QUANTUM KIRWAN MORPHISM AND GROMOV-WITTEN INVARIANTS OF QUOTIENTS II

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Abstract. This is the second in a sequence of papers in which we construct a quantum version of the Kirwan map from the equivariant quantum cohomology $QH_G(X)$ of a smooth polarized complex projective variety X with the action of a connected complex reductive group G to the orbifold quantum cohomology $QH(X/\!\!/G)$ of its geometric invariant theory quotient $X/\!\!/G$, and prove that it intertwines the genus zero gauged Gromov-Witten potential of X with the genus zero Gromov-Witten graph potential of $X/\!\!/G$. In this part we construct virtual fundamental classes on the moduli spaces used in the construction of the quantum Kirwan map and the gauged Gromov-Witten potential.

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We continue with the notation in the first part [55] where we introduced moduli spaces of vortices from the symplectic viewpoint. In order to obtain virtual fundamental classes for the moduli spaces of vortices, we show that the moduli spaces of vortices are homeomorphic to coarse moduli spaces of algebraic stacks equipped with (in good cases) perfect obstruction theories.

4. Stacks of curves and maps

This section contains mostly algebraic preliminaries. In particular, we introduce hom-stacks of morphisms of stacks for which there is unfortunately no general theory yet. However, by a result of Lieblich [29, 2.3.4], the stack of morphisms to a quotient stack by a reductive group is an Artin stack, and this is enough for our purposes. We also must show that the substacks of curves satisfying Mundet's semistability condition are algebraic, and this requires recalling some results of Schmitt [46] who gave a geometric invariant theory construction of a related moduli space.

By our convention a *stack* means a locally Noetherian stack over the fppf site of schemes, following de Jong et al [10] which we take as our standard reference. (Another standard reference on stacks is Laumon-Moret-Bailly [26], with a correction by Olsson [36].) By abuse of terminology we say that a stack is *a scheme resp. algebraic space* if it is the stack associated to a scheme resp. algebraic space. A morphism of stacks $f : \mathcal{X} \to \mathcal{Y}$ is *representable* if for any morphism $g : S \to \mathcal{Y}$ where S is a scheme, the fiber product $S \times_{\mathcal{Y}} \mathcal{X}$ is an algebraic space. An *Artin stack* resp. *Deligne-Mumford stack* $\mathcal{X}$ over a scheme S is a stack for which the diagonal $\mathcal{X} \to \mathcal{X} \times_S \mathcal{X}$ is representable, quasi-compact, and separated, and such that there exists an algebraic space X/S and a smooth resp. étale surjective morphism $X \to \mathcal{X}$. In characteristic zero (here the base field is always the complex numbers) an Artin stack is a Deligne-Mumford stack iff all the automorphism groups are finite, see e.g. [12, Remark 2.1]. A *gerbe* is a locally non-empty, locally connected (between any two objects exists a morphism) Artin stack. The category of Artin resp. Deligne-Mumford stacks is closed under disjoint unions and (category-theoretic) fiber products [26, 4.5]. A morphism of stacks is *proper* if it is separated, of finite type, and universally closed.

4.1. Quotient stacks

The following are examples of stacks:

Example 4.1. (a) (Quot schemes) For integers r, n with $0 < r < n$ let $\mathrm{Gr}(r, n)$ denote the Grassmannian of subspaces of $\mathbb{C}^n$ of dimension r. Any morphism from X to $\mathrm{Gr}(r, n)$ gives rise to a vector bundle $E \to X$ obtained by pull-back of the quotient bundle and a surjective morphism ϕ from the trivial bundle $X \times \mathbb{C}^n$ to E. Grothendieck [20], and later Olsson-Starr [37], studied such pairs in a very general setting, as part of a general program to construct moduli schemes for various functors. Given a scheme $\mathcal{X}/S$ resp. separated Deligne-Mumford stack $\mathcal{X}/S$ and a quasicoherent $\mathcal{O}_{\mathcal{X}}$-module F, let $\mathrm{Quot}_{F/\mathcal{X}/S}$ be the category that assigns to any S-scheme T the set of pairs (E, ϕ) where $\phi : F \times_S T \to E$ is a flat family of quotients. By Grothendieck [20] resp. Olsson-Starr [37], if $\mathcal{X}$ is a scheme projective over S resp. Deligne-Mumford stack then $\mathrm{Quot}_{F/\mathcal{X}/S}$ is a smooth scheme resp. algebraic space, whose connected components are projective resp. quasiprojective if $\mathcal{X}$ is.

(b) (Stack of coherent sheaves) The category of coherent sheaves on a projective scheme carries the structure of an Artin stack. If X is an S-scheme we denote by $\mathrm{Coh}(X/S)$ resp. $\mathrm{Vect}(X/S)$ the category that assigns to any $T \to S$ the category of coherent sheaves resp. vector bundles on $X \times_S T$. By [26, p. 29], see also [29, Theorem 2.1.1], if X is a projective scheme then $\mathrm{Coh}(X)$ is an Artin stack and $\mathrm{Vect}(X)$ an open substack. Charts can be constructed as follows: after suitable twisting any sheaf $E \to X$ can be generated by its global sections, in which case E can be written as a quotient $F \to E$ where $F = X \times H^0(E)^\vee$. Then $\mathrm{Coh}(X)$ is isomorphic locally near E to the quotient of $\mathrm{Quot}_{F/X/S}$ by $\mathrm{Aut}(F)$.

(c) (Stack of bundles, first version) Let G be a reductive group and X an S-scheme. We denote by $\mathrm{Bun}_G(X)$ the category that assigns to any $T \to S$ the category of principal G-bundles on $X \times_S T$. Then for any integer $r > 0$ the stack $\mathrm{Bun}_{GL(r)}(X)$ is canonically isomorphic to the substack $\mathrm{Vect}_r(X)$ of $\mathrm{Vect}(X)$ of vector bundles of rank r. Any principal G-bundle corresponds to a $GL(V)$-bundle $E \to X \times_S T$ together with a reduction of structure group $X \times_S T \to E/G$. If X is projective, then $\mathrm{Vect}(X)$ is an Artin stack, being an open substack of the stack $\mathrm{Coh}(X)$, and the above description gives that $\mathrm{Bun}_G(X)$ is an Artin stack as in Sorger [47, 3.6.6 Corollary].

(d) (Quotient stacks) If G is a reductive group scheme over S then we denote by BG the stack which assigns to an algebraic space $T \to S$ the category of principal G-bundles (torsors) over T. More generally if X is a G-scheme over S then X/G denotes the *quotient stack* which assigns to any morphism $T \to S$ the category of principal G-bundles $P \to T$ together with sections $u : T \to (P \times_S X)/G$, or equivalently equivariant morphisms from P to X, see Laumon-Moret-Bailly [26, 4.6.1]. In particular, $BG = \mathrm{pt}/G$.

4.2. Stacks of curves

Example 4.2. (a) (Stable curves) By a *nodal curve* over the scheme S we mean a flat proper morphism $\pi : C \to S$ of schemes such that the geometric fibers of π are reduced, one-dimensional and have at most ordinary double points (nodes) as singularities; that is, a nodal curve without the connectedness assumption in [7, Definition 2.1]. A nodal curve is *stable* if each fiber has no infinitesimal automorphisms. The category of connected nodal resp. stable marked curves of genus g is then a (non-finite-type) Artin resp. proper Deligne-Mumford stack $\overline{\mathfrak{M}}_{g,n}$ resp. $\overline{\mathcal{M}}_{g,n}$ [11], [7]. Let $\mathfrak{M}_{g,n,\Gamma}$ denote the stack consisting of objects whose combinatorial type in Definition 4.2 is Γ. If Γ is connected then $\mathfrak{M}_{g,n,\Gamma}$ is a locally closed, local complete intersection Artin substack of $\overline{\mathfrak{M}}_{g,n}$ [7].

There is a canonical morphism $\overline{\mathfrak{M}}_{g,n,\Gamma} \to \overline{\mathcal{M}}_{g,n,\Gamma}$ which collapses unstable components, and can be defined as follows. Let $\pi : C \to S$ be an n-marked family of nodal curves. Let $\omega_{C/S}$ denote the relative dualizing sheaf over

S and $\omega_{C/S}[z_1 + \ldots + z_n]$ its twisting by $z_1 + \ldots + z_n$. Consider the curve

$$C^{st} = \operatorname{Proj} \oplus_{n \geq 0} \pi_*((\omega_{C/S}[z_1 + \ldots + z_n])^{\otimes n}).$$

As noted in [7, Section 3], in the case that a family of n-marked curves arises from forgetting a marking of a family of $(n+1)$-marked curves, each fiber C_s^{st} is obtained from C_s by collapsing unstable components. Furthermore, the formation of C^{st} commutes with base change and the map $C^{st} \to S$ is projective and flat. The general case is reduced to this one by adding markings locally. Let $\Upsilon : \Gamma \to \Gamma'$ be a morphism of modular graphs. Then there are morphisms of Artin resp. Deligne-Mumford stacks $\overline{\mathfrak{M}}(\Upsilon) : \overline{\mathfrak{M}}_{g,n,\Gamma} \to \overline{\mathfrak{M}}_{g,n,\Gamma'}$ resp. $\overline{\mathcal{M}}(\Upsilon) : \overline{\mathcal{M}}_{g,n,\Gamma} \to \overline{\mathcal{M}}_{g,n,\Gamma'}$. In the case of forgetting a tail, the morphism $\overline{\mathcal{M}}(\Upsilon)$ can be defined by the composition of the inclusion $\overline{\mathcal{M}}_{g,n,\Gamma} \to \overline{\mathfrak{M}}_{g,n,\Gamma}$, the map $\overline{\mathfrak{M}}(\Upsilon) : \overline{\mathfrak{M}}_{g,n,\Gamma} \to \overline{\mathfrak{M}}_{g,n,\Gamma'}$ and the collapsing map $\overline{\mathfrak{M}}_{g,n,\Gamma'} \to \overline{\mathcal{M}}_{g,n,\Gamma'}$. We denote by $\overline{\mathfrak{C}}_{g,n,\Gamma} \to \overline{\mathfrak{M}}_{g,n,\Gamma}$ resp. $\overline{\mathcal{C}}_{g,n,\Gamma} \to \overline{\mathcal{M}}_{g,n,\Gamma}$ the *universal curve* over $\mathfrak{M}_{g,n,\Gamma}$ resp. $\overline{\mathcal{M}}_{g,n,\Gamma}$ namely the category of n-marked nodal (resp. stable) curves equipped with an additional $(n+1)$-marking which need not be distinct from the first n. In the case of $\overline{\mathcal{M}}_{g,n}$, the forgetful morphism f_{n+1} lifts to a map $\overline{\mathcal{C}}_{g,n+1} \to \overline{\mathcal{C}}_{g,n}$ and the section provided by the $(n+1)$-st marking $\overline{\mathcal{M}}_{g,n+1} \to \overline{\mathcal{C}}_{g,n+1}$ combine to an isomorphism $\overline{\mathcal{M}}_{g,n+1} \to \overline{\mathcal{C}}_{g,n}$. In other words, $\overline{\mathcal{M}}_{g,n+1}$ can be considered the universal curve for $\overline{\mathcal{M}}_{g,n}$.

(b) (Stable parametrized curves) Recall from Section 2.2 that if C is a smooth connected projective curve then a *C-parametrized curve* is a map $u : \hat{C} \to C$ of homology class $[C]$ from a nodal curve $\hat{C}$ to C, and is stable if it has only finitely many automorphisms. The category of nodal resp. stable C-parametrized curves forms an Artin stack $\overline{\mathfrak{M}}_n(C)$ resp. $\overline{\mathcal{M}}_n(C)$. More generally, for any rooted tree Γ we have Artin resp. Deligne-Mumford stacks $\overline{\mathfrak{M}}_{n,\Gamma}(C)$ resp. $\overline{\mathcal{M}}_{n,\Gamma}(C)$; the latter is a special case of the *Fulton-MacPherson compactification* studied in [17]. There is a morphism $\overline{\mathfrak{M}}_{n,\Gamma}(C) \to \overline{\mathcal{M}}_{n,\Gamma}(C)$ which collapses the unstable components. Indeed let $\pi : \hat{C} \to S, u : \hat{C} \to C$ be a family of C-parametrized n-marked nodal curves. Let L_C be an ample line bundle on C, and $\omega_{\hat{C}/S}$ denote the relative dualizing sheaf over S and $\omega_{\hat{C}/S}[z_1 + \ldots + z_n]$ its twisting by $z_1 + \ldots + z_n$. Consider the curve

$$\hat{C}^{st} = \operatorname{Proj} \oplus_{n \geq 0} \pi_*((\omega_{\hat{C}/S}[z_1 + \ldots + z_n] \otimes u^* L_C^{\otimes 3})^{\otimes n}). \tag{21}$$

For families arising by forgetting markings, $\hat{C}^{st}$ is obtained from $\hat{C}$ by collapsing unstable components and the formation of $\hat{C}^{st}$ commutes with base change. The general case is reduced to this one by adding markings locally [7, Section 3].

Any morphism of rooted trees $\Upsilon : \Gamma \to \Gamma'$ of the type collapsing an edge, cutting an edge, forgetting a tail induces a morphism $\overline{\mathfrak{M}}(\Upsilon) : \overline{\mathfrak{M}}_{n,\Gamma}(C) \to \overline{\mathfrak{M}}_{n,\Gamma'}(C)$ resp. $\overline{\mathcal{M}}(\Upsilon) : \overline{\mathcal{M}}_{n,\Gamma}(C) \to \overline{\mathcal{M}}_{n,\Gamma'}(C)$. In the case of forgetting a tail, the morphism $\overline{\mathcal{M}}(\Upsilon)$ can be defined by the composition of the inclusion

$\overline{\mathcal{M}}_{n,\Gamma}(C) \to \overline{\mathfrak{M}}_{n,\Gamma}(C)$, the map $\overline{\mathfrak{M}}(\Upsilon) : \overline{\mathfrak{M}}_{n,\Gamma}(C) \to \overline{\mathfrak{M}}_{n,\Gamma'}(C)$ followed by the collapsing map $\overline{\mathfrak{M}}_{n,\Gamma'}(C) \to \overline{\mathcal{M}}_{n,\Gamma'}(C)$.

(c) (Curves with scalings) Let S be an algebraic space over $\mathbb{C}$ and C a smooth projective nodal curve over S. Recall from e.g. [4, p.95] that the dualizing sheaf $\omega_{C/S}$ is locally free. Factor the projection $\pi : C \to S$ locally into the composition of a regular embedding $i : C \to R$ of relative dimension m and a smooth morphism $j : R \to S$ of relative dimension l. The normal sheaf $N_{C/R}$ of i is locally free of rank m, while the sheaf of relative Kähler differentials $\Omega^1_{R/S}$ is locally free of rank l. The relative dualizing sheaf of π is $\omega_{C/S} := (\Lambda^m N_{C/R})^{-1} \otimes \Lambda^l \Omega^1_{R/S}$. Explicitly, if C is a nodal curve over a point and $\tilde{C}$ denotes the normalization of C (the disjoint union of the irreducible components of C) with nodal points $\{\{w_1^+, w_1^-\}, \ldots, \{w_k^+, w_k^-\}\}$ then ω_C is the sheaf of sections of $\omega_{\tilde{C}} := T^\vee \tilde{C}$ whose residues at the points w_j^+, w_j^- sum to zero, for $j = 1, \ldots, k$. Denote by $\mathbb{P}(\omega_{C/S} \oplus \mathbb{C})$ the fiber bundle obtained by adding in a section at infinity. A *scaling* of C is a section λ of $\mathbb{P}(\omega_{C/S} \oplus \mathbb{C})$. The category of pairs (C, λ) is an Artin stack, with charts given by the forgetful morphisms from stable curves with additional marked points, equipped with scalings.

(d) (Stable scaled affine lines) Let S be an algebraic space over $\mathbb{C}$. A *nodal n-marked scaled affine line*, see [55, Section 2.3], consists of a smooth connected projective nodal curve C over S, an $(n+1)$-tuple $(z_0, \ldots, z_n)$ of sections $S \to C$ (the markings) distinct from the nodes and each other, and a scaling λ of $\mathbb{P}(\omega_{C/S} \oplus \mathbb{C})$, satisfying the following conditions:

   i. (Affine structure on each component on which it is non-degenerate) on each irreducible component C_i of C the form λ is either zero, infinite, or finite except for a single order two pole at a node of C.

   ii. (Monotonicity) on each non-self-crossing path of components from $z_i, i > 0$ to z_0, there is exactly one component on which λ is finite and non-zero; on the components before resp. after, λ vanishes resp. is infinite.

The first condition means that on the complement of the pole, if it exists, there is a canonical affine structure. An n-marked scaled curve is *stable* if it has no infinitesimal automorphisms, or equivalently, each component with degenerate resp. non-degenerate scaling has at least three resp. two special points. We denote by $\overline{\mathcal{M}}_{n,1}(\mathbb{A})$ resp. $\overline{\mathfrak{M}}_{n,1}(\mathbb{A})$ the stack of stable resp. nodal connected affine scaled n-marked curves; this is a proper complex variety resp. Artin stack. The former was constructed in [30]. Charts for the latter are given by forgetful morphisms $\overline{\mathcal{M}}_{m,1}(\mathbb{A}) \to \overline{\mathfrak{M}}_{n,1}(\mathbb{A})$ for $m > n$ defined by forgetting the last $m - n$ points, as in the case without scaling in Behrend-Manin [7].

We also wish to allow *twistings* at nodes of C with infinite scaling, see [40, Section 2] for a precise definition: the node has a cyclic automorphism group μ_r and there exist charts for neighborhoods of the node in each component of the form U/μ_r acting by inverse roots of unity. As in [40, Theorem 1.8], the category $\overline{\mathfrak{M}}^{\mathrm{tw}}_{n,1}(\mathbb{A})$ of scaled twisted marked curves is equivalent to the category of scaled log twisted marked curves, compatibly with base change, and this implies that $\overline{\mathfrak{M}}^{\mathrm{tw}}_{n,1}(\mathbb{A})$ is an Artin stack. For any colored tree Γ we denote by $\overline{\mathfrak{M}}^{\mathrm{tw}}_{n,1,\Gamma}(\mathbb{A})$ resp. $\overline{\mathcal{M}}^{\mathrm{tw}}_{n,1,\Gamma}(\mathbb{A})$ the stack of nodal resp. stable scaled n-marked affine lines of combinatorial type Γ.

There is a canonical morphism $\overline{\mathfrak{M}}_{n,1}(\mathbb{A}) \to \overline{\mathcal{M}}_{n,1}(\mathbb{A})$ defined as follows. Let $(C, \lambda, \underline{z})$ be a family of scaled affine lines over an algebraic space S. Let Λ denote the sheaf over C that assigns to an open subset $U \subset C$ the space of (possibly infinite) sections of $T^\vee U$ given by $f\lambda$ where $f \in \mathcal{O}_C(U)$ is regular on U. Thus Λ is rank one on the components where $\lambda \notin \{0, \infty\}$, and is rank zero otherwise. Denote the sum

$$\omega^\lambda_{C/S}[z_1 + \ldots + z_n] = \omega_{C/S}[z_1 + \ldots + z_n] + \Lambda.$$

In terms of the normalization $\tilde{C}_s$ of any fiber C_s, $\omega^\lambda_{C/S}[z_1 + \ldots + z_n]$ is the sheaf of relative differentials with poles at the markings, nodes, and an additional pole on any component with finite scaling at the node connecting with a component with infinite scaling. Consider the curve

$$C^{st} = \operatorname{Proj} \oplus_{n \geq 0} \pi_*((\omega^\lambda_{C/S}[z_1 + \ldots + z_n])^{\otimes n}). \tag{22}$$

In the case that C arises from a family obtained by forgetting a marked point on a stable scaled affine curve, C^{st} collapses unstable components and its formation commutes with base change. This construction collapses the bubbles that are unstable *furthest away from the root marking*, in particular, any colored component that becomes unstable after forgetting the marking. However, the adjacent component may be destabilized by collapse of this component; it is then necessary to apply the construction again to collapse this component. The forgetful morphism is produced by applying the Proj construction *twice*, in contrast to the case of stable curves where a single application suffices. The general case is reduced to this one by adding markings locally.

Any morphism of colored trees $\Upsilon : \Gamma \to \Gamma'$ of the type *collapsing an edge*, *collapsing edges with relations*, *cutting an edge*, *cutting an edge with relations* or *forgetting a tail* induces morphisms $\overline{\mathfrak{M}}(\Upsilon) : \overline{\mathfrak{M}}_{n,1,\Gamma}(\mathbb{A}) \to \overline{\mathfrak{M}}_{n,1,\Gamma'}(\mathbb{A})$ and $\overline{\mathcal{M}}(\Upsilon) : \overline{\mathcal{M}}_{n,1,\Gamma}(\mathbb{A}) \to \overline{\mathcal{M}}_{n,1,\Gamma'}(\mathbb{A})$. In the case of forgetting a tail, the morphism $\overline{\mathcal{M}}(\Upsilon)$ can be defined by the composition of the inclusion $\overline{\mathcal{M}}_{n,1,\Gamma}(\mathbb{A}) \to \overline{\mathfrak{M}}_{n,1,\Gamma}(\mathbb{A})$, the map $\overline{\mathfrak{M}}(\Upsilon) : \overline{\mathfrak{M}}_{n,1,\Gamma}(\mathbb{A}) \to \overline{\mathfrak{M}}_{n,1,\Gamma'}(\mathbb{A})$ followed by the collapsing map $\overline{\mathfrak{M}}_{n,1,\Gamma'}(\mathbb{A}) \to \overline{\mathcal{M}}_{n,1,\Gamma'}(\mathbb{A})$.

By its construction, the stack $\overline{\mathcal{M}}_{n,1}(\mathbb{A})$ has a *universal curve* $\overline{\mathcal{C}}_{n,1}(\mathbb{A}) \to \overline{\mathcal{M}}_{n,1}(\mathbb{A})$ equipped with universal scaling and markings. The forgetful morphism $\overline{\mathcal{M}}_{n+1,1}(\mathbb{A}) \to \overline{\mathcal{M}}_{n,1}(\mathbb{A})$ is isomorphic to the universal curve, as in the Knudsen case, given by the map $\overline{\mathcal{M}}_{n+1,1}(\mathbb{A}) \to \overline{\mathcal{C}}_{n,1}(\mathbb{A})$. The latter is defined by the n-marked curve (22) with section the $(n+1)$-st marked point. The inverse $\overline{\mathcal{C}}_{n,1}(\mathbb{A}) \to \overline{\mathcal{M}}_{n+1,1}(\mathbb{A})$ is defined by a consideration of various cases: If the extra marked point is a smooth point on a component with infinite scaling, distinct from the other markings, then one adds a bubble with finite scaling with the additional marking to the curve. If the extra marked point is a smooth point on a component with finite or zero scaling, distinct from the other marking, then one adds that point as an additional marking. If the extra marked point coincides with one of the other markings, or with a node, the one adds an additional bubble component with the appropriates scaling, and puts the additional marking on that component. This shows that the morphism $\overline{\mathcal{M}}_{n+1,1}(\mathbb{A}) \to \overline{\mathcal{C}}_n(\mathbb{A})$ induces a bijection of geometric points and is therefore (as a morphism of nodal curves over $\overline{\mathcal{M}}_{n,1}(\mathbb{A})$) an isomorphism.

More generally, one may consider stacks $\overline{\mathfrak{M}}_{n,s}(\mathbb{A})$ of s-scaled n-marked affine lines, that is, curves equipped with markings $z_1, \ldots, z_n$ and scalings $\lambda_1, \ldots, \lambda_s$. By similar arguments, these stacks are Artin and the stacks of stable curves $\overline{\mathcal{M}}_{n,s}(\mathbb{A})$ are Deligne-Mumford.

(e) (Stacks of scaled curves) A *family of nodal C-parametrized curves with finite scaling* consists of $\pi : \hat{C} \to S$ a family of nodal curves, $u : \hat{C} \to C$ a family of nodal maps of homology class $[C]$, a family of sections $z_1, \ldots, z_n : S \to \hat{C}$, and a section $\lambda : \hat{C} \to \mathbb{P}(\omega_{\hat{C}/C} \oplus \mathbb{C})$ of the projectivized relative dualizing sheaf (see 4.2 (c)) satisfying the following conditions:

i. (Finite on any marking) $\lambda(z_i)$ is finite;

ii. (Scaling on each bubble component) on each component C_i of $\hat{C}$ mapping to a point in C such that $\lambda | C_i$ is finite and non-zero, the restriction $\lambda | C_i$ has a unique pole of order 2, at the node connecting C_i with the principal component C_0; and

iii. (Monotonicity) on each non-self-crossing path of components from the principal component to the component containing $z_i, i > 0$, there is exactly one component on which λ is finite and non-zero; on the components before resp. after, λ vanishes resp. is infinite.

The category of such forms an Artin stack $\mathfrak{M}_{\Gamma,n,1}(C)$. There is a "forgetful morphism" from $\overline{\mathfrak{M}}_{n,1}(C)$ to $\overline{\mathfrak{M}}_n(C)$ which forgets the scaling. There is also a morphism $\overline{\mathfrak{M}}_{n,1}(C) \to \overline{\mathcal{M}}_{n,1}(C)$ collapsing the unstable components, whose construction is a combination of the previous cases and left to the reader. Similarly, $\overline{\mathfrak{M}}^{\mathrm{tw}}_{n,1}(C)$ is the stack of scaled marked curves with log structures at the nodes with infinite scaling as in [40, Section 2].

4.3. Stacks of morphisms

Many of our examples will arise as stacks of morphisms between stacks. Fix an algebraic space S. Let $\mathcal{X}$ and $\mathcal{Y}$ be Artin stacks over S. Let $\mathrm{Hom}_S(\mathcal{X}, \mathcal{Y})$ be the fibered category over the category of S-schemes, which to any $T \to S$ associates the groupoid of functors $\mathcal{X} \times_S T \to \mathcal{Y} \times_S T$. Unfortunately, there seems to be no general construction which guarantees that $\mathrm{Hom}_S(\mathcal{X}, \mathcal{Y})$ is an Artin stack, but partial results are given by Olsson [39], Romagny [44], and Lieblich [29, 2.3.4].

Example 4.3. (a) (Hom stacks between schemes) If X, Y are projective schemes over a Noetherian scheme S with X flat over S then $\mathrm{Hom}_S(X, Y)$ is representable by a quasiprojective S-scheme (a subscheme of the Hilbert scheme) by Grothendieck's construction of Hilbert schemes, described in [14].

(b) (Stable maps) Let $\overline{\mathfrak{M}}_{g,n}$ denote the stack of nodal curves with genus g and n markings from Example 4.2, $\overline{\mathfrak{C}}_{g,n} \to \overline{\mathfrak{M}}_{g,n}$ the universal curve, and X a projective variety. Then $\overline{\mathfrak{M}}_{g,n}(X) := \mathrm{Hom}_{\overline{\mathfrak{M}}_{g,n}}(\overline{\mathfrak{C}}_{g,n}, X)$ is the stack of *nodal (or prestable) maps to* X. The locus $\overline{\mathcal{M}}_{g,n}(X)$ of stable maps is defined as the sub-stack of maps with no infinitesimal automorphisms, or equivalently, such that each component on which the map is constant of genus zero (resp. one) has at least three resp. (one) special point. By the constructions in Behrend-Manin [7] and Fulton-Pandharipande [16], $\overline{\mathfrak{M}}_{g,n}(X)$ resp. $\overline{\mathcal{M}}_{g,n}(X)$ is an Artin resp. proper Deligne-Mumford stack. Similarly for any type Γ, let $\overline{\mathfrak{M}}_{g,n,\Gamma}(X) = \mathrm{Hom}_{\overline{\mathfrak{M}}_{g,n,\Gamma}}(\overline{\mathfrak{C}}_{g,n,\Gamma}, X)$ denote the compactified stack of maps of combinatorial type Γ and $\overline{\mathcal{M}}_{g,n,\Gamma}(X)$ the locus of stable maps. Then $\overline{\mathfrak{M}}_{g,n,\Gamma}(X)$ resp. $\overline{\mathcal{M}}_{g,n,\Gamma}(X)$ is an Artin resp. proper Deligne-Mumford stack. There is a canonical morphism from $\overline{\mathfrak{M}}_{g,n,\Gamma}(X)$ to $\overline{\mathcal{M}}_{g,n,\Gamma}(X)$ which collapses unstable components. Indeed, given a family $u : C \to X$ and an ample line bundle $L \to X$ consider the curve

$$C^{st} = \mathrm{Proj} \bigoplus_{n \geq 0} \pi_*(\omega_{C/S}[z_1 + \ldots + z_n] \otimes u^* L^3)^{\otimes n}. \tag{23}$$

For families arising from forgetting markings from a family of stable maps, C^{st} is obtained from C by collapsing unstable components, and the formation of C commutes with base change. The general case reduces to this one by adding markings locally [7].

Any morphism $\Upsilon : \Gamma \to \Gamma'$ of type cutting an edge, collapsing an edge, or forgetting a tail induces morphisms of moduli stacks

$$\overline{\mathfrak{M}}(\Upsilon, X) : \overline{\mathfrak{M}}_{g,n,\Gamma}(X) \to \overline{\mathfrak{M}}_{g,n,\Gamma'}(X), \quad \overline{\mathcal{M}}(\Upsilon, X) : \overline{\mathcal{M}}_{g,n,\Gamma}(X) \to \overline{\mathcal{M}}_{g,n,\Gamma'}(X).$$

In the first case the morphism is induced from fiber product with the morphism $\overline{\mathfrak{M}}(\Upsilon, X) : \overline{\mathfrak{M}}_{g,n,\Gamma} \to \overline{\mathfrak{M}}_{g,n,\Gamma'}$, while $\overline{\mathcal{M}}(\Upsilon, X)$ is defined by composing the inclusion $\overline{\mathcal{M}}(\Upsilon, X) \to \overline{\mathfrak{M}}(\Upsilon, X)$ with $\overline{\mathfrak{M}}(\Upsilon, X)$ and the collapsing morphism to $\overline{\mathcal{M}}_{g,n,\Gamma'}(X)$.

(c) (Stacks of bundles, second version) Let X be an S-scheme and G a reductive group. Let $\mathrm{Hom}(X, BG)$ be the category that assigns to $T \to S$ the groupoid of G-bundles on $X \times_S T$. Then $\mathrm{Hom}(X, BG)$ is a stack, naturally isomorphic to the stack $\mathrm{Bun}_G(X)$ of G-bundles. In particular, if X is a projective S-scheme then $\mathrm{Hom}(X, BG)$ is an Artin stack by Example 4.1.

(d) (Stacks of morphisms to quotient stacks) Let X be an algebraic space over S, G a reductive group and Y a G-scheme, and Y/G the quotient stack. Let $\mathrm{Hom}_S(X, Y/G)$ denote the category that assigns to any $T \to S$, a principal G-bundle P over $X \times_S T$ and a section $X \times_S T \to P \times_G Y$. By Lieblich [29, 2.3.4], $\mathrm{Hom}_S(X, Y/G)$ is an Artin stack. More generally if $f : \mathcal{X} \to \mathcal{Z}$ is a proper morphism of Artin stacks and Y is a separated and finitely-presented G-scheme then let $\mathrm{Hom}_{\mathcal{Z}}(\mathcal{X}, Y/G)$ be the fibered category that associates to any morphism $T \to \mathcal{Z}$ and object X of $\mathcal{X} \times_{\mathcal{Z}} T$ the category of pairs (P, u) where $P \to X$ is a principal G-bundle and section $u : X \to P \times_G Y$. By Olsson [2, Lemma C.5], $\operatorname{Hom}_{\mathcal{Z}}(\mathcal{X}, Y/G)$ is an Artin stack.

(e) (Stacks of morphisms to quotient stacks as quotients) In this example following Schmitt [46] we describe a realization of morphisms to quotients stacks as subschemes of Grothendieck's quot scheme discussed in Example 4.1. First suppose that C is a scheme over S, $G = GL(n)$ and $X = \mathbb{P}^{n-1}$. Any morphism $u : C \to X/G$ corresponds to a vector bundle $E \to C$ together with a section of the projectivization $\mathbb{P}(E)$. There are several equivalent descriptions of this data: (i) a vector bundle E and a line subbundle $L := u^*\mathcal{O}_{\mathbb{P}(E)}(1) \to C$, (ii) a vector bundle $E^\vee$, a line bundle $L^\vee$, and a surjective morphism $E^\vee \to L^\vee$. The latter datum is termed a *swamp* (short for *sheaf with map*) or more generally, a *bump* (short for *bundle with map*) if the group G is arbitrary reductive. Schmitt [46] shows that the functor from schemes to sets which associates to any scheme the set of isomorphism classes of stable bumps, can be realized as a git quotient of a quot scheme. The *type* of a bump is the pair of integers $(\deg(E), \deg(L))$. If S is an arbitrary scheme, then a bump over C parametrized by S with representation V consists of a principal G-bundle P on $S \times C$, a line bundle $L \to S \times C$, and a homomorphism φ from $P(V)$ to L. Since $\operatorname{Bun}_{\mathbb{C}^\times}(C)$ splits non-canonically as $\operatorname{Jac}(C) \times B\mathbb{C}^\times$, this data is equivalent to a morphism $S \to \operatorname{Jac}(C)$ together with a line bundle on the parameter space S, which is the formulation adopted in Schmitt [46].
The idea of Schmitt's construction of the moduli space of semistable bumps is as follows. After suitable twisting, we may assume that E is generated by its global sections, in which case $E^\vee$ is a quotient of a trivial vector bundle F and we obtain a double quotient $F \to E^\vee \to L^\vee$. This gives a quotient $F^2 \to E^\vee \times L^\vee$ with the property that the map to $L^\vee$ factors through $E^\vee$, and so a point in the quot scheme $\operatorname{Quot}_{F^2/X/S}$. Let $\mathfrak{M}^{G,\mathrm{quot},\mathrm{fr}}(C, X, F)$ denote the subscheme of $\operatorname{Quot}_{F^2/X/S}$ arising in this way, and let $\mathfrak{M}^{G,\mathrm{quot}}(C, X, F) = \mathfrak{M}^{G,\mathrm{quot},\mathrm{fr}} / \operatorname{Aut}(F)$ be the quotient by the action of the general linear group $\operatorname{Aut}(F)$. Let $\overline{\mathfrak{M}}^{G,\mathrm{quot}}(C, X, F)$ be its closure in $\operatorname{Quot}_{F^2/X/S} / \operatorname{Aut}(F)$. More generally, for any reductive group G and projective G-variety Y, a choice of representation $G \to GL(V)$ and embedding $Y \to \mathbb{P}(V)$ gives a stack $\overline{\mathfrak{M}}^{\mathrm{quot},G}(C, X, F)$ by encoding the reduction of structure group as a section and taking the closure of $\operatorname{Hom}(X, Y/G)$ (or rather, those maps whose bundles are quotients of F) in $\operatorname{Quot}_{F^2/X/S}$.

(f) (Inertia stacks) The *inertia stack* of a Deligne-Mumford (or Artin) stack $\mathcal{X}$ is

$$I_{\mathcal{X}} := \mathcal{X} \times_{\mathcal{X} \times \mathcal{X}} \mathcal{X}$$

where both maps are the diagonal. The objects of $I_{\mathcal{X}}$ may be identified with pairs (x, g) where x is an object of $\mathcal{X}$ and g is an automorphism of x. For example, if $\mathcal{X} = X/G$ is a global quotient by a finite group then

$$I_{\mathcal{X}} = \cup_{[g] \in G/Ad(G)} X^g / Z_g$$

where $G/\operatorname{Ad}(G)$ denotes the set of conjugacy classes in X and Z_g is the centralizer of g. There is also an interpretation as a hom stack (see e.g. [1])

$$I_{\mathcal{X}} = \cup_{r>0} I_{\mathcal{X},r}, \quad I_{\mathcal{X},r} := \operatorname{Hom}^{\mathrm{rep}}(B\mu_r, \mathcal{X}).$$

(g) (Rigidified inertia stacks) The following stack plays an important role in Gromov-Witten theory of Deligne-Mumford stacks as developed by Abramovich-Graber-Vistoli [1]. If μ_r is the group of r-th roots of unity then $B\mu_r$ is an Deligne-Mumford stack. If $\mathcal{X}$ is a Deligne-Mumford stack then

$$\overline{I}_{\mathcal{X}} = \cup_{r>0} \overline{I}_{\mathcal{X},r}, \quad \overline{I}_{\mathcal{X},r} := I_{\mathcal{X}/r}/B\mu_r.$$

is the *rigidified inertia stack* of representable morphisms from $B\mu_r$ to $\mathcal{X}$, see [1]. There is a canonical quotient cover $\pi : I_{\mathcal{X}} \to \overline{I}_{\mathcal{X}}$ which acts on cohomology as an isomorphism

$$\pi^* H^*(\overline{I}_{\mathcal{X}}, \mathbb{Q}) \to H^*(I_{\mathcal{X}}, \mathbb{Q})$$

so for the purposes of defining orbifold Gromov-Witten invariants, $\overline{I}_{\mathcal{X}}$ can be replaced by $I_{\mathcal{X}}$ at the cost of additional factors of r on the r-twisted sectors. If $\mathcal{X} = X/G$ is a global quotient of a scheme X by a finite group G then

$$\overline{I}_{X/G} = \coprod_{(g)} X^{\mathrm{ss},g}/(Z_g/\langle g\rangle)$$

where $\langle g\rangle \subset Z_g$ is the cyclic subgroup generated by g.

(h) (Rigidified inertia stacks for locally free git quotients) Suppose that X is a polarized smooth projective G-variety such that $X/\!\!/G$ is locally free. Then

$$I_{X/\!\!/G} = \coprod_{(g)} X^{\mathrm{ss},g}/Z_g$$

where $X^{\mathrm{ss},g}$ is the fixed point set of $g \in G$ on X^{ss}, Z_g is its centralizer, and the union is over all conjugacy classes,

$$\overline{I}_{X/\!\!/G} = \coprod_{(g)} X^{\mathrm{ss},g}/(Z_g/\langle g\rangle)$$

where $\langle g\rangle$ is the (finite) group generated by g.

(i) (Stacks of nodal gauged maps) Consider the Artin stack $\overline{\mathfrak{M}}_{g,n}$ of marked nodal curves and X/G the quotient stack associated to the quotient of a projective scheme X by a reductive group G. Then $\operatorname{Hom}_{\overline{\mathfrak{M}}_{g,n}}(\overline{\mathfrak{C}}_{g,n}, X/G)$ is an Artin stack.

(j) (Stacks of parametrized nodal gauged maps) Let C be a curve and X a G-scheme. Then $\operatorname{Hom}_{\overline{\mathfrak{M}}_n(C)}(\overline{\mathfrak{C}}_n(C), X/G)$ is the category that assigns to a morphism $T \to S$ the groupoid of marked nodal curves $\hat{C} \to T \times C$, of class $[C]$ on the second factor, equipped with a principal G-bundle $P \to \hat{C}$ and a morphism $C \to P \times_G X$.

(k) (Stacks of parametrized nodal affine gauged maps) Let X be a G-scheme. Then $\operatorname{Hom}_{\mathfrak{M}_{n,1}(\mathbb{A})}(\overline{\mathfrak{C}}_{n,1}(\mathbb{A}), X/G)$ is the category that assigns to a morphism $T \to S$ the groupoid of marked affine nodal curves $\hat{C} \to T$ equipped with a principal G-bundle $P \to \hat{C}$ and a morphism $\hat{C} \to P \times_G X$. More generally, $\operatorname{Hom}_{\overline{\mathfrak{M}}^{\mathrm{tw}}_{n,1}(\mathbb{A})}(\overline{\mathfrak{C}}^{\mathrm{tw}}_{n,1}(\mathbb{A}), X/G)$ is the hom-stack allowing orbifold singularities in the domain at the nodes with infinite scaling.

4.4. Twisted stable maps

We recall the definitions of twisted curve and twisted stable map to a Deligne-Mumford stack from Abramovich-Graber-Vistoli [1], Abramovich-Olsson-Vistoli [2], and Olsson [40]. These definitions are needed for the construction of the moduli stack of affine gauged maps in the case that $X/\!\!/G$ is an orbifold, but not if the quotient is free. Denote by μ_r the group of r-th roots of unity.

Definition 4.4. (Twisted curves) Let S be a scheme. An *n-marked twisted curve* over S is a collection of data $(f : \mathcal{C} \to S, \{\ddagger_i \subset \mathcal{C}\}_{i=1}^n)$ such that

(a) (Coarse moduli space) $\mathcal{C}$ is a proper stack over S whose geometric fibers are connected of dimension 1, and such that the coarse moduli space of $\mathcal{C}$ is a nodal curve C over S.
(b) (Markings) The $\ddagger_i \subset \mathcal{C}$ are closed substacks that are gerbes over S, and whose images in C are contained in the smooth locus of the morphism $C \to S$.
(c) (Automorphisms only at markings and nodes) If $\mathcal{C}^{ns} \subset \mathcal{C}$ denotes the *non-special locus* given as the complement of the $\ddagger_i$ and the singular locus of $\mathcal{C} \to S$, then $\mathcal{C}^{ns} \to C$ is an open immersion.
(d) (Local form at smooth points) If $p \to C$ is a geometric point mapping to a smooth point of C, then there exists an integer r, equal to 1 unless p is in the image of some $\ddagger_i$, an étale neighborhood $\operatorname{Spec}(R) \to C$ of p and an étale morphism $\operatorname{Spec}(R) \to \operatorname{Spec}_S(\mathcal{O}_S[x])$ such that the pull-back $\mathcal{C} \times_S \operatorname{Spec}(R)$ is isomorphic to $\operatorname{Spec}(R[z]/z^r = x)/\mu_r$.
(e) (Local form at nodal points) If $p \to C$ is a geometric point mapping to a node of C, then there exists an integer r, an étale neighborhood $\operatorname{Spec}(R) \to C$ of p and an étale morphism $\operatorname{Spec}(R) \to \operatorname{Spec}_S(\mathcal{O}_S[x, y]/(xy - t))$ for some $t \in \mathcal{O}_S$ such that the pull-back $\mathcal{C} \times_S \operatorname{Spec}(R)$ is isomorphic to $\operatorname{Spec}(R[z, w]/zw = t', z^r = x, w^r = y)/\mu_r$ for some $t' \in \mathcal{O}_S$.

Let $\mathcal{X}$ be a smooth Deligne-Mumford stack proper over a scheme S over a field of characteristic zero with projective coarse moduli space X, or an open subset thereof.

Definition 4.5. A *twisted stable map* from an n-marked twisted curve $(\pi : \mathcal{C} \to S, (\ddagger_i \subset \mathcal{C})_{i=1}^n)$ over S to $\mathcal{X}$ is a representable morphism of S-stacks $u : \mathcal{C} \to \mathcal{X}$ such that the induced morphism on coarse moduli spaces $u_c : C \to X$ is a stable map in the sense of Kontsevich [23] from the n-pointed curve $(C, \underline{z} = (z_1, \ldots, z_n))$ to X, where z_i is the image of $\ddagger_i$. The *homology class* of a twisted stable curve is the homology class $u_*[\mathcal{C}_s] \in H_2(X, \mathbb{Q})$ of any fiber $\mathcal{C}_s$.

Twisted stable maps naturally form a 2-category, but every 2-morphism is unique and invertible if it exists, and so this 2-category is naturally equivalent to a 1-category which forms a stack over schemes [1].

Theorem 4.6. *([1, 4.2]) The stack $\overline{\mathcal{M}}_{g,n}(\mathcal{X})$ of twisted stable maps from n-pointed genus g curves into $\mathcal{X}$ is a Deligne-Mumford stack. If $\mathcal{X}$ is proper, then for any $c > 0$ the union of substacks $\overline{\mathcal{M}}_{g,n}(\mathcal{X}, d)$ with homology class $d \in H_2(\mathcal{X}, \mathbb{Q})$ satisfying $(d, [\omega]) < c$ is proper.*

The proof uses the equivalence of the category of twisted curves with log-twisted curves. Let $\overline{I}_{\mathcal{X}}$ denote the rigidified inertia stack as in Proposition 4.3 (g). The moduli stack of twisted stable maps admits evaluation maps

$$\mathrm{ev} : \overline{\mathcal{M}}_{g,n}(\mathcal{X}) \to \overline{I}_{\mathcal{X}}^n, \quad \overline{\mathrm{ev}} : \overline{\mathcal{M}}_{g,n}(\mathcal{X}) \to \overline{I}_{\mathcal{X}}^n,$$

where the second is obtained by composing with the involution $\overline{I}_{\mathcal{X}} \to \overline{I}_{\mathcal{X}}$ induced by the map $\mu_r \to \mu_r, \zeta \mapsto \zeta^{-1}$. There is a modification of the definition which produces evaluation maps to the unrigidified moduli stacks: Let $\overline{\mathcal{M}}^{\mathrm{fr}}_{g,n}(\mathcal{X})$ denote the moduli space of *framed* twisted stable maps, that is, twisted stable maps with sections of the gerbes at the marked points [1]. These stacks are $\prod_{i=1}^n r_i$-fold covers of $\overline{\mathcal{M}}_{g,n}(\mathcal{X})$, where $r_i : \overline{\mathcal{M}}_{g,n}(\mathcal{X}) \to \mathbb{Z}_{\geq 0}$ is the order of the isotropy group at the i-th marking, and admit evaluation maps

$$\mathrm{ev}^{\mathrm{fr}} : \overline{\mathcal{M}}^{\mathrm{fr}}_{g,n}(\mathcal{X}) \to I^n_{\mathcal{X}}, \quad \overline{\mathrm{ev}}^{\mathrm{fr}} : \overline{\mathcal{M}}^{\mathrm{fr}}_{g,n}(\mathcal{X}) \to I^n_{\mathcal{X}}.$$

If $\mathcal{C}$ is a finite disjoint union of twisted curves, then a stable map from $\mathcal{C}$ to $\mathcal{X}$ is a stable map of each of its components. For any possibly disconnected combinatorial type Γ, we denote by $\overline{\mathcal{M}}_{g,n,\Gamma}(\mathcal{X})$ resp. $\overline{\mathcal{M}}^{\mathrm{fr}}_{g,n,\Gamma}(\mathcal{X})$ the stack of stable maps resp. framed stable maps whose underlying stable map of schemes has combinatorial type Γ.

Proposition 4.7. (a) (Cutting an edge) If Γ' is obtained from Γ by cutting an edge, there is a morphism

$$\mathcal{G}(\Upsilon, X) : \overline{\mathcal{M}}_{g,n,\Gamma'}(\mathcal{X}) \times_{\overline{I}^2_{\mathcal{X}}} \overline{I}_{\mathcal{X}} \to \overline{\mathcal{M}}_{g,n,\Gamma}(\mathcal{X}) \tag{24}$$

where the second morphism in the fiber product is the diagonal $\Delta : \overline{I}_{\mathcal{X}} \to \overline{I}^2_{\mathcal{X}}$, and an isomorphism

$$\overline{\mathcal{M}}^{\mathrm{fr}}_{g,n,\Gamma'}(\mathcal{X}) \times_{I^2_{\mathcal{X}}} I_{\mathcal{X}} \to \overline{\mathcal{M}}^{\mathrm{fr}}_{g,n,\Gamma}(\mathcal{X}).$$

(b) (Collapsing an edge) If Γ' is obtained from Γ by collapsing an edge then there is an isomorphism

$$\overline{\mathcal{M}}_{g,n,\Gamma'}(\mathcal{X}) \times_{\overline{\mathfrak{M}}_{n,\Gamma'}} \overline{\mathfrak{M}}_{g,n,\Gamma} \to \overline{\mathcal{M}}_{g,n,\Gamma}(\mathcal{X})$$

and similarly for framed twisted stable maps.

Example 4.8. (Inertia stacks for toric orbifolds) Consider a stack $\mathcal{X} = X /\!\!/ G$ obtained as the quotient of a vector space X by a torus G with weights $\mu_1, \ldots, \mu_k$ at a weight $\nu \in \mathfrak{g}^\vee$, see (30). For each subset $\{\mu_i, i \in I\}$ with $\nu \in \operatorname{span}\{\mu_i, i \in I\}$, let Λ_I denote the lattice generated by the $\mu_i, i \in I$, and $G_I = \exp(\Lambda_I^\vee)$ the subgroup generated by the dual lattice. Let X_I denote the span of the weight spaces for $\mu_i, i \in I$ and $X_I /\!\!/ G$ the git quotient of X_I by G. For any element $g \in G$ let $I(g)$ denote the set of i such that $g \in \ker(\exp(\mu_i) : G \to \mathbb{C}^\times)$. Then

$$I_{X /\!\!/ G} = \cup_{g \in G} (X_{I(g)} /\!\!/ G)$$

(finite by the previous paragraph) and is therefore a union of toric stacks. For each $i \in I(g)$, vanishing of the coordinate in $X_{I(g)}$ corresponding to i defines a divisor $\tilde{D}_i$, whose (possibly empty) image in $X_{I(g)}$ is a divisor $D_{i,g}$. The cohomology of $I_{X /\!\!/ G}$ is generated by the classes of the divisors $D_{i,g}, i \in I(g)$.

4.5. Coarse moduli spaces

A *coarse moduli space* for a stack $\mathcal{X}$ is an algebraic space X together with a morphism $\pi : \mathcal{X} \to X$ such that π induces a bijection between the geometric points of X and $\mathcal{X}$ and π is universal for maps to algebraic spaces. By a theorem of Keel-Mori [22], coarse moduli spaces for Artin stacks with finite inertia (in particular, Deligne-Mumford stacks in characteristic zero) exist.

Proposition 4.9. A Deligne-Mumford stack over $\mathbb{C}$ is proper iff its coarse moduli space is proper iff its coarse moduli space is compact and Hausdorff in the analytic topology.

Proof. The first equivalence is e.g. [39, 2.10]. The second equivalence is folklore, see for example [41, Theorem 3.17].

Artin [5] has given conditions for a category fibered in groupoids to be an Artin stack. In particular, each object should admit a versal deformation, universal if the stack is Deligne-Mumford. Versal deformations give a notion of *topological convergence* of a sequence of objects in the category, defined if the corresponding sequence of points s_ν in the parameter space S for a versal deformation converges to a point s, in which case the limit is the equivalence class of objects corresponding to s. In particular, these notions define the underlying C^0 topology of the coarse moduli spaces.

Example 4.10.
(a) (Convergence of nodal curves) Any projective nodal curve has a versal deformation given in the analytic category by a gluing construction in which small balls around the nodes are removed and the components glued together via maps $z \mapsto \delta/z$ [4, p. 176] in local coordinates z near the node.
(b) (Convergence of stable maps) Any map from a projective nodal curve to a projective variety has a versal deformation given by considering its graph as an element in a suitable Hilbert scheme of subvarieties, see for example [14]. The construction of the Hilbert scheme is reduced to the construction of a Quot scheme, which in turn reduces to representability of the

Grassmannian. For the Grassmannian topological convergence of a subbundle implies topological convergence in the sense described above for (uni)versal deformations. It follows that topological convergence for maps is the usual notion of convergence of stable maps discussed in, for example, McDuff-Salamon [31].

(c) (Convergence of bundles) Any vector bundle over a curve has a versal deformation given by considering it, after twisting by a sufficiently positive line bundle, as a quotient of a trivial bundle. A similar statement holds for principal bundles for reductive groups by considering them as vector bundles with reductions. Topological convergence of a sequence of isomorphism classes of bundles is the usual notion of topological convergence of holomorphic bundles, that is, C^0 convergence of the corresponding holomorphic structures on the components after complex gauge transformation.

(d) (Convergence of isomorphism classes of vector bundles) In particular, let C be a smooth projective curve and $\mathcal{M} = \mathrm{Bun}^{\mathrm{ss}}_{GL(r)}(C)$ the moduli stack of semistable bundles of rank $r \geq 0$. By a theorem of Narasimhan-Seshadri [35], if stable=semistable then the coarse moduli space M for $\mathcal{M}$ admits a homeomorphism ϕ to its image in the moduli space of unitary representations of the fundamental group $R = \mathrm{Hom}(\pi_1(C), U(r))/U(r)$, where $\mathrm{Hom}(\pi_1(C), U(r))$ denotes the topological space of representations of $\pi_1(C)$ in $U(r)$. The Hilbert scheme construction, or the construction of universal families in [35], shows that inverse map $R \to M$ is continuous.

5. Stable gauged maps

In this section we identify the moduli space of symplectic vortices as the coarse moduli space of a substack of the moduli stack of gauged maps satisfying a semistability condition introduced by Mundet [33] and further studied by Schmitt [45], [46]. This correspondence of Hitchin-Kobayashi type implies that the moduli space of symplectic vortices, if every vortex has finite automorphism group, is the moduli space of a proper Deligne-Mumford stack.

5.1. Gauged maps

Let G be a complex reductive group, X be a smooth projective G-variety and C a smooth connected projective curve. In this section we construct the stack $\overline{\mathfrak{M}}^G_n(C, X)$ of n-marked gauged maps for integers $n \geq 0$.

Definition 5.1. An *n-marked nodal gauged map* from C to X over a scheme S is a morphism $u : \hat{C} \to C \times X/G$ from a nodal curve $\hat{C}$ over S whose projection onto the first factor has homology class $[C]$, such that if $C_{i,s} \subset \hat{C}_s$ is a component that maps to a point in C, then the bundle corresponding to $u|C_i$ is trivial. More explicitly, such a morphism is given by a datum $(\hat{C}, P, u, \underline{z})$ where

(a) (Nodal curve) $\hat{C} \to S$ is a proper flat morphism with reduced nodal curves as fibers;

(b) (Bundle over the principal component) $P \to C \times S$ is a principal G-bundle;

(c) (Section of the associated fiber bundle) $u : \hat{C} \to P(X) := (P \times X)/G$ is a family of stable maps with base class $[C]$, that is, the composition of u

with the projection $P(X) \to C$ has class $[C]$.

A *morphism* between gauged maps $(S, \hat{C}, P, u)$ and $(S', \hat{C}', P', u')$ consists of a morphism $\beta : S \to S'$, a morphism $\phi : P \to (\beta \times 1)^* P'$, and a morphism $\psi : \hat{C} \to \hat{C}'$ such that the first diagram below is Cartesian and the second and third commute:

$$
\begin{array}{ccc}
\hat{C} & \longrightarrow & S \\
\downarrow{\scriptstyle \psi} & & \downarrow{\scriptstyle \beta} \\
\hat{C}' & \longrightarrow & S'
\end{array}
\qquad
\begin{array}{ccc}
P & \longrightarrow & S \times C \\
\downarrow{\scriptstyle \phi} & & \downarrow{\scriptstyle \mathrm{id}} \\
(\beta \times 1)^* P' & \longrightarrow & S \times C
\end{array}
\qquad
\begin{array}{ccc}
\hat{C} & \xrightarrow{u} & P(X) \\
\downarrow{\scriptstyle \psi} & & \downarrow{\scriptstyle [\phi \times \mathrm{id}_X]} \\
\hat{C}' & \xrightarrow[u']{} & P'(X).
\end{array}
$$

An *n-marked* nodal gauged map is equipped with an n-tuple $(z_1, \ldots, z_n) \in \hat{C}^n$ of distinct smooth points on $\hat{C}$.

Let $\overline{\mathfrak{M}}_n^G(C, X)$ denote the category of n-marked nodal gauged maps, $\overline{\mathfrak{M}}_n^{G,st}(C, X)$ the subcategory where $u : \hat{C} \to P(X)$ is a stable map, and $\mathfrak{M}_n^G(C, X)$ the subcategory where $\hat{C} \to C$ is an isomorphism, that is, the domain is irreducible. The functor from $\overline{\mathfrak{M}}_n^G(C, X)$ to schemes which assigns to any datum $(S, C, P, u, \underline{z})$ the base scheme S makes $\overline{\mathfrak{M}}_n^G(C, X)$ resp. $\overline{\mathfrak{M}}_n^{G,st}(C, X)$ resp. $\mathfrak{M}_n^G(C, X)$ into a category fibered in groupoids. We denote by $\overline{\mathfrak{C}}_n^G(C, X) \to \overline{\mathfrak{M}}_n^G(C, X)$ the universal curve, consisting of a datum $(P \to C, u : \hat{C} \to P(X), \underline{z} : S \to \hat{C}^n, z' : S \to \hat{C})$ with z not necessarily mapping to the smooth locus of $\hat{C}$. The universal curve maps canonically to X/G via evaluation at z':

$$\overline{\mathfrak{C}}_n^G(C, X) \to X/G, \quad (S, C, P, \underline{z}, z') \mapsto (S, (\pi \circ u \circ z')^* P, u \circ z').$$

Theorem 5.2. *$\overline{\mathfrak{M}}_n^G(C, X)$ resp. $\overline{\mathfrak{M}}_n^{G,st}(C, X)$ resp. $\mathfrak{M}_n^G(C, X)$ is a (non-finite-type, non-separated) Artin stack.*

Proof. If $\overline{\mathfrak{C}}_n(C) \to \overline{\mathfrak{M}}_n(C)$ is the universal curve, $\mathrm{Hom}_{\overline{\mathfrak{M}}_n(C)}(\overline{\mathfrak{C}}_n(C), X/G)$ is an Artin stack by the results of Section 4.3 (d). The stack $\overline{\mathfrak{M}}_n^G(C, X)$ is the substack of $\mathrm{Hom}_{\overline{\mathfrak{M}}_n(C)}(\overline{\mathfrak{C}}_n(C), X/G)$ corresponding to morphisms $f : \hat{C} \to X/G$ such that on each component $\hat{C}_i$ mapping to a point in C, the principal G-bundle $P_i \to \hat{C}_i$ defined by f is trivial. Since triviality on the bubbles is an open condition, $\overline{\mathfrak{M}}_n^G(C, X)$ is an Artin stack as well. The condition that $u : \hat{C} \to P(X)$ is stable (has no infinitesimal automorphisms) is an open condition, hence $\overline{\mathfrak{M}}_n^{G,st}(C, X)$ is an open substack, hence also an Artin stack. Similarly the locus $\mathfrak{M}_n^G(C, X)$ where $\hat{C} \cong C$ is open and so also Artin.

Lemma 5.3. (Existence of a morphism collapsing unstable components) *There is a morphism $\overline{\mathfrak{M}}_n^G(C, X) \to \overline{\mathfrak{M}}_n^{G,st}(C, X)$ collapsing unstable components. The composition $\overline{\mathfrak{M}}_n^{G,st}(C, X) \to \overline{\mathfrak{M}}_{n-1}^G(C, X) \to \overline{\mathfrak{M}}_{n-1}^{G,st}(C, X)$ collapsing unstable components is isomorphic to the universal curve $\overline{\mathfrak{C}}_{n-1}^G(C, X) \to \overline{\mathfrak{M}}_{n-1}^G(C, X)$, and in particular proper.*

Proof. Let $\pi : \hat{C} \to S$ be a nodal curve with dualizing sheaf $\omega_{\hat{C}/S}$, a morphism $u : \hat{C} \to C \times P(X)$, an ample G-line bundle $L \to X$, and an ample line bundle $L_C \to C$. The formation of the curve

$$\hat{C}^{st} = \operatorname{Proj} \bigoplus_{n \geq 0} \pi_*(\omega_{\hat{C}/S}[z_1 + \ldots + z_n] \otimes u^*(L_C \boxtimes P(L))^{\otimes 3})^{\otimes n}$$

commutes with base change, in the case that the family arises from a stable family by forgetting a marking. Then u factors through $\hat{C}^{st}$ and this gives the required family in this case. The general case reduces to this one by adding markings locally, see Behrend-Manin [7, Theorem 3.10] and [7, Proposition 4.6].

5.2. Mundet stability

Gauged maps corresponding to solutions of the vortex equations correspond to maps satisfying a semistability condition introduced by Mundet [33]. In this section we construct the stack $\overline{\mathcal{M}}_n^G(C, X)$ of Mundet-semistable gauged maps. These are used later to define gauged Gromov-Witten invariants.

First recall some terminology from the study of moduli spaces of G-bundles, from Ramanathan [43]. We restrict here to the case that G is connected. A subgroup $R \subset G$ is *parabolic* if G/R is complete. Given a parabolic subgroup R, the maximal reductive *Levi subgroup* $L \subset R$ is unique up to conjugation by elements of R. The parabolic R admits a decomposition $R = LU$ where U is a maximal unipotent subgroup. The quotient map will be denoted $p : R \to R/U \cong L$ and the inclusion $i : L \to G$. A *parabolic reduction* of a bundle P to R is a section $\sigma : C \to P/R$.

Definition 5.4. (Associated Graded Bundle) Let P be a principal G-bundle on a curve C.

(a) (As an induced bundle) Given a parabolic reduction $\sigma : C \to P/R$, let $\sigma^* P$ denote the associated R bundle, $p_* \sigma^* P$ the associated L-bundle, and $j : R \to G$ the inclusion. The bundle $\operatorname{Gr}(P) := j_* p_* \sigma^* P$ is the *associated graded* G-bundle for σ.
(b) (As a degeneration) Let $\sigma : C \to P/R$ be parabolic reduction, $Z(L)$ denote the center of the Levi subgroup L, $\mathfrak{z}(\mathfrak{l})$ its Lie algebra, and $\lambda \in \mathfrak{z}(\mathfrak{l})$ a generic antidominant coweight (with respect to the roots of the Lie algebra $\mathfrak{p}$ of P restricted to $\mathfrak{z}(\mathfrak{l})$). For $z \in \mathbb{C}^\times$, the induced family of automorphisms ϕ of R by $z^\lambda = \exp(\ln(z)\lambda)$ by conjugation induces a family of bundles $P_{\sigma,\lambda} := j_*((\sigma^* P \times \mathbb{C}) \times_\phi R)$ over $C \times \mathbb{C}$ with central fiber $\operatorname{Gr}(P)$.

Example 5.5. (Associated graded bundles for vector bundles) If $G = GL(n)$, then a parabolic reduction is equivalent to a filtration of the associated vector bundle and $\operatorname{Gr}(P)$ is the frame bundle of the associated graded vector bundle, see [43]. The degeneration in the second definition above is the one that deforms the "off diagonal" parts of the transition maps of P to zero.

Definition 5.6. (Associated Graded Section) Given a section $u : C \to P(X)$, define the *associated graded section* $\operatorname{Gr}(u) : \hat{C} \to (\operatorname{Gr}(P))(X)$ associated to (σ, λ) as the unique stable limit u_0 of the sections u_z of $P_{\sigma,\lambda}|_{C\times\{z\}}(X)$ given by acting on u by z^λ.

The Mundet stability condition is a collection of inequalities given by integrals over the curve C, analogous to the definition of stability of vector bundles by degrees of sub-bundles. Suppose λ is a weight of $Z(L)$ and so defines a one-dimensional representation $\mathbb{C}_\lambda$. Via the trivialization $\mathfrak{z}(\mathfrak{l}) \cong (p_*\sigma^*P)(\mathfrak{z}(\mathfrak{l})) \subset (\operatorname{Gr}(P))(\mathfrak{g})$ the element λ defines an infinitesimal automorphism of $\operatorname{Gr}(P)$, fixing the principal component $\operatorname{Gr}(u)_0$ of $\operatorname{Gr}(u)$. The polarization $\mathcal{O}_X(1)$ defines a line bundle $P(\mathcal{O}_X(1)) \to P(X)$ and the infinitesimal automorphism defined by λ acts on the fibers over $\operatorname{Gr}(u)_0$ with a weight $\mu_X(\operatorname{Gr}(u)_0,\lambda)$.

Definition 5.7. (Mundet weight) The *Mundet weight* of the pair (σ,λ) as above is defined by

$$\mu(\sigma,\lambda) = \int_{[C]} c_1(p_*\sigma^*P \times_L \mathbb{C}_{-\lambda}) + \mu_X(\operatorname{Gr}(u)_0,-\lambda)[\omega_C]. \tag{25}$$

A gauged map (P,u) is *Mundet stable* iff it satisfies the

$$\text{(Weight Condition)} \quad \mu(\sigma,\lambda) < 0 \tag{26}$$

for all (σ,λ), *Mundet unstable* if there exists a *de-stabilizing pair* (σ,λ) violating (26) with strict inequality, *Mundet semistable* if it is not unstable, and *Mundet polystable* if it is semistable but not stable and (P,u) is isomorphic to its associated graded for any pair (σ,λ) satisfying the above with equality. A gauged map is *semistable* if it is Mundet semistable with stable section, and *stable* if it is semistable and has finite automorphism group.

Remark 5.8. (a) (Connection with stability of bundles) In the case that X is trivial and G is semisimple, Mundet stability is the same as Ramanathan stability of principal G-bundles [43].

(b) (Definition in terms of the moment map) If $P(K)$ is a smooth principal K-bundle so that $P = P(K)\times_K G$ is a smooth principal G-bundle, then via the correspondence between complex structures on $P(G)$ and connections on P we may view $\operatorname{Gr}(P)$ as a limiting connection on $P(K)$, and the section $\operatorname{Gr}(u)$ as a stable section of $P(K)\times_K X$. Then the weight $\mu_X(\operatorname{Gr}(u)_0,\lambda)$ can be expressed in terms of the moment map as

$$\mu_X(\operatorname{Gr}(u)_0,\lambda) = ((P(K))(\Phi)\circ \operatorname{Gr}(u)_0,\lambda),$$

by the usual correspondence between moment maps and linearizations of actions.

(c) (Dependence on choices) The stability condition depends on the cohomology class $[\omega_C] \in H^2(C)$, in addition to the metric on $\mathfrak{k}$ and the choice of moment map (or polarization) on X. Rescaling the metric on $\mathfrak{k}$ is equivalent to rescaling $[\omega_C]$ or to rescaling the moment map. Allowing a varying curve C equipped with a cohomology class $[\omega_C]$ leads to various properness issues, see Mundet-Tian [34].

(d) (Comparison with Mundet's definition) We have chosen the definition to generalize that of Ramanathan [43] for principal G-bundles. Mundet's definition in [33, Section 4] is slightly different: For a parabolic reduction σ and possibly irrational antidominant $\lambda \in \mathfrak{z}$, identified with an infinitesimal gauge transformation,

$$\mu(\sigma, \lambda) = \inf_t \int_C (F_{e^{it\lambda} A}, -\lambda) + ((e^{it\lambda} u)^* P(\Phi), -\lambda)\omega_C.$$

Then $\mu(\sigma, \lambda)$ agrees with the previous definition in the case that λ is a coweight, since in this case the infimum equals the limit as $t \to -\infty$, the right-hand-side of (25). To see that the two definitions are the same, it suffices to check that if (26) is violated by some irrational λ it is also violated for rational λ. For λ' sufficiently close to λ and defining the same parabolic reduction, we have

$$\lim_{t\to\infty} e^{it\lambda} A = \lim_{t\to\infty} e^{it\lambda'} A =: A_\infty, \quad \lim_{t\to\infty} F_{e^{it\lambda} A} = \lim_{t\to\infty} F_{e^{it\lambda'} A} = F_{A_\infty} \quad (27)$$

uniformly in all derivatives. Furthermore, by Gromov compactness $e^{it\lambda} u$ Gromov converges to some limit $u_\infty : \hat{C} \to P(X)$ as $t \to \infty$, with principal component $u_{0,\infty} : \hat{C} \to P(X)$ and

$$\lim_{t\to\infty} \int_C ((e^{it\lambda} u)^* P(\Phi), -\lambda)\omega_C \to \int_C (u_{0,\infty}^* P(\Phi), -\lambda)\omega_C.$$

Let $\lambda' \in \mathfrak{k}(P)_{(A_\infty, u_\infty)}$ commute with λ. It follows from the local slice theorem for holomorphic actions that if in addition λ' is sufficiently close to λ then for z not in the bubbling set

$$\lim_{t\to\infty} e^{it\lambda'} u(z) = u_{0,\infty}(z) = \lim_{t\to\infty} e^{it\lambda'} u(z). \quad (28)$$

Indeed after passing to a maximal torus containing both λ, λ' the equation (28) holds iff the weights for the action at $u_{0,\infty}(z)$ have the same sign on λ and λ'. Since rational Lie algebra vectors are dense in the Lie algebra of any closed subgroup, we may find λ' rational satisfying (27),(28) and so violating semistability if λ does. Mundet also allows a correction coming from the center of $\mathfrak{g}$ on the right-hand-side of (26), so that in the case X trivial and $G = GL(n)$ the definition of semistability agrees with that for vector bundles.

Theorem 5.9 below gives the equivalence of the stability condition with the existence of a solution to the vortex equations. We denote by $\mathcal{G}(P)$ the group of *complex gauge transformations* of P. There is a one-to-one correspondence between the space $\mathcal{J}(P(G))$ of complex structures on $P(G)$ and connections $\mathcal{A}(P)$ on P. The identification $\mathcal{A}(P) \to \mathcal{J}(P(G))$ is equivariant for the action of $\mathcal{K}(P)$, in the sense that $J_{kA} = Dk \circ J_A \circ Dk^{-1}$. Thus, the identification defines an extension of the $\mathcal{K}(P)$ action on $\mathcal{A}(P)$ to an action of $\mathcal{G}(P)$. The group $\mathcal{G}(P)$ acts on $\mathcal{H}(P, X)$ by composition on the second factor. A pair (A, u) is *simple* if no element of $\mathcal{G}(P)$ semisimple at each point in C leaves (A, u) fixed, see [33, Definition 2.17].

Theorem 5.9 (Mundet's Hitchin-Kobayashi correspondence [33]). *Let $P \to C$ be a principal K-bundle. A simple pair $(A,u) \in \mathcal{H}(P,X)$ defines a Mundet-stable gauged map if and only if there exists a complex gauge transformation $g \in \mathcal{G}(P)$ such that $g(A,u)$ is a vortex.*

Remark 5.10. (Analytic Mundet stability) Mundet's proof depends on the convexity of the functional $\mathcal{I}(P)$ (depending on the choice of (A,u)) obtained by integrating the one form determined by the moment map

$$\mathcal{I}(P) : \mathcal{G}(P)/\mathcal{K}(P) \to \mathbb{R}, \quad [\exp(it\xi)] \mapsto \int_0^1 \langle F_{\exp(it\xi)(A,u)}, \xi \rangle \mathrm{d}t. \tag{29}$$

If (A,u) is complex gauge equivalent to a symplectic vortex, then $\mathcal{I}(P)$ is bounded from below. On the other hand, if $\mathcal{I}(P)$ is not bounded from below then Mundet (using previous results of Uhlenbeck-Yau [50]) constructs a direction $\xi \in \mathfrak{k}(P)$ in which $\lim_{t\to\infty} \exp(-it\xi)(A,u)$ exists and

$$\lim_{t\to\infty} (F_{\exp(-it\xi)(A,u)}, \xi) \geq 0$$

and shows that the corresponding parabolic reduction violates the stability condition. A pair (A,u) is Mundet unstable resp. semistable resp. stable iff the Mundet functional $\mathcal{I}(P)$ is not bounded from below resp. bounded from below resp. attains its minimum in $\mathcal{G}(P)/\mathcal{K}(P)$.

The algebraic moduli spaces arising from the Mundet semistability condition are investigated in Schmitt [46]. Let $\mathcal{M}_n^G(C,X) \subset \mathfrak{M}_n^{G,st}(C,X)$ denote the category of n-marked gauged maps to X with irreducible domain that are Mundet semistable. We wish to show that $\mathcal{M}_n^G(C,X)$ is an Artin stack, for which it suffices to show that the semistability condition is open. Recall from Section 4.3 (d) Schmitt's compactification of the moduli space of gauged maps by *projective bumps*. Schmitt [46] defines a semistability condition for projective bumps which generalizes Mundet semistability for gauged maps. Let $\overline{\mathfrak{M}}^{G,\mathrm{quot}}(C,X,F)$ denote the moduli stack of projective bumps from 4.3 (e), let $\overline{\mathcal{M}}^{G,\mathrm{quot}}(C,X,F)$ denote the subcategory of $\overline{\mathfrak{M}}^{G,\mathrm{quot}}(C,X,F)$ consisting of families of Mundet semistable bumps. For $d \in H_2^G(X,\mathbb{Z})$ and let $\overline{\mathcal{M}}^{G,\mathrm{quot}}(C,X,F,d)$ denote the moduli substack of $\overline{\mathcal{M}}^{G,\mathrm{quot}}(C,X,F)$ of semistable bumps with class d. The semistable locus $\overline{\mathcal{M}}^{G,\mathrm{quot}}(C,X,F)$ is independent of F for F of sufficiently large rank, by [46, Proposition 2.7.2.9], and will be denoted $\overline{\mathcal{M}}^{G,\mathrm{quot}}(C,X)$. Recall that X is equipped with the equivariant class $[\omega_{X,G}] \in H_G^2(X)$.

Theorem 5.11. *Let X be a smooth polarized projective G-variety. The moduli stack of projective bumps $\overline{\mathfrak{M}}^{G,\mathrm{quot}}(C,X,F)$ resp. Mundet semistable projective bumps $\overline{\mathcal{M}}^{G,\mathrm{quot}}(C,X,F)$ is an Artin stack locally of finite type, containing $\mathcal{M}^G(C,X)$ as an open substack. More precisely each $\overline{\mathfrak{M}}^{G,\mathrm{quot}}(C,X,F,d)$ resp. $\overline{\mathcal{M}}^{G,\mathrm{quot}}(C,X,F,d)$ has a presentation as a quotient of a closed subscheme of a quot scheme resp. semistable locus in a closed subscheme of a quot scheme. If stable=semistable for projective bumps then for each constant $c > 0$, the union of components $\overline{\mathcal{M}}^{G,\mathrm{quot}}(C,X,F,d)$ with $(d,[\omega_{X,G}]) < c$ is a proper Deligne-Mumford stack with projective coarse moduli space.*

Proof. Schmitt [46] avoids the language of stacks, but the construction is the same: $\overline{\mathfrak{M}}^{G,\mathrm{quot}}(C,X,F)$ is the quotient of a rigidified moduli space $\overline{\mathfrak{M}}^{G,\mathrm{quot},\mathrm{fr}}(C,X,F)$, and $\overline{\mathcal{M}}^{G,\mathrm{quot}}(C,X,F)$ is the quotient of the git semistable locus [46, Theorem 2.7.1.4]. The necessary local quot scheme is constructed as follows, in the case $G = GL(n)$. Let $E \to C$ be a vector bundle and $u : E \to L$ a quotient corresponding to a section of $\mathbb{P}(E^\vee)$. After suitable twisting, we may assume that E is generated by its global sections, in which case E is a quotient of a trivial vector bundle F. We then obtain a double quotient $F \to E \to L$. Such a double quotient can be considered as a quotient $F^2 \to E \times L$. Let $\mathfrak{M}^{G,\mathrm{fr},\mathrm{quot}}(C,X,F)$ denote the open subscheme of the quot scheme $\mathrm{Quot}_{F^2/C}$ consisting of such quotients. Let $\mathfrak{M}^{G,\mathrm{quot}}(C,X,F) = \mathfrak{M}^{G,\mathrm{fr},\mathrm{quot}}(C,X,F)/\operatorname{Aut}(F)$ be the quotient stack by the action of the general linear group $\operatorname{Aut}(F)$. Let $\overline{\mathfrak{M}}^{G,\mathrm{quot}}(C,X,F)$ its closure in $\mathrm{Quot}_{F^2/C}/\operatorname{Aut}(F)$. Schmitt [46, Section 2.7] shows that a suitable canonical polarization on $\mathrm{Quot}_{F^2/C}$ gives a semistability condition which reproduces Mundet semistability. The git construction shows that each substack $\overline{\mathcal{M}}^{G,\mathrm{quot}}(C,X,F,d)$ is proper. On the other hand, the set of classes d such that $(d,[\omega_{X,G}]) < c$ and $\overline{\mathcal{M}}^{G,\mathrm{quot}}(C,X,F,d)$ is non-empty, is finite, since, as one may check, $(d,[\omega_{X,G}])$ is the degree of the line bundle L in Schmitt's construction.

On the other hand, there is a natural Kontsevich-style moduli space which allows bubbling in the fibers satisfying a stability condition. Denote by $\overline{\mathcal{M}}_n^{\mathrm{pre},G}(C,X)$ the category of n-marked nodal gauged maps (that is, not necessarily stable sections) that are Mundet semi-stable. Let $\overline{\mathcal{M}}_n^G(C,X)$ denote the subcategory of $\overline{\mathcal{M}}_n^{\mathrm{pre},G}(C,X)$ consisting of Mundet semistable gauged maps that are semistable, that is, have sections that are stable maps, and $\mathcal{M}_n^G(C,X)$ the subcategory where $\hat{C} \cong C$. The relationship between the Kontsevich-style compactification $\overline{\mathcal{M}}^G(C,X)$ and the Grothendieck-style compactification $\overline{\mathcal{M}}^{G,\mathrm{quot}}(C,X)$ is given by relative version of Givental's collapsing morphism [18, p. 646]:

Proposition 5.12. Let X be a smooth polarized projective G-variety. Then $\overline{\mathcal{M}}_n^G(C,X)$ is an Artin stack equipped with a proper Deligne-Mumford morphism to $\overline{\mathcal{M}}^{G,\mathrm{quot}}(C,X)$.

Proof. By Givental's main lemma [18, p. 646], see [28, Lemma 2.6], [15, Section 8], [8], [42] for any smooth projective variety X embedded in a projective space $\mathbb{P}(V)$, there exists a proper morphism $\overline{\mathcal{M}}_{g,0}(C \times X) \to \mathrm{Quot}_{F/C}$ where $\mathrm{Quot}_{F/C}$ is the quot scheme of the trivial bundle $F = C \times V^\vee$ compactifying $\operatorname{Hom}(C,\mathbb{P}(V))$. We apply this as follows, continuing Example 4.2 (e): Consider the forgetful morphism $\overline{\mathcal{M}}_0^G(C,X) \to \operatorname{Hom}(C,BG)$. As in Narasimhan-Seshadri [35] for $G = GL(n,\mathbb{C})$ or Ramanathan [43], Sorger [47] in general, the stack $\operatorname{Hom}(C,BG)$ admits a local presentation as a quotient $\mathfrak{M}^{\mathrm{fr},\mathrm{quot}}(C,F)/\operatorname{Aut}(F)$ where $\mathfrak{M}^{\mathrm{fr},\mathrm{quot}}(C,F)$ is a quasiprojective scheme of bundles whose associated vector bundle is equipped with a presentation as a quotient of F. The space $\mathfrak{M}^{\mathrm{fr},\mathrm{quot}}(C,F)$ has a universal G-bundle $\mathfrak{U}^{\mathrm{fr},\mathrm{quot}}(C,F) \to C \times \mathfrak{M}^{\mathrm{fr},\mathrm{quot}}(C,F)$ equipped with a G-equivariant $\operatorname{Aut}(F)$-action. Let $\tilde{d} \in H_2(\mathfrak{U}^{\mathrm{quot}}(C,F) \times_G X)$ be the class corresponding to d that is, whose push-forward under $\mathfrak{U}^{\mathrm{fr},\mathrm{quot}}(C,F) \times_G X \to \mathfrak{M}^{\mathrm{quot}}(C,F)$ is zero and whose fiber class is determined by d. The stack $\mathfrak{M}^G(C,X,F,d)$ is a category of bundles with section, and so is isomorphic to the quotient of the rigidified moduli space $\mathcal{M}_{g,0}(\mathfrak{U}^{\mathrm{fr,quot}}(C,F) \times_G X, \tilde{d})$ by the action of $\mathrm{Aut}(F)$. Let $\overline{\mathfrak{M}}^{\mathrm{fr,quot}}(C, \overline{\mathfrak{U}}^{\mathrm{quot}}(C,F) \times_G X, \tilde{d})$ be the subscheme of $\mathrm{Quot}_{F^2/C}$ compactifying morphisms $C \to \overline{\mathfrak{U}}^{\mathrm{quot}}(C,F) \times_G X$ of class $\tilde{d}$. By the relative version of Givental's lemma [42, Theorem, p.4] there exists a proper morphism $g : \overline{\mathcal{M}}_{g,0}(\overline{\mathfrak{U}}^{\mathrm{quot}}(C,F) \times_G X, \tilde{d}) \to \overline{\mathfrak{M}}^{\mathrm{fr,quot}}(C, \overline{\mathfrak{U}}^{\mathrm{quot}}(C,F) \times_G X, \tilde{d})$ mapping each stable map to the corresponding quotient. The morphism $\overline{\mathfrak{M}}^G(C,X,F,d) \to \overline{\mathfrak{M}}^{G,\mathrm{quot}}(C,X,F,d)$ is the quotient of g by the action of $\mathrm{Aut}(F)$. Since g is proper and of Deligne-Mumford type, so is the quotient. After restricting to the semistable locus, we may assume that F is sufficiently large so that every bundle occurs as a quotient of F. Then $\overline{\mathcal{M}}_0^G(C,X)$ is the inverse image of the open substack $\overline{\mathcal{M}}^{G,\mathrm{quot}}(C,X)$ and so also an Artin stack. Furthermore $\overline{\mathcal{M}}_n^G(C,X)$ is the inverse image of $\overline{\mathcal{M}}_0^G(C,X)$ under the forgetful morphism obtained by iterating Lemma 5.3, and so an Artin stack. Since the forgetful morphism and g are both Deligne-Mumford and proper, the claim follows.

Corollary 5.13. Let X be a smooth polarized projective G-variety. Suppose that every Mundet semistable gauged map is stable. For each constant $c > 0$, the union of components $\overline{\mathcal{M}}_n^G(C,X,d)$ with $(d, [\omega_{X,G}]) < c$ is a proper Deligne-Mumford stack.

Proof. By Theorem 5.11 and Proposition 5.12, the morphisms $\overline{\mathcal{M}}^{G,\mathrm{quot}}(C,X,d) \to \mathrm{pt}$ and $\overline{\mathcal{M}}_n^G(C,X,d) \to \overline{\mathcal{M}}^{G,\mathrm{quot}}(C,X,d)$ are proper and Deligne-Mumford, hence so is their composition, and similarly for the union of components satisfying the bound in the Corollary.

There is another approach to the properness result above which uses symplectic geometry rather than the git constructions in Schmitt [46]. Let K be a maximal compact subgroup of G.

Theorem 5.14. *Let X be a smooth polarized projective G-variety or a G-vector space with a proper moment map. Suppose that every semistable gauged map is stable. The map assigning to any stable gauged map the corresponding vortex defines a homeomorphism Z from the coarse moduli space of $\overline{\mathcal{M}}_n^G(C,X,d)$ to the moduli space of vortices $\overline{M}_n^K(C,X,d)$.*

Proof. That the map Z is a bijection follows from Mundet's Theorem 5.9 applied to the principal component. We check that the map is a homeomorphism. The topology on the coarse moduli space $\overline{M}_n^G(C,X,d)$ is induced from specialization in families: For any convergent sequence $[(P_\nu, u_\nu)] \to [(P,u)]$ there exists an analytic family $\hat{C}$ of nodal curves over a connected complex manifold S, a family of holomorphic G-bundles $P \to C \times S$, a family of maps $\hat{C} \to P(X)$, and a convergent sequence $s_\nu \in s$ such that (P_ν, u_ν) resp. (P,u) is isomorphic to the fiber over s_ν resp. s. Fixing a reduction of structure group to K and using the correspondence between holomorphic structures and connections gives a family

$(A_s \in \mathcal{A}(P), u_s : \hat{C}_s \to P(X))$ of connections and sections on a fixed K-bundle P. If $s_\nu \in S$ is a sequence converging to $s \in S$ as $\nu \to \infty$, then $A_{s_\nu} \to A$ uniformly in all derivatives and u_{s_ν} Gromov converges to u_s. In particular, the principal component $u_{s_\nu,0}$ converges to $u_{s,0}$ uniformly in all derivatives on compact subsets of the complement of the bubbling set. Then $u_{s_\nu}^* P(\Phi) \to u_s^* P(\Phi)$ in the L^p topology, for $p > 2$, and uniformly on compact subsets of the complement of the bubbling set. So $F_{A_{s_\nu}} + u_{s_\nu}^* P(\Phi)\omega_C \to F_{A_s} + u_s^* P(\Phi)\omega_C$ in the L^p-topology on $\Omega^2(C, P(\mathfrak{k}))$ and uniformly on compact subsets of the complement of the bubbling set. Let $\xi_{A,u}$ denote the unique global minimum of $\mathcal{I}(P)$, so that the correspondence is given by $(A,u) \mapsto \exp(i\xi_{A,u})(A,u)$. Then $F_{\exp(i\xi_{A,u})(A_s,u_s)}$ converges to $F_{\exp(i\xi_{A,u})(A,u)}$ in L^p. By the implicit function theorem, there exists a unique complex gauge transformation of the form $\exp(i\xi_\nu')$ such that $\exp(i\xi_\nu')\exp(i\xi_{A,u})(A_\nu, u_\nu)$ is a vortex, with $\xi_\nu' \to 0$ in $W^{1,p}$. Since $\exp(i\xi_\nu')\exp(i\xi_{A,u}) = \exp(i\xi_{A_\nu,u_\nu})$ mod $\mathcal{K}(P)$, this implies $\xi_{A_s,u_s} \to \xi_{A,u}$ in $W^{1,p}$. In particular, for $p > 2$ this implies $\xi_{A_s,u_s} \to \xi_{A,u}$ in C^0, which implies that Z is continuous. Continuity of the inverse map $\overline{M}_n^K(C,X) \to \overline{M}_n^G(C,X)$ follows from the fact that $\overline{M}_n^G(C,X)$ is a coarse moduli space for C^0 families of gauged maps. This in turn follows from its construction via Quot scheme methods as in Section 4.5. Namely, for each bundle one finds a point in the Grassmannian corresponding to a realization of the bundle as a quotient; the construction of this point depends continuously on the connection and curve chosen.

Lemma 5.15. $\overline{\mathcal{M}}_n^{\mathrm{pre},G}(C,X)$ *is an Artin stack equipped with a morphism* $\overline{\mathcal{M}}_n^{\mathrm{pre},G}(C,X) \to \overline{\mathcal{M}}_n^G(C,X)$ *collapsing unstable components.*

Proof. $\overline{\mathcal{M}}_n^{\mathrm{pre},G}(C,X)$ is the pre-image of $\overline{\mathcal{M}}_n^G(C,X)$ under the morphism of Lemma 5.3.

The assignment $X \to \overline{\mathcal{M}}_n^G(C,X)$ is functorial in the following sense, generalizing functoriality of the stacks of stable map in Behrend-Manin [7].

Definition 5.16. The category of *smooth polarized varieties with reductive group actions* has

(a) (Objects) are data (G,X,L) consisting of a reductive group G, a smooth polarized G-variety X, and an ample G-line bundle $L \to X$;
(b) (Morphisms) from (G_0,X_0,L_0) to (G_1,X_1,L_1) consist of pairs of a morphism $\varphi : X_0 \to X_1$ a *surjective* homomorphism $\psi : G_0 \to G_1$ and an *injective* right inverse $\iota : G_1 \to G_0$ such that φ preserves Hilbert-Mumford weights, that is, if x_0 is fixed by one-parameter subgroup $\mathbb{C}^\times \to \iota(G_1)$ then x_1 has the same weight as $\varphi(x_0)$.

Remark 5.17. The definition of morphism implies that G_0 is a product of G_1 with the kernel of ψ, and that the semistable locus in X_0 maps to the semistable locus in X_1.

Proposition 5.18. $X \mapsto \overline{\mathcal{M}}_n^G(C,X)$ extends to a functor from the category of smooth polarized varieties with reductive group actions to (Artin stacks, equivalence classes of morphisms of Artin stacks).

Proof. Consider the composition $\overline{\mathfrak{M}}_n^{G_0}(C, X_0) \to \overline{\mathfrak{M}}_n^{G_1}(C, X_1)$. Let (P_0, u_0) be an object of $\overline{\mathfrak{M}}_n^{G_0}(C, X_0)$. Any parabolic reduction of $P_0 \times_{G_0} G_1$ to a parabolic subgroup R_1 defines a parabolic reduction of P_1 to $R_0 = \psi^{-1}(R_1)$, via the isomorphism $P \times_{G_0} G_1/R_1 \to P/R_0$, and the associated graded bundles $\mathrm{Gr}(P_0)$. Any character of the center of R_1 defines a character of the center of R_0. The image of the associated graded section $\mathrm{Gr}(u_0) : C \to P(X_0)$ is the associated graded section of the image of u_0 under $P(X_0) \to P(X_1)$. Since the Hilbert-Mumford weights are preserved, the Mundet weight is the same and the image of the Mundet semistable locus $\overline{\mathcal{M}}_n^{G_0}(C, X_0)$ lies in $\overline{\mathcal{M}}_n^{G_1,\mathrm{pre}}(C, X_1)$. By restriction we obtain a morphism from $\overline{\mathcal{M}}_n^{G_0}(C, X_0)$ to $\overline{\mathcal{M}}_n^{G_1,\mathrm{pre}}(C, X_1)$, and by composition with the collapse map, to $\overline{\mathcal{M}}_n^{G_1}(C, X_1)$. The functor axioms (identity, composition) are immediate from the definition of the collapse maps.

In particular taking X_1 and G_1 in the lemma above to be trivial gives:

Corollary 5.19. There exists a forgetful morphism $f : \overline{\mathcal{M}}_n^G(C, X) \to \overline{\mathcal{M}}_n(C)$ which maps $(\hat{C}, P, u, \underline{z})$ to the stable map to C obtained from $(C, \pi \circ u, \underline{z})$ by composing with the projection $C \times X/G \to C$ and collapsing unstable components as in (21).

In order to investigate splitting properties of the gauged Gromov-Witten invariants we introduce moduli spaces whose combinatorial type is a rooted forest (finite collection of trees) Γ. Denote by $\overline{\mathfrak{M}}_{n,\Gamma}^G(C, X)$, resp. $\overline{\mathfrak{M}}_{n,\Gamma}^{G,st}(C, X)$, resp. $\overline{\mathcal{M}}_{n,\Gamma}^{G,\mathrm{pre}}(C, X)$, resp. $\overline{\mathcal{M}}_{n,\Gamma}^G(C, X)$ the stacks of nodal gauged maps resp. nodal gauged maps with stable sections resp. Mundet semistable maps resp. Mundet semistable maps with stable sections of combinatorial type Γ defined as follows:

Definition 5.20. (Stacks of gauged maps with disconnected combinatorial type) Suppose that $\Gamma = \Gamma_0 \cup \Gamma_1 \ldots \cup \Gamma_l$ is a disjoint union of trees $\Gamma_0, \ldots, \Gamma_l$ equipped with a root vertex $v_0 \in \mathrm{Vert}(\Gamma_0)$ and a homology class $d = d_0 + \ldots + d_l \in H_2^G(X, \mathbb{Z})$. Let $\overline{\mathcal{M}}_{n_0,\Gamma_0}^G(C, X)$ be defined as above and for $i \geq 1$ (not containing the root vertex) let

$$\overline{\mathcal{M}}_{n_i,\Gamma_i}^G(C, X, d_i) := \overline{\mathcal{M}}_{0,n_i,\Gamma_i}(X, d_i)/G$$

(the quotient of the moduli stack of parametrized stable maps by the G-action). Let $\overline{\mathcal{M}}_{n,\Gamma}^G(C, X, d)$ be the product of moduli stacks $\overline{\mathcal{M}}_{n,\Gamma_i}^G(C, X, d_i)$.

Let $<$ denote the partial ordering on combinatorial types, so that $\Gamma < \Gamma'$ if Γ' is obtained from Γ by collapsing edges. Denote by $\overline{\mathcal{M}}_{n,\Gamma}(C) = \cup_{\Gamma' \leq \Gamma} \mathcal{M}_{n,\Gamma'}(C)$ the "compactified" stack of nodal curves of combinatorial type Γ and $\overline{\mathcal{C}}_{n,\Gamma}(C) \to \overline{\mathcal{M}}_{n,\Gamma}(C)$ the universal curve. These stacks of various combinatorial types are related as follows, in the language of tree morphisms [7].

Proposition 5.21. Any morphism of rooted trees $\Upsilon : \Gamma \to \Gamma'$ induces a morphism of moduli spaces of nodal resp. stable gauged maps

$$\overline{\mathfrak{M}}(\Upsilon, X) : \overline{\mathfrak{M}}_{n,\Gamma}^G(C) \to \overline{\mathfrak{M}}_{n,\Gamma'}^G(C), \quad \overline{\mathcal{M}}(\Upsilon, X) : \overline{\mathcal{M}}_{n,\Gamma}^G(C) \to \overline{\mathcal{M}}_{n,\Gamma'}^G(C).$$

In particular,

(a) (Cutting an edge) If $\Upsilon : \Gamma \to \Gamma'$ is a morphism cutting an edge, then $\overline{\mathcal{M}}^G_{n,\Gamma}(C,X)$ may be identified with the fiber product $\overline{\mathcal{M}}^G_{n,\Gamma'}(C,X) \times_{(X/G)^2} (X/G)$ over the diagonal $\Delta : (X/G) \to (X/G)^2$ and $\overline{\mathcal{M}}(\Upsilon, X)$ is projection of the fiber product on the first factor.
(b) (Collapsing an edge) If Γ' is obtained from Γ by collapsing an edge then $\overline{\mathcal{M}}^G_{n,\Gamma}(C,X)$ is isomorphic to $\overline{\mathcal{M}}^G_{n,\Gamma'}(C,X) \times_{\overline{\mathfrak{M}}_{n,\Gamma'}(C)} \overline{\mathfrak{M}}_{n,\Gamma}(C)$ and $\overline{\mathcal{M}}(\Upsilon, X)$ is projection on the first factor.

5.3. Toric quotients and quasimaps

In this section we treat the case that X is a vector space equipped with a linear action of a torus G.

Remark 5.22. (Quotients of vector spaces by tori) Suppose X has weights $\mu_1, \ldots, \mu_k \in \mathfrak{g}^\vee$. A moment map for the G-action on X is given by

$$(z_1, \ldots, z_k) \mapsto \nu - \left(\sum_{i=1}^k \mu_i |z_i|^2 / 2 \right)$$

where $\nu \in \mathfrak{g}^\vee$ is a constant. Assuming ν is rational, the choice of this constant determines a polarization $\mathcal{O}_X(1) \to X$ given by twisting the trivial bundle with the rational character corresponding to ν. The semistable locus is then

$$X^{\mathrm{ss}} = \{ (z_1, \ldots, z_k) |\, \mathrm{span}\{\mu_i, z_i \neq 0\} \ni \nu \} . \tag{30}$$

The git quotient $X /\!\!/ G$ is a toric stack with residual action of the torus $(\mathbb{C}^\times)^k / G$. One has stable=semistable if $\mu_i(\nu) \neq 0$ for all i. If so, the git quotient $X /\!\!/ G$ is proper if the weights $\mu_1, \ldots, \mu_k$ are contained in an open half-space in the real part. Note that $X /\!\!/ G$ depends on the choice of ν. The components of the complements of the hyperplanes $\ker \mu_i$ are called *chambers* for ν.

Example 5.23. (The projective plane and its blow-up as a quotient of affine four-space) Suppose that $X = \mathbb{C}^4$ and $G = (\mathbb{C}^\times)^2$ acting with weights $(1,0), (1,0), (1,1), (0,1)$.

(a) For $\nu = (1,2)$ the unstable locus has a component given by the sum of the weight spaces with weights $(1,0), (1,1)$ and a component equal to the weight space with weight $(0,1)$. The quotient $X /\!\!/ G$ is isomorphic to $\mathbb{P}^2$ via the map $[x_1, x_2, x_3, x_4] \mapsto [x_1, x_2, x_3 x_4^{-1}] \in \mathbb{P}^2$.
(b) For $\nu = (2,1)$, the unstable locus has a component given by the sum of the weight spaces with weights $(0,1), (1,1)$ and a component with weight $(0,1)$. The quotient $X /\!\!/ G$ is isomorphic to the blow-up of $\mathbb{P}^2$ with the map to $\mathbb{P}^2$ blowing down the exceptional divisor given by $[x_1, x_2, x_3, x_4] \mapsto [x_1, x_2, x_3 x_4^{-1}]$.

See Figure 13.

Morphisms from a curve C to the git quotient $X /\!\!/ G$ are closely related to objects in the *stack of quasimaps* $H^0(C, P \times_G X)/G$ as follows. If $u \in H^0(C, P \times_G X)$ takes values in the semistable locus then it defines a map to $X /\!\!/ G$, and any map to

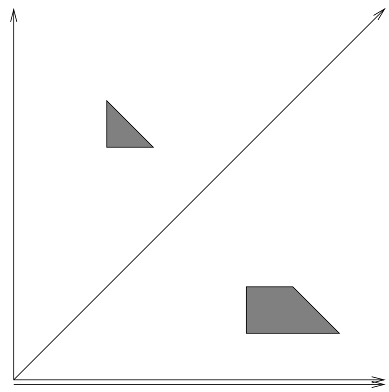

FIGURE 13. Quotients for the $(\mathbb{C}^\times)^2$ action on $\mathbb{C}^4$

$X/\!\!/G$ arises in this way. If C has genus zero, $P \to C$ has $c_1(P) = d$ and X_j denotes the weight space with weight μ_j then there is an isomorphism of G-modules

$$H^0(C, P \times_G X) \to X(d) := \bigoplus_j X_j^{\oplus \max(0,(d,\mu_j)+1)}. \tag{31}$$

Any polarization $\mathcal{O}_X(1)$ of X induces a polarization $\mathcal{O}_{X(d)}(1)$ by taking the moment map resp. polarization to be given by $\nu \in \mathfrak{g}^\vee$ We say that a quasimap $u \in H^0(C, P \times_G X)$ is *(semi)stable* if it is (semi)stable for the polarization $\mathcal{O}_{X(d)}(1)$.

Proposition 5.24. For any $d \in H_2^G(X, \mathbb{Z})$, there exists a constant ρ_0 such that if stable=semistable for the G-action on $X(d)$ and $\rho > \rho_0$ then a gauged map (P, u) of class d is ρ-semistable iff $u \in H^0(C, P \times_G X)$ is semistable for the action of G, so that there is an isomorphism of stacks

$$\mathcal{M}^G(C, X, d) \cong H^0(C, P \times_G X)/\!\!/G = X(d)/\!\!/G.$$

Proof. Since G is abelian, there are no parabolic reductions and Mundet's criterion for semistability becomes

$$\mu(\sigma, \lambda) = \int_{[C]} (c_1(P) + \rho P(\Phi) \circ \mathrm{Gr}(u)_0[\omega_C], -\lambda) \leq 0$$

where λ represents an infinitesimal automorphism of the bundle P, that is, an element of the group G. For ρ sufficiently large, we may ignore the term involving $c_1(P)$ and obtain the stability condition for the action of G on $H^0(C, P \times_G X)$.

Example 5.25. (a) (Projective Space) Let $X = \mathbb{C}^k$ with $G = \mathbb{C}^\times$ acting diagonally. Identify $H_2^G(X, \mathbb{Z}) \cong \mathbb{Z}$. Then $X(d) = \mathbb{C}^{kd}$ and $\mathcal{M}^G(C, X, d) = \mathbb{C}^{kd}/\!\!/\mathbb{C}^\times = \mathbb{P}^{kd-1}$. A polynomial $[u] \in \mathcal{M}^G(C, X, d)$ defines a map to $\mathbb{P}^{k-1}$ of degree d iff its components have no common zeroes.

(b) (The projective plane and its blow-up as a quotient by a two-torus) Suppose that $X = \mathbb{C}^4$ and $G = (\mathbb{C}^\times)^2$ acting with weights $(1,0)$, $(1,0)$, $(1,1)$, $(0,1)$. With $d = (1,0)$, we have $X(d) = \mathbb{C}_{(0,1)} \oplus \mathbb{C}^{\oplus 2}_{(1,1)} \oplus \mathbb{C}^{\oplus 4}_{(1,0)}$. The moduli spaces of gauged maps are $\mathbb{P}^5$ for $\nu = (1,2)$ or $\mathrm{Bl}_{\mathbb{P}^1}(\mathbb{P}^5)$ for $\nu = (2,1)$. For example, by Thaddeus [49] the two quotients are related by blow-up along $\mathbb{C}^{\oplus 2}_{(1,1)} /\!\!/ G = \mathbb{P}^1$.

The comparison between the vortex equations and quasimaps has been investigated from the symplectic point of view by J. Wehrheim [54], based on earlier work of Cieliebak-Salamon [9]. The space of quasimaps appears in the work of Morrison-Plesser [32], Givental [18], Lian-Liu-Yau [28] etc. on mirror symmetry as an algebraic model for the space of stable maps to the quotient $X/\!\!/G$.

5.4. Affine gauged maps

Let X be a G-variety as above. In this section we construct the stack $\overline{\mathcal{M}}^G_{n,1}(\mathbb{A}, X)$ of affine gauged maps. These are used later to construct the quantum Kirwan morphism. The following extends [55, Definition 1.2] to the case of orbifold target $X/\!\!/G$.

Definition 5.26. (Affine gauged maps) An *n-marked affine gauged map* to X over a scheme S consists of

(a) (Projective weighted line) a weighted projective line $C = \mathbb{P}[1, r]$ for some $r > 0$
(b) (Marking) an n-tuple of distinct points $(z_1, \ldots, z_n) : S \to (C - \{\infty\})^n$, where $\infty := B\mu_r$ is the stacky point at infinity;
(c) (Scaling) a non-zero meromorphic one-form $\lambda \in H^0(C \times S, T^\vee_C(2\infty))$ and
(d) (Representable morphism) a representable morphism $u : C \times S \to X/G$ such that $u(\infty, s) \in X/\!\!/G$ for all $s \in S$.

A *morphism* of n-marked affine gauged maps $(\underline{z}_j, \lambda_j, u_j)$ consists of an automorphism $\psi : C \to C$ mapping $z_{0,i}$ to $z_{1,i}$ and pulling back λ_1 to λ_0 and an isomorphism of $u_1 \circ \psi$ with u_0.

The complement of ∞ in C has the structure of an affine line determined by λ, hence the use of the terminology *affine*. The category $\mathcal{M}^G_{n,1}(\mathbb{A}, X)$ of n-marked affine gauged maps to X/G has the structure of an Artin stack: In the case that $X/\!\!/G$ is a free quotient, it is an open substack of the stack $\mathrm{Hom}_{\overline{\mathfrak{M}}_{n,1}(\mathbb{A})}(\overline{\mathfrak{C}}_{n,1}(\mathbb{A}), X/G)$ considered in Example 4.3. More generally, in the case that $X/\!\!/G$ has orbifold singularities, $\mathcal{M}^G_{n,1}(\mathbb{A}, X)$ is an open substack of $\mathrm{Hom}_{\overline{\mathfrak{M}}^{\mathrm{tw}}_{n,1}(\mathbb{A})}(\overline{\mathfrak{C}}^{\mathrm{tw}}_{n,1}(\mathbb{A}), X/G)$ where $\overline{\mathfrak{M}}^{\mathrm{tw}}_{n,1}(\mathbb{A})$ is defined in Definition 4.2.

Definition 5.27. (Nodal affine gauged maps) A *nodal n-marked affine gauged map* to X over a scheme S consists of a nodal marked scaled affine curve $C = (C, \lambda, z_0, \ldots, z_n)$ over S, possibly twisted at the nodes of infinite scaling and the root marking, and a representable morphism $u : C \to X/G$. In addition we require that

(a) (Root marking is target-stable) $u(z_0) \in I_{X/\!\!/G}$;
(b) (Infinite area components are target-stable) on any component such that λ is infinite, u takes values in the stable locus $X/\!\!/G$;

(c) (Zero area components are bundle-stable) the bundle is stable, hence trivializable, on the locus on which the scaling is zero.

A *morphism* of affine gauged maps $(C, \lambda, \underline{z}, u : C \to X/G)$ to $(C', \lambda', \underline{z}', u' : C' \to X/G)$ is a morphism $\phi : C \to C'$ of scaled curves from $(C, \lambda, \underline{z})$ to $(C', \lambda', \underline{z}')$ such that $u = u' \circ \phi$. The *homology class* of $u : C \to X/G$ is $u_*[C] \in H_2^G(X, \mathbb{Q})$ (integral in the absence of orbifold singularities on the curve C). A affine gauged map over S is *stable* if every fiber $u_s : C_s \to X$ admits only finitely many automorphisms, or equivalently, every component on which u has zero homology class has at least three special points or two special points and a non-trivial scaling.

Denote by $\overline{\mathcal{M}}^G_{n,1}(\mathbb{A}, X, d)$ resp. $\overline{\mathfrak{M}}^G_{n,1}(\mathbb{A}, X, d)$ the stack of stable resp. not-necessarily stable scaled curves of genus zero and homology class d and by $\overline{\mathcal{M}}^G_{n,1}(\mathbb{A}, X)$ the sum over homology classes.

Theorem 5.28. *$\overline{\mathfrak{M}}^G_{n,1}(\mathbb{A}, X, d)$ resp. $\overline{\mathcal{M}}^G_{n,1}(\mathbb{A}, X, d)$ is an Artin stack resp. proper Deligne-Mumford stack.*

Proof. It follows from Example 4.3 that the hom-stack $\mathrm{Hom}_{\overline{\mathfrak{M}}^{\mathrm{tw}}_{n,1,\Gamma}(\mathbb{A})}(\overline{\mathfrak{C}}^{\mathrm{tw}}_{n,1,\Gamma}(\mathbb{A}), X/G)$ is an Artin stack, since $\overline{\mathcal{C}}^{\mathrm{tw}}_{n,1,\Gamma}(\mathbb{A}) \to \overline{\mathfrak{M}}^{\mathrm{tw}}_{n,1,\Gamma}(\mathbb{A})$ is proper and X is smooth. The conditions defining $\overline{\mathcal{M}}^G_{n,1}(\mathbb{A}, X/G, d)$ (values in the semistable locus where $\lambda = \infty$) are open and so $\overline{\mathcal{M}}^G_{n,1}(\mathbb{A}, X/G, d)$ is an open substack of $\mathrm{Hom}_{\overline{\mathfrak{M}}^{\mathrm{tw}}_{n,1,\Gamma}(\mathbb{A})}(\overline{\mathfrak{C}}^{\mathrm{tw}}_{n,1,\Gamma}(\mathbb{A}), X/G)$. Furthermore, by assumption G acts freely on the semistable locus in X and so $\overline{\mathcal{M}}^G_{n,1}(\mathbb{A}, X/G, d)$ has finite automorphism groups, and so is Deligne-Mumford. Properness is equivalent to properness of the underlying coarse moduli space by Proposition 4.9. This in turn follows from the compactness [55, Theorem 3.20] and Theorem 5.29 below.

Theorem 5.29. *Suppose that X is either a smooth polarized projective G-variety or a polarized vector space with linear action of G and proper moment map. The coarse moduli space of $\overline{\mathcal{M}}^G_{n,1}(\mathbb{A}, X)$ is homeomorphic to the moduli space of affine symplectic vortices $\overline{M}^K_{n,1}(\mathbb{A}, X)$.*

Proof. This is mostly proved in [52] using the heat flow for gauged maps in [51]. We sketch the proof: Any morphism $u : C \to X/G$ with $u(z_0) \in X/\!\!/G$ determines, by restriction, a pair (A, u) on $\mathbb{A} \cong C - \{z_0\}$ taking values in the semistable locus, which can be complex-gauge-transformed (using the implicit function theorem) to a pair satisfying the vortex equations outside of a sufficiently large ball. The heat flow for gauged maps provides a complex gauge transform to a symplectic vortex; the convexity of Mundet's functional implies that the complex gauge transformation is unique up to unitary gauge transformation. Let $C \to S, u : C \to X/G, \lambda, \underline{z} : S \to C^n$ be a family of stable affine gauged maps and $s_0 \in S$. After restricting to a neighborhood of s_0 we may assume that the bundles are obtained from a fixed principal K-bundle on C_{s_0} and family of connections on C_{s_0} via gluing. For each $s \in S$, there is a unique-up-to-unitary gauge transformation $g_s \in \mathcal{G}(P)$ such that $g_s(A_s, u_s)$ is a vortex, obtained as the minimum of a functional $\tilde{\psi}_s$ obtained by integrating the moment map. We have

$F_{g_s(A_s,u_s)} \to F_{g_s(A_{s_0},u_{s_0})_j}$ as $s \to s_0$ in L^p, by convergence away from the bubbling set, for any component $(A_{s_0}, u_{s_0})_j$ of (A_{s_0}, u_{s_0}) with finite scaling. By the implicit function theorem g_s converges to g_{s_0} in a suitable Sobolev $1,p$-space for $p > 2$, hence in C^0. Continuity of the inverse map $\overline{M}^K_{n,1}(\mathbb{A}, X) \to \overline{M}^G_{n,1}(\mathbb{A}, X)$ follows from the fact that $\overline{M}^G_{n,1}(\mathbb{A}, X)$ is a coarse moduli space for C^0 families of gauged maps, by its construction via Quot scheme methods as in Section 4.5. Let (C_s, P_s, A_s, u_s) be a family of nodal affine vortices over a topological space S. (C_s, P_s, A_s) defines a continuous family of holomorphic bundles, denoted $(C_s, P_s^{\mathbb{C}})$. Any such bundle is the pull-back of the universal deformation $P^{\mathrm{univ}} \to C^{\mathrm{univ}}$ of $(C_{s_0}, P^{\mathbb{C}}_{s_0})$ by some continuous map $S \to S^{\mathrm{univ}}$, where $P^{\mathbb{C}}_{s_0}$ is the holomorphic bundle defined by A_0. Consider u_s as a continuous family of holomorphic maps to $P^{\mathrm{univ}}(X)$, with Gromov limit $u_0 : C_0 \to P^{\mathrm{univ}}(X)$. The latter is also the limit in the algebraic sense of the maps u_s, that is, the limit of the corresponding points $[u_s]$ in the moduli space of stable maps to $P^{\mathrm{univ}}(X)$. Taking the universal deformation of u_0 realizes u_0 as an algebraic specialization of u_s, which shows that that map $\overline{M}^K_{n,1}(\mathbb{A}, X) \to \overline{M}^G_{n,1}(\mathbb{A}, X)$ is continuous.

Following Behrend-Manin [7] in the case of stable maps, we show that the moduli stacks of affine gauged maps are functorial for suitable morphisms of G-varieties.

Proposition 5.30. There is a canonical morphism $\overline{\mathcal{M}}^{G,\mathrm{pre}}_{n,1}(\mathbb{A}, X) \to \overline{\mathcal{M}}^G_{n,1}(\mathbb{A}, X)$, given by (recursively) collapsing unstable components.

Proof. Given a family $u : C \to X/G$ of affine gauged maps to X and an ample G-line bundle $L \to X$ define

$$C^{st} = \operatorname{Proj} \bigoplus_{n \geq 0} \pi_*(\omega^{\lambda}_{C/S}[z_1 + \ldots + z_n] \otimes u^* L^3)^{\otimes n}.$$

The map u factors through C^{st} and commutes with base change, in the case that the family arises from forgetting a marking from a stable family by the similar arguments to those in Behrend-Manin [7]. As in the case of (22), it is necessary to perform this construction *twice* in order to produce a stable affine gauged map. The general case reduces to this one, by adding markings locally. The orbifold case is as in [3, Section 9], by taking the proj relative to the target stack.

Recall the category of *smooth polarized G-varieties* from Definition 5.16.

Corollary 5.31. $X \mapsto \overline{\mathcal{M}}^G_{n,1}(\mathbb{A}, X)$ extends to a functor from the category of smooth polarized G-varieties to Deligne-Mumford stacks.

Proof. Given morphisms $\phi : X_0 \to X_1, G_0 \to G_1$ we obtain a morphism from $\overline{\mathcal{M}}^{G_0}_{n,1}(\mathbb{A}, X_0)$ to $\overline{\mathcal{M}}^{G_1,\mathrm{pre}}_{n,1}(\mathbb{A}, X_1)$ by composing u with ϕ. Composing with the collapsing morphism 5.30 gives the required morphism of moduli stacks.

Example 5.32. (Affine gauged maps in the toric case) Suppose that G is a torus and X a vector space with weights $\mu_1, \ldots, \mu_k$. Then $\mathcal{M}^G_{1,1}(\mathbb{A}, X) = \mathrm{Hom}(\mathbb{A}, X)^{\mathrm{ss}}/G$ where $\mathrm{Hom}(\mathbb{A}, X)^{\mathrm{ss}}$ is the space of morphisms from $\mathbb{A}$ to X that are generically semistable, that is, $u : \mathbb{A} \to X$ such that $u^{-1}(X^{\mathrm{ss}}) \subset \mathbb{A}$ is non-empty.

(a) (Projective space quotient) If $X = \mathbb{C}^k$ with $G = \mathbb{C}^\times$ acting diagonally, the component of homology class $d \in H_2^G(X,\mathbb{Z}) \cong \mathbb{Z}$ is

$$\operatorname{Hom}(\mathbb{A}, X, d)^{\mathrm{ss}}/G = \left\{ \sum_{e \le d} (a_{e,1}, \ldots, a_{e,k}) z^e \ \mid (a_{d,1}, \ldots, a_{d,k}) \neq 0 \right\} /G$$

For example, if $d = 1$ and $k = 2$ then

$$\mathcal{M}_{1,1}^G(\mathbb{A}, X, 1) \cong \{ z \mapsto (a_{1,1} z + a_{0,1}, a_{1,2} z + a_{0,2}), (a_{1,1}, a_{1,2}) \neq 0 \}/G$$

is the total space of $\mathcal{O}_{\mathbb{P}}(1)^{\oplus 2}$. Its boundary is isomorphic to $\overline{\mathcal{M}}_{0,2}(\mathbb{P}, [\mathbb{P}]) \cong \mathbb{P}^2$, the moduli space of twice-marked stable maps of degree $[\mathbb{P}]$, by the map which attaches a trivial affine gauged map at the marking z_1.

(b) (Point quotient) The case of $X = \mathbb{C}$ is studied from the point of view of vortices in Taubes [48, Theorem 1] and Jaffe-Taubes [21]. To describe this classification, let $\operatorname{Sym}^d(\mathbb{A}) = \mathbb{A}^d/S_d$ denote the symmetric product. The references [], [21] show that the map

$$\mathcal{M}_{1,1}^G(\mathbb{A}, X, d) \to \operatorname{Sym}^d(\mathbb{A}), \ [u] \mapsto u^{-1}(0)$$

is a homeomorphism on coarse moduli spaces, which is obvious from the algebraic description given here.

(c) (Weighted projective line quotient) The following is an example with orbifold singularities in the quotient $X/\!\!/G$. Let $\mathbb{C}_2$ resp. $\mathbb{C}_3$ denote the weight space for $G_{\mathbb{C}} = \mathbb{C}^\times$ with weight 2 resp. 3 so that $X = \mathbb{C}_2 \oplus \mathbb{C}_3$ and $X/\!\!/G = \mathbb{P}[2,3]$. Identifying $H_2^G(X,\mathbb{Q}) \cong \mathbb{Q}$ so that $H_2^G(X,\mathbb{Z}) \cong \mathbb{Z}$ we see that for complex numbers $a_0, b_0, a_1, b_1, \ldots$

$$\begin{aligned}
\mathcal{M}_{1,1}^G(\mathbb{A}, X, 0) &= \{(a_0, b_0) \neq 0\}/G \cong \mathbb{P}[2,3] \\
\mathcal{M}_{1,1}^G(\mathbb{A}, X, 1/3) &= \{(a_0, b_1 z + b_0), b_1 \neq 0\}/G \cong \mathbb{C}^2/\mathbb{Z}_3 \\
\mathcal{M}_{1,1}^G(\mathbb{A}, X, 1/2) &= \{(a_1 z + a_0, b_1 z + b_0), a_1 \neq 0\}/G \cong \mathbb{C}^3/\mathbb{Z}_2 \\
\mathcal{M}_{1,1}^G(\mathbb{A}, X, 2/3) &= \{(a_1 z + a_0, b_2 z^2 + b_1 z + b_0), b_2 \neq 0\}/G \cong \mathbb{C}^4/\mathbb{Z}_3 \\
\mathcal{M}_{1,1}^G(\mathbb{A}, X, 1) &= \{(a_2 z^2 + a_1 z + a_0, b_3 z^3 + b_2 z^2 + b_1 z + b_0), (a_2, b_3) \neq 0\}/G.
\end{aligned}$$

The stacks $\overline{\mathcal{M}}_{n,\Gamma}^G(\mathbb{A}, X)$ satisfy functoriality with respect to morphisms of colored trees, similar to Proposition 4.7, with the caveat that because we allow stacky points in the domain, in the case that $X/\!\!/G$ is only locally free, the gluing maps will not be isomorphisms:

Proposition 5.33. Any morphism of colored trees $\Upsilon : \Gamma \to \Gamma'$ induces a morphism of moduli spaces

$$\overline{\mathfrak{M}}(\Upsilon, X) : \overline{\mathfrak{M}}_{n,1,\Gamma}^G(\mathbb{A}, X) \to \overline{\mathfrak{M}}_{n,1,\Gamma'}^G(\mathbb{A}, X), \quad \overline{\mathcal{M}}(\Upsilon, X) : \overline{\mathcal{M}}_{n,1,\Gamma}^G(\mathbb{A}, X) \to \overline{\mathcal{M}}_{n,1,\Gamma'}^G(\mathbb{A}, X).$$

In particular,

(a) (Cutting an edge or edges with relations) If $\Upsilon : \Gamma' \to \Gamma$ is a morphism corresponding to cutting an edge of Γ, then there is a gluing morphism

$$\mathcal{G}(\Upsilon, X) : \overline{\mathcal{M}}^G_{n,1,\Gamma'}(\mathbb{A}, X) \times_{\overline{I}^{2m}_{X/G}} \overline{I}^m_{X/G} \to \overline{\mathcal{M}}^G_{n,1,\Gamma}(\mathbb{A}, X) \tag{32}$$

where m is the number of cut edges and the second morphism is the diagonal

$$\Delta : \overline{I}^m_{X/G} \to \overline{I}^{2m}_{X/G}$$

which is an isomorphism in the absence of stacky points in the domain, that is, if $X /\!\!/ G$ is a variety, and in general is an isomorphism after passing to finite covers. The morphism $\overline{\mathcal{M}}(\Upsilon, X)$ is given by projection on the first factor.

(b) (Collapsing an edge) If Γ' is obtained from Γ by collapsing an edge then there is an isomorphism

$$\overline{\mathcal{M}}^G_{n,1,\Gamma'}(\mathbb{A}, X) \times_{\overline{\mathfrak{M}}_{n,1,\Gamma'}(\mathbb{A})} \overline{\mathfrak{M}}_{n,1,\Gamma}(\mathbb{A}) \to \overline{\mathcal{M}}^G_{n,1,\Gamma}(\mathbb{A}, X)$$

and $\overline{\mathcal{M}}(\Upsilon, X)$ is given by projection on the first factor.

Proof. In the case that $X /\!\!/ G$ is a variety, these claims are immediate from the definitions. In the case that $X /\!\!/ G$ is a Deligne-Mumford stack, the gluing maps are isomorphisms after passing to the stack $\overline{\mathcal{M}}^{\mathrm{fr},G}_{n,1,\Gamma}(\mathbb{A}, X)$ of stable affine gauged maps with sections at the stacky points.

5.5. Scaled gauged maps

In this section, we construct moduli stacks of scaled maps with projective domain. These are used later to relate the gauged graph potential of X with the graph potential of the quotient $X /\!\!/ G$. Let X be a smooth projectively embedded G-variety and C a smooth connected projective curve.

Definition 5.34. A *nodal scaled gauged map* from C to X consists of a twisted nodal scaled n-marked curve $(\hat{C}, \underline{z}, \omega)$ as in [55, Definition 2.40], with orbifold structures only at the nodes with infinite scaling, together with a morphism $\hat{C} \to C \times X/G$ consisting of a bundle $P \to \hat{C}$ and a representable morphism $u : \hat{C} \to C \times P(X)$. Such a map is *stable* iff

(a) (Finite scaling) if the scaling ω is finite on the principal component then u is stable for the large area chamber;
(b) (Infinite scaling) if the scaling ω is infinite on the principal component then C admits a decomposition into not-necessarily-irreducible components $C = C_0 \cup \ldots \cup C_r$ where $u_0 = u|C_0$ is an r-marked stable map $C_0 \to C \times (X /\!\!/ G)$ and $u_i = u|C_i : C_i \to C \times X/G$ are stable affine gauged maps.

A morphism of scaled Mundet-stable curves $(\hat{C}, \lambda, \underline{z}, u)$ to $(\hat{C}', \lambda', \underline{z}', u')$ is an isomorphism of the underlying scaled curves $\phi : \hat{C} \to \hat{C}'$ intertwining the scalings, markings, and morphisms. A nodal gauged map is *stable* if it is Mundet stable and has finitely many automorphisms, that is, each non-principal component with non-degenerate scaling resp. degenerate scaling has at least two resp. three special points.

Let $\overline{\mathfrak{M}}^G_{n,1}(C,X)$ denote the stack of nodal Mundet-semistable scaled gauged maps, and $\overline{\mathcal{M}}^G_{n,1}(C,X)$ the stack of semistable scaled gauged maps. $\overline{\mathcal{M}}^G_{n,1}(C,X)$ is the union of stacks $\overline{\mathcal{M}}^G_{n,1}(C,X)_{<\infty}$ consisting of gauged vortices for large area chamber and a scaling on the underlying curve, and a stack $\overline{\mathcal{M}}^G_{n,1}(C,X,d)_\infty$ consisting of maps from C to $X/\!\!/G$ and collections of affine maps to X/G. By forgetting the affine maps one obtains a fiber bundle

$$\overline{\mathcal{M}}^G_{n,1}(C,X,d)_\infty \to \overline{\mathcal{M}}_r(C,X/\!\!/G) \tag{33}$$

with fiber $\Pi_{j=1}^r \overline{\mathcal{M}}^G_{i_j,1}(\mathbb{A},X)$.[1] Note that $\overline{\mathcal{M}}^G_{n,1}(C,X)$ contains $\overline{\mathcal{M}}^G_n(C,X)$ as the zero section.

Proposition 5.35. $\overline{\mathfrak{M}}^G_{n,1}(C,X)$ is an Artin stack. $\overline{\mathcal{M}}^G_{n,1}(C,X)$ is an open substack equipped with a morphism $\rho : \overline{\mathcal{M}}^G_{n,1}(C,X) \to \overline{\mathcal{M}}_{0,1}(C) \cong \mathbb{P}$ with the property that there exist isomorphisms $\overline{\mathcal{M}}^G_n(C,X,d) \to \rho^{-1}(0)$ (with stability on the domain given by the large area chamber $\rho \to 0$) and $\overline{\mathcal{M}}^G_{n,1}(C,X,d)_\infty \to \rho^{-1}(\infty)$. The coarse moduli space of $\overline{\mathcal{M}}^G_{n,1}(C,X)$ is homeomorphic to the moduli space of scaled vortices $\overline{M}^K_{n,1}(C,X)$.

Proof. That $\overline{\mathfrak{M}}^G_{n,1}(C,X)$ is an Artin stack follows from Example 4.3, since the universal scaled curve is proper over $\overline{\mathfrak{M}}^G_{n,1}(C,X)$ and $\overline{\mathfrak{M}}^G_{n,1}(C,X)$ is the hom-stack of representable morphisms from the universal scaled curve to X/G. To see that $\overline{\mathcal{M}}^G_{n,1}(C,X,d)$ is an Artin substack we must show that $\overline{\mathfrak{M}}^G_{n,1}(C,X,d)_{<\infty} \backslash \overline{\mathcal{M}}^G_{n,1}(C,X,d)_{<\infty}$ is closed in $\overline{\mathfrak{M}}^G_{n,1}(C,X,d)$. An argument which uses the Hitchin-Kobayashi correspondences above goes as follows: Suppose that $u : \hat{C} \to C \times X/G$ is a family of scaled maps over a parameter space S with central fiber u_0 an infinite-area gauged map, that is, a map $C_0 \to X/\!\!/G$ together with a collection of affine gauged maps $C_i \to X/G$, $i = 1,\ldots,k$, and for $s \neq 0$, the map u_s ρ_s-unstable with $\rho_s \to \infty$ as $s \to 0$. In particular the principal component of u_s can be represented as a pair (A_s, v_s) with (A_s, v_s) flowing under the heat flow in Venugopalan [51] to a limit (A'_s, v'_s) that is reducible. Since K acts locally freely on the zero level set $\Phi^{-1}(0)$ there exists a constant $c > 0$ such that

$$\forall x \in X,\ \dim(K_x) > 0 \implies \|\Phi(x)\| > c.$$

Using the energy-area identity there exist constants c_0, c_1 such that

$$\|\rho_s^{-1} F_{A'_s} + \rho_s (v'_s)^* P(\Phi)\|_{L^2} \geq c_0 + c_1 |\rho_s|$$

which implies the same estimate for (A_s, v_s). Now suppose that (A_s, u_s) converges to some (A_0, v_0) with $v_0^* P(\Phi) = 0$ on the principal component. Since $\|F_{A_s}\|_{L^2}$ is

[1] Equation (33) is corrected from the published version.

bounded, we must have $\|v_s^* P(\Phi)\|_{L^2} \to \infty$. This contradicts $\|v_0^* P(\Phi)\| = 0$. Hence u_0 is not in the closure of the unstable locus in $\overline{\mathfrak{M}}_{n,1}^G(C,X)_{<\infty}$.

If every polystable scaled gauged map is stable then $\overline{\mathcal{M}}_{n,1}^G(C,X)$ is Deligne-Mumford. The homeomorphism of the coarse moduli space to the moduli space of vortices is already established for curves with finite scaling or curves with infinite scaling via Mundet's correspondence and its version for affine curves in Theorem 5.29. It remains to show that the homeomorphisms on these subsets glue together to a homeomorphism on the entire space, that is, that the bijection and its inverse are continuous. The first issue is the continuity of the complex gauge transformation used in the definition of the correspondence under specialization. Let $C \to S, u : C \to X/G, \lambda, \underline{z} : S \to C^n$ be a family of stable scaled gauged maps and $s_0 \in S$. After restricting to a neighborhood of s_0 we may assume that the bundles are obtained by applying the gluing construction to a principal K-bundle on C_{s_0} to family of connections on C_{s_0}. For each $s \in S$, there is a unique-up-to-unitary gauge transformation $g_s \in \mathcal{G}(P)$ such that $g_s(A_s, u_s)$ is a vortex, obtained as the minimum of the Mundet functional. On any component, say j-th, with finite scaling we have $F_{g_{s_0}(A_s,u_s)} \to F_{g_{s_0}(A_{s_0},u_{s_0})_j}$ as $s \to s_0$ in Lebesgue space $L^p, p > 2$. By an argument using the implicit function theorem, g_s converges to g_{s_0} in $W^{1,p}$, hence in C^0 on the complement of the bubbling set. (Note that the L^p convergence does not hold at the bubbling points and indeed there is not C^0 convergence of the complex gauge transformation on the principal component.) A similar discussion holds on any of the bubbles on which the limiting scaling is finite: Namely suppose that $\phi_s : B_{r_s}(0) \to C$ is a sequence of embeddings of balls of radius $r_s \to \infty$ such that $\phi_s^*(A_s, u_s)$ converges to an affine vortex (A, u). Then $\phi_s^* F_{A_s,u_s}$ converges to $F_{A,u}$ in L^p. This implies that the gauge transformations g_s converge in C^0 on the compact subsets of the affine line. Continuity of the inverse map $\overline{M}_{n,1}^K(C,X) \to \overline{M}_{n,1}^G(C,X)$ follows from the fact that $\overline{M}_{n,1}^G(C,X)$ is a coarse moduli space for C^0 families of gauged maps, which is similar to the case $C = \mathbb{A}$.

6. Virtual fundamental classes

The virtual fundamental class theory of Behrend-Fantechi [6] (which is a version of earlier approach of Li-Tian [27]) constructs a Chow class from a perfect relative obstruction theory. Stacks of representable morphisms to quotient stacks by reductive groups have canonical relative obstruction theories, by the same construction in [6] and a deformation result of Olsson [38]. In this section, we construct virtual fundamental classes for stacks of gauged (resp. gauged affine, gauged scaled) maps.

6.1. Sheaves on stacks

Any Artin stack $\mathcal{X}$ comes equipped with a canonical *structure sheaf* of rings $\mathcal{O}_{\mathcal{X}}$ in any of the standard Grothendieck topologies on $\mathcal{X}$. A *sheaf* on an Artin stack $\mathcal{X}$ will mean a sheaf of $\mathcal{O}_{\mathcal{X}}$-modules over the lisse-étale site of $\mathcal{X}$, see Olsson [36], de Jong et al [10]. A sheaf E is *coherent* if for every object U of the lisse-étale site, the restriction $E|U$ admits presentations $(\O_{\mathcal{X}}^n|U) \to (E|U)$ of finite type, and furthermore any such map has kernel of finite type. The *derived category* *of bounded complexes of sheaves with coherent cohomology* $D^b\operatorname{Coh}(\mathcal{X})$ is the subcategory of the derived category of complexes of coherent sheaves with coherent bounded cohomology groups. It is a triangulated category obtained by inverting quasi-isomorphisms in the category of complexes of sheaves with coherent cohomology.

Example 6.1. (Examples of complexes of coherent sheaves on Artin stacks)

(a) (Equivariant sheaves) If $\mathcal{X} = X/G$ is the quotient stack associated to a group action of a group G on a scheme G, then the category of sheaves on $\mathcal{X}$ is equivalent to the category of G-equivariant sheaves on X, by an argument involving simplicial spaces [26, 12.4.5].

(b) (Cotangent complex) Any morphism of Artin stacks $f : \mathcal{X} \to \mathcal{Y}$ defines a *cotangent complex* $L_{\mathcal{X}/\mathcal{Y}} \in D^b\operatorname{Coh}(\mathcal{X})$ satisfying the expected properties [36], for example, if $g : \mathcal{Y} \to \mathcal{Z}$ is another morphism of Artin stacks then there is a distinguished triangle in $D^b\operatorname{Coh}(\mathcal{X})$ $\ldots \to L_{\mathcal{X}/\mathcal{Z}} \to L_{\mathcal{X}/\mathcal{Y}} \to Lf^*L_{\mathcal{Y}/\mathcal{Z}}[1] \to \ldots$.

6.2. Cycles on stacks

The notion of *rational Chow group* $A(\mathcal{X})$ of a Deligne-Mumford stack $\mathcal{X}$ is developed in Vistoli [53], and further improved in Kresch [24]. A *cycle* of dimension k on $\mathcal{X}$ is an element of the free abelian group $Z_k(\mathcal{X})$ generated by all integral closed substacks of dimension k so that the group of cycles is

$$Z(\mathcal{X}) = \bigoplus_k Z_k(\mathcal{X}).$$

A *cycle with rational coefficients* of dimension k is an element of the group $Z_k(\mathcal{X})\otimes\mathbb{Q}$. The *group of rational equivalences* on cycles of dimension k on $\mathcal{X}$ is

$$W_k(\mathcal{X}) = \bigoplus_{\mathcal{Y}} \mathbb{C}(\mathcal{Y})^*$$

the sum of the spaces of non-zero rational functions on substacks $\mathcal{Y}$ of $\mathcal{X}$ of dimension $k+1$. Set

$$W(\mathcal{X}) = \oplus_k W_k(\mathcal{X}), \quad W(\mathcal{X})_{\mathbb{Q}} = W(\mathcal{X}) \otimes \mathbb{Q}.$$

If X is a scheme, there is a homomorphism $\partial_X : W(X) \to Z(X)$ that takes a rational function on a subvariety of X to the cycle associated to its Weil divisor. For a stack $\mathcal{X}$, the functors Z, W define sheaves on the étale site of $\mathcal{X}$, the maps ∂_X define a morphism of sheaves and hence a morphism of spaces of global sections $\partial_{\mathcal{X}} : W(\mathcal{X}) \to Z(\mathcal{X})$. The *Chow group* is the cokernel

$$A(\mathcal{X}) := \operatorname{coker}(\partial_{\mathcal{X}} : W(\mathcal{X}) \to Z(\mathcal{X}))$$

and the *rational Chow group* is $A(\mathcal{X})_{\mathbb{Q}} = A(\mathcal{X}) \otimes \mathbb{Q}$.

Let $f : \mathcal{X} \to \mathcal{Y}$ be a morphism of Deligne-Mumford stacks. If f is flat, then there is a *flat pull-back* $f^* : Z(\mathcal{Y}) \to Z(\mathcal{X})$. If f is proper, then there is a *proper* *push-forward* $f_* : Z(\mathcal{X})_{\mathbb{Q}} \to Z(\mathcal{Y})_{\mathbb{Q}}$ given for finite flat morphisms by $f_*[\mathcal{X}'] = \deg(\mathcal{X}'/f(\mathcal{X}'))[f(\mathcal{X}')]$; note that for stacks the degree is a rational number, see Vistoli [53, Section 2]. These maps pass to rational equivalences, so that we obtain maps

$$f^* : A(\mathcal{Y}) \to A(\mathcal{X}) \quad f \quad \text{flat}, \quad f_* : A(\mathcal{X})_{\mathbb{Q}} \to A(\mathcal{Y})_{\mathbb{Q}} \quad f \quad \text{proper}.$$

If $f : \mathcal{X} \to \mathcal{Y}$ is a regular local embedding of codimension d and $Z \to \mathcal{Y}$ is a morphism from a scheme V, then there is a Gysin homomorphism

$$f^! : Z(\mathcal{Y}) \to A(\mathcal{X} \times_{\mathcal{Y}} V) \quad f \quad \text{regular local embedding}$$

defined by local intersection products. Vistoli [53, Theorem 3.11] proves that this passes to rational equivalence. The Gysin homomorphisms satisfy the usual functorial properties with respect to proper and flat morphisms: For any fiber diagram

$$\begin{array}{ccc} \mathcal{X}'' & \longrightarrow & \mathcal{Y}'' \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle q} \\ \mathcal{X}' & \longrightarrow & \mathcal{Y}' \\ \downarrow & & \downarrow \\ \mathcal{X} & \xrightarrow{f} & \mathcal{Y} \end{array}$$

where $\mathcal{Y}', \mathcal{Y}''$ are schemes and f is a regular local embedding, (i) if q is proper then $f^!q_* = p_*f^{|}$ and (ii) if q is flat then $f^!q^* = p^*f^!$.

If $f : X \to Y$ is a morphism of schemes then there is a *bivariant Chow group* $A^\vee(X \to Y)$, whose elements α of degree l associate to any morphism $U \to Y$ and each class $u \in A_k(U)$ a class, denoted $\alpha \cap u$, in $A_{k-l}(X \times_Y U)$ satisfying compatibility with flat pull-back, proper push-forward, and Gysin homomorphisms for regular local embeddings. The definition of bivariant Chow groups extends to *representable* morphisms of stacks $f : \mathcal{X} \to \mathcal{Y}$ [53, Section 5], and the action on Chow groups of schemes extends to an action on Chow groups of stacks equipped with morphisms to $\mathcal{Y}$.

The theory of Gromov-Witten invariants requires bivariant Chow theory for representable morphisms of Artin stacks. As explained by Behrend-Fantechi [6, Section 7]

Proposition 6.2. (a) If $\mathcal{X} \to \mathcal{Y}$ is a representable morphism of Artin stacks, then there exists a bivariant Chow group $A^\vee(\mathcal{X} \to \mathcal{Y})$.
(b) If $\mathcal{X} \to \mathcal{Y}$ is a regular local immersion then there exists a canonical element $[f] \in A^\vee(\mathcal{X} \to \mathcal{Y})$ whose action on Chow cycles is denoted $f^!$.
(c) If $\mathcal{X} \to \mathcal{Y}$ is flat then there is a canonical *orientation class* $[f] \in A^\vee(\mathcal{X} \to \mathcal{Y})$.

Now we define derived categories and Chow groups for G-stacks. Let $\mathcal{X}$ be a G-stack with multiplication $\mu : G \times \mathcal{X} \to \mathcal{X}$ and projection on the right factor

$\rho : G \times \mathcal{X} \to \mathcal{X}$. and $F \to \mathcal{X}$ a sheaf. A *G-linearization* of F is an isomorphism of sheaves $\phi : \mu^* F \to \rho^* F$ which is compatible with multiplication in the sense that $(\mu \times \mathrm{Id}_{\mathcal{X}})^* \phi$ is equal to $(\mathrm{Id}_G \times \mu)^* \phi$. A *$G$-sheaf* on $\mathcal{X}$ is a sheaf together with a linearization. Any G-sheaf F descends to a sheaf F/G on the quotient stack $\mathcal{X}/G$, so that the cohomology of F/G is the invariant part of the cohomology of F.

The *equivariant derived category* $D^b \operatorname{Coh}^G(\mathcal{X})$ is the derived category of the quotient stack $D^b \operatorname{Coh}(\mathcal{X}/G)$. In particular, any complex of G-sheaves defines an object in $D^b \operatorname{Coh}^G(\mathcal{X})$. Note that if $\mathcal{X}/G$ is Deligne-Mumford, then $D^b \operatorname{Coh}^G(\mathcal{X})$ is the usual derived category of bounded complexes of coherent sheaves, otherwise one needs more complicated constructions involving Cartesian sheaves [36]. The *equivariant cotangent complex* is the cone $L^G_{\mathcal{X}} := \operatorname{Cone}(L_{\mathcal{X}} \to \mathfrak{g}^\vee)$ on the morphism $L_{\mathcal{X}} \to \mathfrak{g}^\vee$ induced by the action of G. By the exact triangle for cotangent complexes, if the action of G on $\mathcal{X}$ is locally free, so that $\mathcal{X}/G$ is again a Deligne-Mumford stack then $L^G_{\mathcal{X}}$ descends to $L_{\mathcal{X}/G}$.

Suppose that G is a reductive group, and $\mathcal{X}$ is a proper Deligne-Mumford stack $\mathcal{X}$ of dimension n equipped with an action of G. The *equivariant Chow groups* $A^G(\mathcal{X})$ are defined by Edidin-Graham (for schemes) [13] and Graber-Pandharipande (for stacks) [19] as follows. Let V be an l-dimensional representation of G such that V has an open subset U on which G acts freely and whose complement has codimension more than $n - i$. Let

$$A^G_i(\mathcal{X}) = A_{i+l-g}(U \times_G \mathcal{X})$$

be the *i-th equivariant Chow group*. By [13, Proposition 1] (for schemes; the argument for Deligne-Mumford stacks is the same) $A^G_i(\mathcal{X})$ is independent of the choice of V and U. It satisfies the following properties

(a) (Functoriality) If $\mathcal{X}, \mathcal{Y}$ are Deligne-Mumford stacks equipped with actions of G then any G-equivariant proper resp. flat morphism $f : \mathcal{X} \to \mathcal{Y}$ induces a map $f_* : A^G(\mathcal{X}) \to A^G(\mathcal{Y})$ resp. $f^* : A^G(\mathcal{Y}) \to A^G(\mathcal{X})$.
(b) (Free actions) If the action of G on $\mathcal{X}$ is locally free then $A^G(\mathcal{X}) \to A(\mathcal{X}/G)$ is an isomorphism, where $\mathcal{X}/G$ is the quotient stack. (This is [19, Lemma 6] in the case $G = \mathbb{C}^\times$).

More generally, Kresch has introduced a notion of Chow groups for Artin stacks [25], so that $A(\mathcal{X}/G)$ is isomorphic to $A^G(\mathcal{X})$, for a not-necessarily-free action of a reductive group G on a Deligne-Mumford stack $\mathcal{X}$.

6.3. Obstruction theories

Often a stack $\mathcal{X}$ is given (at least locally) as a zero locus of a vector bundle $\mathcal{E} \to \mathcal{Y}$. In such a case, $\mathcal{X}$ is a complete intersection and so carries a fundamental class. The notion of *perfect obstruction theory* for $\mathcal{X}$ keeps some of this information and is enough to reconstruct a virtual fundamental class for $\mathcal{X}$.

Definition 6.3. An *obstruction theory* for an Deligne-Mumford stack $\mathcal{X}$ is a pair (E, ϕ) where $E \in \operatorname{Ob}(D^b \operatorname{Coh}(\mathcal{X}))$ is an object in the derived category of coherent sheaves in the étale topology and $\phi : E \to L_{\mathcal{X}}$ is a morphism in the derived category of coherent sheaves such that

(a) $h^i(E) = 0, i > 0$;
(b) $h^0(\phi)$ is an isomorphism;
(c) $h^{-1}(\phi)$ is surjective.

The rank of E is the *virtual dimension* of $\mathcal{X}$. A *relative obstruction theory* for a morphism of stacks $f : \mathcal{X} \to \mathcal{Y}$ is defined similarly, but replacing the cotangent complex $L_{\mathcal{X}}$ with its relative version L_f. An obstruction theory (E, ϕ) is *perfect* if E has amplitude in $[-1, 0]$, that is, non-vanishing cohomology only in degrees $0, -1$. Behrend-Fantechi [6] furthermore assume that there is a *global resolution* of E, that is, a complex of vector bundles $F = [F^{-1} \to F^0]$ together with an isomorphism of F to E in $D^b \operatorname{Coh}(\mathcal{X})$, but this assumption is removed in Kresch [25].

There is an equivariant version of obstruction theory in the sense of Behrend-Fantechi [6], given as follows. Let $\mathcal{X}$ be a proper Deligne-Mumford G-stack where G is a reductive group, and let U be a free G-variety as in the definition of the equivariant Chow ring above. Let $L_\pi \in \operatorname{Ob}(D^b \operatorname{Coh}(\mathcal{X} \times_G U))$ be the relative cotangent complex for $\pi : \mathcal{X} \times_G U \to U/G$. An *equivariant obstruction theory* is a pair (E, ϕ) where $E \in \operatorname{Ob}(D^b \operatorname{Coh}(\mathcal{X} \times_G U))$ and ϕ is a morphism in $D^b \operatorname{Coh}(\mathcal{X} \times_G U)$ to L_π. We suppose that E admits a global presentation $E^{-1} \to E^0$.

Example 6.4. (Examples of Obstruction Theories)

(a) (Stacks of morphisms to projective schemes [6]) Let C, X be projective schemes such that C is Gorenstein, and $\operatorname{Hom}(C, X)$ the scheme of morphisms from C to X. Let $u : C \times X \to X$ be the universal morphism and $p : C \times X \to C$ the projection. Let

$$E = Rp_*(u^* L_X \otimes \omega) = (Rp_* u^* T_X)^\vee.$$

Then E forms part of an obstruction theory for X, perfect if X is smooth and C is a curve [6, 6.3]. Indeed, by the functorial properties of the cotangent complex there is a homomorphism

$$e : u^* L_X \to L_{C \times \operatorname{Hom}(C,X)} \to L_{C \times \operatorname{Hom}(C,X)/C} \cong \pi^* L_{\operatorname{Hom}(C,X)}.$$

Then e induces a homomorphism $\phi := \pi_*(e^\vee)^\vee : E^\vee \to L_X^\vee$. The map ϕ is an obstruction theory [6, 6.3].

(b) (Stacks of morphisms to Artin stacks) The construction of an obstruction theory extends to the case that X is an Artin S-stack and $\operatorname{Hom}_S(C, X)$ is replaced by a Deligne-Mumford substack of the stack of *representable* morphisms $\operatorname{Hom}_S^{\mathrm{rep}}(C, X)$, as long as one can show that $\operatorname{Hom}_S^{\mathrm{rep}}(C, X)$ is also an Artin S-stack and E has amplitude in $[-1, 0]$; see [38, Theorem 1.1] for the extension of basic results about deformation theory of morphisms of schemes to the setting of stacks. That the Hom-stack $\operatorname{Hom}_S(C, X)$ is an Artin stack, if C, X are, is not known in general, but holds as long as $X = Y/G$ is a quotient stack for action of a reductive group G on a projective variety Y by Example 4.3 (d). In this case, if X is smooth than E has amplitude in $-1, 0$; cohomology below degree -1 vanishes since $T(Y/G)$ has amplitude in $0, 1$ while vanishing in degree 1 follows from the assumption that the substack is Deligne-Mumford and $H^1(E) = \mathrm{Ext}^{-1}(E, \mathbb{C})$ is the sheaf of infinitesimal automorphisms [38, Theorem 1.5].

(c) (Moduli stacks of bundles) Let C be a projective scheme and G a reductive group so that $\mathrm{Hom}(C, BG)$ is the moduli stack of principal G-bundles on C. By Examples 4.1 (c) and 4.3 (c) $\mathrm{Hom}(C, BG)$ has an obstruction theory with $E = (Rp_* \underline{\mathfrak{g}}[1])^\vee$ where $\underline{\mathfrak{g}}$ denotes the trivial sheaf with fiber $\mathfrak{g}$. If C is a projective curve then this obstruction theory is perfect on the substack of irreducible bundles. In fact $\mathrm{Hom}(C, BG)$ is a smooth Artin stack and the obstruction theory coincides with the cotangent complex [47, 3.6.8].

(d) (Hom-stacks over stacks) Continuing 4.3 (d) let $\mathcal{X}$ be a Gorenstein Deligne-Mumford curve over an Artin stack $\mathcal{Z}$, $\mathcal{Y}$ an Artin stack over $\mathcal{Z}$ and suppose that $\mathrm{Hom}^{\mathrm{rep}}_{\mathcal{Z}}(\mathcal{X}, \mathcal{Y})$ is an Artin stack, and $\mathrm{Hom}^{\mathrm{rep},0}_{\mathcal{Z}}(\mathcal{X}, \mathcal{Y})$ the sub-stack of $\mathrm{Hom}^{\mathrm{rep}}_{\mathcal{Z}}(\mathcal{X}, \mathcal{Y})$ with finite automorphism group. The restriction of the relative obstruction theory to $\mathrm{Hom}^{\mathrm{rep},0}_{\mathcal{Z}}(\mathcal{X}, \mathcal{Y})$ is perfect.

(e) (Moduli stacks of gauged maps) In particular, for any type Γ and non-negative integer n, the moduli stack $\overline{\mathcal{M}}^G_{n,\Gamma}(C, X)$ has a relative obstruction theory over $\overline{\mathfrak{M}}_{n,\Gamma}(C)$ with complex given by $(Rp_* u^* T(X/G))^\vee$.

(f) (Moduli stacks of affine gauged maps) For any type Γ and non-negative integer n, the moduli stack $\overline{\mathcal{M}}^G_{n,\Gamma}(\mathbb{A}, X)$ is an open substack of $\mathrm{Hom}^{\mathrm{rep}}_{\overline{\mathfrak{M}}_{n,\Gamma}(\mathbb{A})}(\overline{\mathfrak{C}}_{n,\Gamma}(\mathbb{A}), X/G)$ and has a relative obstruction theory over $\overline{\mathfrak{M}}_{n,\Gamma}(\mathbb{A})$ with complex given by $(Rp_* u^* T(X/G))^\vee$.

(g) (Moduli stacks of stable maps, equivariant case) If a group G acts on a smooth projective variety X, then for any type Γ and non-negative integers g, n the moduli stack of stable maps $\overline{\mathcal{M}}_{g,n,\Gamma}(X)$ admits an equivariant perfect relative obstruction theory over $\overline{\mathfrak{M}}_{g,n,\Gamma}$ as in Graber-Pandharipande [19] and so an equivariant virtual fundamental class.

6.4. Definition of the virtual fundamental class

Behrend-Fantechi [6] and Kresch [24] construct for any Deligne-Mumford stack $\mathcal{X}$ an *intrinsic normal cone* of pure dimension zero $C_{\mathcal{X}}$, defined by patching together the quotients $C_{U/M}/f^* T_M$ for local embeddings $f : U \to M$. (See [24, Theorem 1] for a correction to the argument in [6].) If (E, ϕ) is a perfect obstruction theory with $E = (E^{-1} \to E^0)$ then the morphism ϕ induces a morphism of cone stacks $C_{\mathcal{X}} \to E^{\vee,1}/E^{\vee,0}$. Let C_E denote the fiber product of $E^{\vee,1}$ and $C_{\mathcal{X}}$ over $E^{\vee,1}/E^{\vee,0}$. In the rest of the paper, we denote by $A(\mathcal{X})$ etc. the rational Chow group of $\mathcal{X}$.

Definition 6.5. (Virtual fundamental classes)

(a) (Non-equivariant case) The *virtual fundamental class* $[\mathcal{X}]$ (depending on (E, ϕ)) is the intersection of C_E with the zero section of $E^{\vee,1}$ in $A(\mathcal{X})$. By [6, 5.3], $[\mathcal{X}]$ is independent of the choice of global resolution used to construct it.

(b) (Equivariant virtual fundamental classes) In the equivariant case, the morphism $\pi : U \times_G \mathcal{X} \to U/G$ is of Deligne-Mumford type and gives an intrinsic normal cone $[C_{\mathcal{X}}] \in A^G_0(\mathcal{X}) = A_0(\mathcal{X} \times_G U)$. One obtains a virtual fundamental class in $A^G(\mathcal{X})$.

(c) (Relative virtual fundamental classes) Let $f : \mathcal{X} \to \mathcal{Y}$ be a representable morphism of algebraic stacks, and $A^\vee(\mathcal{X} \to \mathcal{Y})$ the bivariant Chow group constructed by Vistoli [53]. If f is flat or a regular immersion, one denotes by $[f] \in A^\vee(\mathcal{X} \to \mathcal{Y})$ the orientation class of 6.2, and by $f^!$ its action on Chow groups of Deligne-Mumford stacks. Given a relative perfect obstruction theory let $[\mathcal{X}] \in A_{\dim(\mathcal{Y})+\mathrm{rk}(E)}(\mathcal{X})$ be the *relative virtual fundamental class* given by intersecting C_E with the zero section of $E^{\vee,1}$ [6], [24].

Example 6.6. (a) (Stable Maps) Let X be a smooth projective variety and g, n non-negative integers. For any type Γ, the moduli stack $\overline{\mathcal{M}}_{g,n,\Gamma}(X,d)$ with genus g and n markings is a proper Deligne-Mumford stack equipped with a perfect relative obstruction theory over $\overline{\mathfrak{M}}_{g,n,\Gamma}$ and so has a virtual fundamental class $[\overline{\mathcal{M}}_{g,n,\Gamma}(X,d)]$. More generally if X is proper smooth G-stack with G a reductive group then the obstruction theory is equivariant and $\overline{\mathcal{M}}_{g,n}(X,d)$ has a G-equivariant virtual fundamental class $[\overline{\mathcal{M}}_{g,n,\Gamma}(X,d)] \in A_G(\overline{\mathcal{M}}_{g,n,\Gamma}(X,d))$. Even more generally let $\mathcal{X}$ be a smooth proper Deligne-Mumford stack and Γ a combinatorial type. The moduli space of twisted stable maps $\overline{\mathcal{M}}_{g,n,\Gamma}(\mathcal{X})$ discussed in Section 4 has a canonical perfect relative obstruction theory and hence a virtual fundamental class $[\overline{\mathcal{M}}_{g,n,\Gamma}(\mathcal{X},d)]$.

(b) (Stable gauged maps) For a type Γ and non-negative integer n if $\overline{\mathcal{M}}^G_{n,\Gamma}(C,X)$ is a Deligne-Mumford substack (equivalently in characteristic zero, all automorphism groups are finite) then it has a virtual fundamental class $[\overline{\mathcal{M}}^G_{n,\Gamma}(C,X)] \in A(\overline{\mathcal{M}}^G_{n,\Gamma}(C,X))$, by Example 6.4 (e).

(c) Recall that $\overline{\mathcal{M}}^G_{n,1}(\mathbb{A},X)$ admits a forgetful morphism to $\overline{\mathfrak{M}}^{\mathrm{tw}}_{n,1}(\mathbb{A})$ (where the superscript *tw* indicates that we allow orbifold structures at the nodes with infinite scaling, in the case that $X/\!\!/G$ is only locally free) and to $\overline{\mathcal{M}}_{n,1}(\mathbb{A})$, the latter collapsing components that become unstable after forgetting the morphism to X/G. $\overline{\mathcal{M}}^G_{n,1}(\mathbb{A},X)$ has a canonical perfect relative obstruction theory over $\overline{\mathfrak{M}}^{\mathrm{tw}}_{n,1}(\mathbb{A})$, whose complex is dual to the push-forward of $u^*T(X/G)$ over the universal curve over $\overline{\mathcal{M}}^G_{n,1}(\mathbb{A},X)$, by Example 6.4 and so a virtual fundamental class $[\overline{\mathcal{M}}^G_{n,1}(\mathbb{A},X)]$.

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