QUILTED FLOER COHOMOLOGY

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Abstract. We generalize Lagrangian Floer cohomology to sequences of Lagrangian correspondences. For sequences related by the geometric composition of Lagrangian correspondences we establish an isomorphism of the Floer cohomologies. We give applications to calculations of Floer cohomology, displaceability of Lagrangian correspondences, and transfer of displaceability under geometric composition.

Contents

1. Introduction

Lagrangian Floer cohomology associates to a pair of Lagrangian manifolds a chain complex whose differential counts pseudoholomorphic strips with boundary values in the given Lagrangians. In this paper we generalize Floer cohomology to include Lagrangian correspondences. Recall that if (M_0, ω_0) and (M_1, ω_1) are symplectic manifolds, then a *Lagrangian correspondence* L_{01} from M_0 to M_1 is a Lagrangian submanifold $L_{01} \subset M_0^- \times M_1$, where $M_0^- := (M_0, -\omega_0)$. These were introduced by Weinstein [26] in an attempt to create a symplectic category with morphisms between not necessarily symplectomorphic manifolds. So we also denote a Lagrangian correspondence by $M_0 \xrightarrow{L_{01}} M_1$. With this notation we can

view a pair of Lagrangian submanifolds $L, L' \subset M$ as sequence of Lagrangian correspondences pt $\xrightarrow{L}$ M $\xrightarrow{L'}$ pt from the point via M back to the point. This is a special case of a *cyclic sequence of Lagrangian correspondences*

$$M_0 \xrightarrow{L_{01}} M_1 \xrightarrow{L_{12}} M_2 \ \dots M_r \xrightarrow{L_{r(r+1)}} M_{r+1} = M_0,$$

for which we will define a *quilted Floer cohomology*

$$HF(L_{01}, L_{12}, \dots, L_{r(r+1)}). \tag{1}$$

The quilted differential counts tuples of pseudoholomorphic strips $(u_j : \mathbb{R}\times[0,1] \to M_j)_{j=0,\dots,r}$ whose boundaries match up via the Lagrangian correspondences, $(u_j(s,1), u_{j+1}(s,0)) \in L_{j(j+1)}$ for $j = 0, \dots, r$. These tuples are examples of pseudoholomorphic quilts with the strips thought of as patches and the boundary matching conditions thought of as seams. The theory of quilts is developed in higher generality in [22].

In this paper, we next investigate the effect of geometric composition on Floer cohomology. The *geometric composition* of two Lagrangian correspondences $L_{01} \subset M_0^- \times M_1$, $L_{12} \subset M_1^- \times M_2$ is

$$L_{01} \circ L_{12} := \big\{(x_0, x_2) \in M_0 \times M_2 \,\big|\, \exists x_1 : (x_0, x_1) \in L_{01}, (x_1, x_2) \in L_{12}\big\}. \tag{2}$$

In general, this will be a singular subset of $M_0^- \times M_2$, with isotropic tangent spaces. However, if we assume transversality of the intersection $L_{01} \times_{M_1} L_{12} := \big(L_{01} \times L_{12}\big) \cap \big(M_0^- \times \Delta_{M_1} \times M_2\big)$, then the restriction of the projection $\pi_{02} : M_0^- \times M_1 \times M_1^- \times M_2 \to M_0^- \times M_2$ to $L_{01} \times_{M_1} L_{12}$ is automatically an immersion. We will study the class of *embedded* geometric compositions, for which in addition π_{02} is injective, and hence $L_{01} \circ L_{12}$ is a smooth Lagrangian correspondence. If the composition $L_{(\ell-1)\ell} \circ L_{\ell(\ell+1)}$ is embedded, then we obtain under suitable monotonicity assumptions a canonical isomorphism

$$HF(\dots, L_{(\ell-1)\ell}, L_{\ell(\ell+1)}, \dots) \cong HF(\dots, L_{(\ell-1)\ell} \circ L_{\ell(\ell+1)}, \dots). \tag{3}$$

For the precise monotonicity and admissibility conditions see Section 5.4. The proof proceeds in two steps. First, we allow for varying widths $(\delta_j > 0)_{j=0,\dots,k-1}$ of the pseudoholomorphic strips $(u_j : \mathbb{R} \times [0, \delta_j] \to M_j)_{j=0,\dots,k-1}$ defining the differential. Section 5.3 of this paper shows that Floer cohomology is independent of the choice of widths. (These domains are not conformally equivalent due to the identification between boundary components that is implicit in the seam conditions.) The second (hard analytic) part is to prove that with the width $\delta_\ell > 0$ sufficiently close to zero, the k-tuples of holomorphic strips with seam conditions in $(\dots, L_{(\ell-1)\ell}, L_{\ell(\ell+1)}, \dots)$ are in one-to-one correspondence with the $(k-1)$-tuples of holomorphic strips with seam conditions in $(\dots, L_{(\ell-1)\ell} \circ L_{\ell(\ell+1)}, \dots)$. This analysis is completely analogous to [21], where we establish the bijection for the Floer trajectories of the special cyclic sequence pt $\xrightarrow{L_0}$ M_0 $\xrightarrow{L_{01}}$ M_1 $\xrightarrow{L_{12}}$ M_2 $\xrightarrow{L_2}$ pt when $\delta_1 \to 0$. The monotonicity assumptions are crucial for this part since the exclusion of a novel "figure eight bubble" in [21] hinges on a strict energy-index proportionality.

In section 6 we provide a number of new tools for the calculation of Floer cohomology (and hence detection of non-displaceability), arising as direct consequences of (3) or from a conjectural generalization of Perutz' long exact Gysin sequence [13]. As first specific example we confirm the calculation $HF(T^n_{\mathrm{Cl}}, T^n_{\mathrm{Cl}}) \cong H_*(T^n)$ of Cho [3] for the Clifford torus in $\mathbb{CP}^n$, and we calculate some further Floer cohomologies in CP^n using reduction at pairs of transverse level sets. Next, we prove Hamiltonian non-displaceability of the Lagrangian

3-sphere $\Sigma \subset (\mathbb{CP}^1)^- \times \mathbb{CP}^2$ arising from reduction at the level set of an S^1 action on $\mathbb{CP}^2$ containing T_{Cl}. The latter follows from the nontriviality of $HF(T^2_{\mathrm{Cl}}, T^2_{\mathrm{Cl}})$ together with our isomorphism

$$HF(S^1 \times T_{\mathrm{Cl}}, \Sigma) \cong HF(T_{\mathrm{Cl}}, T_{\mathrm{Cl}}).$$

Finally, we generalize this non-displaceability result to certain Lagrangian embeddings $\Sigma \subset (\mathbb{CP}^{k-1})^- \times \mathbb{CP}^n$ of $(S^1)^{n-k} \times S^{2k-1}$ for any $2 \leq k \leq n$.

Another consequence of our results is a general prescription for defining topological invariants by decomposing into simple pieces. For example, let Y be a compact manifold and $f : Y \to \mathbb{R}$ a Morse function giving a decomposition $Y = Y_{01} \cup \ldots \cup Y_{(k-1)k}$ into simple cobordisms by cutting along non-critical level sets $X_1, \ldots, X_{k-1}$. First one associates to each X_j a monotone symplectic manifold $M(X_j)$, and to each $Y_{(j-1)j}$ with $\partial Y_{(j-1)j} = X_{j-1}^- \sqcup X_j$ a smooth monotone Lagrangian correspondence $L(Y_{(j-1)j}) \subset M(X_{j-1})^- \times M(X_j)$ (taking $M(X_0)$ and $M(X_k)$ to be points.) Second, one checks that the basic moves described by Cerf theory (cancellation or change of order of critical points) change the sequence of Lagrangian correspondences by replacing adjacent correspondences with an embedded composition, or vice-versa. In other words, the equivalence class of sequences of Lagrangian correspondences by embedded compositions $[L(Y_{01}), \ldots, L(Y_{(k-1)k})]$ does not depend on the choice of the Morse function f. Then the results of this paper provide a group-valued invariant of Y, by taking the Floer homology of the sequence of Lagrangian correspondences. For example, in [24] we investigate the theory which uses as symplectic manifolds the moduli spaces of flat bundles with compact structure group on three-dimensional cobordisms containing tangles.

Notation and Organization: We will frequently refer to the standing assumptions (M1-2), (L1-3), and (G1-2) that can be found on pages 19 - 20.

Section 2 is a detailed introduction to Lagrangian correspondences, geometric composition, and sequences of correspondences, which also provides the basic framework for the sequels [20, 22] to this paper. In Section 3 we generalize gradings to Lagrangian correspondences and establish their behaviour under geometric composition, so that the isomorphism (3) becomes an isomorphism of graded groups. Section 4 provides a review of monotonicity and Floer cohomology and gives a first definition of the Floer cohomology (1) by building a pair of Lagrangians in the product $M_0 \times M_1 \times \ldots M_{k-1}$. The latter is however unsatisfactory since it does not provide an approach to the isomorphism (3). Section 5 gives the general definition of quilted Floer cohomology (1) and finalizes the proof of the isomorphism (3). Finally, Section 6 gives a number of direct symplectic applications of the isomorphism (3).

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2. Lagrangian correspondences

Let M be a smooth manifold. A *symplectic form* on M is a closed, non-degenerate two-form ω. A *symplectic manifold* is a smooth manifold equipped with a symplectic form. If (M_1, ω_1) and (M_2, ω_2) are symplectic manifolds, then a diffeomorphism $\varphi : M_1 \to M_2$ is a *symplectomorphism* if $\varphi^* \omega_2 = \omega_1$. Let Symp denote the category whose objects are symplectic manifolds and whose morphisms are symplectomorphisms. The following operations give Symp a structure similar to that of a tensor category.

(a) (Duals) If $M = (M, \omega)$ is a symplectic manifold, then $M^- = (M, -\omega)$ is a symplectic manifold, called the *dual* of M.

(b) (Sums) If $M_j = (M_j, \omega_j), j = 1, 2$ are symplectic manifolds, then the disjoint union $M_1 \cup M_2$ equipped with the symplectic structure ω_1 on M_1 and ω_2 on M_2, is a symplectic manifold. The empty set $\emptyset$ is a unit for the disjoint union.

(c) (Products) Let $M_j = (M_j, \omega_j), j = 1, 2$ be symplectic manifolds, then the Cartesian product $(M_1 \times M_2, \pi_1^* \omega_1 + \pi_2^* \omega_2)$ is a symplectic manifold. (Here $\pi_j : M_1 \times M_2 \to M_j$ denotes the projections.) The symplectic manifold pt, consisting of a single point, is a unit for the Cartesian product.

Clearly the notion of symplectomorphism is very restrictive; in particular, the symplectic manifolds must be of the same dimension. A more flexible notion of morphism is that of Lagrangian correspondence, defined as follows [27, 26, 7]. Let $M = (M, \omega)$ be a symplectic manifold. A submanifold $L \subset M$ is *isotropic*, resp. *coisotropic*, resp. *Lagrangian* if the ω-orthogonal complement TL^ω satisfies $TL^\omega \subseteq TL$ resp. $TL^\omega \supseteq TL$ resp. $TL^\omega = TL$.

Definition 2.0.1. Let M_1, M_2 be symplectic manifolds. A *Lagrangian correspondence from* M_1 *to* M_2 is a Lagrangian submanifold $L_{12} \subset M_1^- \times M_2$.

Example 2.0.2. The following are examples of Lagrangian correspondences:

(a) (Trivial correspondence) The one and only Lagrangian correspondence between $M_1 = \emptyset$ and any other M_2 is $L_{12} = \emptyset$.

(b) (Lagrangians) Any Lagrangian submanifold $L \subset M$ can be viewed both as correspondence $L \subset \mathrm{pt}^- \times M$ from the point to M and as correspondence $L \subset M^- \times \mathrm{pt}$ from M to the point.

(c) (Graphs) If $\varphi_{12} : M_1 \to M_2$ is a symplectomorphism then its graph

$$\mathrm{graph}(\varphi_{12}) = \{(m_1, \varphi_{12}(m_1)) \mid \ m_1 \in M_1\} \subset M_1^- \times M_2$$

is a Lagrangian correspondence.

(d) (Fibered coisotropics) Suppose that $\iota : C \to M$ is a coisotropic submanifold. Then the null distribution TC^ω is integrable, see e.g. [10, Lemma 5.30]. Suppose that TC^ω is in fact fibrating, that is, there exists a symplectic manifold (B, ω_B) and a fibration $\pi : C \to B$ such that $\iota^* \omega$ is the pull-back $\pi^* \omega_B$. Then

$$(\iota \times \pi) : C \to M^- \times B$$

maps C to a Lagrangian correspondence.

(e) (Level sets of moment maps) Let G be a Lie group with Lie algebra $\mathfrak{g}$. Suppose that G acts on M by Hamiltonian symplectomorphisms generated by a moment map $\mu : M \to \mathfrak{g}^*$. (That is μ is equivariant and the generating vector fields $\mathfrak{g} \to \mathrm{Vect}(M), \xi \mapsto \xi_M$ satisfy $\iota(\xi_M)\omega = -d(\mu, \xi)$.) If G acts freely on $\mu^{-1}(0)$, then $\mu^{-1}(0)$ is a smooth coisotropic fibered over the symplectic quotient $M/\!\!/G = \mu^{-1}(0)/G$, which is a symplectic manifold. Hence we have a Lagrangian correspondence

$$(\iota \times \pi) : \ \mu^{-1}(0) \to M^- \times (M/\!\!/G).$$

The symplectic two-form $\omega_{M/\!\!/G}$ on $M/\!\!/G$ is the unique form on $M/\!\!/G$ satisfying $\pi^* \omega_{M/\!\!/G} = \iota^* \omega$.

Definition 2.0.3. Let M_0, M_1, M_2 be symplectic manifolds and $L_{01} \subset M_0^- \times M_1$, $L_{12} \subset M_1^- \times M_2$ Lagrangian correspondences.

(a) The *dual* Lagrangian correspondence of L_{01} is

$$(L_{01})^t := \{(m_1, m_0) \mid (m_0, m_1) \in L_{01}\} \subset M_1^- \times M_0.$$

(b) The *geometric composition* of L_{01} and L_{12} is

$$L_{01} \circ L_{12} := \left\{ (m_0, m_2) \in M_0^- \times M_2 \;\middle|\; \exists m_1 \in M_1 : \begin{array}{c} (m_0, m_1) \in L_{01} \\ (m_1, m_2) \in L_{12} \end{array} \right\} \subset M_0^- \times M_2.$$

Geometric composition and duals of Lagrangian correspondences satisfy the following:

(a) (Graphs) If $\varphi_{01} : M_0 \to M_1$ and $\varphi_{12} : M_1 \to M_2$ are symplectomorphisms, then

$$\mathrm{graph}(\varphi_{01}) \circ \mathrm{graph}(\varphi_{12}) = \mathrm{graph}(\varphi_{12} \circ \varphi_{01}),$$

$$\mathrm{graph}(\varphi_{01})^t = \mathrm{graph}(\varphi_{01}^{-1}).$$

(b) (Zero) Composition with $\emptyset$ always yields $\emptyset$, that is for any Lagrangian correspondence $L_{01} \subset M_0^- \times M_1$ we have

$$\emptyset \circ L_{01} = \emptyset, \qquad L_{01} \circ \emptyset = \emptyset.$$

(c) (Identity) If $L_{01} \subset M_0^- \times M_1$ is a Lagrangian correspondence and $\Delta_j \subset M_j^- \times M_j, j = 0, 1$ are the diagonals, then

$$L_{01} = \Delta_0 \circ L_{01} = L_{01} \circ \Delta_1.$$

(d) (Associativity) If $L_{01} \subset M_0^- \times M_1, L_{12} \subset M_1^- \times M_2, L_{23} \subset M_2^- \times M_3$ are Lagrangian correspondences, then

$$(L_{01} \circ L_{12}) \circ L_{23} = L_{01} \circ (L_{12} \circ L_{23}),$$

$$(L_{01} \circ L_{12})^t = (L_{12})^t \circ (L_{01})^t.$$

The geometric composition can equivalently be defined as $L_{01} \circ L_{12} = \pi_{02}(L_{01} \times_{M_1} L_{12})$, the image under the projection $\pi_{02} : M_0^- \times M_1 \times M_1^- \times M_2 \to M_0^- \times M_2$ of

$$L_{12} \times_{M_1} L_{01} := (L_{01} \times L_{12}) \cap (M_0^- \times \Delta_1 \times M_2).$$

Here $\Delta_1 \subset M_1^- \times M_1$ denotes the diagonal. $L_{01} \circ L_{12} \subset M_0^- \times M_2$ is an immersed Lagrangian submanifold if $L_{01} \times L_{12}$ intersects $M_0^- \times \Delta_1 \times M_2$ transversally. In general, the geometric composition of smooth Lagrangian submanifolds may not even be immersed. We will be working with the following class of compositions, for which the resulting Lagrangian correspondence is in fact a smooth submanifold, as will be seen in Lemma 2.0.5 below.

Definition 2.0.4. We say that the composition $L_{01} \circ L_{12}$ is *embedded* if $L_{12} \times_{M_1} L_{01}$ is cut out transversally (i.e. $(L_{01} \times L_{12}) \pitchfork (M_0^- \times \Delta_1 \times M_2)$) and the projection $\pi_{02} : L_{12} \times_{M_1} L_{01} \to L_{01} \circ L_{12} \subset M_0^- \times M_2$ is injective.

By some authors (e.g. [7]) geometric composition of Lagrangian correspondences is more generally defined under clean intersection hypotheses. This extension is not needed in the present paper, because the quilted Floer cohomology is invariant under Hamiltonian isotopy, and after such an isotopy transversality may always be achieved. However, transverse intersection only yields an immersed[1] Lagrangian correspondence, as the following Lemma shows.

[1]One can not necessarily remove all self-intersections of the immersed composition by Hamiltonian isotopy on one correspondence. A basic example is the composition of transverse Lagrangian submanifolds $L, L' \subset M$. Identifying $M \cong M \times \{\mathrm{pt}\} \cong \{\mathrm{pt}\} \times M$ the projection $L \times_M L' \to L \circ L' \subset \{\mathrm{pt}\} \times \{\mathrm{pt}\}$ maps the (finite) intersection $L \pitchfork L'$ to a point.

Lemma 2.0.5. *Let $L_{01} \subset M_0^- \times M_1$, $L_{12} \subset M_1^- \times M_2$ be Lagrangian correspondences such that the intersection $(L_{01} \times L_{12}) \pitchfork (M_0^- \times \Delta_1 \times M_2)$ is transverse. Then the projection $\pi_{02} : L_{12} \times_{M_1} L_{01} \to L_{01} \circ L_{12} \subset M_0^- \times M_2$ is an immersion.*

In particular, if the intersection is transverse and the projection is injective, then the composition $L_{01} \circ L_{12} =: L_{02}$ is embedded.

Proof. This is a consequence of the fact that the geometric composition of linear Lagrangian correspondences is always well defined (i.e. yields another linear Lagrangian correspondence). Fix a point $\underline{x} = (x_0, x_1, x_1, x_2) \in L_{01} \times_{M_1} L_{12}$ then we need to check that $\ker \mathrm{d}_{\underline{x}}\pi_{02} = \{0\}$ for the projection restricted to $L_{12} \times_{M_1} L_{01}$. In fact, we will show that

$$\ker \mathrm{d}_{\underline{x}}\pi_{02} \cong \frac{T_{\underline{x}}(M_0 \times M_1 \times M_1 \times M_2)}{(T_{(x_0,x_1)}L_{01} \times T_{(x_1,x_2)}L_{12}) + (T_{x_0}M_0 \times T_{(x_1,x_1)}\Delta_1 \times T_{x_2}M_2)}, \tag{4}$$

which is zero by transversality. To simplify notation we abbreviate $\Lambda_{01} := T_{(x_0,x_1)}L_{01}$, $\Lambda_{12} := T_{(x_1,x_2)}L_{12}$, and $V_i := T_{x_i}M_i$. Now (4) follows as in [7, Section 4.1]. For completeness we recall the precise argument: We identify

$$\begin{aligned} \ker \mathrm{d}_{\underline{x}}\pi_{02} &= (\Lambda_{01} \times_{V_1} \Lambda_{12}) \cap (\{0\} \times V_1 \times V_1 \times \{0\}) \\ &\cong \left\{ v_1 \in V_1 \,\middle|\, (0, v_1) \in \Lambda_{01}, (v_1, 0) \in \Lambda_{12} \right\} \;=\; \ker \Lambda_{01}^t \cap \ker \Lambda_{12}, \end{aligned} \tag{5}$$

where $\ker \Lambda_{12} := \{v_1 \in V_1 \,|\, (v_1, 0) \in \Lambda_{12}\} \subset V_1$ and similarly $\ker \Lambda_{01}^t \subset V_1$. On the other hand, we use the symplectic complements with respect to $\omega_{0112} := (-\omega_0) \oplus \omega_1 \oplus (-\omega_1) \oplus \omega_2$ on $V_0 \times V_1 \times V_1 \times V_2$ to identify

$$\begin{aligned} &(V_0 \times V_1 \times V_1 \times V_2) \,/\, (\Lambda_{01} \times \Lambda_{12}) + (V_0 \times \Delta_{V_1} \times V_2) \\ &\quad\cong (\Lambda_{01} \times \Lambda_{12})^{\omega_{0112}} \cap (V_0 \times \Delta_{V_1} \times V_2)^{\omega_{0112}} \\ &\quad= \left\{ (0, v_1, v_1, 0) \,\middle|\, (0, v_1) \in \Lambda_{01}^{(-\omega_0)\oplus\omega_1}, (v_1, 0) \in \Lambda_{12}^{(-\omega_1)\oplus\omega_2} \right\} \\ &\quad\cong (\operatorname{im} \Lambda_{01})^{\omega_1} \cap (\operatorname{im} \Lambda_{12}^t)^{\omega_1}, \end{aligned} \tag{6}$$

where $\operatorname{im} \Lambda_{01} := \pi_{V_1}(\Lambda_{01}) \subset V_1$, similarly $\operatorname{im} \Lambda_{12}^t \subset V_1$, and we used the equivalence

$$(-\omega_0 \oplus \omega_1)((0, v_1), (\lambda_0, \lambda_1)) = 0 \;\; \forall (\lambda_0, \lambda_1) \in \Lambda_{01} \quad \Leftrightarrow \quad \omega_1(v_1, \lambda_1) = 0 \;\; \forall \lambda_1 \in \pi_{V_1}(\Lambda_{01}).$$

Now the two vector spaces in (5) and (6) are identified by the dualities $\ker \Lambda_{01}^t = (\operatorname{im} \Lambda_{01})^{\omega_1}$ and $\ker \Lambda_{12} = (\operatorname{im} \Lambda_{12}^t)^{\omega_1}$, which follow from the Lagrangian property of Λ_{01} resp. Λ_{12}^t,

$$(0, v_1) \in \Lambda_{01} \quad \Leftrightarrow \quad (0, v_1) \in \Lambda_{01}^{-\omega_0 \oplus \omega_1} \quad \Leftrightarrow \quad v_1 \in (\operatorname{im} \Lambda_{01})^{\omega_1}.$$

This proves (4) and hence finishes the proof of the Lemma. □

Remark 2.0.6. Suppose that the composition $L_{01} \circ L_{12} =: L_{02}$ is embedded.

(a) By the injectivity, for every $(x_0, x_2) \in L_{02}$ there is a unique solution $x_1 \in M_1$ to $(x_0, x_1, x_1, x_2) \in L_{01} \times L_{12}$. Due to the transversality assumption, this solution is given by a smooth map $\ell_1 : L_{02} \to M_1$.

(b) If L_{01} and L_{12} are compact, oriented, and equipped with a relative spin structure, then L_{02} is also compact and inherits an orientation and relative spin structure, see [23]. The orientation is induced from the canonical orientation of the diagonal, see Remark 3.0.5(b), and the splitting

$$T(M_0 \times M_2 \times M_1 \times M_1) = \big(TL_{02} \times \{0\}\big) \oplus \big(\{0\} \times (T\Delta_1)^\perp\big) \oplus T(L_{01} \times L_{12})^\perp.$$

(c) If $\pi_1(L_{01})$ and $\pi_1(L_{12})$ are torsion, then $\pi_1(L_{02})$ is torsion. If moreover M_0 and M_2 are monotone with the same monotonicity constant, then L_{02} is monotone, see Section 4.1.

2.1. **Generalized Lagrangian correspondences.** A simple resolution of the composition problem is given by passing to sequences of Lagrangian correspondences and composing them by concatenation. In [20] we employ these to define a symplectic category containing all smooth Lagrangian correspondences as composable morphisms, yet retaining geometric composition in cases where it is well defined.

Definition 2.1.1. Let M, M' be symplectic manifolds. A *generalized Lagrangian correspondence* $\underline{L}$ from M to M' consists of

(a) a sequence $N_0, \ldots, N_r$ of any length $r+1 \geq 2$ of symplectic manifolds with $N_0 = M$ and $N_r = M'$,
(b) a sequence $L_{01}, \ldots, L_{(r-1)r}$ of compact Lagrangian correspondences with $L_{(j-1)j} \subset N_{j-1}^- \times N_j$ for $j = 1, \ldots, r$.

Definition 2.1.2. Let $\underline{L}$ from M to M' and $\underline{L}'$ from M' to M'' be two generalized Lagrangian correspondences. Then we define composition

$$(\underline{L}, \underline{L}') := \left(L_{01}, \ldots, L_{(r-1)r}, L'_{01}, \ldots, L'_{(r'-1)r'}\right)$$

as a generalized Lagrangian correspondence from M to M''. Moreover, we define the dual

$$\underline{L}^t := \left(L^t_{(r-1)r}, \ldots, L^t_{01}\right).$$

as a generalized Lagrangian correspondence from M' to M.

We conclude this subsection by mentioning special cases of generalized Lagrangian correspondences. The first is the case $M = M'$, which we will want to view separately as a cyclic correspondence, without fixing the "base point" M.

Definition 2.1.3. A *cyclic generalized Lagrangian correspondence* $\underline{L}$ consists of

(a) a cyclic sequence $M_0, M_1, \ldots, M_r, M_{r+1} = M_0$ of symplectic manifolds of any length $r+1 \geq 1$,
(b) a sequence $L_{01}, \ldots, L_{r(r+1)}$ of compact Lagrangian correspondences with $L_{j(j+1)} \subset M_j^- \times M_{j+1}$ for $j = 0, \ldots, r$.

The second special case is $M = \{pt\}$, which generalizes the concept of Lagrangian submanifolds. Namely, note that any Lagrangian submanifold $L \subset M'$ can be viewed as correspondence $L \subset \{pt\}^- \times M'$.

Definition 2.1.4. Let M' be a symplectic manifold. A *generalized Lagrangian submanifold* $\underline{L}$ of M' is a generalized Lagrangian correspondence from a point $M = \{pt\}$ to M'. That is, $\underline{L}$ consists of

(a) a sequence $N_{-r}, \ldots, N_0$ of any length $r \geq 0$ of symplectic manifolds with $N_{-r} = \{\text{pt}\}$ a point and $N_0 = M'$,
(b) a sequence $L_{(-r)(-r+1)}, \ldots, L_{(-1)0}$ of compact Lagrangian correspondences $L_{(i-1)i} \subset N_{i-1}^- \times N_i$.

3. Gradings

The purpose of this section is to review the theory of graded Lagrangians and extend it to generalized Lagrangian correspondences. It can be skipped at first reading.

Following Kontsevich and Seidel [15] one can define graded Lagrangian subspaces as follows. Let V be a symplectic vector space and let $\mathrm{Lag}(V)$ be the Lagrangian Grassmannian of V. An *N-fold Maslov covering for V* is a $\mathbb{Z}_N$-covering $\mathrm{Lag}^N(V) \to \mathrm{Lag}(V)$ associated to the Maslov class in $\pi_1(\mathrm{Lag}(V))$. A *grading* of a Lagrangian subspace $\Lambda \in \mathrm{Lag}(V)$ is a lift to $\tilde{\Lambda} \in \mathrm{Lag}^N(V)$.

Remark 3.0.5. (a) For any basepoint $\Lambda_0 \in \mathrm{Lag}(V)$ we obtain an N-fold Maslov cover $\mathrm{Lag}^N(V, \Lambda_0)$ given as the homotopy classes of paths $\tilde{\Lambda} : [0,1] \to \mathrm{Lag}(V)$ with base point $\tilde{\Lambda}(0) = \Lambda_0$, modulo loops of Maslov index N. The covering is $\tilde{\Lambda} \mapsto \tilde{\Lambda}(1)$. The base point has a canonical grading given by the constant path $\tilde{\Lambda}_0 \equiv \Lambda_0$. Any path between basepoints Λ_0, Λ_0' induces an identification $\mathrm{Lag}^N(V, \Lambda_0) \to \mathrm{Lag}^N(V, \Lambda_0')$.

(b) For the diagonal $\Delta \subset V^- \times V$ we fix a canonical grading and orientation as follows. We identify the Maslov coverings $\mathrm{Lag}^N(V^- \times V, \Lambda^- \times \Lambda)$ and $\mathrm{Lag}^N(V^- \times V, \Delta)$ by concatenation of the paths

$$(e^{Jt}\Lambda^- \times \Lambda)_{t\in[0,\pi/2]}, \quad (\{(tx + Jy, x + tJy) | x, y \in \Lambda\})_{t\in[0,1]}, \tag{7}$$

where $J \in \mathrm{End}(V)$ is an ω-compatible complex structure on V (i.e. $J^2 = -\mathrm{Id}$ and $\omega(\cdot, J\cdot)$ is symmetric and positive definite). In particular, this induces the *canonical grading* on the diagonal Δ with respect to any Maslov covering $\mathrm{Lag}^N(V^- \times V, \Lambda^- \times \Lambda)$, by continuation. Any identification $\mathrm{Lag}^N(V^- \times V, \Lambda_0^- \times \Lambda_0) \to \mathrm{Lag}^N(V^- \times V, \Lambda_1^- \times \Lambda_1)$ induced by a path in $\mathrm{Lag}^N(V)$ maps the graded diagonal to the graded diagonal, since the product $\gamma^- \times \gamma$ of any loop $\gamma : S^1 \to \mathrm{Lag}(V)$ has Maslov index 0. Similarly, we define a *canonical orientation* on Δ by choosing any orientation on Λ, giving the product $\Lambda^- \times \Lambda$ the product orientation (which is well defined), and extending the orientation over the path (7). This is related to the orientation induced by projection of the diagonal on the second factor by a sign $(-1)^{n(n-1)/2}$, where $\dim(M) = 2n$.

Let M be a symplectic manifold and let $\mathrm{Lag}(M) \to M$ be the fiber bundle whose fiber over $m \in M$ is the space $\mathrm{Lag}(T_mM)$ of Lagrangian subspaces of T_mM. An *N-fold Maslov covering* of M is an N-fold cover $\mathrm{Lag}^N(M) \to \mathrm{Lag}(M)$ whose restriction to each fiber is an N-fold Maslov covering $\mathrm{Lag}^N(T_mM) \to \mathrm{Lag}(T_mM)$. Any choice of Maslov cover for $\mathbb{R}^{2n}$ induces a one-to-one correspondence between N-fold Maslov covers of M and $\mathrm{Sp}^N(2n)$-structures on M. Here $2n = \dim M$ and $\mathrm{Sp}^N(2n)$ is the N-fold covering group of $\mathrm{Sp}(2n)$ associated to the Maslov class in $\pi_1(\mathrm{Sp}(2n))$. (Explicitly, this is realized by using the identity as base point.) An *$\mathrm{Sp}^N(2n)$-structure* on M is an $\mathrm{Sp}^N(2n)$-bundle $\mathrm{Fr}^N(M) \to M$ together with an isomorphism $\mathrm{Fr}^N(M) \times_{\mathrm{Sp}^N(2n)} \mathrm{Sp}(2n) \simeq \mathrm{Fr}(M)$ to the symplectic frame bundle of M. It induces the N-fold Maslov covering

$$\mathrm{Lag}^N(M) = \mathrm{Fr}^N(M) \times_{\mathrm{Sp}^N(2n)} \mathrm{Lag}^N(\mathbb{R}^{2n}).$$

Graded symplectic manifolds (i.e. equipped with Maslov coverings) form a structure similar to that of a tensor category, that is, the notions of duals, disjoint union, and Cartesian product extend naturally to the graded setting. The dual $\mathrm{Lag}^N(M^-)$ of a Maslov covering $\mathrm{Lag}^N(M) \to \mathrm{Lag}(M)$ is the same space with the inverted $\mathbb{Z}_N$-action. We denote this

identification by

$$\mathrm{Lag}^N(M) \to \mathrm{Lag}^N(M^-), \qquad \tilde{\Lambda} \mapsto \tilde{\Lambda}^-. \tag{8}$$

For Sp^N-structures $\mathrm{Fr}^N(M_0)$ and $\mathrm{Fr}^N(M_1)$ the embedding

$$\mathrm{Sp}^N(2n_0) \times_{\mathbb{Z}_N} \mathrm{Sp}^N(2n_1) \to \mathrm{Sp}^N(2n_0 + 2n_1)$$

induces an $\mathrm{Sp}^N(2n_0+2n_1)$-structure $\mathrm{Fr}^N(M_0 \times M_1)$ on the product and an equivariant map

$$\mathrm{Fr}^N(M_0) \times \mathrm{Fr}^N(M_1) \to \mathrm{Fr}^N(M_0 \times M_1) \tag{9}$$

covering the inclusion $\mathrm{Fr}(M_0) \times \mathrm{Fr}(M_1) \to \mathrm{Fr}(M_0 \times M_1)$. The corresponding product of N-fold Maslov covers on $M_0 \times M_1$ is the N-fold Maslov covering

$$\mathrm{Lag}^N(M_0 \times M_1) := \big(\mathrm{Fr}^N(M_0) \times \mathrm{Fr}^N(M_1)\big) \times_{\mathrm{Sp}^N(2n_0)\times\mathrm{Sp}^N(2n_1)} \mathrm{Lag}^N(\mathbb{R}^{2n_0} \times \mathbb{R}^{2n_1}).$$

Combining this product with the dual yields a Maslov covering for $M_0^- \times M_1$ which we can identify with

$$\mathrm{Lag}^N(M_0^- \times M_1) = \big(\mathrm{Fr}^N(M_0) \times \mathrm{Fr}^N(M_1)\big) \times_{\mathrm{Sp}^N(2n_0)\times\mathrm{Sp}^N(2n_1)} \mathrm{Lag}^N(\mathbb{R}^{2n_0,-} \times \mathbb{R}^{2n_1}).$$

Finally, the inclusion $\mathrm{Lag}(M_0) \times \mathrm{Lag}(M_1) \to \mathrm{Lag}(M_0 \times M_1)$ lifts to a map

$$\mathrm{Lag}^N(M_0) \times \mathrm{Lag}^N(M_1) \to \mathrm{Lag}^N(M_0 \times M_1), \quad (\tilde{L}_0, \tilde{L}_1) \mapsto \tilde{L}_0 \times^N \tilde{L}_1 \tag{10}$$

with fiber $\mathbb{Z}_N$. It is defined by combining the product (9) with the basic product of the linear Maslov cover $\mathrm{Lag}^N(\mathbb{R}^{2n_0}) \times \mathrm{Lag}^N(\mathbb{R}^{2n_1}) \to \mathrm{Lag}^N(\mathbb{R}^{2n_0} \times \mathbb{R}^{2n_1})$.

Definition 3.0.6. (a) Let M_0, M_1 be two symplectic manifolds equipped with N-fold Maslov covers and let $\phi : M_0 \to M_1$ be a symplectomorphisms. A *grading* of ϕ is a lift of the canonical isomorphism $\mathrm{Lag}(M_0) \to \mathrm{Lag}(M_1)$ to an isomorphism $\phi^N : \mathrm{Lag}^N(M_0) \to \mathrm{Lag}^N(M_1)$, or equivalently, a lift of the canonical isomorphism $\mathrm{Fr}(M_0) \to \mathrm{Fr}(M_1)$ of symplectic frame bundles to an isomorphism $\mathrm{Fr}^N(M_0) \to \mathrm{Fr}^N(M_1)$.

(b) Let $L \subset M$ be a Lagrangian submanifold and M be equipped with an N-fold Maslov cover. A *grading* of L is a lift $\sigma_L^N : L \to \mathrm{Lag}^N(M)$ of the canonical section $\sigma_L : L \to \mathrm{Lag}(M)$.

Remark 3.0.7. (a) The set of graded symplectomorphisms forms a group under composition. In particular, the identity on M has a canonical grading, given by the identity on $\mathrm{Lag}^N(M)$.

(b) Given a one-parameter family ϕ_t of symplectomorphisms with $\phi_0 = \mathrm{Id}_M$, we obtain a grading of ϕ_t by continuity.

(c) Any choice of grading on the diagonal $\tilde{\Delta} \in \mathrm{Lag}^N(\mathbb{R}^{2n,-} \times \mathbb{R}^{2n})$ induces a bijection between gradings of a symplectomorphism $\phi : M_0 \to M_1$ and gradings of its graph $\mathrm{graph}(\phi) \subset M_0^- \times M_1$ with respect to the induced Maslov cover $\mathrm{Lag}^N(M_0^- \times M_1)$. Indeed, the graph of the grading, $\mathrm{graph}(\phi^N) \subset (\mathrm{Fr}^N(M_0) \times \mathrm{Fr}^N(M_1))|_{\mathrm{graph}(\phi)}$ is a principal bundle over $\mathrm{graph}(\phi)$ with structure group $\mathrm{Sp}^N(2n)$, $2n = \dim M_0 = \dim M_1$. The graded diagonal descends under the associated fiber bundle construction with $\mathrm{graph}(\phi^N)$ to a section of $\mathrm{Lag}^N(M_0^- \times M_1)|_{\mathrm{graph}(\phi)}$ lifting $\mathrm{graph}(\phi)$. Moreover, this construction is equivariant for the transitive action of $H^0(M_0, \mathbb{Z}_N)$ on both the set of gradings of ϕ and the set of gradings of $\mathrm{graph}(\phi)$.

We will refer to this as the *canonical bijection* when using the canonical grading $\tilde{\Delta} \in \mathrm{Lag}^N(\mathbb{R}^{2n,-} \times \mathbb{R}^{2n})$ in Remark 3.0.5. In particular, the diagonal in $M^- \times M$ has

a canonical grading induced by the canonical bijection from the canonical grading of the identity on M.

(d) Any grading σ_L^N of a Lagrangian submanifold $L \subset M$ induces a grading of $L \subset M^-$ via the diffeomorphism $\mathrm{Lag}^N(M^-) \to \mathrm{Lag}^N(M)$.

(e) Given graded Lagrangian submanifolds $L_0 \subset M_0, L_1 \subset M_1$, the product $L_0 \times L_1 \subset M_0 \times M_1$ inherits a grading from (10).

(f) Given a graded symplectomorphism $\phi : M_0 \to M_1$ and a graded Lagrangian submanifold $L \subset M_0$, the image $\phi(L) \subset M_1$ inherits a grading by composition $\sigma^N_{\phi(L)} = \phi^N \circ \sigma^N_L$.

Example 3.0.8. (a) Let $\mathrm{Lag}^2(M)$ be the bundle whose fiber over m is the space of oriented Lagrangian subspaces of T_mM. Then $\mathrm{Lag}^2(M) \to \mathrm{Lag}(M)$ is a 2-fold Maslov covering. A $\mathrm{Lag}^2(M)$-grading of a Lagrangian $L \subset M$ is equivalent to an orientation on L.

(b) By [15, Section 2], any symplectic manifold M with $H^1(M) = 0$ and minimal Chern number N_M admits an N-fold Maslov covering $\mathrm{Lag}^N(M)$ iff N divides $2N_M$. Any Lagrangian with minimal Maslov number N_L admits a $\mathrm{Lag}^N(M)$-grading iff N divides N_L. In particular, if L is simply connected, then $N_L = 2N_M$ and L admits a $\mathrm{Lag}^{2N_M}(M)$ grading.

(c) Suppose that $[\omega]$ is integral, $[\omega] = (1/l)c_1(TM)$, and $\mathcal{L}$ is a line bundle with connection ∇ and curvature $\mathrm{curv}(\nabla) = (2\pi/i)\omega$. This induces a $2l$-fold Maslov cover $\mathrm{Lag}^{2l}(M) \to \mathrm{Lag}(M)$, see [15, Section 2b]. Let $L \subset M$ be a Bohr-Sommerfeld monotone Lagrangian as in Remark 4.1.4. A grading of $\mathcal{L}$ is equivalent to a choice of (not necessarily horizontal) section of $\mathcal{L}|L$ whose l-th tensor power is $\phi_L^{\mathcal{K}}$; that is, a choice of the section $\exp(2\pi i\psi)\phi_L^{\mathcal{L}}$ in (18).

Definition 3.0.9. Let $\Lambda_0, \Lambda_1 \subset V$ be a transverse pair of Lagrangian subspaces in a symplectic vector space V and let $\tilde\Lambda_0, \tilde\Lambda_1 \in \mathrm{Lag}^N(V)$ be gradings. The *degree* $d(\tilde\Lambda_0, \tilde\Lambda_1) \in \mathbb{Z}_N$ is defined as follows. Let $\tilde\gamma_0, \tilde\gamma_1 : [0,1] \to \mathrm{Lag}^N(V)$ be paths with common starting point $\tilde\gamma_0(0) = \tilde\gamma_1(0)$ and end points $\tilde\gamma_j(1) = \tilde\Lambda_j$. Let $\gamma_j : [0,1] \to \mathrm{Lag}(V)$ denote their image under the projection $\mathrm{Lag}^N(V) \to \mathrm{Lag}(V)$ and define

$$d(\tilde\Lambda_0, \tilde\Lambda_1) := \tfrac{1}{2}\dim(\Lambda_0) + I(\gamma_0, \gamma_1) \mod N, \tag{11}$$

where $I(\gamma_0, \gamma_1)$ denotes the Maslov index for the pair of paths as in [18, 14].

Let us recall from [14] that the Maslov index for a pair of paths with regular crossings (in particular with a finite set of crossings $\mathcal{C} := \{s \in [0,1] \,|\, \gamma_0(s) \cap \gamma_1(s) \neq \{0\}\}$) is given by the sum of crossing numbers with the endpoints weighted by $1/2$,

$$I(\gamma_0, \gamma_1) = \frac{1}{2} \sum_{s \in \mathcal{C} \cap \{0,1\}} \mathrm{sign}(\Gamma(\gamma_0, \gamma_1, s)) + \sum_{s \in \mathcal{C} \cap (0,1)} \mathrm{sign}(\Gamma(\gamma_0, \gamma_1, s)).$$

Each crossing operator $\Gamma(\gamma_0, \gamma_1, s)$ is defined on $v \in \gamma_0(s) \cap \gamma_1(s)$ by fixing Lagrangian complements $\gamma_0(s)^c$, $\gamma_1(s)^c$ of $\gamma_0(s), \gamma_1(s)$ and setting

$$\Gamma(\gamma_0, \gamma_1, s)v = \tfrac{d}{dt}\big|_{t=0}\omega(v, w(t) - w'(t)) \tag{12}$$

where $w(t) \in \gamma_0(s)^c$ such that $v + w(t) \in \gamma_0(s+t)$ and $w'(t) \in \gamma_1(s)^c$ such that $v + w'(s+t) \in \gamma_1(s)$.

Remark 3.0.10. The degree can alternatively be defined by fixing $\tilde{\gamma}_0 \equiv \tilde{\Lambda}_0$ and choosing a path $\tilde{\gamma} : [0,1] \to \mathrm{Lag}^N(V)$ from $\tilde{\gamma}(0) = \tilde{\Lambda}_0$ to $\tilde{\gamma}(1) = \tilde{\Lambda}_1$ such that the crossing form $\Gamma(\gamma, \Lambda_0, 0)$ of the underlying path $\gamma : [0,1] \to \mathrm{Lag}(V)$ is positive definite at $s = 0$. Then the degree

$$d(\tilde{\Lambda}_0, \tilde{\Lambda}_1) = \tfrac{\dim \Lambda_0}{2} + I(\Lambda_0, \gamma) = -\sum_{s \in (0,1)} \mathrm{sign}(\Gamma(\gamma, \Lambda_0, s)) = -I'(\gamma, \Lambda_0) \qquad \mathrm{mod}\ N$$

is given by the Maslov index I' of $\gamma|_{(0,1)}$ (not counting the endpoints) relative to Λ_0. Equivalently, we have

$$d(\tilde{\Lambda}_0, \tilde{\Lambda}_1) = I'(\gamma^{-1}, \Lambda_0) \qquad \mathrm{mod}\ N$$

for the reversed path $\gamma^{-1} : [0,1] \to \mathrm{Lag}(V)$ from $\gamma^{-1}(0) = \Lambda_1$ to $\gamma^{-1}(1) = \Lambda_0$ such that the crossing form $\Gamma(\gamma^{-1}, \Lambda_0, 1)$ is negative definite at $s = 1$.

Lemma 3.0.11. *(Index theorem for once-punctured disks) Let $\Lambda_0, \Lambda_1 \subset V$ be a transverse pair of Lagrangian subspaces with gradings $\tilde{\Lambda}_0, \tilde{\Lambda}_1 \in \mathrm{Lag}^N(V)$. Then for any smooth path of graded Lagrangian subspaces $\tilde{\Lambda} : [0,1] \to \mathrm{Lag}^N(V)$ with endpoints $\tilde{\Lambda}(j) = \tilde{\Lambda}_j$, $j = 0, 1$ we have*

$$d(\tilde{\Lambda}_0, \tilde{\Lambda}_1) = \mathrm{Ind}(D_{V,\Lambda}) \mod N.$$

Here $D_{V,\Lambda}$ is any Cauchy-Riemann operator in V on the disk D with one outgoing strip-like end $(0,\infty) \times [0,1] \hookrightarrow D$ and with boundary conditions given by Λ (the projection of $\tilde{\Lambda}$ to $\mathrm{Lag}(V)$) such that $\Lambda(j) = \Lambda_j$ is the boundary condition over the boundary components $(0,\infty) \times \{j\}$, $j = 0, 1$ of the end.

Proof. It suffices to prove the index identity for a fixed path $\tilde{\Lambda}$. Indeed, if $\tilde{\Lambda}'$ is any other path with the same endpoints then we have $\mathrm{Ind}(D_{V,\Lambda}) - \mathrm{Ind}(D_{V,\Lambda'}) = \mathrm{Ind}(D_{V,\Lambda}) + \mathrm{Ind}(D_{V,-\Lambda'}) = \mathrm{Ind}(D_{V,\Lambda\#(-\Lambda')})$ by gluing. Here the last Cauchy-Riemann operator is defined on the disk with no punctures and with boundary conditions given by the loop $\Lambda\#(-\Lambda')$. Since the loop lifts to a loop $\tilde{\Lambda}\#(-\tilde{\Lambda}')$ in $\mathrm{Lag}^N(V)$, its Maslov index (and thus index) is 0 modulo N.

By Remark 3.0.10, the degree can be defined by a path $\tilde{\Lambda}$ from $\tilde{\Lambda}_1$ to $\tilde{\Lambda}_0$ whose projection Λ has negative definite crossing form at $s = 1$. The sum of crossing numbers in $d(\tilde{\Lambda}_0, \tilde{\Lambda}_1) = \sum_{s \in (0,1)} \mathrm{sign}(\Gamma(\Lambda, \Lambda_0, s))$ is the Maslov index $I_H(\Lambda)$ in [17, Lemma 11.11] and hence equals to the Fredholm index $\mathrm{Ind}(D_{V,\Lambda})$ over the half space, or the conformally equivalent disk with strip-like end. This conformal isomorphism takes the boundary ends $(-\infty, -1)$ resp. $(1,\infty)$ in the half space $\{\mathrm{Im}\, z \geq 0\}$ (over which Λ equals to Λ_1 resp. Λ_0) to $\{1\} \times (1,\infty)$ resp. $\{0\} \times (1,\infty)$ in the strip-like end. □

Lemma 3.0.12. *The degree map satisfies the following properties.*

(a) *(Additivity) If $V = V' \times V''$ then*

$$d(\tilde{\Lambda}'_0 \times^N \tilde{\Lambda}''_0, \tilde{\Lambda}'_1 \times^N \tilde{\Lambda}''_1) = d(\tilde{\Lambda}'_0, \tilde{\Lambda}'_1) + d(\tilde{\Lambda}''_0, \tilde{\Lambda}''_1)$$

for $\tilde{\Lambda}'_j, \tilde{\Lambda}''_j$ graded Lagrangian subspaces in V', V'' respectively, $j = 0, 1$.

(b) *(Multiplicativity) For $\tilde{\Lambda}_0, \tilde{\Lambda}_1$ graded Lagrangian subspaces and any $c \in \mathbb{Z}_N$*

$$d(\tilde{\Lambda}_0, c \cdot \tilde{\Lambda}_1) = c + d(\tilde{\Lambda}_0, \tilde{\Lambda}_1).$$

(c) *(Skewsymmetry) For $\tilde{\Lambda}_0, \tilde{\Lambda}_1$ graded Lagrangian subspaces*

$$d(\tilde{\Lambda}_0, \tilde{\Lambda}_1) + d(\tilde{\Lambda}_1, \tilde{\Lambda}_0) = \dim \Lambda_0 = d(\tilde{\Lambda}_0, \tilde{\Lambda}_1) + d(\tilde{\Lambda}_0^-, \tilde{\Lambda}_1^-).$$

(d) *(Diagonal) For a transverse pair $\tilde\Lambda_0, \tilde\Lambda_1$ of graded Lagrangian subspaces in V and $\tilde\Delta$ the canonically graded diagonal in $V^- \times V$*

$$d(\tilde\Delta, \tilde\Lambda_0^- \times^N \tilde\Lambda_1) = d(\tilde\Lambda_0, \tilde\Lambda_1).$$

Proof. The first three properties are standard, see [15, Section 2d]. We prove the diagonal property to make sure all our sign conventions match up. For that purpose we fix $\tilde L \in \mathrm{Lag}^N(V)$ and choose the following paths $\tilde\gamma_{\cdot,\cdot}$ of graded Lagrangian subspaces (with underlying paths $\gamma_{\cdot,\cdot}$ of Lagrangian subspaces):

- $\tilde\gamma_0 : [-1,1] \to \mathrm{Lag}^N(V)$ from $\tilde\gamma_0(-1) = \tilde L$ to $\tilde\gamma_0(1) = \tilde\Lambda_0$ such that $\tilde\gamma_0|_{[-1,0]} \equiv \tilde L$,
- $\tilde\gamma_1 : [-1,1] \to \mathrm{Lag}^N(V)$ from $\tilde\gamma_1(-1) = \tilde L$ to $\tilde\gamma_1(1) = \tilde\Lambda_1$, such that $\gamma_1|_{[-1/2,0]} \equiv JL \pitchfork L$ and $\gamma_1|_{[-1,-1/2]}$ is a smoothing of $t \mapsto e^{\pi(1+t)J}L$.
- $\tilde\gamma : [-1,1] \to \mathrm{Lag}^N(V^- \times V)$ starting with $\tilde\gamma|_{[-1,-1/2]} = (\tilde\gamma_1^- \times^N \tilde\gamma_0)|_{[-1,-1/2]}$, ending at $\tilde\gamma|_{[0,1]} \equiv \tilde\Delta$, and such that $\gamma|_{[-\frac12,0]}$ is a smoothing of $t \mapsto \{((2t+1)x + Jy, x + (2t+1)Jy) | x, y \in L\}$. (The lift to graded subspaces matches up since $\gamma|_{[-1,0]}$ is exactly the path of (7) which defines $\tilde\Delta$ by connecting it to $\tilde L^- \times \tilde L$.)

Note that we have $I(\gamma_0,\gamma_1)|_{[-1,0]} = -\frac12 \dim \Lambda_0$ and $I(\gamma, \gamma_0^- \times \gamma_1)|_{[-1,0]} = I(\gamma_1^-, \gamma_0^-)|_{[-1,0]} + I(\gamma_0,\gamma_1)|_{[-1,0]} = -\dim \Lambda_0$ since $\gamma|_{[-\frac12,0]}$ is transverse to $L^- \times JL$. With these preparations we can calculate

$$\begin{aligned} d(\tilde\Lambda_0, \tilde\Lambda_1) &= \tfrac12 \dim \Lambda_0 + I(\gamma_0,\gamma_1) = I(\gamma_0,\gamma_1)\big|_{[0,1]} \\ &= I(\Delta, \gamma_0^- \times \gamma_1)\big|_{[0,1]} \\ &= \dim \Lambda_0 + I(\gamma, \gamma_0^- \times \gamma_1) = d(\tilde\Delta, \tilde\Lambda_0^- \times^N \tilde\Lambda_1). \end{aligned}$$

Here the identity of the Maslov indices over the interval $[0,1]$ follows from identifying the intersections $K(s) := \gamma_0 \cap \gamma_1 \cong \Delta \cap (\gamma_0^- \times \gamma_1)$ and the crossing forms $\Gamma(s), \hat\Gamma(s) : K(s) \to \mathbb{R}$ at regular crossings $s \in [0,1]$ (after a homotopy of the paths to regular crossings). Fix Lagrangian complements $\gamma_0(s)^c$ and $\gamma_1(s)^c$, then for $v \in K(s)$ pick $w_i(t) \in \gamma_i(s)^c$ such that $v + w_i(t) \in \gamma_i(s+t)$. For the corresponding vector $\hat v = (v,v) \in \Delta \cap (\gamma_0^- \times \gamma_1)$ we can pick $\hat w(t) = (0,0) \in \Delta^c$ satisfying $\hat v + \hat w(t) \in \Delta$ and $\hat w'(t) = (w_0, w_1) \in \gamma_0(s)^c \times \gamma_1(s)^c$ satisfying $\hat v + \hat w'(t) \in (\gamma_0 \times \gamma_1)(s+t)$ to identify the crossing forms

$$\begin{aligned} \hat\Gamma(s)\hat v &= \tfrac{d}{dt}\big|_{t=0}(-\omega \oplus \omega)(\hat v, \hat w(t) - \hat w'(t)) \\ &= \tfrac{d}{dt}\big|_{t=0}\big(-\omega(v, -w_0(t)) + \omega(v, -w_1(t))\big) \\ &= \tfrac{d}{dt}\big|_{t=0}\omega(v, w_0(t) - w_1(t)) = \Gamma(s)v. \end{aligned}$$

□

If $L_0, L_1 \subset M$ are $\mathrm{Lag}^N(M)$-graded Lagrangians and intersect transversally then one obtains a *degree map*

$$\mathcal{I}(L_0,L_1) := L_0 \cap L_1 \to \mathbb{Z}_N, \qquad x \mapsto |x| := d(\sigma^N_{L_0}(x), \sigma^N_{L_1}(x)).$$

More generally, if L_0, L_1 do not necessarily intersect transversally, then we can pick a Hamiltonian perturbation $H : [0,1] \times M \to \mathbb{R}$ such that its time 1 flow $\phi_1 : M \to M$ achieves transversality $\phi_1(L_0) \pitchfork L_1$. Then the Hamiltonian isotopy and the grading on L_0 induce a grading on $\phi_1(L_0)$, which is transverse to L_1. The degree map is then defined on the perturbed intersection points, $d : \mathcal{I}(L_0,L_1) := \phi_1(L_0) \cap L_1 \to \mathbb{Z}_N$.

3.1. **Graded generalized Lagrangian correspondences.** In this section we extend the grading and degree constructions to generalized Lagrangian correspondences and discuss their behaviour under geometric composition and insertion of the diagonal.

Definition 3.1.1. Let M and M' be symplectic manifolds equipped with N-fold Maslov coverings. Let $\underline{L} = (L_{01}, \ldots, L_{(r-1)r})$ be a generalized Lagrangian correspondence from M to M' (i.e. $L_{(j-1)j} \subset M_{j-1}^- \times M_j$ for a sequence $M = M_1, \ldots, M_r = M'$ of symplectic manifolds). A *grading* on $\underline{L}$ consists of a collection of N-fold Maslov covers $\mathrm{Lag}^N(M_j) \to M_j$ and gradings of the Lagrangian correspondences $L_{(j-1)j}$ with respect to $\mathrm{Lag}^N(M_{j-1}^- \times M_j)$, where the Maslov covers on $M_1 = M$ and $M_r = M'$ are the fixed ones.

A pair of graded generalized Lagrangian correspondences $\underline{L}_1$ and $\underline{L}_2$ from M to M' (with fixed Maslov coverings) defines a cyclic Lagrangian correspondence $\underline{L}_1 \# (\underline{L}_2)^t$, which is graded in the following sense.

Definition 3.1.2. Let $\underline{L} = (L_{01}, \ldots, L_{r(r+1)})$ be a cyclic generalized Lagrangian correspondence (i.e. $L_{j(j+1)} \subset M_j^- \times M_{j+1}$ for a cyclic sequence $M_0, M_1, \ldots, M_{r+1} = M_0$ of symplectic manifolds). An *N-grading* on $\underline{L}$ consists of a collection of N-fold Maslov covers $\mathrm{Lag}^N(M_j) \to M_j$ and gradings of the Lagrangian correspondences $L_{j(j+1)}$ with respect to $\mathrm{Lag}^N(M_j^- \times M_{j+1})$.

In the following, we will consider a cyclic generalized Lagrangian correspondence $\underline{L}$ and assume that it intersects the generalized diagonal transversally, i.e.

$$\big(L_{01} \times L_{12} \times \ldots \times L_{r(r+1)}\big) \pitchfork \big(\Delta_{M_0}^- \times \Delta_{M_1}^- \times \ldots \times \Delta_{M_r}^-\big)^T, \tag{13}$$

where $\Delta_M^- \subset M \times M^-$ denotes the (dual of the) diagonal and $M_0 \times M_0^- \times M_1 \times \ldots \times M_r^- \to M_0^- \times M_1 \times \ldots \times M_r^- \times M_0$, $Z \mapsto Z^T$ is the transposition of the first to the last factor. In section 4.3 this transversality will be achieved by a suitable Hamiltonian isotopy. It ensures that the above transverse intersection cuts out a finite set, which we identify with the *generalized intersection points*

$$\begin{aligned}\mathcal{I}(\underline{L}) &:= \times_{\Delta_{M_0}} \big(L_{01} \times_{\Delta_{M_1}} L_{12} \ldots \times_{\Delta_{M_r}} L_{r(r+1)}\big) \\ &= \big\{\underline{x} = (x_0, \ldots, x_r) \in M_0 \times \ldots \times M_r \,\big|\, (x_0, x_1) \in L_{01}, \ldots, (x_r, x_0) \in L_{r(r+1)}\big\}.\end{aligned}$$

Remark 3.1.3. Consider two cyclic generalized Lagrangian correspondences

$$\begin{aligned}\underline{L} &= (L_{01}, \ldots, L_{(j-1)j}, L_{j(j+1)}, \ldots, L_{r(r+1)}), \\ \underline{L}' &= (L_{01}, \ldots, L_{(j-1)j} \circ L_{j(j+1)}, \ldots, L_{r(r+1)})\end{aligned}$$

such that the composition $L_{(j-1)j} \circ L_{j(j+1)}$ is embedded in the sense of Definition 2.0.4. Then the generalized intersection points

$$\begin{aligned}\mathcal{I}(\underline{L}) &= \big\{(\ldots, x_{j-1}, x_j, x_{j+1}, \ldots) \in \ldots \times M_{j-1} \times M_j \times M_{j+1} \ldots \,\big| \\ &\qquad\qquad \ldots, (x_{j-1}, x_j) \in L_{(j-1)j}, (x_j, x_{j+1}) \in L_{j(j+1)}, \ldots\big\} \\ &= \big\{(\ldots, x_{j-1}, x_{j+1}, \ldots) \in \ldots \times M_{j-1} \times M_{j+1} \ldots \,\big| \\ &\qquad\qquad \ldots, (x_{j-1}, x_{j+1}) \in L_{(j-1)j} \circ L_{j(j+1)}, \ldots\big\} = \mathcal{I}(\underline{L}')\end{aligned}$$

are canonically identified, since the intermediate point $x_j \in M_j$ with $(x_{j-1}, x_j) \in L_{(j-1)j}$ and $(x_j, x_{j+1}) \in L_{j(j+1)}$ is uniquely determined by the pair $(x_{j-1}, x_{j+1}) \in L_{(j-1)j} \circ L_{j(j+1)}$.

Now an N-grading on $\underline{L}$ induces an N-fold Maslov covering on $M := M_0^- \times M_1 \times \ldots \times M_r \times M_r^- \times M_0$ and a grading of $L := L_{01} \times L_{12} \times \ldots \times L_{r(r+1)}$. In addition, we have a grading on $\Delta^T := (\Delta_{M_0}^- \times \Delta_{M_1}^- \times \ldots \times \Delta_{M_r}^-)^T$ from the canonical grading on each factor. In order to define a degree we then identify generalized intersection points $\underline{x} = (x_0, x_1, \ldots, x_r)$ with the actual intersection points $x = (x_0, x_1, x_1, \ldots, x_r, x_r, x_0) \in L \cap \Delta^T$.

Definition 3.1.4. Let $\underline{L}$ be a graded cyclic generalized Lagrangian correspondence $\underline{L}$ that is transverse to the diagonal (13). Then the *degree* is

$$\mathcal{I}(\underline{L}) \to \mathbb{Z}_N, \qquad \underline{x} \mapsto |\underline{x}| = d(\sigma_L^N(x), \sigma_{\Delta^T}^N(x)).$$

Lemma 3.1.5. *Alternatively, the degree is defined as follows:*

(a) *Pick any tuple of Lagrangian subspaces $\Lambda_i' \in \mathrm{Lag}(T_{x_i} M_i)$, $\Lambda_i'' \in \mathrm{Lag}(T_{x_i} M_i^-)$, $i = 0, \ldots, r$ whose product is transverse to the diagonal, $\Lambda_i' \times \Lambda_i'' \pitchfork \Delta_{T_{x_i} M_i}$. Then there exists a path (unique up to homotopy) $\gamma : [0,1] \to \mathrm{Lag}(T_x M)$ from $\gamma(0) = T_x L$ to $\gamma(1) = \Lambda_0'' \times \Lambda_1' \times \ldots \times \Lambda_r' \times \Lambda_r'' \times \Lambda_0'$ that is transverse to the diagonal at all times, $\gamma(t) \pitchfork T_x \Delta^T$. We lift the grading $\sigma_L^N(x) \in \mathrm{Lag}^N(T_x M)$ along this path and pick preimages under the graded product map (10) to define $\tilde{\Lambda}_i' \in \mathrm{Lag}^N(T_{x_i} M_i)$ and $\tilde{\Lambda}_i'' \in \mathrm{Lag}^N(T_{x_i} M_i^-)$. Then*

$$|\underline{x}| = \sum_{i=0}^{r} d(\tilde{\Lambda}_i', \tilde{\Lambda}_i''^-).$$

(b) *If $\underline{L}$ has even length $r+1 \in 2\mathbb{N}$ then it defines an N-fold Maslov cover on $\widetilde{M} := M_0^- \times M_1 \times M_2^- \times \ldots \times M_r$ and a pair of graded Lagrangian submanifolds,*

$$L_{(0)} := L_{01} \times L_{23} \times \ldots \times L_{(r-1)r} \subset \widetilde{M},$$
$$L_{(1)} := (L_{12} \times L_{34} \times \ldots \times L_{r(r+1)})^T \subset \widetilde{M}^-,$$

where we denote by $M_1^- \times \ldots \times M_r^- \times M_0 \to M_0 \times M_1^- \times \ldots \times M_r^-$, $Z \mapsto Z^T$ the transposition of the last to the first factor. If $\underline{L}$ has odd length $r+1 \in 2\mathbb{N}+1$ we insert the diagonal $\Delta_{M_0} \subset M_0^- \times M_0 = M_{r+1}^- \times M_0$ (with its canonical grading) before defining a pair of graded Lagrangian submanifolds as above. By (13) the Lagrangians intersect transversally $L_{(0)} \pitchfork L_{(1)}^-$, and this intersection is canonically identified with $\mathcal{I}(\underline{L})$. Then for $\underline{x} \in \mathcal{I}(\underline{L})$ corresponding to $y \in L_{(0)} \cap L_{(1)}^-$ we have

$$|\underline{x}| = |y| = d(\sigma_{L_{(0)}}^N(y), \sigma_{L_{(1)}}^N(y)^-).$$

Proof. In (a) we use the fact that the path γ has zero Maslov index to rewrite

$$d(\sigma_L^N(x), \sigma_{\Delta^T}^N(x)) = d(\tilde{\Lambda}_0' \times^N \tilde{\Lambda}_0'' \times^N \ldots \times^N \tilde{\Lambda}_r' \times^N \tilde{\Lambda}_r'', \tilde{\Delta}_{T_{x_0} M_0}^- \times^N \ldots \times^N \tilde{\Delta}_{T_{x_r} M_r}^-),$$

where we moreover transposed the factors. Now by Lemma 3.0.12 the right hand side can be written as the sum over $d(\tilde{\Lambda}_i' \times^N \tilde{\Lambda}_i'', \tilde{\Delta}_{T_{x_i} M_i}^-) = d(\tilde{\Lambda}_i', \tilde{\Lambda}_i''^-)$.

In (b) note that a reordering of the factors identifies the pair of graded Lagrangians $(L_{(0)} \times L_{(1)}, \Delta_{\widetilde{M}}^-)$ with (L, Δ^T) for r odd. So Lemma 3.0.12 implies

$$d(\sigma_L^N(x), \sigma_{\Delta^T}^N(x)) = d(\sigma_{L_{(0)}}^N(y) \times^N \sigma_{L_{(1)}}^N(y), \tilde{\Delta}_{T_{(y,y)}\widetilde{M}}^-) = d(\sigma_{L_{(0)}}^N(y), \sigma_{L_{(1)}}^N(y)^-).$$

For r even the same argument proves

$$d(\sigma_{L_{(0)}}^N(y), \sigma_{L_{(1)}}^N(y)^-) = d\big(\sigma_L^N(x) \times^N \tilde{\Delta}_{T_{x_0} M_0}, (\tilde{\Delta}_{T_{x_0} M_0}^- \times \ldots \times \tilde{\Delta}_{T_{x_r} M_r}^- \times \tilde{\Delta}_{T_{x_0} M_0}^-)^T\big),$$

which equals to $d(\sigma_L^N(x), \sigma_{\Delta^T}^N(x))$ by Lemma 3.1.6 (b) below. □

The following Lemma describes the effect of inserting a diagonal on the grading of generalized Lagrangian correspondences. Part (a) addresses noncyclic correspondences, whereas (b) applies to cyclic correspondences with $\Lambda = T_{(x_0,x_1,\ldots,x_r,x_0)}(L_{01} \times L_{12} \times \ldots \times L_{r(r+1)})$, $K = T_{(x_0,x_0,x_1,\ldots,x_r)}(\Delta^-_{M_0} \times \Delta^-_{M_1} \times \ldots \times \Delta^-_{M_r})$, $V_0 = T_{x_0}M_0$, and $V_1 = T_{(x_1,\ldots,x_r)}(M_1 \times M_1^- \times \ldots \times M_r \times M_r^-)$.

Lemma 3.1.6. *Let V_0, V_1, V_2 be symplectic vector spaces.*

(a) *Let $\tilde\Lambda_0 \subset \mathrm{Lag}^N(V_0)$, $\tilde\Lambda_{01} \subset \mathrm{Lag}^N(V_0^- \times V_1)$, $\tilde\Lambda_{12} \subset \mathrm{Lag}^N(V_1^- \times V_2)$, and $\tilde\Lambda_2 \subset \mathrm{Lag}^N(V_2^-)$ be graded Lagrangian subspaces. If the underlying Lagrangian subspaces are transverse then*

$$d(\tilde\Lambda_0 \times^N \tilde\Lambda_{12}, \tilde\Lambda_{01}^- \times^N \tilde\Lambda_2^-) = d(\tilde\Lambda_0 \times^N \tilde\Delta_1 \times^N \tilde\Lambda_2, \tilde\Lambda_{01}^- \times^N \tilde\Lambda_{12}^-).$$

(b) *Let $\tilde\Lambda \subset \mathrm{Lag}^N(V_0^- \times V_1 \times V_0)$ and $\tilde K \subset \mathrm{Lag}^N(V_0 \times V_0^- \times V_1)$ be graded Lagrangian subspaces. If the underlying Lagrangian subspaces are transverse then*

$$d(\tilde\Lambda \times^N \tilde\Delta_0, (\tilde K \times^N \tilde\Delta_0^-)^T) = d(\tilde\Lambda, \tilde K^T),$$

with the transposition $V_0 \times W \to W \times V_0$, $Z \mapsto Z^T$.

Proof. To prove (a) pick a path $\gamma_{0112} : [0,1] \to \mathrm{Lag}(V_0 \times V_1^- \times V_1 \times V_2^-)$ from $\gamma_{0112}(0) = \Lambda_{01}^- \times \Lambda_{12}^-$ to a split Lagrangian subspace $\gamma_{0112}(1) = \Lambda_0' \times \Lambda_1' \times \Lambda_1'' \times \Lambda_2'$ that is transverse to $\Lambda_0 \times \Delta_1 \times \Lambda_2$ at all times and hence has Maslov index $I(\gamma_{0112}, \Lambda_0 \times \Delta_1 \times \Lambda_2) = 0$. We can homotope this path with fixed endpoints to $\gamma_{0112} = \gamma_{01} \times \gamma_{12} : [0,1] \to \mathrm{Lag}(V_0 \times V_1^-) \times \mathrm{Lag}(V_1 \times V_2^-)$ that may intersect $\Lambda_0 \times \Delta_1 \times \Lambda_2$ but still has vanishing Maslov index. We lift the grading along the paths γ_{01} and γ_{12} and pick preimages under the graded product map (10) to obtain gradings $\tilde\Lambda_0' \in \mathrm{Lag}^N(V_0)$, $\tilde\Lambda_1' \in \mathrm{Lag}^N(V_1^-)$, $\tilde\Lambda_1'' \in \mathrm{Lag}^N(V_1)$, $\tilde\Lambda_2' \in \mathrm{Lag}^N(V_2^-)$. With these we calculate, using Lemma 3.0.12

$$\begin{aligned}
d(\tilde\Lambda_0 \times^N \tilde\Lambda_{12}, \tilde\Lambda_{01}^- \times^N \tilde\Lambda_2^-) &= d(\tilde\Lambda_0 \times^N \tilde\Lambda_1''^- \times^N \tilde\Lambda_2'^-, \tilde\Lambda_0' \times^N \tilde\Lambda_1' \times^N \tilde\Lambda_2^-) \\
&= d(\tilde\Lambda_0, \tilde\Lambda_0') + d(\tilde\Lambda_1''^-, \tilde\Lambda_1') + d(\tilde\Lambda_2'^-, \tilde\Lambda_2^-) \\
&= d(\tilde\Lambda_0, \tilde\Lambda_0') + d(\tilde\Delta_1, \tilde\Lambda_1' \times^N \tilde\Lambda_1'') + d(\tilde\Lambda_2, \tilde\Lambda_2') \\
&= d(\tilde\Lambda_0 \times^N \tilde\Delta_1 \times^N \tilde\Lambda_2, \tilde\Lambda_0' \times^N \tilde\Lambda_1' \times^N \tilde\Lambda_1'' \times^N \tilde\Lambda_2') \\
&= d(\tilde\Lambda_0 \times^N \tilde\Delta_1 \times^N \tilde\Lambda_2, \tilde\Lambda_{01}^- \times^N \tilde\Lambda_{12}^-).
\end{aligned}$$

The first and last degree identity are due to the vanishing of the Maslov index

$$0 \;=\; I(\Lambda_0 \times \Delta_1 \times \Lambda_2, \gamma_{01} \times \gamma_{12}) \;=\; I(\Lambda_0 \times \gamma_{12}^-, \gamma_{01} \times \Lambda_2^-) = 0.$$

The identity of these Maslov indices follows from identifying the intersections $K(s) := (\Lambda_0 \times \gamma_{12}^-(s)) \cap (\gamma_{01}(s) \times \Lambda_2^-) \cong (\Lambda_0 \times \Delta_1 \times \Lambda_2) \cap (\gamma_{01} \times \gamma_{12})$ and the crossing form $\Gamma(s), \hat\Gamma(s) : K(s) \to \mathbb{R}$ given by (12) at regular crossings $s \in [0,1]$. Fix Lagrangian complements $\gamma_{01}(s)^c \subset V_0 \times V_1^-$ and $\gamma_{12}(s)^c \subset V_1 \times V_2^-$, then for $v = (v_0, v_1, v_2) \in K(s)$ we can pick $(w_1, w_2)(t) \in \gamma_{12}(s)^c$ such that $v + (0, w_1, w_2)(t) \in \Lambda_0 \times \gamma_{12}(s+t)$ and $(w_0', w_1')(t) \in \gamma_{01}(s)^c$ such that $v + (w_0', w_1', 0)(t) \in \gamma_{01}(s+t) \times \Lambda_2$. For the corresponding vector $\hat v = (v_0, v_1, v_1, v_2) \in (\Lambda_0 \times \Delta_1 \times \Lambda_2^-) \cap (\gamma_{01}^- \times \gamma_{12}^-)$ we have $\hat v + (0,0,0,0) \in (\Lambda_0 \times \Delta_1 \times \Lambda_2)$ and $\hat v + (w_0', w_1', w_1, w_2)(t) \in$

$(\gamma_{01} \times \gamma_{12})(s+t)$. With this we identify the crossing forms

$$\begin{aligned}\hat\Gamma(s)\hat v &= \tfrac{d}{dt}\big|_{t=0}(\omega_0 \oplus -\omega_1 \oplus \omega_1 \oplus -\omega_2)\big(\hat v, (0,0,0,0) - (w_0', w_1', w_1, w_2)(t)\big)\\ &= \tfrac{d}{dt}\big|_{t=0}\big(-\omega_0(v_0, w_0') - \omega_1(v_1, w_1 - w_1') + \omega_2(v_2, w_2)\big)\\ &= \tfrac{d}{dt}\big|_{t=0}(\omega_0 \oplus -\omega_1 \oplus \omega_2)\big(v, (0, w_1, w_2)(t) - (w_0', w_1', 0)(t)\big) = \Gamma(s)v.\end{aligned}$$

This proves (a). To prove (b) we pick a path $\gamma : [0,1] \to \mathrm{Lag}(V_0^- \times V_1 \times V_0)$ from $\gamma(0) = \Lambda$ to a split Lagrangian subspace $\gamma(1) = \Lambda_0^- \times \Lambda_1 \times \Lambda_0' \in \mathrm{Lag}(V_0^-) \times \mathrm{Lag}(V_1) \times \mathrm{Lag}(V_0)$ that is transverse to K^T at all times and hence has Maslov index

$$0 = I(\gamma, K^T) = I(\gamma \times \Delta_0, (K \times \Delta_0^-)^T).$$

Here the equality of Maslov follows directly from the identification of the trivial intersections $(\gamma \times \Delta_0) \cap (K \times \Delta_0^-)^T \cong \gamma \cap K^T = \{0\}$. Now we can lift the grading along γ to obtain gradings $\tilde\Lambda_0 \in \mathrm{Lag}^N(V_0)$, $\tilde\Lambda_1 \in \mathrm{Lag}^N(V_1)$, $\tilde\Lambda_0' \in \mathrm{Lag}^N(V_0)$. With these we calculate, using part (a) and the fact that gradings are invariant under simultaneous transposition of both factors

$$\begin{aligned}d(\tilde\Lambda \times^N \tilde\Delta_0, (\tilde K \times^N \tilde\Delta_0^-)^T) &= d(\tilde\Lambda_0^- \times^N \tilde\Lambda_1 \times^N \tilde\Lambda_0' \times^N \tilde\Delta_0, (\tilde K \times^N \tilde\Delta_0^-)^T)\\ &= d(\tilde\Lambda_0' \times^N \tilde\Delta_0 \times^N \tilde\Lambda_0^- \times^N \tilde\Lambda_1, \tilde\Delta_0^- \times^N \tilde K)\\ &= d(\tilde\Lambda_0' \times^N \tilde K^-, \tilde\Delta_0^- \times^N (\tilde\Lambda_0^- \times^N \tilde\Lambda_1)^-)\\ &= d(\tilde\Delta_0 \times^N (\tilde\Lambda_0^- \times^N \tilde\Lambda_1), \tilde\Lambda_0'^- \times^N \tilde K)\\ &= d(\tilde K^-, \tilde\Lambda_0'^- \times^N (\tilde\Lambda_0^- \times^N \tilde\Lambda_1)^-)\\ &= d(\tilde\Lambda_0' \times^N \tilde\Lambda_0^- \times^N \tilde\Lambda_1, \tilde K)\\ &= d(\tilde\Lambda_0^- \times^N \tilde\Lambda_1 \times^N \tilde\Lambda_0', \tilde K^T) \;=\; d(\tilde\Lambda, \tilde K^T)\end{aligned}$$

□

In the rest of this section we investigate the effect of geometric composition on the grading of Lagrangian correspondences. This requires a generalization of Viterbo's index calculations [18].

First, we lift the composition map to Maslov covers. Let M_0, M_1, M_2 be symplectic manifolds equipped with N-fold Maslov coverings $\mathrm{Lag}^N(M_j), j = 0,1,2$. We equip the products $M_i^- \times M_j$ and $M_0^- \times M_1 \times M_1^- \times M_2$ with the induced Maslov coverings $\mathrm{Lag}^N(M_i^- \times M_j)$ resp. $\mathrm{Lag}^N(M_0^- \times M_1 \times M_1^- \times M_2)$. We denote by

$$\mathcal{T}(M_1) \subset \mathrm{Lag}(M_0^- \times M_1 \times M_1^- \times M_2)\big|_{M_0 \times \Delta_{M_1} \times M_2}$$

the subbundle whose fibre over (m_0, m_1, m_1, m_2) consists of the Lagrangian subspaces $\Lambda_{0112} \subset T_{(m_0,m_1,m_1,m_2)}(M_0^- \times M_1 \times M_1^- \times M_2)$ that are transverse to the diagonal $\Delta_{0112} := T_{m_0}M_0 \times \Delta_{T_{m_1}M_1} \times T_{m_2}M_2$. The linear composition of Lagrangian subspaces extends a smooth map

$$\circ : \mathcal{T}(M_1) \to \mathrm{Lag}(M_0^- \times M_2), \qquad \Lambda_{0112} \mapsto \pi_{M_0 \times M_2}\big(\Lambda_{0112} \cap \Delta_{0112}\big).$$

The preimage of $\mathcal{T}(M_1)$ in the Maslov cover will be denoted by

$$\mathcal{T}^N(M_1) \subset \mathrm{Lag}^N(M_0^- \times M_1 \times M_1^- \times M_2)\big|_{M_0 \times \Delta_{M_1} \times M_2}.$$

Finally, recall that we have a canonical grading of the diagonal $\tilde{\Delta}_{M_1} \in \mathrm{Lag}^N(M_1^- \times M_1)$ and its dual $\tilde{\Delta}^-_{M_1} \in \mathrm{Lag}^N(M_1 \times M_1^-)$, and let us denote another exchange of factors by $\mathrm{Lag}^N(M_0^- \times M_2 \times M_1 \times M_1^-) \to \mathrm{Lag}^N(M_0^- \times M_1 \times M_1^- \times M_2)$, $\tilde{\Lambda} \mapsto \tilde{\Lambda}^T$.

Lemma 3.1.7. *The linear composition $\circ : \mathcal{T}(M_1) \to \mathrm{Lag}(M_0^- \times M_2)$ lifts to a unique smooth map $\circ^N : \mathcal{T}^N(M_1) \to \mathrm{Lag}^N(M_0^- \times M_2)$ with the property that*

$$\circ^N \left(\left(\tilde{\Lambda}_{02} \times^N \tilde{\Lambda}_{11} \right)^T \right) = d(\tilde{\Lambda}_{11}, \tilde{\Delta}^-_{M_1}) \cdot \tilde{\Lambda}_{02}. \tag{14}$$

for all graded Lagrangians $\tilde{\Lambda}_{02} \in \mathrm{Lag}^N(M_0^- \times M_2)$ and $\tilde{\Lambda}_{11} \in \mathrm{Lag}^N(M_1 \times M_1^-)$, such that the underlying Lagrangian $\Lambda_{11} \in \mathrm{Lag}(M_1 \times M_1^-)$ is transverse to the diagonal.

Proof. We denote by $\mathrm{Lag}(\mathbb{R}^{2n})$ the Lagrangian Grassmannian in $\mathbb{R}^{2n}$, write $\dim M_i = 2n_i$, and abbreviate $\mathbb{R}_{0112} := \mathbb{R}^{2n_0,-} \times \mathbb{R}^{2n_1} \times \mathbb{R}^{2n_1,-} \times \mathbb{R}^{2n_2}$. Let $\mathcal{T} \subset \mathrm{Lag}(\mathbb{R}_{0112})$ be the subset of Lagrangian subspaces meeting the diagonal $\mathbb{R}^{2n_0} \times \Delta_{\mathbb{R}^{2n_1}} \times \mathbb{R}^{2n_2}$ transversally. The linear composition map

$$\mathrm{Lag}(\mathbb{R}_{0112}) \supset \mathcal{T} \to \mathrm{Lag}(\mathbb{R}^{2n_0,-} \times \mathbb{R}^{2n_2}), \qquad \Lambda \mapsto \pi_{\mathbb{R}^{2n_0} \times \mathbb{R}^{2n_2}} \left(\Lambda \cap (\mathbb{R}^{2n_0} \times \Delta_{\mathbb{R}^{2n_1}} \times \mathbb{R}^{2n_2}) \right)$$

is $\mathrm{Sp}(2n_0) \times \mathrm{Sp}(2n_1) \times \mathrm{Sp}(2n_2)$-equivariant, and lifts to a unique $\mathrm{Sp}^N(2n_0) \times \mathrm{Sp}^N(2n_1) \times \mathrm{Sp}^N(2n_2)$-equivariant map

$$\mathrm{Lag}^N(\mathbb{R}_{0112}) \supset \mathcal{T}^N \to \mathrm{Lag}^N(\mathbb{R}^{2n_0,-} \times \mathbb{R}^{2n_2}) \tag{15}$$

with the property (14). On the other hand, the restriction of $\mathrm{Fr}(M_0) \times \mathrm{Fr}(M_1) \times \mathrm{Fr}(M_1) \times \mathrm{Fr}(M_2)$ to $M_0 \times \Delta_{M_1} \times M_2$ admits a reduction of the structure group to $\mathrm{Sp}(2n_0) \times \mathrm{Sp}(2n_1) \times \mathrm{Sp}(2n_2)$, and similarly the restriction

$$\mathrm{Fr}^N_{0112} := \left(\mathrm{Fr}^N(M_0) \times \mathrm{Fr}^N(M_1) \times \mathrm{Fr}^N(M_1) \times \mathrm{Fr}^N(M_2) \right) \big|_{M_0 \times \Delta_{M_1} \times M_2}$$

admits a reduction of the structure group to $\mathrm{Sp}^N(2n_0) \times \mathrm{Sp}^N(2n_1) \times \mathrm{Sp}^N(2n_2)$. This group acts on $\mathrm{Lag}^N(\mathbb{R}_{0112})$ by the diagonal action of $\mathrm{Sp}^N(2n_1)$ on $\mathbb{R}^{2n_1} \times \mathbb{R}^{2n_1,-}$. Finally, we use the associated fiber bundle construction to identify

$$\begin{aligned}
&\mathrm{Lag}^N(M_0^- \times M_1 \times M_1^- \times M_2)\big|_{M_0 \times \Delta_{M_1} \times M_2} \\
&\cong \mathrm{Fr}^N_{0112} \times_{\mathrm{Sp}^N(2n_0) \times \mathrm{Sp}^N(2n_1) \times \mathrm{Sp}^N(2n_1) \times \mathrm{Sp}^N(2n_2)} \mathrm{Lag}^N(\mathbb{R}_{0112}) \\
&\cong \left(\mathrm{Fr}^N(M_0) \times \mathrm{Fr}^N(M_1) \times \mathrm{Fr}^N(M_2) \right) \times_{\mathrm{Sp}^N(2n_0) \times \mathrm{Sp}^N(2n_1) \times \mathrm{Sp}^N(2n_2)} \mathrm{Lag}^N(\mathbb{R}_{0112})
\end{aligned}$$

and

$$\mathrm{Lag}^N(M_0^- \times M_2) = \left(\mathrm{Fr}^N(M_0^-) \times \mathrm{Fr}^N(M_2) \right) \times_{\mathrm{Sp}^N(2n_0) \times \mathrm{Sp}^N(2n_2)} \mathrm{Lag}^N(\mathbb{R}^{2n_0,-} \times \mathbb{R}^{2n_2}).$$

Then the forgetful map on the first factor and the equivariant map (15) on the second factor define the unique lift $\circ^N$. □

Now consider two graded Lagrangian correspondences $L_{01} \subset M_0^- \times M_1$ and $L_{12} \subset M_1^- \times M_2$ and suppose that the composition $L_{01} \circ L_{12} =: L_{02} \subset M_0^- \times M_2$ is smooth and embedded. The canonical section $\sigma_{L_{02}} : L_{02} \to \mathrm{Lag}(M_0^- \times M_2)$ is given by the linear composition $\circ$ applied to $(\sigma_{L_{01}} \times \sigma_{L_{12}})|_{L_{01} \times_{\Delta_{M_1}} L_{12}}$. The gradings $\sigma^N_{L_{01}}$, $\sigma^N_{L_{12}}$ induce a grading on L_{02},

$$\sigma^N_{L_{02}} := \circ^N \left(\sigma^N_{L_{01}} \times^N \sigma^N_{L_{12}} \right) \big|_{L_{01} \times_{\Delta_{M_1}} L_{12}}, \tag{16}$$

where the map $\times^N$ is defined in (10) and we identify $L_{02} \cong L_{01} \times_{\Delta_{M_1}} L_{12}$.

Proposition 3.1.8. *Let $L_0 \subset M_0$, $L_{01} \subset M_0^- \times M_1$, $L_{12} \subset M_1^- \times M_2$, and $L_2 \subset M_2^-$ be graded Lagrangians such that the composition $L_{01} \circ L_{12} =: L_{02}$ is embedded. Then, with respect to the induced grading on L_{02}, the degree map $\mathcal{I}(L_0 \times L_2, L_{02}) \to \mathbb{Z}_N$ is the pull-back of the degree map $\mathcal{I}(L_0 \times L_{12}, L_{01} \times L_2) \to \mathbb{Z}_N$ under the canonical identification [2] of intersection points.*

Proof. Suppose for simplicity that Hamiltonian perturbations have been applied to the Lagrangians L_0, L_2 such that $\mathcal{I}(L_0 \times L_2, L_{02})$ (and hence also $\mathcal{I}(L_0 \times L_{12}, L_{01} \times L_2)$) is the intersection of transverse Lagrangians. We need to consider $(m_0, m_1, m_2) \in (L_0 \times L_{12}) \cap (L_{01} \times L_2)$, which corresponds to $(m_0, m_2) \in (L_0 \times L_2) \cap L_{02}$. We abbreviate the tangent spaces of the Lagrangians by $\Lambda_j = T_{m_j} L_j$, $\Lambda_{ij} = T_{(m_i,m_j)} L_{ij}$, and $\Delta_1 = \Delta_{T_{m_1} M_1}$ and their graded lifts by $\tilde{\Lambda}_j = \sigma^N_{L_j}(m_j)$, $\tilde{\Lambda}_{ij} = \sigma^N_{L_{ij}}(m_i, m_j)$, and $\tilde{\Delta}_1 = \tilde{\Delta}_{T_{m_1} M_1}$. We claim that

$$\begin{aligned} d(\tilde{\Lambda}_0 \times^N \tilde{\Lambda}_{12}, \tilde{\Lambda}_{01}^- \times^N \tilde{\Lambda}_2^-) &= d(\tilde{\Lambda}_0 \times^N \tilde{\Delta}_1 \times^N \tilde{\Lambda}_2, \tilde{\Lambda}_{01}^- \times^N \tilde{\Lambda}_{12}^-) \\ &= d(\tilde{\Lambda}_0 \times^N \tilde{\Lambda}_2, \tilde{\Lambda}_{01}^- \circ^N \tilde{\Lambda}_{12}^-). \end{aligned} \tag{17}$$

The first identity is Lemma 3.1.6. To prove (17) we begin by noting the transverse intersection $\Lambda_{02} \pitchfork \Lambda_0 \times \Lambda_2$. We denote $\tilde{\Lambda}_{02} := \tilde{\Lambda}_{01} \circ^N \tilde{\Lambda}_{12}$ (hence $\tilde{\Lambda}_{02}^- = \tilde{\Lambda}_{01}^- \circ^N \tilde{\Lambda}_{12}^-$) and pick a path $\tilde{\gamma}_{02} : [0,1] \to \mathrm{Lag}^N(T_{m_0} M_0^- \times T_{m_2} M_2)$ from $\tilde{\gamma}_{02}(0) = \tilde{\Lambda}_0 \times^N \tilde{\Lambda}_2$ to $\tilde{\gamma}_{02}(1) = \tilde{\Lambda}_{02}^-$ whose crossing form with $\Lambda_0 \times \Lambda_2$ at $s = 0$ is positive definite and hence by Remark 3.0.10

$$d(\tilde{\Lambda}_0 \times^N \tilde{\Lambda}_2, \tilde{\Lambda}_{02}^-) = -I'(\gamma_{02}, \Lambda_0 \times \Lambda_2).$$

Here I' denotes the Maslov index of a pair of paths (the second one is constant), not counting crossings at the endpoints. Next, fix a complement $L_{11} \in \mathrm{Lag}(T_{(m_1,m_1)} M_1 \times M_1^-)$ of the diagonal. Then both $(\Lambda_{02} \times^N L_{11})^T$ and $\Lambda_{01} \times \Lambda_{12}$ are transverse to $T_{m_0} M_0 \times \Delta_1 \times T_{m_2} M_2$ and their composition is Λ_{02}. By Lemma 3.1.9 below we then find a path γ_{0112} and lift it to $\tilde{\gamma}_{0112} : [0,1] \to \mathrm{Lag}^N(T_{(m_0,m_1,m_1,m_2)} M_0^- \times M_1 \times M_1^- \times M_2)$ from $\tilde{\gamma}_{0112}(0) = [\tilde{\Lambda}_{02} \times^N \tilde{L}_{11}]^T$ to $\tilde{\gamma}_{0112}(1) = \tilde{\Lambda}_{01} \times^N \tilde{\Lambda}_{12}$ whose composition $\circ(\gamma_{0112}) = \Lambda_{02}$ is constant and that has no crossings with $\Lambda_0 \times \Delta_1 \times \Lambda_2$ (by the transversality $\gamma_{0112} \cap (\Lambda_0 \times \Delta_1 \times \Lambda_2) = \Lambda_{02} \cap (\Lambda_0 \times \Lambda_2) = \{0\}$). Here the grading of $\tilde{L}_{11}$ is determined by continuation along this path. Since the composition $\circ(\gamma_{0112})$ is constant this continuation yields

$$\tilde{\Lambda}_{02} = \circ^N(\tilde{\gamma}_{0112}) = \circ^N((\tilde{\Lambda}_{02} \times^N \tilde{L}_{11})^T) = d(\tilde{L}_{11}, \tilde{\Delta}_1^-) \cdot \tilde{\Lambda}_{02}.$$

Here we also used (14), and we deduce that $d(\tilde{L}_{11}, \tilde{\Delta}_1^-) = 0 \mod N$. Furthermore, we fix a path $\tilde{\gamma}_{11} : [0,1] \to \mathrm{Lag}^N(T_{(m_1,m_1)} M_1^- \times M_1)$ from $\tilde{\gamma}_{11}(0) = \tilde{\Delta}_1$ to $\tilde{\gamma}_{11}(1) = \tilde{L}_{11}^-$ whose crossing form with Δ_1 at $s = 0$ is positive definite, and thus

$$-I'(\gamma_{11}, \Delta_1) = d(\tilde{\Delta}_1, \tilde{L}_{11}^-) = d(\tilde{L}_{11}, \tilde{\Delta}_1^-) = 0 \mod N.$$

Now the concatenated path $(\tilde{\gamma}_{02} \times \tilde{\gamma}_{11})^T \# \tilde{\gamma}_{0112}^-$ connects $\tilde{\Lambda}_0 \times^N \tilde{\Delta}_1 \times^N \tilde{\Lambda}_2$ to $\tilde{\Lambda}_{01}^- \times^N \tilde{\Lambda}_{12}^-$ with positive definite crossing form at $s = 0$, and (17) can be verified,

$$\begin{aligned} d(\tilde{\Lambda}_0 \times^N \tilde{\Delta}_1 \times^N \tilde{\Lambda}_2, \tilde{\Lambda}_{01}^- \times^N \tilde{\Lambda}_{12}^-) &= -I'((\gamma_{02} \times \gamma_{11})^T \# \gamma_{0112}^-, \Lambda_0 \times \Delta_1 \times \Lambda_2) \\ &= -I'(\gamma_{02}, \Lambda_0 \times \Lambda_2) - I'(\gamma_{11}, \Delta_1) - I'(\gamma_{0112}^-, \Lambda_0 \times \Delta_1 \times \Lambda_2) \\ &= -I'(\gamma_{02}, \Lambda_0 \times \Lambda_2) = d(\tilde{\Lambda}_0 \times^N \tilde{\Lambda}_2, \tilde{\Lambda}_{02}^-). \end{aligned}$$

□

[2] Here it suffices to allow for Hamiltonian perturbation on M_0 and M_2, i.e. replacing L_0, L_2 with $L_0' := \phi_1^{H_0}(L_0)$, $L_2' := (\phi_1^{H_2})^{-1}(L_2)$. Then for every $(m_0, m_2) \in (L_0' \times L_2') \cap L_{02}$ there is a unique $m_1 \in M_1$ such that $(m_0, m_1) \in L_{01}$, $(m_1, m_2) \in L_{12}$, and hence $(m_0, m_1, m_2) \in (L_0' \times L_{12}) \cap (L_{01} \times L_2')$.

Lemma 3.1.9. *Let V_0, V_1, V_2 be symplectic vector spaces, $\Lambda_{02} \subset V_0^- \times V_2$ a Lagrangian subspace, and denote by*

$$\mathcal{T}_{\Lambda_{02}} \subset \mathrm{Lag}(V_0^- \times V_1 \times V_1^- \times V_2)$$

the subset of Lagrangian subspaces $\Lambda \subset V_0^- \times V_1 \times V_1^- \times V_2$ with $\Lambda \pitchfork (V_0 \times \Delta_{V_1} \times V_2) =: \hat{\Lambda}_{02}$ and $\pi_{02}(\hat{\Lambda}_{02}) = \Lambda_{02}$. Then $\mathcal{T}_{\Lambda_{02}}$ is contractible.

Proof. We fix metrics on V_0, V_1, and V_2. Then we will construct a contraction $(\rho_t)_{t\in[0,1]}$, $\rho_t : \mathcal{T}_{\Lambda_{02}} \to \mathcal{T}_{\Lambda_{02}}$ with $\rho_0 = \mathrm{Id}$ and $\rho_1 \equiv \Psi(\Lambda_{02} \times (\Delta_1)^\perp)$, where $\Psi : V_0^- \times V_2 \times V_1 \times V_1^- \to V_0^- \times V_1 \times V_1^- \times V_2$ exchanges the factors. To define $\rho_t(\Lambda)$ we write $\Lambda = \hat{\Lambda}_{02} \oplus \hat{\Lambda}_{11}$, where $\hat{\Lambda}_{11}$ is the orthogonal complement of $\hat{\Lambda}_{02}$ in Λ. Now $\hat{\Lambda}_{02}$ is the image of $(\mathrm{Id}_{V_0}, i_1, i_1, \mathrm{Id}_{V_2}) : \Lambda_{02} \to V_0^- \times V_1 \times V_1^- \times V_2$ for a linear map $i_1 : \Lambda_{02} \to V_1$ and $\hat{\Lambda}_{11}$ is the image of $(j_0, \mathrm{Id}_{V_1} + j_1, -\mathrm{Id}_{V_1} + j_1, j_2) : V_1 \to V_0^- \times V_1 \times V_1^- \times V_2$ for linear maps $j_i : V_1 \to V_i$. One can check that

$$\rho_t(\Lambda) := \mathrm{im}\big(\mathrm{Id}_{V_0}, t \cdot i_1, t \cdot i_1, \mathrm{Id}_{V_2}\big) \oplus \mathrm{im}\big(t \cdot j_0, \mathrm{Id}_{V_1} + t^2 \cdot j_1, -\mathrm{Id}_{V_1} + t^2 \cdot j_1, t \cdot j_2\big)$$

is an element of $\mathcal{T}_{\Lambda_{02}}$ for all $t \in [0,1]$ and defines a smooth contraction. □

4. Floer cohomology

The main content of this section is a review of the construction of graded Floer cohomology for pairs of Lagrangian submanifolds in monotone and exact cases by Floer, Oh, and Seidel. In 4.3 we then extend Floer cohomology to generalized Lagrangian correspondences, which in Section 5 will be reformulated in terms of pseudoholomorphic quilts.

4.1. **Monotonicity.** Let (M, ω) be a symplectic manifold. Let $\mathcal{J}(M, \omega)$ denote the space of compatible almost complex structures on (M, ω). Any $J \in \mathcal{J}(M, \omega)$ gives rise to a complex structure on the tangent bundle TM; the first Chern class $c_1(TM) \in H^2(M, \mathbb{Z})$ is independent of the choice of J. Throughout, we will use the following standing assumptions on all symplectic manifolds:

(M1): (M, ω) is *monotone*, that is for some $\tau \geq 0$

$$[\omega] = \tau c_1(TM).$$

(M2): If $\tau > 0$ then M is compact. If $\tau = 0$ then M is (necessarily) noncompact but satisfies "bounded geometry" assumptions as in [17].

Note here that we treat the exact case $[\omega] = 0$ as special case of monotonicity (with $\tau = 0$). Next, we denote the index map by

$$c_1 : \pi_2(M) \to \mathbb{Z}, \quad u \mapsto (c_1, u_*[S^2]).$$

The *minimal Chern number* $N_M \in \mathbb{N}$ is the positive generator of its image.

Associated to a Lagrangian submanifold $L \subset M$ are the Maslov index and action (i.e. symplectic area) maps

$$I : \pi_2(M, L) \to \mathbb{Z}, \qquad A : \pi_2(M, L) \to \mathbb{R}.$$

Our standing assumptions on all Lagrangian submanifolds are the following:

(L1): L is *monotone*, that is

$$2A(u) = \tau I(u) \quad \forall u \in \pi_2(M, L)$$

where the $\tau \geq 0$ is (necessarily) that from (M1).

(L2): L is compact and oriented.

Any homotopy class $[u] \in \pi_2(M, L)$ that is represented by a nontrivial J-holomorphic disk $u : (D, \partial D) \to (M, L)$ has positive action $A([u]) = \int u^*\omega > 0$. Monotonicity with $\tau > 0$ then implies that the index is also positive. So, for practical purposes, we define the (effective) *minimal Maslov number* $N_L \in \mathbb{N}$ as the generator of $I(\{[u] \in \pi_2(M, L) | A([u]) > 0\}) \subset \mathbb{N}$. If M and L are exact ($\tau = 0$), then $A \equiv 0$, so we have $N_L = \infty$.

If the Lagrangian submanifold L is oriented then $I(u)$ is always even since it is the Maslov index of a loop of oriented Lagrangian subspaces. So the orientation and monotonicity assumption on L imply $N_L \geq 2$, i.e. any nontrivial holomorphic disk must have $I(u) \geq 2$, which excludes disk bubbling in transverse moduli spaces of index 0 and 1.

In order for the Floer cohomology groups to be well defined we will also have to make the following additional assumption.

(L3): L has minimal Maslov number $N_L \geq 3$.

Moreover, we will restrict our considerations to Maslov coverings and gradings that are compatible with orientations, that is we make the following additional assumptions on the grading of the symplectic manifolds M and Lagrangian submanifolds $L \subset M$. (In the case $N = 2$ these assumptions reduce to (L2).)

(G1): M is equipped with a Maslov covering $\mathrm{Lag}^N(M)$ for N even, and the induced 2-fold Maslov covering $\mathrm{Lag}^2(M)$ is the one described in Example 3.0.8 (i).

(G2): L is equipped with a grading $\sigma_L^N : L \to \mathrm{Lag}^N(M)$, and the induced 2-grading $L \to \mathrm{Lag}^2(M)$ is the one given by the orientation of L.

In the following we discuss topological situations which ensure monotonicity.

Lemma 4.1.1. *Suppose that M is monotone and $L \subset M$ is a Lagrangian such that $\pi_1(L)$ is torsion (that is, every element has finite order). Then L is monotone and the minimal Maslov number is at least $2N_M/k$ where k is the maximum of orders of elements of $\pi_1(L)$.*

Proof. Let $u : (D, \partial D) \to (M, L)$ and let $k(u)$ be the order of the restriction of u to the boundary in $\pi_1(L)$. After passing to a $k(u)$-fold cover $\tilde{u}$, we may assume that the restriction of $\tilde{u}$ to ∂D is homotopically trivial in L. By adding the homotopy we obtain a sphere $v : S^2 \to M$ with $k(u)I(u) = I(\tilde{u}) = 2c_1(v)$ divisible by $2N_M$. For the relation between the first Chern class and the Maslov index see e.g. [10, Appendix]. The similar identity for the actions (due to $\omega|_L = 0$) completes the proof. □

In practice, we will need the action-index relation not only for disks as in (L1) but also for other surfaces with several boundary components mapping to several Lagrangians. (This really only becomes relevant in [22] for the definition of relative invariants from surfaces with striplike ends.) In particular, to define Floer cohomology for a pair of Lagrangians (and especially later to prove the isomorphism (3)) we need the action-index for annuli between the two Lagrangians. This provides the energy-index relation in Remark 4.2.2. In fact, it also implies monotonicity (L1) for both Lagrangians as long as M is connected.

Definition 4.1.2. (a) We say that a tuple $(L_e)_{e \in \mathcal{E}}$ is *monotone* with monotonicity constant $\tau \geq 0$ if the following holds: Let Σ be any connected compact surface with nonempty boundary $\partial\Sigma = \sqcup_{e \in \mathcal{E}} C_e$ (with C_e possibly empty or disconnected). Then for every map $u : \Sigma \to M$ satisfying $u(C_e) \subset L_e$ we have the action-index relation

$$2 \int u^*\omega = \tau \cdot I(u^*TM, (u^*TL_e)_{e \in \mathcal{E}}),$$

where I is the sum of the Maslov indices of the totally real subbundles $(u|_{C_e})^*TL_e$ in some fixed trivialization of u^*TM.

(b) We say that a pair (L_0, L_1) is *monotone for Floer theory* if (a) holds for the annulus $\Sigma = [0,1] \times S^1$ and every map u with boundary values $u(\{j\} \times S^1) \subset L_j$ for $j = 0, 1$.

The following is a minor generalization of [11, Proposition 2.7].

Lemma 4.1.3. *Suppose that M is monotone.*

(a) *If each $L_e \subset M$ is monotone, and the image of each $\pi_1(L_e)$ in $\pi_1(M)$ is torsion, then the tuple $(L_e)_{e\in\mathcal{E}}$ is monotone.*

(b) *If both $L_0, L_1 \subset M$ are monotone, and the image of $\pi_1(L_0)$ or $\pi_1(L_1)$ in $\pi_1(M)$ is torsion, then the pair (L_0, L_1) is monotone for Floer theory.*

Proof. To check (a) consider $u : \Sigma \to M$ satisfying $u(C_e) \subset L_e$. By assumption we have integers $N_e \in \mathbb{N}$ such that $N_e u|_{C_e}$ is contractible in M. Let $N = \prod_{e\in\mathcal{E}} N_e$, so that $Nu|_{C_e}$ is contractible for all boundary components C_e of Σ. Let $\tilde{\Sigma} \to \Sigma$ be a finite N-cover defined by a representation $\rho : \pi_1(\Sigma) \to \mathbb{Z}_N$ with $\rho([C_e]) = [N/N_e]$, so that each component of the inverse image $\tilde{C}_e$ of C_e is an N_e-fold cover. The pull-back $\tilde{u} : \tilde{\Sigma} \to M$ of $u : \Sigma \to M$ has restrictions to the boundary $\tilde{u}|_{\tilde{C}_e}$ that are homotopically trivial in M. Thus $\tilde{u}$ is homotopic to the union of some maps $v_e : (D, \partial D) \to (M, L_e)$ and a map $v : S \to M$ on a closed surface S. We can now use the closedness of ω and the monotonicity of M and each L_e to deduce

$$\begin{aligned} 2N \int u^*\omega \;=\; 2\int \tilde{u}^*\omega \;&=\; 2\int v^*\omega + \sum_{e\in\mathcal{E}} 2\int v_e^*\omega \\ &= 2\tau c_1(v^*TM) + \sum_{e\in\mathcal{E}} \tau I(v_e) \;=\; \tau I(\tilde{u}) \;=\; \tau N I(u), \end{aligned}$$

using properties of the Maslov index explained in [10, Appendix].

In the case of (b) we can take a multiple cover of the annulus such that one boundary loop is contractible in M, and hence the multiply covered annulus is homotopic to two disks to which we can apply monotonicity of the single Lagrangians. □

In the exact case, with $\omega = d\lambda$, any tuple of exact Lagrangians $(L_e)_{e\in\mathcal{E}}$, that is with $[\lambda|_{L_e}] = 0 \in H^1(L_e)$, is automatically monotone. Moreover, note that monotonicity is invariant under Hamiltonian isotopies of one or several Lagrangians.

Remark 4.1.4. Another situation in which one naturally has monotonicity is the Bohr-Sommerfeld setting, as pointed out to us by P. Seidel. Suppose that the cohomology class $[\omega]$ is integral. Let $(\mathcal{L}, \nabla) \to (M, \omega)$ be a unitary line-bundle-with-connection having curvature $(2\pi/i)\omega$. The restriction of $(\mathcal{L}, \nabla)$ to any Lagrangian $L \subset M$ is flat. L is *Bohr-Sommerfeld* if the restriction of $(\mathcal{L}, \nabla)$ to L is trivial, that is, there exists a non-zero horizontal section $\phi_L^{\mathcal{L}}$. The section $\phi_L^{\mathcal{L}}$ is unique up to a collection of phases $U(1)^{\pi_0(L)}$. Suppose that M is monotone, $[\omega] = \lambda c_1(M)$ for some $\lambda > 0$. Since $c_1(M)$ and $[\omega]$ are integral, we must have $\lambda = k/l$ for some integers $k, l > 0$. Let $\mathcal{K}^{-1} \to M$ denote the anticanonical bundle, $\mathcal{K}_m^{-1} = \Lambda^{\mathrm{top}}_{\mathbb{C}}(T^{0,1}_m M)$, which satisfies $kc_1(\mathcal{K}^{-1}) = l\frac{i}{2\pi}[\mathrm{curv}(\nabla)] = lc_1(\mathcal{L})$. Hence there exists an isomorphism

$$\Phi : (\mathcal{K}^{-1})^{\otimes k} \to \mathcal{L}^{\otimes l}.$$

Let $L \subset M$ be an oriented Lagrangian submanifold. The restriction of $\mathcal{K}^{-1}$ to L has a natural non-vanishing section $\phi_L^{\mathcal{K}}$ given by the orientation and the isomorphisms

$$\Lambda^{\mathrm{top}}_{\mathbb{R}} TL \to \Lambda^{\mathrm{top}}_{\mathbb{C}} T^{0,1}M|_L, \quad v_1 \wedge \ldots \wedge v_n \mapsto (v_1 + iJv_1) \wedge \ldots \wedge (v_n + iJv_n).$$

We say that L is *Bohr-Sommerfeld monotone* with respect to $(\mathcal{L}, \nabla, \Phi)$ if the sections $(\phi_L^{\mathcal{L}})^{\otimes l}$ and $\Phi \circ (\phi_L^{\mathcal{K}})^{\otimes k}$ are homotopic, that is, there exists a function $\psi : L \to \mathbb{R}$ such that

$$(\exp(2\pi i\psi)\phi_L^{\mathcal{L}})^{\otimes l} = \Phi \circ (\phi_L^{\mathcal{K}})^{\otimes k}. \tag{18}$$

Lemma 4.1.5. *Let $(L_e)_{e\in\mathcal{E}}$ be a collection of Lagrangians such that each is Bohr-Sommerfeld monotone with respect to $(\mathcal{L}, \nabla, \Phi)$. Then $(L_e)_{e\in\mathcal{E}}$ is monotone.*

Proof. Let Σ be a compact Riemann surface with boundary components $(C_e)_{e\in\mathcal{E}}$. Let $u : \Sigma \to M$ be a map with boundary $u(C_e) \subset L_e$. The index $I(u)$ is the sum of Maslov indices of the bundles $(u|_{C_e})^*TL_e$, with respect to some fixed trivialization of u^*TM. Equivalently, $I(u)$ is the sum of winding numbers of the sections $\phi_{L_e}^{\mathcal{K}}$ with respect to the induced trivialization of $u^*\mathcal{K}^{-1}$. Since each L_e is Bohr-Sommerfeld, $kI(u)$ is the sum of the winding numbers of the sections $(\phi_{L_e}^{\mathcal{L}})^{\otimes l}$, with respect to the induced trivialization of $u^*\mathcal{L}^{\otimes l}$. Write $u^*\nabla^{\otimes l} = \mathrm{d} + \alpha$ for some $\alpha \in \Omega^1(\Sigma)$ in this trivialization, so that $u^* \operatorname{curv}(\nabla^{\otimes l}) = \mathrm{d}\alpha$. Since the sections are horizontal, we have

$$kI(u) = (i/2\pi)\int_{\partial\Sigma} \alpha = (i/2\pi)\int_{\Sigma} u^* \operatorname{curv}(\nabla^{\otimes l}) = lA(u).$$

□

4.2. **Graded Floer cohomology for pairs of Lagrangians.** Let $L_0, L_1 \subset M$ be compact Lagrangian submanifolds. For a time-dependent Hamiltonian $H \in C^\infty([0,1] \times M)$ let $(X_t)_{t\in[0,1]}$ denote the family of Hamiltonian vector fields for $(H_t)_{t\in[0,1]}$, and let $\phi_{t_0,t_1} : M \to M$ denote its flow. (That is, $\phi_{t_0,t_1}(y) = x(t_1)$, where $x : [0,1] \to M$ satisfies $\dot{x} = X_t(x)$, $x(t_0) = y$.) We will abbreviate $\phi_1 := \phi_{0,1}$ for the time 1 flow from $t_0 = 0$ to $t_1 = 1$. Let $\mathrm{Ham}(L_0, L_1)$ be the set of $H \in C^\infty([0,1] \times M)$ such that $\phi_1(L_0)$ intersects L_1 transversally. Then we have a finite set of perturbed intersection points

$$\mathcal{I}(L_0, L_1) := \big\{\gamma : [0,1] \to M \,\big|\, \gamma(t) = \phi_{0,t}(\gamma(0)),\ \gamma(0) \in L_0,\ \gamma(1) \in L_1\big\}.$$

It is isomorphic to the intersection $\phi_1(L_0) \pitchfork L_1$. If we assume that M and L_0, L_1 are graded as in (G1-2), then we obtain a degree map from Section 3,

$$\mathcal{I}(L_0, L_1) \to \mathbb{Z}_N, \qquad x \mapsto |x| = d(\sigma_{L_0}^N(x), \sigma_{L_1}^N(x)).$$

Since N is even the sign $(-1)^{|x|}$ is well-defined. It agrees with the usual sign in the intersection number, given by the orientations of $\phi_1(L_0)$ and L_1, which also determine the mod 2 grading by assumption.

Next, we denote the space of time-dependent ω-compatible almost complex structures by

$$\mathcal{J}_t(M, \omega) := \mathcal{C}^\infty([0,1], \mathcal{J}(M, \omega)).$$

For any $J \in \mathcal{J}_t(M, \omega)$ and $H \in \mathrm{Ham}(L_0, L_1)$ we say that a map $u : \mathbb{R} \times [0,1] \to M$ is (J, H)-*holomorphic* with Lagrangian boundary conditions if

$$\overline{\partial}_{J,H} u := \partial_s u(s,t) + J_{t,u(s,t)}(\partial_t u(s,t) - X_t(u(s,t))) = 0, \tag{19}$$

$$u(\mathbb{R}, 0) \subset L_0, \quad u(\mathbb{R}, 1) \subset L_1. \tag{20}$$

The (perturbed) energy of a solution is

$$E_H(u) := \int_{\mathbb{R}\times[0,1]} |\partial_s u|^2 = \int_{\mathbb{R}\times[0,1]} u^*\omega + \mathrm{d}(H(u)\mathrm{d}t).$$

The following exponential decay lemma will be needed later and is part of the proof of Theorem 4.2.3 below.

Lemma 4.2.1. *Let $H \in \mathrm{Ham}(L_0, L_1)$ and $J \in \mathcal{J}_t(M,\omega)$. Then for any (J,H)-holomorphic strip $u : \mathbb{R}\times[0,1] \to M$ with Lagrangian boundary conditions in L_0, L_1 the following are equivalent:*

(a) *u has finite energy $E_H(u) = \int_{\mathbb{R}\times[0,1]} |\partial_s u|^2 < \infty$;*
(b) *There exist $x_\pm \in \mathcal{I}(L_0,L_1)$ such that $u(s,\cdot)$ converges to $x_\pm$ exponentially in all derivatives as $s \to \pm\infty$.*

For any $x_\pm \in \mathcal{I}(L_0,L_1)$ we denote by

$$\mathcal{M}(x_-,x_+) := \left\{u : \mathbb{R}\times[0,1] \to M \;\middle|\; (19),(20), E_H(u) < \infty, \lim_{s\to\pm\infty} u(s,\cdot) = x_\pm\right\}/\mathbb{R}$$

the space of finite energy (J,H)-holomorphic maps modulo translation in $s \in \mathbb{R}$. It is isomorphic to the moduli space of finite energy J'-holomorphic maps with boundary conditions in $\phi_1(L_0)$ and L_1, and without Hamiltonian perturbation. Here $J' \in \mathcal{J}_t(M,\omega)$ arises from J by pullback with $\phi_{t,1}$.

Remark 4.2.2. Suppose that the pair (L_0,L_1) is monotone, then for any $x_\pm \in \mathcal{I}(L_0,L_1)$ there exists a constant $c(x_-,x_+)$ such that for all $u \in \mathcal{M}(x_-,x_+)$ the energy-index relation holds:

$$2E_H(u) = \tau\cdot \mathrm{Ind}(D_u) + c(x_-,x_+), \tag{21}$$

where D_u denotes the linearized operator at u of the Cauchy-Riemann equation (19) on the space of sections of u^*TM satisfying the linearized Lagrangian boundary conditions from (20). Its Fredholm index is given by the Maslov-Viterbo index of u. This monotonicity ensures energy bounds for the moduli spaces of fixed index and thus compactness up to bubbling.

Theorem 4.2.3. *(Floer,Oh) Let $L_0, L_1 \subset M$ be a monotone pair of Lagrangian submanifolds satisfying (L1-2) and (M1-2). For any $H \subset \mathrm{Ham}(L_0,L_1)$, there exists a subset $\mathcal{J}_t^{\mathrm{reg}}(L_0,L_1;H) \subset \mathcal{J}_t(M,\omega)$ of Baire second category, such that the following holds for all $x_\pm \in \mathcal{I}(L_0,L_1)$.*

(a) *$\mathcal{M}(x_-,x_+)$ is a smooth manifold whose dimension near a nonconstant solution u is given by the formal dimension $\mathrm{Ind}(D_u) - 1$.*
(b) *The component $\mathcal{M}(x_-,x_+)_0 \subset \mathcal{M}(x_-,x_+)$ of formal dimension zero is finite.*
(c) *Suppose that L_0 and L_1 have minimal Maslov numbers $N_{L_k} \geq 3$. Then the one-dimensional component $\mathcal{M}(x_-,x_+)_1 \subset \mathcal{M}(x_-,x_+)$ has a compactification as one-dimensional manifold with boundary*

$$\partial\overline{\mathcal{M}(x_-,x_+)_1} \cong \bigcup_{x\in\mathcal{I}(L_0,L_1)} \mathcal{M}(x_-,x)_0 \times \mathcal{M}(x,x_+)_0 \tag{22}$$

(d) *If (L_0,L_1) is relatively spin (as defined in e.g. [23]), then there exists a coherent set of orientations on $\mathcal{M}(x_-,x_+)_0, \mathcal{M}(x_-,x_+)_1$ for all $x_\pm \in \mathcal{I}(L_0,L_1)$, that is, orientations compatible with (22).*

For the proofs of (a-c) we refer to Oh's paper [11] and the clarifications [12], [9]. For the exact case see [17]. The proof of (d) is contained in [23] loosely following [5]. From (d) we obtain a map

$$\epsilon : \mathcal{M}(x_-, x_+)_0 \to \{\pm 1\}$$

defined by comparing the given orientation to the canonical orientation of a point.

Now let M be a monotone symplectic manifold satisfying (M1-2) and equipped with an N-fold Maslov covering. Let $L_0, L_1 \subset M$ be a monotone, relative spin pair of graded Lagrangian submanifolds satisfying (L1-3), and let $H \in \mathrm{Ham}(L_0, L_1)$. The *Floer cochain group* is the $\mathbb{Z}_N$-graded group

$$CF(L_0, L_1) = \bigoplus_{d \in \mathbb{Z}_N} CF^d(L_0, L_1), \qquad CF^d(L_0, L_1) = \bigoplus_{x \in \mathcal{I}(L_0, L_1), |x| = d} \mathbb{Z}\langle x \rangle,$$

and the *Floer coboundary operator* is the map of degree 1,

$$\partial^d : \ CF^d(L_0, L_1) \to CF^{d+1}(L_0, L_1),$$

defined by

$$\partial^d \langle x_- \rangle := \sum_{x_+ \in \mathcal{I}(L_0, L_1)} \Big(\sum_{u \in \mathcal{M}(x_-, x_+)_0} \epsilon(u) \Big) \langle x_+ \rangle.$$

Here we choose some $J \in \mathcal{J}_t^{\mathrm{reg}}(L_0, L_1; H)$. If an isolated trajectory $u \in \mathcal{M}(x_-, x_+)_0$ exists, then the degree identity $|x_+| = |x_-| + 1$ can be seen by concatenating the paths $\tilde{\gamma_0}, \tilde{\gamma_1}$ of graded Lagrangians in the definition of $|x_-|$ with the unique graded lifts of u^*TL_0, u^*TL_1 to obtain paths of graded Lagrangians defining $|x_+|$ (using a trivialization of u^*TM over the strip, compactified to a disk). By additivity of the Maslov index this shows $|x_+| = |x_-| + I(u^*TL_0, u^*TL_1) = |x_-| + 1$. It follows from Theorem 4.2.3 that $\partial^2 = 0$. Now the *Floer cohomology* [3]

$$HF(L_0, L_1) := \bigoplus_{d \in \mathbb{Z}_N} HF^d(L_0, L_1), \qquad HF^d(L_0, L_1) := \ker(\partial^d)/\mathrm{im}(\partial^{d-1})$$

is $\mathbb{Z}_N$-graded. It is independent of the choice of H and J; a generalization of this fact is proved in Section 5.3 below. If the gradings moreover satisfy (G1-2), then we have a well defined splitting

$$HF(L_0, L_1) = HF^{\mathrm{even}}(L_0, L_1) \oplus HF^{\mathrm{odd}}(L_0, L_1).$$

Remark 4.2.4. In a suitable derived sense the Floer cohomology $HF(L_1, L_0)$ for the switched pair is the dual space $\mathrm{Hom}(HF(L_0, L_1), \mathbb{Z})$, see [25] for details.

4.3. **Floer cohomology for generalized Lagrangian correspondences.** The goal of this section is to define a first version of Floer cohomology for a cyclic generalized Lagrangian correspondence $\underline{L}$ as in Definition 2.1.3. So we consider $\underline{L} = (L_{01}, \ldots, L_{r(r+1)})$, a sequence of smooth Lagrangian correspondences $L_{(j-1)j} \subset M_{j-1}^- \times M_j$ between a sequence $M_0, M_1, \ldots, M_{r+1} = M_0$ of symplectic manifolds. For example, we could consider a non-cyclic sequence of Lagrangians $L_{01} \subset M_1$, $\big(L_{(i-1)i} \subset M_{i-1}^- \times M_i\big)_{i=2,\ldots,r}$, $L_{r0} \subset M_r^-$, which is a special case of the cyclic setup with $M_0 = \{\mathrm{pt}\}$. The usual Floer cohomology for pairs of Lagrangians fits into this case with $r = 1$ and Lagrangian submanifolds $L_{01}, L_{10} \subset M_1$.

[3] Note that our conventions differ from Seidel's definition of graded Floer cohomology in [15] in two points which cancel each other: The roles of x_- and x_+ are interchanged and we switched the sign of the Maslov index in the definition of the degree (11).

We assume that $\underline{L}$ satisfies (M1-2,L1-3), i.e. each M_j satisfies (M1-2) and each $L_{(j-1)j}$ satisfies (L1-3) with a fixed monotonicity constant $\tau \geq 0$. We moreover assume that $\underline{L}$ is graded in the sense of Definition 3.1.2 and equipped with a relative spin structure in the following sense.

Definition 4.3.1. Let $\underline{L} = (L_{01}, \ldots, L_{r(r+1)})$ be a cyclic generalized Lagrangian correspondence (i.e. $L_{j(j+1)} \subset M_j^- \times M_{j+1}$ for a cyclic sequence $M_0, M_1, \ldots, M_{r+1} = M_0$ of symplectic manifolds). A *relative spin structure* on $\underline{L}$ consists of a collection of background classes $b_j \in H^2(M_j, \mathbb{Z}_2)$ for $j = 0, \ldots, r+1$ and relative spin structures on $L_{j(j+1)}$ with background classes $-\pi_j^* b_j + \pi_{j+1}^* b_{j+1}$. The cyclic requirement on the background classes $b_0 \in H^2(M_0, \mathbb{Z}_2)$ and $b_{r+1} \in H^2(M_{r+1}, \mathbb{Z}_2) = H^2(M_0, \mathbb{Z}_2)$ is $b_{r+1} = b_0$ for r odd and $b_{r+1} = b_0 + w_2(M_0)$ for r even.[4]

Eventually, in Section 5, we will define the Floer cohomology $HF(\underline{L})$ directly, using "quilts of pseudoholomorphic strips". In this section however we define $HF(\underline{L})$ as a special case of the Floer cohomology for pairs of Lagrangian submanifolds – which are constructed from the sequence $\underline{L}$ as follows. If $\underline{L}$ has even length $r+1 \in 2\mathbb{N}$ we define a pair of graded Lagrangian submanifolds,

$$\begin{aligned} L_{(0)} &:= (L_{01} \times L_{23} \times \ldots \times L_{(r-1)r}) \\ L_{(1)} &:= (L_{12} \times L_{34} \times \ldots \times L_{r(r+1)})^T \end{aligned} \quad \subset M_0^- \times M_1 \times M_2^- \times \ldots \times M_r =: \widetilde{M}.$$

Here we denote by $M_1^- \times \ldots \times M_r^- \times M_0 \to M_0^- \times M_1 \times \ldots \times M_r$, $Z \mapsto Z^T$ the transposition of the last to the first factor, combined with an overall sign change in the symplectic form. If $\underline{L}$ has odd length $r+1 \in 2\mathbb{N}+1$ we insert the diagonal $\Delta_0 \subset M_0^- \times M_0 = M_{r+1}^- \times M_0$ (equipped with its canonical grading) into $\underline{L}$ before arranging it into a pair of Lagrangian submanifolds as above, yielding

$$\begin{aligned} L_{(0)} &= (L_{01} \times L_{23} \times \ldots \times L_{r(r+1)}) \\ L_{(1)} &= (L_{12} \times L_{34} \times \ldots \times L_{(r-1)r} \times \Delta_0)^T \end{aligned} \quad \subset M_0^- \times M_1 \times \ldots \times M_r^- \times M_{r+1} = \widetilde{M}$$

In the case of a noncyclic correspondence with $M_0 = M_{r+1} = \{\mathrm{pt}\}$ the transposition as well as insertion of the diagonal are trivial operations. Note that, beyond the grading, also the monotonicity, compactness, and orientation assumptions (L1-2) on $\underline{L}$ transfer directly to properties (L1-2) for $L_{(0)}$ and $L_{(1)}$. Similarly, a relative spin structure on $\underline{L}$ induces compatible relative spin structures on $L_{(0)}$ and $L_{(1)}$, see [23]. Moreover, we say that $\underline{L}$ is *monotone* if the pair of Lagrangians $(L_{(0)}, L_{(1)})$ is monotone in the sense of Definition 4.1.2(b). If this is the case, then a graded Floer cohomology for $\underline{L}$ can be defined by

$$HF(\underline{L}) := HF(L_{(0)}, L_{(1)}).$$

Remark 4.3.2. To see that $HF(L_{(0)}, L_{(1)})$ is well defined we need to make sure that $L_{(0)}$ and $L_{(1)}$ also satisfy (L3), or that the bubbling of holomorphic discs is otherwise excluded. Note that the assumption (L3) on the factors of $\underline{L}$ does not directly transfer to the product Lagrangians $L_{(0)}$ and $L_{(1)}$ since a difference of Maslov numbers greater than 3 could give a total Maslov number less than 3. However, if we use a split almost complex structure $\widetilde{J} = J_0 \oplus \ldots \oplus J_r$ on $\widetilde{M}$, induced from compatible almost complex structures J_k on each M_k, then any nonconstant holomorphic disc in $\widetilde{M}$ with boundary on $L_{(0)}$ or $L_{(1)}$ will simply

[4] This shift is necessary in order to fit in the canonical relative spin structure for the diagonal Δ_0, see [20] for details.

be a product of J_k-holomorphic discs. Pairs of these discs take boundary values in the Lagrangian correspondences $L_{(k-1)k}$ which satisfy the monotonicity assumptions as well as (L3). Hence each of these double discs must have nonnegative area and hence index, and at least one of them has positive area and hence Maslov index at least 3.

The proof that transversality can be achieved with an almost complex structure (and also Hamiltonian perturbation) of split type can be found in Theorem 5.2.3 and Proposition 5.2.1 below. This excludes bubbling such that $\partial^2 = 0$ for this specific choice of perturbation data (and hence for any other choice of regular perturbation data). So the Floer cohomology $HF(L_{(0)}, L_{(1)})$ is indeed well defined.

In the case of a non-cyclic sequence the Floer cohomology $HF(\underline{L})$ specializes to

$$HF(L_1, L_{12}, \ldots, L_{(r-1)r}, L_r) = HF(L_1 \times L_{23} \times \ldots, L_{12} \times L_{34} \times \ldots).$$

In particular we reproduce the definition of Floer cohomology for a pair of Lagrangians $L_0, L_1 \subset M$, viewed as cyclic correspondence $\{pt.\} \xrightarrow{L_0} M \xrightarrow{L_1} \{pt.\}$. We moreover define a Floer cohomology for any Lagrangian $L \subset M^- \times M$, viewed as cyclic correspondence $M \xrightarrow{L} M$, in particular for graphs $L = \mathrm{graph}(\phi)$ of symplectomorphisms $\phi : M \to M$. By definition, this invariant is $HF(L) := HF(L, \Delta_M)$, which reproduces the Floer cohomology $HF(\mathrm{graph}(\phi)) = HF(\mathrm{graph}(\phi), \Delta_M) = HF(\phi)$ of a symplectomorphism.

5. Quilted Floer cohomology

The purpose of this section is to reformulate the definition of Floer cohomology for generalized Lagrangian correspondences in terms of quilted surfaces (consisting of strips). As in Section 4.3 consider a cyclic generalized Lagrangian correspondence $\underline{L}$, that is, a sequence of symplectic manifolds $M_0, M_1, \ldots, M_r, M_{r+1}$ with $M_0 = M_{r+1}$ for $r \geq 0$, and a sequence of Lagrangian correspondences

$$L_{01} \subset M_0^- \times M_1, \quad L_{12} \subset M_1^- \times M_2, \quad \ldots, \quad L_{r(r+1)} \subset M_r^- \times M_{r+1}.$$

5.1. **Unfolding of Floer cohomology in products.** We defined the Floer cohomology $HF(\underline{L})$ as the standard Floer cohomology in the product manifold $\widetilde{M} = M_0^- \times M_1 \times M_2^- \times \ldots$ of a pair of Lagrangians $L_{(0)}, L_{(1)}$ that is built from the cyclic sequence $\underline{L}$. We will show how quilts arise naturally from "unfolding" this construction and phrasing it in terms of tuples of holomorphic curves in the M_j.

Informally, $HF(\underline{L})$ can be viewed as the Morse homology on the path space

$$\mathcal{P}(L_{(0)}, L_{(1)}) = \left\{ y : [0,1] \to \widetilde{M} \,\middle|\, y(0) \in L_{(0)}, y(1) \in L_{(1)} \right\}$$

of the (potentially multivalued) symplectic action functional

$$\mathcal{A}_H(y) = \int_{[0,1]\times[0,1]} v^* \omega_{\widetilde{M}} + \int_0^1 H(t, y(t)) \mathrm{d}t.$$

Here $v : [0,1] \to \mathcal{P}(L_{(0)}, L_{(1)})$ is a smooth homotopy from a fixed $v(0) = y_0 \in \mathcal{P}(L_{(0)}, L_{(1)})$ (in a given connected component) to $v(1) = y$, which can also be viewed as map $v : [0,1] \times [0,1] \to \widetilde{M}$ satisfying Lagrangian boundary conditions on $\{0\} \times [0,1]$ and $\{1\} \times [0,1]$.

Suppose for now that r is odd, then the path space can be identified with the set of tuples of paths in the manifolds M_j, connected via $L_{j(j+1)}$-matching conditions at the ends,

$$\mathcal{P}(\underline{L}) = \left\{ \underline{x} = \left(x_j : [0,1] \to M_j\right)_{j=0,\ldots,r} \,\middle|\, (x_j(1), x_{j+1}(0)) \in L_{j(j+1)} \right\}.$$

Here and throughout we will use the index $j \in \{0, \dots, r\}$ modulo $r+1$, so e.g. $x_{r+1} := x_0$ and the matching condition for $j = r+1$ is $(x_r(1), x_0(0)) \in L_{r(r+1)}$. We make the identification with $\mathcal{P}(L_{(0)}, L_{(1)})$ by $y(t) = \big(x_0(1-t), x_1(t), x_2(1-t), \dots, x_r(t)\big)$, then the unperturbed $(H = 0)$ symplectic action functional on $\mathcal{P}(\underline{L})$ becomes

$$\mathcal{A}_0(\underline{x}) = \sum_{j=0}^{r} \int_{[0,1]\times[0,1]} v_j^* \omega_{M_j}.$$

Here $v_j : [0,1] \times [0,1] \to M_j$ interpolate between fixed paths $v_j(0,\cdot)$ and $v_j(1,\cdot) = x_j$, and satisfy what we will call "seam conditions" $(v_j(s,1), v_{j+1}(s,0)) \in L_{j(j+1)}$ for all $s \in [0,1]$. Next, assume that the almost complex structure on $\widetilde{M}$ is of time-independent split form $J = (-J_0) \oplus J_1 \oplus (-J_2) \oplus \ldots \oplus J_r$, given by a tuple $J_j \in \mathcal{J}(M_j, \omega_j)$ of almost complex structures on the factors of $\widetilde{M}$. This defines a metric on the path space, and the gradient flow lines, viewed as PDE's are the J-holomorphic strips $w : \mathbb{R} \times [0,1] \to \widetilde{M}$ with boundary values in $L_{(0)}$ and $L_{(1)}$. They are in one-to-one correspondence with $(r+1)$-tuples of J_j-holomorphic maps $u_j : \mathbb{R} \times [0,1] \to M_j$ satisfying the seam conditions

$$(u_j(s,1), u_{j+1}(s,0)) \in L_{j(j+1)}, \qquad \text{for all } j = 0, \dots, r,\ s \in \mathbb{R}.$$

Here we again use cyclic notation $u_{r+1} := u_0$, and the correspondence is given by $w(s,t) = \big(u_0(s, 1-t), u_1(s,t), u_2(s, 1-t), \dots, u_r(s,t)\big)$.

For r even there is a slight modification of the previous correspondence. The product manifold $\widetilde{M}$ has two factors M_0 and $M_{r+1} = M_0$ matched up via the diagonal. So the path space can be identified with the generalized path space $\mathcal{P}(\underline{L})$ as above with the exception that the path $x_0 : [0,2] \to M_0$ in $M_0 = M_{r+1}$ is parametrized by an interval of length 2 and satisfies the matching condition $(x_0(2), x_1(0)) \in L_{01}$ at its end. Similarly, a J-holomorphic strip $w : \mathbb{R} \times [0,1] \to \widetilde{M}$ corresponds via $w(s,t) = \big(u_0(s, 2-t), u_1(s,t), u_2(s, 1-t) \dots, u_r(s, 1-t), u_0(s,t)\big)$ to a tuple of J_j-holomorphic strips as above, with the exception that the strip $u_0 : \mathbb{R} \times [0,2] \to M_0$ has width 2. This tuple $(u_j)_{j=0,\dots,r}$ is the first instance of a nontrivial pseudoholomorphic quilt – containing strips of different widths.

When r is even, the Floer trajectories of the pair $L_{(0)}, L_{(1)}$ in fact cannot be identified with an $(r+1)$-tuple of pseudoholomorphic maps, all defined on strips of width 1, with seam conditions in $L_{j(j+1)}$. Conformal rescaling $\tilde{u}_0(s,t) := u_0(2s, 2t)$ would result in a "time-shifted" matching condition $(\tilde{u}_0(s,1), u_1(2s,0)) \in L_{01}$ unless u_1 is rescaled, too, which would result in $\tilde{u}_0$ having width 1 but all other strips having width $\frac{1}{2}$. In fact, only simultaneous rescaling of all components in these pseudoholomorphic quilts preserves holomorphicity and seam conditions (unless the Lagrangian correspondences are of split type, e.g. $L_{01} = L_0 \times L_1$ for Lagrangians $L_j \subset M_j$). It cannot change the relative widths of strips.

By a reparametrization of the path in M_0, one could identify $\mathcal{P}(L_{(0)}, L_{(1)})$ and the action functional with the generalized path space $\mathcal{P}(\underline{L})$ and a corresponding action functional, where all paths are parametrized by $[0,1]$. However, the reparametrized $\hat{u}_0(s,t) := u_0(s, 2t)$ now satisfies $\partial_s \hat{u}_0 + \frac{1}{2} J_0 \partial_t \hat{u}_0 = 0$ with a no longer complex structure $\frac{1}{2} J_0$ that squares to $-\frac{1}{4}$. This is due to the fact that the pullback of the metric on $\mathcal{P}(L_{(0)}, L_{(1)})$ to $\mathcal{P}(\underline{L})$ is the L^2-metric on each factor with respect to $\omega_j(\cdot, J_j \cdot)$ for $j = 1, \dots, r$ but $\frac{1}{2}\omega_0(\cdot, J_0 \cdot)$ on M_0. We could drop the factor $\frac{1}{2}$ in the metric on M_0 to obtain J_j-holomorphic strips of width 1 in each factor as trajectories, however these would be the "gradient flow lines" with respect to a different metric. In general, it is not know how Floer homology behaves under a change of

metric. However, we will show that it is independent of the choice of weights $\delta_j^{-1}\omega_j(\cdot, J_j\cdot)$ in the L^2-metric on $\mathcal{P}(\underline{L})$. This setup is equivalent to defining the generalized path space with varying widths $x_j : [0, \delta_j] \to M_j$ but fixing the standard L^2-metric induced by ω_j and J_j on each factor.

5.2. **Construction of quilted Floer cohomology.** In the quilted setup for $HF(\underline{L})$ we fix widths $\underline{\delta} = (\delta_j > 0)_{j=0,\dots,r}$ and consider the generalized path space

$$\mathcal{P}(\underline{L}) := \big\{\underline{x} = \big(x_j : [0,\delta_j] \to M_j\big)_{j=0,\dots,r} \,\big|\, (x_j(\delta_j), x_{j+1}(0)) \in L_{j(j+1)}\big\}.$$

We define a perturbed symplectic action functional on $\mathcal{P}(\underline{L})$ by picking a homotopy $\underline{v} = (v_j)_{j=0,\dots,r} : [0,1] \to \mathcal{P}(\underline{L})$ from a fixed $\underline{v}(0)$ to $\underline{v}(1) = \underline{x}$ and setting

$$\mathcal{A}_{\underline{H}}(\underline{x}) = \sum_{j=0}^{r}\bigg(\int_{[0,1]\times[0,\delta_j]} v_j^*\omega_{M_j} + \int_0^{\delta_j} H_j(t, x_j(t))\mathrm{d}t\bigg),$$

using a tuple of Hamiltonian functions

$$\underline{H} = \big(H_j \in C^\infty([0,\delta_j]\times M_j)\big)_{j=0,\dots,r}.$$

By folding and rescaling as in the previous section, this is equivalent to the path space $\mathcal{P}(L_{(0)}, L_{(1)})$ with symplectic action functional perturbed by a Hamiltonian of split type, e.g. $H = \sum_{j=0}^r (-1)^{j+1}\delta_j \tilde{H}_j$ for r odd, where $\tilde{H}_j(t,x) = H_j(\delta_j t, x)$ for j odd and $\tilde{H}_j(t,x) = H_j(\delta_j(1-t), x)$ for j even. Here the critical points correspond to the perturbed intersection points $\phi_1^H(L_{(0)}) \cap L_{(1)}$, where ϕ_1^H is the time-one flow of H. In the quilted setup, the critical points of $\mathcal{A}_{\underline{H}}$ are tuples of Hamiltonian chords,

$$\mathcal{I}(\underline{L}) := \left\{\underline{x} = \big(x_j : [0,\delta_j] \to M_j\big)_{j=0,\dots,r} \;\middle|\; \begin{array}{r} \dot{x}_j(t) = X_{H_j}(x_j(t)), \\ (x_j(\delta_j), x_{j+1}(0)) \in L_{j(j+1)} \end{array}\right\}.$$

$\mathcal{I}(\underline{L})$ is canonically identified with $\times_{\phi^{H_0}_{\delta_0}}\big(L_{01} \times_{\phi^{H_1}_{\delta_1}} L_{12} \dots \times_{\phi^{H_r}_{\delta_r}} L_{r(r+1)}\big)$, the set of points

$$\big\{(m_0,\dots,m_r) \in M_0 \times \dots \times M_r \,\big|\, (\phi^{H_j}_{\delta_j}(m_j), m_{j+1}) \in L_{j(j+1)}\big\},$$

where $\phi^{H_j}_{\delta_j}$ is the time δ_j flow of the Hamiltonian H_j. In this setting we can check that Hamiltonians of split type suffice to achieve transversality for the intersection points.

Proposition 5.2.1. *There is a dense open subset* $\mathrm{Ham}(\underline{L}) \subset \oplus_{j=0}^r C^\infty([0,\delta_j]\times M_j)$ *such that for every* $(H_0,\dots,H_r) \in \mathrm{Ham}(\underline{L})$ *the set* $\times_{\phi^{H_0}_{\delta_0}}\big(L_{01} \times_{\phi^{H_1}_{\delta_1}} L_{12} \dots \times_{\phi^{H_r}_{\delta_r}} L_{r(r+1)}\big)$ *is smooth and finite, that is, the defining equations are transversal.*

Proof. By assumption $L_{j(j+1)}$ is an embedded submanifold of $M_{j(j+1)} := M_j^- \times M_{j+1}$ and so locally $L_{j(j+1)}$ is the zero set of a submersion $\psi_{j(j+1)} : M_{j(j+1)} \to \mathbb{R}^{n_j+n_{j+1}}$. The defining equations for $\times_{\phi^{H_0}_{\delta_0}}\big(L_{01} \times_{\phi^{H_1}_{\delta_1}} L_{12} \dots \times_{\phi^{H_r}_{\delta_r}} L_{r(r+1)}\big)$ are

$$\psi_{j(j+1)}\big(\phi^{H_j}_{\delta_j}(m_j), m_{j+1}\big) = 0 \qquad \text{for all } j = 0,\dots,r. \tag{23}$$

Consider the universal moduli $\mathcal{U}$ space of data $(H_0,\dots,H_r,m_0,\dots,m_r)$ satisfying (23), where now each H_j has class C^ℓ for some $\ell \geq 2$. The linearized equations for $\mathcal{U}$ are

$$D\psi_{j(j+1)}(D\phi^{H_j}_{\delta_j}(h_j,v_j), v_{j+1}) = 0 \qquad \text{for all } j = 0,\dots,r. \tag{24}$$

for $v_j \in T_{m_j} M_j$ (with $v_{r+1} := v_0$) and $h_j \in C^\ell([0,1] \times M_j)$. The map

$$C^\ell([0,1] \times M_j) \to T_{\phi^{H_j}_{\delta_j}(m_j)} M_j, \quad h_j \mapsto D\phi^{H_j}_{\delta_j}(h_j, 0)$$

is surjective, which shows that the product of the operators on the left-hand side of (24) is also surjective. By the implicit function theorem $\mathcal{U}$ is a smooth Banach manifold, and we consider its projection to $\oplus_{k=0}^r C^\ell([0,\delta_k] \times M_k)$. By the Sard-Smale theorem, the set of regular values (the set of functions $H = (H_0, \dots, H_r)$ such that the perturbed intersection is transversal) is dense in $\oplus_{k=0}^r C^\ell([0,\delta_k] \times M_k)$. On the other hand, the set of regular values is clearly open. Hence the set of smooth functions that are regular values is open and dense. □

With this split Hamiltonian perturbation we have a canonical bijection of critical points $\phi_1^H(L_{(0)}) \cap L_{(1)} \cong \mathcal{I}(\underline{L})$, and hence the (graded) Floer chain group $CF(L_{(0)}, L_{(1)})$ is identified with

$$CF(\underline{L}) := \bigoplus_{d \in \mathbb{Z}_N} CF^d(\underline{L}), \qquad CF^d(\underline{L}) := \bigoplus_{\underline{x} \in \mathcal{I}(\underline{L}), |\underline{x}| = d} \mathbb{Z}\langle \underline{x} \rangle.$$

The grading is defined as in Section 3.1,

$$\mathcal{I}(\underline{L}) \cong \phi_1^H(L_{(0)}) \cap L_{(1)} \to \mathbb{Z}_N, \qquad \underline{x} \cong y \mapsto |y| = |\underline{x}|.$$

Next, fix a tuple of almost complex structures

$$\underline{J} = (J_j)_{j=0,\dots,r} \in \oplus_{j=0}^r \mathcal{C}^\infty([0,\delta_j], \mathcal{J}(M_j, \omega_j)) =: \mathcal{J}_t(\underline{L})$$

and equip $\mathcal{P}(\underline{L})$ with the L^2-metric induced by the t-dependent metric $\omega_j(\cdot, J_j \cdot)$ on each factor M_j. Then the Floer trajectories (obtained by reformulating the gradient flow as PDE) are $(r+1)$-tuples of maps $u_j : \mathbb{R} \times [0,\delta_j] \to M_j$ that are (J_j, H_j)-holomorphic,

$$\overline{\partial}_{J_j, H_j} u_j = \partial_s u_j + J_j\big(\partial_t u_j - X_{H_j}(u_j)\big) = 0 \qquad \forall j = 0, \dots r, \tag{25}$$

and satisfy the seam conditions

$$(u_j(s,\delta_j), u_{j+1}(s,0)) \in L_{j(j+1)} \qquad \forall j = 0, \dots r,\ s \in \mathbb{R}. \tag{26}$$

For a Floer trajectory to be counted towards the differential between critical points $\underline{x}^\pm \in \mathcal{I}(\underline{L})$ we moreover require finite energy and limits

$$E(\underline{u}) := \sum_{j=0}^r \int_{\mathbb{R}\times[0,\delta_j]} u_j^* \omega_j + \mathrm{d}(H_j(u_j)\mathrm{d}t) < \infty, \qquad \lim_{s \to \pm\infty} u_j(s,\cdot) = x_j^\pm \quad \forall j = 0, \dots, r. \tag{27}$$

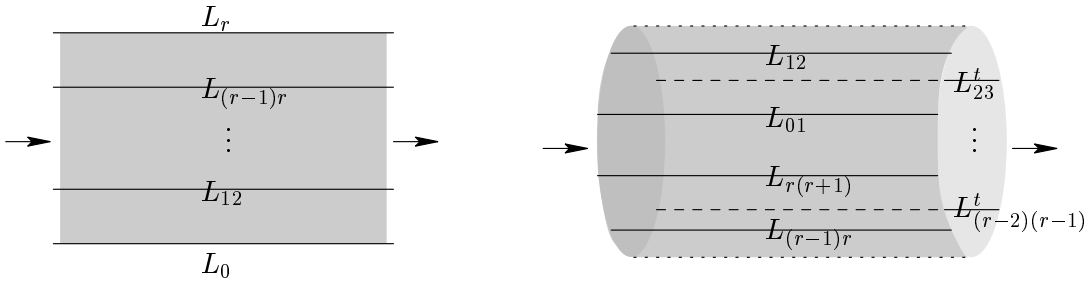

FIGURE 1. Quilted Floer trajectories for $M_0 = \{\mathrm{pt}\}$ and in general

As in standard Floer theory, the moduli spaces of "quilted holomorphic strips"

$$\mathcal{M}(\underline{x}^-,\underline{x}^+) := \big\{\underline{u} = \big(u_j : \mathbb{R}\times[0,\delta_j] \to M_j\big)_{j=0,\dots,r} \,\big|\, (25),(26),(27)\big\}/\mathbb{R}$$

arise from quotienting out by simultaneous $\mathbb{R}$-shift in all components u_j. (Separate shifts will not preserve the seam condition unless the correspondences are of split type.) We will see that they have the same Fredholm, exponential decay, and compactness properties as usual for Floer trajectories. For that purpose we restrict ourselves to the monotone case.

Remark 5.2.2. The "monotonicity for Floer theory" assumption for the pair $(L_{(0)}, L_{(1)})$ in Definition 4.1.2 can be phrased directly for $\underline{L}$ in the language of [22]: "$\underline{L}$ is a monotone boundary condition for the quilted cylinder". That is, the action-index relation

$$2\sum_{j=0}^{r}\int u_j^*\omega_j = \tau\cdot I\big((u_j^*TM_j)_{j=0,\dots,r}, (s_{j(j+1)}^*TL_{j(j+1)})_{j=0,\dots,r}\big)$$

holds for each tuple of maps $u_j : S^1\times[0,\delta_j]\to M_j$ that satisfies the seam conditions $s_{j(j+1)}(s) := (u_j(s,\delta_j), u_{j+1}(s,0)) \in L_{j(j+1)}$ for $j=0,\dots,r$. Here the topological index I is defined by choosing a trivialization for each $u_j^*TM_j$ and then summing over the Maslov indices of the loops $s_{j(j+1)}^*TL_{j(j+1)}$ of Lagrangian subspaces with respect to these trivializations.

Note that the monotonicity condition for $\underline{L}$ is independent of the width δ_j of the annuli that parametrize the maps u_j. Moreover, it implies monotonicity for the sequence $\underline{L}' = (L_{01},\dots,L_{(j-1)j}\circ L_{j(j+1)},\dots,L_{r(r+1)})$ obtained from an embedded composition $L_{(j-1)j}\circ L_{j(j+1)}$. To see the latter note that any seam condition $s_{(j-1)(j+1)} : S^1\to L_{(j-1)j}\circ L_{j(j+1)}$ induces a smooth map $u_j : S^1\times[0,1]\to M_j$ that is constant in $[0,1]$, fits the seam conditions for $\underline{L}$, but contributes zero to both energy and Maslov index. Hence the action index relation for $\underline{L}$ implies the same relation for $\underline{L}'$.

Theorem 5.2.3. *Suppose that the symplectic manifolds M_j satisfy (M1-2) with the same value of the monotonicity constant τ, the Lagrangian correspondences $L_{j(j+1)}$ satisfy (L1-2), and $\underline{L}$ satisfies the monotonicity assumption of Remark 5.2.2.*

For any choice of widths $\underline{\delta}$ and regular Hamiltonian perturbations $\underline{H}\subset \mathrm{Ham}(\underline{L})$, there exists a subset $\mathcal{J}_t^{\mathrm{reg}}(\underline{L};\underline{H})\subset\mathcal{J}_t(\underline{L})$ of Baire second category, such that the following holds for all $\underline{x}_\pm\in\mathcal{I}(\underline{L})$.

(a) *$\mathcal{M}(\underline{x}_-,\underline{x}_+)$ is a smooth manifold whose dimension near a nonconstant solution $\underline{u}$ is given by the formal dimension, equal to $\mathrm{Ind}(D_{\underline{u}})-1$. Here $D_{\underline{u}}$ is the linearized operator at $\underline{u}$ of (25) on the space of sections satisfying the linearized boundary- and seam conditions of (26).*

(b) *The component $\mathcal{M}(\underline{x}_-,\underline{x}_+)_0\subset\mathcal{M}(\underline{x}_-,\underline{x}_+)$ of formal dimension zero is finite.*

(c) *Suppose that each $L_{j(j+1)}$ has minimal Maslov number $N_{L_{j(j+1)}}\geq 3$. Then the one-dimensional component $\mathcal{M}(\underline{x}_-,\underline{x}_+)_1\subset\mathcal{M}(\underline{x}_-,\underline{x}_+)$ has a compactification as one-dimensional manifold with boundary*

$$\partial\overline{\mathcal{M}(\underline{x}_-,\underline{x}_+)_1} \cong \bigcup_{\underline{x}\in\mathcal{I}(\underline{L})} \mathcal{M}(\underline{x}_-,\underline{x})_0\times\mathcal{M}(\underline{x},\underline{x}_+)_0 \tag{28}$$

(d) *If $\underline{L}$ is relatively spin (as defined in Definition 4.3.1), then there exists a coherent set of orientations on $\mathcal{M}(\underline{x}_-,\underline{x}_+)_0, \mathcal{M}(\underline{x}_-,\underline{x}_+)_1$ for all $\underline{x}_\pm\in\mathcal{I}(\underline{L})$, that is, orientations compatible with (28).*

Proof. Suppose for simplicity that r is odd. (For even r we can insert a diagonal into the sequence $\underline{L}$, then the quilted holomorphic strips of widths $\underline{\delta}$ can be identified with quilted holomorphic strips for the new sequence with widths $(\frac{\delta_0}{2}, \delta_1, \ldots, \delta_r, \frac{\delta_0}{2})$.) Then the quilted moduli space $\mathcal{M}(\underline{x}_-, \underline{x}_+)$ is canonically identified with the moduli space of $(J_{\underline{\delta}}, H)$-holomorphic maps $w : \mathbb{R} \times [0,1] \to \widetilde{M}$ with boundary conditions $w(\mathbb{R}, 0) \subset L_{(0)}$, $w(\mathbb{R}, 1) \subset L_{(1)}$, finite energy $E_H(w) < \infty$, and limits $\lim_{s\to\pm\infty} w(s, \cdot) = \underline{x}_\pm \in \mathcal{I}(L_{(0)}, L_{(1)})$.

The correspondence is by

$$w(s,t) = \big(u_0(s, \delta_0(1-t)), u_1(s, \delta_1 t), u_2(s, \delta_2(1-t)), \ldots, u_r(s, \delta_r t)\big),$$

where $H = \sum_{j=0}^r (-1)^{j+1} \delta_j \tilde{H}_j$ as above and

$$J_{\underline{\delta}} := \big(-\delta_0^{-1} J_0(\delta_0(1-t)), \delta_1^{-1} J_1(\delta_1 t), \ldots, \delta_r^{-1} J_r(\delta_r t)\big)$$

satisfies all properties of a t-dependent $\omega_{\widetilde{M}}$-compatible almost complex structure except that it squares to the negative definite diagonal matrix $J_{\underline{\delta}}^2 = -(\delta_0^{-2}\mathrm{Id}_{TM_0} \oplus \ldots \oplus \delta_r^{-2}\mathrm{Id}_{TM_r})$.

This still presents an elliptic boundary value problem of the form $\partial_s + \mathcal{D}$, where the linearizations of $\mathcal{D}$ are self-adjoint operators. Hence the Fredholm property, energy-index relation, exponential decay, compactness, and gluing properties carry over directly from the case of holomorphic curves. One crucial component is the spectrum of the operator $J_{\underline{\delta}}\partial_t$ on $L^2([0,1], T\widetilde{M})$ with boundary conditions in $T\phi_1^H(L_{(0)}), TL_{(1)}$. Since the boundary conditions are transverse on the ends, we have a spectral gap ensuring exponential decay. In fact, the decay rate is uniform for bounded (but possibly small) widths, as proven in [21]. Spectral crossings correspond to intersections of the Lagrangian subspaces, as in the standard case, and hence index calculations reproduce the Maslov index. For the compactness of index 0 and 1 moduli spaces it suffices to exclude bubbling by considering a single blow-up point for the gradient. This analysis is local, in the interior of one component u_j (leading to a J_j-holomorphic sphere in M_j) or near a seam, where we can consider $u_j(s,-t) \times u_{j+1}(s,t)$ as $(-J_j) \oplus J_{j+1}$-holomorphic curve with boundary condition in $L_{j(j+1)}$. The latter type of bubbling hence leads to a holomorphic disc in $M_j^- \times M_{j+1}$ with boundary on $L_{j(j+1)}$, which are excluded by our assumptions on the minimal Maslov index.

To see that there exist regular "deformed complex structures" $J_{\underline{\delta}} \in \mathcal{J}_t^{reg}(L_{(0)}, L_{(1)}; H)$ of split form, arising from a choice of $\underline{J}$, we note that the unique continuation theorem [4, Theorem 4.3] applies to the interior of every single nonconstant strip $u_j : \mathbb{R} \times (0, \delta_j) \to M_j$. It implies that the set of regular points, $(s_0, t_0) \in \mathbb{R} \times (0, \delta_j)$ with $\partial_s u_j(s_0, t_0) \neq 0$ and $u_j^{-1}(u_j(\mathbb{R} \cup \{\pm\infty\}), t_0) = \{(s_0, t_0)\}$, is open and dense. These points can be used to prove surjectivity of the linearized operator for a universal moduli space of solutions with respect to split almost complex structures. (The constant solutions are automatically transverse due to the previously ensured transversality of the intersection points $\phi_1^H(L_{(0)}) \pitchfork L_{(1)}$.) The existence of a Baire second category set of regular $\underline{J}$ then follows from the usual Sard-Smale argument as in [10].

Orientations can be defined as in the standard Floer theory since the linearized operator canonically deforms through Fredholm operators to a standard Cauchy-Riemann operator. The deformation transfers to the setup with $H = 0$ (changing $L_{(0)}$ and also $J_{\underline{\delta}}$ by a Hamiltonian transformation) and then deforms $J_{\underline{\delta}}$ through the endomorphisms $\big(\exp(\tau \ln \delta_0)\mathrm{Id}_{TM_0}, \ldots, \exp(\tau \ln \delta_r)\mathrm{Id}_{TM_r}\big) \circ J_{\underline{\delta}}$ to a true almost complex structure at $\tau = 1$. □

Now, assuming monotonicity and choosing regular $\underline{H}$ and $\underline{J}$ we can define the Floer homology $HF(\underline{L})$ just as in the standard case: The Floer coboundary operator $\partial^d : CF^d(\underline{L}) \to CF^{d+1}(\underline{L})$ is defined by

$$\partial^d \langle \underline{x}_- \rangle := \sum_{\underline{x}_+ \in \mathcal{I}(\underline{L})} \Big(\sum_{\underline{u} \in \mathcal{M}(\underline{x}_-, \underline{x}_+)_0} \epsilon(\underline{u}) \Big) \langle \underline{x}_+ \rangle,$$

where the signs $\epsilon : \mathcal{M}(\underline{x}_-, \underline{x}_+)_0 \to \{\pm 1\}$ are defined by comparing the given orientation to the canonical orientation of a point. It follows from Theorem 5.2.3 (c) that $\partial^2 = 0$, and ∂ is a map of degree 1 by index calculations as in the standard case. This defines the *quilted Floer cohomology*

$$HF(\underline{L}) := \bigoplus_{d \in \mathbb{Z}_N} HF^d(\underline{L}), \qquad HF^d(\underline{L}) := \ker(\partial^d)/\mathrm{im}(\partial^{d-1})$$

as $\mathbb{Z}_N$-graded group. It is independent of the choice of $\underline{H}$ and $\underline{J}$ by a standard construction of continuation maps. The same construction also allows for a deformation of the widths $\underline{\delta}$, in the folded setup of the above proof, where the δ_j are merely scale factors in the endomorphism $J_{\underline{\delta}}$. For a more conceptional proof based on quilts interpolating between strips of different widths see Section 5.3 below.

Remark 5.2.4. One can also allow the sequence $\underline{L}$ to have length zero (that is, the empty sequence) as a generalized correspondence from M to M; this is the case $r = -1$ in the previous notation. In this case we define $HF(\underline{L}) = HF(\mathrm{Id}_M)$, the cylindrical Floer homology. This would be the case without seams in Figure 1.

5.3. **Invariance of quilted Floer cohomology and relative quilt invariants.** The purpose of this section is to prove the independence of quilted Floer cohomology from the choice of perturbation data, in particular the choice of widths.

Consider a cyclic generalized Lagrangian correspondence $\underline{L} = (L_{k(k+1)})_{k=0,\dots,r}$ satisfying the monotonicity conditions of Theorem 5.2.3. Fix a tuple of widths $\underline{\delta} = (\delta_k)_{k=0,\dots,r}$. Then Proposition 5.2.1 and Theorem 5.2.3 provide tuples of Hamiltonians $\underline{H} = (H_k)_{k=0\dots,r}$ and almost complex structures $\underline{J} = (J_k)_{k=0,\dots,r}$ such that the Floer homology $HF(\underline{L})$ can be defined by counting quilted Floer trajectories $\underline{u} \in \mathcal{M}(\underline{x}^-, \underline{x}^+)$ between generalized intersection points $\underline{x}^\pm \in \mathcal{I}(\underline{L})$.

In the language of quilted surfaces developed in [22] the Floer trajectories correspond to the *holomorphic quilted cylinders* $\underline{u} \in \mathcal{M}_{\underline{Z}}(\underline{x}^-, \underline{x}^+)$ with $\underline{K} = (H_k \mathrm{d}t)_{k=0,\dots,r}$ and $\underline{J} = (J_k)_{k=0,\dots,r}$. Here the quilted surface is the quilted cylinder $\underline{Z}$ whose patches are strips $(S_k = \mathbb{R} \times [0, \delta_k])_{k=0,\dots,r}$ of the given widths with the canonical complex structure and the obvious (up to a shift chosen as ± 1) ends $\epsilon_{k,e_\pm} : \mathbb{R}^\pm \times [0, \delta_k] \to S_k, (s,t) \mapsto (s, \pm 1 + t)$. The seams are $\sigma_k = \{(k, \mathbb{R} \times \{\delta_k\}), (k+1, \mathbb{R} \times \{0\})\}$ for $k = 0, \dots, r$ modulo $(r+1)$, with seam maps $\phi_{\sigma_k} : \partial S_k \supset (s, \delta_k) \mapsto (s, 0) \subset \partial S_{k+1}$. This quilted surface is shown on the right in Figure 1. There are no remaining boundary components except for in the special case of a noncyclic sequence with $M_0 = \{\mathrm{pt}\}$, which is indicated on the left in Figure 1. In that case $\underline{Z}$ has no seam σ_r between S_r and S_0 but true boundary components $(0, \mathbb{R} \times \{0\})$ and $(r, \mathbb{R} \times \{\delta_r\})$. The ends of the quilted surface are the incoming $\underline{e}_- = \big((0, e_-), (1, e_-), \dots, (r, e_-)\big)$ and the outgoing $\underline{e}_+ = \big((0, e_+), (1, e_+), \dots, (r, e_+)\big)$. Note however that the perturbation data $(\underline{J}, \underline{K})$ is $\mathbb{R}$-invariant and the count for the Floer differential is modulo simultaneous $\mathbb{R}$-shift of all maps u_k. That is, unlike in the definition of relative quilt invariants in [22], where no

symmetries are divided out and index 0 solutions are counted, we here count the isolated solutions $\mathcal{M}(\underline{x}^-,\underline{x}^+)_0 = \mathcal{M}_{\underline{Z}}(\underline{x}^-,\underline{x}^+)_1/\mathbb{R}$, which are pseudoholomorphic quilts of index 1.

The proof of independence of Floer cohomology from the choice of perturbations and particularly the widths goes somewhat beyond the proof for standard Floer theory. It is best formulated by using quilted surfaces that are not obtained by "unfolding of strips". With Proposition 5.3.1 below in place we can in particular identify the two definitions of Floer cohomology $HF(\underline{L}) \cong HF(L_{(0)}, L_{(1)})$ for a cyclic sequence in Sections 4.3 and 5.2. For that purpose one chooses special widths in the quilted setup of Section 5.2, namely those that correspond by the discussion in Section 5.1 to the "folded" Floer trajectories of $HF(L_{(0)}, L_{(1)})$. The proof of the Proposition however uses the notation and construction of relative quilt invariants in [22].

Proposition 5.3.1. *$HF(\underline{L})$ is independent, up to isomorphism of $\mathbb{Z}_N$-graded groups, of the choice of perturbation data $(\underline{H},\underline{J})$ and widths $\underline{\delta}$ of the strips.*

Proof. Suppose that $(\underline{H}^i,\underline{J}^i,\underline{\delta}^i)$ are two different choices for $i = 0, 1$. For $\{i,l\} = \{0,1\}$ let $\underline{Z}_{il}$ be the quilted cylinder as before, but with complex structures j_k on each strip $S_k \cong \mathbb{R}\times[0,1]$ that interpolate between the two widths δ_k^i at the end (k,e_-) and δ_k^l at the end (k,e_+). In order for the seams to be real analytic we pick the standard complex structure near the boundary components $\mathbb{R}\times\{0,1\} \subset \partial S_k$ and only in the interior of S_k scale to the appropriate width and interpolate. Figure 2 shows the example for $r = 3$ and $M_0 = M_4 = \{\mathrm{pt}\}$. We moreover interpolate the perturbation data on the two ends by

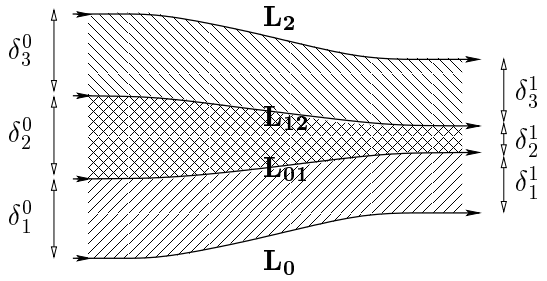

FIGURE 2. Interpolating between two widths

some regular $(\underline{K}_{il},\underline{J}_{il})$ on $\underline{Z}_{il}$. The relative invariants, constructed in [22] from the zero-dimensional moduli spaces, then provide maps between the corresponding Floer cohomology groups

$$\Phi_{\underline{Z}_{01}} : HF(\underline{L})^0 \to HF(\underline{L})^1, \qquad \Phi_{\underline{Z}_{10}} : HF(\underline{L})^1 \to HF(\underline{L})^0.$$

The surface $\underline{Z}_{01}\#\underline{Z}_{10}$ that is glued at $\{\underline{e}_-\} = \mathcal{E}_-(\underline{Z}_{01})$ and $\{\underline{e}_+\} = \mathcal{E}_+(\underline{Z}_{10})$ can be deformed with fixed ends to the infinite strip with translationally invariant perturbation data $(\underline{H}^1,\underline{J}^1,\underline{\delta}^1)$. The invariant defined by the latter is the identity on $HF(\underline{L})^1$ since only constant strips can contribute (all nonconstant solutions lie in at least 1-dimensional moduli spaces due to the nontriviality of the $\mathbb{R}$-action). Since the relative quilt invariants are independent of the above choices, the relative invariant $\Phi_{\underline{Z}_{01}\#\underline{Z}_{10}}$ is the identity on $HF(\underline{L})^1$ (and similarly for $\Phi_{\underline{Z}_{10}\#\underline{Z}_{01}}$). Then by the gluing theorem for relative quilt invariants [22] (where the sign is positive) we have

$$\Phi_{\underline{Z}_{01}} \circ \Phi_{\underline{Z}_{10}} = \Phi_{\underline{Z}_{01}\#\underline{Z}_{10}} = \mathrm{Id}, \qquad \Phi_{\underline{Z}_{10}} \circ \Phi_{\underline{Z}_{01}} = \Phi_{\underline{Z}_{10}\#\underline{Z}_{01}} = \mathrm{Id}.$$

This proves that the Floer cohomology groups $HF(\underline{L})^0$ and $HF(\underline{L})^1$ arising from the different choices of data are isomorphic. □

5.4. **Geometric composition and quilted Floer cohomology.** In this section we prove and discuss the isomorphism (3), more precisely stated as follows.

Theorem 5.4.1. *Let $\underline{L} = (L_{01}, \ldots, L_{r(r+1)})$ be a cyclic sequence of Lagrangian correspondences between symplectic manifolds $M_0, \ldots, M_{r+1} = M_0$ as in Definition 2.1.3. Suppose*

(a) *the symplectic manifolds all satisfy (M1-2) with the same monotonicity constant τ,*
(b) *the Lagrangian correspondences all satisfy (L1-3),*
(c) *the sequence $\underline{L}$ is monotone, relatively spin, and graded in the sense of Section 4.3,*
(d) *the composition $L_{(j-1)j} \circ L_{j(j+1)}$ is embedded in the sense of Definition 2.0.4.*

Then with respect to the induced relative spin structure, orientation, and grading [5] *on the modified sequence $\underline{L}' = (L_{01}, \ldots, L_{(j-1)j} \circ L_{j(j+1)}, \ldots, L_{r(r+1)})$ there exists a canonical isomorphism of graded groups*

$$HF(\underline{L}) = HF(\ldots L_{(j-1)j}, L_{j(j+1)} \ldots) \xrightarrow{\sim} HF(\ldots L_{(j-1)j} \circ L_{j(j+1)} \ldots) = HF(\underline{L}'),$$

induced by the canonical identification of intersection points in Remark 3.1.3.

Before summarizing the proof let us mention the (im)possibility of various generalizations.

Remark 5.4.2. (a) The relative spin structures are only needed to define the Floer cohomology groups with $\mathbb{Z}$ coefficients. Here we only prove the isomorphism with $\mathbb{Z}_2$ coefficients. The full result then follows from a comparison of signs in [23].

(b) There should also be versions of this result for Floer cohomology with coefficients in flat vector bundles, and Novikov rings, using an understanding of their behaviour under geometric composition, similar to the theory presented here for gradings. The gradings on the Lagrangians can be dropped if one wants only an isomorphism of ungraded groups.

(c) Note that the geometric composition $L_{(j-1)j} \circ L_{j(j+1)}$ could be a smooth Lagrangian despite the composition not being embedded. If this failure is in the transversality, then our approach does not apply (as e.g. for a G-invariant Lagrangian $L \subset \mu^{-1}(0)$ in the zero set of the moment map, whose composition with Σ_μ is the smooth projection $L \circ \Sigma_\mu = \pi(L)$ despite L not being transverse to $\mu^{-1}(0)$). (For such Lagrangians one would expect a correspondence between holomorphic curves in $M/\!\!/G$ and symplectic vortices in M, in the spirit of [6] and the Lagrangian version of the Atiyah-Floer conjecture [19].) However, when $L_{(j-1)j} \times_{M_j} L_{j(j+1)}$ is transverse but a k-fold cover of $L_{(j-1)j} \circ L_{j(j+1)}$, then the map of intersection points $\mathcal{I}(\underline{L}) \to \mathcal{I}(\underline{L}')$ is a k-to-1 map as well. In this case our analysis still applies and gives a k-to-1 map of moduli spaces. This may lead to further calculation tools for Floer cohomology but needs to be investigated on a case-by-case basis.

(d) The assumption (L3) on the minimal Maslov numbers is needed only to achieve $\partial^2 = 0$ and thus make the Floer cohomology well-defined. In the absence of (L3) we have $\partial^2 = w\mathrm{Id}$, where $w = w(L_{01}) + \ldots + w(L_{r(r+1)})$ is the sum of numbers of pseudoholomorphic disks through a generic point on each Lagrangian. So instead of (L3), which implies $w(L_{(j-1)j}) = 0$ for each j, we could assume $w = 0$. In case $w \neq 0$, Theorem 5.4.1 generalizes to an isomorphism in the derived category of matrix factorization, see [25].

[5] The grading of $L_{(j-1)(j+1)}$ is given by (16), the orientation is given by Remark 2.0.6(b), and for the relative spin structure see [23].

(e) The monotonicity assumptions (M1) and (L1) cannot simply be replaced by other tools which allow the definition of Floer cohomology (such as Novikov rings, twisted coefficients, obstructions, or deformations). This is since a new type of bubbling can occur in the strip shrinking that we use to prove the isomorphism. We have called it the "figure eight bubble" and describe it in [21]. However, we are lacking the construction of a moduli space of figure eight bubbles. Our present method for excluding these bubbles hinges on strict monotonicity with nonnegative constant $\tau \geq 0$ as well as the 2-grading assumption implied by orientations. In general, we expect figure eight bubbles to be a codimension 1 phenomenon in a 1-parameter family of strip widths approaching zero. We hence expect the isomorphism to fail in more general settings, except for special topological assumptions restricting the expected dimension of figure eight bubbles. Eventually, we expect to construct obstruction classes and an A_∞-type structure from moduli spaces of figure eight bubbles, and to replace the isomorphism by a morphism of A_∞-modules. However, all of this depends on a basic removable singularity result for figure eight bubbles, which has not yet been accomplished.

Theorem 5.4.1 is fairly obvious if one of the composed Lagrangians correspondences is the graph of a symplectomorphism. It suffices to observe that symplectomorphisms map pseudoholomorphic curves to pseudoholomorphic curves. However, there is no corresponding effect for more general Lagrangian correspondences. Here the natural approach to a proof is to degenerate the holomorphic curve equation in M_j until solutions become constant across the strip (or, equivalently, shrink the width of that strip to zero). This limit corresponds to geometric composition of the two Lagrangian correspondences attached to the strip. Clearly, most difficulties in this proof are localized near the degenerating strip. We thus banished the analysis to [21], where we prove the special case $HF(L_0, L_{01}, L_{12}, L_2) \xrightarrow{\sim} HF(L_0, L_{01} \circ L_{12}, L_2)$ of Theorem 5.4.1 by establishing a bijection between the Floer trajectories for (L_0, L_{02}, L_2) on strips of width $(1, 1)$ and those for $(L_0, L_{01}, L_{12}, L_2)$ on strips of width $(1, \delta, 1)$ for sufficiently small width δ of the middle strip. These quilted Floer trajectories are shown in Figure 3. The missing piece of proof in

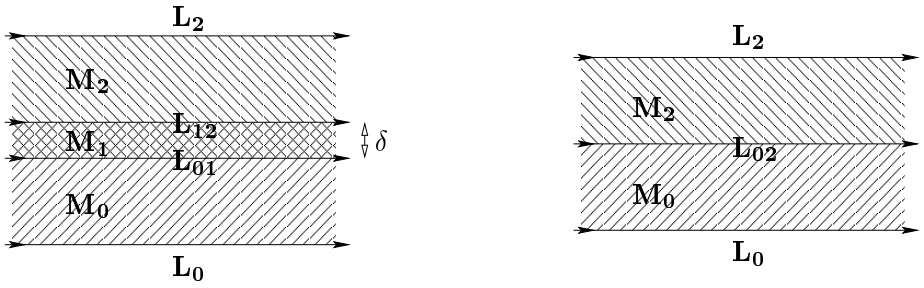

FIGURE 3. Shrinking the middle strip

[21] is the independence of the Floer cohomology from the choice of $\delta > 0$, which we here established in Proposition 5.3.1.

Summary of proof of Theorem 5.4.1. For a start, the assumptions of the Theorem only guarantee that $HF(\underline{L})$ is well-defined. By Remark 5.2.2 the monotonicity of $\underline{L}$ also implies monotonicity of $\underline{L}'$ and hence monotonicity in the sense of (L1) for $L_{(j-1)j} \circ L_{j(j+1)}$.

(Assuming the symplectic manifolds to be connected, any disk can be extended to a quilted cylinder.) Compactness and orientation (L2) also holds for the composed correspondence, but the minimal Maslov index condition (L3) may not. However, this only affects the question whether $w = 0$ in $\partial^2 = w\mathrm{Id}$ on the Floer chain group for $\underline{L}'$. To define the latter we choose some widths $\underline{\delta}'$, Hamiltonian perturbations $\underline{H}'$ to make the intersection $\mathcal{I}(\underline{L}')$ transverse, and almost complex structures $\underline{J}'$ to make the moduli spaces of Floer trajectories for $\underline{L}'$ regular. Thanks to Proposition 5.3.1 we may then choose the same widths $\underline{\delta}$ except for some small $\delta_j > 0$, the same Hamiltonian perturbations $\underline{H}$ except for the additional $H_j \equiv 0$, and the same almost complex structures $\underline{J}$ except for some additional time-independent $J_j \in \mathcal{J}(M_j, \omega_j)$, to define $HF(\underline{L})$. We only need to make sure that this choice makes the intersection points $\mathcal{I}(\underline{L})$ and the moduli spaces of Floer trajectories for $\underline{L}$ regular. The first is automatically the case by the transversality assumption for $L_{(j-1)j} \times_{M_j} L_{j(j+1)}$, the latter is true for $\delta_j > 0$ sufficiently small and is proven as part of the adiabatic limit analysis [21]. (Actually, precisely following the constructions of [21], we can achieve transversality for $\underline{L}'$ with J'_{j-1} and J'_{j+1} being time-independent near the seam; then J_{j-1} and J_{j+1} are obtained by a slight linear dilation and constant extension near the new seams.) With these choices, the Maslov index assumptions imply $\partial^2 = 0$ on $CF(\underline{L})$. Next, the injectivity assumption for the composition $L_{(j-1)j} \times_{M_j} L_{j(j+1)}$ provides a canonical bijection of generalized intersection points $\mathcal{I}(\underline{L}') \cong \mathcal{I}(\underline{L})$ as in Remark 3.1.3. In [21] we establish bijections between the corresponding moduli spaces of Floer trajectories for $\delta_j > 0$ sufficiently small. This means that the Floer differentials on $CF(\underline{L}')$ and $CF(\underline{L})$ agree under the canonical identification of generators. In particular that implies $\partial^2 = 0$ on $CF(\underline{L}')$. Hence both Floer cohomologies are well-defined and isomorphic as claimed. □

Remark 5.4.3. To see that the assumption that the composition $L_{(j-1)j} \circ L_{j(j+1)}$ is embedded is necessary, consider the case that $r = 2$ and M_0, M_2 are points. In this case, if $v : \mathbb{R} \times [0,1] \to M_1$ is a Floer trajectory of index one with limits $x^+ \neq x^-$, we can consider the rescaled maps $v_\delta : \mathbb{R} \times [0, \delta] \to M_1$. In this case a figure eight bubble always develops in the limit $\delta \to 0$. This shows that the bijection between trajectories fails in this case.

6. Applications

6.1. Direct computation of Floer cohomology.

Theorem 6.1.1. *Let $L_{01} \subset M_0^- \times M_1$ be a Lagrangian correspondence and suppose that the Lagrangian submanifolds $L_0 \subset M_0$ and $L_1 \subset M_1$ are such that both $L_0 \circ L_{01}$ and $L_{01} \circ L_1$ are embedded compositions. Assume that M_0, M_1 satisfy (M1-2), L_0, L_1, L_{01} satisfy (L1-3), and $(L_0 \times L_1, L_{01})$ is a monotone pair in the sense of Definition 4.1.2 (b). Then there exists a canonical isomorphism*

$$HF(L_0 \circ L_{01}, L_1) \xrightarrow{\sim} HF(L_0, L_{01} \circ L_1).$$

Proof. By Theorem 5.4.1 both Floer cohomologies are isomorphic to the quilted Floer cohomology $HF(L_0, L_{01}, L_1) = HF(L_0 \times L_1, L_{01})$. □

Example 6.1.2. We begin with a "warm-up" example. Let N be a compact, simply-connected, monotone symplectic manifold. The submanifold $\Delta_{ij} := \{(x_1, x_2, x_3, x_4) | x_i = x_j\} \subset N^- \times N \times N^- \times N$ is coisotropic for appropriate choices of $1 \leq i < j \leq 4$. Then we can identify

$$HF(\Delta_{14} \cap \Delta_{23}, \Delta_{12} \cap \Delta_{34}) \cong HF(\Delta_N, \Delta_N) \cong H(N) \tag{29}$$

with the homology of N. This follows from Theorem 6.1.1 applied to $L_0 = \Delta_N \subset N^- \times N = M_0$, $L_1 = \Delta_{12} \cap \Delta_{34} \subset N^- \times N \times N^- \times N = M_1$, and $L_{01} = \{(w, z; w, x, x, z) | w, x, z \in N\} \subset M_0^- \times M_1$. Then the compositions $L_0 \circ L_{01} = \Delta_{14} \cap \Delta_{23}$ and $L_{01} \circ L_1 = \Delta_N$ are clearly embedded. Monotonicity together with simply connectedness ensures the monotonicity of all the Lagrangians and pairs of Lagrangians. Since N is orientable, all minimal Maslov indices are at least 2. The reader can easily verify the identification (29) using the fact that the components of a holomorphic trajectory for $(\Delta_{14} \cap \Delta_{23}, \Delta_{12} \cap \Delta_{34})$ fit together to a holomorphic cylinder $v : S^1 \times [0, 1] \to N$.

The following is a more non-trivial example of Theorem 6.1.1.

Example 6.1.3. Let M_n be the moduli space of Euclidean n-gons of edge length 1, as in for example Kirwan [8]:

$$M_n = (S^2)^n /\!\!/ SO(3) = \{(v_1, \ldots, v_n) \in (S^2)^n | v_1 + \ldots + v_n = 0\}/SO(3).$$

We take on S^2 the standard symplectic form ω with volume 4π so that $c_1(S^2) = [\omega]$. For $n \geq 5$ odd M_n is a monotone symplectic manifold with minimal Chern number 1 and monotonicity constant 1. For example, M_3 is a point and M_5 is diffeomorphic to the fourth del Pezzo surface, given by blowing-up of $\mathbb{P}^2$ at four points [25]. For $i \neq j$ the submanifold $\Delta_{ij} = \{[v_1, \ldots, v_n] \in M_n \,|\, v_i = -v_j\}$ is a coisotropic, spherically fibered over M_{n-2} by the map that forgets v_i, v_j. The image of Δ_{ij} in $M_{n-2}^- \times M_n$ is a Lagrangian correspondence, also denoted Δ_{ij}. For i, j, k distinct the composition $\Delta_{ij} \circ \Delta_{jk}^t$ is embedded and yields the graph of a permutation on M_{n-2}. For $k = i \pm 1$ or for $i, k = j \pm 1$ this permutation is trivial, so we have

$$\Delta_{ij} \circ \Delta_{j(i\pm1)}^t = \Delta_{M_{n-2}}, \quad \Delta_{(j\pm1)j} \circ \Delta_{j(j\mp1)}^t = \Delta_{M_{n-2}}.$$

Now let $L \subset M_{n-2}$ be a compact, oriented, monotone Lagrangian, and $L_{ij} = L \circ \Delta_{ij}$ be its inverse image in M_n. This composition is embedded and we can also identify it with $L_{ij} = \Delta_{ij}^t \circ L$. The latter allows to calculate

$$\Delta_{ij} \circ L_{jk} = \Delta_{ij} \circ \Delta_{jk}^t \circ L :$$

For i, j, k distinct it is an embedded composition, which yields the image of L under permutation. Suppose that the pair (L, L) is monotone, so that $HF(L, L)$ is well-defined. Using Theorem 6.1.1 we can also (define and) calculate

$$HF(L_{ij}, L_{j(i\pm1)}) = HF(L \circ \Delta_{ij}, L_{j(i\pm1)}) = HF(L, \Delta_{ij} \circ L_{j(i-1)}) = HF(L, L).$$

and similarly $HF(L_{(j\pm1)j}, L_{j(j\mp1)}) = HF(L, L)$.

The papers [24], [25] give many other similar examples.

6.2. **Computations in $\mathbb{CP}^n$.** In this section we demonstrate, at the example of $\mathbb{CP}^n$, how some Floer cohomologies in toric symplectic varieties can be calculated by reduction. We equip $\mathbb{CP}^n = \{[\underline{z}] = [z_0 : z_1 : \ldots : z_n]\}$ with Fubini-Study symplectic form and moment maps $\mu_j([\underline{z}]) = \pi |z_j|^2 / |\underline{z}|^2$ for $j = 1, \ldots, n$. We denote by

$$\Sigma_j := \left\{ \left([\ldots z_{j-1} : z_{j+1} : \ldots], [z_0 : \ldots : z_n]\right) \,\middle|\, \mu_j([z_0 : \ldots : z_n]) = \tfrac{\pi}{n+1} \right\} \subset (\mathbb{CP}^{n-1})^- \times \mathbb{CP}^n$$

the Lagrangian sphere arising from reduction at the level set

$$\mu_j^{-1}(\tfrac{\pi}{n+1}) = \left\{ [z_0 : \ldots : z_n] \,\middle|\, z_j = \tfrac{1}{\sqrt{n+1}}, \textstyle\sum_{i \neq j} |z_i|^2 = \tfrac{n}{n+1} \right\}.$$

Note that the reduced space, e.g. $\mu_n^{-1}(\frac{\pi}{n+1})/S^1 = \{[z_0 : \ldots z_{n-1}] \,\big|\, \sum_i |z_i|^2 = \frac{n}{n+1}\}/S^1$ for $j = n$, is $\mathbb{CP}^{n-1}$ with Fubini-Study form scaled by $\frac{n}{n+1}$, hence has the same monotonicity

constant $\tau = n^{-1}\frac{n\pi}{n+1} = (n+1)^{-1}\pi$ as $\mathbb{CP}^n$. (Recall that the generator $\mathbb{CP}^1 \subset \mathbb{CP}^n$ of $\pi_2(\mathbb{CP}^n)$ has Fubini-Study symplectic area π and Chern number $n+1$.)

More generally, for each $1 < k \leq n$ a Lagrangian correspondence

$$\Sigma_{(k,\ldots,n)} := \left\{ \left([z_0 : \ldots : z_{k-1}], [z_0 : \ldots : z_n]\right) \,\middle|\, \mu_j([\underline{z}]) = \tfrac{\pi}{n+1} \ \forall j \geq k \right\} \subset (\mathbb{CP}^{k-1})^- \times \mathbb{CP}^n$$

arises from reduction at the level set

$$(\mu_k \times \ldots \times \mu_n)^{-1}(\tfrac{\pi}{n+1}, \ldots, \tfrac{\pi}{n+1}) = \left\{ [\underline{z}] \,\middle|\, z_k = |z_{k+1}| = \ldots = |z_n| = \tfrac{1}{\sqrt{n+1}}, \sum_{i=0}^{k-1} |z_i|^2 = \tfrac{k}{n+1} \right\}.$$

Here again the reduced spaces $\mathbb{CP}^{k-1} = \{[z_0 : \ldots : z_{k-1}] \mid \sum_i |z_i|^2 = \frac{k}{n+1}\}/S^1$ carry scaled Fubini-Study forms with monotonicity constant $\tau = \frac{\pi}{n+1}$. Moreover, note that $\Sigma_{(k,\ldots,n)}$, diffeomorphic to the product of an $(n-k)$-torus with a $(2k-1)$-sphere, can be viewed as Lagrangian embedded in $(\mathbb{CP}^{k-1})^- \times \mathbb{CP}^n$ and also as coisotropic submanifold of $\mathbb{CP}^n$. One can check explicitly that the Lagrangians $\Sigma_{(k,\ldots,n)}$ are monotone, and we will see in Corollary 6.3.3 below that they are nondisplaceable by Hamiltonian diffeomorphisms. The reason is that as coisotropic they contain the nondisplaceable Clifford torus

$$T^n_{\mathrm{Cl}} = (\mu_1 \times \ldots \mu_n)^{-1}(\tfrac{\pi}{n+1}, \ldots, \tfrac{\pi}{n+1}) = \left\{ [\underline{z}] \,\middle|\, z_0 = |z_1| = \ldots = |z_n| = \tfrac{1}{\sqrt{n+1}} \right\} \subset \mathbb{CP}^n.$$

That T^n_{Cl} is the only nondisplaceable fibre of the torus fibration is known by e.g. [2]. Its Floer cohomology was calculated by Cho [3] with all possible spin structures. Here we reproduce this calculation for the standard spin structure, employing the above Lagrangian correspondences and the isomorphism of Floer cohomology under embedded geometric composition (Theorem 5.4.1). This approach also allows for a direct computation of Floer cohomology for any pair of nonstandard spin structures on T^n_{Cl}, which we will discuss in[23].

Theorem 6.2.1. [3] *For any $n \in \mathbb{N}$ with the standard spin structure (given by* [3, Prp.8.1]*)*

$$HF(T^n_{\mathrm{Cl}}, T^n_{\mathrm{Cl}}) \cong H_*(T^n) \cong \mathbb{Z}^{2n}.$$

Proof. The isomorphism between the Floer cohomology and the homology of the Clifford n-torus follows inductively from the following chain of isomorphisms:

$$\begin{aligned} HF(T^n_{\mathrm{Cl}}, T^n_{\mathrm{Cl}}) &= HF(T^1_{\mathrm{Cl}} \circ \Sigma_{(2,\ldots,n)}, \Sigma_1^t \circ T^{n-1}_{\mathrm{Cl}}) \\ &\cong HF(T^1_{\mathrm{Cl}}, \Sigma_{(2,\ldots,n)}, \Sigma_1^t, T^{n-1}_{\mathrm{Cl}}) \\ &\cong HF(T^1_{\mathrm{Cl}}, \Sigma_{(2,\ldots,n)} \circ \Sigma_1^t, T^{n-1}_{\mathrm{Cl}}) \\ &\cong HF(T^1_{\mathrm{Cl}}, T^1_{\mathrm{Cl}} \times T^{n-1}_{\mathrm{Cl}}, T^{n-1}_{\mathrm{Cl}}) \\ &\cong HF(T^1_{\mathrm{Cl}}, T^1_{\mathrm{Cl}}) \otimes HF(T^{n-1}_{\mathrm{Cl}}, T^{n-1}_{\mathrm{Cl}}). \end{aligned} \tag{30}$$

Let us go through this step by step: The geometric composition $T^1_{\mathrm{Cl}} \circ \Sigma_{(2,\ldots,n)} = T^n_{\mathrm{Cl}}$ is the preimage of T^1_{Cl} under the projection $(\mu_2 \times \ldots \times \mu_n)^{-1}(\frac{\pi}{n+1}, \ldots, \frac{\pi}{n+1}) \to \mathbb{CP}^1$, hence automatically embedded in the sense of Definition 2.0.4. Similarly, $T^{n-1}_{\mathrm{Cl}} \circ \Sigma_1 = T^n_{\mathrm{Cl}}$ is the preimage of T^{n-1}_{Cl} under the projection $\mu_1^{-1}(\frac{\pi}{n+1}) \to \mathbb{CP}^{n-1}$, and by transposition we obtain the embedded composition $\Sigma_1^t \circ T^{n-1}_{\mathrm{Cl}} = T^n_{\mathrm{Cl}}$. Finally, the intersection

$$\Sigma_{(2,\ldots,n)} \times_{\mathbb{CP}^n} \Sigma_1^t \ \cong\ (\mu_2 \times \ldots \times \mu_n)^{-1}(\tfrac{\pi}{n+1}, \ldots, \tfrac{\pi}{n+1}) \cap \mu_1^{-1}(\tfrac{\pi}{n+1}) \ =\ T^n_{\mathrm{Cl}} \ \subset\ \mathbb{CP}^n$$

is transverse and embeds to

$$\Sigma_{(2,\ldots,n)} \circ \Sigma_1^t = \left\{ \left([z_0 : z_1], [z_0 : z_2 : \ldots : z_n]\right) \,\middle|\, [\underline{z}] \in T^n_{\mathrm{Cl}} \right\} = T^1_{\mathrm{Cl}} \times T^{n-1}_{\mathrm{Cl}} \subset \mathbb{CP}^1 \times \mathbb{CP}^{n-1}.$$

To make sure that Theorem 5.4.1 indeed implies all the above isomorphisms of Floer cohomology, it remains to ensure that the maximally decomposed tuple $(T^1_{\mathrm{Cl}}, \Sigma_{(2,\dots,n)}, \Sigma_j^t, T^{n-1}_{\mathrm{Cl}})$ is monotone. That follows from the monotonicity of all factors together with the torsion fundamental groups of the symplectic manifolds involved. Moreover, it turns out that we need not worry about the minimal Maslov indices 2. This is since the proof of Theorem 5.4.1 provides, for certain choices of perturbations, a canonical identification of the Floer chain group $CF(T_{\mathrm{Cl}}, T_{\mathrm{Cl}})$ with each of the other chain groups in (30), under which the Floer differentials agree. Since we have $\partial^2 = 0$ on the first chain group (see [11]), the differentials on all the other chain groups also square to zero, making the Floer cohomologies well defined. So, strictly speaking, our calculation uses the derived version of Theorem 5.4.1.

Moreover, we need to fix spin structures on T^{n-1}_{Cl} and Σ_1 as well as on T^1_{Cl} and $\Sigma_{(2,\dots,n)}$ such that the induced spin structure on the composition, T^n_{Cl} is the standard one. For the Clifford tori we pick the standard spin structure given by the trivialization of $TT^k_{\mathrm{Cl}} \subset \mathbb{C}^k$ in the coordinate chart $\mathbb{C}^k \cong \{z_0 = \frac{1}{\sqrt{n+1}}\} \subset \mathbb{CP}^k$. On the sphere $\Sigma_1 \subset \mathbb{CP}^n$ we fix the spin structure given by the standard orientation in the chart $\{z_1 = \frac{1}{\sqrt{n+1}}\}$. (The orientation provides a trivialization over the 0-skeleton, which coincides with the 1- and 2-skeleton of this sphere of dimension ≥ 3; see [3] or [23] for more details on spin structures.) We can read off the standard spin structure induced on T^n_{Cl} from the identification

$$\mathrm{T}T^n_{\mathrm{Cl}} \cong \mathrm{pr}^*\mathrm{T}T^{n-1}_{\mathrm{Cl}} \oplus E, \qquad E = (\mathrm{pr}^*\mathrm{T}\mathbb{CP}^{n-1})^\perp \subset \mathrm{T}\Sigma_1|_{\mathrm{T}^n_{\mathrm{Cl}}}.$$

Here $\mathrm{T}\mathbb{CP}^{n-1}|_{T^{n-1}_{\mathrm{Cl}}} = \mathrm{T}T^{n-1}_{\mathrm{Cl}} \oplus i\mathrm{T}T^{n-1}_{\mathrm{Cl}}$ inherits a trivialization from T^{n-1}_{Cl}, so the orientation of Σ_1 induces a trivialization of the line bundle E (given by the linearized action of μ_1).

For the spin structure on $\Sigma_{(2,\dots,n)} \subset \mathbb{CP}^n$we identify $\Sigma_{(2,\dots,n)} \cong T^{n-2} \cdot S^3$ with the orbit of the sphere $S^3 = \left\{[z_0 : z_1 : \frac{1}{\sqrt{n+1}} : \dots : \frac{1}{\sqrt{n+1}}] \,\middle|\, |z_0|^2 + |z_1|^2 = \frac{2}{n+1}\right\} \subset \mathbb{CP}^n$ under the action of the torus $T^{n-2} \subset \mathbb{C}^{n-2}$ in the $z_3, \dots, z_n$-coordinates. If we pick the standard trivialization of T^{n-2} and the standard orientation of $S^3 \subset \mathbb{C}^2$ in the above chart, then again the standard spin structure is induced on T^n_{Cl} by the identification

$$\mathrm{T}_{\underline{z}}T^n_{\mathrm{Cl}} \cong \mathrm{T}_{\mathrm{pr}(\underline{z})}T^1_{\mathrm{Cl}} \oplus \mathrm{T}_{\underline{z}}(T^{n-2}\underline{z}) \oplus F_{(z_0,z_1)}, \qquad F = (\mathrm{pr}^*\mathrm{T}\mathbb{CP}^1)^\perp \subset \mathrm{T}S^3|_{T^1_{\mathrm{Cl}}}.$$

Here $\mathrm{T}\mathbb{CP}^1|_{T^1_{\mathrm{Cl}}} = \mathrm{T}T^1_{\mathrm{Cl}} \oplus i\mathrm{T}T^1_{\mathrm{Cl}}$ inherits a trivialization from T^1_{Cl}, so the orientation of S^3 induces a trivialization of the line bundle F.

Finally, we know from elementary curve counts (see e.g. [3]) that, with the standard spin structure on both factors, $HF(T^1_{\mathrm{Cl}}, T^1_{\mathrm{Cl}}) \cong \mathbb{Z} \oplus \mathbb{Z} \cong H_*(S^1 = T^1)$. Since the homology $H_*(T^n)$ satisfies the same inductive relation (30) as the Floer cohomology that proves the theorem. □

This Floer cohomology calculation directly generalizes when replacing $T^1_{\mathrm{Cl}} \subset \mathbb{CP}^1$ with another Lagrangian submanifold in a possibly higher dimensional complex projective space.

Theorem 6.2.2. *Let $1 \leq k < n$ and let $L \subset \mathbb{CP}^k$ be an oriented, monotone Lagrangian submanifold. Denote by $\mathrm{pr} : (\mu_{k+1} \times \dots \times \mu_n)^{-1}(\frac{\pi}{n+1}, \dots, \frac{\pi}{n+1}) \to \mathbb{CP}^k$ the reduction of $\mathbb{CP}^n$ by the $(S^1)^{n-k}$-action at the monotone level set. Then $\mathrm{pr}^{-1}(L) \subset \mathbb{CP}^n$ is a monotone Lagrangian submanifold and*

$$HF(\mathrm{pr}^{-1}(L), T^n_{\mathrm{Cl}}) \cong HF(L, T^k_{\mathrm{Cl}}) \otimes H_*(T^{n-k}).$$

Here we assume that $\partial^2 = 0$ on $CF(L, T^k_{\mathrm{Cl}})$; otherwise the above isomorphism holds in the category of derived matrix factorizations, see [25].

Proof. Denote by $\Sigma_{(1,\dots,k)} \subset (\mathbb{CP}^{n-k})^- \times \mathbb{CP}^n$ the Lagrangian correspondence arising from reduction at the level set $(\mu_1 \times \ldots \times \mu_k)^{-1}(\frac{\pi}{n+1}, \dots, \frac{\pi}{n+1}) \subset \mathbb{CP}^n$. Then

$$\Sigma_{(k+1,\dots,n)} \times_{\mathbb{CP}^n} \Sigma^t_{(1,\dots,k)} \cong (\mu_{k+1} \times \ldots \mu_n)^{-1}(\tfrac{\pi}{n+1}, \dots) \cap (\mu_1 \times \ldots \mu_k)^{-1}(\tfrac{\pi}{n+1}, \dots) = T^n_{\mathrm{Cl}}$$

is transverse and embeds to $\Sigma_{(k+1,\dots,n)} \circ \Sigma^t_{(1,\dots,k)} = T^k_{\mathrm{Cl}} \times T^{n-k}_{\mathrm{Cl}} \subset \mathbb{CP}^k \times \mathbb{CP}^{n-k}$. Now in complete analogy to the proof of Theorem 6.2.1 above, we have a chain of isomorphisms

$$\begin{aligned} HF(\mathrm{pr}^{-1}(L), T^n_{\mathrm{Cl}}) &= HF(L \circ \Sigma_{(k+1,\dots,n)}, \Sigma^t_{(1,\dots,k)} \circ T^{n-k}_{\mathrm{Cl}}) \\ &\cong HF(L, \Sigma_{(k+1,\dots,n)}, \Sigma^t_{(1,\dots,k)}, T^{n-k}_{\mathrm{Cl}}) \\ &\cong HF(L, \Sigma_{(k+1,\dots,n)} \circ \Sigma^t_{(1,\dots,k)}, T^{n-k}_{\mathrm{Cl}}) \\ &\cong HF(L, T^k_{\mathrm{Cl}} \times T^{n-k}_{\mathrm{Cl}}, T^{n-k}_{\mathrm{Cl}}) \\ &\cong HF(L, T^k_{\mathrm{Cl}}) \otimes HF(T^{n-k}_{\mathrm{Cl}}, T^{n-k}_{\mathrm{Cl}}). \end{aligned}$$

This finishes the proof since $HF(T^{n-k}_{\mathrm{Cl}}, T^{n-k}_{\mathrm{Cl}}) \cong H_*(T^{n-k})$ by [3] or Theorem 6.2.1. □

Theorem 6.2.2 applies, for example, to $\mathbb{RP}^1 \subset \mathbb{CP}^1$ and yields another Lagrangian torus $\mathrm{pr}^{-1}(\mathbb{RP}^1) \subset \mathbb{CP}^n$ with

$$HF(\mathrm{pr}^{-1}(\mathbb{RP}^1), T^n_{\mathrm{Cl}}) \cong HF(\mathbb{RP}^1, T^k_{\mathrm{Cl}}) \otimes H_*(T^{n-1}) \cong H_*(T^n).$$

More generally, Theorem 6.2.2 applies to odd real projective spaces $\mathbb{RP}^k \subset \mathbb{CP}^k$ for $k = 2\ell - 1 \geq 3$ with $\mathbb{Z}_2$-coefficients. [6] By explicit calculation due to Alston [1] the underlying Floer cohomology is

$$HF(\mathbb{RP}^{2\ell-1}, T^{2\ell-1}_{\mathrm{Cl}}) \cong \mathbb{Z}_2^{2^\ell}.$$

Now our calculations in Theorem 6.2.2 provide with $\mathbb{Z}_2$-coefficients

$$HF(\mathrm{pr}^{-1}(\mathbb{RP}^{2\ell-1}), T^{2\ell-1}_{\mathrm{Cl}}) \cong HF(\mathbb{RP}^{2\ell-1}, T^{2\ell-1}_{\mathrm{Cl}}) \otimes H_*(T^{n-2\ell+1}_{\mathrm{Cl}}) \cong \mathbb{Z}_2^{2^\ell + 2(n-2\ell+1)}.$$

6.3. **Detecting nontrivial Floer cohomology of a Lagrangian correspondence.** In this section we provide a tool for deducing nontriviality of Floer cohomology and hence nondisplaceability of a Lagrangian correspondence itself (as Lagrangian submanifold).

Theorem 6.3.1. *Let $L_{01} \subset M_0^- \times M_1$ be a Lagrangian correspondence. Suppose that there exists a Lagrangian submanifold $L_1 \subset M_1$ such that $L_0 := L_{01} \circ L_1$ is an embedded composition and $HF(L_0, L_0) \neq 0$. Assume that M_0, M_1 satisfy (M1-2), L_0, L_1, L_{01} satisfy (L1-3), and $(L_0 \times L_1, L_{01})$ is a monotone pair in the sense of Definition 4.1.2 (b). Then the Lagrangian $L_{01} \subset M_0^- \times M_1$ has nonzero Floer cohomology $HF(L_{01}, L_{01}) \neq 0$.*

Proof. The assumptions guarantee that $HF(L_0, L_0) \cong HF(L_0, L_{01}, L_1) = HF(L_0 \times L_1, L_{01})$ are all well-defined and isomorphic, hence nonzero. Now $HF(L_0 \times L_1, L_{01})$ is a module over $HF(L_{01}, L_{01})$, where the multiplication is defined by counting pseudoholomorphic 3-gons, see e.g. [17] or [20]. The unit $1_{L_{01}} \in HF(L_{01}, L_{01})$ is defined by counting pseudoholomorphic 1-gons; it is nontrivial since it acts as identity on a nontrivial group. Hence $HF(L_{01}, L_{01})$ contains a nonzero element, as claimed. □

[6] The number of holomorphic discs through a generic point is 0 for $\mathbb{RP}^k$ (which has minimal Maslov number $k+1 \geq 3$ for $k \geq 2$) and it is $k+1$ for T^k_{Cl} by [3], hence $\partial^2 = 0$ on $CF(\mathbb{RP}^k, T^k_{\mathrm{Cl}})$ only holds for odd k and with $\mathbb{Z}_2$ coefficients.

Corollary 6.3.2. *Let $\Sigma \subset M$ be the level set of the moment map of a Hamiltonian G-action. Suppose that Σ contains a G-invariant Lagrangian submanifold $L \subset M$ such that $HF(L,L) \neq 0$ and $\mathrm{pr}(L) \subset \Sigma/G = M/\!\!/G$ is smooth. Assume that $M, M/\!\!/G$ satisfy (M1-2), $\Sigma, L, \mathrm{pr}(L)$ satisfy (L1-3), and $(\mathrm{pr}(L) \times L, \Sigma)$ is a monotone pair in the sense of Definition 4.1.2 (b). Then $HF(\Sigma,\Sigma) \neq 0$.*

Proof. This is a case of Theorem 6.3.1, where $L_1 = \mathrm{pr}(L)$, and the composition $\Sigma \circ \mathrm{pr}(L) = \mathrm{pr}^{-1}(\mathrm{pr}(L)) = L$ is automatically embedded. □

The following example in case $k = n = 2$ was initially pointed out to us in 2006 by Paul Seidel; we since learned of alternative proof methods by Biran-Cornea and Fukaya-Oh-Ono-Ohta. We use the notation of Section 6.2; in particular $\Sigma_{(n)} \subset (\mathbb{CP}^{n-1})^- \times \mathbb{CP}^n$ is a Lagrangian $2n-1$-sphere arising from reduction at the level set $\mu_n^{-1}(\frac{\pi}{n+1})$.

Corollary 6.3.3. *For every $2 \leq k \leq n$ the Lagrangian embedding $\Sigma_{(k,\ldots,n)} \subset (\mathbb{CP}^{k-1})^- \times \mathbb{CP}^n$ of $(S^1)^{n-k} \times S^{2k-1}$ is Hamiltonian non-displaceable.*

Proof. By construction $\Sigma_{(k,\ldots,n)}$ is the correspondence arising from the level set of $\mu_k \times \ldots \times \mu_n$ at the level $(\frac{\pi}{n+1}, \ldots, \frac{\pi}{n+1})$ which contains the nondisplaceable Clifford torus $T^n_{\mathrm{Cl}} \subset \mathbb{CP}^n$. The projection $\mathrm{pr}(T^n_{\mathrm{Cl}}) = T^n_{\mathrm{Cl}} \circ \Sigma_{(k,\ldots,n)}$ is the Clifford torus $T^{k-1}_{\mathrm{Cl}} \subset \mathbb{CP}^{k-1}$. The Clifford tori as well as $\Sigma_{(k,\ldots,n)}$ are monotone with minimal Maslov number 2 (except for $k = n$ when $\Sigma_{(n)}$ is simply connected), but as in Section 6.2 we need not worry about its minimal Maslov index or Oh's number of disks through a generic point. The derived version of Theorem 5.4.1 provides an isomorphism between $HF(T^{k-1}_{\mathrm{Cl}}, \Sigma_{(k,\ldots,n)}, T^n_{\mathrm{Cl}})$ and $HF(T^n_{\mathrm{Cl}}, T^n_{\mathrm{Cl}})$, which is well defined by [11] and nonzero by [3] or our calculation in Section 6.2. This proves that $HF(T^{k-1}_{\mathrm{Cl}}, \Sigma_{(k,\ldots,n)}, T^n_{\mathrm{Cl}}) \neq 0$ is well defined; now the rest of the proof proceeds as in Theorem 6.3.1. □

6.4. **Gysin sequence for spherically fibered Lagrangian correspondence.** In this section, we give a conjectural relation between Floer cohomology $HF(L,L')$ for $L, L' \subset M_0$ and the Floer cohomology $HF(L_{01} \circ L, L_{01} \circ L')$ for the images in M_1 under a correspondence $L_{01} \subset M_0^- \times M_1$. Results of this type can be viewed as *transfer of non-displaceability* results, in the sense that non-triviality of $HF(L,L')$ implies non-triviality of $HF(L_{01} \circ L, L_{01} \circ L')$ and hence non-displaceability of $L_{01} \circ L$ from $L_{01} \circ L'$ by Hamiltonian perturbation.

In our example, the Lagrangian correspondence arises from a spherically fibered coisotropic $\iota : C \to M$ with projection $\pi : C \to B$. The image of C under $\iota \times \pi$ is a Lagrangian correspondence from M to B, also denoted C. Our standing assumptions are compactness, orientability, and monotonicity, i.e. M, B, and C satisfy (M1-2) and (L1-2) with a fixed $\tau \geq 0$. Perutz [13] proved the following analogue of the Gysin sequence.

Theorem 6.4.1. *Suppose that the minimal Maslov number of C is at least $\mathrm{codim}(C \subset M) + 2$. Then there exists a long exact sequence*

$$\ldots \to HF(C,C) \to HF(\mathrm{Id}) \to HF(\mathrm{Id}) \to HF(C,C) \to \ldots$$

where the map $HF(\mathrm{Id}) \to HF(\mathrm{Id})$ is quantum multiplication by the Euler class of π.

One naturally conjectures the following relative version (for example, compare the Seidel triangle in [16] with the relative version in [16].)

Conjecture 6.4.2. *Let $L_0, L_1 \subset B$ be a monotone pair of Lagrangian submanifolds satisfying (L1-3). Suppose that the minimal Maslov number of C is at least* $\operatorname{codim}(C \subset M) + 2$ *Then there exists a long exact sequence*

$$\ldots \to HF(L_0, C^t, C, L_1) \to HF(L_0, L_1) \to HF(L_0, L_1) \to HF(L_0, C^t, C, L_1) \to \ldots$$

where the middle map is Floer theoretic multiplication [7] *by the Euler class of π.*

The compositions $C \circ L_1$ and $L_0 \circ C^t = (C \circ L_0)^t$ are clearly embedded. Hence Conjecture 6.4.2 together with Theorem 5.4.1 implies the following.

Corollary 6.4.3. *Under the same assumptions as in Conjecture 6.4.2 there exists a long exact sequence*

$$\ldots \to HF(C \circ L_0, C \circ L_1) \to HF(L_0, L_1) \to HF(L_0, L_1) \to HF(C \circ L_0, C \circ L_1) \to \ldots$$

In particular, we obtain a 'transfer of non-displaceability' result if the Euler class vanishes.

Corollary 6.4.4. *With the same assumptions as in Corollary 6.4.2, if the Euler class of $\pi : C \to B$ is zero, then $HF(C \circ L_0, C \circ L_1)$ is isomorphic to two copies of $HF(L_0, L_1)$.*

Example 6.4.5. Suppose that M is a monotone Hamiltonian $G = SU(2)$ manifold, with moment map Φ, and $\Phi^{-1}(0)$ is an $SU(2)$-bundle over the symplectic quotient $M/\!\!/G$.

Let (L_0, L_1) be a monotone pair of G-invariant Lagrangians contained in the zero level set and with minimal Maslov number at least three. Necessarily each L_j is a principal $SU(2)$ bundle over $L_j/G \subset M/\!\!/G$. Suppose that the minimal Maslov number of $\Phi^{-1}(0)$, considered as a Lagrangian in $M^- \times M/\!\!/G$, is at least 5. Then there is a long exact sequence

$$\ldots \to HF(L_0, L_1) \to HF(L_0/G, L_1/G) \to HF(L_0/G, L_1/G) \to HF(L_0, L_1) \to \ldots.$$

In particular, if $M \to M/\!\!/G$ is a trivial G-bundle, then $HF(L_0, L_1)$ is isomorphic to two copies of $HF(L_0/G, L_1/G)$.

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