(C, X) /\!\!/ GL(l)$ defines a compactification $\overline{\mathcal{M}}^{G,\mathrm{quot}}(C, X)$ of $\mathcal{M}^G(C, X)$ independent of the choice of l as long as l is sufficiently large. A version of the quot-scheme compactification with markings is obtained by adding tuples of points to the data. That is,

$$\overline{\mathcal{M}}_n^{G,\mathrm{quot}}(C, X) := \overline{\mathcal{M}}^{G,\mathrm{quot}}(C, X) \times \overline{\mathcal{M}}_n(C)$$

where we recall that $\overline{\mathcal{M}}_n(C)$ is the moduli stack of stable maps $p : \hat{C} \to C$ of class $[C]$ with n markings and genus that of C. The orbit-equivalence relation in can be described more naturally in terms of *S-equivalence*: Given a family (P_S, u_S) of semistable gauged maps over a scheme S, such that the generic fiber is isomorphic to some fixed (P,u), then we declare (P,u) to be S-equivalent to (P_s,u_s) for any $s \in S$. Any equivalence class of semistable gauged maps has a unique representative that is polystable, by the git construction in Schmitt [46, Remark 2.3.5.18]. From the construction evaluation at the markings defines maps to the quotient stack

$$\overline{\mathcal{M}}_n^{G,\mathrm{quot}}(C,X,d) \to (V/\mathbb{C}^\times)^n, \quad ((p\circ z_i)^*L, j\circ p\circ z_i)$$

rather than to the git quotient $X^n \subset \mathbb{P}(V)^n$.[2]

Example 5.2. (Mundet semistable maps in the toric case) If G is a torus and $X = \mathbb{P}(V)$ then we can given an explicit description of Schmitt's quot-scheme compactification $\overline{\mathcal{M}}^{G,\mathrm{quot}}(C,X,d)$ of Mundet semistable maps [46].

Specifically let $X = \mathbb{P}(V)$ where V is a k-dimensional vector space and

$$V = \bigoplus_{i=1}^k V_i \tag{18}$$

is the decomposition of V into weight spaces V_i with weight $\mu_i \in \mathfrak{g}_{\mathbb{Z}}^\vee$.

A point of $\mathcal{M}^G(C,X,d)$ is a pair (P,u):

$$P \to C \qquad u\colon C \to P(X),$$

where P is a G-bundle and u is a section. We consider u as a morphism $\widetilde{u}\colon L \to P(V)$ with $L \to C$ a line bundle. Via the decomposition of V, we can write $\widetilde{u}$ as a k-tuple:

$$(\widetilde{u}_1,\dots,\widetilde{u}_k) \in \bigoplus_{i=1}^k H^0(P(V_i)\otimes L^\vee).$$

The compactification $\overline{\mathcal{M}}^{G,\mathrm{quot}}(C,X,d)$ is obtained by allowing the $\widetilde{u}_i$ to have simultaneous zeros:

$$\widetilde{u}_1^{-1}(0)\cap\cdots\cap\widetilde{u}_k^{-1}(0) \neq \emptyset$$

We make use of this example later one so we collect a few results about $\overline{\mathcal{M}}^{G,\mathrm{quot}}(C,X,d)$ below.

Recall (10) there is a projection $H_G^2(X) \to H^2(BG) = \mathfrak{g}_{\mathbb{Z}}^\vee$ and similarly we have $H_2^G(X) \to H_2(BG) = \mathfrak{g}_{\mathbb{Z}}$. Associated to $v = (P,u)$ is the discrete data:

$v_*[C] = d \in H_2^G(X,\mathbb{Z})$ and its image $d(P) \in H_2(BG)$
$c_1^G(\widetilde{X}) \in H_G^2(X)$ and its image $\theta \in H^2(BG)$
$d(u) := -c_1(L) \in H^2(C,\mathbb{Z}) \cong \mathbb{Z}$.

Note $d(P)$ is the degree of P; that is, $d(P) = c_1(P) \in H^2(C,\mathfrak{g}_{\mathbb{Z}}) \cong \mathfrak{g}_{\mathbb{Z}}$. We can now state the following.

[2] The Ciocan-Fontanine-Kim-Maulik [13] moduli space of *stable quotients* remedies this defect by imposing a stability condition at the marked points $z_1,\dots,z_n \in C$. The moduli stack then admits a morphism to $\overline{\mathcal{I}}_{X/\!\!/G}^n$ by evaluation at the markings.

Lemma 5.3. *Let G be a torus acting on a vector space V. Let $V = \bigoplus_{i=1}^k V_i$ be its decomposition into weight spaces with weights $\mu_1, \ldots, \mu_k$.*

(a) *The Mundet semistable locus consists of pairs (P, u) such that*

$$\mathrm{hull}(\{-d(P)^\vee + \mu_i | \tilde{u}_i \neq 0\}) \ni \theta. \tag{19}$$

(b) *let $W = \bigoplus_{i=1}^k H^0(P(V_i) \otimes L^\vee)$ and let W^{ss} consist of $(\widetilde{u}_1, \ldots, \widetilde{u}_k)$ such that (19) holds. Then $\overline{\mathcal{M}}^{G,\mathrm{quot}}(C, X, d) \cong W^{ss}/G$.*

(c) *If $\widetilde{u}_i \neq 0$ then $(\mu_i, d(P)^\vee) + d(u) \geq 0$. If moreover (19) holds then*

$$(\theta - d(P)^\vee, d(P)) + d(u) \geq 0.$$

(d) $(v_*[C], c_1(P(\widetilde{X})) = (\theta, d(P)) + d(u)$.

Proof. Since G is abelian, $\mathrm{Gr}(P) = P$ for any pair (λ, σ). It follows that for any $\lambda \in \mathfrak{g}_{\mathbb{Q}}$, the Mundet weight is

$$\mu(\sigma, \lambda) := \{\min_i (d(P)^\vee, \lambda) - \mu_i(\lambda) + \theta(\lambda), \tilde{u}_i \neq 0\}.$$

Hence the semistable locus is the space of pairs (P, u) where

$$\mathrm{hull}(\{-d(P)^\vee + \mu_i | \tilde{u}_i \neq 0\}) \ni \theta.$$

This proves (a). The description (b) follows immediately. For (c), if $\widetilde{u}_i \neq 0$ then $\deg \mathrm{div}(\widetilde{u}_i) \geq 0$. But we also have

$$(\mu_i, d(P)^\vee) + d(u) = c_1(P(V_i) \otimes L^\vee) = \deg \mathrm{div}(\widetilde{u}_i) \geq 0. \tag{20}$$

In particular $-d(P)^\vee + \theta \in \mathrm{hull}(\{\mu_i | \tilde{u}_i \neq 0\})$. Together with (20) this shows $(\theta + d(P)^\vee, d(P)) \geq 0$.

To prove (d) we use that the sections $\widetilde{u}_i$ above are homotopic to the zero section

$$\tilde{u}_0 : C \to P(V) \otimes L^\vee, \quad z \mapsto (z, 0)$$

and $\tilde{X}$ is induced from an equivariant line bundle on V with character θ at the fixed point at zero. Therefore we have

$$(v_*[C], c_1(P(\tilde{X}))) = (u_{0,*}[C], c_1(P(\mathbb{C}_\theta) \otimes L^\vee)) = (\theta, d(P)) + d(u). \tag{21}$$

□

For an explicit example, if $G = \mathbb{C}^\times$ and $V = \mathbb{C}^k$ then

$$\deg(P(V_i) \otimes L^\vee) = \deg(P(V_i)) - \deg(L) = d(P) + d(u), \quad i = 1, \ldots, k.$$

It follows that the moduli stack admits an isomorphism

$$\overline{\mathcal{M}}^{G,\mathrm{quot}}(C, X, d) \cong \mathbb{C}^{k(d(P)+d(u)+1),\times}/\mathbb{C}^\times \cong \mathbb{P}^{k(d(P)+d(u)+1)-1}.$$

This moduli stack is substantially simpler in topology than the moduli space of stable maps to CX/G, despite the dramatically more complicated stability condition. This ends the example.

A *Kontsevich-style compactification* of the stack of Mundet-semistable gauged maps which admits evaluation maps as well as a Behrend-Fantechi virtual fundamental class [24] is defined as follows. The objects in this compactification allow *stable sections*, that is, stable maps $u : \hat{C} \to P(X)$ whose composition with $P(X) \to C$ has class $[C]$. Thus objects of $\overline{\mathcal{M}}_n^G(C,X)$ are triples $(P, \hat{C}, u, \underline{z})$ consisting of a G-bundle $P \to C$, a projective nodal curve $(\hat{C}, \underline{z})$, and a stable map $u : \hat{C} \to P \times_G X$ whose class projects to $[C] \in H_2(C,\mathbb{Z})$. Morphisms are the obvious diagrams. To see that this category forms an Artin stack, note that the moduli stack of bundles $\mathrm{Hom}(C, BG)$ has a universal bundle

$$U \to C \times \mathrm{Hom}(C, BG).$$

Consider the associated X-bundle

$$U \times_G X \to C \times \mathrm{Hom}(C, BG).$$

The stack $\overline{\mathcal{M}}_n^G(C,X)$ is a substack of the stack of stable maps to $U \times_G X$, and is an Artin stack by e.g. Lieblich [33, 2.3.4], see [54] for more details. Note that hom-stacks are not in general algebraic [8].

Properness of the Kontsevich-style compactification follows from a combination of Schmitt's construction and the Givental map. A proper relative Givental map is described in Popa-Roth [44], and in this case gives a morphism

$$\overline{\mathcal{M}}^G(C,X,d) \to \overline{\mathcal{M}}^{G,\mathrm{quot}}(C,X,d). \tag{22}$$

For each fixed bundle this map collapses bubbles of the section u and replaces them with base points with multiplicity given by the degree of the bubble tree. Since the Givental morphism (22), the forgetful morphism $\overline{\mathcal{M}}_n^G(C,X,d) \to \overline{\mathcal{M}}^G(C,X,d)$ and the quot-scheme compactification $\overline{\mathcal{M}}_n^{G,\mathrm{quot}}(C,X,d)$ are all proper, so is $\overline{\mathcal{M}}_n^G(C,X,d)$.

5.4. **Energy positivity.** A natural notion of *energy* of a gauged map is defined as follows. For a gauged map $v = (P,u)$ the energy is given by the pairing with the equivariant first Chern class of the linearization

$$\mathcal{E}(v) := (d, c_1(P(\tilde{X}))) \in \mathbb{Z}, \quad d = v_*[C] \in H_2^G(X,\mathbb{Z}).$$

From Mundet's correspondence [40] it is immediate that the energy is non-negative, since in the symplectic definition the energy is defined as an integral of a non-negative function (the *energy density*) over the domain curve. Here we give an algebraic proof:

Lemma 5.4. *For any Mundet-semistable gauged map $v = (P,u)$ from a smooth projective genus zero curve C with class $d = u_*[C] \in H_2^G(X,\mathbb{Z})$, the pairing $\mathcal{E}(v) = (d, c_1(P(\tilde{X}))) \in \mathbb{Z}$ is non-negative. The energy $\mathcal{E}(v)$ vanishes only if the bundle P is trivializable and u constant in some trivialization of $P(X)$ induced by a trivialization of P.*

Proof. We give two proofs. By a special case of the Drinfeld-Simpson theorem [18], [13, Lemma 3.2.7], P admits a reduction to a Borel subgroup $B \subset G$. Let $\pi_B : B \to T$ be the projection (12), and $\mathrm{Gr}(P)$ the associated graded. Since the map π_Q is T-equivariant, the section u induces a section $\mathrm{Gr}(u) : \mathcal{C} \to \mathrm{Gr}(P)(X)$ that is also T-semistable. Therefore it suffices to consider the case $G = T$. Let $k = \dim(V)$ and $V = \bigoplus_{i=1}^k V_i$ the decomposition of V into weight spaces V_i with weight μ_i.

We use the notation introduced in example 5.2. In particular the first Chern class $c_1^G(\tilde{X})$ becomes identified, up to positive scalar multiple with a pair

$$c_1^G(\tilde{X}) \mapsto (1, \theta) \in \mathbb{Z} \oplus \mathfrak{g}_{\mathbb{Z}}^\vee.$$

The Mundet semistability criterion for one-parameter subgroups of $\mathrm{Aut}(P) \cong T$ has Mundet weights equal to

$$\mu_{BG}(\sigma, \lambda) = (d(P)^\vee, \lambda), \quad \mu_M(\sigma, \lambda) = (d(P)^\vee, \lambda) + \mu_X(\sigma, \lambda).$$

By lemma 5.3 (c) we have

$$(\theta - d(P)^\vee, d(P)) + d(u) \geq 0.$$

This implies

$$\begin{aligned}
(23)\quad (v_*[C], c_1(P(\tilde{X}))) &= (\theta, d(P)) + d(u) \\
(24)\quad &= (\theta - d(P)^\vee, d(P)) + d(u) + (d(P)^\vee, d(P)) \\
(25)\quad &\geq (\theta - d(P)^\vee, d(P)) + d(u) \geq 0
\end{aligned}$$

as claimed. If $(d(P), \mathrm{d}(P)^\vee)$ is zero then we must have $d(P) = 0$, hence P is trivializable. Hence

$$P(\tilde{X}) = C \times \tilde{X}, \quad (\pi \circ u)_*[C] = 0 \in H_2(X)$$

where $\pi : P(X) \cong C \times X \to X$ is the projection on the second factor. This implies that u is constant.

In the second proof we evaluate the Mundet weight for a carefully chosen one-parameter subgroup. As before assume $G = T = (\mathbb{C}^\times)^r$ and $X = \mathbb{P}(V)$. Consider $v \colon C \to X/G$ as a pair $(P \to C, P \xrightarrow{\alpha} \mathbb{P}(V))$ with α a T-equivariant map. The energy of v is the pairing

$$(v_*[C], c_1^G(\tilde{X})), \quad \text{where } v_*[C] \in \mathbb{Z} \oplus \mathfrak{t}_{\mathbb{Z}}, \quad c_1^G(\tilde{X}) \in \mathbb{Z} \oplus \mathfrak{t}_{\mathbb{Z}}^\vee.$$

The latter is up to positive scalar equal to $(1, \theta)$; the former is equal to $(\lambda, d(P))$ for an appropriate integer λ. The energy is equivalently

$$\mathcal{E}(v) = v^* c_1^G(\tilde{X}) \in H^2(C) = \mathbb{Z}.$$

Hence $\lambda = \deg(\alpha)$. The following is readily verified. If $d(P) = (d_1, \ldots, d_r)$ then P is the frame bundle of the vector bundle E defined by

$$E := \oplus_{i=1}^r \mathcal{O}_C(d_i).$$

The map α is given by global sections $u_0, \ldots, u_m \in H^0(\alpha^* \mathcal{O}_{\mathbb{P}(V)}(1))$ which are weight vectors for G. Consider the weight space decomposition $V = \oplus_i V_i$

where G acts on V_i with weight μ_i. Equivariance implies that $-\mu_i$ is the weight of u_i. We claim $\deg(\alpha) \geq (-\mu_i, d(P))$. To see this let

$$|E| = \underline{\mathrm{Spec}}(\mathrm{Sym}^*(E^\vee))$$

be the total space of E. Via the clutching construction T is given by gluing of trivializations in coordinate charts near $[1,0],[0,1] \in \mathbb{P}^1$,

$$|E| = \mathrm{Spec}\,\mathbb{C}[z, x_1, \ldots, x_r] \cup \mathrm{Spec}\,\mathbb{C}[z^{-1}, y_1, \ldots, y_r]$$

with $y_i = z^{-d_i} x_i$. The space of global sections $H^0(\alpha^* \mathcal{O}_{\mathbb{P}(V)}(1))$ has a basis of pairs $(z^j \prod_i x_i^{n_i}, z^{-k} \prod_i y_i^{n_i})$ that transform as follows

$$z^j \prod_i x_i^{n_i} = z^j \prod_i z^{n_i d_i} y_i^{n_i} = z^{j+(-\mu_i, d(P)) - \deg(\alpha))} \prod_i y_i^{n_i}$$

That is $-k = j + (-\mu_i, d(P)) - \deg(\alpha) \leq 0$. As $j \geq 0$ we conclude $(-\mu_i, d(P)) \leq \deg(\alpha)$. Mundet stability for the one-parameter subgroup generated by λ is defined by a limiting equivariant map $\alpha^0 \colon P \to \mathbb{P}(V)$ given by sections u_i^0 whose image is fixed by z^λ. The stability condition is

$$\min_{i, u_i^0 \neq 0} (d(P)^\vee, \lambda) + (\theta - \mu_i, \lambda) \leq 0.$$

Substitute in $\lambda = -d(P)^\vee$ and multiply by -1 to obtain

$$(d(P), d(P)^\vee) + (\theta, d(P)^\vee) + (-\mu_i, d(P)^\vee) \geq 0.$$

Therefore

$$\mathcal{E}(v) = (\theta, d(P)^\vee) + \deg(\alpha) \geq (\theta, d(P)^\vee) + (-\mu_i, d(P)^\vee) \geq 0.$$

For equality to hold we need $d(P) = 0$ and $\deg(\alpha) = 0$. The first condition together with equivariance says α factors through C; the second condition says α is constant. □

Remark 5.5. The following gives an alternative proof of non-negativity of the energy in the case that any Mundet semistable map (P, u) has non-empty semistable locus $u^{-1}(P(X^{\mathrm{ss}}))$, see Corollary 6.2 below. In this case an invariant ample divisor $D \subset X$ is given by choosing an invariant section of the ample bundle $\tilde{X}^k$ for k large, as in (9). Let

$$D /\!\!/ G := (X^{\mathrm{ss}} \cap D)/G \subset X /\!\!/ G$$

denote the associated divisor in the git quotient. We may assume that the divisor $D /\!\!/ G \subset X /\!\!/ G \subset X/G$ does not contain $v(C)$, since $D /\!\!/ G$ is ample. Since the divisor D is G-invariant and ample, D contains the unstable locus, that is, $D \supset (X - X^{\mathrm{ss}})$. The divisor D then induces a divisor

$$P(D) = P \times_G D \subset P(X).$$

Let $u^{X /\!\!/ G} : C \to X /\!\!/ G$ denote the induced map to the symplectic quotient. Since $u(C)$ is not contained in $P(D)$, the pairing is the number of intersection points counted with multiplicity:

$$(v_*[C], c_1(P \times_G \tilde{X})) = \# u^{-1}(P(D)).$$

If the pairing is zero then the image of u is contained in the semistable locus, and u induces a constant map to $X /\!\!/ G$. Hence the bundle and section are trivializable.

5.5. **Convex targets.** The definition of Mundet semistability also gives good moduli spaces in the cases of some affine targets. A finite dimensional complex G-vector space V is said to be *convex* if there exists a central one-parameter subgroup $\phi_\xi : \mathbb{C}^\times \to G$ such that V has positive weights for the induced action of ϕ_ξ,

$$V = \bigoplus_\mu V_i, \quad \mu_i(\xi) > 0, \quad i = 1, \dots, k.$$

Given a convex G-vector space, the *projectivization* of V is the quotient

$$\overline{V} = ((V \times \mathbb{C})^\times - \{(0,0)\})/\mathbb{C}_\xi^\times$$

where $\mathbb{C}^\times$ acts on $\mathbb{C}$ with weight one. Thus $\overline{V}$ is a weighted projective space (in a particular a smooth Deligne-Mumford stack) and contains V as an open substack. A quasiprojective G-variety X is *convex* if there exists a projective embedding $\pi : X \to V$ to a convex G-vector space V whose image intersects the locus $V - \{0\}$. The following is a simple application of the technique called *symplectic cutting* in the literature [31]:

Lemma 5.6. *Any convex G-variety X embeds as a dense open substack of a Deligne-Mumford stack $\overline{X}$ with complement a prime $\mathbb{C}_\xi^\times$-fixed divisor isomorphic to $(X - \{0\})/\mathbb{C}_\xi^\times$.*

Proof. Let $\tilde{X} \to X$ denote the given linearization on X and $\tilde{X}(l)$ the linearization on $X \times \mathbb{C}$ obtained by twisting by the $\mathbb{C}^\times$-character with weight l. Consider the git quotient

$$\overline{X} = (X \times \mathbb{C}) /\!\!/ \mathbb{C}_\xi^\times .$$

The inverse image of $(0,0) \in V \times \mathbb{C}$ is unstable, for sufficiently large d. Thus the proper morphism $X \to V$ induces a proper morphism $\overline{X}$ to $\overline{V}$. In particular, this implies that $\overline{X}$ is also proper. The G action on $X \times \mathbb{C}$ given by $g(x,z) = (gx, z)$ descends to a G-action on $\overline{X}$, and restricts to the given action on the open substack $X \subset \overline{X}$. □

In the following we will refer to $\overline{\mathcal{M}}_n^G(C, \overline{X}, d)$ allowing $\overline{X}$ to be a smooth Deligne-Mumford stack without further comment; we do not allow stacky structures on the domain curves since we are only interested in defining $\overline{\mathcal{M}}_n^G(C, X, d)$ in which case the target X is a variety.

Corollary 5.7. *Let $d \in H_G^2(\overline{X})$ be a class that pairs trivially with the divisor class $[\overline{X} - X] \in H_2^G(\overline{X})$. Then there exists a constant $l(E)$ such that if the energy bound $\mathcal{E}(d) < E$ holds and $l \geq l(E)$ then the moduli stack $\overline{\mathcal{M}}_n^G(C, \overline{X}, d)$ consists of maps whose images are disjoint from $(\overline{X} - X)/G$.*

Proof. The intersection number of any curve $u : \mathbb{P}^1 \to \overline{V}$ contained in $\overline{V} - V$ with $\overline{V} - V$ is non-negative. Indeed $\overline{V} - V \cong \mathbb{P}[\mu_1, \ldots, \mu_k]$ has ample normal bundle $\mathcal{O}_{\mathbb{P}[\mu_1,\ldots,\mu_k]}(1)$. On the other hand, there are no stable gauged maps $C \to X/G$ with image in $(\overline{V} - V)/G$ for sufficiently large $l > l(d)$. The trivial reduction σ together with the generator ξ of the one-parameter subgroup $\mathbb{C}^\times$ has weight $\mu(\sigma, \xi) \to \infty$ as $d \to \infty$, while (23) implies that $(d(P)^\vee, \lambda)$ is bounded in terms of the energy. Combining these observations let $v : \hat{C} \to \overline{V}/G$ be a stable gauged map intersecting $(\overline{V} - V)/G$. The intersection number $\#u^{-1}(P(\overline{V} - V)) > 0$ is positive and equal to the pairing $(d, [\overline{V} - V]) \in \mathbb{Q}$ of $d \in H_2^G(X, \mathbb{Q})$ with $[\overline{V} - V] \in H^2_G(\overline{V}, \mathbb{Q})$. The latter vanishes by assumption, a contradiction. □

Lemma 5.8. *Suppose that (P, u) is a map from C to X/G with X convex. Any destabilizing pair (σ, λ) has associated graded $(\mathrm{Gr}(P), \mathrm{Gr}(u))$ disjoint from the divisor at infinity $P(\overline{X} - X)$, for l sufficiently large.*

Proof. As in the proof of Lemma 5.4, by choosing a Borel structure refining the parabolic structure and passing to the associated graded we may assume that G is a torus. Suppose that (σ, λ) has associated graded $\mathrm{Gr}(P), \mathrm{Gr}(u)$ intersecting $P(\overline{X} - X)$. By invariance of intersection number, the limit $\mathrm{Gr}(u)$ must take values in $P(\overline{X} - X)$, since otherwise the intersection number with $P(\overline{X} - X)$ would be positive. We suppose that $\mu_1, \ldots, \mu_k$ are the weights of G on V, so that u has components $u_1, \ldots, u_k$. From the description of the associated graded, if λ satisfies $\mu_i(\lambda) \geq 0$ for all $i = 1, \ldots, k$ such that u_i is non-zero, then the associated graded takes values in $P(X)$. Hence $\mu_i(\lambda) > 0$ for non-empty subset $I \subset \{1, \ldots, k\}$ of indices such that u_i is non-zero. Let

$$m = \min_{i \in I} \frac{\mu_i(\lambda)}{\mu_i(\xi)}.$$

The minimum m is negative since some $\mu_i(\lambda) < 0$ and $\mu_i(\xi) > 0$ for all $i = 1, \ldots, k$. The associated graded section is then given by the collection of sections $u_i, i \in I$ with $\mu_i(\lambda)/\mu_i(\xi) = m$. The corresponding Hilbert-Mumford weight is the weight of the action of $\mathbb{C}^\times_\lambda$ on the fibers of $\tilde{X}/\mathbb{C}^\times_\xi$, and is equal to ml. Therefore, the weight $\mu(x, \lambda)$ is positive. For l sufficiently large the pair is not destabilizing. □

As a result, for convex target it suffices to check semistability for pairs such that the associated graded exists without compactification.

6. Variation of polarization

The moduli space of Mundet-semistable gauged maps depends on the linearization. Changing the linearization leads to wall-crossing in which loci of bundles with the same associated graded are flipped [24] as is standard in variation of git as explained in e.g Thaddeus [50]. Consider the family of linearizations $\tilde{X}^k$ given by the k-th tensor product of the given one $\tilde{X}$ for k a positive integer. While taking tensor products does not change the definition of semistability for X, it does change the definition of Mundet semistability.

Lemma 6.1. *For any fixed degree $d \in H_2^G(X)$, there are at most finitely many changes in the stability condition as k varies. That is, there exist*

$$-\infty = k_0 < k_1 < \ldots < k_l = \infty \in \mathbb{Q} \cup \{-\infty, \infty\}$$

such that if $k', k'' \in (k_i, k_{i+1})$ then the semistable loci for k', k'' are equal.

Proof. Denote by $\mu_k(\sigma, \lambda)$ the Mundet weight corresponding to $\tilde{X}^k$. Changes in the definition of stability correspond to pairs (P, u) such that for some pair (σ, λ) and $k_-, k_+ \in \mathbb{Q}$,

$$\mu_{k_-}(\sigma, \lambda) < 0, \quad \mu_{k_+}(\sigma, \lambda) > 0$$

while for some $k \in (k_-, k_+)$, $\mu_k(\sigma, \lambda) = 0$ so that the pair (P, u) is semistable but not stable. As in 5.3(1), the wall-crossings arise from pairs (P, u) such that

$$\dim(\mathrm{hull}(\{\mu_i | \tilde{u}_i \neq 0\})) < \mathrm{rank}(G), \quad \mathrm{hull}(\{\mu_i | \tilde{u}_i \neq 0\}) \ni \theta + d(P)/k. \tag{26}$$

Suppose there are infinitely many wall-crossings. Let (P_k, u_k) denote the corresponding reducible gauged maps for some k in an unbounded set $\mathcal{W}(d) \subset \mathbb{Q}$. The equation (26) implies that $\|d(P_k)\| > ck$ for some positive constant c and all $k \in \mathcal{W}(d)$. On the other hand, as in (23)

$$\begin{aligned} (u_{k,*}[C], c_1(P_k(\tilde{X}))) &= (\theta, d(P_k)) + d(u_k) \\ &= (\theta - d(P_k)/k, d(P_k)) + d(u_k) + (d(P_k)/k, d(P_k)) \\ &\geq (d(P_k), d(P_k))/k \geq c^2 k \end{aligned}$$

for $k \in \mathcal{W}(d)$. Since $\mathcal{W}(d)$ is unbounded, this implies that the homology class $v_{k,*}[C] \in H_2^G(X)$ is also unbounded, a contradiction. $\square$

Denote by $\overline{\mathcal{M}}^G(C, X, d, k)$ the moduli space of Mundet semistable maps using the linearization $\tilde{X}^k$. By the finite-ness above in Lemma 6.1, we have the following:

Corollary 6.2. *For any $d \in H_2^G(X, \mathbb{Q})$ there exists $k(d)$ such that for $k \geq k(d)$, the stack $\overline{\mathcal{M}}^G(C, X, d, k)$ consists of those bundles that are Mundet semistable for all $k \geq k(d)$, that is,*

$$(k_1, k_2 \geq k(d)) \implies (\overline{\mathcal{M}}^G(C, X, d, k_1) = \overline{\mathcal{M}}^G(C, X, d, k_2)).$$

More precisely, an object $(\hat{C}, P, u, \underline{z})$ is destabilized by (σ, λ) for some $k \geq k(d)$ iff it is destabilized for all $k \geq k(d)$.

The following describes the Mundet semistability condition for large k.

Lemma 6.3. *For any $d \in H_2^G(X, \mathbb{Q})$ there exists $k(d)$ such that for $k \geq k(d)$, the stack $\overline{\mathcal{M}}^G(C, X, d, k)$ has objects given by tuples $(P, \hat{C}, u, \underline{z})$ for which $(u|\hat{C}_0)^{-1}(X^{\mathrm{ss}}/G)$ is non-empty.*

Proof. It suffices to show that if $v : C \to X/G$ is a Mundet unstable gauged map with class d for large k, then $v(C)$ is contained in some Kirwan-Ness stratum X_λ/G and vice versa. Let (σ, λ) be a pair destabilizing u:

$$\mu(\sigma, \lambda) = \mu_{BG}(\sigma, \lambda) + k\mu_X(\sigma, \lambda) > 0.$$

By Corollary 6.2, we may assume that $v = (P, u)$ is Mundet destabilized by (σ, λ) for all $k \geq k(d)$. Then the Hilbert Mumford weight $\mu_X(\sigma, \lambda) > 0$ must be positive. The associated graded $\mathrm{Gr}(u)$ is contained in the fixed point set Z_λ, that is, $\mathrm{Gr}(u)(\hat{C}_0) \subset P(Z_\lambda)$, and $(\mathrm{Gr}(P), \mathrm{Gr}(u))$ has positive Hilbert-Mumford weight with respect to (σ, λ). Thus u is generically unstable.

Conversely, suppose that $v = (P, u) : \hat{C} \to X/G$ takes values in some stratum X_μ/G generically on the root component $\hat{C}_0 \subset \hat{C}$. The stratum fibers

$$X_\mu = G \times_{Q_\mu} Y_\mu \to G/Q_\mu \tag{27}$$

as in (13). By composition with the map $P(X) \to P(Q_\mu) = P/Q_\mu$ arising from (27) we obtain a map $\sigma : (u|\hat{C}_0 \cap u^{-1}(X_\mu)) \to P/Q_\mu$. Locally P is trivial, and so in a neighborhood of any point in $u^{-1}(X_\mu)$ the map is given by a map to G/Q_μ. By completeness of G/Q_μ this map extends to $\sigma : \hat{C}_0 \to P/Q_\mu$, by definition a parabolic reduction σ. Consider the one-parameter subgroup generated by a positive coweight μ. The associated graded $\mathrm{Gr}(u)$ maps to Z_μ on the root component. The Hilbert-Mumford weight $\mu_X(\sigma, \lambda)$ is positive, by construction. Hence $\mu_{BG}(\sigma, \lambda) + k\mu_X(\sigma, \lambda)$ is positive for large k, and the pair (P, u) is Mundet unstable for large k. □

7. Scaled gauged maps

The Mundet semistable moduli spaces have a large linearization limit which includes both stable maps to the git quotient as well as what we called affine gauged maps. This is an algebraic version of a limit that was first studied in the symplectic context by Gaio-Salamon [22].

Definition 7.1. (Scaled gauged maps) *A prestable* scaled gauged map *is a datum $(P, \hat{\mathcal{C}}, u, \delta, \underline{z})$ consisting of a prestable scaled curve $(\hat{\mathcal{C}}, \delta, \underline{z})$ and pair $(P \to C, u : \hat{\mathcal{C}} \to P(X))$ giving a map to the quotient stack $\hat{C} \to X/G$. In the case that $X/\!\!/G$ is an orbifold, the domain $\hat{C}$ is allowed to have a twisted stacky structure $\hat{\mathcal{C}}$ so that the points with non-trivial automorphism are nodes with infinite scaling and the data above gives a representable morphism $v : \hat{\mathcal{C}} \to X/\!\!/G$ as in [2]. Denote by*

$$D_\infty = \mathbb{P}(\omega_{\hat{\mathcal{C}}/C}) \subset \mathbb{P}(\omega_{\hat{\mathcal{C}}/C} \oplus \mathcal{O}_{\hat{\mathcal{C}}}), \quad \textit{resp.} \quad D_0 = \mathbb{P}(\mathcal{O}_{\hat{\mathcal{C}}}) \subset \mathbb{P}(\omega_{\hat{\mathcal{C}}/C} \oplus \mathcal{O}_{\hat{\mathcal{C}}})$$

the divisor at infinity resp. the zero section. The datum $(P, \hat{\mathcal{C}}, u, \delta)$ is semistable *if either*

(a) *the scaling $\delta|\hat{C}_0$ is finite, and the datum $(P, \hat{\mathcal{C}}, u)$ is Mundet semistable; here we are interested in the chamber $k \geq k(d)$ from Lemma 6.3, or*

(b) *the scaling* $\delta|\hat{\mathcal{C}}_0$ *on* $\hat{\mathcal{C}}_0$ *is infinite, and* $\delta^{-1}(D_\infty) \subset \hat{\mathcal{C}}$ *maps to the semistable locus in* X/G*,*

A semistable scaled gauged map is stable *if it has finitely many automorphisms.*

We introduce the following notation for moduli stacks. Denote by $\overline{\mathcal{M}}_{n,1}^G(C,X)$ the moduli of stable marked scaled gauged maps. The existence of a universal scaled curve implies that again, $\overline{\mathcal{M}}_{n,1}^G(C,X)$ is a hom stack from a Deligne-Mumford stack to a quotient stack of a variety by a reductive group, and so Artin by [33, Proposition 2.3.4].

The moduli stack of stable scaled curves defines a cobordism using the following forgetful morphism. Forgetting everything besides the map δ defines a morphism

$$\rho : \overline{\mathcal{M}}_{n,1}^G(C,X,d) \to \overline{\mathcal{M}}_{0,1} \cong \mathbb{P}^1, \quad (C,v,\delta,\underline{z} = (z_1,\ldots,z_n)) \mapsto \delta|_{\hat{C}_0 \cong C}.$$

The fiber of ρ over any non-infinite point $\alpha \in \mathbb{P}^1 - \{\infty\}$ is

$$\rho^{-1}(\alpha) \cong \overline{\mathcal{M}}_n^G(C,X,d)$$

the space of Mundet semistable gauged maps in the chamber $k \geq k(d)$. On the other hand, the fiber over infinity consists of stable maps to $C \times X/\!\!/G$ of degree $(1,d)$ together with bubble trees which call *affine gauged maps* because of the affine structure given by the one-form. Affine gauged maps were introduced first in a symplectic context by Ziltener [57]; a Narasimhan-Seshadri correspondence which relates that viewpoint with the one given here is in Venugopalan-Woodward [52].

Definition 7.2. *An affine gauged map is a datum*

$$(C,\delta,\underline{z} = (z_0,\ldots,z_n),v)$$

where $(C,\delta,\underline{z})$ *is an affine scaled marked curve from Definition 4, and* $v = (P,u) : C \to X/G$ *is a morphism to the quotient stack such that*

(a) $v(\delta^{-1}(D_\infty)) \subset X^{\mathrm{ss}}/G$*. In other words, on the locus* $u^{-1}(\delta^{-1}(D_\infty))$*, the map has image in the* X*-semistable locus; and*
(b) *on the locus* $v^{-1}(\delta^{-1}(D_0))$*, the bundle is trivial.*

In the case that $X/\!\!/G$ *is an orbifold,* C *is equipped with an twisted stacky structure* $\mathcal{C}$ *with non-trivial automorphism groups only at the nodes and marking with infinite scaling as in [2] and data above defines a representable morphism* $v : \mathcal{C} \to X/G$*. Such a datum is* stable *if there exist only finitely many automorphisms* $\varphi \in \mathrm{Aut}(C,v,\delta,\underline{z})$*, or in other words, if there each component* C_i *on which the map* $v_*[C_i] = 0 \in H_2^G(X,\mathbb{Q})$ *the scaling is finite and non-zero (resp. zero or infinity) has at least two (resp. three) special points.*

We introduce the following notation for moduli spaces of affine gauged maps. Let $\overline{\mathcal{M}}_{n,1}^G(\mathbb{A},X)$ denote the moduli space of affine gauged maps with group G and target X with n markings in addition to the marking at infinity.

For each $d \in H_2^G(X,\mathbb{Q})$, let $\overline{\mathcal{M}}^G_{n,1}(\mathbb{A},X,d)$ denote the locus of maps with $v_*[C] = d \in H_2^G(X,\mathbb{Q})$. The moduli stack admits natural evaluation maps

$$\mathrm{ev} \times \mathrm{ev}_\infty : \overline{\mathcal{M}}^G_{n,1}(\mathbb{A},X) \to (X/G)^n \times \overline{\mathcal{I}}_{X/\!\!/G}$$

given by evaluation at the markings $z_i, i = 1,\ldots,n$ and z_0. Also define, for ease of notation,

$$\overline{\mathcal{M}}_n(C, X/\!\!/G, d) = \overline{\mathcal{M}}_{g,n}(C \times X/\!\!/G, (1,d))$$

the so called *graph space* of stable maps to $X/\!\!/G \times C$ of degree $(1,d)$. Generalizing the positivity of energy of stable gauged maps in Lemma 5.4 we have the following:

Proposition 7.3. *Any object $(P,\hat{\mathcal{C}},u,\underline{z})$ of $\overline{\mathcal{M}}^G_{n,1}(C,X)$ or object $(\mathcal{C},\underline{z},\delta,P,u)$ of $\overline{\mathcal{M}}^G_{n,1}(\mathbb{A},X)$ has non-negative energy, and vanishing energy only if the bundle and section are trivializable on each component.*

Proof. Each irreducible component of the domain carries either a Mundet-semistable map, a map to X, a map to $X/\!\!/G$, or an affine gauged map which is necessarily generically semistable. The statement of the proposition follows from applying Lemma 5.4 and Remark 5.5 to each component. □

Later we will need a bound on the number of irreducible components of the domain in terms of the energy.

Corollary 7.4. *Let k be an integer such that if x is any object of $\mathrm{Hom}(\mathrm{pt}, X/\!\!/G)$, then the order of the automorphism group $|\mathrm{Aut}(x)|$ of x divides k. Any object $(P,\hat{\mathcal{C}},u,\underline{z},\delta)$ of $\overline{\mathcal{M}}^G_{n,1}(C,X,d)$ or object $(P,\mathcal{C},u,\underline{z},\delta)$ of $\overline{\mathcal{M}}^G_{n,1}(\mathbb{A},X,d)$ with non-zero energy has energy at least $1/k$.*

Proof. By Proposition 7.3, any component with non-zero energy has positive energy defined as the pairing of $u^*c_1(P(\tilde{X}))$ with $[\mathcal{C}]$. By [12, Theorem 1.187 part (iii)], $ku^*c_1(P(\tilde{X}))$ is represented by an integral divisor, and so an integral cohomology class. The statement of the Corollary follows. □

8. Properness for trivial actions

In this section we show properness of the moduli stack of gauged scaled maps in the case that the group acting is trivial.

Proposition 8.1. *Let $\mathcal{X}$ be a smooth proper Deligne-Mumford stack with projective coarse moduli space X and ample line bundle on the coarse moduli space $\tilde{X} \to X$. For any $n > 0, E > 0$, the union of the stacks $\overline{\mathcal{M}}_{n,1}(C,\mathcal{X},d)$ for $(d, c_1(\tilde{X})) < E$ is proper.*

Proof. By properness of moduli stacks of stable maps to Deligne-Mumford stacks [1, Section 6], the union of the components

$$\bigcup_d \overline{\mathcal{M}}_n(C,\mathcal{X},d) := \overline{\mathcal{M}}_{g,n}(C \times \mathcal{X},(1,d)), \quad (d, c_1(\tilde{X})) < E$$

is proper. Therefore it suffices to show that the forgetful morphism

$$f : \overline{\mathcal{M}}_{n,1}(C, \mathcal{X}, d) \to \overline{\mathcal{M}}_n(C, \mathcal{X}, d) := \overline{\mathcal{M}}_{g,n}(C \times \mathcal{X}, (1, d))$$

obtained by forgetting δ and collapsing unstable components is proper. Let $[u] \in \overline{\mathcal{M}}_n(C, \mathcal{X}, d)$ with representative $u : \hat{C} \to \mathcal{X}$. Since $\mathcal{X}$ is projective, Bertini implies that there exists a divisor $D \subset X$ transverse u and meeting each non-constant component of u transversally and disjoint from the markings and images of unstable components of the domain. Let $\mathcal{D} = D \times_X \mathcal{X}$ and $U \subset \overline{\mathcal{M}}_{n,1}(C, \mathcal{X}, d)$ be the open substack of maps such that each component meets $\mathcal{D}$ transversally and in a set of distinct points disjoint from the markings and ghost components. By taking a divisor D of sufficiently large degree, we may assume that for each component $\hat{C}_i \subset \hat{C}$, the map u restricted to $\hat{C}_i$ meets the divisor in at least three points:

$$\#(u|_{\hat{C}_i})^{-1}(\mathcal{D}) \geq 3, \quad \forall \hat{C}_i \subset \hat{C}.$$

Choose an ordering of the additional points $u^{-1}(\mathcal{D})$ meeting u. Let U denote the substack of $\overline{\mathcal{M}}_{n+k,1}(C, \mathcal{X}, d)$ so that the last k points represent transverse intersections with $\mathcal{D}$. The map forgetting the last k points gives an étale morphism from U to $\overline{\mathcal{M}}_n(C, \mathcal{X}, d)$, see for example [21, Proposition 4]. The map

$$\overline{\mathcal{M}}_{n,1}(C, \mathcal{X}, d) \supset U \to \overline{\mathcal{M}}_{n+k,1}(C), \quad (u : \hat{\mathcal{C}} \to \mathcal{X}, \underline{z}) \mapsto (u : \hat{\mathcal{C}} \to \mathcal{X}, \underline{z} \cup u^{-1}(\mathcal{D}))$$

fits into a Cartesian diagram

$$\begin{tikzcd}
\overline{\mathcal{M}}_{n,1}(C, \mathcal{X}, d) \arrow[d] & U \arrow[l] \arrow[r] \arrow[d] & \overline{\mathcal{M}}_{n+k,1}(C, \mathcal{X}, d) \arrow[r] \arrow[d] & \overline{\mathcal{M}}_{n+k,1}(C) \arrow[d] \\
\overline{\mathcal{M}}_n(C, \mathcal{X}, d) & f(U) \arrow[l] \arrow[r] & \overline{\mathcal{M}}_{n+k}(C, \mathcal{X}, d) \arrow[r] & \overline{\mathcal{M}}_{n+k}(C)
\end{tikzcd}$$

where the right-hand vertical arrow is proper. Since the pull-back of proper morphisms is proper and properness is étale local in the target, the left-hand-arrow is also proper. Since $\overline{\mathcal{M}}_n(C, \mathcal{X}, d)$ is proper, $\overline{\mathcal{M}}_{n,1}(C, \mathcal{X}, d)$ is proper as well. □

Proposition 8.2. *For any $E, n > 0$ the union of moduli stacks $\overline{\mathcal{M}}_{n,1}(\mathbb{A}, \mathcal{X}, d)$ with $(d, c_1(\tilde{\mathcal{X}})) < E$ is proper.*

Proof. Consider the forgetful map $f : \overline{\mathcal{M}}_{n,1}(\mathbb{A}, \mathcal{X}, d) \to \overline{\mathcal{M}}_{0,n+1}(\mathcal{X}, d)$ defined by composing $\overline{\mathcal{M}}_{n,1}(\mathbb{A}, \mathcal{X}, d) \to \overline{\mathfrak{M}}_{0,n+1}(\mathcal{X}, d)$ with the stabilization map $\overline{\mathfrak{M}}_{0,n+1}(\mathcal{X}, d) \to \overline{\mathcal{M}}_{0,n+1}(\mathcal{X}, d)$ [7, Proposition 3.10], [1, Proposition 9.1.1]. As before, choose $[u] \in \overline{\mathcal{M}}_{0,n+1}(\mathcal{X}, d)$ with representative $u : \mathcal{C} \to \mathcal{X}$ and a divisor $\mathcal{D} \subset \mathcal{X}$ meeting u transversally away from the markings and ghost components. Properness of $\mathcal{M}_{n+k,1}(\mathbb{A})$ and the Cartesian diagram

$$\begin{tikzcd} \overline{\mathcal{M}}_{n,1}(\mathbb{A},\mathcal{X},d) \arrow[d] & U \arrow[l] \arrow[r] \arrow[d] & \overline{\mathcal{M}}_{n+k,1}(\mathbb{A}) \arrow[d] \\ \overline{\mathcal{M}}_{0,n}(\mathcal{X},d) & f(U) \arrow[l] \arrow[r] & \overline{\mathcal{M}}_{0,n+k} \end{tikzcd}$$

imply that $\overline{\mathcal{M}}_{n,1}(\mathbb{A},\mathcal{X},d)$ is proper over $\overline{\mathcal{M}}_{0,n}(\mathcal{X},d)$. Since $\overline{\mathcal{M}}_{0,n}(\mathcal{X},d)$ is itself proper by [1, Theorem 1.4.1], $\overline{\mathcal{M}}_{n,1}(\mathbb{A},\mathcal{X},d)$ is itself proper. $\square$

9. Boundedness

In this section we show that the moduli space of gauged scaled maps with fixed numerical invariants is finite type. The results of Ciocan-Fontanine-Kim-Maulik [13, Section 3.2] imply such a result in the case of a vector space target. We extend the argument here to the case of projective spaces.

Theorem 9.1. *(c.f. [13, Theorem 3.2.5]) Let $E > 0$ and $\mathcal{C}$ a twisted prestable curve. Let V be a finite-dimensional complex vector space with an action of G via a representation $G \to GL(V)$ with finite kernel and $X = \mathbb{P}(V)$. Suppose that the semistable locus X^{ss} is non-empty and equal to the stable locus. Then the following family of gauged maps is bounded: pairs $v = (P,u)$ consisting of a principal G-bundle $P \to \mathcal{C}$ and representable section $u : \mathcal{C} \to P(X)$ such that the energy $\mathcal{E}(v) < E$ and the section u sends the generic point of $\mathcal{C}$ to $P(X^{\mathrm{ss}})$.*

Proof. A similar theorem with $\mathbb{P}(V)$ replaced by V is given by Ciocan-Fontanine-Kim-Maulik [13, Theorem 3.2.5].

First we assume that $\mathcal{C}$ is an ordinary curve and G is a torus. By lemma 5.3(4) we have

$$\mathcal{E}(v) = (u_*[C], c_1(P(\tilde{X}))) = (\theta, d(P)) + d(u) \in [0,E]. \tag{28}$$

By the Hilbert-Mumford criterion the semistable locus in $\mathbb{P}(X)$ is

$$X^{\mathrm{ss}} = \mathbb{P}(V)^{\mathrm{ss}} = \{[x_1,\dots,x_k] \in \mathbb{P}(V) \mid \operatorname{hull}(\{\mu_i | x_i \neq 0\}) \ni \theta\}.$$

Let $u : C \to P(X)$ be a section that is generically semistable. Recall example 5.2 and that u is given by a k-tuple $(\tilde{u}_1,\dots,\tilde{u}_k)$. The condition that u is generically semistable means each $\tilde{u}_i \neq 0$ hence by lemma 5.3(3)

$$(\mu_i, d(P)^\vee) + d(u) \geq 0, \quad \forall i = 1,\dots,k.$$

The same holds for any vector near θ, since the condition of lying in the convex hull is open in in the interior. Choose a basis

$$\xi_1,\dots,\xi_r \in \mathfrak{g}_{\mathbb{Q}}^\vee, \quad \operatorname{hull}(\xi_1,\dots,\xi_r) \ni \theta$$

of points near θ so that

$$(\xi_i, d(P)^\vee) - d(u) \geq 0, \quad i = 1,\dots,r. \tag{29}$$

Combining (29) and (28) shows that the possible degrees $d(P)$ lie in a finite set.

Next we consider the case that $\mathcal{C}$ is an ordinary curve and G is an arbitrary compact connected reductive complex group. By a simple case of the Drinfeld-Simpson theorem [18], [13, Lemma 3.2.7], the bundle P admits a reduction to a Borel subgroup $B \subset G$. Let $\pi_B : B \to T$ be the projection (12), and $\operatorname{Gr}(P)$ the associated graded. Since the map π_Q is T-equivariant, the section u induces a section $\operatorname{Gr}(u) : \mathcal{C} \to \operatorname{Gr}(P)(X)$ that is also T-semistable. A G-bundle corresponds via a faithful representation $G \to GL(r)$ to a vector bundle $F \to C$ together with a reduction of the structure group, given by a section of an affine bundle $GL(r)/G$. Equation (29) shows that splitting type of the associated graded $\operatorname{Gr}(F)$ is uniformly bounded given a fixed $d \in H_2^G(X, \mathbb{Z})$, and furthermore the first Chern class $d(P)$ has bounded pairing with $c_1^G(\tilde{X})$.

Given this bound on the splitting type of the associated graded, a standard argument (see for example the boundedness arguments in [26, 3.3]) shows that after twisting by a sufficiently positive bundle depending only on a bound on the splitting type, any vector bundle $F \to C$ as above is generated by their global sections and has no higher cohomology. Indeed the long exact sequence in cohomology shows that, for any locally free subsheaf $F' \subset F$ appearing as a summand in the associated graded $\operatorname{Gr}(F)$ we have an exact sequence

$$H^0(C, F') \to H^0(C, F) \to H^0(C, F/F') \to H^1(C, F').$$

From this and the corresponding sequence for the twist $F(-z), z \in C$ one obtains that if $F', F/F'$ are generated by their global sections and have no higher cohomology then F has the same property. An induction shows that $F \to C$ is a quotient of a fixed trivial bundle $\mathcal{O}_C^{\oplus l}$ for $l \geq k(E)$ where $k(E)$ is a constant depending only on the energy bound E. Thus the family of bundles is bounded.

To show that the families of sections are bounded, note that any two sections $u_0, u_1 : \hat{C} \to P(X)$ have homology classes that differ by an element of $H_2(X, \mathbb{Q}) \subset H_2^G(X, \mathbb{Q})$. The homomorphism

$$(\cdot, c_1^G(\tilde{X})) \in \operatorname{Hom}(H_2^G(X, \mathbb{Q}), \mathbb{Q})$$

restricts to the standard pairing on $H_2(X, \mathbb{Q})$ corresponding to the hyperplane class, the energy bound (28) implies that $d(u)$ is bounded from above and below. Now the difference of homology classes of any sections of $P(\mathbb{P}(V))$ lies in the kernel of the map $H_2(P(\mathbb{P}(V)) \to H_2(C)$ and so, since $\mathbb{P}(V)$ is simply-connected, lie in the image of the inclusion $H_2(\mathbb{P}(V)) \to H_2(P(\mathbb{P}(V))$ of a fiber. It follows that the degree also classifies homology classes of sections:

$$(u_1 \cong u_2) \iff (d(u_1) = d(u_2)). \qquad (30)$$

By (30) the possible homology classes of the sections are bounded as well. This shows that the family of maps to X/G is bounded.

Finally suppose that $\mathcal{C}$ is a twisted curve and G is complex connected reductive. Let $\hat{\mathcal{C}} \to S$ be a family of twisted curves, $P \to \hat{\mathcal{C}}$ a family of bundles and $u : \mathcal{C} \to P(X)$ a family of sections as above. By [42, Theorem 1.14], after étale cover there exists a finite flat morphism $\pi : Z \to \hat{\mathcal{C}}$ from a projective scheme $Z \to S$ to $\hat{\mathcal{C}}$; the proof in fact shows that π is surjective. By faithfully flat descent, sheaves on $\mathcal{C}$ may be described in terms of descent data as sheaves E on $Z \leftleftarrows Z \times_{\mathcal{C}} Z$. Such data consists of the bundle Z together with isomorphisms $\varphi : \pi_1^* E \to \pi_2^* E$, where π_1, π_2 are the projections onto the factors of $Z \times_{\mathcal{C}} Z$ see Tag 03O6 in [16]. A principal G-bundle is given via an embedding $GL(r)$ as descent data for a locally free sheaf of rank r together with a reduction given by a section of the associated $GL(r)/G$ bundle and an isomorphism $\varphi : \pi_1^* E \to \pi_2^* E$ preserving the G-reduction. Any such substack may be realized via quot scheme techniques as a quotient of a variety by a reductive group action as above. □

Corollary 9.2. *For any real $E > 0$, the union of components $\overline{\mathcal{M}}_{n,1}^G(C, X, d)$, $\overline{\mathcal{M}}_n^G(C, X, d)$, and $\overline{\mathcal{M}}_{n,1}^G(\mathbb{A}, X, d)$ with $(d, c_1^G(\tilde{X})) < E$ is finite type.*

Proof. We consider only $\overline{\mathcal{M}}_{n,1}^G(C, X, d)$; the proof for the other moduli spaces is similar. We first show that only finitely many combinatorial types are possible for a given energy bound. Let $v = (P \to C, u : \hat{\mathcal{C}} \to P(X))$ be a gauged map of class d and energy $\mathcal{E}(v) = (d, c_1^G(\tilde{X}))$. The energy of any component with non-zero energy $\langle v_*[\mathcal{C}_i], c_1^G(\tilde{X}) \rangle$ is at least $1/k$ for some integer k, by Corollary 7.4. It follows that the number of irreducible components $\hat{\mathcal{C}}_i$ of the domain $\hat{\mathcal{C}}$ with positive energy is bounded by $k\mathcal{E}(v) + n$.

The bound on the number of components with positive energy gives a bound on the total number of components as follows. Any component $\hat{\mathcal{C}}_i$ of $\hat{\mathcal{C}}$ on which the map $v : \hat{\mathcal{C}} \to X/G$ is trivial and has trivial scaling has at least three special points. Removing the component $\hat{\mathcal{C}}_i$ defines a curve $\hat{\mathcal{C}} - \hat{\mathcal{C}}_i$ with at least three connected components, and so a partition of the markings $\{z_1, \ldots, z_n\}$ and irreducible components of $\hat{\mathcal{C}}$ with non-trivial energy into three non-empty subsets. Thus the number of components of the domain with trivial scaling is also bounded by the number of partitions of $k\mathcal{E}(d) + n$. On the other hand, by the monotonicity condition the number of components with scaling is at most the number of terminal components, since there is at most one component with finite, non-zero scaling on the any path from a root component to the terminal component. That is, the number of vertices $\mathrm{Vert}(\Gamma)$ of the combinatorial type Γ of $\hat{\mathcal{C}}$ is bounded by an integer $v = v(d)$. There are finitely many trees Γ satisfying this bound, hence finitely many possibilities for Γ.

Given an energy bound, the possible homology classes of each component form a finite set by Theorem 9.1 and the requirement that $d(P) \in H_2(\mathcal{C}, \mathbb{Z}/k)$. It follows that there are only finitely many possible labellings

$\mathrm{Vert}(\Gamma) \to H_2^G(X,\mathbb{Q})$ of the given graphs by degree two homology classes with the given energy bound; hence finitely many combinatorial types as claimed. It follows that the image of $\overline{\mathcal{M}}_{n,1}^G(C,X,d) \to \overline{\mathfrak{M}}_{n,1}(C)$ is contained in an Artin stack of finite type for each $d \in H_2^G(X,\mathbb{Z})$.

We now use boundedness of the splitting type to prove that the moduli stack is finite type. As in the proof of Theorem 9.1, we describe bundles on stacky curves in terms of descent data. Let $\hat{\mathcal{C}} \to S$ be a family of stacky curves, $P \to \hat{\mathcal{C}}$ a family of bundles and $u : \mathcal{C} \to P(X)$ a family of sections as above. By [42, Theorem 1.14], after étale cover there exists a finite flat morphism $\pi : Z \to \hat{\mathcal{C}}$ from a projective scheme $Z \to S$ to $\hat{\mathcal{C}}$; the proof in fact shows that π is surjective. By faithfully flat descent, sheaves on $\mathcal{C}$ may be described in terms of descent data as sheaves E on $Z \leftleftarrows Z \times_{\mathcal{C}} Z$; that is, Z together with isomorphisms $\varphi : \pi_1^* E \to \pi_2^* E$, where π_1, π_2 are the projections onto the factors of $Z \times_{\mathcal{C}} Z$ see Tag 03O6 in [16]. A principal G-bundle is given via an embedding $GL(r)$ as descent data for a locally free sheaf of rank r together with a reduction given by a section of the associated $GL(r)/G$ bundle and an isomorphism $\varphi : \pi_1^* E \to \pi_2^* E$ preserving the G-reduction. Because the splitting type of $P \to \mathcal{C}$ is bounded, the splitting type of E is bounded as well, and the image of P in $\mathrm{Hom}(Z, BG)$ is contained in a substack of finite type. Any such substack may be realized by standard constructions via quot scheme techniques as a quotient of a variety by a reductive group action as in [33, 2.3.4]. Since the homology class $d \in H_2^G(X,\mathbb{Q})$ is bounded, the image of S in $\mathrm{Hom}(Z, X/G)$ consists of sections with bounded homology class, and so also lies in a substack of finite type by [1, Theorem 1.4.1], see also Lieblich [33]. It follows that $\overline{\mathcal{M}}_{n,1}^G(C,X)$ is covered by finitely many stacks of finite type, and so is itself finite type. □

10. Universal closure

In this section we show that the moduli stack of scaled gauged maps is universally closed using the valuative criterion and Schmitt's git construction [46].

10.1. **Removal of singularities for bundles on surfaces.** We begin with the following theorem of J.-L. Colliot-Thélène and J.-J. Sansuc, [14] describes extensions of bundles on complements of finite subsets of surfaces, see also Ciocan-Fontanine-Kim-Maulik [13]: For any scheme X and reductive group G a *principal G-bundle* is a scheme P with a free right action of G that is locally trivial in the fpqc topology on X. If X is smooth, then this is equivalent to local triviality in the étale topology.

Theorem 10.1. *Let X be a smooth complex variety of dimension two and G a connected reductive group. If $U \subset X$ is the complement of a finite set of non-singular points on X, then any principal G-bundle on U is the restriction of a principal G-bundle on X, unique up to isomorphism.*

We briefly recall the main point of the proof. First, we prove the corresponding result for vector bundles. let $F \to U$ be a locally free sheaf and $i : F \to X$ the inclusion. The sheaf $i_* F \cong i_* F^{\vee\vee}$ is reflexive by e.g. Hartshorne [25, Corollary 1.7], and reflexive sheaves on surfaces are locally free by e.g. Hartshorne [25, Corollary 1.4]. This shows that F has an extension. The extension is unique up to isomorphism by Hartog's theorem. If $F_1, F_2 \to X$ are two such locally free extensions then the given isomorphism $\varphi \in H^0(U, \mathrm{Hom}(F_1, F_2))$ extends over X since $X - U$ is a finite set.

Next we consider the case of arbitrary reductive groups. Fix an embedding $G \to GL(r)$ for some $r \geq 1$ and let $P \to U$ be a principal G-bundle. Using the embedding, we obtain a vector bundle E, which extends uniquely to X. Let Q denote the frame bundle of F. The bundle P corresponds to a reduction of structure group $\sigma : U \to Q/P$. Since G is connected reductive, it follows from Matsushima's criterion [36], [9] that the homogeneous space $GL(r)/G$ is a smooth affine variety. Moreover, by assumption, Q/G admits a section σ over U. Thus, by Hartogs' theorem, σ extends over X.

10.2. **Existence of limits for families with finite scaling.** First we show properness over the space of finite scalings using Schmitt's git construction [46] the Keel-Mori theorem [28].

Lemma 10.2. *The forgetful morphism to the moduli space of finite scalings*

$$\overline{\mathcal{M}}^G_{n,1}(C, X, d) \supset \rho^{-1}(\mathcal{M}_{0,1}) \to \mathcal{M}_{0,1} \subset \overline{\mathcal{M}}_{0,1}, \quad [\hat{\mathcal{C}}, u, P, \underline{z}, \delta] \mapsto \delta$$

is proper.

Proof. This follows an argument given with González [24]: there is a proper relative Givental map described in Popa-Roth [44]

$$\overline{\mathcal{M}}^G(C, X, d) \to \overline{\mathcal{M}}^{G,\mathrm{quot}}(C, X, d).$$

For each fixed bundle, this map collapses bubbles of the section u and replaces them with base points with multiplicity given by the degree of the bubble tree. On the other hand, $\overline{\mathcal{M}}^{G,\mathrm{quot}}(C, X, d)$ has a git construction given in (17) and so has a proper coarse moduli space. Finally $\overline{\mathcal{M}}^G_n(C, X, d) \to \overline{\mathcal{M}}^G(C, X, d)$ is proper, each forgetful map being isomorphic to a universal curve. Under the stable=semistable assumption, the Luna slice theorem [34] implies that $\overline{\mathcal{M}}^{G,\mathrm{quot}}_n(C, X, d)$ is étale-locally the quotient of a smooth variety by a finite group and so has finite inertia stack. By the Keel-Mori theorem [28], explicitly stated in [15, Theorem 1.1], the morphism from $\overline{\mathcal{M}}^{G,\mathrm{quot}}_n(C, X, d)$ to its coarse moduli space is proper, so $\overline{\mathcal{M}}^G_n(C, X, d)$ is proper as well. Hence

$$\rho^{-1}(\mathbb{C}) \cong \overline{\mathcal{M}}^G_n(C, X, d) \times \mathbb{C} \to \mathbb{C}$$

is proper. Schmitt's construction [47, Section 2.7.2] implies that if stable=semistable then the automorphism groups of objects in $\overline{\mathcal{M}}^{G,\mathrm{quot}}_n(C, X, d)$

are finite, and so the moduli stack is Deligne-Mumford. Since quot schemes are projective, and moduli spaces of stable maps to projective schemes are projective, the moduli spaces $\overline{\mathcal{M}}_n^{G,\mathrm{quot}}(C, X, d)$ have projective coarse moduli spaces and so are proper. $\square$

Next we show the valuative criterion for universal closure in the case that the scalings go to infinity. This is a combination of properness for stable maps to the targets, its quotient, and removal of singularities for bundles on surfaces.

Theorem 10.3. *Given a family of scaled Mundet-semistable gauged maps over a punctured curve S with finite scaling δ*

$$(P, \hat{C}, u, \underline{z}, \delta) \to S = \overline{S} - \{\infty\}$$

there exists an extension over $\overline{S}$, after étale cover.

Proof. It suffices, by the Lemma 10.2, to consider the case that the scaling δ becomes infinite. We first consider the case that $\hat{\mathcal{C}} \cong C$. There are three steps, in which we construct the central fiber curve and a scaled gauged map by stages. First we construct a limit

$$\hat{\mathcal{C}}_\infty^{X/\!\!/G} \to C, \quad v_\infty^{X/\!\!/G} : \hat{\mathcal{C}}_\infty^{X/\!\!/G} \to X/\!\!/G, \quad \underline{z}_\infty \subset \hat{\mathcal{C}}_\infty^{X/\!\!/G}$$

by properness of the moduli space of stable maps to $X/\!\!/G$. However, the limiting domain $\hat{\mathcal{C}}_\infty^{X/\!\!/G}$ is not the one we want because there may be bubbling in X that is not captured by bubbling in $X/\!\!/G$. Forgetting some of the components of $\hat{C}_\infty^{X/\!\!/G}$ and using removal of singularities for bundles on surfaces gives a curve and map

$$\hat{\mathcal{C}}_{BG} \to C, \quad \phi : \hat{\mathcal{C}}_{BG} \to BG$$

where the map ϕ is a classifying map for an extension of the bundle P over $\hat{\mathcal{C}}_{BG}$. Then we apply properness of the moduli stack of sections of $P(X)$ to obtain the desired limiting curve

$$\hat{\mathcal{C}}_\infty \to C, \quad u_\infty : \hat{\mathcal{C}}_\infty \to P(X), \quad \delta_\infty : \hat{\mathcal{C}}_\infty \to \mathbb{P}(\omega_{\hat{\mathcal{C}}_\infty/C} \oplus \mathcal{O}_{\hat{\mathcal{C}}_\infty}), \quad \underline{z}_\infty \subset \hat{\mathcal{C}}_\infty, .$$

Here are the details:

Step 1: Construct the part with infinite scaling. We first introduce the following notation for the maps to the git quotient. By definition of $k(d)$ and Lemma 6.3, the maps $u : C \to P(X)$ are generically semistable, and so defines a curve and map

$$C^{X/\!\!/G} := u^{-1}P(X^{\mathrm{ss}}) \neq \emptyset, \quad u^{X/\!\!/G} = (u^{X/\!\!/G} : C^{X/\!\!/G} \to X/\!\!/G).$$

By properness of $X/\!\!/G$, $u^{X/\!\!/G}$ extends to a family of stable maps with domain $C \times S$. Order the base points so that they give sections

$$\zeta_i : S \to C, \quad \zeta_i(S) \subset P(X^{\mathrm{us}}), i = 1, \ldots, l.$$

Denote by $\underline{z} \cup \underline{\zeta} : S \to C^{n+l}$ the family of sections obtained by adding the base points and removing duplicates; that is, after restricting to an open subvariety we may assume that any two sections that coincide in one fiber, coincide everywhere; then we just remove one of the duplicate sections. Because the domain is irreducible, the datum $(C, u^{X/\!\!/G}, \underline{z} \cup \underline{\zeta}, \delta)$ is a stable scaled map to the smooth Deligne-Mumford stack $X/\!\!/G$.

Properness for the moduli space of scaled maps with trivial group action in Proposition 8.1 implies that the family extends over the central fiber: Since $\overline{\mathcal{M}}_{n,1}(C, X/\!\!/G, d)$ is proper, the map $u^{X/\!\!/G}$ extends over the central fiber to a stable scaled map

$$\left(\hat{\mathcal{C}}_\infty^{X/\!\!/G} \to C, \quad u_\infty^{X/\!\!/G} : \hat{\mathcal{C}}_\infty^{X/\!\!/G} \to X/\!\!/G, \quad \underline{z}_\infty \cup \underline{\zeta}_\infty, \quad \delta_\infty\right).$$

In particular, the markings $\underline{\zeta}_\infty$ lie on the locus of $\hat{\mathcal{C}}_\infty^{X/\!\!/G}$ with finite scaling $\hat{\mathcal{C}}_\infty^{X/\!\!/G} - \delta_\infty^{-1}(D_\infty)$.

Step 2: Construct the part with finite scaling. Let Γ be the combinatorial type of the limit $\hat{\mathcal{C}}_\infty^{X/\!\!/G}$ in the previous step and Γ' the combinatorial type obtained by forgetting the components of $\hat{\mathcal{C}}_\infty^{X/\!\!/G}$ on which the scale $\delta|\hat{\mathcal{C}}_\infty^{X/\!\!/G}$ is zero. More precisely, choose a family of sections $S \to (\hat{\mathcal{C}}_\infty^{X/\!\!/G})^k$ taking values in the locus with non-zero scaling with the property that the components with non-zero scaling become stable. By e.g. Behrend-Manin [7, Lemma 3.12], there exists a proper family $\hat{\mathcal{C}}_\infty^{BG}$ of stable curves with a morphism from $\hat{\mathcal{C}}_\infty^{X/\!\!/G}$ collapsing the components with zero scaling. The family $\hat{\mathcal{C}}_\infty^{BG}$ consists of a collection of components on which the scaling is finite and non-zero, or infinite, with a morphism

$$\varphi : \hat{\mathcal{C}}_\infty^{X/\!\!/G} \to \hat{\mathcal{C}}_\infty^{BG}.$$

The scaling δ_∞ is finite at the base points $\underline{\zeta}_\infty$ and markings $\underline{z}_\infty$. The image of the base points $\underline{\zeta}_\infty$ under the morphism φ are denoted $\underline{\zeta}_\infty^{BG} = \varphi(\underline{\zeta}_\infty)$. The points $\underline{\zeta}_\infty^{BG}$ are no longer necessarily distinct from each other and the markings. Because the scalings $\delta_\infty^{X/\!\!/G}$ are finite at $\underline{\zeta}_\infty$, the scalings δ_∞^{BG} are finite at $\underline{\zeta}_\infty^{BG}$, that is, $\delta_\infty^{BG}(\underline{\zeta}_\infty^{BG}) < \infty$. In particular, all of the points $\underline{\zeta}_\infty^{BG}$ are non-singular, since the only nodes in $\hat{\mathcal{C}}_\infty^{BG}$ are contained in $(\delta_\infty^{BG})^{-1}(\infty)$.

Removal of singularities for bundles on surfaces implies that the bundle extends over the central fiber. The morphism $u^{X/\!\!/G}$ induces an extension of the bundle P_∞^{BG} over the complement of the base points $\underline{\zeta}_\infty^{BG}$, given by pull-back of

$$P_\infty^{BG} \to \hat{\mathcal{C}}_\infty^{BG} - \underline{\zeta}_\infty^{BG}, \quad P_\infty^{BG} := (u_\infty^{X/\!\!/G}|\hat{\mathcal{C}}_\infty^{BG} - \underline{\zeta}_\infty^{BG})^*(X^{\mathrm{ss}} \to X^{\mathrm{ss}}/G)$$

under $u_\infty^{X/\!\!/G}$. By construction, the points $\zeta_{i,\infty}$ are non-singular points in $\hat{C}_\infty^{BG}$. By removal of singularities for bundles Theorem 10.1, the bundle $P^{BG} \to \hat{\mathcal{C}}^{BG}$ given by P_∞^{BG} over the central fiber has a unique extension over the points $\zeta_{i,\infty}$. This implies the existence of a limiting bundle $P_\infty \to \hat{\mathcal{C}}_\infty^{BG}$

with classifying map

$$\phi_\infty : \hat{\mathcal{C}}^{BG}_\infty \to BG, \quad P_\infty := \phi_\infty^*(EG \to BG).$$

Denote by $\hat{C}^{BG}$ the resulting family over $\mathbb{C}^\times$, and P the resulting bundle over $\hat{\mathcal{C}}^{BG}$.

Step 3: Construct the full limit. In the last step we apply properness for the moduli stack of stable sections. The associated fiber bundle $P(X) \to \hat{\mathcal{C}}^{BG}$ is projective, since X is projective and $\hat{\mathcal{C}}^{BG}$ is projective. Then it follows from properness of stable maps to $P(X)$ that there exists a limit $u_\infty : \hat{\mathcal{C}}_\infty \to P(X)$ extending u. The scaling naturally extends to a scaling δ_∞, possibly after adding additional components with finite scaling and trivial maps.

We check that the limit constructed above satisfies the axioms of a stable scaled gauged map. The monotonicity condition on the scalings is guaranteed by the description of the one-form in (7). Furthermore, on the locus $\delta^{-1}(D_\infty)$, the map u_∞ agrees with the pull-back of $u_\infty^{X/\!\!/G}$ and so takes values in the semistable locus. The locus $\delta^{-1}(D_0)$ is a union of components that map to points in $\hat{\mathcal{C}}^{BG}_\infty$. This implies that bundle P is trivial on $\delta^{-1}(D_0)$. Finally, the inequality $\delta(z_{i,\infty}) < \infty$ is automatically satisfied since the scaling on $\hat{\mathcal{C}}_\infty^{X/\!\!/G}$ is finite at the markings, and the forgetful map maps all components with zero scaling to loci where the scaling is finite. Each component on which the scaling and gauged are trivial has at least three special points, since the limit u_∞ is a stable section. Each component with finite, non-zero scaling has at least two special points contains either the limit of a marking or a base point. If trivial such a component is attached to a component with trivial scaling, and so has at least two special points. Finally each component with infinite scaling occurs in the domain $u^{X/\!\!/G}$ and so has at least three special points.

Finally we consider the general case that domains of the family are nodal. That is, we have a family of gauged maps $(P, \hat{\mathcal{C}}, u, \underline{z}, \delta) \to S = \overline{S} - \{\infty\}$ such that every $\hat{\mathcal{C}}_s$ is a nodal projective curve. In this case we repeat the first two steps for the family obtained by restricting to the root component $\hat{\mathcal{C}}_0 \subset \hat{\mathcal{C}}$. In the last stage, properness for stable maps to $P(X)$ implies the existence of a limit of $u : \hat{\mathcal{C}} \to P(X)$. □

Almost exactly the same argument shows that the moduli stack of affine gauged maps is universally closed:

Lemma 10.4. *Given a family of stable affine gauged maps over a punctured curve S,*

$$(P, \hat{\mathcal{C}}, u, \underline{z}, \delta) \to S := \overline{S} - \{\infty\}$$

there exists an extension over $\overline{S}$, after étale cover.

We use this and the statement of the Lemma above, which dealt with irreducible domain, to prove universal closure of the moduli space of scaled gauged maps:

Theorem 10.5. *The moduli stack $\overline{\mathcal{M}}_n^G(C,X)$ is universally closed.*

Proof. Let $(P,\hat{C},v,\underline{z},\delta) \to S = \overline{S} - \{\infty\}$ be a family of scaled gauged maps over a curve S. We may assume that either δ is finite or infinite on the root component $\hat{C}_0$ for all $s \in S$, after possibly replacing S with an open subscheme. In the finite case, the existence of a central extension over the central fiber follows from 10.3. In the infinite case, the central fiber is a collection of affine gauged maps and maps to $X /\!\!/ G$ by gluing at the nodes. That is, the curve $\hat{C}$ is a union of components $\hat{C} = \hat{C}_0 \cup \hat{C}_1 \cup \ldots \hat{C}_r$ where $\hat{C}_0$ is a family of curves with infinite scaling and $\hat{C}_1,\ldots,\hat{C}_r$ are families of affine scaled curves. By Lemma 10.4, the restriction of the families $(P,\hat{C},v,\underline{z},\delta)$ to $\hat{C}_1,\ldots,\hat{C}_r$ have extension over the central fiber. Similarly properness of the moduli stack of stable maps to $X /\!\!/ G$ implies the existence of a limit of the restriction of the family to $\hat{C}_0$. By closure of the diagonal, these families glue together to a scaled gauged map on the central fiber. □

11. Separation

In this section we check the valuative criterion for separatedness. This is again a combination of separatedness of the moduli stack of stable maps to the target, its quotient, and uniqueness of extensions on bundles on surfaces.

Proposition 11.1. *For $i = 0,1$ let $v^i := (\hat{C}^i \to \overline{S}, P^i \to \hat{C}^i, u^i : \hat{C}^i \to X/G, \delta^i, \underline{z}^i : S \to \hat{C}^{i,n})$ be families of stable scaled gauged maps over a curve $\overline{S}$ that are isomorphic over the punctured curve $S = \overline{S} - \{\infty\}$. Then v^0 is isomorphic to v^1 over $\overline{S}$.*

Proof. Again this follows from a three-step process, in which we show that the maps agree on parts of the limit corresponding to infinite, finite, and zero scaling.

Step 1: The maps agree on the part with non-zero scaling. First we introduce for notation the induced map to the git quotient. Let $\hat{\mathcal{C}}_\infty^{i,X/\!\!/G}$ denote the union of components $\hat{\mathcal{C}}_\infty^i$ with infinite or finite, non-zero scaling $\delta_\infty^i | \hat{\mathcal{C}}_\infty^i$. The inverse image of the semistable locus $X/\!\!/G$ is dense in $\hat{\mathcal{C}}_\infty^{i,X/\!\!/G}$ by Lemma 6.3. By properness of the stack $X/\!\!/G$ we obtain maps

$$u_\infty^{i,X/\!\!/G} : \hat{\mathcal{C}}_\infty^{i,X/\!\!/G} \to X/\!\!/G.$$

We would like to apply separatedness for the moduli stack of stable scaled maps to the git quotient to show that these maps are isomorphic. However, the maps $u_\infty^{i,X/\!\!/G}$ may not be stable since they may contain unstable components. To remedy this, choose an ample invariant divisor $D \subset X$ as in (9). Let $\underline{\zeta}^i = u^{-1}(P(D))$ denote the points mapping to $P(D)$. Consider the scaled curve to $X/\!\!/G$ given by $(u^{i,X/\!\!/G}, \delta^i, \underline{z}^i \cup \underline{\zeta}^i)$ where $\underline{z}^i \cup \underline{\zeta}^i$ denotes the union obtained by adding the intersection points with the divisor and deleting duplicates and nodes. After restricting to an open subscheme of

S containing the central fiber, we may assume that $\underline{z}^i \cup \underline{\zeta}^i$ are distinct, non-singular points.

We claim that the tuples constructed in the previous paragraph are stable scaled maps. Any component of $\hat{\mathcal{C}}^i$ with finite, non-zero scaling that becomes a unstable component after passing to $X /\!\!/ G$ must either be stable, or correspond to a map to X/G that lies generically in a fiber of $X^{\mathrm{ss}} \to X^{\mathrm{ss}}/G$. Such maps always intersect D since D is ample. Hence any such component has at least two special points and a non-trivial scaling, and so is stable.

Separation for the moduli stack of stable scaled maps to the git quotient implies that the central fibers are isomorphic: By Proposition 8.1 there exists an isomorphism

$$(u_\infty^{0,X/\!\!/G} : \hat{\mathcal{C}}_\infty^{0,X/\!\!/G} \to X/\!\!/G, \quad \delta^0, \underline{z}^0 \cup \underline{\zeta}^0) \cong (u_\infty^{1,X/\!\!/G} : \hat{\mathcal{C}}_\infty^{1,X/\!\!/G} \to X/\!\!/G, \quad \delta^1, \underline{z}^1 \cup \underline{\zeta}^1). \tag{31}$$

Step 2: The limits agree on the part with non-zero scaling. This step is an application of uniqueness of removal of singularities for bundles on surfaces in Theorem 10.1. Let $\hat{\mathcal{C}}_\infty^{i,X/\!\!/G}$ denote the curves from Step 1. The classifying maps $\phi_i : \hat{\mathcal{C}}_\infty^{i,X/\!\!/G} \to BG$ are isomorphic (that is, the corresponding bundles are isomorphic) except at finitely many non-singular points, the base points, since the maps $u_i^{X/\!\!/G}$ agree. Furthermore, the base points $\underline{\zeta}_i$ are contained in components of $\hat{\mathcal{C}}_\infty^{i,X/\!\!/G}$ with infinite scaling, since D contains the unstable locus X^{us}. By uniqueness of the completion of bundles on surfaces in Theorem 10.1, the bundles P_0, P_1 are isomorphic over $\hat{\mathcal{C}}^{i,X/\!\!/G}$.

Step 3: The limits agree entirely. Finally we apply separatedness for families of stable sections to show that the limiting sections are isomorphic. Separation of stable maps to $P_\infty(X)$ where $P_\infty := P_i|_{\hat{\mathcal{C}}_\infty^{i,X/\!\!/G}}$ implies that there exists an isomorphism,

$$(u^0 : \hat{\mathcal{C}}^0 \to P_\infty(X), \underline{z}^0) \cong (u^1 : \hat{\mathcal{C}}^1 \to P_\infty(X), \underline{z}^1).$$

Since the scaled curves appearing in the limit already agree, this implies that the stable scaled maps

$$(\hat{C}^i \to \overline{S}, P_i \to \hat{C}^i, u^i : \hat{C}^i \to P_i(X), \delta^i, \underline{z}^i : \overline{S} \to \hat{C}^{i,n}), \quad i = 0,1$$

are isomorphic.

The existence of a unique limit in the case of a family with infinite scaling is similar and left to the reader. □

This proves the valuative criterion for separatedness. Universal closure and of finite-type was shown in previous sections. This completes the proof of properness of $\overline{\mathcal{M}}_{n,1}^G(C, X, d)$ in Theorem 1.1. Finally we complete the proof of the properness of moduli stacks of affine gauged maps.

Corollary 11.2. *For any $E > 0$, the union of moduli stacks of affine gauged maps $\overline{\mathcal{M}}^G_{n,1}(\mathbb{A}, X, d)$ with $(d, c_1^G(\tilde{X})) \le E$ is proper.*

Proof. The proof is by an embedding argument. The stack $\overline{\mathcal{M}}^G_{n,1}(\mathbb{A}, X, d)$ embeds in $\overline{\mathcal{M}}^G_n(C, X, y)$ as follows: Given an affine gauged map $(C_0, \delta, \underline{z}, u)$ and a point $z \in C$ define $\hat{C} := (C_0 \sqcup C)/(z_0 \sim z)$ and extend the map u so that it is constant on the root component $\hat{C}_0 \cong C$. Since any closed substack of a proper stack is proper, $\overline{\mathcal{M}}^G_{n,1}(\mathbb{A}, X, d)$ is proper. □

Remark 11.3. (Convex targets) We conclude by describing results for convex targets: For the moduli stack of gauged maps to a convex variety X defined in Section 5.5, the conclusion of Theorem 1.1 also holds. As explained in Corollary 5.7, Mundet stable maps to X are equivalent to maps to $\overline{X}$ as long as the linearization $\tilde{X}$ is chosen so that the linearization is obtained from $\tilde{X}(l)$ for l sufficiently large.

Conversely, for any class d which pairs trivially with the class of the divisor at infinity, affine gauged maps to $\overline{X}/\!\!/G$ and maps to $X/\!\!/G$ of class d are equivalent: the intersection number between the map and divisor is zero and any such map cannot have a component mapping entirely to the divisor. In the case of maps to the quotient, any point in $\overline{X}$ is unstable for $\tilde{X}(l)$ for l sufficiently large, and so $\overline{X}/\!\!/G$ is isomorphic to $X/\!\!/G$. It follows that the inclusion $\overline{\mathcal{M}}^G_{n,1}(C, X) \to \overline{\mathcal{M}}^G_{n,1}(C, \overline{X})$ is an isomorphism, as claimed, and in particular the union of components $\overline{\mathcal{M}}^G_{n,1}(C, X, d)$ with $\mathcal{E}(d) < E$ is proper for any energy bound $E > 0$.

12. Table of notation

This section contains a table of the notations for the different moduli stacks of stable maps to quotient stacks used in the paper.

Notation	Moduli stack	Page Number
$\overline{\mathcal{M}}_{g,n}(X)$	Stable maps of genus g with n markings	19
$\overline{\mathcal{M}}^G_n(C, X)$	Mundet-semistable gauged maps with n markings	28
$\overline{\mathcal{M}}_n(C, X/\!\!/G)$	Stable sections of $C \times X/\!\!/G \to C$	3
$\overline{\mathcal{M}}^{G,\mathrm{quot,lev}}_n(C, X)$	Gauged maps with level structure	25
$\overline{\mathcal{M}}^{G,\mathrm{quot}}_n(C, X)$	Quot-scheme compactification of gauged maps	25
$\overline{\mathcal{M}}^G_{n,1}(\mathbb{A}, X)$	Scaled affine gauged maps	35
$\overline{\mathcal{M}}^G_{n,1}(C, X)$	Scaled gauged maps with domain C, n markings	34

References

[1] D. Abramovich and A. Vistoli. Compactifying the space of stable maps. *J. Amer. Math. Soc.*, 15(1):27–75 (electronic), 2002.

[2] D. Abramovich, T. Graber, and A. Vistoli. Gromov-Witten theory of Deligne-Mumford stacks. *Amer. J. Math.*, 130(5):1337–1398, 2008.

[3] D. Abramovich, M. Olsson, and A. Vistoli. Twisted stable maps to tame Artin stacks. *J. Algebraic Geom.*, 20(3):399–477, 2011.
[4] E. Arbarello, M. Cornalba, and P. A. Griffiths. *Geometry of algebraic curves. Volume II*, volume 268 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer, Heidelberg, 2011. With a contribution by Joseph Daniel Harris.
[5] K. Behrend. Gromov-Witten invariants in algebraic geometry. *Invent. Math.*, 127(3):601–617, 1997.
[6] K. Behrend and B. Fantechi. The intrinsic normal cone. *Invent. Math.*, 128(1):45–88, 1997.
[7] K. Behrend and Yu. Manin. Stacks of stable maps and Gromov-Witten invariants. *Duke Math. J.*, 85(1):1–60, 1996.
[8] B. Bhatt. A non-algebraic hom-stack. Note. http://www-personal.umich.edu/∼bhattb/math/hom-stack-example.pdf
[9] A. Bialynicki-Birula. On Homogeneous Affine Spaces of Linear Algebraic Groups. *American Journal of Mathematics*, 85: 577–582
[10] A. Borel. *Linear algebraic groups*. Springer-Verlag, New York, second edition, 1991.
[11] W. Chen and Y. Ruan. Orbifold Gromov-Witten theory. In *Orbifolds in mathematics and physics (Madison, WI, 2001)*, volume 310 of *Contemp. Math.*, pages 25–85. Amer. Math. Soc., Providence, RI, 2002.
[12] K. Behrend. Introduction to Algebraic stacks. In *Moduli Spaces*, London Mathematical Society Lecture Notes 441. Edited by Leticia Brambila Paz and Peter Newstead. 2014.
[13] I. Ciocan-Fontanine, B. Kim, and D. Maulik. Stable quasimaps to GIT quotients. *J. Geom. Phys.*, 75:17–47, 2014.
[14] J.-L. Colliot-Thélène and J.-J. Sansuc. Fibr es quadratiques et composantes connexes r eelles. *Math. Ann.*, 244(2):105–134, 1979.
[15] B. Conrad. The Keel-Mori theorem via stacks. http://math.stanford.edu/∼conrad/papers/coarsespace.pdf.
[16] J. de Jong et al. The Stacks Project.
[17] P. Deligne and D. Mumford. The irreducibility of the space of curves of given genus. *Inst. Hautes Études Sci. Publ. Math.*, (36):75–109, 1969.
[18] V. G. Drinfeld and C. Simpson. B-structures on G-bundles and local triviality. *Math. Res. Lett.*, 2(6):823–829, 1995.
[19] The Moduli Space in the Gauged Linear Sigma Model. H. Fan, T. Jarvis and Y. Ruan. arXiv:1603.02666.
[20] E. Frenkel, C. Teleman, A. J. Tolland. Gromov-Witten gauge theory I. arxiv:0904.4834.
[21] W. Fulton and R. Pandharipande. Notes on stable maps and quantum cohomology. In *Algebraic geometry—Santa Cruz 1995*, pages 45–96. Amer. Math. Soc., Providence, RI, 1997.
[22] A. R.-P. Gaio and D. A. Salamon. Gromov-Witten invariants of symplectic quotients and adiabatic limits. *J. Symplectic Geom.*, 3(1):55–159, 2005.
[23] A. B. Givental. Equivariant Gromov-Witten invariants. *Internat. Math. Res. Notices*, (13):613–663, 1996.
[24] E. González and C. Woodward. Quantum Witten localization and abelianization for qde solutions. arxiv:0811.3358.
[25] R. Hartshorne. Stable reflexive sheaves. *Math. Ann.*, 254(2):121–176, 1980.
[26] D. Huybrechts and M. Lehn. *The geometry of moduli spaces of sheaves*. Friedr. Vieweg & Sohn, Braunschweig, 1997.
[27] Wim H. Hesselink. Desingularizations of varieties of nullforms. *Invent. Math.*, 55(2):141–163, 1979.

[28] S. Keel and S. Mori. Quotients by groupoids. *Ann. of Math. (2)*, 145(1):193–213, 1997.

[29] F. C. Kirwan. *Cohomology of Quotients in Symplectic and Algebraic Geometry*, volume 31 of *Mathematical Notes*. Princeton Univ. Press, Princeton, 1984.

[30] M. Kontsevich. Enumeration of rational curves via torus actions. In *The moduli space of curves (Texel Island, 1994)*, pages 335–368. Birkhäuser Boston, Boston, MA, 1995.

[31] E. Lerman. Symplectic cuts. *Math. Res. Letters*, 2:247–258, 1995.

[32] B. H. Lian, K. Liu, and S.-T. Yau. Mirror principle. I. *Asian J. Math.*, 1(4):729–763, 1997.

[33] M. Lieblich. Remarks on the stack of coherent algebras. *Int. Math. Res. Not.*, pages Art. ID 75273, 12, 2006.

[34] D. Luna. Slices étales. In *Sur les groupes algbriques*, Bull. Soc. Math. France, Paris, Mmoire 33, Socit Mathmatique de France, pp. 81–105.

[35] S. Ma'u and C. Woodward. Geometric realizations of the multiplihedra. *Compos. Math.*, 146(4):1002–1028, 2010.

[36] Y. Matsushima. Espaces homogenes de Stein des groupes de Lie complexes. *Nagoya Math J.* 16:205–218, 1960.

[37] G. Laumon and L. Moret-Bailly. *Champs algébriques*, volume 39 of *Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics [Results in Mathematics and Related Areas. 3rd Series. A Series of Modern Surveys in Mathematics]*. Springer-Verlag, Berlin, 2000.

[38] D. R. Morrison and M. R. Plesser. Summing the instantons: quantum cohomology and mirror symmetry in toric varieties. *Nuclear Phys. B*, 440(1-2):279–354, 1995.

[39] D. Mumford, J. Fogarty, and F. Kirwan. *Geometric Invariant Theory*, volume 34 of *Ergebnisse der Mathematik und ihrer Grenzgebiete, 2. Folge*. Springer-Verlag, Berlin-Heidelberg-New York, third edition, 1994.

[40] I. Mundet i Riera. A Hitchin-Kobayashi correspondence for Kähler fibrations. *J. Reine Angew. Math.*, 528:41–80, 2000.

[41] L. Ness. A stratification of the null cone via the moment map. *Amer. J. Math.*, 106(6):1281–1329, 1984. with an appendix by D. Mumford.

[42] M. C. Olsson. (Log) twisted curves. *Compos. Math.*, 143(2):476–494, 2007.

[43] A. Ott. Removal of singularities and Gromov compactness for symplectic vortices. *J. Symplectic Geom.* 12:257–311, 2014.

[44] M. Popa and M. Roth. Stable maps and Quot schemes. *Invent. Math.*, 152(3):625–663, 2003.

[45] A. Ramanathan. Moduli for principal bundles over algebraic curves. I. *Proc. Indian Acad. Sci. Math. Sci.*, 106(3):301–328, 1996.

[46] A. Schmitt. A universal construction for moduli spaces of decorated vector bundles over curves. *Transform. Groups*, 9(2):167–209, 2004.

[47] A. Schmitt. *Geometric invariant theory and decorated principal bundles*. Zurich Lectures in Advanced Mathematics. European Mathematical Society (EMS), Zürich, 2008.

[48] J. Stasheff. *H-spaces from a homotopy point of view*. Lecture Notes in Mathematics, Vol. 161. Springer-Verlag, Berlin, 1970.

[49] C. Teleman. The quantization conjecture revisited. *Ann. of Math. (2)*, 152(1):1–43, 2000.

[50] M. Thaddeus. Geometric invariant theory and flips. *J. Amer. Math. Soc.*, 9(3):691–723, 1996.

[51] G. Tian and G. Xu. Virtual cycles of gauged Witten equation. arXiv:1602.07638

[52] S. Venugopalan and C. Woodward. Classification of affine vortices. arXiv:1301.7052. To appear in *Duke Math. Journal*.

[53] Chris T. Woodward. Quantum Kirwan morphism and Gromov-Witten invariants of quotients III. Transformation Groups 1–39. First online: 21 September 2015. arXiv:1408.5869.

[54] Chris T. Woodward. Quantum Kirwan morphism and Gromov-Witten invariants of quotients II. Transformation Groups 20 (2015) 881–920. arXiv:1408.5864.

[55] Chris T. Woodward. Quantum Kirwan morphism and Gromov-Witten invariants of quotients I. Transformation Groups 20 (2015) 507–556. arXiv:1204.1765.

[56] Guanbo Xu. Moduli space of twisted holomorphic maps with Lagrangian boundary condition: compactness arXiv:1202.4096.

[57] F. Ziltener. *A Quantum Kirwan Map: Bubbling and Fredholm Theory for Symplectic Vortices over the Plane*, volume 230 of *Mem. Amer. Math. Soc.* no. 1082, 2014.

Department of Mathematics, University of Massachusetts Boston, 100 William T. Morrissey Boulevard, Boston, MA 02125, U.S.A.

E-mail address: eduardo@math.umb.edu

Department of Mathematics, California Institute of Technology, 1200 East California Boulevard, Pasadena, CA 91125, U.S.A.

E-mail address: pablos@caltech.edu

Mathematics-Hill Center, Rutgers University, 110 Frelinghuysen Road, Piscataway, NJ 08854-8019, U.S.A.

E-mail address: ctw@math.rutgers.edu