QUANTUM KIRWAN MORPHISM AND GROMOV-WITTEN INVARIANTS OF QUOTIENTS III

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Abstract. This is the third in a sequence of papers in which we construct a quantum version of the Kirwan map from the equivariant quantum cohomology $QH_G(X)$ of a smooth polarized complex projective variety X with the action of a connected complex reductive group G to the orbifold quantum cohomology $QH(X/\!\!/G)$ of its geometric invariant theory quotient $X/\!\!/G$, and prove that it intertwines the genus zero gauged Gromov-Witten potential of X with the genus zero Gromov-Witten graph potential of $X/\!\!/G$. We also give a formula for a solution to the quantum differential equation on $X/\!\!/G$ in terms of a localized gauged potential for X. These results overlap with those of Givental [14], Lian-Liu-Yau [21], Iritani [20], Coates-Corti-Iritani-Tseng [11], and Ciocan-Fontanine-Kim [7], [8].

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We continue with the notation in the previous part [27], where we defined perfect obstruction theories and virtual fundamental classes for the moduli stacks of stable gauged maps. In this part we define the resulting gauged invariants, which come in various flavors (gauged invariants for fixed scaling, affine gauged invariants, and invariants with varying scaling). We show the splitting axioms for the invariants and deduce the main results of the series: Let a complex reductive group G act on a smooth polarized projective (or in some cases, quasiprojective) variety X with only finite stabilizers on the semistable locus, and let Λ_X^G be the Novikov field for $H_G^2(X, \mathbb{Q})$. We construct a quantum Kirwan map

$$\kappa_X^G : QH_G(X) \to QH_{S^1}(X/\!\!/G)$$

and prove the adiabatic limit theorem that the quantum Kirwan map intertwines the gauged graph potential and graph potential of the git quotient

$$\tau_X^G : QH_G(X) \to \Lambda_X^G, \quad \tau_{X/\!\!/G} : QH(X/\!\!/G) \to \Lambda_X^G$$

in the limit of large area. We end with a partial computation of the quantum Kirwan map in the toric case, that is, when X is a vector space with a linear action of a torus G.

We thank I. Ciocan-Fontanine and B. Kim for pointing out a missing circle-equivariant term in Example 9.15.

7. Gauged Gromov-Witten invariants

7.1. Equivariant Gromov-Witten theory for smooth varieties

First we recall the definition equivariant Gromov-Witten invariants for a smooth projective target using the Behrend-Fantechi machinery [4], as explained in Graber-Pandharipande [17]. We adopt the perspective on the splitting axiom adopted in Behrend [3]: Invariants are defined for any possibly disconnected combinatorial type, and the splitting axiom can be broken down into *cutting edges* and *collapsing edges* axiom. In preparation for studying the properties of the virtual fundamental classes, suppose as in Behrend-Fantechi [4, p. 51] that there is a diagram of Deligne-Mumford stacks

$$\begin{array}{ccc} \mathcal{X}' & \xrightarrow{u} & \mathcal{X} \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle f} \\ \mathcal{Z}' & \xrightarrow{v} & \mathcal{Z} \end{array}$$

where $v : \mathcal{Z}' \to \mathcal{Z}$ is a local complete intersection morphism with finite unramified diagonal over a stack $\mathcal{Y}$. Let $E \to L_{\mathcal{X}}$ and $F \to L_{\mathcal{X}'}$ be perfect relative obstruction theories for $\mathcal{X}$ and $\mathcal{X}'$ over $\mathcal{Y}$, respectively. A *compatibility datum* for E and F is a triple of morphisms in $D(\mathcal{O}_{\mathcal{X}'})$ giving rise to a morphism of distinguished triangles

$$\begin{array}{ccccccc} u^*E & \xrightarrow{\phi} & F & \xrightarrow{\psi} & g^*L_{\mathcal{Z}'/\mathcal{Z}} & \xrightarrow{\chi} & u^*E[1] \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ u^*L_{\mathcal{X}/\mathcal{Y}} & \longrightarrow & L_{\mathcal{X}'/\mathcal{Y}} & \longrightarrow & L_{\mathcal{X}'/\mathcal{X}} & \longrightarrow & u^*L_{\mathcal{X}/\mathcal{Y}}[1] \end{array}$$

We say that E, F are *compatible perfect relative obstruction theories* if there exists a compatibility datum. By [4, 7.5] if E, F are compatible perfect relative obstruction theories, and $\mathcal{Z}'$ and $\mathcal{Z}$ as above are smooth then $v^![\mathcal{X}] = [\mathcal{X}']$.

Example 7.1. (Cutting an edge for stable maps) Let $\Upsilon : \Gamma \to \Gamma'$ be a morphism of graphs disconnecting an edge, that is, replacing an edge in Γ with a pair of semi-infinite edges in Γ'. We have a morphism of stacks of stable curves $\overline{\mathfrak{M}}(\Upsilon) : \overline{\mathfrak{M}}_{g,n,\Gamma} \to \overline{\mathfrak{M}}_{g,n+2,\Gamma'}$ obtained by identifying the two additional markings, and an induced isomorphism $\overline{\mathcal{M}}(\Upsilon) : \overline{\mathcal{M}}_{g,n,\Gamma} \to \overline{\mathcal{M}}_{g,n+2,\Gamma'}$, except in the case that there exists an automorphism of a curve of combinatorial type Γ' interchanging the two markings, in which case it is a double cover.

The stack of stable maps $\overline{\mathcal{M}}_{g,n,\Gamma}(X)$ may be identified (up to a possible automorphism) with the sub-stack of $\overline{\mathcal{M}}_{g,n+2,\Gamma'}(X)$ consisting of objects with $u(z_{n+1}) = u(z_{n+2})$, where z_{n+1}, z_{n+2} are the new markings. That is, we have a Cartesian diagram

$$\begin{array}{ccc} \overline{\mathcal{M}}_{g,n,\Gamma}(X) & \longrightarrow & \overline{\mathcal{M}}_{g,n+2,\Gamma'}(X) \\ \downarrow & & \downarrow \\ \overline{\mathfrak{M}}_{g,n,\Gamma} \times X & \xrightarrow{\Delta} & \overline{\mathfrak{M}}_{g,n+2,\Gamma'} \times X \times X \end{array}$$

where Δ combines the identification of the moduli stacks with the diagonal embedding of X. As explained in Behrend [3, p.8] for the case of stable maps, the two perfect relative obstruction theories are compatible which implies

$$[\overline{\mathcal{M}}_{g,n,\Gamma}(X)] = \Delta^![\overline{\mathcal{M}}_{g,n+2,\Gamma'}(X)].$$

Indeed if Γ' is obtained from Γ by cutting an edge then we check that the obstruction theories are compatible over Δ. Consider the Cartesian diagram

$$\begin{array}{ccc} \overline{\mathcal{M}}_{g,n,\Gamma}(X) & \xrightarrow{\overline{\mathcal{M}}(\Upsilon,X)} & \overline{\mathcal{M}}_{g,n+2,\Gamma'}(X) \\ \Psi\downarrow & & \downarrow \\ \overline{\mathfrak{M}}_{g,n,\Gamma} \times X & \longrightarrow & \overline{\mathfrak{M}}_{g,n+2,\Gamma'} \times X \times X \end{array}$$

Let $\pi : \mathcal{C} \to \overline{\mathcal{M}}_{g,n,\Gamma}(X)$ denote the universal curve, and let $\mathcal{C}'' = \overline{\mathcal{M}}(\Upsilon, X)^*\mathcal{C}'$ be the curve over $\overline{\mathcal{M}}_{g,n+2,\Gamma'}(X)$ obtained by normalizing at the node corresponding to the edge, with $p : \mathcal{C}'' \to \mathcal{C}$ the projection, and $\mathrm{ev}'' : \mathcal{C}'' \to X, \quad \mathrm{ev} : \mathcal{C} \to X$ the universal maps. So $\mathcal{C}$ is obtained from $\mathcal{C}''$ by identifying the two sections x_1, x_2 of $\mathcal{C}''$, and is equipped with a section x induced from x_1, x_2. We have a short exact sequence of complexes relating the push-forward on $\mathcal{C}, \mathcal{C}''$,

$$0 \to \mathrm{ev}^* TX \to p_* p^* \mathrm{ev}^* TX \to x_* x^* \mathrm{ev}^* TX \to 0,$$

and so an exact triangle

$$R\pi_* \mathrm{ev}^* TX \to R\pi''_* p^* \mathrm{ev}^* TX \to x^* \mathrm{ev}^* TX \to R\pi_* \mathrm{ev}^* TX[1]. \qquad (35)$$

Now if $E_\Gamma(X) := (R\pi_* \operatorname{ev}^* TX)^\vee$ then

$$\overline{\mathfrak{M}}(\Upsilon)^* E_{\Gamma'}(X) = (R\pi''_* \operatorname{ev}^{'',*} TX)^\vee = (R\pi''_* p^* \operatorname{ev}^* TX)^\vee.$$

Moreover we have an exact triangle

$$\Psi^* L_\Delta[-1] \to \overline{\mathcal{M}}(\Upsilon, X)^* E_{\Gamma''}(X) \to E_\Gamma(X) \to \Psi^* L_\Delta.$$

This gives rise to a homomorphism of distinguished triangles

$$\begin{array}{ccccc} \overline{\mathfrak{M}}(\Upsilon)^* E_{\Gamma'}(X) & \longrightarrow & E_\Gamma(X) & \longrightarrow & \Psi^* L_\Delta \\ \downarrow & & \downarrow & & \downarrow \\ \overline{\mathfrak{M}}(\Upsilon)^* L_{\overline{\mathcal{M}}_{g,n+2,\Gamma'}(X)/\overline{\mathfrak{M}}_{g,n+2,\Gamma'}} & \longrightarrow & L_{\overline{\mathcal{M}}_{g,n,\Gamma}(X)/\overline{\mathfrak{M}}_{g,n,\Gamma}} & \longrightarrow & L_{\overline{\mathcal{M}}(\Upsilon,X)}. \end{array}$$

Example 7.2. (Collapsing an edge for stable maps) Let $\Upsilon : \Gamma' \to \Gamma$ be a morphism of modular graphs given by collapsing an edge. Associated to Υ are morphisms of Artin resp. Deligne-Mumford stacks

$$\overline{\mathfrak{M}}(\Upsilon) : \overline{\mathfrak{M}}_{g,n,\Gamma'} \to \overline{\mathfrak{M}}_{g,n,\Gamma}, \quad \overline{\mathcal{M}}(\Upsilon) : \overline{\mathcal{M}}_{g,n,\Gamma'} \to \overline{\mathcal{M}}_{g,n,\Gamma}.$$

The inclusion of $\overline{\mathfrak{M}}_{g,n,\Gamma}(X)$ to $\overline{\mathfrak{M}}_{g,n,\Gamma'}(X)$ induces an *isomorphism* of perfect relative obstruction theories. As in Behrend [3], the relative obstruction theories for $\overline{\mathcal{M}}_{g,n,\Gamma}(X), \overline{\mathcal{M}}_{g,n,\Gamma'}(X)$ are related by pull-back. Letting $s(\Gamma), s(\Gamma')$ denote the stabilizations of Γ, Γ' consider the diagram from [3, p. 15]

$$\begin{array}{ccccc} \sqcup_{d' \to d} \overline{\mathcal{M}}_{g,n,\Gamma'}(X, d') & \longrightarrow & \overline{\mathcal{M}}_{g,n,s(\Gamma')} \times_{\overline{\mathcal{M}}_{g,n,s(\Gamma)}} \overline{\mathcal{M}}_{g,n,\Gamma}(X,d) & \longrightarrow & \overline{\mathcal{M}}_{g,n,\Gamma}(X,d) \\ \downarrow & & \downarrow & & \downarrow \\ \overline{\mathfrak{M}}_{g,n,\Gamma'} & \longrightarrow & \overline{\mathcal{M}}_{g,n,s(\Gamma')} \times_{\overline{\mathcal{M}}_{g,n,s(\Gamma)}} \overline{\mathfrak{M}}_{g,n,\Gamma} & \longrightarrow & \overline{\mathfrak{M}}_{g,n,\Gamma} \\ & \searrow & \downarrow & & \downarrow \\ & & \overline{\mathcal{M}}_{g,n,s(\Gamma')} & \longrightarrow & \overline{\mathcal{M}}_{g,n,s(\Gamma)}. \end{array}$$

All the squares are Cartesian and it follows as in [3] (see especially [3, Proposition 8], which uses bivariant Chow theory for representable morphisms of Artin stacks) that

$$\overline{\mathcal{M}}(\Upsilon)^! [\overline{\mathcal{M}}_{g,n,\Gamma}(X,d)] = \overline{\mathcal{M}}(\Upsilon, X)_* \sum_{d' \mapsto d} [\overline{\mathcal{M}}_{g,n,\Gamma'}(X, d')]$$

where

$$\overline{\mathcal{M}}(\Upsilon, X) : \overline{\mathcal{M}}_{g,n,\Gamma'}(X, d') \to \overline{\mathcal{M}}_{g,n,\Gamma'} \times_{\overline{\mathcal{M}}_{g,n,\Gamma}} \overline{\mathcal{M}}_{g,n,\Gamma}(X, d)$$

is the identification with the fiber product.

It follows that the virtual fundamental classes $[\overline{\mathcal{M}}_{g,n,\Gamma}(X,d)] \in A^G(\overline{\mathcal{M}}_{g,n,\Gamma}(X,d))$ satisfy the following properties as in Behrend [3]:

Proposition 7.3. (a) (Constant maps) If $d = 0$ then $[\overline{\mathcal{M}}_{g,n,\Gamma}(X,d)]$ is obtained by cap product of $[X \times \overline{\mathcal{M}}_{g,n,\Gamma}]$ with the Euler class of $R^1 p_* e^* TX$.

(b) (Cutting edges) If $\Upsilon : \Gamma \to \Gamma'$ is a morphism of modular graphs of type cutting an edge then $[\overline{\mathcal{M}}_{g,n,\Gamma}(X,d)] = \Delta^! [\overline{\mathcal{M}}_{g,n+2,\Gamma'}(X,d')]$ where $\Delta : X \to X \times X$ is the diagonal.

(c) (Collapsing edges) If $\Upsilon : \Gamma \to \Gamma'$ is a morphism of graphs of type collapsing an edge then

$$\mathcal{M}(\Upsilon)^! [\overline{\mathcal{M}}_{g,n,\Gamma'}(X,d')] = \overline{\mathcal{F}}(\Upsilon, X)_* \sum_{d \mapsto d'} [\overline{\mathcal{M}}_{g,n,\Gamma}(X,d)]$$

where

$$\overline{\mathcal{F}}(\Upsilon, X) : \overline{\mathcal{M}}_{g,n,\Gamma}(X,d) \to \overline{\mathcal{M}}_{g,n,s(\Gamma)} \times_{\overline{\mathcal{M}}_{g,n,s(\Gamma')}} \overline{\mathcal{M}}_{g,n,\Gamma'}(X,d').$$

(d) (Forgetting Tails) If $\Upsilon : \Gamma \to \Gamma'$ is a morphism of graphs of type forgetting a tail then

$$\overline{\mathcal{M}}(\Upsilon, X)^! [\overline{\mathcal{M}}_{g,n,\Gamma'}(X,d)] = [\overline{\mathcal{M}}_{g,n+1,\Gamma}(X,d)]$$

where $\overline{\mathcal{M}}(\Upsilon, X)$ was defined in [27, Example 4.3].

We now pass from Chow groups/rings to homology/cohomology with rational coefficients. (One can work with more general theories here, as in Behrend-Manin [5].) For any cohomology classes $\alpha \in H_G(X, \mathbb{Q})^n$ and $\beta \in H(\overline{M}_{g,n,\Gamma}, \mathbb{Q})$ (if $2g+n \geq 3$) pairing with the virtual fundamental class $[\overline{\mathcal{M}}_{g,n,\Gamma}(X,d)] \in H(\overline{M}_{g,n,\Gamma}(X,d))$ defines a *Gromov-Witten invariant*

$$\langle \alpha; \beta \rangle_{\Gamma,d} = \int_{[\overline{\mathcal{M}}_{g,n,\Gamma}(X,d)]} \operatorname{ev}^* \alpha \cup f^* \beta \in H(BG).$$

These invariants satisfy axioms for morphisms of modular graphs:

Proposition 7.4. (a) (Cutting edges) If Γ' is obtained from Γ by cutting an edge then

$$\langle \alpha; \beta \rangle_{\Gamma,d} = \sum_{i=1}^{N} \langle \alpha, \delta_i, \delta^i; \overline{\mathcal{M}}(\Upsilon)^* \beta \rangle_{\Gamma',d}$$

where $(\delta_i)_{i=1}^N, (\delta^i)_{i=1}^N$ are dual bases for $H_G(X)$ over $H(BG)$.

(b) (Collapsing edges) If $\Upsilon : \Gamma \to \Gamma'$ is a morphism collapsing an edge then

$$\langle \alpha; \beta \cup \gamma \rangle_{\Gamma',d'} = \sum_{d \mapsto d'} \langle \alpha; \overline{\mathcal{M}}(\Upsilon)^* \beta \rangle_{\Gamma,d}$$

where $\gamma \in H^2(\overline{\mathcal{M}}_{g,n,\Gamma'})$ is the dual class for $\overline{\mathcal{M}}(\Upsilon) : \overline{\mathcal{M}}_{g,n,\Gamma} \to \overline{\mathcal{M}}_{g,n,\Gamma'}$.

(c) (Forgetting tails) If $\Upsilon : \Gamma \to \Gamma'$ is a morphism forgetting a tail then for $\alpha' \in H^2_G(X)$,

$$\langle \alpha, \alpha'; \overline{\mathcal{M}}(\Upsilon)^* \beta \rangle_{\Gamma,d} = (d, \alpha') \langle \alpha; \beta \rangle_{\Gamma',d'}$$

Proof. These follow from Proposition 7.3 as in (the more abstract formulation) in Behrend-Manin [5, Theorem 9.2] to which we refer the reader for more detail.

Definition 7.5. (Novikov field) The Novikov field Λ_X for X is the set of all maps $a : H_2(X) := H_2(X,\mathbb{Z})/\operatorname{torsion} \to \mathbb{Q}$ such that for every constant c, the set of classes

$$\{d \in H_2(X,\mathbb{Z})/\operatorname{torsion},\ \langle [\omega], d\rangle \le c, \quad a(d) \neq 0\}$$

on which a is non-vanishing is finite. The delta function at d is denoted q^d. Addition is defined in the usual way and multiplication is convolution, so that $q^{d_1} q^{d_2} = q^{d_1+d_2}$.

Define as vector space the *quantum cohomology of* X

$$QH_G(X) := H_G(X,\mathbb{Q}) \otimes \Lambda_X.$$

Define genus g correlators

$$\langle \cdot, \cdot \rangle_{g,n} = \sum_{d \in H_2(X)} q^d \langle \cdot, \cdot \rangle_{\Gamma,d}$$

where Γ is a genus g graph with one vertex and n semi-infinite edges. By Proposition 7.4,

Theorem 7.6. *([14], [3]) After tensoring with the field of fractions of $H(BG)$ the space $QH_G(X)$ equipped with the maps $\langle \cdot, \cdot \rangle_{g,n}$ forms a cohomological field theory.*

Restricting to genus zero we obtain a CohFT algebra: Maps

$$\mu^n : QH_G(X)^n \times H(\overline{M}_{0,n+1}, \mathbb{Q}) \to QH_G(X)$$

defined by

$$(\mu^n(\alpha_1, \ldots, \alpha_n; \beta), \alpha_0) = \sum_{d \in H_2(X,\mathbb{Z})} q^d \langle \alpha_0, \ldots, \alpha_n; \beta \rangle_{0,d} \in \Lambda_X.$$

Here $(\ , \ )$ denotes the pairing on $QH_G(X)$ induced by cup product and integration over $H_G(X)$.

A related collection of invariants is expressed as the integrals over *parametrized* stable maps to X. Let $\operatorname{Hom}(C, X, d) \subset \operatorname{Hom}(C, X)$ denote the subscheme of maps of class $d \in H_2(X,\mathbb{Z})$. Compactifications of $\operatorname{Hom}(C, X, d)$ are provided by so-called *graph spaces*

$$\overline{\mathcal{M}}_n(C, X, d) := \overline{\mathcal{M}}_{g,n}(C \times X, (1, d))$$

of stable maps $u : \hat{C} \to C \times X$ of degree $(1, d)$. Each stable map $u = (u_C, u_X) : \hat{C} \to C \times X$ has a single component $\hat{C}_0 \subset \hat{C}$ that maps isomorphically onto C via u_C, with all other components mapping to points. We denote by

$$\operatorname{ev} : \overline{\mathcal{M}}_n(C, X) \to X^n, \quad \operatorname{ev}_C : \overline{\mathcal{M}}_n(C, X) \to C^n \tag{36}$$

the evaluation maps followed by projection on the second, resp. first factor. The stacks $\overline{\mathcal{M}}_n(C,X,d)$ have equivariant relatively perfect obstruction theories over $\overline{\mathfrak{M}}_n(C)$ with complex given by Rp_*e^*TX, where $p : \overline{\mathcal{C}}_n(C,X) \to \overline{\mathcal{M}}_n(C,X)$ is the universal curve and $e : \overline{\mathcal{C}}_n(C,X) \to C$ the evaluation map. For any cohomology classes $\alpha \in H_G(X,\mathbb{Q})^n$ and $\beta \in H(\overline{\mathcal{M}}_{n,\Gamma}(C),\mathbb{Q})$ pairing with the virtual fundamental class $[\overline{\mathcal{M}}_{n,\Gamma}(C,X,d)] \in H(\overline{\mathcal{M}}_{n,\Gamma}(C,X,d))$ defines a *graph Gromov-Witten invariant*

$$\langle \alpha;\beta\rangle_{C,\Gamma,d} = \int_{[\overline{\mathcal{M}}_{n,\Gamma}(C,X,d)]} \operatorname{ev}^*\alpha \cup f^*\beta \in H(BG). \tag{37}$$

These invariants satisfy axioms for morphisms of *rooted* modular trees, similar to those in Proposition 7.4 which we omit to save space. Define

$$\tau_X^n : QH_G(X)^n \times H(\overline{M}_n(C)) \to \Lambda_X^G \otimes H(BG),\ (\alpha,\beta) \mapsto \sum_d q^d \langle \alpha;\beta\rangle_{C,d}. \tag{38}$$

The splitting axiom for these invariants implies:

Theorem 7.7. *The maps $(\tau_X^n)_{n\geq 0}$ define a CohFT trace on the CohFT algebra $QH_G(X)$.*

The Gromov-Witten invariants have a natural circle-invariant extension given by interpreting the equivariant parameter as the first Chern class of the tangent lines at the markings. Let $\psi_i \in H^2(\overline{\mathcal{M}}_n^G(C,X))$ denote the Chern class of the cotangent line at the i-th marking. Given $\alpha_i(\zeta) \in QH_{G\times S^1}(X) \cong QH_G(X)[\zeta]$ for $I = 1,\ldots,n$ define

$$\langle \alpha(\zeta);\beta\rangle_{\Gamma,d} = \int_{[\overline{\mathcal{M}}_{g,n,\Gamma}(X,d)]} \operatorname{ev}_1^*\alpha_1|_{\zeta=-\psi_1} \ldots \operatorname{ev}_n^*\alpha_n|_{\zeta=-\psi_n} \cup f^*\beta \in H(BG). \tag{39}$$

This circle-invariant extension will play a role in the adiabatic limit theorem given later.

7.2. Gromov-Witten theory for smooth Deligne-Mumford stacks

We review the orbifold Gromov-Witten theory developed by Chen-Ruan [6] and Abramovich-Graber-Vistoli [1], needed in our case if the geometric invariant theory quotient $X/\!\!/G$ is an orbifold. For simplicity, we restrict to the case without group action.

Let $\mathcal{X}$ be a proper smooth Deligne-Mumford stack. The moduli stack of twisted stable maps admits evaluation maps

$$\operatorname{ev} : \overline{\mathcal{M}}_{g,n}(\mathcal{X}) \to \overline{I}_{\mathcal{X}}^n, \quad \overline{\operatorname{ev}} : \overline{\mathcal{M}}_{g,n}(\mathcal{X}) \to \overline{I}_{\mathcal{X}}^n,$$

where the second is obtained by composing with the involution of the rigidified inertia stack $\overline{I}_{\mathcal{X}} \to \overline{I}_{\mathcal{X}}$ induced by the automorphism of the group μ_r of r-th roots of unity $\mu_r \to \mu_r, \varphi \mapsto \varphi^{-1}$. (See [27, Section 4.3] for the definition of the rigidified inertia stack.) The virtual fundamental classes satisfy splitting axioms for morphisms of modular graphs, in particular, for cutting an edge in which case one of the evaluation maps is taken to be with respect to opposite signs on the pair of marked points created by the cutting. Given a homology class $d \in H_2(\mathcal{X}, \mathbb{Q})$ and non-negative integers g, n, let $\overline{\mathcal{M}}_{g,n}(\mathcal{X}, d)$ denote the moduli stack of stable maps to $\mathcal{X}$ with class d. The virtual fundamental classes $[\overline{\mathcal{M}}_{g,n,\Gamma}(\mathcal{X}, d)]$ satisfy the splitting axioms for morphisms of modular graphs similar to those in the case that X is a variety. Orbifold Gromov-Witten invariants are defined by virtual integration of pull-back classes using the evaluation maps above. For non-negative integers $n_- + n_+ = n$ denote by $\mathrm{ev}^*_{n_+}$ resp. $\overline{\mathrm{ev}}^*_{n_-}$ the untwisted resp. twisted evaluation map on the first n_+ resp. last n_- markings. Define *Gromov-Witten invariants*

$$\begin{aligned} &H(\overline{I}_{\mathcal{X}})^{n_+} \times H(\overline{I}_{\mathcal{X}})^{n_-} \times H(\overline{M}_{g,n}) \to \mathbb{Q}, \\ &\quad (\alpha_+, \alpha_-, \beta) \mapsto \langle \alpha_+, \alpha_-, \beta \rangle_{\Gamma,d} = \int_{[\overline{\mathcal{M}}_{g,n,\Gamma}(\mathcal{X},d)]} \mathrm{ev}^*_{n_+} \alpha_+ \cup \overline{\mathrm{ev}}^*_{n_-} \overline{\alpha}_- \cup f^* \beta. \end{aligned} \tag{40}$$

The orbifold Gromov-Witten invariants satisfy properties similar to those for usual Gromov-Witten invariants, with the notable exception [1, 6.1.4] that if Γ' is obtained from Γ by cutting an edge then

$$\langle \alpha; \beta \rangle_{\Gamma,d} = \sum_k \langle \alpha, \delta_k, \delta^k; \overline{\mathcal{M}}(\Upsilon)^* \beta \rangle_{\Gamma',d} \tag{41}$$

where δ_k, δ^k are dual bases of $H(\overline{I}_{\mathcal{X}})$ with respect to a different inner product: the inner product defined using re-scaled integration $([\overline{I}_{\mathcal{X}}], r\cdot)$ where $r : \overline{I}_{\mathcal{X}} \to \mathbb{Z}_{\geq 0}$ is the order of the isotropy group. The definition of orbifold Gromov-Witten invariants leads to the definition of orbifold quantum cohomology as follows.

Definition 7.8. (Orbifold quantum cohomology) To each component $\mathcal{X}_i$ of $\overline{I}_{\mathcal{X}}$ is assigned a rational number $\mathrm{age}(\mathcal{X}_i)$ as follows. Let (x, g) be an object of $\mathcal{X}_i$. The element g acts on $T_x\mathcal{X}$ with eigenvalues $\alpha_1, \ldots, \alpha_n$ with $n = \dim(\mathcal{X})$. Let r be the order of g and define $s_j \in \{0, \ldots, r-1\}$ by $\alpha_j = \exp(2\pi i s_j / r)$. The *age* is defined by

$$\mathrm{age}(\mathcal{X}_i) = (1/r) \sum_{j=1}^n s_j.$$

Let $\Lambda_{\mathcal{X}} \subset \mathrm{Hom}(H_2(X, \mathbb{Q}), \mathbb{Q})$ denote the Novikov field of linear combinations of formal symbols $q^d, d \in H_2(\mathcal{X}, \mathbb{Q})$ where for each c, only finitely many q^d with $(d, [\omega]) < c$ have non-zero coefficient. Let

$$QH(\mathcal{X}) = H(\overline{I}_{\mathcal{X}}) \otimes \Lambda_{\mathcal{X}}$$

denote the *orbifold quantum cohomology* equipped with the *age grading*

$$QH^\bullet(\mathcal{X}) = \bigoplus_{\mathcal{X}_i \subset \overline{I}_{\mathcal{X}}} H^{\bullet - 2\,\mathrm{age}(\mathcal{X}_i)}(\mathcal{X}_i) \otimes \Lambda_{\mathcal{X}}.$$

Theorem 7.9. *The orbifold Gromov-Witten invariants define the structure of a CohFT on $QH(\mathcal{X})$, in particular, a CohFT algebra structure on $QH(\mathcal{X})$ and the graph invariants define a trace on $QH(\mathcal{X})$.*

Proof. This follows from the splitting axiom (41), and the analogous splitting axiom for the graph invariants whose proof is similar.

7.3. Twisted Gromov-Witten invariants

We also describe twisted versions of Gromov-Witten invariants arising from vector bundles on the target, see for example Coates-Givental [9]. Under suitable positivity assumptions, these invariants are equal to the Gromov-Witten invariants of hypersurfaces defined by sections.

Definition 7.10. (Twisting class and twisted Gromov-Witten invariants) Let E be a G-equivariant complex vector bundle over a smooth projective G-variety X. Pull-back under the evaluation map $e : \overline{\mathcal{C}}_{g,n}(X) \to X$ on the universal curve gives rise to a vector bundle $\operatorname{ev}^* E \to \overline{\mathcal{C}}_{g,n}(X)$, which we can push down to an *index*

$$\operatorname{Ind}_G(E) := Rp_* e^* E$$

in the derived category of bounded complexes of equivariant coherent sheaves on $\overline{\mathcal{M}}_{g,n}(X)$. Since p is a local complete intersection morphism, $\operatorname{Ind}_G(E)$ admits a resolution by vector bundles, see [9, Appendix], and we may consider the (invertible) equivariant Euler class

$$\epsilon(E) := \operatorname{Eul}_{G\times\mathbb{C}^\times}(\operatorname{Ind}_G(E)) \in H_G(\overline{\mathcal{M}}_{g,n}(X)) \otimes \mathbb{Q}[\varphi, \varphi^{-1}]$$

where φ is the parameter for the action of $\mathbb{C}^\times$ by scalar multiplication in the fibers. The *twisted equivariant Gromov-Witten invariants* associated to $E \to X$ and type Γ are the maps

$$H_G(X)^n \times H(\overline{M}_{g,n,\Gamma}) \to H_G(\mathrm{pt}, \mathbb{Q}) \otimes \mathbb{Q}[\varphi, \varphi^{-1}]$$
$$\langle \alpha; \beta \rangle_{\Gamma,E,d} = \int_{[\overline{\mathcal{M}}_{n,\Gamma}(C,X)]} \operatorname{ev}^* \alpha \cup f^* \beta \cup \epsilon(E). \quad (42)$$

Proposition 7.11. The twisted invariants satisfy the properties:

(a) (Collapsing edges) If $\Upsilon : \Gamma \to \Gamma'$ is of type collapsing an edge then for any labelling d' of Γ',

$$\langle \alpha; \beta \cup \gamma \rangle_{\Gamma',d',E} = \sum_{d \mapsto d'} \langle \alpha; \overline{\mathcal{M}}(\Upsilon)^* \beta \rangle_{\Gamma,d,E}$$

where γ is the dual class to $\overline{\mathcal{M}}(\Upsilon)$.

(b) (Cutting edges) If $\Upsilon : \Gamma \to \Gamma'$ is of type cutting an edge then

$$\langle \alpha; \beta \rangle_{\Gamma,d,E} = \sum_k \langle \alpha, \delta_k \cup \operatorname{Eul}_{G\times\mathbb{C}^\times}(E), \delta^k; \overline{\mathcal{M}}(\Upsilon)^* \beta \rangle_{\Gamma',d,E}.$$

(c) (Forgetting tails) If $\Upsilon : \Gamma \to \Gamma'$ is a morphism forgetting a tail, which corresponds to the last marking z_n then for $\alpha' \in H^2_G(X), \alpha \in H_G(X)^{n-1}$,

$$\langle \alpha, \alpha'; \overline{\mathcal{M}}(\Upsilon)^* \beta \rangle_{\Gamma,d,E} = (d, \alpha') \langle \alpha; \beta \rangle_{\Gamma',d',E}$$

Proof. We discuss only the cutting-edge axiom; the rest are similar to those in the untwisted case. Let $p : \mathcal{C}' \to \mathcal{C}$ be the normalization and x the section given by the node corresponding to the cut edge. The short exact sequence of sheaves

$$0 \to E \to p^* p_* E \to x^* x_* E \to 0$$

gives rise to an exact triangle in the derived category of bounded complexes of coherent sheaves

$$R\pi_* \operatorname{ev}^* E \to R\pi''_* p^* \operatorname{ev}^* E \to x^* \operatorname{ev}^* E \to R\pi_* \operatorname{ev}^* E[1].$$

Taking Euler classes gives the result.

A receptacle for twisted Gromov-Witten invariants is the equivariant cohomology of a point with the equivariant parameter inverted. This means that twisted composition maps take values in the equivariant cohomology of the space with equivariant parameter inverted. Define

$$QH_{G\times\mathbb{C}^\times}(X,\mathbb{Q}) = QH_G(X,\mathbb{Q}) \otimes \mathbb{Q}[\varphi,\varphi^{-1}]$$

and define twisted composition maps

$$\mu_E^{g,n} : QH_{G\times\mathbb{C}^\times}(X,\mathbb{Q})^n \times H(\overline{\mathcal{M}}_{g,n+1},\mathbb{Q}) \to QH_{G\times\mathbb{C}^\times}(X,\mathbb{Q}),$$

$$(\mu_E^{g,n}(\alpha_1,\ldots,\alpha_n;\beta),\alpha_0) := \sum_{d\in H_2(X)} q^d \langle \alpha_0 \cup \operatorname{Eul}_{G\times\mathbb{C}^\times}(E),\ldots,\alpha_n;\beta\rangle_{\Gamma,d,E}$$

where Γ is a genus g graph with a single vertex. Discussion of twisted composition maps can be found in e.g. Pandharipande [24].

Theorem 7.12. (Equivariant twisted Gromov-Witten invariants define a CohFT algebra) *Suppose that X is a smooth projective G-variety and $E \to X$ is a G-equivariant vector bundle. The datum $(QH_{G\times\mathbb{C}^\times}(X,\mathbb{Q}),(\mu_E^{g,n})_{g,n\geq 0})$ form a CohFT algebra, denoted $QH_G(X,E)$.*

7.4. Gauged Gromov-Witten invariants

In this section we define gauged Gromov-Witten invariants. As in Behrend [3], invariants are defined for any possibly disconnected combinatorial type, and the splitting axiom can be broken down into *cutting edges* and *collapsing edges* axiom. However, the definition for disconnected type requires an additional datum, of an assignment of each non-root component to a semi-infinite edge of a root component. The equivariant virtual classes for the non-root components combine with the non-equivariant virtual classes for the root component to a non-equivariant virtual class for moduli space for disconnected type.

Definition 7.13. (Virtual fundamental classes for moduli stacks of gauged maps) Let X be a smooth projective G-variety.

(a) (Combinatorial type with a single vertex) We already remarked in [27, Example 6.6] that if $\overline{\mathcal{M}}_n^G(C,X)$ is a Deligne-Mumford stack, then it has a perfect obstruction theory, given by the dual of the derived push-forward of the pull-back of the tangent complex $(Rp_* e^* T(X/G))^\vee$ where $p : \overline{\mathcal{C}}_n^G(X,d) \to \overline{\mathcal{M}}_n^G(X,d)$ is the universal curve, $e : \overline{\mathcal{C}}_n^G(X,d) \to X/G$ the universal stable gauged map, and $T(X/G)$ the tangent complex to X/G. Hence one obtains a virtual fundamental class of expected dimension

$$[\overline{\mathcal{M}}_n^G(C,X,d)] \in A(\overline{\mathcal{M}}_n^G(C,X,d)).$$

(b) (Connected combinatorial type) More generally, given any connected rooted tree Γ we denote by $\overline{\mathcal{M}}_{n,\Gamma}^G(C,X,d)$ resp. $\overline{\mathcal{M}}_{n,\Gamma}^{G,\mathrm{fr}}(C,X,d)$ the moduli stack of stable gauged maps resp. with framings at the markings of combinatorial type Γ and class d. Under the assumption that $\overline{\mathcal{M}}_n^G(C,X,d)$ is Deligne-Mumford, the action of G^n on $\overline{\mathcal{M}}_{n,\Gamma}^{G,\mathrm{fr}}(C,X,d)$ is locally free. The same construction gives a virtual fundamental class

$$[\overline{\mathcal{M}}_{n,\Gamma}^G(C,X,d)] \in A(\overline{\mathcal{M}}_{n,\Gamma}^G(C,X,d)) \cong A^{G^n}(\overline{\mathcal{M}}_{n,\Gamma}^{G,\mathrm{fr}}(C,X,d)).$$

(c) (Disconnected combinatorial type) Suppose $\Gamma = \Gamma_0 \cup \ldots \cup \Gamma_l$ with Γ_0 containing the root vertex is given the *additional datum* of a map from the non-root components to the root edges: Suppose that $\Gamma = \Gamma_0 \cup \Gamma_1 \cup \cdots \cup \Gamma_l$ is a disconnected rooted $H_2^G(X)$-labelled graph such that Γ_j has semi-infinite edges I_j, and for each $j = 1,\ldots,l$ is given a semi-infinite edge $e(j)$ of Γ_0. We denote by

$$\overline{\mathcal{M}}_{n,\Gamma}^{G,f}(C,X,d) = (\overline{\mathcal{M}}_{n_0,\Gamma_0}^{G,\mathrm{fr}}(C,X,d_0) \times \prod_{j=1}^{l} \overline{\mathcal{M}}_{0,n_j,\Gamma_j}(X,d_j))/G^{n_0}$$

where the action of the i-th factor in G^{n_0} acts at the i-th framing on the principal component, and diagonally on the components corresponding to Γ_j with $e(j) = i$. We have virtual fundamental classes

$$[\overline{\mathcal{M}}_{n_0,\Gamma_0}^G(C,X,d_0)] \in A(\overline{\mathcal{M}}_{n_0,\Gamma_0}^G(C,X,d)) \cong A_{G^{n_0}}(\overline{\mathcal{M}}_{n_0,\Gamma_0}^{G,\mathrm{fr}}(C,X,d))$$

$$[\overline{\mathcal{M}}_{0,n_j,\Gamma_j}(X,d_j)] \in A_G(\overline{\mathcal{M}}_{0,n_j,\Gamma_j}(X,d_j))$$

and so a virtual fundamental class

$$[\overline{\mathcal{M}}_{n,\Gamma}^G(C,X,d)] = \cup_{d=d_0+\ldots+d_l} [\overline{\mathcal{M}}_{n_0,\Gamma_0}^{G,\mathrm{fr}}(C,X,d_0)] \times \prod_{j=1}^{l} [\overline{\mathcal{M}}_{0,n_j,\Gamma_j}(X,d_j)]$$

in

$$A_{G^{n_0}}(\overline{\mathcal{M}}_{n_0,\Gamma_0}^{G,\mathrm{fr}}(C,X,d) \times \prod_j \overline{\mathcal{M}}_{0,n_j}(X,d_j)) \cong A(\overline{\mathcal{M}}_{n,\Gamma}^{G,f}(C,X,d)).$$

These classes satisfy the following properties similar to those in [3]:

Proposition 7.14. (a) (Constant maps) If $d = 0$ and genus$(C) = 0$ then $\overline{\mathcal{M}}^G_{\Gamma,n}(C, X, d) = (X/\!\!/G) \times \overline{\mathcal{M}}_{\Gamma,n}(C)$ and $[\overline{\mathcal{M}}^G_{n,\Gamma}(C, X, d)] = [X/\!\!/G \times \overline{\mathcal{M}}_{n,\Gamma}(C)]$.

(b) (Cutting edges) If Γ' is obtained from Γ by cutting an edge then (with the obvious labelling of the additional component) $[\overline{\mathcal{M}}^G_{n,\Gamma}(C, X, d)] = \Delta^![\overline{\mathcal{M}}^G_{n+2,\Gamma'}(C, X, d)]$.

(c) (Collapsing edges) If $\Upsilon : \Gamma' \to \Gamma$ is a morphism collapsing an edge then $\overline{\mathcal{M}}(\Upsilon)^![\overline{\mathcal{M}}^G_{n,\Gamma}(C, X, d)]$ is the push-forward of $\sum_{d' \mapsto d}[\overline{\mathcal{M}}^G_{n,\Gamma'}(C, X, d')]$ under

$$\overline{\mathcal{F}}(\Upsilon, X) : \overline{\mathcal{M}}^G_{n,\Gamma'}(C, X, d') \to \overline{\mathcal{M}}_{n,s(\Gamma')}(C) \times_{\overline{\mathcal{M}}_{n,s(\Gamma)}(C)} \overline{\mathcal{M}}^G_{n,\Gamma}(C, X, d).$$

(d) (Forgetting tails) If $\Upsilon : \Gamma \to \Gamma'$ is a morphism forgetting a tail then

$$\overline{\mathcal{M}}(\Upsilon)^![\overline{\mathcal{M}}^G_{n,\Gamma'}(C, X, d)] = [\overline{\mathcal{M}}^G_{n+1,\Gamma}(C, X, d)].$$

Proof. The items (a), (b) and (d) are similar to the ordinary Gromov-Witten case considered in Behrend [3] and left to the reader. For a morphism Υ cutting an edge for gauged maps, recall from [27, Proposition 5.21] that $\overline{\mathcal{M}}^G_{n,\Gamma}(C, X)$ may be identified with the fiber product $\overline{\mathcal{M}}^G_{n,\Gamma'}(C, X) \times_{(X/G)^2} (X/G)$ over the diagonal $\Delta : (X/G) \to (X/G)^2$. We denote by

$$\overline{\mathcal{M}}(\Upsilon, X) : \overline{\mathcal{M}}^G_{n,\Gamma}(C, X) \to \overline{\mathcal{M}}^G_{n,\Gamma'}(C, X)$$

the resulting morphism. We check that the obstruction theories $E_{\Gamma'}$ and E_Γ are compatible over Δ. Let $\mathcal{C}$ denote the universal curve over $\overline{\mathcal{M}}^G_{n,\Gamma}(C, X)$, similarly for $\mathcal{C}'$ and Γ'. Let $\mathcal{C}'' = \mathcal{M}(\Upsilon, X)^*\mathcal{C}'$ be the curve over $\overline{\mathcal{M}}^G_{n,\Gamma'}(C, X)$ obtained by normalizing at the node corresponding to the edge, with $f : \mathcal{C}'' \to \mathcal{C}$ the projection and $e'' : \mathcal{C}'' \to X/G, \quad e : \mathcal{C} \to X/G$ the universal maps. So $\mathcal{C}$ is obtained from $\mathcal{C}''$ by identifying the two sections x_1, x_2 of $\mathcal{C}''$, and is equipped with a section x induced from x_1, x_2. The short exact sequence of complexes of coherent sheaves

$$0 \to e^*T_{X/G} \to f_*f^*e^*T_{X/G} \to x_*x^*e^*T_{X/G} \to 0$$

(viewing $T_{X/G}$ as a two-term complex) induces an exact triangle in the derived category

$$Rp_*e^*T_{X/G} \to Rp''_*f^*e^*T_{X/G} \to x^*e^*T_{X/G} \to Rp_*e^*T_{X/G}[1].$$

We have relative obstruction theories with complexes

$$E_\Gamma := (Rp_*e^*T_{X/G})^\vee, \quad \overline{\mathcal{M}}(\Upsilon, X)^*E_{\Gamma'} = (Rp''_*e^{'',*}T_{X/G})^\vee = (Rp''_*f^*e^*T_{X/G})^\vee.$$

Note that $(x^*e^*T_{X/G})^\vee = \Psi^*L_\Delta$, where $\Delta : (X/G) \to (X/G)^2$ is the diagonal and Ψ is evaluation at the node. We have an exact triangle

$$\Psi^*L_\Delta[-1] \to \overline{\mathcal{M}}(\Upsilon, X)^*E_{\Gamma'} \to E_\Gamma \to \Psi^*L_\Delta.$$

This gives rise to a morphism of exact triangles as in Example 7.1. By compatibility, the virtual fundamental classes are related by $[\overline{\mathcal{M}}^G_{n,\Gamma}(C,X)] = \Delta^![\overline{\mathcal{M}}^G_{n+2,\Gamma'}(C,X)]$.

For collapsing an edge for gauged maps, let $\Upsilon : \Gamma' \to \Gamma$ be a morphism of rooted graphs given by *collapsing an edge*. Associated to Υ are morphisms of Artin resp. Deligne-Mumford stacks

$$\overline{\mathfrak{M}}(\Upsilon) : \overline{\mathfrak{M}}_{n,\Gamma'}(C) \to \overline{\mathfrak{M}}_{n,\Gamma}(C), \quad \overline{\mathcal{M}}(\Upsilon) : \overline{\mathcal{M}}_{n,\Gamma'}(C) \to \overline{\mathcal{M}}_{n,\Gamma}(C).$$

The first is a regular local immersion, and so defines a class in the bivariant Chow group $[\overline{\mathfrak{M}}(\Upsilon)] \in A^\vee(\overline{\mathfrak{M}}_{n,\Gamma'}(C) \to \overline{\mathfrak{M}}_{n,\Gamma}(C))$. As in Behrend [3], the relative obstruction theories for $\overline{\mathcal{M}}^G_{n,\Gamma}(C,X), \overline{\mathcal{M}}^G_{n,\Gamma'}(C,X)$ are related by pull-back. Following we have [3, p. 15]

$$\overline{\mathfrak{M}}(\Upsilon)^![\overline{\mathcal{M}}^G_{n,\Gamma}(C,X,d)] = \overline{\mathcal{F}}(\Upsilon,X)_* \sum_{d' \mapsto d} [\overline{\mathcal{M}}^G_{n,\Gamma'}(C,X,d')]$$

as claimed.

We now pass to homology/cohomology. (One could also consider the quantum Chow ring etc.) Pairing with the virtual fundamental class gives a map

$$\int_{[\overline{\mathcal{M}}^G_n(C,X)]} : H(\overline{\mathcal{M}}^G_n(C,X),\mathbb{Q}) \to \mathbb{Q}.$$

Evaluation at the marked points gives a morphism

$$\mathrm{ev} : \overline{\mathcal{M}}^G_n(C,X) \to (X/G)^n, \quad (P,\hat{C},u,z_1,\ldots,z_n) \mapsto (z_j^*P, u \circ z_j)_{j=1}^n.$$

Forgetting the bundle and curve and collapsing any unstable components defines a forgetful morphism from [27, Corollary 5.19] $f : \overline{\mathcal{M}}^G_n(C,X) \to \overline{\mathcal{M}}_n(C)$.

Definition 7.15. (Gauged Gromov-Witten invariants of a given combinatorial type)

(a) (Invariants for a tree with a single vertex) The *gauged Gromov-Witten invariants* associated to X are the maps

$$H_G(X,\mathbb{Q})^n \times H(\overline{\mathcal{M}}_n(C),\mathbb{Q}) \to \mathbb{Q}, \quad (\alpha,\beta) \mapsto \langle \alpha,\beta \rangle_d$$

$$\langle \alpha;\beta \rangle_d := \int_{[\overline{\mathcal{M}}^G_n(C,X,d)]} \mathrm{ev}^*\alpha \cup f^*\beta.$$

(b) (Invariants for a connected tree) The invariant for a *connected* rooted $H_2^G(X)$-labelled tree Γ and G-equivariant vector bundle $E \to X$ is the integral $\langle \alpha,\beta \rangle_{E,\Gamma,d} \in \mathbb{Q}$ of $\mathrm{ev}^*\alpha \cup f^*\beta \cup \epsilon(E)$ over the moduli stack $\overline{\mathcal{M}}^G_{n,\Gamma}(C,X)$ of stable gauged maps of combinatorial type Γ.

(c) (Invariants for forests) Invariants for possibly $H_2^G(X)$-labelled rooted forests are defined as follows, given the *additional datum* of a map from the non-root components to the root edges: Suppose that $\Gamma = \Gamma_0 \cup \Gamma_1 \cup \cdots \cup \Gamma_l$ is a rooted $H_2^G(X)$-labelled forest such that each tree Γ_j has semi-infinite edges

I_j, and for each $j = 1, \ldots, l$ is given a semi-infinite edge $e(j)$ of Γ_0. We define gauged Gromov-Witten invariants for Γ by fiber integration over the map $\overline{\mathcal{M}}^G_{n,\Gamma}(C,X) \to \overline{\mathcal{M}}^G_{n_0,\Gamma_0}(C,X)$ whose fibers are moduli stack of stable maps of type Γ_j for $j > 0$: set

$$\langle \alpha; \beta \rangle_{\Gamma,d} := \langle (\alpha_i')_{i \in I_0}; \beta \rangle_{\Gamma_0, d_0}$$

where for each semi-infinite edge i of Γ_0

$$\alpha_i' = \left(\prod_{i=e(j)} \langle (\alpha_e)_{e \in I_j}, \beta_e \rangle_{\Gamma_j, d_j} \right) \alpha_i,$$

using the $H(BG)$-module structure on $H_G(X)$, and $\beta_j \in H(\overline{M}_{n_j,\Gamma_j}(C))$ is the component of β in the decomposition $\overline{\mathcal{M}}_{n,\Gamma}(C) = \prod_j \overline{\mathcal{M}}_{n_j,\Gamma_j}(C)$.

Remark 7.16. It is not possible to define invariants for forests (as opposed to trees) as purely a product over the tree components, since the non-root components resp. root component defines invariants with values in $H(BG) \otimes \Lambda_X^G$ resp. Λ_X^G.

These invariants (with or without cohomological twisting) satisfy axioms for morphisms of rooted trees:

Proposition 7.17. (a) (Cutting edges) If Γ' is obtained from Γ by cutting an edge then

$$\langle \alpha; \beta \rangle_{\Gamma,d} = \sum_{i=1}^{\dim(H(X))} \langle \alpha, \delta_i, \delta^i; \overline{\mathcal{M}}(\Upsilon)^* \beta \rangle_{\Gamma',d}$$

where δ_i, δ^i are dual bases for $H_G(X)$ over $H(BG)$;

(b) (Collapsing edges) If $\Upsilon : \Gamma \to \Gamma'$ is a morphism collapsing an edge then

$$\langle \alpha; \beta \cup \gamma \rangle_{\Gamma',d'} = \sum_{d' \mapsto d} \langle \alpha; \overline{\mathcal{M}}(\Upsilon)^* \beta \rangle_{\Gamma,d}$$

where γ is the dual class for $\overline{\mathcal{M}}(\Upsilon) : \overline{\mathcal{M}}_{n,\Gamma}(C) \to \overline{\mathcal{M}}_{n,\Gamma'}(C)$.

(c) (Forgetting tails) If $\Upsilon : \Gamma \to \Gamma'$ is a morphism forgetting a tail then for $\alpha' \in H^2_G(X), \alpha \in H_G(X)^n$,

$$\langle \alpha, \alpha'; \overline{\mathcal{M}}(\Upsilon)^* \beta \rangle_{\Gamma,d} = (d, \alpha') \langle \alpha; \beta \rangle_{\Gamma',d'}$$

Proof. By Proposition 7.14 and the same arguments in the Gromov-Witten case, see Behrend-Manin [5, Theorem 9.2].

Definition 7.18. Denote by Λ_X^G the *equivariant Novikov field* for X, the set of all maps $a : H_2^G(X) := H_2^G(X, \mathbb{Q}) \to \mathbb{Q}$ such that for every constant c, the set of classes

$$\left\{ d \in H_2^G(X), \ \langle [\omega_{X,G}], d \rangle \leq c \right\}$$

on which a is non-vanishing is finite. Addition is defined in the usual way and multiplication is convolution.

From now on, we denote by $QH_G(X) = H_G(X) \otimes \Lambda_X^G$ the quantum cohomology over the Novikov field Λ_X^G. Summing over equivariant homology classes gives a map

$$\tau_{X,n}^G : QH_G(X)^n \times H(\overline{\mathcal{M}}_n(C)) \to \Lambda_X^G, \quad (\alpha,\beta) \mapsto \sum_{d \in H_2^G(X,\mathbb{Z})} q^d \langle \alpha, \beta \rangle_d .$$

By Proposition 7.17,

Theorem 7.19. *If $\overline{\mathcal{M}}_n^G(C,X)$ is a Deligne-Mumford stack (that is, if stable=semistable) then the maps $(\tau_{X,n}^G)_{n\geq 0}$ form a trace on the CohFT algebra $QH_G(X)$.*

Twisted gauged Gromov-Witten invariants are defined as follows.

Definition 7.20. (Twisting class and twisted gauged invariants) Let $E \to X$ be a G-equivariant complex vector bundle, inducing a vector bundle on X/G. Pullback under the evaluation map $e : \overline{\mathcal{C}}_n^G(C,X) \to X/G$ gives rise to a vector bundle $e^*E \to \overline{\mathcal{C}}_n^G(C,X)$, which we can push down to an *index* complex

$$\operatorname{Ind}(E) := Rp_* e^* E$$

in the derived category of bounded complexes of coherent sheaves on $\overline{\mathcal{M}}_n^G(C,X)$. As in [9, Appendix], $\operatorname{Ind}(E)$ admits a resolution by vector bundles and we may define the $\mathbb{C}^\times$-equivariant Euler class

$$\epsilon(E) := \operatorname{Eul}_{\mathbb{C}^\times}(\operatorname{Ind}(E)) \in H_G(\overline{\mathcal{M}}_n^G(C,X)) \otimes \mathbb{Q}[\varphi,\varphi^{-1}]$$

where φ is the parameter for the action of $\mathbb{C}^\times$ by scalar multiplication in the fibers. The *twisted gauged Gromov-Witten invariants* associated to $E \to X, C$ are the maps

$$\begin{aligned} &H_G(X)^n \times H(\overline{M}_n(C)) \to \mathbb{Q}[\varphi,\varphi^{-1}], \\ &\qquad (\alpha,\beta) \mapsto \langle \alpha,\beta\rangle_{\Gamma,E,d} = \int_{[\overline{\mathcal{M}}_n^G(C,X,d)]} \operatorname{ev}^*\alpha \cup f^*\beta \cup \epsilon(E). \end{aligned} \tag{43}$$

Example 7.21. Recall that in the case that X is a vector space, G is a torus, $\overline{\mathcal{M}}^G(C,X)$ is the toric variety $X(d)$, see [27, (31)]. Integrals over toric varieties may be computed via residues, as in for example Szenes-Vergne [25]. Some sample computations are computed in Morrison-Plesser [23, Section 4], who made contact with the Gelfand-Kapranov-Zelevinsky theory of hypergeometric functions. We return to this case in Example 9.15.

8. Quantum Kirwan morphism and the adiabatic limit theorem

In this section we explain how to "quantize" the classical Kirwan morphism in order to obtain a morphism of CohFT algebras to the quantum cohomology of the quotient. The existence of such a morphism was noted under "sufficiently positive" conditions on the first Chern class in Gaio-Salamon [13]. The quantum Kirwan morphism relates small quantum cohomologies under suitable positivity assumptions. We also give a partial computation of the quantum Kirwan map in the toric case.

8.1. Affine gauged Gromov-Witten invariants

We first define gauged affine Gromov-Witten invariants by integrating pull-back and universal classes over the moduli stack of affine gauged maps. As in Behrend [3], we separate the splitting axiom into a *cutting edges* and *collapsing edges* axiom. The main difference with Behrend [3] is that one cannot cut an arbitrary edge and still have a colored tree if the edge separates some of the colored vertices from the root edge and not others, so there is a new *cutting edges with relations* axiom which cuts several edges at once. There is also a difference in the *collapsing edges* axiom: because the source moduli space $\overline{\mathcal{M}}_{n,1}(\mathbb{A})$ is not smooth, not every boundary divisor is Cartier and so there is a new *collapsing edges with relations* axiom which holds for combinations of boundary divisors that are Cartier. Let X be a smooth polarized quasiprojective variety such that the git quotient $X/\!\!/G$ is a (necessarily smooth) Deligne-Mumford stack.

Definition 8.1. (Virtual fundamental classes for affine gauged maps)

(a) (Virtual fundamental class for a colored tree with a single vertex) The construction in [4, Chapter 7] gives a virtual fundamental class $[\overline{\mathcal{M}}^G_{n,1}(\mathbb{A}, X, d)] \in A(\overline{\mathcal{M}}^G_{n,1}(\mathbb{A}, X, d))$.

(b) (Virtual fundamental class for a connected colored tree) More generally, for any combinatorial type of colored tree Γ we have a virtual fundamental class
$$[\overline{\mathcal{M}}^G_{n,1,\Gamma}(\mathbb{A}, X, d)] \in A(\overline{\mathcal{M}}^G_{n,1,\Gamma}(\mathbb{A}, X, d)).$$

(c) (Virtual fundamental class for a disconnected colored forest) Suppose that $\Gamma = \Gamma_0 \cup \ldots \cup \Gamma_l$ with Γ_0 a possibly disconnected union of components each with at least one colored vertex, and $\Gamma_1, \ldots, \Gamma_l$ connected components with $\mathrm{Vert}(\Gamma_j) \subset \mathrm{Vert}^0(\Gamma)$. Suppose that for each component Γ_j we are given a non-root edge $e(j)$ of Γ_0. We denote by
$$\overline{\mathcal{M}}^G_{n,\Gamma}(\mathbb{A}, X, d) := \cup_{d=d_0+\ldots+d_l} \left(\overline{\mathcal{M}}^{G,\mathrm{fr}}_{n_0,\Gamma_0}(\mathbb{A}, X, d_0) \times_{G^{n_0}} \prod_{j=1}^{l} \overline{\mathcal{M}}_{0,n,\Gamma_j}(X, d_j) \right)$$
the fiber product determined by the mapping e above. We have virtual fundamental classes
$$[\overline{\mathcal{M}}^G_{n_0,\Gamma_0}(\mathbb{A}, X, d_0)] \in A(\overline{\mathcal{M}}^G_{n_0,\Gamma_0}(\mathbb{A}, X, d)) \cong A_{G^{n_0}}(\overline{\mathcal{M}}^{G,\mathrm{fr}}_{n_0,\Gamma_0}(\mathbb{A}, X, d))$$
given the product of virtual fundamental classes of the components and equivariant virtual fundamental classes
$$[\overline{\mathcal{M}}_{0,n_j,\Gamma_j}(X, d_j)] \in A_G(\overline{\mathcal{M}}_{0,n_j,\Gamma_j}(X, d_j)).$$
These give a virtual fundamental class
$$[\overline{\mathcal{M}}^G_{n,\Gamma}(\mathbb{A}, X, d)] = \cup_{d=d_0+\ldots+d_l} [\overline{\mathcal{M}}^{G,\mathrm{fr}}_{n_0,\Gamma_0}(\mathbb{A}, X, d_0)] \times \prod_{j=1}^{l} [\overline{\mathcal{M}}_{0,n,\Gamma_j}(X, d_j)]$$

in

$$A_{G^{n_0}}(\overline{\mathcal{M}}^{G,\mathrm{fr}}_{n_0,\Gamma_0}(\mathbb{A},X,d) \times \prod_j \overline{\mathcal{M}}_{0,n_j}(X,d_j)) \cong A(\overline{\mathcal{M}}^G_{n,\Gamma}(\mathbb{A},X,d)).$$

Note that it is not possible to define the virtual fundamental classes without the additional labelling, since the virtual fundamental classes for the components Γ_j are equivariant while that for Γ_0 is not.

These classes satisfy the following properties:

Proposition 8.2. (a) (Collapsing edges) If Γ' is obtained from Γ by collapsing an edge and $\Upsilon : \Gamma \to \Gamma'$ is the corresponding morphism of colored trees then

$$\overline{\mathfrak{M}}(\Upsilon)^![\overline{\mathcal{M}}^G_{n,1,\Gamma'}(\mathbb{A},X,d')] = \overline{\mathcal{F}}(\Upsilon,X)_* \sum_{d\mapsto d'} [\overline{\mathcal{M}}^G_{n,1,\Gamma}(\mathbb{A},X,d)]$$

where

$$\overline{\mathcal{F}}(\Upsilon,X) : \overline{\mathcal{M}}^G_{n,1,\Gamma}(\mathbb{A},X,d) \to \overline{\mathcal{M}}_{n,1,s(\Gamma)}(\mathbb{A}) \times_{\overline{\mathcal{M}}_{n,1,s(\Gamma')}(\mathbb{A})} \overline{\mathcal{M}}^G_{n,1,\Gamma'}(\mathbb{A},X,d')$$

is the identification with the fiber product;

(b) (Collapsing edges with relations) If $\Gamma_0,\ldots,\Gamma_r$ are obtained from Γ by collapsing edges with relations and $\Upsilon : \Gamma_0 \sqcup \ldots \sqcup \Gamma_r \to \Gamma$ is the corresponding morphism of colored trees so that $\cup_{i=1}^r \overline{\mathfrak{M}}^{\mathrm{tw}}_{n,1,\Gamma_i}(\mathbb{A}) \to \overline{\mathfrak{M}}^{\mathrm{tw}}_{n,1,\Gamma'}(\mathbb{A})$ is a regular local immersion (that is, is a Cartier divisor) then

$$\overline{\mathfrak{M}}(\Upsilon)^![\overline{\mathcal{M}}^G_{n,1,\Gamma}(\mathbb{A},X,d)] = \overline{\mathcal{F}}(\Upsilon,X)_* \sum_{d\mapsto d',i=1,\ldots,r} [\overline{\mathcal{M}}^G_{n,1,\Gamma}(\mathbb{A},X,d)].$$

(c) (Cutting edges or edges with relations) If $\Upsilon : \Gamma \to \Gamma'$ is a morphism of trees of type cutting an edge or edges with relations then

$$\mathcal{G}(\Upsilon,X)_*[\overline{\mathcal{M}}^G_{n,\Gamma'}(\mathbb{A},X,d')] = \Delta^![\overline{\mathcal{M}}^G_{n,\Gamma}(\mathbb{A},X,d)]$$

where $\Delta : \overline{I}^m_{X/G} \to \overline{I}^{2m}_{X/G}$ is the diagonal and $\mathcal{G}(\Upsilon,X)$ is the gluing morphism in [27, (32)].

(d) (Forgetting tails) If $\Upsilon : \Gamma \to \Gamma'$ is a morphism forgetting a tail then

$$\overline{\mathcal{M}}(\Upsilon)^![\overline{\mathcal{M}}^G_{n,\Gamma'}(\mathbb{A},X,d)] = [\overline{\mathcal{M}}^G_{n+1,\Gamma}(\mathbb{A},X,d)].$$

Proof. Cutting an edge is similar to the case of gauged maps from projective curves covered in Proposition 7.14 and omitted. For collapsing an edge, Let $\Upsilon : \Gamma' \to \Gamma$ be a morphism of edge-rooted colored trees given by *collapsing an edge* connecting vertices of the same color. Associated to Υ are morphisms of Artin resp. Deligne-Mumford stacks

$$\overline{\mathfrak{M}}(\Upsilon) : \overline{\mathfrak{M}}^{\mathrm{tw}}_{n,1,\Gamma'}(\mathbb{A}) \to \overline{\mathfrak{M}}^{\mathrm{tw}}_{n,1,\Gamma}(\mathbb{A}), \quad \overline{\mathcal{M}}(\Upsilon) : \overline{\mathcal{M}}_{n,1,\Gamma'}(\mathbb{A}) \to \overline{\mathcal{M}}_{n,1,\Gamma}(\mathbb{A}).$$

As in Behrend [3], the relative obstruction theories for $\overline{\mathcal{M}}^G_{n,1,\Gamma}(\mathbb{A},X), \overline{\mathcal{M}}^G_{n,1,\Gamma'}(\mathbb{A},X)$ are related by pull-back:

$$\overline{\mathfrak{M}}(\Upsilon)^![\overline{\mathcal{M}}^G_{n,1,\Gamma'}(\mathbb{A},X,d')] = \overline{\mathcal{F}}(\Upsilon,X)_* \sum_{d\mapsto d'} [\overline{\mathcal{M}}^G_{n,1,\Gamma}(\mathbb{A},X,d)].$$

For collapsing several edges, let $\Gamma_0,\ldots,\Gamma_r$ be colored trees obtained from Γ by collapsing edges by morphisms $\Upsilon_1,\ldots,\Upsilon_r$ so that $\cup_{i=1}^r \overline{\mathfrak{M}}^{\mathrm{tw}}_{n,1,\Gamma_i}(\mathbb{A}) \to \overline{\mathfrak{M}}^{\mathrm{tw}}_{n,1,\Gamma'}(\mathbb{A})$ is a regular local immersion (that is, is a Cartier divisor). Then

$$\overline{\mathfrak{M}}(\Upsilon)^![\overline{\mathcal{M}}^G_{n,1,\Gamma'}(\mathbb{A},X,d')] = \overline{\mathcal{F}}(\Upsilon,X)_* \sum_{d\mapsto d'} [\overline{\mathcal{M}}^G_{n,1,\Gamma}(\mathbb{A},X,d)]$$

as in Example 7.2. The last item is left to the reader.

To define invariants, note that evaluation at the marked points defines a map

$$\mathrm{ev}\times\mathrm{ev}_\infty : \overline{\mathcal{M}}^G_{n,1}(\mathbb{A},X) \to (X/G)^n \times \overline{I}_{X/\!\!/G}.$$

By integration over the moduli stacks of affine gauged maps we obtain affine gauged Gromov-Witten invariants defining the quantum Kirwan morphism of CohFT algebras from $QH_G(X)$ to $QH(X/\!\!/G)$.

Definition 8.3. (Affine gauged Gromov-Witten invariants)

(a) (Invariants for a connected colored tree) The *affine gauged Gromov-Witten invariants* for a connected colored tree Γ are the maps

$$\begin{aligned} &H_G(X)^n \times H(X/\!\!/G) \times H(\overline{M}_{n,1}(\mathbb{A})) \to \mathbb{Q}, \\ &(\alpha,\alpha_\infty,\beta) \mapsto \langle \alpha;\alpha_\infty;\beta\rangle_{\Gamma,d} := \int_{[\overline{\mathcal{M}}^G_{n,1,\Gamma}(\mathbb{A},X,d)]} \mathrm{ev}^*\alpha \cup f^*\beta \cup \mathrm{ev}_\infty^*\alpha_\infty. \end{aligned} \tag{44}$$

(b) (Invariants for a colored forest) Invariants for possibly disconnected $H_2^G(X)$-labelled colored forests are defined as follows, given the *additional datum* of a map from the non-root components to the root edges: Suppose that $\Gamma = \Gamma_0 \cup \Gamma_1 \cup \cdots \cup \Gamma_l$ is a disconnected colored $H_2^G(X)$-labelled tree such that each component of Γ_0 has at least one vertex in $\mathrm{Vert}^0(\Gamma)$ or $\mathrm{Vert}^1(\Gamma)$, for $j > 1$ the tree Γ_j has semi-infinite edges labelled I_j, and for each $j = 1,\ldots,l$ is given a semi-infinite edge $e(j)$ of Γ_0. Let $\mathrm{Edge}(\Gamma) = \mathrm{Edge}^0(\Gamma) \cup \mathrm{Edge}^\infty(\Gamma)$ denote the partition corresponding to nodes mapping to X/G or $I_{X/\!\!/G}$, that is, edges connecting $\mathrm{Vert}^0(\Gamma)$ with $\mathrm{Vert}^0(\Gamma)\cup\mathrm{Vert}^1(\Gamma)$ or edges connecting $\mathrm{Vert}^1(\Gamma)\cup\mathrm{Vert}^\infty(\Gamma)$ with $\mathrm{Vert}^\infty(\Gamma)$ as in [26, Remark 2.25]. We suppose that we have a labelling of the semi-infinite edges by classes $\alpha_e \in H_G(X), e \in \mathrm{Edge}^0(\Gamma)$ and $\alpha_e \in H(I_{X/\!\!/G}), e \in \mathrm{Edge}^\infty(\Gamma)$. We define gauged Gromov-Witten invariants for Γ by fiber integration over the map

$\overline{\mathcal{M}}^G_{n,\Gamma}(\mathbb{A}, X) \to \overline{\mathcal{M}}^G_{n_0,\Gamma_0}(\mathbb{A}, X)$ whose fibers are moduli stacks of stable maps of type Γ_j for $j > 0$: set

$$\langle \alpha; \beta \rangle_{\Gamma,d} := \langle (\alpha'_j)_{j \in I_0}; \beta_0 \rangle_{\Gamma_0, d_0}$$

where for each semi-infinite edge i of Γ_0 connecting to a vertex in $\operatorname{Vert}^0(\Gamma_0)$ or $\operatorname{Vert}^1(\Gamma_0)$,

$$\alpha'_i = \left(\prod_{i=e(j)} \langle (\alpha_e)_{e \in I_j}, \beta_j \rangle_{\Gamma_j, d_j} \right) \alpha_i,$$

using the $H(BG)$-module structure on $H_G(X)$, and β_j is the Künneth component of β for the component Γ_j.

(c) (Twisted affine Gromov-Witten invariants) Twisted invariants $\langle \alpha; \beta \rangle_{\Gamma,d,E}$ associated to G-equivariant vector bundles $E \to X$ are defined by inserting Euler classes of indices $\epsilon(E)$ into the integrands.

The properties of the affine Gromov-Witten invariants are similar to those for the projective case:

Proposition 8.4. (a) (Collapsing an edge) If Γ' is obtained from Γ by collapsing an edge then for any labelling d' of Γ', and $\overline{\mathcal{M}}(\Upsilon) : \overline{\mathcal{M}}^{\mathrm{tw}}_{n,1,\Gamma}(\mathbb{A}) \to \overline{\mathcal{M}}^{\mathrm{tw}}_{n,1,\Gamma'}(\mathbb{A})$ has dual class γ then

$$\langle \alpha; \beta \cup \gamma \rangle_{\Gamma',d',E} = \sum_{d \mapsto d'} \langle \alpha; \overline{\mathcal{M}}(\Upsilon)^* \beta \rangle_{\Gamma,d,E}$$

(b) (Collapsing edges with relations) More generally, if $\Gamma_0, \ldots, \Gamma_r$ are each obtained from Γ by collapsing edges with relations and $\Upsilon : \Gamma_0 \sqcup \ldots \sqcup \Gamma_r \to \Gamma$ is the corresponding morphism of colored trees so that

$$\cup_{i=1}^r \overline{\mathcal{M}}^{\mathrm{tw}}_{n,1,\Gamma_i}(\mathbb{A}) \to \overline{\mathcal{M}}^{\mathrm{tw}}_{n,1,\Gamma'}(\mathbb{A}) \tag{45}$$

is a regular local immersion (that is, is a Cartier divisor) with dual class γ then

$$\langle \alpha; \beta \cup \gamma \rangle_{\Gamma',d',E} = \sum_{d \mapsto d', i=1,\ldots,r} \langle \alpha; \iota^*_{\Gamma_i,\Gamma} \beta \rangle_{\Gamma_i,d,E}$$

where $\iota^*_{\Gamma_i,\Gamma}$ are the components of (45).

(c) (Cutting an edge) If Γ' is obtained from Γ by cutting an edge or edges with relations then

$$\langle \alpha; \beta \rangle_{\Gamma,d,E} = \sum_k \langle \alpha, \delta_k \cup \operatorname{Eul}_{G \times \mathbb{C}^\times}(E), \delta^k; \overline{\mathcal{M}}(\Upsilon)^* \beta \rangle_{\Gamma',d,E}$$

where $(\delta_k), (\delta^k), k = 1, \ldots, \dim(H(I^m_{X/\!\!/G}))$ are dual bases for $H(I^m_{X/\!\!/G})$ resp. $H_G(X)$ if the cut edges lie in $\operatorname{Edge}^\infty(\Gamma)$ resp. $\operatorname{Edge}^0(\Gamma)$.

(d) (Forgetting tails) If $\Upsilon : \Gamma \to \Gamma'$ is a morphism forgetting a tail then

$$\langle \alpha, \alpha'; \overline{\mathcal{M}}(\Upsilon)^* \beta \rangle_{\Gamma,d,E} = (d, \alpha') \langle \alpha; \beta \rangle_{\Gamma',d,E}$$

where (d, α') is the pairing between $d \in H_2^G(X, \mathbb{Q})$ and $\alpha' \in H^2_G(X, \mathbb{Q})$.

Proof. By Proposition 8.2; the cutting edges case follows from an integration over the fiber $\overline{\mathcal{M}}^G_{n,\Gamma}(\mathbb{A},X) \to \overline{\mathcal{M}}^G_{n,\Gamma_0}(\mathbb{A},X)$ with fibers $\prod_{j>0} \overline{\mathcal{M}}_{0,n,\Gamma_j}(X)$. The collapsing edges and forgetting tails properties are left to the reader.

8.2. Quantum Kirwan morphism

In this section we use the affine gauged Gromov-Witten invariants to define the quantum Kirwan morphism from $QH_G(X)$ to $QH(X/\!\!/G)$. For simplicity, we restrict to the case E trivial, that is, the untwisted case. We remind that here $QH(X/\!\!/G)$ is defined over the equivariant Novikov ring, that is, $QH(X/\!\!/G) = H(X/\!\!/G) \otimes \Lambda^G_X$.

Definition 8.5. (Quantum Kirwan morphism) Suppose that X is a smooth polarized projective G-variety or a vector space with a linear action of G and proper moment map such that the git quotient $X/\!\!/G$ is a Deligne-Mumford stack, so that the moduli stacks $\overline{\mathcal{M}}^G_n(\mathbb{A},X)$ are proper Deligne-Mumford stacks. The *quantum Kirwan morphism* is the collection of maps

$$\kappa^{G,n}_X : QH_G(X)^n \times H(\overline{\mathcal{M}}_{n,1}(\mathbb{A})) \to QH(X/\!\!/G), n \geq 0$$

given by pull-back to $\overline{\mathcal{M}}^G_{n,1}(\mathbb{A},X)$ and push-forward to $X/\!\!/G$. That is, for $\alpha \in H_G(X)^n, \alpha_\infty \in H_G(\overline{I}_{X/\!\!/G}), \beta \in H^*(\overline{\mathcal{M}}_{n,1}(\mathbb{A}))$ let

$$(\kappa^{G,n}_X(\alpha,\beta),\alpha_\infty) = \sum_{d \in H^G_2(X,\mathbb{Q})} q^d \langle \alpha;\alpha_\infty;\beta\rangle_d$$

using Poincaré duality; the pairing on the left is given by cup product and integration over $\overline{I}_{X/\!\!/G}$. Define $\kappa^0_G \in H^*(\overline{\mathcal{M}}_{n,1}(\mathbb{A}))$ similarly, by integrating the unit.

Theorem 8.6. *The collection $\kappa^G_X = (\kappa^{G,n}_X)_{n\geq 0}$ satisfies the axioms of a morphism of CohFT algebras.*

Proof. First note that the splitting axiom is well-defined: Note that $\kappa^{G,0}_X$ has contributions with coefficients q^d with $(d,[\omega_{X,G}]) > 0$, since trivial maps with no finite markings are unstable. It follows that the sum on the right-hand-side of [26, (11)] is finite modulo terms with coefficient q^a and higher, for any $a \in \mathbb{R}$. The equation [26, (11)] now follows from parts (a)-(c) of Proposition 8.4.

Remark 8.7. (a) (Equivariant quantum Kirwan morphism) If the action of G extends to an action of a group $\tilde{G}$ containing G as a normal subgroup, there is a map

$$QH_{\tilde{G}}(X)^n \times H(\overline{\mathcal{M}}^G_{n,1}(\mathbb{A},X)) \to QH_{\tilde{G}/G}(X/\!\!/G)$$

defined by the same formula. After extending the coefficient ring of $QH_{\tilde{G}/G}(X/\!\!/G)$ from $\Lambda_{X/\!\!/G}$ to Λ^G_X we have a morphism of CohFT algebras

$$(\kappa^{\tilde{G},G,n}_X)_{n\geq 0} : QH_{\tilde{G}}(X) \to QH_{\tilde{G}/G}(X/\!\!/G). \tag{46}$$

(b) (Flatness of the quantum Kirwan morphism in the positive case) Suppose that $c_1^G(X)$ is semipositive in the sense that $(c_1^G(X), d) \geq 0$ for the homology class d of any gauged affine map. In this case, the "quantum corrections" in any $\kappa_X^{G,n}(\alpha_1, \ldots, \alpha_n)$ are of degree at most $\deg(\alpha_1) + \ldots + \deg(\alpha_n) + 2 - 2n$. In particular, the element $\kappa_X^{G,0}(1)$ can be written as the sum of elements of degree 0 and 2 with respect to the grading induced by the grading on $H(I_{X/\!\!/G})$. If $c_1^G(X)$ is positive, then the dimension count shows that $\kappa_X^{G,0}$ is an element of degree 0 in $H(I_{X/\!\!/G})$, times an element of Λ_X^G, that is, a multiple of the point class. If $(c_1^G(X), d)$ is at least two whenever $(d, [\omega_{X,G}]) > 0$ then $\kappa_X^{G,0}$ vanishes.

We end this section with a partial computation of the quantum Kirwan morphism in the toric case. Suppose that $X \cong \mathbb{C}^k$ is a vector space equipped with a linear action of a torus G with Lie algebra $\mathfrak{g}$ and weights $\mu_1, \ldots, \mu_k \in \mathfrak{g}^\vee$ in the sense that G acts on the j-th factor by the character $\exp(\mu_j)$. We denote by $\tilde{G} = (\mathbb{C}^\times)^k$ the torus acting on X by scalar multiplication on each factor. Let $v_1, \ldots, v_k$ be the standard coordinates on the Lie algebra $\tilde{\mathfrak{g}}$ so that

$$QH_{\tilde{G}}(X) = \mathbb{Q}[v_1, \ldots, v_k] \otimes \Lambda_X^{\tilde{G}}.$$

However, for the purposes of this section it suffices to tensor with the G-equivariant Novikov field Λ_X^G. The inclusion $G \to \tilde{G}$ induces a map $r : QH_{\tilde{G}}(X) \to QH_G(X)$, which after identification of the equivariant cohomology with symmetric functions $QH_G(X) \cong \mathrm{Sym}(\mathfrak{g}^\vee) \otimes \Lambda_X^G$ is the restriction map induced by the inclusion $\mathfrak{g} \to \tilde{\mathfrak{g}}$. Let $l(v_j), j = 1, \ldots, k$ denote the divisor classes in $H(I_{X/\!\!/G})$ defined by v_j, see [27, Example 4.8].

Lemma 8.8. *Let G be a torus acting on a vector space X as above. For any $d \in H_2^G(X, \mathbb{Z})$ such that the polarization vector ν lies in $\mathrm{span}\{-\mu_j, \mu_j(d) \geq 0\}$ (see [27, (30)]) we have*

$$\kappa_X^{G,1}\left(\prod_{\mu_j(d)\geq 0} r(v_j)^{\mu_j(d)}\right) = q^d \prod_{\mu_j(d)\leq 0} l(v_j)^{-\mu_j(d)} + \text{ higher order}$$

where higher order means terms with coefficient $q^{d'}$ with $(d', [\omega_{X,G}]) > (d, [\omega_{X,G}])$.

Proof. We show

$$\int_{[\overline{\mathcal{M}}_{1,1}^G(\mathbb{A}, X, d)]} \prod_{\mu_j(d)\geq 0} r(v_j)^{\mu_j(d)} \cup \mathrm{ev}_\infty^* \alpha = \int_{[X/\!\!/G]} \prod_{\mu_j(d)\leq 0} l(v_j)^{-\mu_j(d)} \cup \alpha. \tag{47}$$

We compute the left-hand-side by interpreting the first factor as an Euler class

$$\prod_{\mu_j(d)\geq 0} r(v_j)^{\mu_j(d)} = \mathrm{ev}^* \,\mathrm{Eul}\left(\bigoplus_{\mu_j(d)\geq 0} \mathbb{C}^{\mu_j(d)}\right)$$

and counting the zeros of a section. Identifying framed maps with a single marking with maps $u : \mathbb{A} \to X$, consider the map

$$\sigma : \overline{\mathcal{M}}^G_{1,1}(\mathbb{A}, X, d) \to \mathrm{ev}_1^* \prod_{\mu_j(d) \geq 0} \mathbb{C}^{\mu_j(d)}_{\mu_j}, \quad u \mapsto (u_i^{(j)}(0))_{i=1,j=1}^{k,\mu_j(d)-1}$$

whose components are the derivatives of the map at the finite marking. On the stratum $\mathcal{M}^G_{1,1}(\mathbb{A}, X, d)$ of curves with irreducible domain, the intersection $\sigma^{-1}(0)$ maps injectively into $X /\!\!/ G \subset \overline{I}_{X/\!\!/G}$ via ev_∞. Indeed the assumption on the span of $\mu_j, \mu_j(d) \geq 0$ implies that $\mathcal{M}^G_{1,1}(\mathbb{A}, X, d)$ is non-empty, the equation $\mathrm{ev}_\infty(u) = \mathrm{pt}_{X/\!\!/G}$ fixes the leading order terms (see Examples [27, 5.32] and 8.9) and $\sigma(u) = 0$ fixes the lower order terms in u. Since ev_∞ maps smoothly onto $\bigoplus_{\mu_j(d)\geq 0} \mathbb{C}_{\mu_j} \cap X^{\mathrm{ss}}/G$, the integral (47) is equal to

$$\int_{[X/\!\!/G]} \alpha \cup \prod_{\mu_j(d)<0} l(v_j)^{-\mu_j(d)}$$

where the virtual integration of $[u]$ is with respect to the virtual fundamental class induced from that on the moduli stack. Taking into account the *obstruction bundle*

$$R^1 p_* e^* T(X/G) = \mathrm{ev}_1^* \bigoplus_{\mu_j(d)<0} \mathbb{C}^{-\mu_j(d)-1}_{\mu_j}$$

we see that $\kappa^{X,1}_G(\prod r(v_j)^{\max(0,\mu_j(d))})$ contains a term of the form $q^d \prod l(v_j)^{\max(0,-\mu_j(d))}$ plus contributions from other strata and components of the moduli space of other homology classes.

We check next that there are no contributions from boundary strata. On the boundary with curves of reducible domain, each map u consists of a component $u_1 : C_1 \to X/G$ consisting of an affine scaled map of homology class d' with $(d', [\omega_{X,G}]) < (d, [\omega_{X,G}])$ connecting the marking z_1 to the infinite marking z_0, together with bubbles in $X/\!\!/G$ and possibly other affine scaled maps. The vanishing $\sigma(u) = 0$ implies that, in particular, the $\mu_j(d')$-th derivative of u_1 is zero if $\mu_j(d')$ is integral and less than some non-negative $\mu_j(d)$. The same conclusion holds if $\mu_j(d') < \mu_j(d)$ is negative, since in this case the j-th component of u_1 vanishes identically. On the other hand, since $(d-d', [\omega_{X,G}]) > 0$, the set of points in $X/\!\!/G$ whose j-th coordinate vanishes if $\mu_j(d-d') > 0$, is unstable, see [27, (30)]. Thus, $\sigma^{-1}(0)$ is empty on the boundary strata and the only contribution to the integral above arises from the component of maps with irreducible domain.

Example 8.9. (a) (Projective Space Quotient) If $G = \mathbb{C}^\times$ acts on $X = \mathbb{C}^k$ with all weights one, so that $X/\!\!/G = \mathbb{P}^{k-1}$, then $\mathcal{M}^G_{1,1}(\mathbb{A}, X)$ may be identified with the space of k-tuples of polynomials $(p_1(z), \ldots, p_n(z))$ with $(p_1(z), \ldots, p_k(z))$ non-zero for z generic. We obtain a section

$$\sigma : \overline{\mathcal{M}}^G_{1,1}(\mathbb{A}, X, 1) \to \mathrm{ev}_1^*(X \times X \to X)$$

by evaluating the polynomials at 0. This section has no zeroes other than at $[c_1 z, \ldots, c_k z]$ for $(c_1, \ldots, c_k) \neq 0$, which lies in the open stratum of maps with irreducible domain. In particular,

$$\int_{[\overline{\mathcal{M}}^G_{1,1}(\mathbb{A},X,1)]} \mathrm{ev}_1^*(X \times X \to X) \cup \mathrm{ev}_\infty^*([\mathrm{pt}_{X/\!\!/G}]) = 1$$

which implies that $\kappa_X^{G,1}(\xi^k) = q$ where ξ is the generator of $QH_G(X) \cong \Lambda_X^G[\xi]$.

(b) (Weighted Projective Line Quotient) Let $X = \mathbb{C}_2 \oplus \mathbb{C}_3$ and $G = \mathbb{C}^\times$ so that $X/\!\!/G = \mathbb{P}[2,3]$. Let θ_1 resp. θ_2 resp. θ_3 resp. θ_3^2 denote the generator of the component of $QH(X/\!\!/G) \cong H(\overline{I}_{X/\!\!/G}) \otimes \Lambda_{X/\!\!/G}$ with trivial isotropy resp. $\mathbb{Z}_2$ isotropy resp. corresponding to $\exp(\pm 2\pi i/3) \in \mathbb{Z}_3$. Let $\xi \in H_G^2(X)$ denote the integral generator. One has

$$\kappa_X^{G,1}(1) = 1, \quad \kappa_X^{G,1}(\xi) = \theta_1, \quad \kappa_X^{G,1}(\xi^2) = q^{1/3}\theta_3/6,$$
$$\kappa_X^{G,1}(\xi^3) = q^{1/2}\theta_2/18, \quad \kappa_X^{G,1}(\xi^4) = q^{2/3}\theta_3^2/36, \quad \kappa_X^{G,1}(\xi^5) = q/108.$$

In particular, we see that $\kappa_X^{G,1}$ is surjective and the kernel is $\xi^5 - q/108$, hence

$$QH(\mathbb{P}[2,3]) = \mathbb{Q}[\xi] \otimes \Lambda_X^G/(\xi^5 - q/108)$$

which is a special case of Coates-Lee-Corti-Tseng [10].

Remark 8.10. (a) (Quantum Kirwan surjectivity) We conjecture the quantum analog of Kirwan surjectivity, namely that $\kappa_X^{G,1}$ is surjective onto the orbifold quantum cohomology $QH(X/\!\!/G)$ of the quotient $X/\!\!/G$. We have worked out some special cases with Gonzalez in [16].

(b) (Quantum reduction in stages) One naturally expects a quantum analog of the reduction in stages theorem: If $G' \subset G$ is a normal subgroup then $\kappa_{X/\!\!/G'}^{G/G'} \circ \kappa_X^{G,G'} = \kappa_X^G : QH_G(X) \to QH(X/\!\!/G)$. That is, we have a commutative diagram of CohFT algebras

$$\begin{array}{ccc} QH_G(X) & \xrightarrow{\kappa_X^G} & QH(X/\!\!/G) \\ & {\scriptstyle \kappa_X^{G,G'}} \searrow \quad \nearrow {\scriptstyle \kappa_{X/\!\!/G'}^{G/G'}} & \\ & QH_{G/G'}(X/\!\!/G') & \end{array} .$$

There is a $\mathbb{C}^\times$-equivariant extension of the quantum Kirwan map

$$\kappa_X^G : QH_G(X) \to QH_{\mathbb{C}^\times}(X/\!\!/G)$$

where $QH_{\mathbb{C}^\times}(X/\!\!/G)$ denotes the completed $\mathbb{C}^\times$-equivariant cohomology, isomorphic to $QH(X/\!\!/G)[[\zeta]]$ where ζ is the equivariant parameter. The construction is by pushing-forward over $\overline{\mathcal{M}}_n^G(\mathbb{A}, X)$ $\mathbb{C}^\times$-equivariantly, as follows. Choose a base point in $\mathbb{A}$, inducing an identification $\mathbb{A} \to \mathbb{C}$ and so a $\mathbb{C}^\times$-action on $\mathbb{A}$. The action induces an action on $\mathcal{M}_n^G(\mathbb{A}, X)$, given by pre-composing each morphism with the action. Equivalently, the action is given by acting by scalar multiplication on the one-form λ, and so extends to the compactification $\overline{\mathcal{M}}_n^G(\mathbb{A}, X)$. As a result, the virtual fundamental class for $\overline{\mathcal{M}}_n^G(\mathbb{A}, X)$ has a $\mathbb{C}^\times$-equivariant extension which gives rise to a $\mathbb{C}^\times$-equivariant extension of κ_X^G.

8.3. The adiabatic limit theorem

We show the adiabatic limit [26, Theorem 1.5], using a divisor class relation relating curves with finite and infinite scaling. Note that divisor class relations in one-dimensional source moduli spaces have already been used to prove the associativity of the quantum products, as well as the homomorphism property of the quantum Kirwan morphism.

Recall from [27, Theorem 5.35] the stack $\overline{\mathcal{M}}_{n,1}^G(C,X)$ of scaled gauged maps from C to X. Under the stable=semistable assumption it has a perfect relative obstruction theorem over $\overline{\mathfrak{M}}_{n,1}^{\mathrm{tw}}(C)$, whose complex is dual to $Rp_*e^*T(X/G)$, and so a virtual fundamental class.

Definition 8.11. If every polystable gauged map is stable then the *scaled gauged Gromov-Witten invariants* for $\alpha \in H_G(X)^n, \beta \in H(\overline{M}_{n,1}(C))$ are

$$\langle \alpha, \beta \rangle_{d,1,E} = \int_{[\overline{\mathcal{M}}_{n,1}^G(C,X,d)]} \mathrm{ev}^* \alpha \cup f^* \beta \cup \epsilon(E). \tag{48}$$

Define

$$\phi^n : QH_G(X)^n \times H(\overline{M}_{n,1}(C)) \to \Lambda_X^G, \quad (\alpha, \beta) \mapsto \sum_{d \in H_2^G(X,\mathbb{Q})} q^d \langle \alpha, \beta \rangle_{d,1,E}$$

for $\alpha \in H_G(X)^n$, extended to $QH_G(X)^n$ by linearity.

More generally there are invariants for arbitrary combinatorial type that satisfy the splitting axioms as in 7.14. The adiabatic limit theorem [26, 1.5] follows from the divisor class relation

$$[\overline{\mathcal{M}}_n(C)] = [\cup_{r,[I_1,\ldots,I_r]} D_{I_1,\ldots,I_r}] \in H(\overline{\mathcal{M}}_{n,1}(C)) \tag{49}$$

from [26, Proposition 2.43]. Indeed[1] the locus of gauged maps of $D_{I_1,\ldots,I_r}$ admits as in [27, (33)] a fibration to $\overline{\mathcal{M}}_{r,1}^G(C,X)$ with fiber a subset of $\Pi_{j=1}^r \overline{\mathcal{M}}_{i_j,1}^G(\mathbb{A},X)$. To describe this fibration, let $\overline{\mathcal{M}}_{r,1}^{G,\mathrm{fr}}(C,X)$ denote the stack of gauged maps equipped with trivializations $T_{z_i}(\hat{C}) \to \mathbb{C}$ of the tangent lines at the framings. Then $D_{I_1,\ldots,I_r}$ is obtained as the $(\mathbb{C}^\times)^r$-quotient of the diagonal action

$$D_{I_1,\ldots,I_r} \cong (\overline{\mathcal{M}}_{r,1}^{G,\mathrm{fr}}(C,X) \times \Pi_{j=1}^r \overline{\mathcal{M}}_{i_j,1}^G(\mathbb{A},X))/(\mathbb{C}^\times)^r.$$

We have a commutative diagram

$$\begin{array}{ccc} (\overline{\mathcal{M}}_{r,1}^{G,\mathrm{fr}}(C,X) \times_{(I_{X /\!\!/ G})^r} \Pi_{j=1}^r \overline{\mathcal{M}}_{i_j,1}^G(\mathbb{A},X)) & \longrightarrow & \overline{\mathcal{M}}_{r,1}^{G,\mathrm{fr}}(C,X) \\ \downarrow & & \downarrow \\ D_{I_1,\ldots,I_r} & \longrightarrow & \overline{\mathcal{M}}_{r,1}^G(C,X) \end{array} .$$

[1] This explanation was added after publication.

The integral of $\mathrm{ev}^*(\alpha)$ over $D_{I_1,\ldots,I_r}$ can be computed by first pulling back to $\overline{\mathcal{M}}_{r,1}^{G,\mathrm{fr}}(C,X) \times_{(I_{X/\!\!/G})^r} \Pi_{j=1}^r \overline{\mathcal{M}}_{i_j,1}^G(\mathbb{A},X)$, pushing forward to $\overline{\mathcal{M}}_{r,1}^{G,\mathrm{fr}}(C,X)$ and taking the quotient. The equivariant push-forward produces the cohomology class

$$\kappa_X^{G,i_1}(\alpha_{I_1}) \otimes \ldots \otimes \kappa_X^{G,i_r}(\alpha_{I_r})$$

where $\alpha_{I_k} := \otimes_{i \in I_k} \alpha_i$. Taking the quotient identifies the equivariant parameter ζ with the Chern class of the tangent line, that is $\zeta = -\psi$. Thus the contribution from $D_{I_1,\ldots,I_r}$ is

$$\tau_{X/\!\!/G}^r(\kappa_X^{G,i_1}(\alpha_{I_1})|_{\zeta=-\psi}, \ldots, \kappa_X^{G,i_r}(\alpha_{I_r})|_{\zeta=-\psi}).$$

The divisor class formula gives the relation

$$\tau_X^{G,n}(\alpha_1,\ldots,\alpha_n) = \sum_{I_1,\ldots,I_r} \tau_{X/\!\!/G}^r(\kappa_X^{G,i_1}(\alpha_{I_1})|_{\zeta=-\psi}, \ldots, \kappa_X^{G,i_r}(\alpha_{I_r})|_{\zeta=-\psi})$$

which is the precise form of the adiabatic limit theorem. More generally, the divisor class relations from $\overline{M}_{n,1}(C)$ pull back to relations in $\overline{\mathcal{M}}_{n,1}^G(C,X)$ giving a 2-morphism between $\tau_{X/\!\!/G} \circ \kappa_X^G$ and τ_X^G.

9. Localized graph potentials

In this section we make contact with the hypergeometric functions appearing in the work of Givental [14], Lian-Liu-Yau [21], Iritani [20] and others. These results compute a fundamental solution of the quantum differential equation of the quotient by studying the contributions to the localization formula for the circle action on the moduli spaces of gauged maps on the projective line. Note that in contrast to [14], [21] etc., the target can be an arbitrary projective (or in some cases, quasiprojective) G-variety. The virtual localization formula expresses the result as a sum over fixed point contributions, and comparing the contributions to the adiabatic limit [26, Theorem 1.5] one obtains a stronger result which is closely related to the "mirror theorems" of [14], [21], [20], [11], [7].

9.1. Liouville insertions

First we introduce a "Liouville class" in the definition of the graph potential. This is mostly for historical reasons, to compare with the results of Givental [14]. We first consider the case of ordinary Gromov-Witten theory with target X. Denote the universal curve and evaluation map

$$\begin{array}{ccc} \overline{\mathcal{C}}_n(C,X) & \xrightarrow{e \times e_C} & X \times C \\ \downarrow p & & \\ \overline{\mathcal{M}}_n(C,X) & & \end{array} .$$

Let $[\omega_C] \in H^2(C)$ denote a generator.

Definition 9.1. (Liouville class and invariants with Liouville insertions) Let $\gamma \in H^2(X)$. The *Liouville class* associated to γ is

$$\lambda(\gamma) := \exp(p_*(e^*\gamma \cup e_C^*[\omega_C])) \in H(\overline{M}_n(C,X)).$$

The graph invariants with Liouville insertions are maps

$$H(X)^n \times H(\overline{M}_n(C)) \otimes H^2(X) \to \mathbb{Q}[\varphi, \varphi^{-1}],$$
$$\langle \alpha, \beta, \gamma \rangle_{E,d} = \int_{[\overline{\mathcal{M}}_n(C,X,d)]} \operatorname{ev}^* \alpha \cup f^*\beta \cup \lambda(\gamma) \cup \epsilon(E) \quad (50)$$

where φ is the equivariant parameter for scalar multiplication.

Similarly for gauged Gromov-Witten invariants, any class $\gamma \in H^2_G(X)$ gives rise to a *gauged Liouville class*

$$\lambda(\gamma) = \exp(p_*(e^*\gamma \cup e_C^*[\omega_C])) \in H(\overline{M}_n^G(C,X)).$$

Define invariants

$$H(X)^n \times H(\overline{M}_n(C)) \otimes H^2(X) \to \mathbb{Q}[\varphi, \varphi^{-1}],$$
$$\langle \alpha, \beta, \gamma \rangle_{E,d} = \int_{[\overline{\mathcal{M}}_n^G(C,X,d)]} \operatorname{ev}^* \alpha \cup f^*\beta \cup \lambda(\gamma) \cup \epsilon(E). \quad (51)$$

9.2. Localized equivariant graph potentials

In this section we discuss the extraction of a fundamental solution to the quantum differential equation from the graph potential, following e.g. Givental [14]. Let X be a smooth projective variety, or more generally, a smooth proper Deligne-Mumford stack with projective coarse moduli space. Let $C = \mathbb{P}$ be equipped with the standard $\mathbb{C}^\times$ action with fixed points $0, \infty \in \mathbb{P}$. Denote by ζ the equivariant parameter corresponding to the $\mathbb{C}^\times$-action. The graph potential τ_X has a natural $\mathbb{C}^\times$-equivariant generalization

$$\tau_X^{\mathbb{C}^\times} : QH(X) \times H_{\mathbb{C}^\times}(\overline{M}_n(\mathbb{P})) \times H^2(X) \to \Lambda_X[[\zeta]].$$

For simplicity we restrict to the untwisted case, that is, E trivial. The class $[\omega_{\mathbb{P}}] \in H^2(\mathbb{P})$ with integral one has a unique equivariant extension $[\omega_{\mathbb{P},\mathbb{C}^\times}] \in H^2_{\mathbb{C}^\times}(\mathbb{P})$ taking values 0 resp. ζ in $H_{\mathbb{C}^\times}(\mathrm{pt}) \cong \mathbb{Q}[\zeta]$ after restriction to 0 resp. ∞ in $\mathbb{P}$. The following is well-known, see for example Givental [14].

Proposition 9.2. (Fixed points for the $\mathbb{C}^\times$-action on graph spaces) The induced action of $\mathbb{C}^\times$ on $\overline{\mathcal{M}}_n(\mathbb{P}, X, d)$ has fixed points given by configurations $[u]$ consisting of a principal component $C_0 \cong \mathbb{P}$ on which the map u is constant, an $(n_- + 1)$-marked stable map $u_- : C_- \to X$ of degree d_- attached to $0 \in \mathbb{P}$, and an $(n_+ + 1)$-marked stable map $u_+ : C_+ \to X$ of degree d_+ attached to $\infty \in \mathbb{P}$ with $d_- + d_+ = d$ and $n_- + n_+ = n$.

We denote by $F_n(d_-, d_+)$ the locus of the fixed point set with stable maps of classes d_-, d_+ respectively attached at $0, \infty \in \mathbb{P}$. It has a canonical map $F_n(d_-, d_+) \to X$ given by evaluation at any point on the component where the map is constant. We denote by $\overline{\gamma}$ the pull-back of γ to $F_n(d_-, d_+)$.

Lemma 9.3. (Restriction of Liouville class to fixed points) *The restriction of* $\lambda(\gamma)$ *to* $F_n(d_-, d_+)$ *is equal to* $\exp(\overline{\gamma} + (d_+, \gamma)\zeta)$.

Proof. The restriction of $[\omega_{\mathbb{P},\mathbb{C}^\times}]$ to the fixed point 0 resp. ∞ in $\mathbb{P}$ is 0 resp. ζ. Hence the restriction of $e^*\gamma \cup e_C^*[\omega_{\mathbb{P},\mathbb{C}^\times}]$ to a fixed map as in Proposition 9.2 is given by

$$e^*\gamma \cup e_C^*[\omega_{\mathbb{P},\mathbb{C}^\times}]|_{C_+} = \zeta e^*\gamma$$

for the components C_+ attached to ∞, and

$$e^*\gamma \cup e_C^*[\omega_{\mathbb{P},\mathbb{C}^\times}]|_{C_-} = 0$$

for the components C_- attached to 0. The push-forward $p_*(e^*\gamma \cup e_C^*[\omega_{\mathbb{P},\mathbb{C}^\times}])$ is given by integration over the union $C_- \cup C_0 \cup C_+$, and the integrals may be computed separately over each component. There are two components on which the integrand is non-zero: over the components C_+ attached at ∞ the integral is $(\gamma, d_+)\zeta$, while the integral over the constant component is $\overline{\gamma}$, since the integral of $[\omega_{\mathbb{P},\mathbb{C}^\times}]$ over $\mathbb{P}$ is 1 by definition. Hence

$$p_*(e^*\gamma \cup e_C^*[\omega_{\mathbb{P},\mathbb{C}^\times}]) = (\gamma, d_+)\zeta + \overline{\gamma}$$

and the Liouville class is $\exp(\overline{\gamma} + (d_+, \gamma)\zeta)$.

Definition 9.4. (Localized graph potentials) Define the *localized graph potentials* (also known as the one-point descendent potential)

$$\tau_{X,\pm} : QH(X) \to QH(X)[[\zeta^{-1}]]$$

by push-pull over the fixed point component given by $\overline{\mathcal{M}}_{0,n+1}(X, d)$

$$\tau_{X,\pm}(\alpha, q, \zeta) := \sum_{n \geq 0} (1/n!) \tau^n_{X,\pm}(\alpha, \ldots, \alpha, q, \zeta)$$

where for $n \neq 1$

$$\tau^n_{X,\pm}(\alpha_1, \ldots, \alpha_n, q, \zeta) = \sum_{d \in H_2^G(X,\mathbb{Z})} q^d \operatorname{ev}_{n+1,*}\left(\mp(\zeta(\mp\zeta - \psi_{n+1}))^{-1} \bigcup_{i=1}^n \operatorname{ev}_i^* \alpha_i \right),$$

$\psi_{n+1} \in H^2(\overline{\mathcal{M}}_{0,n+1}(X, d))$ is the cotangent line at the $(n+1)$-st marked point, and the inverted Euler class is expanded as a power series in ζ^{-1} as usual in equivariant localization. For $n = 1$ there is an additional term, equal to α_1, arising from the situation that there is no bubble component attached at 0.

Let $\pi : I_X \to X$ denote the canonical projection and $\pi^*\gamma \in H^2(I_X)$ the pull-back of the class $\gamma \in H^2(X)$.

Lemma 9.5. (Properties of localized graph potentials) *For* $\alpha \in H(X)$, $\gamma \in H^2_G(X)$,

(a) (Duality) $\tau_{X,+}(\alpha, q, \zeta) = \tau_{X,-}(\alpha, q, -\zeta)$.
(b) (Pairing) $\tau_X^{\mathbb{C}^\times}(\alpha, \gamma, q, \zeta) = \int_{I_X} \tau_{X,-}(\alpha, q, \zeta) \cup \tau_{X,+}(\alpha, q e^{\zeta\gamma}, \zeta) \cup \exp(\pi^*\gamma)$.

Proof. The pairing formula follows from the virtual localization formula [17] for the $\mathbb{C}^\times$-action on $\overline{\mathcal{M}}_{0,n}(\mathbb{P}, X)$ applied to $\tau_X^{\mathbb{C}^\times}(\alpha, \gamma, q, \zeta)$. In order to apply the virtual localization formula, one needs to know that $\overline{\mathcal{M}}_{0,n}(\mathbb{P}, X)$ embeds in a non-singular Deligne-Mumford stack; this follows by taking a projective embedding of X. Each fixed point component is described in Proposition 9.2. The integral over the fixed point component corresponding to components of degrees d_-, d_+ attached at $0, \infty$ is

$$\int_{I_X} \mathrm{ev}_{n_-+1,-,*}\left(\mp(\zeta(\mp\zeta - \psi_{n_-+1}))^{-1} \bigcup_{i=1}^{n_-} \mathrm{ev}_{i,-}^* \alpha_i\right) \cup \exp((d_+, \gamma)\zeta + \pi^*\gamma)$$
$$\cup\, \mathrm{ev}_{n_++1,+,*}\left(\mp(\zeta(\mp\zeta - \psi_{n_++1}))^{-1} \bigcup_{i=1}^{n_+} \mathrm{ev}_i^* \alpha_{i,+}\right). \quad (52)$$

Indeed, the normal complex of each such configuration is $T_{w_+}^\vee C \otimes T_{w_-}^\vee \hat{C}^\rho \oplus T_{w_+} C$, corresponding to deformations of the node and attaching point to C respectively; take the inverse Euler class gives the factor $(\mp\zeta(\mp\zeta - \psi_{n_\pm+1}))^{-1}$. Summing over all possible classes d_-, d_+ and markings n_-, n_+ one obtains

$$\int_{I_X} \tau_{X,-}(\alpha, q, \zeta) \cup \tau_{X,+}(\alpha, qe^{\zeta\gamma}, \zeta) \cup \exp(\pi^*\gamma)$$

as claimed.

The components of $\tau_{X,\pm}$ give solutions to the quantum differential equation [26, (4)] for the Frobenius manifold associated to the Gromov-Witten theory of X because of the topological recursion relations, see Pandharipande [24].

9.3. Localized gauged graph potentials

In this section we define a gauged version of the localized graph potential. We show that the gauged graph potential factorizes as a pairing between contributions arising from the fixed points of the $\mathbb{C}^\times$-action on $C = \mathbb{P}$ at 0 and ∞. We begin by introducing the gauged version of the Liouville class, which was introduced in special cases in [14].

Definition 9.6. (Liouville class and invariants with Liouville insertions) Any class $\gamma \in H_G^2(X)$ gives rise to an equivariant class

$$\lambda(\gamma) := \exp(p_*(e^*\gamma \cup e_C^*[\omega_{C,\mathbb{C}^\times}])) \in H_{\mathbb{C}^\times}(\overline{\mathcal{M}}_n^G(C, X)). \quad (53)$$

(Here $H_{\mathbb{C}^\times}(\overline{\mathcal{M}}_n^G(C, X))$ denotes equivariant cohomology with formal power series coefficients, so that the exponential is well defined.) Inserting the class (53) in the integrals gives rise to gauged trace maps with Liouville insertions

$$\tau_X^{G,\mathbb{C}^\times,n} : H_G(X)^n \times H_{\mathbb{C}^\times}(\overline{\mathcal{M}}_n(C)) \times H_G^2(X) \to \mathbb{Q}[[\zeta]]$$
$$(\alpha, \beta, \gamma) \to \sum_{d \in H_2^G(X,\mathbb{Z})} q^d \int_{[\overline{\mathcal{M}}_n^G(C,X,d)_{\mathbb{C}^\times} \to B\mathbb{C}^\times]} \mathrm{ev}^* \alpha \cup f^*\beta \cup \lambda(\gamma).$$

The resulting potential, as in Givental [14], admits a "factorization" in terms of contributions to the fixed point formula near 0 and ∞ in $\mathbb{P}$; the statement and proof take the remainder of this subsection. First we describe the fixed point locus of the action.

Definition 9.7 (Clutching construction for gauged maps from $\mathbb{P}$). We give a clutching construction for gauged maps, generalizing that of bundles over the projective line. Below we will show that all $\mathbb{C}^\times$-fixed points arise from this clutching construction. Given one-parameter subgroups $\phi_\pm : \mathbb{C}^\times \to G$ let $X^{\phi_\pm}$ denote the locus of points with limits,

$$X^{\phi_\pm} := \left\{ x \in X \mid \exists \lim_{z \to 0} \phi_\pm(z)x \right\}.$$

If X is projective then $X^{\phi_\pm} = X$ but if X is linear then $X^{\phi_\pm}$ is the sum of the positive weight spaces. Let $P(\phi_+, \phi_-)$ denote the bundle over $\mathbb{P}^1$ formed from trivial bundles over $\mathbb{C}$ with clutching function $\phi_+(z)\phi_-(z^{-1})^{-1}$,

$$P(\phi_+, \phi_-) = (\mathbb{C} \times G) \cup_{\phi_+\phi_-^{-1}} (\mathbb{C} \times G).$$

For $x \in X^{\phi_+} \cap X^{\phi_-}$ let $u(\phi_+, \phi_-, x)$ denote the section of $P(\phi_+, \phi_-) \times_G X$ given by

$$(r_\pm^* u(\phi_+, \phi_-, x))(z) = \phi_\pm(z)x, z \in \mathbb{C}^\times$$

where $r_\pm$ is restriction to the open subsets isomorphic to $\mathbb{C}$ near 0 resp. ∞.

A more general construction is necessary to handle orbifold case, which involves rational one-parameter subgroups. Suppose that k is an integer, $\pi : \mathbb{C}^\times \to \mathbb{C}^\times$ is a k-fold cover, θ is a k-th root of unity, and $\widetilde{\phi}_\pm : \mathbb{C}^\times \to G$ are one-parameter subgroups such that $\widetilde{\phi}_\pm(\theta^i)$ fixes x for all i, $\widetilde{\phi}_+\widetilde{\phi}_-^{-1}$ admits a k-th square root $\phi : \mathbb{C}^\times \to G$. Then

$$P(\widetilde{\phi}_+, \widetilde{\phi}_-) = (\mathbb{C} \cup G) \cup_\phi (\mathbb{C} \cup G), \quad (r_\pm^* u(\phi_+, \phi_-, x))(z) = \widetilde{\phi}_\pm(z^{1/k})x$$

define a bundle-with-section fixed up to automorphism by the $\mathbb{C}^\times$-action.

Lemma 9.8. (Every fixed point arises from clutching) *Any $\mathbb{C}^\times$-fixed element $[P, u] \in \mathcal{M}^G(\mathbb{P}, X)^{\mathbb{C}^\times}$ such that $u(z)$ has finite stabilizer for generic z is of the form $P = P(\widetilde{\phi}_+, \widetilde{\phi}_-), u = u(\widetilde{\phi}_+, \widetilde{\phi}_-, x)$ for some $\widetilde{\phi}_+, \widetilde{\phi}_-, x \in X^{\widetilde{\phi}_-} \cap X^{\widetilde{\phi}_+}$ as in Definition 9.7.*

Proof. Suppose that x has generic stabilizer of order k and let $P \to \mathbb{P}$ be a bundle with section $u : \mathbb{P}^1 \to P \times_G X$ that is $\mathbb{C}^\times$-fixed up to automorphism. For any $w \in \mathbb{C}^\times$ let $m(w) : \mathbb{P} \to \mathbb{P}$ denote the action of w. By assumption for any $w \in \mathbb{C}^\times$ there exists an isomorphism $\phi(w) \in \operatorname{Hom}(P, m(w)^*P)$ so that (denoting $\phi(w) : P(X) \to m(w)^*P(X)$ with the same notation) we have $\phi(w) \circ u = m(w) \circ u$. The automorphism $\phi(w)$ is unique up to an element of the finite order stabilizer of u in each fiber. In local trivializations of P near $0, \infty$ the automorphism is given by a map $\phi_\pm : \mathbb{C} \to G$ and the section is given by $u(z) = m(z)u(1) = \phi_\pm(z)u(1)$. Furthermore, $\phi_\pm(z)$ is unique up to an element of the stabilizer of $u(1)$ which implies that $\phi_\pm$ lifts to one-parameter subgroups $\widetilde{\phi}_\pm : \widetilde{\mathbb{C}}^\times \to G$ for some finite cover $\pi : \widetilde{\mathbb{C}}^\times \to \mathbb{C}^\times$. Hence $u(z) = \widetilde{\phi}_\pm(\tilde{z})u(1), \tilde{z} \in \pi^{-1}(z)$. The transition map between these two trivializations preserves the section and is therefore of the form $(z, g) \mapsto (z, \widetilde{\phi}_+(\tilde{z})\widetilde{\phi}_-^{-1}(\tilde{z})g)$. The statement of the Lemma follows.

To investigate the stability of a map formed by the clutching construction in Lemma 9.8, we restrict to the case that G is a torus with Lie algebra $\mathfrak{g}$ and weight lattice $\Lambda^\vee \subset \mathfrak{g}^\vee$.

Lemma 9.9. (Semistability of gauged maps formed by clutching) *Suppose that $\widetilde{\phi}_\pm : \widetilde{\mathbb{C}}^\times \to G$ and $x \in X^{\widetilde{\phi}_+} \cap X^{\widetilde{\phi}_-}$. For ρ sufficiently large, the pair $(P = P(\widetilde{\phi}_-, \widetilde{\phi}_+), u = u(\widetilde{\phi}_-, \widetilde{\phi}_+, x))$ given by the clutching construction is Mundet semistable iff x is semistable.*

Proof. With (P, u) as in the statement of the Lemma, the slope inequality $\mu(\sigma, \lambda) \le 0$ holds for all σ, λ for ρ sufficiently large iff u is semistable at a generic point in the domain. Since u is $\mathbb{C}^\times$-fixed, it suffices to check the semistability of u at $z = 1$ in the local chart near 0, hence the condition in the Lemma.

Corollary 9.10. (Clutching description of the circle-fixed gauged maps) Suppose that G is a torus. For ρ sufficiently large:

(a) Each component of $\mathcal{M}_2^G(\mathbb{P}, X, d)^{\mathbb{C}^\times}$ with markings at $0, \infty$ is isomorphic to a subset of $X /\!\!/ G$ with evaluation maps given by $\lim_{z \to 0} \widetilde{\phi}_\pm(z) x$ for some one-parameter subgroups $\widetilde{\phi}_\pm : \mathbb{C}^\times \to G$.

(b) The fixed point set $\overline{\mathcal{M}}_n^G(\mathbb{P}, X, d)^{\mathbb{C}^\times}$ is isomorphic to a union of quotients

$$\left(\overline{\mathcal{M}}_{0,n_-+1}(X, d_-) \times_{\overline{I}_{X/\!\!/G}} \mathcal{M}_2^{G,\mathrm{fr}}(\mathbb{P}, X, d_0)^{\mathbb{C}^\times} \times_{\overline{I}_{X/\!\!/G}} \overline{\mathcal{M}}_{0,n_++1}(X, d_+) \right) /\!\!/ G^2 \tag{54}$$

for some $d_- + d_0 + d_+ = d$ and $n_- + n_+ = n$, where the stability condition is induced from that on the middle factor.

(c) The restriction of $\lambda(\gamma)$ to a fixed point component of $\overline{\mathcal{M}}_n^G(\mathbb{P}, X)$ in (54) is equal to $\exp(\overline{\gamma} + (d_+ + \phi_+, \gamma)\zeta)$ where $\overline{\gamma}$ is the image of γ under $H_G^2(X) \to H(\overline{I}_{X/\!\!/G})$ and $\phi_+ \in \mathfrak{g}_\mathbb{Q} \cong H_2^G(X, \mathbb{Q})$ is considered an element of $H_2^G(X)$ via the push-forward $H(BG) \to H_2^G(X)$.

Proof. (a) By Lemma 9.8, the fixed points correspond to data $(\widetilde{\phi}_+, \widetilde{\phi}_-, x)$ such that the corresponding sections $\widetilde{\phi}_\pm(z)x$ extend over 0. By Lemma 9.9 the value of the section over the open orbit must be semistable, which proves the claim. The description in (b) includes the components C_-, C_+ attached to $0, \infty$ and is immediate. For (c) the class $[\omega_{C,\mathbb{C}^\times}]$ restricts to $0, \zeta$ respectively at $0, \infty$. The integral over the component C_+ attached to ∞ is therefore $(\gamma, d_+)\zeta$. The integral over the principal component C_0 can be computed by $\mathbb{C}^\times$-localization: Since $\mathbb{C}^\times$ acts on the fiber at ∞ via the one-parameter subgroup ϕ_+, the restriction of $e^*\gamma \cup e_C^*[\omega_{C,\mathbb{C}^\times}]$ to the node at ∞ in the universal curve is $(\gamma + (\gamma, \phi_+)\zeta)\zeta$. After dividing by the Euler class, one obtains that the integral over the point ∞ is $\gamma + (\gamma, \phi_+)\zeta$. Part (c) follows.

Example 9.11. (Projective space quotient) Let $X = \mathbb{C}^k$ with $G = \mathbb{C}^\times$ acting diagonally. There are no holomorphic curves in X, hence the classes $d_\pm$ of the bubble components attached to $0, \infty$ always vanish. The moduli stack of gauged maps of class $d \in H_2^G(X, \mathbb{Z}) \cong \mathbb{Z}$ is isomorphic to $\mathbb{P}^{kd+k-1}$, the projective space of k-tuples of polynomials in two variables of degree d. The group $\mathbb{C}^\times$ acts by pull-back, with fixed point set $\mathcal{M}(\mathbb{P}, X, d)^{\mathbb{C}^\times}$ the union of projective spaces of k-tuples of homogeneous polynomials of some degree $i = 0, \ldots, d$, each isomorphic to $\mathbb{P}^{k-1}$. Identifying $H_2^G(X) \cong \mathbb{Z}$ we have $\phi_- = i, \phi_+ = d - i$, and the isomorphism is given by evaluation at a generic point. The Liouville class is the usual Liouville class on

$\mathbb{P}^{k-1}$, times an equivariant correction $\exp((\gamma, \phi_+)\zeta))$; this class already appears in Givental [14].

Let us reformulate the description of the fixed point set in Corollary 9.10 following the "factorization philosophy" as follows. Given $\tilde{\phi}_\pm, d_\pm$ as above and x with order of stabilizer $k_\pm$, let

$$F^G_{n_\pm}(\widetilde{\phi}_\pm, d_\pm) := \{([u], x) \in \overline{\mathcal{M}}_{0,n_\pm+1}(X, d_-) \times X \mid u(z_{n_\pm+1}) = \lim_{z\to\pm\infty} \widetilde{\phi}_\pm(z)x\}/\!\!/G.$$

Since x is stabilized by $\widetilde{\phi}_\pm(\theta)$, where θ is a k-th root of unity, we have natural maps

$$F^G_{n_\pm}(\widetilde{\phi}_\pm, d_\pm) \to \overline{I}_{X/\!\!/G}, \quad (u, x) \mapsto [x, \widetilde{\phi}_\pm(\theta)]. \tag{55}$$

Denote by

$$F^G_{n_\pm}(d) := \cup_{\phi_\pm + d_\pm = d} F^G_{n_\pm}(\widetilde{\phi}_\pm, d_\pm). \tag{56}$$

Corollary 9.10 implies

$$\overline{\mathcal{M}}^G_n(\mathbb{P}, X, d)^{\mathbb{C}^\times} \cong \bigcup_{d_- + d_+ = d} \bigcup_{n_- + n_+ = n} F^G_{n_-}(d_-) \times_{\overline{I}_{X/\!\!/G}} F^G_{n_+}(d_+). \tag{57}$$

We may view the factorization of the fixed point sets as a nodal degeneration as follows. Consider a degeneration of $\mathbb{P}^1$ to a a nodal curve with two components, each projective weighted lines $\mathbb{P}(1, k)$ with node at the orbifold singularity $B\mathbb{Z}_k$. Let $P(\phi), P(\phi_+), P(\phi_-)$ denote the (possibly orbifold) bundles defined by clutching maps ϕ, ϕ_+, ϕ_-. Then $P(\phi)$ degenerates to a principal bundle over the nodal line with restrictions $P(\phi_+)$ and $P(\phi_-)$. Each $\mathbb{C}^\times$-fixed section u degenerates to a pair of sections (u_-, u_+) of $P(\phi_-) \cup P(\phi_+)$, given by x in the trivializations near the node and $\phi_\pm(z)x$ in the trivializations near 0 in the two copies of $\mathbb{P}(1, k)$. The degeneration description implies the following splitting of the normal complex. Let

$$N(\overline{\mathcal{M}}^G_n(\mathbb{P}, X, d)^{\mathbb{C}^\times}) \text{ resp. } N_- := N(F^G_{n_-}(d_-)) \text{ resp. } N_+ := N(F^G_{n_+}(d_+))$$

denote the normal complex of $\overline{\mathcal{M}}^G_n(\mathbb{P}, X, d)^{\mathbb{C}^\times}$ resp. $F^G_{n_-}(d_-)$ resp. $F^G_{n_+}(d_+)$ in $\overline{\mathcal{M}}^G_n(\mathbb{P}, X, d)$ resp. $\overline{\mathcal{M}}^G_{n_-}(\mathbb{P}(1, k), X, d_-)$ resp. $\overline{\mathcal{M}}^G_{n_+}(\mathbb{P}(1, k), X, d_+)$. Deforming the node gives rise to an embedding $F^G_{n_-}(d_-) \times_X F^G_{n_+}(d_+) \to \overline{\mathcal{M}}^G_n(\mathbb{P}, X, d)^{\mathbb{C}^\times}$, and the pullback of the K-class of the normal complex is independent of the deformation parameter. It follows that there is an isomorphism in K-theory

$$[N(\overline{\mathcal{M}}^G_n(\mathbb{P}, X, d)^{\mathbb{C}^\times})] = [N_-] \oplus [N_+]. \tag{58}$$

Explicitly for any type with more than two components in the domain, the normal complex receives contributions from deformations of the map, deformations of the node at the principal component, and deformations of the attaching point to the principal component, assuming there is some non-trivial component attached: In K-theory

$$N_\pm \cong (Rp_* e^*(T(X/G))))^{\mathrm{mov}} \oplus \left(T^\vee_{w_+} C \otimes T^\vee_{w_-} \hat{C}^\rho \right) \oplus T_{w_+} C \tag{59}$$

where $(Rp_* e^*(T(X/G)))^{\mathrm{mov}}$ is the moving part (under the action of $\mathbb{C}^\times$) of the index of the tangent complex of X/G and $w_\pm \in \hat{C}^\rho$ are the preimages of the node connecting to the principal component at 0 in the normalization $\hat{C}^\rho$, so that $w_+ = 0$ in the principal component identified with C. The first factor in (59) represents deformations of the map, the second deformation of the node, and the third the deformation of the attaching point to the principal component. The Euler class is

$$\mathrm{Eul}(N_\pm) = \mathrm{Eul}((Rp_* e^*(T(X/G))))^{\mathrm{mov}})(\mp\zeta)(\mp\zeta - \psi)$$

where ψ is the cotangent line of the node of the component attached at $0 \in \mathbb{P}$.

We define the localized gauged graph potentials by twisted integration over the fixed point sets above. Pushforward over the map (55) induces a map in equivariant cohomology

$$\mathrm{ev}_\infty^*(\mathrm{ev}_1^* \times \ldots \times \mathrm{ev}_n^*) : H_G(X)^{\otimes n} \to H_{\mathbb{C}^\times}(I_{X/\!\!/G}). \tag{60}$$

Example 9.12. (Vector spaces) In the case that X a vector space, the map to X is homotopically trivial and (60) may be identified with the map

$$\Psi^{\phi_\pm} : H_G(X)^n \to H_{\mathbb{C}^\times}(X/\!\!/G) \tag{61}$$

given by cup product, pull-back $H_{G\times\mathbb{C}^\times}(X) \to H_G(\mathrm{pt}) \to H_{G\times\mathbb{C}^\times}(X)$ under the map induced by the constant $\mathbb{C}^\times$-invariant map given by multiplication by zero and the Kirwan map $H_{G\times\mathbb{C}^\times}(X) \to H_{\mathbb{C}^\times}(X/\!\!/G)$. The map (61) may be computed explicitly using naturality of the quotient construction as follows. Composition of the action with the group homomorphism

$$\varphi_\pm : \ G \times \mathbb{C}^\times \to G \times \mathbb{C}^\times, \quad (g, z) \mapsto (z^{\phi_\pm} g, z)$$

makes the action of $\mathbb{C}^\times$ on X trivial and maps the subgroup $G\times\{1\}$ to $G\times\{1\}$. The quotient map $H_{G\times\mathbb{C}^\times}(X) \to H_{\mathbb{C}^\times}(X/\!\!/G)$ is, for this twisted action, independent of the $\mathbb{C}^\times$-equivariant parameter. By naturality, (61) is equal to the composition of the maps

$$H_G(X)^n \to H(B(G \times \mathbb{C}^\times)) \to H(B(G \times \mathbb{C}^\times)) \to H_{G\times\mathbb{C}^\times}(X) \to H_{\mathbb{C}^\times}(X/\!\!/G)$$

where the second map is induced by $\varphi_\pm$ and the action on $H_{G\times\mathbb{C}^\times}(X)$ is trivial. After the identifications

$$H(BG) \cong \mathrm{Sym}(\mathfrak{g}^\vee), \quad H(B(G \times \mathbb{C}^\times)) \cong \mathrm{Sym}(\mathfrak{g}^\vee \oplus \mathbb{C})$$

we obtain a description of (61) as the map

$$\mathrm{Sym}(\mathfrak{g}^\vee)^n \to H_{\mathbb{C}^\times}(X/\!\!/G), \quad (p_1, \ldots, p_n)(\cdot) \mapsto (\kappa_X^G|_{q=0})(p_1 \ldots p_n)(\cdot + \phi_\pm)$$

where $(\cdot + \phi_\pm)$ denotes translation by $\phi_\pm$ and $\kappa_X^G|_{q=0}$ is the classical Kirwan map.

Definition 9.13. (Localized Gauged Graph Potentials) The *localized gauged graph potentials* $\tau^G_{X,\pm}$ are the integrals over $F^G_{n_\pm}(d_\pm)$ of (56)

$$\tau^G_{X,\pm} : QH_G(X) \to QH(X/\!\!/_\rho G)[\zeta,\zeta^{-1}]], \quad \tau^G_{X,\pm}(\alpha,q,\zeta) = \sum_{n\geq 0}(1/n!)\tau^{G,n}_{X,\pm}(\alpha,\ldots,\alpha,q,\zeta)$$

$$\tau^{G,n}_{X,\pm}(\alpha_1,\ldots,\alpha_n,q,\zeta) = \sum_d q^d \operatorname{ev}_{\infty,*}(\operatorname{ev}_1^*\alpha_1 \cup \ldots \cup \operatorname{ev}_n^*\alpha_n \cup \operatorname{Eul}(N_\pm)^{-1}). \quad (62)$$

Proposition 9.14. (Properties of localized gauged potentials)

(a) (Duality) $\tau^G_{X,+}(q,\alpha,\zeta) = \tau^G_{X,-}(q,\alpha,-\zeta)$.

(b) (Pairing) $\lim_{\rho\to\infty}\tau^{G,\mathbb{C}^\times}_X(\alpha,\gamma,q,\zeta) = \int_{\overline{I}_{X/\!\!/G}} \tau^G_{X,-}(\alpha,q,\zeta)\cup\tau^G_{X,+}(\alpha,qe^{\gamma\zeta},\zeta))\cup \exp(\overline{\gamma})$.

Here the action $e^{\zeta\gamma}$ on $\Lambda^G_X[[\zeta]]$ is

$$\left(f(q) = \sum c_d q^d\right) \mapsto \left(f(q\exp(\zeta\gamma)) = \sum c_d q^d \exp(\zeta(\gamma,d))\right).$$

Proof. (a) The fixed point sets $F^G_n(\phi_-,d_-)$ and $F^G_n(\phi_+,d_+)$ are isomorphic, and $\mathbb{C}^\times$-equivariantly so after twisting by the automorphism $z\mapsto 1/z$. Similarly the complexes N_-,N_+ are isomorphic up to this twisting. The first claim follows. (b) is a consequence of virtual localization applied to the stack $\overline{\mathcal{M}}^G_n(\mathbb{P},X)$ and (57), (58), and part (c) of Corollary (9.10). In order to apply the virtual localization formula one needs to know that $\overline{\mathcal{M}}^G_n(\mathbb{P},X)$ embeds in a non-singular Deligne-Mumford stack. For this consider an embedding $G\to GL(k)$ and G-equivariant embedding $X\to\mathbb{P}^{l-1}$ for some l. As in the proof of [27, Proposition 5.12], $\overline{\mathcal{M}}^G_n(\mathbb{P},X)$ embeds in the moduli stack $\overline{\mathcal{M}}_{g,0}(\mathfrak{U}^{\mathrm{fr,quot}}(C,F)\times_G X,\tilde{d})/\operatorname{Aut}(F)$ for a suitable sheaf F. The latter embeds in $\overline{\mathcal{M}}_{g,0}(\mathfrak{U}^{\mathrm{fr,quot}}(C,F)\times_G \mathbb{P}^{l-1},\tilde{d})/\operatorname{Aut}(F)$. Now $\mathfrak{U}^{\mathrm{fr,quot}}(C,F)\times_G\mathbb{P}^{l-1}$ is an $\operatorname{Aut}(F)$-equivariant quasiprojective scheme and so embeds in $\mathbb{P}^N$ for sufficiently large N, hence $\overline{\mathcal{M}}_{g,0}(\mathfrak{U}^{\mathrm{fr,quot}}(C,F)\times_G\mathbb{P}^{l-1},\tilde{d})/\operatorname{Aut}(F)$ embeds $\operatorname{Aut}(F)$-equivariantly in $\mathcal{M}_{g,0}(\mathbb{P}^N)$. Since $\overline{\mathcal{M}}_{g,0}(\mathbb{P}^N)/\operatorname{Aut}(F)$ is a non-singular Deligne-Mumford stack, the claim follows.

Example 9.15. (Localized gauged graph potential for toric quotients) Let G be a torus acting on a vector space X is a vector space with weights $\mu_1,\ldots,\mu_k$ and weight spaces $X_1,\ldots,X_k$ with free quotient $X/\!\!/G$. Let $D_j\in H^2(X/\!\!/G)$ denote the divisor class corresponding to μ_j. For any given class $\phi\in H^G_2(X,\mathbb{Z})\cong\mathfrak{g}_\mathbb{Z}$, the loci X^ϕ are sums of weight spaces

$$X^\phi := \left\{x \mid \exists \lim_{z\to 0}\phi(z)x\right\} = \oplus_{\mu_j(\phi)\geq 0}X_j.$$

Since there are no non-constant stable maps to X, $F^G(\phi,0)$ is isomorphic to $X^\phi/\!\!/G$ under evaluation at any generic point. The domain of any gauged map without markings is irreducible and the normal complex to $F^G(\phi,0)$ is the moving part $Rp_*e^*(T(X/G))^{\mathrm{mov}}$. This splits as a sum of $\mu_j(\phi)$ copies of X_j with weights $1,\ldots,\mu_j(\phi)$ for $\mu_j(\phi)$ negative, and $-\mu_j(\phi)-2$ copies of the normal complex for

$\mu_j(\phi) \leq 0$ with weights $\mu_j(\phi)+1,\ldots,-1$. Putting this together with the normal bundle of $X^\phi /\!\!/ G$ in $X/\!\!/G$ and replacing ϕ with d we obtain

$$\tau^{G,0}_{X,-}(\zeta,q) = \sum_{d \in H_2^G(X)} q^d \frac{\prod_{j=1}^k \prod_{m=-\infty}^{0}(D_j + m\zeta)}{\prod_{j=1}^k \prod_{m=-\infty}^{\mu_j(d)}(D_j + m\zeta)}. \tag{63}$$

Note that the terms with $X^d/\!\!/G = \emptyset$ contribute zero in the above sum, since in this case the factor in the numerator $\prod_{\mu_j(d)<0} D_j$ vanishes. The function $\tau^G_{X,-}(\alpha,\zeta,q)$ for $\alpha \in H_G(X)$ can be computed as follows. Since there are no non-constant holomorphic spheres in X, the evaluation maps $\mathrm{ev}_1,\ldots,\mathrm{ev}_n$ are equal on $\overline{\mathcal{M}}^G_n(\mathbb{C}_\pm, X)^{\mathbb{C}^\times}$. It follows that the pushforward of $\mathrm{ev}^* \alpha^{\otimes n}/\mp\zeta(\mp\zeta - \psi)$ is equal to

$$\alpha^n(\zeta)^{-2}\int_{[\overline{M}_{0,n+1}]}(\psi_{n+1}/\mp\zeta)^{n-2}. \tag{64}$$

This integral can be computed iteratively by pushing forward under the maps f_i forgetting the i-th marked point for $i \leq n$. We have the relation for the first Chern class ψ_{n+1} of the cotangent line at the last marked point in $\overline{\mathcal{M}}_{0,n+1}$

$$\psi_{n+1} = f_i^*\psi_n + [D_{\{0,i,n+1\}\cup\{1,\ldots,\hat{i},\ldots,n\}}] \in H(\overline{M}_{0,n+1}).$$

The divisor class is degree one in any fiber of the forgetful map f_i and it follows that the integral (64) is equal to $(\alpha/\zeta)^n$. By Example 9.12 this implies that for $\alpha \in H_G(X)$

$$\tau^G_{X,-}(\alpha,\zeta,q) = \sum_{d\in H_2^G(X)} q^d \exp(\Psi^d(\alpha)/\zeta)\frac{\prod_{j=1}^k \prod_{m=-\infty}^{0}(D_j + m\zeta)}{\prod_{j=1}^k \prod_{m=-\infty}^{\mu_j(d)}(D_j + m\zeta)}. \tag{65}$$

Thus $\tau^G_{X,-}$ is the generalization of Givental's *I-function*, see [15], considered in Ciocan-Fontanine-Kim [8, Section 5.3].

9.4. Localized adiabatic limit theorem

We prove the refinement [26, Theorem 1.6] of the adiabatic limit Theorem [26, Theorem 1.5] by comparing the fixed point contributions to the graph potentials. Recall the stack $\overline{\mathcal{M}}^G_{n,1}(\mathbb{P},X)$ of curves with scaling from (48). We claim the invariants of (48) also have $\mathbb{C}^\times$-equivariant extensions:

Lemma 9.16. (Existence of a circle action on the master space) *The $\mathbb{C}^\times$-action on $\overline{\mathcal{M}}^G_n(\mathbb{P},X)$ extends to $\overline{\mathcal{M}}^G_{n,1}(\mathbb{P},X)$. The action on the substack $\overline{\mathcal{M}}^G_n(\mathbb{P},X)_\infty$ is the action induced from the action of $\mathbb{C}^\times$ on the factors $\overline{\mathcal{M}}^G_{i_j,1}(\mathbb{A},X)$ and $\overline{\mathcal{M}}^G_r(\mathbb{P},X/\!\!/G)$.*

Proof. The $\mathbb{C}^\times$-action on $\mathbb{P}$ induces an action on stable maps to $\mathbb{P}\times X$, by composition, and on the relative dualizing sheaf. Hence $\mathbb{C}^\times$ acts on $\overline{\mathfrak{M}}^G_{n,1}(\mathbb{P},X)$. The stability condition is unchanged, since the Hilbert-Mumford weights for two objects related by the $\mathbb{C}^\times$-action are in one-to-one correspondence. This implies that

$\mathbb{C}^\times$ acts on the semistable locus $\overline{\mathcal{M}}^G_{n,1}(\mathbb{P}, X)$. For the substack $\overline{\mathcal{M}}^G_n(\mathbb{P}, X)_\infty$ consisting of fiber products of $\overline{\mathcal{M}}^G_{i_j,1}(\mathbb{A}, X)$ and $\overline{\mathcal{M}}^G_r(\mathbb{P}, X/\!\!/G)$, the action is given by pull-back of sections under the action by rotation on $\mathbb{P}$. The relative dualizing sheaf is identified with the dualizing sheaf by trivialization of the cotangent line $T_0^\vee\mathbb{P}$. Since $\mathbb{C}^\times$ acts with weight one on $T_0^\vee\mathbb{P}$, after trivialization of the cotangent line it acts by scalar multiplication on the sections of the projectivized dualizing sheaf in the objects of $\overline{\mathcal{M}}^G_{i_j,1}(\mathbb{A}, X)$, as claimed.

The fixed point set for the action factorizes as follows. Let $F^G_{n,1}(d) \subset \overline{\mathcal{M}}^G_{n,1}(\mathbb{P}, X, d)$ denote the fixed point set of class $d \in H_2^G(X)$. We wish to express $F^G_{n,1}(d)$ as a fiber product of "contributions from $0, \infty$". Let $\overline{\mathcal{M}}^G_{n_\pm,1}(\mathbb{P}(1,k), X, d)^{\mathbb{C}^\times}$ denote the $\mathbb{C}^\times$-fixed locus in $\overline{\mathcal{M}}^G_{n_\pm,1}(\mathbb{P}(1,k), X)$. Let

$$F^G_{n_\pm,1}(d) \subset \overline{\mathcal{M}}^G_{n_\pm,1}(\mathbb{P}(1,k), X, d)^{\mathbb{C}^\times}$$

denote the locus of bundles with sections that are constant in some trivialization in a neighborhood of $B\mathbb{Z}_k$ and $n_\pm$ markings map to $0 \in \mathbb{P}(1,k)$ under the projection $\mathbb{P}(1,k) \times X \to \mathbb{P}(1,k)$. Both conditions are preserved under limits. It follows that $F^G_{n_\pm,1}(d)$ is a proper Deligne-Mumford stack. It admits a relative perfect obstruction theory over $\overline{\mathcal{M}}_{n_\pm,1}(\mathbb{P}(1,k))$ induced from the relative perfect obstruction theory on $\overline{\mathcal{M}}^G_{n_\pm,1}(\mathbb{P}(1,k), X, d)^{\mathbb{C}^\times}$. Consider the evaluation map $F^G_{n_\pm,1}(d) \to \overline{I}_{X/\!\!/G}$ at $B\mathbb{Z}_k \subset \mathbb{P}(1,k)$. For any $g \in G$ of finite order k denote by $F^G_{n_\pm,1}(d,[g]) \subset F^G_{n_\pm,1}(d)$ the locus of bundles-with-sections whose sections take values in the twisted sector corresponding to $[g] \in G/\operatorname{Ad}(G)$ at $B\mathbb{Z}_k$. Consider the map

$$\pi_\pm : F^G_{n_\pm,1}(d_\pm,[g]) \to \mathcal{M}_{0,1}(\mathbb{P}(1,k)) \cong \mathbb{P}.$$

The fiber $\pi_\pm^{-1}(\rho)$ of $F^G_{n_\pm,1}(d,[g])$ over a non-zero scaling ρ is by Section 9.2

$$\pi^{-1}(\rho) = \bigcup_{d_- + \phi_- = d, \widetilde{\phi}_-(\theta) = g} F^G_{n_\pm}(d_-, \phi_-) \tag{66}$$

where θ is a k-th root of unity. The locus $\pi_\pm^{-1}(\infty)$ in $F^G_{n_\pm,1}(d,[g])$ with infinite scaling is by Section 9.3 a union of fiber products

$$\left(\prod_{j=1}^r \overline{\mathcal{M}}^G_{i_j,1}(\mathbb{A}, X) \right) \times_{(\overline{I}_{X/\!\!/G})^r} \times \overline{\mathcal{M}}_{0,r+1}(X/\!\!/G, d) \times_{\overline{I}_{X/\!\!/G}} (X^g/\!\!/Z_g). \tag{67}$$

The identification is given by the map which attaches affine gauged maps to $0 \in \mathbb{P}(1,k)$. From the descriptions in Sections 9.2, 9.3 we obtain

$$F^G_{n,1}(d) = \bigcup_{d_- + d_+ = d, n_- + n_+ = n} \ \bigcup_{[g] \in G/\operatorname{Ad}(G)} F^G_{n_-,1}(d_+,[g]) \times_{\overline{I}_{X/\!\!/G} \times \mathbb{P}} F^G_{n_+,1}(d_-,[g])$$

where the map to $\mathbb{P}$ is given by the scaling. We can now complete the proof of the localized version of the adiabatic limit theorem, [26, Theorem 1.6].

Proof. (of [26, Theorem 1.6]) The divisor class relation $[\pi^{-1}(0)] = [\pi^{-1}(\infty)]$ implies that the integrals over (66), (67) are equal. Hence for any $\alpha_\infty \in H(X^g /\!\!/ Z_g) \subset H(I_{X/\!\!/G})$, we have

$$\sum_{[I_1,\ldots,I_r]} \int_{[I_{X/\!\!/G}]} (\tau^r_{X/\!\!/G,-}(\kappa_X^{G,|I_1|}(\alpha_{I_1},1),\ldots,\kappa_X^{G,|I_r|}(\alpha_{I_r},1),1) \cup \alpha_\infty) = \int_{[I_{X/\!\!/G}]} \tau_{X,-}^{G,n}(\alpha_1,\ldots,\alpha_n,1) \cup \alpha_\infty. \quad (68)$$

Summing over n with $\alpha_1 = \ldots \alpha_n = \alpha$ gives

$$\int_{[I_{X/\!\!/G}]} (\tau_{X/\!\!/G,-} \circ \kappa_X^G)(\alpha) \cup \alpha_\infty = \int_{[I_{X/\!\!/G}]} \tau_{X,-}^G(\alpha) \cup \alpha_\infty.$$

Since this holds for any g and any α_∞, it follows [26, Theorem 1.6]

$$(\tau_{X/\!\!/G,-} \circ \kappa_X^G)(\alpha) = \tau_{X,-}^G(\alpha) \in QH(X/\!\!/G).$$

The localized adiabatic limit [26, Theorem 1.6] allows us to deduce relations in the quantum cohomology algebras, although these relations are rather non-explicit unless κ_X^G is known. Namely, recall that any differential operator annihilating the localized graph potential $\tau_{X/\!\!/G,-}$ defines relations, see for example [14], [18]. Let $V \subset QH_G(X)$ be a submanifold such that the restriction of κ_X^G to V is an embedding. In particular, any differential operator on V pushes forward to a differential operator on $(\kappa_X^G)_*(V)$. We especially have in mind the case that $V \cong QH_G^2(X)$ and is isomorphic to $QH^2(X/\!\!/G)$.

Corollary 9.17. (Relations on quantum cohomology algebras) Suppose that $\Box$ is a differential operator on V.

(a) (Annihilation at a point) If $\Box$ annihilates the restriction of $\tau_{X,-}^G$ at $\alpha \in \widetilde{QH(X/\!\!/G)}$, then $(\kappa_X^G)_*\Box$ annihilates $\tau_{X/\!\!/G,-}$ at $\kappa_X^G(\alpha)$ and the symbol $\Sigma((\kappa_X^G)_*\Box)$ of $(\kappa_X^G)_*\Box$ satisfies $\Sigma((\kappa_X^G)_*\Box) \star_{\kappa_X^G(\alpha)} \tau_{X/\!\!/G,-}(\alpha) = 0$.
(b) (Annihilation on small quantum cohomology) If $V = QH_G^2(X)$ and the quantum cohomology $T_{\kappa_X^G(\alpha)}QH(X/\!\!/G)$ is generated by $T_{\kappa_X^G(\alpha)}QH^2(X/\!\!/G)$, then $\Sigma((\kappa_X^G)_*\Box)(\alpha)$ is a relation in $T_{\kappa_X^G(\alpha)}QH(X/\!\!/G)$.

Proof. Suppose $\Box$ is a differential operator as in the Corollary part (a). That $(\kappa_X^G)_*\Box$ annihilates $\tau_{X/\!\!/G,-}$ follows from [26, Theorem 1.6]. The relation on the principal symbol follows from the fact that $\tau_{X/\!\!/G,-}$ is a fundamental solution to the quantum differential equation [26, (4)], and so differential operators transform into quantum multiplications at $\kappa_X^G(\alpha)$. That the principal symbol defines relations follows as in [14], [18, Theorem 2.4], using that the components of $\tau_{X/\!\!/G,-}$ generate the quantum cohomology resp. derivatives of $\tau_{X/\!\!/G,-}$ in the directions $(D\kappa_X^G)(\alpha), \alpha \in QH_G^2(X)$ under the $QH_G^2(X)$-generation hypothesis.

In the remainder of the section we discuss the toric case, that is, X is a complex vector space and G is a torus acting on X so that $X/\!\!/G$ is a smooth proper Deligne-Mumford stack. In this case, the identity in [26, Theorem 1.6] seems to be essentially the same as the "mirror theorems" in [14], [21], [20], [11], [7]. Regarding relations in the quantum cohomology of toric varieties, the following is introduced in Batyrev [2].

Definition 9.18. The *quantum Stanley-Reisner ideal* is the ideal $QSR_X^G \subset T_0 QH_G(X), \alpha \in QH_G^2(X)$ generated by the elements

$$\prod_{\mu_j(d)>0} r(v_j)^{\mu_j(d)} - q^d \prod_{\mu_j(d)<0} r(v_j)^{\mu_j(d)} \in T_0 QH_G(X)$$

for $d \in H_2^G(X,\mathbb{Z}) \cong \mathfrak{g}_{\mathbb{Z}}$. Similarly the *equivariant quantum Stanley-Reisner ideal* is the ideal $QSR_X^{\tilde{G},G} \subset T_0 QH_{\tilde{G}}(X)$ generated by the elements

$$\prod_{\mu_j(d)>0} v_j^{\mu_j(d)} - q^d \prod_{\mu_j(d)<0} v_j^{\mu_j(d)} \in T_0 QH_{\tilde{G}}(X),$$

that is, without restriction to $\mathfrak{g} \subset \tilde{\mathfrak{g}}$.

Example 9.19. (a) (Projective spaces) For the usual action of $G = \mathbb{C}^\times$ on $X = \mathbb{C}^k$, we have $H_G^2(X,\mathbb{Z}) = \mathbb{Z}$ with positive generator $d = 1$ and $r(v_j) = u$ the coordinate on $\mathfrak{g}$ for $j = 1, \ldots, k$. The quantum Stanley-Reisner ideal QSR_X^G has generator

$$\prod_{j=1}^{k} r(v_j) - q = u^k - q$$

while the equivariant Stanley-Reisner ideal $QSR_X^{\tilde{G},G}$ has generator $\prod_{j=1}^k v_j - q$.

(b) (Weighted projective line) Suppose that $G = \mathbb{C}^\times$ acts on $X = \mathbb{C}^2$ with weights $\mu_1 = 2, \mu_2 = 3$. Then $H_2^G(X) \cong \mathbb{Z}$ with generator $d = 1$. We have $r(v_1) = 2u$ while $r(v_2) = 3u$. The quantum Stanley-Reisner ideal QSR_X^G has generator

$$\prod_{j=1}^{2} r(v_j)^{\mu_j(1)} - q = (2u)^2(3u)^3 - q$$

while the equivariant Stanley-Reisner ideal $QSR_X^{\tilde{G},G}$ has generator

$$\prod_{j=1}^{2} v_j^{\mu_j(1)} - q = v_1^2 v_2^3 - q.$$

Recall the definition of $\kappa_X^{\tilde{G},G}$ from Remark 8.7.

Theorem 9.20. *The kernel of* $D_0\kappa_X^{G,1} : T_0 QH_G(X) \to T_{\kappa_X^G(0)} QH(X/\!\!/G)$ *resp.* $D_0\kappa_X^{\tilde{G},G} : T_0 QH_G(X) \to T_{\kappa_X^{\tilde{G},G}(0)} QH_T(X/\!\!/G)$ *contains* QSR_X^G *resp.* $QSR_X^{\tilde{G},G}$.

Proof. For $d \in H_2^G(X,\mathbb{Z}) \cong H(BG,\mathbb{Z}) \cong \mathfrak{g}_{\mathbb{Z}}^\vee$ let $\Box_d$ denote the differential operator on $QH_2^G(X,\mathbb{R})$ corresponding to d,

$$\Box_d = \prod_{\mu_j(d)\geq 0} \partial_j^{\mu_j(d)} - q^d \prod_{\mu_j(d)\leq 0} \partial_j^{-\mu_j(d)}.$$

We may identify the coordinates on $QH_2^G(X,\mathbb{R})$ with the quantum parameters, using the divisor equation. Then the operator $\Box_d$ annihilates the function of Example 9.15, see for example Iritani [20], Cox-Katz [12, (11.92)]. It follows from Corollary 9.17 that the corresponding product of the tangent vectors maps to zero in $T_{\kappa_X^G(0)} QH(X/\!\!/G)$, and so lies in the kernel of $D_0\kappa_X^G$.

Remark 9.21. (Isomorphism with the quantum Stanley-Reisner ring) In joint work with Gonzalez [16], we show that $\kappa_X^{G,1}$ is surjective and QSR_X^G is exactly its kernel, after passing to a suitable formal version of $QH_G(X)$, so that $T_{\kappa_X^G(0)} QH(X/\!\!/G)$ is canonically isomorphic to the quantum Stanley-Reisner ring. Related computations can be found in McDuff-Tolman [22] and Iritani [19].

References

[1] D. Abramovich, T. Graber, and A. Vistoli. Gromov-Witten theory of Deligne-Mumford stacks. *Amer. J. Math.*, 130(5):1337–1398, 2008.

[2] V. V. Batyrev. Quantum cohomology rings of toric manifolds. *Astérisque*, (218):9–34, 1993. Journées de Géométrie Algébrique d'Orsay (Orsay, 1992).

[3] K. Behrend. Gromov-Witten invariants in algebraic geometry. *Invent. Math.*, 127(3):601–617, 1997.

[4] K. Behrend and B. Fantechi. The intrinsic normal cone. *Invent. Math.*, 128(1):45–88, 1997.

[5] K. Behrend and Yu. Manin. Stacks of stable maps and Gromov-Witten invariants. *Duke Math. J.*, 85(1):1–60, 1996.

[6] W. Chen and Y. Ruan. Orbifold Gromov-Witten theory. In *Orbifolds in mathematics and physics (Madison, WI, 2001)*, volume 310 of *Contemp. Math.*, pages 25–85. Amer. Math. Soc., Providence, RI, 2002.

[7] I. Ciocan-Fontanine and B. Kim. Wall-crossing in genus zero quasimap theory and mirror maps. arXiv:1304.7056

[8] I. Ciocan-Fontanine and B. Kim. Big I-functions. arxiv:1401.7417.

[9] T. Coates and A. Givental. Quantum Riemann-Roch, Lefschetz and Serre. *Ann. of Math. (2)*, 165(1):15–53, 2007.

[10] T. Coates, Y.-P. Lee, A. Corti, and H.-H. Tseng. The quantum orbifold cohomology of weighted projective spaces. *Acta Math.*, 202(2):139–193, 2009.

[11] T. Coates, A. Corti, H. Iritani, H. -H. Tseng. A Mirror Theorem for Toric Stacks. arXiv:1310.4163.

[12] D. A. Cox and S. Katz. *Mirror symmetry and algebraic geometry*, volume 68 of *Mathematical Surveys and Monographs.* American Mathematical Society, Providence, RI, 1999.

[13] A. R. P. Gaio and D. A. Salamon. Gromov-Witten invariants of symplectic quotients and adiabatic limits. *J. Symplectic Geom.*, 3(1):55–159, 2005.

[14] A. B. Givental. Equivariant Gromov-Witten invariants. *Internat. Math. Res. Notices*, (13):613–663, 1996.

[15] A. Givental. A mirror theorem for toric complete intersections. In *Topological field theory, primitive forms and related topics (Kyoto, 1996)*, volume 160 of *Progr. Math.*, pages 141–175. Birkhäuser Boston, Boston, MA, 1998.

[16] E. Gonzalez and C. Woodward. Quantum cohomology and toric minimal model programs. arXiv:1207.3253.

[17] T. Graber and R. Pandharipande. Localization of virtual classes. *Invent. Math.*, 135(2):487–518, 1999.

[18] H. Iritani. Quantum d-modules and equivariant Floer theory for free loop spaces. *Math Z.*, 252(3):577–622, 2006.

[19] H. Iritani. Convergence of quantum cohomology by quantum Lefschetz. *J. Reine Angew. Math.*, 610:29–69, 2007.

[20] H. Iritani. Quantum D-modules and generalized mirror transformations. *Topology*, 47(4):225–276, 2008.

[21] B. H. Lian, K. Liu, and S.-T. Yau. Mirror principle. I. *Asian J. Math.*, 1(4):729–763, 1997.

[22] D. McDuff and S. Tolman. Topological properties of Hamiltonian circle actions. *IMRP Int. Math. Res. Pap.*, pages 72826, 1–77, 2006.

[23] D. R. Morrison and M. R. Plesser. Summing the instantons: quantum cohomology and mirror symmetry in toric varieties. *Nuclear Phys. B*, 440(1-2):279–354, 1995.

[24] R. Pandharipande. Rational curves on hypersurfaces (after A. Givental). *Astérisque*, (252):Exp. No. 848, 5, 307–340, 1998. Séminaire Bourbaki. Vol. 1997/98.

[25] A. Szenes and M. Vergne. Toric reduction and a conjecture of Batyrev and Materov. *Invent. Math.*, 158(3):453–495, 2004.

[26] C. Woodward. Quantum Kirwan morphism and Gromov-Witten invariants of quotients I. *Transformation Groups*, 20:507–556, 2015. arxiv:1204.1765.

[27] C. Woodward. Quantum Kirwan morphism and Gromov-Witten invariants of quotients II. arxiv:1408.5864.