QUANTUM KIRWAN MORPHISM AND GROMOV-WITTEN INVARIANTS OF QUOTIENTS I

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Abstract. This is the first in a sequence of papers in which we construct a quantum version of the Kirwan map from the equivariant quantum cohomology $QH_G(X)$ of a smooth polarized complex projective variety X with the action of a connected complex reductive group G to the orbifold quantum cohomology $QH(X/\!\!/G)$ of its geometric invariant theory quotient $X/\!\!/G$, and prove that it intertwines the genus zero gauged Gromov-Witten potential of X with the genus zero Gromov-Witten graph potential of $X/\!\!/G$. In this part we introduce the moduli spaces used in the construction of the quantum Kirwan morphism.

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*Partially supported by NSF grant DMS0904358 and the Simons Center for Geometry and Physics

1. Introduction

This is the first in a sequence of papers in which we construct a quantum version of the morphism studied by Kirwan [27], which maps the equivariant cohomology of a Hamiltonian group action to the cohomology of the symplectic quotient. The existence of a quantum version was suggested by Salamon and Ziltener [54], [55]. Here we work under the assumption that the target is a smooth projectively-embedded variety with a connected reductive group action such that the stable locus is equal to the semistable locus; this allows us to use the virtual fundamental cycle machinery of Behrend-Fantechi [3]. We prove that the quantum Kirwan map intertwines the Gromov-Witten graph potential of the quotient with the *gauged Gromov-Witten* potential of the action in the large area limit. In physics language, the quantum Kirwan map relates correlators of a (possibly non-linear, non-abelian) gauged sigma model with those of the sigma model of the quotient. As such, the results overlap with those of Givental [20], Lian-Liu-Yau [31], Iritani [24] and others. The connection to mirror symmetry is explained in the paper of Hori-Vafa [23]: because mirror symmetry for vector spaces is rather trivial, the non-trivial change of coordinates arises when passing from a gauged linear sigma model to the sigma model for the quotient. Since the quantum Kirwan map is defined geometrically, it can be rather difficult to compute and the algebraic approach in [20], [31] is more effective in cases where it applies. The geometric approach pursued here has the advantage that are no semipositivity assumptions on $X/\!\!/G$ or abelian-ness assumptions on the group G. Also, X can be a projective variety rather than a vector space. At the time that we started the project, there were rather few papers about these situations; however in the meantime papers such as Ciocan-Fontanine-Kim [9], [10] and Coates-Corti-Iritani-Tseng [14] substantially extend the hypotheses of the previous theorems. However, the approaches are still quite different: In [14], the fundamental solution to the quantum differential equation is expressed using a characterization of Givental's Lagrangian cone for toric stacks, while in [9], [10] the relationship to the Gromov-Witten invariants of the quotient is given by a wall-crossing formula rather than an adiabatic limit, and the proof involves localization using an auxiliary group action.

To explain the gauged Gromov-Witten potential, recall that the *equivariant cohomology* of a G-space X is the cohomology of the homotopy quotient $X_G = EG \times_G X$ where $EG \to BG$ is a universal G-bundle. Equivariant quantum cohomology should count maps $u : C \to X_G$ where C is a curve equipped with some additional data and u is a holomorphic map. Any such map can be viewed as a map to $BG = EG/G$ together with a lift to X_G. Holomorphic maps to BG correspond to holomorphic G-bundles, and so a holomorphic map to X_G is given by a holomorphic G-bundle $P \to C$ together with a holomorphic section of the associated X bundle $u : C \to P(X) := P \times_G X$. Givental [20] had earlier introduced an *equivariant Gromov-Witten theory*, base on equivariant counts of maps from a curve to X. These counts give rise to a family of products on the *equivariant quantum cohomology* $QH_G(X) = H_G(X) \otimes \Lambda_X^G$ where convergence issues are solved by the introduction of the *Novikov field* $\Lambda_X^G \subset \mathrm{Hom}(H_2^G(X,\mathbb{Z}),\mathbb{Q})$. In the language of maps to the classifying space X_G, Givental's equivariant Gromov-Witten theory corresponds to counting maps from C to X_G whose image lies in a single fiber of the projection $X_G \to BG$, that is, such that the G-bundle is trivial. In order to distinguish the theory here from that of Givental, we will call the theory with non-trivial bundles *gauged Gromov-Witten theory*, and call the map $u : C \to X_G$ a *gauged* map to X.

Gauged Gromov-Witten invariants should be defined as integrals over moduli spaces of gauged maps. In order to obtain proper moduli spaces one needs to impose a *stability* or *moment map* condition as well as compactify the moduli space by, for example, allowing *bubbling*. Mundet and Salamon's work, see [8], provides a symplectic approach to the moduli spaces of gauged maps and construction of invariants in the case that the target is a vector space. Mundet's thesis [39] connects the symplectic approach to an algebraic stability condition, which combines the Ramanathan stability condition for principal bundles with the Hilbert-Mumford stability for the action. Schmitt [44], [45] had earlier constructed a Grothendieck-style compactification in the case that the target is a smooth projective variety, while in the symplectic setting Mundet [40] and Ott [42] show that the connected components of the moduli spaces of semistable gauged maps have a Kontsevich-style compactification.

In general, one needs virtual fundamental cycles to define integration over the moduli spaces. Here we restrict to the case that the target X is a projective G-variety and note that a pair $(P \to C, u : C \to P(X))$ is by definition a morphism from C to the *quotient stack* X/G introduced by Deligne-Mumford [15]. We then use the theory of virtual fundamental classes developed by Behrend-Fantechi [3], based on earlier work of Li-Tian [30], to define gauged Gromov-Witten invariants. The symplectic geometry is then only used as motivation, and to show that the Deligne-Mumford stacks that arise are proper. Of course it is desirable to have semistable reduction theorems to show properness but from the algebraic perspective even the stability conditions are somewhat obscure and we prefer the symplectic route to properness. A good example is the moduli space of semistable bundles on a curve, where properness is immediate from the Narasimhan-Seshadri description as unitary representations of the fundamental group but semistable reduction is more involved.

The theory of gauged Gromov-Witten invariants in genus zero fits into an algebraic formalism that is a "complexification" of the theory of *homotopy associative*, or A_∞ , algebras introduced by Stasheff [46]. In the A_∞ story, which roughly corresponds to "open strings" in the mathematical physics language, one has notions of A_∞ algebras, A_∞ morphisms, and A_∞ traces. These notions are associated with different polytopes called the associahedra, multiplihedra, and cyclohedra respectively. The complexifications of these spaces are the Grothendieck-Knudsen space $\overline{\mathcal{M}}_{0,n}$ of stable n-marked genus 0 curves, Ziltener compactification $\overline{\mathcal{M}}_{n,1}(\mathbb{A})$ of n-marked 1-scaled affine lines, and the Fulton-MacPherson space $\overline{\mathcal{M}}_n(\mathbb{P}) := \overline{\mathcal{M}}_{0,n}(\mathbb{P}, [\mathbb{P}])$ of stable n-marked, genus 0 maps of class $[\mathbb{P}]$ to the projective line $\mathbb{P}$ respectively. The first space leads to the notion of *genus zero cohomological field theory (CohFT)*, in particular, a *CohFT algebra* given by the invariants with multiple incoming markings and a single outgoing marking. The second space is associated to the notion of *morphism of CohFT algebras*, and the third to the notion of *trace on a CohFT algebra*. The following is proved in Gonzalez-Woodward

[21], and is reviewed in [53, Theorem 7.20].

Theorem 1.1 (Gauged Gromov-Witten invariants). *[21] Let X be a smooth polarized projective G-variety and C a smooth connected projective curve. Suppose that every semistable gauged map is stable; then the category of stable gauged maps is a proper Deligne-Mumford stack equipped with a perfect obstruction theory. The gauged Gromov-Witten invariants $(\tau_X^{G,n})_{n\geq 0} : QH_G(X)^n \times H_n(\overline{\mathcal{M}}_n(\mathbb{P})) \to \Lambda_X^G$ define a trace on $QH_G(X)$.*

The gauged Gromov-Witten invariants are also defined for polarized quasiprojective varieties under suitable properness assumptions for the moduli stacks of gauged maps, for example, if the polarization corresponds to an equivariant symplectic form with proper moment map convex at infinity. In particular, gauged Gromov-Witten invariants are defined for a vector space X equipped with the action of a torus G such that the weights are contained in an open half-space in the space of rational weights; this condition implies in particular that the quotient $X/\!\!/G$ is proper under the stable=semistable condition.

One can organize the gauged Gromov-Witten invariants into a formal *gauged Gromov-Witten potential*

$$\tau_X^G : QH_G(X) \to \mathbb{Q}, \quad \alpha \mapsto \sum_{n\geq 0} \tau_X^{G,n}(\alpha, \ldots, \alpha; 1)/n!.$$

Here "formal" means that the sum may not converge and we treat τ_X^G as a Taylor series. The splitting axiom implies that the bilinear form constructed from the second derivatives of the potential is compatible with the quantum product $\star_\alpha$ on $T_\alpha QH_G(X)$ in the usual sense. However this bilinear form will usually be degenerate and so will not define a family of Frobenius algebra structures. It is convenient to throw into the definition of the trace certain *Liouville classes* on the moduli space of gauged maps, in which case, if G is trivial, τ_X^G becomes equivalent to the *graph potential* considered in Givental [20].

From the definition of Mundet stability one expects the gauged Gromov-Witten invariants to be related in the limit that the equivariant symplectic class $[\omega_{X,G}]$ approaches infinity to the Gromov-Witten invariants of the quotient $X/\!\!/G$, or more precisely, to the genus zero *graph potential* for the geometric invariant theory quotient $X/\!\!/G$

$$\tau_{X/\!\!/G} : QH(X/\!\!/G) \to \Lambda_{X/\!\!/G}.$$

For our purposes, it is more natural to work over the larger Novikov field Λ_X^G.

However one sees quickly that the two potentials above cannot be equal, most obviously because they have different domains. Gaio-Salamon [18] showed that the limiting process in which the area of the domain is taken to infinity involves various kinds of bubbling which one hopes to incorporate into a description of the relationship. Salamon and Ziltener [54] suggested that a map $\kappa_X^G : QH_G(X) \to QH(X/\!\!/G)$ might by defined by counting the "affine vortices" that arise in the large area limit. By a Hitchin-Kobayashi correspondence with Venugopalan [49], [50] these maps correspond to the following algebraic objects:

Definition 1.2. (Affine gauged maps) In the case that $X/\!\!/G$ is a free quotient, an *affine gauged map* to X consists of a tuple

$$(p : P \to \mathbb{P}, \lambda : \mathbb{P} \to T^\vee\mathbb{P} \otimes \mathcal{O}(2\infty), u : \mathbb{P} \to P(X), \underline{z} \in (\mathbb{P})^n)$$

consisting of

(a) (Scaling form) a meromorphic one-form λ on $\mathbb{P}$ with only a double pole at $\infty \in \mathbb{P}$ (hence inducing an affine structure on $\mathbb{P} - \{\infty\}$);
(b) (Morphism to the quotient stack, stable at infinity) a morphism $\mathbb{P} \to X/G$, consisting of a G-bundle $p : P \to \mathbb{P}$ and a section $u : \mathbb{P} \to P(X)$ such that $u(\infty)$ is contained in the open subvariety $P(X^{\rm ss})$ associated with the semistable locus $X^{\rm ss} \subset X$; and
(c) (Markings) an n-tuple of distinct points $\underline{z} = (z_1, \ldots, z_n) \in (\mathbb{P})^n$.

In the case that the action of G on the semistable locus of X is only locally free we also allow a stacky structure at infinity. That is, for $r > 0$ an integer we denote by μ_r the group of r-th complex roots of unity and by

$$\mathbb{P}[1, r] := (\mathbb{C}^2 - \{0\})/\mathbb{C}^\times$$

the weighted projective line with a μ_r-singularity at ∞, where $\mathbb{C}^\times$ acts on $\mathbb{C}^2$ with weights $1, r$. The morphism u is then required to be a representable morphism from some $\mathbb{P}[1, r]$ to X/G.

Theorem 1.3 (Quantum Kirwan Morphism). *Suppose that X is a projective G-variety equipped with a polarization such that every semistable point is stable. Integrating over a compactified stack of affine gauged maps defines a morphism of CohFT algebras*

$$\kappa_X^G : QH_G(X) \to QH(X/\!\!/G).$$

By definition a morphism of CohFT algebras consists of a sequence of maps

$$\kappa_X^{G,n} : QH_G(X)^n \times H(\overline{\mathcal{M}}_{n,1}(\mathbb{A})) \to QH(X/\!\!/G), \quad n \geq 0$$

satisfying a splitting axiom that guarantees that the formal map

$$\kappa_X^G : QH_G(X) \to QH(X/\!\!/G), \quad \alpha \mapsto \sum_{n \geq 0} \kappa_X^{G,n}(\alpha, \ldots, \alpha; 1)/n!$$

induces a $\star$-homomorphism on each tangent space. In the case that the *curvature* $\kappa_X^{G,0}$ of the quantum Kirwan morphism vanishes, one obtains in particular a morphism of small quantum cohomology rings

$$\kappa_X^{G,1} : T_0 QH_G(X) \to T_0 QH(X/\!\!/G).$$

In this sense, morphisms of CohFT algebras can be considered as non-linear generalizations of algebra homomorphisms.

More precisely the quantum Kirwan morphism is defined by virtual integration over a compactification $\overline{\mathcal{M}}_{n,1}^G(\mathbb{A}, X)$ of the moduli stack of affine gauged maps equipped with evaluation and forgetful maps

$$\mathrm{ev} \times \mathrm{ev}_\infty : \overline{\mathcal{M}}_{n,1}^G(\mathbb{A}, X) \to (X/G)^n \times \overline{I}_{X/\!\!/G}, \quad f : \overline{\mathcal{M}}_{n,1}^G(\mathbb{A}, X) \to \overline{\mathcal{M}}_{n,1}(\mathbb{A})$$

where $\overline{I}_{X/\!\!/G}$ is the rigidified inertia stack appearing in orbifold Gromov-Witten theory. Properness of this moduli space follows from compactness results of Ziltener [54], [55] (for affine vortices) and a Hitchin-Kobayashi correspondence for affine vortices due to Venugopalan and the author [50]. This space again has a perfect relative obstruction theory, and pull-push using the virtual fundamental class gives rise to the maps $\kappa_{X,n}^G$. From the physics point of view, Witten [51] explained the relationship between correlators in gauged sigma models and sigma models with target the symplectic quotient as a kind of "renormalization" given by "counting pointlike instantons". The "quantum Kirwan map" is a precise mathematical meaning for part of this statement for arbitrary gauged (possibly non-linear) sigma models.

Despite the complicated-looking definition, the stack of affine gauged maps is easy to understand in simple cases.

Example 1.4. (Toric orbifolds) Let X a vector space and G a torus with Lie algebra $\mathfrak{g}$ acting with weights $\mu_1, \ldots, \mu_k \in \mathfrak{g}^\vee$ contained in a half-space a real form $\mathfrak{g}_\mathbb{R}^\vee$. Assume that stable=semistable, so that the quotient $X/\!\!/G$ is a proper toric Deligne-Mumford stack. Then $\mathcal{M}_{1,1}^G(\mathbb{A}, X)$ is isomorphic to the stack of morphisms u from $\mathbb{A}$ to X (that is, X-valued polynomials in a single variable) satisfying the following conditions for any $d \in \mathfrak{g}_\mathbb{Q}$:

(a) (Degree Restriction) the j-th component u_j of u has degree at most (d, μ_j), $j = 1, \ldots, k$, that is,

$$u_j(z) = a_{j,0} + a_{j,1}z + \ldots + a_{j,\lfloor (d,\mu_j) \rfloor} z^{\lfloor (d,\mu_j) \rfloor}$$

for some constants $a_{j,k} \in \mathbb{C}$. If (d, μ_j) is not an integer let $a_{j,(d,\mu_j)} = 0$.

(b) (Stability condition) The collection of leading order coefficients

$$u(\infty) := (a_{j,(d,\mu_j)})_{j=1}^k \in X^{\exp(d)}$$

lies in the semistable locus X^{ss} of X and so defines a point in the substack of the inertia stack given by $\{\exp(d) \times X^{\exp(d),\mathrm{ss}}\}/G \subset I_{X/\!\!/G}$.

If $X/\!\!/G$ is Fano with minimal Chern number at least two then $\kappa_X^G(0) = 0$ and $D_0\kappa_X^G$ is an algebra homomorphism of small quantum cohomologies. Thus knowing the map $D_0\kappa_X^G$ allows to give a presentation of the small quantum cohomology of $X/\!\!/G$.

(a) (Projective space) If $G = \mathbb{C}^\times$ acts by scalar multiplication on $X = \mathbb{C}^k$ then there is a unique morphism $u : \mathbb{A} \to X$ such that all components have degree 1,

$$u(z) = (a_{1,0} + a_{1,1}z, \ldots, a_{k,0} + a_{k,1}z)$$

with marking at $z_1 = 0$, given limit at infinity

$$\mathrm{ev}_\infty u = [a_{1,1}, \ldots, a_{k,1}] \in \mathbb{P}^{k-1} = X /\!\!/ G$$

and vanishing value

$$\mathrm{ev}_0 u = u(0) = (a_{1,0}, \ldots, a_{k,0}) \in \mathbb{C}^k = X$$

at $z_1 = 0 \in \mathbb{C} \cong \mathbb{A}$. Interpreting ξ^k as the Euler class of $\mathbb{C}^k$ this implies that

$$D_0 \kappa_X^G(\xi^k) = q$$

where ξ is the generator of $QH_G(X)$ and $k = \dim(X)$. See Lemma 8.8 of [53] for a justification that the higher codimension strata may be ignored. This gives the standard presentation

$$QH(X /\!\!/ G) = \Lambda_X^G[\xi]/(\xi^k - q)$$

of quantum cohomology of projective space.

(b) (The teardrop orbifold, a weighted projective line) Suppose that $G = \mathbb{C}^\times$ acts on $X = \mathbb{C}^2$ with weights $1, 2$, so that $X /\!\!/ G = \mathbb{P}[1, 2]$ is the teardrop orbifold. Morphisms of class $d \in \mathbb{Q} \cong H_2^G(X, \mathbb{Q})$ exist only if $d \in \mathbb{Z}/2$, and if so are given by pairs of polynomials

$$u(z) = (a_{1,0} + \ldots + a_{1,\lfloor d \rfloor} z^{\lfloor d \rfloor}, a_{2,0} + \ldots + a_{2,2d} z^{2d}).$$

In zero degree, $\overline{M}_{1,1}^G(\mathbb{A}, X, 0) = X /\!\!/ G$ implies that $D_0 \kappa_X^G(1) = 1$. We may interpret $(2\xi)^{2d} \xi^{\lfloor d \rfloor}$ as the Euler class of the vector bundles corresponding to the derivatives of u at 0. Integrating the Euler class amounts to counting such maps whose derivatives $j! a_{1,j}, j < d$, $j! a_{2,j}, j < 2d$ are zero at the marking and, if d is even, have semistable leading order terms

$$\mathrm{ev}_\infty u = [a_{1,d}, a_{2,2d}] \in \mathbb{P}[1, 2]$$

see Lemma 8.8 of [53]. There is a unique such $[u]$ for a given $\mathrm{ev}_\infty u$. For $d = 1/2$ this implies

$$D_0 \kappa_X^G(\xi^2) = 1_{\mathbb{Z}_2}/2$$

half the generator $1_{\mathbb{Z}_2}$ of the twisted sector. For $d = 1$ this implies that

$$D_0 \kappa_X^G(\xi^3) = q/4.$$

Hence the presentation of the quantum cohomology of the teardrop orbifold

$$QH(\mathbb{P}[1, 2]) = \Lambda_X^G[\xi]/(q - 4\xi^3)$$

which is a special case of Coates-Lee-Corti-Tseng [13].

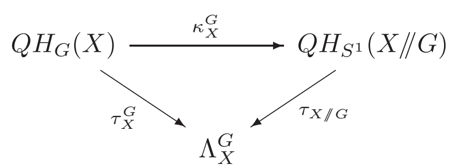
See Example 5.32 for more details. A similar strategy gives relations for the quantum cohomology of any proper toric Deligne-Mumford stack with projective moduli space. In addition, the setup also applies to quotients by non-abelian groups such as quiver varieties.

One of the main results of this paper is that in the large area limit the graph potentials are naturally related via the quantum Kirwan morphism in the following sense. The quantum Kirwan map has an S^1-equivariant extension $\kappa_{X,G} : QH_G(X) \to QH(X/\!\!/G)[[\hbar]]$ (which for any fixed power of q, involves only finitely many powers of $\hbar$). The trace $\tau_{X/\!\!/G}$ has a natural S^1-equivariant extension given by interpreting the $\hbar$ as the first Chern class of the tangent line at the marking (that is, the opposite of the ψ-class). We fix a symplectic class $[\omega_{X,G}] \in H^2_G(X)$ and consider the stability conditions corresponding to the classes $\rho[\omega_{X,G}]$ as $\rho \in (0,\infty)$ varies.

Theorem 1.5 (Adiabatic Limit Theorem). *Suppose that C is a smooth projective curve and X a polarized projective G-variety such that stable=semistable for gauged maps from C to X of sufficiently large ρ. Then*

$$\tau_{X/\!\!/G} \circ \kappa_X^G = \lim_{\rho\to\infty} \tau_X^G. \tag{1}$$

In other words, the diagram



commutes in the limit $\rho \to \infty$.[1] The equality, or commutativity of the diagram, holds in the space of distributions in q, in other words, for each power of q separately. The result has some analogies with the "quantization commutes with reduction" theorem of Guillemin-Sternberg [22]. However, in the intervening years the use of "quantum" has changed so that now it often refers to holomorphic curves. The terminology "adiabatic" arises from the fact that stable gauged maps correspond to minima of an energy function depending on ρ, so the theorem relates minima in the limit $\rho \to \infty$.

One is often interested not in the graph Gromov-Witten invariants but rather in the *localized graph potential* that arises as the fixed point contributions for a circle acting on the domain. As in the work of Givental [20] this localization in the case of Gromov-Witten invariants gives rise to a solution $\tau_{X/\!\!/G,\pm}$ for the quantum differential equation for $X/\!\!/G$: for $\alpha \in QH(X/\!\!/G), \nu \in T_\alpha QH(X/\!\!/G)$ after localization of the equivariant parameter $\hbar$: A formal map

$$\tau_{X/\!\!/G,\pm} : QH(X/\!\!/G) \to QH(X/\!\!/G)[[\hbar^{-1}]], \quad (\pm\hbar)\partial_\nu \tau_{X/\!\!/G,\pm}(\alpha) = \nu \star_\alpha \tau_{X/\!\!/G,\pm}(\alpha).$$

[1] The S^1-subscript was omitted in the published version.

The components of $\tau_{X/\!\!/G,\pm}$ satisfy a version of the Picard-Fuchs equations which play an important role in mirror symmetry [20]. There are similar gauged versions (again formal)

$$\tau_{X,\pm}^G : QH_G(X) \to QH(X/\!\!/G)[[\hbar^{-1}]]$$

which capture the contributions from $0, \infty \in \mathbb{P}$ to the localization formula applied to the *gauged* graph potential. The factorization of the graph potentials generalizes to the gauged setting and we prove a localized adiabatic limit Theorem 1.6 below for the contributions to the fixed point formula.

Theorem 1.6 (Localized adiabatic limit theorem). *Under the assumptions of Theorem 1,* $\tau_{X/\!\!/G,\pm} \circ \kappa_X^G = \tau_{X,\pm}^G$.

In other words, after composition with the quantum Kirwan map the localized graph potential is equal to the localized gauged potential. The result also holds in the "twisted case", that is, after inserting the Euler class of the index of an equivariant bundle on the target, which in good cases describes the localized graph potentials of complete intersections. In this way one obtains formula similar to the "mirror formulas" of Givental [20], Lian-Liu-Yau [31], Iritani [24] and others for localized graph potentials [20] in the toric case, but now for arbitrary geometric invariant theory quotients. However, the approach here is different from that of [20], [31] etc. in that the "mirror map" is expressed as integrals over moduli spaces, while the approach of [20], [31] etc. solves for the "mirror map" as the solution to an algebraic equation. Recently other authors have given geometric interpretations of the mirror map [29], [10].

Various applications have been developed jointly with E. Gonzalez. We were rather surprised to discover that many of the "standard formulas" from classical equivariant symplectic geometry generalize to the quantum case by substituting the quantum Kirwan morphism for the classical Kirwan map; this is rather unexpected since all of these formulas involve functoriality of cohomology in some way which is generally lacking in the quantum setting. For example, there is (i) a wall-crossing "quantum Kalkman" formula for Gromov-Witten invariants under variation of quotient, including invariance in the case of crepant flops (ii) an abelianization "quantum Martin" formula relating Gromov-Witten invariants of quotients by connected reductive groups and their maximal tori, first suggested by Hori-Vafa [23, Appendix] and Bertram-Ciocan-Fontanine-Kim [5] and (iii) a quantum version of Witten's non-abelian localization principle, relating the equivariant quantum cohomology correlators for X with the quantum cohomology correlators for $X/\!\!/G$.

We remark that Ciocan-Fontanine-Kim-Maulik [12] have introduced a different notion of gauged Gromov-Witten invariants of quotients of affine varieties which works in any genus. A symplectic interpretation of these invariants has recently been given by Venugopalan [48]. From this point of view it seems that the Ciocan-Fontanine-Kim-Maulik invariants are a higher genus generalization of the "large area chamber" gauged Gromov-Witten invariants investigated here. See [11], [9] for more recent work.

2. Traces and morphisms of cohomological field theory algebras

To state the main result precisely, we explain what it means to have a "commutative diagram" of cohomological field theories. In this section we describe the moduli spaces of stable curves (complexified associahedron in genus zero), stable parametrized curves (complexified cyclohedron) and stable affine scaled curves (complexified multiplihedron), which lead to the notion of CohFT algebra, trace on a CohFT algebra, and morphism of CohFT algebras respectively, in analogy with the theory of A_∞ spaces, morphisms, and traces. Then we introduce notions of compositions of morphisms and traces, or morphisms of CohFT algebras, which are analogous to the composition of the corresponding A_∞ notions. This makes CohFT algebras into a kind of ∞-category. Notably, we do not have a version of complexified multiplihedron for higher genus curves, which is why the theory here is restricted to genus zero.

2.1. Complexified associahedron and CohFT algebras

Moduli spaces of stable curves were introduced by Mayer and Mumford [38], and further studied in Deligne-Mumford [15]. Moduli of stable marked curves were studied by Grothendieck in 1968 and later by Knudsen [28]. In this section we describe these compactifications and the notion of *cohomological field theory* introduced by Kontsevich-Manin, see [32]. We remark that since the notion of stack is not introduced until [52, Section 4], we avoid it until then and adopt the point of view that the moduli spaces are just topological spaces. First we describe stable curves.

Definition 2.1. Let $n \geq 0$ be an integer.

(a) (Nodal curves) An *n-marked nodal curve* consists of a projective nodal curve C with an n-tuple of distinct, non-singular points $\underline{z} = (z_1, \ldots, z_n) \in C^n$. An *isomorphism* of n-marked nodal curves $(C, \underline{z}), (C', \underline{z}')$ is an isomorphism $\phi : C \to C'$ such that $\phi(z_i) = z_i'$ for $i = 1, \ldots, n$.

(b) (Stable curves) A nodal n-marked curve $C = (C, z)$ is *stable* iff C has finite automorphism group. That is, each genus zero component has at least three *special points* (nodes or markings) and each genus one component has at least one special point. Note that we do not require C to be connected.

(c) (Modular graphs) The combinatorial type Γ of a stable curve is a *modular graph*:

   i. (Graph) an unoriented graph $\Gamma = (\mathrm{Vert}(\Gamma), \mathrm{Edge}(\Gamma))$ whose vertices correspond to irreducible components of C, finite edges $\mathrm{Edge}_{<\infty}(\Gamma)$ to nodes, and semi-infinite edges $\mathrm{Edge}_\infty(\Gamma)$ to markings, equipped with a

   ii. (Genus function) $\mathbb{Z}_{\geq 0}$-valued function $g : \mathrm{Vert}(\Gamma) \to \mathbb{Z}_{\geq 0}$ recording the genus of each irreducible component of C and

   iii. (Labelling of semi-infinite edges) a bijection $l : \mathrm{Edge}_\infty(\Gamma) \to \{1, \ldots n\}$ of the semi-infinite edges with labels $1, \ldots, n$.

   A modular graph is *stable* if it corresponds to a stable curve. That is, each vertex with label 0 resp. 1 has valence at least 3 resp 1.

Any stable curve C has a *universal deformation* given by a family of curves $\pi : C_S \to S$ over a parameter space S uniquely defined up to isomorphism, see for example [2, p. 184]. Let $\overline{M}_{g,n}$ denote the set of isomorphism classes of connected genus g, n-marked stable curves. More naturally one should consider the moduli *stack* of stable curves but we put off discussion of stacks to [52, Section 4]. In the case of genus zero curves the universal deformation, and topology on $\overline{M}_{g,n}$, have a simple description [2, p. 184], [37, Appendix D]: For any marking z_i and irreducible component C_j of C we denote by z_i^j the node in C_j connecting to the irreducible component of C containing z_i, or z_i if z_i is contained in C_j.

Definition 2.2. (Convergence of a sequence of stable curves) A sequence $[C_\nu]$ converges to $[C]$ in $\overline{M}_{g,n}$ if C_ν is isomorphic to $\pi^{-1}(s_\nu)$ for a sequence s_ν converging to s in the base S of the universal deformation. Explicitly, if $g = 0$, a sequence $[(C_\nu, z_{1,\nu}, \ldots, z_{n,\nu})]$ with smooth domain C_ν converges to $[(C, z_1, \ldots, z_n)]$ if there exists, for each irreducible component C_j of C, a sequence of holomorphic isomorphisms $\phi_{j,\nu} : C_j \to C_\nu$ such that

(a) (Limit of a marking) for all i, j, $\lim_{\nu\to\infty} \phi_{j,\nu}^{-1}(z_{i,\nu}) = z_i^j$; and
(b) (Limit of a different parametrization) for all $j \neq k$, $\lim_{\nu\to\infty} \phi_{j,\nu}\phi_{k,\nu}^{-1}$ has limit the constant map with value the node of C_j connecting to C_k.

With the topology induced by this notion of convergence, $\overline{M}_{g,n}$ is compact and Hausdorff (and in fact, a projective variety [15].) For any possible disconnected graph Γ we denote by $M_{g,n,\Gamma}$ the space of isomorphism classes of curves of combinatorial type Γ with n semiinfinite edges and total genus g, and $\overline{M}_{g,n,\Gamma}$ its closure. If $\Gamma = \Gamma_0 \sqcup \Gamma_1$ is a disjoint union then $\overline{M}_{g,n,\Gamma} = \overline{M}_{g,n,\Gamma_0} \times \overline{M}_{g_1,n_1,\Gamma_1}$. The moduli spaces of stable marked curves $\overline{M}_{n,\Gamma}$ satisfy a natural functoriality with respect to morphisms of modular graphs Γ.

Definition 2.3. (Morphisms of modular graphs) A *morphism* of modular graphs $\Upsilon : \Gamma \to \Gamma'$ is a surjective morphism of the set of vertices $\mathrm{Vert}(\Gamma) \to \mathrm{Vert}(\Gamma')$ obtained by combining the following: (these are called *extended isogenies* in Behrend-Manin [4])

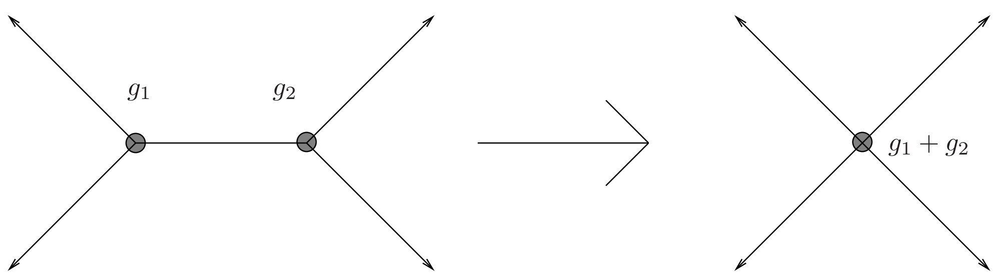


FIGURE 1. Collapsing an edge

(a) Υ *collapses an edge* if the map on vertices $\mathrm{Vert}(\Upsilon) : \mathrm{Vert}(\Gamma) \to \mathrm{Vert}(\Gamma')$ is a bijection except for a single vertex $v' \in \mathrm{Vert}(\Gamma')$ which has two preimages connected by an edge in $\mathrm{Edge}(\Gamma)$, and $\mathrm{Edge}(\Gamma') \cong \mathrm{Edge}(\Gamma) - \{e\}$.

The genus function on Γ' is obtained by push-forward that is, $g(v') = \sum_{\Upsilon(v)=v'} g(v)$.

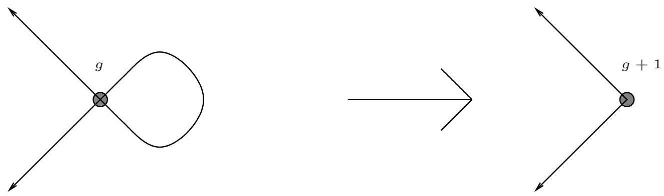


FIGURE 2. Collapsing a loop

(b) Υ *collapses a loop* if the map on vertices is a bijection and $\mathrm{Edge}(\Gamma') \cong \mathrm{Edge}(\Gamma) - \{e\}$ where e is an edge connecting a vertex v to itself. Then the genus functions on Γ, Γ' are identical except at v where $g'(v) = g(v) + 1$.

(c) Υ *cuts an edge* $e \in \mathrm{Edge}(\Gamma)$ if the map on vertices is a bijection, but $\mathrm{Edge}(\Gamma') \cong \mathrm{Edge}(\Gamma) - \{e\} + \{e_+, e_-\}$ where $e_\pm$ are semiinfinite edges attached to the vertices contained in e. Any morphism $\Upsilon : \Gamma \to \Gamma'$ cutting an edge induces an isomorphism $\overline{M}(\Upsilon) : \overline{M}_{g,n,\Gamma'} \to \overline{M}_{g,n,\Gamma}$ obtained by identifying the markings corresponding to the edges $e_\pm$ in Γ'.

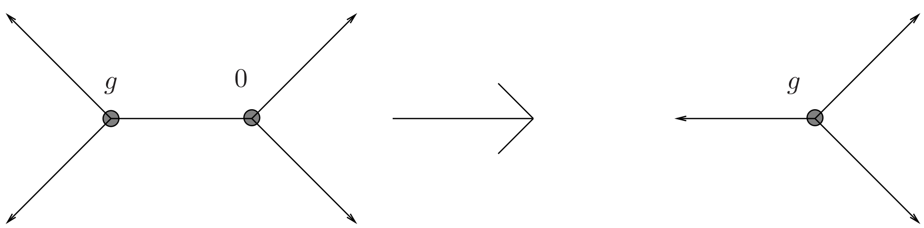


FIGURE 3. Forgetting a tail

(d) Υ *forgets a tail* (semiinfinite edge) $e \in \mathrm{Edge}^\infty(\Gamma)$ if the map on vertices is a bijection, but $\mathrm{Edge}(\Gamma') \cong \mathrm{Edge}(\Gamma) - \{e\}$. In this case there is a morphism $\overline{M}(\Upsilon) : \overline{M}_{g,n,\Gamma} \to \overline{M}_{g,n,\Gamma'}$ obtained by forgetting the corresponding marking and collapsing any unstable components.

Proposition 2.4. The boundary of $M_{g,n,\Gamma}$ in $\overline{M}_{g,n,\Gamma}$ is the union of spaces $M_{g,n,\Gamma'}$ such that Γ is obtained from Γ' by collapsing edges and forgetting loops. The boundary of $\overline{M}_{g,n}$ is a union of the following subspaces (which will be *divisors* with respect to the algebraic structure of the moduli space introduced later)

(a) (Non-separating node) if $2g + n > 3$, a subspace

$$\iota_{g-1,n+2} : D_{g-1,n+2} \to \overline{M}_{g,n}$$

equipped with an isomorphism

$$\varphi_{g-1,n+2} : D_{g-1,n+2} \to \overline{M}_{g-1,n+2}.$$

The inclusion is obtained by identifying the last two marked points.

(b) (Separating node) for each splitting $g = g_1 + g_2, \{1,\ldots,n\} = I_1 \cup I_2$ with $2g_j + |I_j| \geq 3, j = 1,2$, a subspace

$$\iota_{g_1+g_2,I_1\cup I_2} : D_{g_1+g_2,I_1\cup I_2} \to \overline{M}_{g,n}$$

corresponding to the formation of a separating node, splitting the surface into pieces of genus g_1, g_2 with markings I_1, I_2, equipped with an isomorphism

$$\varphi_{g_1+g_2,I_1\cup I_2} : D_{g_1+g_2,I_1\cup I_2} \to \overline{M}_{g_1,|I_1|+1} \times \overline{M}_{g_2,|I_2|+1}$$

(except that on the level of orbifolds in the case $I_1 = I_2 = \emptyset$ and $g_1 = g_2$ there is an additional automorphism exchanging the components.)

The pull-back $\iota^*_{g_1+g_2,I_1\cup I_2}\beta$ of any class $\beta \in H(\overline{M}_{g,n})$ has a Künneth decomposition

$$\iota^*_{g_1+g_2,I_1\cup I_2}\beta = \sum_{j\in J} \beta_{1,j} \otimes \beta_{2,j} \tag{2}$$

for some index set J and classes $\beta_{k,j} \in H(\overline{M}_{g_k,|I_k|+1})$. Each boundary divisor $D_{g-1,n+2}$ or $D_{g_1+g_2,I_1\cup I-2}$ has a homology class $\overline{M}_{g,n}$, and since $\overline{M}_{g,n}$ is a rational homology manifold each of these homology classes has a dual class $\gamma_{g-1,n+2}$ resp. $\gamma_{g_1+g_2,I_1\cup I_2}$ in $H^2(\overline{M}_{g,n},\mathbb{Q})$. For the following, see Manin [32].

Definition 2.5. A *cohomological field theory* (CohFT) with values in a field Λ is a $\mathbb{Z}_2$-graded vector space V equipped with a symmetric non-degenerate bilinear form $V \times V \to \Lambda$ and collection of S_n-invariant (with Koszul signs) *correlators*

$$V^n \times H(\overline{M}_{g,n}) \to \Lambda, \quad (\alpha,\beta) \mapsto \langle \alpha;\beta\rangle_{g,n}, \;\; g,n \geq 0$$

where by convention $H(\overline{M}_{g,n}) = \Lambda$ if $\overline{M}_{g,n} = \emptyset$, satisfying the following two splitting axioms:

(a) (Non-separating node) if $g \geq 1$ then

$$\langle \alpha; \beta \cup \gamma_{g-1,n+2}\rangle_{g,n} = \sum_k \langle \alpha, \delta_k, \delta^k; \iota^*_{g-1,n+2}\beta\rangle_{g-1,n+2}$$

where δ_k, δ^k are dual bases for V;

(b) (Separating node) if $2g + n \geq 4$, $I_1 \cup I_2$ is a partition of $\{1,\ldots,n\}$, and $g = g_1 + g_2$ with $2g_i + |I_i| \geq 3$ for $i = 1,2$ then

$$\langle \alpha; \beta \cup \gamma_{g_1+g_2,I_1\cup I_2}\rangle_{g,n} = \sum_k \langle (\alpha_i)_{i\in I_1}, \delta_k; \cdot\rangle_{g_1,|I_1|+1} \langle (\alpha_i)_{i\in I_2}, \delta^k; \cdot\rangle_{g_2,|I_2|+1} (\iota^*_{g_1+g_2,I_1\cup I_2}\beta)$$

where the dots indicate insertion of the Künneth components of $\iota^*_{g_1+g_2,I_1\cup I_2}\beta$, δ_k, δ^k are dual bases for V, and there is an additional factor of 2 in the exceptional case $g_1 = g_2$, $I_1 = I_2 = \emptyset$ arising from the additional automorphism.

That is, if β is as in (2) then

$$\langle \alpha; \beta \cup \gamma_{g_1+g_2, I_1 \cup I_2} \rangle_{g,n} = \sum_{j \in J, k} \langle (\alpha_i)_{i \in I_1}, \delta_k; \beta_{1,j} \rangle_{g_1, |I_1|+1} \langle (\alpha_i)_{i \in I_2}, \delta^k; \beta_{2,j} \rangle_{g_2, |I_2|+1}.$$

The correlators of any cohomological field theory define a family of associative algebra structures. In the standard axiomatization these are part of the associated *Frobenius manifold structure* on V [32]. However for the purposes of functoriality it is helpful to keep the "algebra" and "metric" parts of this structure separate, and instead we define the following:

Definition 2.6. A *CohFT algebra* consists of a $\mathbb{Z}_2$-graded vector space V and a collection of S_n-invariant (with Koszul signs) multilinear maps

$$\mu^n : V^n \times H(\overline{M}_{0,n+1}) \to V, n \geq 2$$

satisfying the splitting axiom for any subset $I \subset \{1, \ldots, n\}$ of order at least two:

$$\mu^n(\alpha_1, \ldots, \alpha_n; \gamma_{0, I \cup (I^c \cup \{0\})} \cup \beta) = \sum_j \mu^{n-|I|+1}(\alpha_i, i \notin I, \mu^{|I|}(\alpha_i, i \in I; \beta_{1,j}); \beta_{2,j}) \tag{3}$$

where $\iota_I^* \beta = \sum_j \beta_{1,j} \otimes \beta_{2,j}$ is the Künneth decomposition of the restriction of β as in (2).

Proposition 2.7. Any CohFT gives rise to a CohFT algebra.

Proof. By restricting to genus zero, and using duality to put one factor of V on the right. The splitting axiom (3) is a special case of the splitting axiom in Definition 2.5.

Remark 2.8.
(a) Manin [32] terms a set of such maps a Comm_∞-*structure* on V. However we avoid this terminology since it isn't clear what this is a homotopy version of.
(b) The CohFT algebra structure does not require a metric, so a CohFT may be thought of roughly speaking as a CohFT algebra plus a metric. We remark that a more natural definition of CohFT may be obtained by allowing incoming and outgoing markings. However, we prefer to write the classes on the left to emphasize the analogy with A_∞ spaces.
(c) The various relations in $\overline{M}_{0,n+1}$ give rise to relations on the maps μ^n. In particular the map $\mu^2 : V \times V \to V$ is associative, by the splitting axiom and the relation $[D_{0,\{0,3\} \cup \{1,2\}}] = [D_{0,\{0,1\} \cup \{2,3\}}] \in H^2(\overline{M}_{0,4})$.
(d) The notion of CohFT algebra is the "complex analog" of the notion of A_∞ algebra in the following imprecise sense. Denote by $\overline{M}_{0,n}^{\mathbb{R}}$ the moduli space of projective nodal curves C equipped with an anti-holomorphic involution fixing the markings, that is, the moduli space of *stable n-marked disks* where the markings are not necessarily in cyclic order. The symmetric group S_n acts canonically on $\overline{M}_{0,n}^{\mathbb{R}}$ by permuting the markings. The quotient of $\overline{M}_{0,n}^{\mathbb{R}}$ by the action of S_n is homeomorphic to the $(n-1)$-st associahedron Assoc_{n-1} introduced in Stasheff [46]. An A_∞ *space* is a space

X equipped with a collection of maps $X^n \times \mathrm{Assoc}_n \to X, n \geq 2$ satisfying a splitting axiom for the restriction to the boundary, and A_∞ *algebras* arise as spaces of chains on A_∞ spaces. To obtain the notion of CohFT algebra we replace X by a vector space and Assoc_n by the cohomology of its complexification. One could also imagine a cochain-level version but there are reasons to expect that this gives nothing new, see Teleman [47]

(e) The notion of a genus zero CohFT can be repackaged in terms of a non-linear structure called a *Frobenius manifold*, which is a non-linear generalization of the notion of *Frobenius algebra* (unital algebra with compatible metric.) Any genus zero CohFT (with unit and grading) gives rise to a Frobenius manifold whose potential is

$$f : V \to \Lambda, \quad f(v) = \sum_{n \geq 3} \frac{1}{n!} \langle v, \ldots, v; 1 \rangle_{0,n}.$$

The third derivatives of f give rise to a family of algebra structures

$$\star_v : T_v V^2 \to T_v V.$$

These give rise to a family of connections depending on a parameter $\hbar$,

$$\nabla_\hbar : \Omega^0(V, TV) \to \Omega^1(V, TV), \quad \nabla_{\hbar,\xi} \sigma(v) = (\mathrm{d}\sigma(\xi))(v) - (1/\hbar) \xi \star_v \sigma(v).$$

The associativity of $\star$ translates into the flatness of $\nabla_\hbar$, so that locally there exist sections $\sigma : V \to TV$ satisfying

$$\text{(Quantum Differential Equation)} \quad \hbar \partial_\xi \sigma(v) = \xi \star_v \sigma, \forall v, \xi \in V. \tag{4}$$

For a full discussion of the correspondence between Frobenius manifolds and CohFT's the reader is referred to Manin [32].

2.2. Complexified cyclohedron and traces on CohFT algebras

In this section we study the moduli spaces of stable marked *parametrized* curves. These are a special case of moduli spaces of stable maps (the degree one case) but we prefer to view them in a different way, as a special case of the Fulton-MacPherson construction [17]. We then discuss the associated notion of *trace* on a CohFT algebra. Let C be a smooth connected projective curve.

Definition 2.9. (a) (Parametrized nodal curves) A *C-parametrized nodal curve* is a (possibly disconnected) nodal curve $\hat{C}$ equipped with a morphism $u : \hat{C} \to C$ of homology class $u_*[\hat{C}] = [C]$ and with the same arithmetic genus. That is, $\hat{C}$ is the union of irreducible components $C_0, \ldots, C_r$ where u maps the *principal component* C_0 isomorphically onto C and u maps the other irreducible *bubble components* $C_1, \ldots, C_r$ onto points. Since the arithmetic genus of $\hat{C}$ is the same as that of C, the bubble components must be rational. A *marking* of a C-parametrized curve is an n-tuple $\underline{z} = (z_1, \ldots, z_n)$ of points in $\hat{C}^n$ distinct from the nodes and each other.

An *isomorphism* of such curves is an isomorphism of nodal curves which induces the identity on C.

(b) (Stable parametrized curves) A C-parametrized curve is *stable* if it has no infinitesimal automorphisms, that is, each non-principal irreducible component of $\hat{C}$ has at least three marked or nodal points.

(c) (Rooted forests) Any C-parametrized curve has a *combinatorial type* which is a forest Γ (finite collection of trees) with a distinguished *root vertex* corresponding to the principal component and a labelling of the semiinfinite edges given by a bijection $l : \mathrm{Edge}_\infty(\Gamma) \to \{1, \ldots, n\}$. A rooted forest is *stable* if it corresponds to a stable parametrized curve, that is, each non-root vertex has valence at least three.

The set $\overline{M}_n(C)$ of isomorphism classes of connected stable C-parametrized curves has a natural topology, similar to that of $\overline{M}_{0,n}$ in genus zero: The following can be taken as a definition or a proposition using a suitable construction of the universal deformation of a stable map to C:

Definition 2.10. (Convergence of a sequence of parametrized stable curves) Suppose C has genus 0. A sequence $[(\hat{C}_\nu, u_\nu)]$ with smooth domain $\hat{C}_\nu$ *converges* to $[(\hat{C}, u)]$ if there exists, for each irreducible component $\hat{C}_j$ of the limit $\hat{C}$, a sequence of holomorphic isomorphisms $\phi_{j,\nu} : \hat{C}_j \to \hat{C}_\nu$ such that

(a) (Limit of a marking) for all i, j, $\lim_{\nu\to\infty} \phi_{j,\nu}^{-1}(z_{i,\nu}) = z_i^j$, the node in C_j connecting to the irreducible component of C containing z_i, or z_i if z_i is contained in C_j;

(b) (Limit of a different parametrization) for all $j \neq k$, $\lim_{\nu\to\infty} \phi_{j,\nu}\phi_{k,\nu}^{-1}$ has limit the constant map with value the node of C_j connecting to C_k; and

(c) (Limit of the map) for all j, $\lim \phi_{j,\nu}^* u_\nu = u|C_j$.

Convergence for nodal domains $\hat{C}_\nu$ is defined similarly, by considering convergence on each irreducible component separately.

The definition for arbitrary genus is similar, but the maps $\phi_{j,\nu}$ exist only after removing small neighborhoods of the nodes. The topology on $\overline{M}_n(C)$ induced by this notion of convergence is compact and Hausdorff, and a special case of the Fulton-MacPherson compactification of configuration spaces considered in [17]. The open stratum $M_n(C)$ of $\overline{M}_n(C)$ is the configuration space $\mathrm{Conf}_n(C)$ of n-tuples of distinct points on C.

More generally for any rooted forest Γ with n semiinfinite edges we denote by $M_{n,\Gamma}(C)$ the moduli space of isomorphism classes with type Γ and by $\overline{M}_{n,\Gamma}(C)$ its closure. The moduli spaces of stable marked curves $\overline{M}_{n,\Gamma}(C)$ satisfy a natural functoriality with respect to morphisms of rooted forests Γ. We say that a *morphism of rooted forests* is a morphism of modular graphs corresponding to the rooted forests mapping the root vertex to the root vertex.

Proposition 2.11. (Morphisms of moduli spaces associated to morphisms of forests)

(a) Any morphism of rooted forests $\Upsilon : \Gamma \to \Gamma'$ induces a morphism of moduli spaces $\overline{M}(\Upsilon) : \overline{M}_{n,\Gamma}(C) \to \overline{M}_{n,\Gamma'}(C)$.

(b) The boundary of $M_{n,\Gamma}(C)$ is the union of spaces $M_{n,\Gamma'}(C)$ such that there is a morphism of rooted forests $\Gamma' \to \Gamma$ collapsing an edge.
(c) If Γ' is obtained from Γ by cutting an edge, then there is an isomorphism $\overline{M}_{n,\Gamma'}(C) \to \overline{M}_{n,\Gamma}(C)$ identifying the vertices corresponding to the additional semi-infinite edges.

The proof is standard from properties of moduli spaces of stable maps. Note that if $\Gamma' = \Gamma_0 \cup \Gamma_1$ is disconnected where Γ_0 contains the root vertex then $\overline{M}_{n,\Gamma}(C) \cong \overline{M}_{n_0,\Gamma_0}(C) \times \overline{M}_{0,n_1,\Gamma_1}$ where n_j is the number of semi-infinite edges of Γ_j. Thus the boundary of $\overline{M}_{n,\Gamma}$ is the union of products of lower-dimensional moduli spaces of C-parametrized stable curves and stable curves.

Remark 2.12. (Relation to the cyclohedron) Let C be a projective line. Any anti-holomorphic involution of C induces an anti-holomorphic involution of $\overline{M}_n(C)$, with fixed point set $\overline{M}_n(C)^{\mathbb{R}}$ identified with the moduli space of stable parametrized n-marked disks. The symmetric group S_n acts by permutation, and the quotient by S_{n-1} is isomorphic to the subset $\overline{M}_n(C)^{\mathbb{R},+}$ of $\overline{M}_n(C)^{\mathbb{R}}$ such that the marked points $z_0, \ldots, z_n$ occur in cyclic order around the boundary of the disk. The action of S^1 by rotation preserves $\overline{M}_n(C)^{\mathbb{R},+}$ and the quotient is the cyclohedron Cycl_n, that is, the moduli space of points on the circle compactified by allowing bubbling, see Markl [34]. In this sense it is slight abuse of terminology to call $\overline{M}_n(C)$ the complexification of Cycl_n; rather, $\overline{M}_n(C)$ is the complexification of a circle bundle over Cycl_n.

The boundary structure of the moduli space $\overline{M}_n(C)$ is described in the following.

Proposition 2.13. The boundary of $\overline{M}_n(C)$ is the union of the following subspaces (which will be divisors once the algebraic structure on $\overline{M}_n(C)$ is introduced): For each subset $I \subset \{1, \ldots, n\}$ of order at least two a subspace $\iota_I : D_I \to \overline{M}_n(C)$ where the markings for $i \in I$ have bubbled off onto an (unparametrized) sphere bubble. The subspace D_I admits a homeomorphism $\varphi_I : D_I \to \overline{M}_{0,|I|+1} \times \overline{M}_{n-|I|+1}(C)$.

For any $\beta \in \overline{M}_n(C)$, the pull-back $\iota_I^* \beta$ to a subspace D_I has a Künneth decomposition

$$\iota_I^* \beta = \sum_{j \in J} \beta_{1,j} \otimes \beta_{2,j} \tag{5}$$

for some index set J and classes $\beta_{1,j} \in H(\overline{M}_{0,|I|+1})$ and $\beta_{2,j} \in \overline{M}_{n-|I|+1}(C)$. In general, the moduli space of stable maps is not smooth. However, the space $\overline{M}_n(C)$, as a special case of Fulton-MacPherson [17], is a compact smooth manifold. In particular, any subset D_I has a homology class $[D_I] \in H_2(\overline{M}_n(C), \mathbb{Z})$ and a dual class $\gamma_I \in H^2(\overline{M}_n(C), \mathbb{Z})$, although we work with rational coefficients below. Let Λ be a vector space.

Definition 2.14. (Trace on a CohFT algebra) A (C-based, Λ-valued) *trace* on a CohFT algebra V is a collection of S_n-invariant (with Koszul signs) multilinear maps

$$\tau^n : V^n \times H(\overline{M}_n(C)) \to \Lambda, \quad n \geq 0$$

satisfying a splitting axiom for any $I \subset \{1, \ldots, n\}$

$$\tau^n(\alpha; \beta \cup \gamma_I) = \tau^{n-|I|+1}(\alpha_i, i \notin I, \mu^{|I|}(\alpha_i, i \in I; \cdot); \cdot)(\iota_I^* \beta)$$

where γ_I is the dual class to D_I and the $\cdot$'s denote insertion of the Künneth components of β. That is, with β as in (5),

$$\tau^n(\alpha; \beta \cup \gamma_I) = \sum_{j \in J} \tau^{n-|I|+1}(\alpha_i, i \notin I, \mu^{|I|}(\alpha_i, i \in I; \beta_{1,j}); \beta_{2,j}).$$

Remark 2.15. (a) In our main application, gauged Gromov-Witten invariants will define a Λ_X^G-valued trace on $QH_G(X)$, which is a CohFT defined over the field of fractions of $H(BG) \otimes \Lambda_X^G$. In other words, the space Λ above need not be the ring or field over which the CohFT algebra or CohFT is defined.

(b) One should compare the notion of trace with that for A_∞ algebras described in [33, Proposition 2.14]. The corresponding notion for an A_∞ space X consists of a sequence of maps

$$X^n \times \mathrm{Cycl}_n \to Y, \quad n \geq 0$$

to an ordinary space Y, satisfying a suitable splitting axiom.

Any trace $(\tau^n)_{n \geq 0}$ on a CohFT algebra V defines a formal map

$$\tau : V \to \Lambda, \quad v \mapsto \sum_n \frac{1}{n!} \tau^n(v, \ldots, v; 1) \tag{6}$$

often called a *potential*. The splitting axiom implies that the second derivatives of τ with two point classes inserted define a Λ-valued family of bilinear forms on $T_v V$ compatible with the multiplications $\star_v$ on $T_v V$:

2.3. Complexified multiplihedron and morphisms of CohFT algebras

In this section we review a construction of Ma'u-Woodward [36], based on earlier work of Ziltener [54], which introduces a compactification of the moduli space of distinct points on the affine line up to translation. This compactification "complexifies" the multiplihedron in the same way that the Grothendieck-Knudsen space and the Fulton-MacPherson spaces complexify the associahedron and cyclohedron respectively. We then discuss the associated notion of *morphism* of CohFT algebras. Let $\mathbb{A}$ denote an affine line over $\mathbb{C}$, unique up to isomorphism. We denote by

$$\Omega^1(\mathbb{A}, \mathbb{C})^{\mathbb{C}} = \{w \mathrm{d}z | w \in \mathbb{C}\} \cong \mathbb{C}$$

the space of $\mathbb{C}$-invariant one-forms on $\mathbb{A}$.

Definition 2.16. (Scaled affine line) A *scaling* of an affine line $\mathbb{A}$ is a translation-invariant, non-zero one form $\lambda \in \Omega^1(\mathbb{A}, \mathbb{C})^{\mathbb{C}}$. A *scaled affine line* is an affine line equipped with a scaling. An *n-marking* of an affine line is an n-tuple $\underline{z} = (z_1, \ldots, z_n)$ of distinct points in $\mathbb{A}^n$. An *isomorphism* of scaled n-marked affine lines is an affine isomorphism $\psi : C_0 \to C_1$, such that $\psi^* \lambda_1 = \lambda_0$ and $\psi(z_{0,i}) = z_{1,i}, i = 1, \ldots, n$.

Let $M_{n,1}(\mathbb{A})$ denote the moduli space of isomorphism classes of scaled n-marked affine lines. If $\mathbb{A}$ is a scaled affine line then the group of automorphisms of $\mathbb{A}$ preserving the scaling is the additive group $\mathbb{C}$ acting on $\mathbb{A}$ by translation. Thus

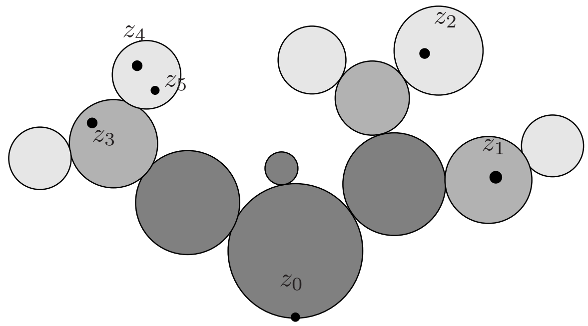


Figure 4. An affine scaled curve

Proposition 2.17. The moduli space $M_{n,1}(\mathbb{A})$ may be identified with the configuration space $\mathrm{Conf}_n(\mathbb{A})$ of n-tuples of distinct points on $\mathbb{A}$ up to the action of $\mathbb{C}$ by translation, $M_{n,1}(\mathbb{A}) \cong \mathrm{Conf}_n(\mathbb{A})/\mathbb{C}$.

Proof. For any tuple $(z_1, \ldots, z_n, \lambda)$ we take the unique rescaling so that $\lambda = \mathrm{d}z$ and then take the associated configuration. Conversely, any configuration defines an affine scaled map by taking the scaling $\lambda = \mathrm{d}z$ to be standard.

Remark 2.18. (Two-forms instead of one-forms) The moduli space $M_{n,1}(\mathbb{A})$ can be viewed in a different way: Any scaling λ gives rise to a real area form $\omega_{\mathbb{A}} := \lambda \wedge \overline{\lambda}$ on $\mathbb{A}$. Replacing λ with $\omega_{\mathbb{A}}$ amounts to forgetting a complex phase; thus, one can view $M_{n,1}(\mathbb{A})$ as the moduli space of data $(z_1, \ldots, z_n, \omega, \phi)$ where $z_1, \ldots, z_n \in \mathbb{A}$ are distinct points, $\omega_{\mathbb{A}} \in \Omega^2(\mathbb{A}, \mathbb{R})$ is a translationally-invariant area form, and $\phi \in U(1)$ is a phase. Any automorphism ψ of $\mathbb{A}$ has a well-defined *argument* $\arg(\psi) \in U(1)$ giving the angle of rotation, and ψ acts on $(\omega_{\mathbb{A}}, \phi)$ by $(\psi^* \omega_{\mathbb{A}}, \arg(\psi)\phi)$. This is the point of view taken in Ziltener's thesis [54].

The moduli space $M_{n,1}(\mathbb{A})$ has a natural compactification obtained by allowing bubbles with degenerate scalings.

Definition 2.19. (Stable nodal scaled affine lines) A *possibly degenerate scaling* on an affine line $\mathbb{A}$ is an element of the set

$$\overline{\Omega}^1(\mathbb{A}, \mathbb{C})^{\mathbb{C}} = \Omega^1(\mathbb{A}, \mathbb{C})^{\mathbb{C}} \cup \{\infty\} \cong \mathbb{P}.$$

A possibly degenerate scaling λ is *degenerate* if $\lambda = 0$ or $\lambda = \infty$ and is *non-degenerate* otherwise. The action of the group of automorphisms $\mathrm{Aut}(\mathbb{A})$ on $\Omega^1(\mathbb{A}, \mathbb{C})^{\mathbb{C}}$ by pull-back extends naturally to an action on $\overline{\Omega}^1(\mathbb{A}, \mathbb{C})$, with fixed points $\{0\}, \{\infty\}$.

Let C be a nodal curve and ω_C the dualizing sheaf, whose sections consist of one-forms possibly with poles at the nodes of C whose residues on either side of a node are equal. Let $\mathbb{P}(\omega_C \oplus C)$ denote the projectivization of ω_C. Let $\lambda : C \to \mathbb{P}(\omega_C \oplus \mathbb{C})$ be a section. A component C_i of C will be called *colored* if the the restriction of λ to C_i is finite and non-zero. A *nodal marked scaled affine line* is a datum $(C, z_0, \ldots, z_n, \lambda)$ such that the following holds:

(Monotonicity) on any non-self-crossing path from a marking z_i to the root marking z_0, there is exactly one *colored irreducible component* with finite scaling; and the irreducible components before (resp. after) this irreducible component have infinite (resp. zero scaling).

See Figure 4, where irreducible components with infinite resp. finite, non-zero resp. zero scaling are shown with dark resp. grey resp. light grey shading. An *isomorphism* of nodal marked scaled affine lines $(C_j, \underline{z}_j, \lambda_j), j = 0, 1$ is an isomorphism of nodal curves $\phi : C_0 \to C_1$ intertwining the (possibly degenerate) scalings and markings in the sense that $\phi^* \lambda_1 = \lambda_0$ and $\phi(z_{0,i}) = z_{1,i}$. A nodal marked scaled affine line is *stable* if it has no automorphisms, or equivalently, if each irreducible component with finite scaling has at least two special points, and each irreducible component with degenerate scaling has at least three special points.

The space $\overline{M}_{n,1}(\mathbb{A})$ of isomorphism classes of connected stable scaled n-marked lines has a natural topology, similar to the topology on the moduli space of stable curves. Given a stable affine scaled curve $(C, z_1, \ldots, z_n, \lambda)$, for any marking z_i and irreducible component C_j we denote by z_i^j the node in C_j connecting to the irreducible component of C containing z_i, or z_i if z_i is contained in C_j. The following can be taken as a definition or a proposition with a suitable notion of family of stable scaled marked lines, see [52, Example 4.2].

Definition 2.20. (Convergence of a sequence of nodal scaled affine lines) A sequence $[(C_\nu, \underline{z}_\nu, \lambda_\nu)]$ with smooth domain C_ν *converges* to $[(C, \underline{z}, \lambda)]$ if there exists, for each irreducible component C_j of the limit C, a sequence of holomorphic isomorphisms $\phi_{j,\nu} : C_j \to C_\nu$ such that

(a) (Limit of the scaling) $\lim_{\nu\to\infty} \phi_{j,\nu}^* \lambda_\nu = \lambda | C_j$;
(b) (Limit of a marking) $\lim_{\nu\to\infty} \phi_{j,\nu}^{-1}(z_{i,\nu}) = z_i^j$; and
(c) (Limit of a different parametrization) $\lim_{\nu\to\infty} \phi_{j,\nu}\phi_{i,\nu}^{-1}$ has limit the constant map with value the node of C_j connecting to C_i.

Convergence for sequences with nodal domain is defined similarly.

Example 2.21. (Two markings converging) If $C_\nu = \mathbb{P} = \mathbb{A} \cup \{\infty\}$ and two points $z_{1,\nu}, z_{2,\nu}$ come together in the sense that $\lim_{\nu\to\infty} z_{1,\nu} - z_{2,\nu} \to 0$, then there exists a sequence of holomorphic maps $\phi_\nu : \mathbb{P} \to C_\nu$ such that $\phi_\nu^{-1}(z_{1,\nu}), \phi_\nu^{-1}(z_{2,\nu})$ converge to distinct points, and the scaling $\phi_\nu^*(\lambda_\nu)$ converges to zero. The limiting configuration consists of a irreducible component with two markings and one node with zero scaling, and an irreducible component with finite scaling, one node, and the root marking z_0. See Figure 5.

Example 2.22. (Two markings diverging) If $C_\nu = \mathbb{P}$ for all ν with constant scaling λ_ν and two points $z_{1,\nu}, z_{2,\nu}$ go to infinity in $\mathbb{A} \subset \mathbb{P}$ in different directions, then for $k \in \{1, 2\}$ there exists (i) a sequence of holomorphic maps $\phi_{k,\nu} : \mathbb{C} \to C_\nu$ such that $\phi_{k,\nu}^{-1}(z_{k,\nu})$ and $\phi_{k,\nu}^* \lambda_\nu$ converge for $k \in \{1, 2\}$ and (ii) a sequence $\phi_{12,\nu} : \mathbb{C} \to C_\nu$ such that $\phi_{12,\nu}^{-1} z_{k,\nu}$ for $k = 1, 2$ converge to distinct points and $\phi_{12,\nu}^* \lambda_\nu$ converges to infinity. The limiting configuration consists of two components with a single marking and node and finite scaling, and a component with two nodes, the root marking, and infinite scaling. See Figure 6.

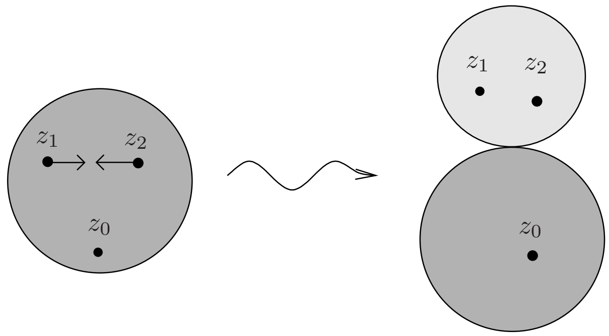


FIGURE 5. Two markings converging

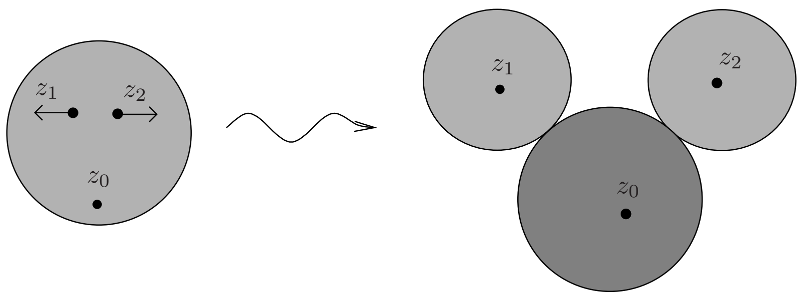


FIGURE 6. Two markings diverging

We denote by $\overline{M}_{n,1}(\mathbb{A})$ the space of isomorphism classes of connected nodal scaled lines, equipped with the topology above. By Ma'u-Woodward [36] $\overline{M}_{n,1}(\mathbb{A})$ is a compact Hausdorff space.

Definition 2.23. (Combinatorial types of nodal scaled affine lines) The *combinatorial type* of a connected scaled affine line is a *colored tree* consisting of a tree $\Gamma = (\mathrm{Vert}(\Gamma), \mathrm{Edge}(\Gamma))$ together with a partition of the vertices

$$\mathrm{Vert}(\Gamma) = \mathrm{Vert}^0(\Gamma) \cup \mathrm{Vert}^1(\Gamma) \cup \mathrm{Vert}^\infty(\Gamma)$$

and a labelling of its semi-infinite edges given by a bijection $\mathrm{Edge}_\infty(\Gamma) \to \{1, \ldots, n\}$ that satisfies the combinatorial version of the monotonicity condition:

> (Monotonicity) on any non-self-crossing path from a semi-infinite edge labelled j to the semi-infinite edge labelled 0, there is exactly one vertex in $\mathrm{Vert}^1(\Gamma)$, all vertices before resp. after are in $\mathrm{Vert}^0(\Gamma)$ resp. $\mathrm{Vert}^\infty(\Gamma)$.

A colored tree is *stable* if it corresponds to a stable affine scaled line, that is, each vertex $v \in \mathrm{Vert}^0(\Gamma)$ resp. $\mathrm{Vert}^\infty(\Gamma)$ resp. $\mathrm{Vert}^1(\Gamma)$ has valence at least 3 resp. 3 resp. 2.

We call the vertices in $\mathrm{Vert}^1(\Gamma)$ the *colored vertices*. Colored trees can be pictured as trees where part of the tree containing the root semi-infinite edge edge is "below water" and part "above water"; the colored vertices in $\mathrm{Vert}^1(\Gamma)$ are those "at the water level". The monotonicity condition then says that the path from any "above water" semiinfinite edge to the unique "below water" semiinfinite edge passes through the water surface exactly once. However, in our trees we adopt the standard convention of having the root edge (which corresponds to an outgoing marking) at the top of the picture.

More generally we allow disconnected curves where each connected component is either a nodal affine curve, or a nodal curve with infinite or zero scaling.

Definition 2.24. (Colored forests) A colored forests Γ is a union of components that are either colored trees, or ordinary trees with all vertices in $\mathrm{Vert}^0(\Gamma)$ or all vertices in $\mathrm{Vert}^\infty(\Gamma)$.

For any colored forest Γ with n semiinfinite edges we denote by $M_{n,1,\Gamma}(\mathbb{A})$ the space of isomorphism classes of scaled lines of combinatorial type Γ, and $\overline{M}_{n,1,\Gamma}(\mathbb{A})$ its closure.

Definition 2.25. (Morphisms of colored forests) A *morphism of colored forests* from Γ to Γ' is a combination of the following simple morphisms:

(a) (Collapsing edges without relations) $\Upsilon : \Gamma \to \Gamma'$ *collapses an edge* if Υ is injective except that it maps two vertices in $\mathrm{Vert}^0(\Gamma) \cup \mathrm{Vert}^1(\Gamma)$ to the same vertex in $\mathrm{Vert}(\Gamma')$ or two vertices in $\mathrm{Vert}^\infty(\Gamma)$ with the same vertex in $\mathrm{Vert}^\infty(\Gamma)$. (In other words, any edge except those connecting $\mathrm{Vert}^1(\Gamma)$ with $\mathrm{Vert}^\infty(\Gamma)$).

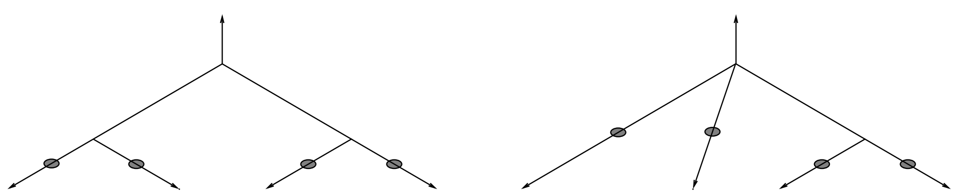


FIGURE 7. Collapsing an edge connecting two vertices of the same type

(b) (Collapsing edges with relations) $\Upsilon : \Gamma \to \Gamma'$ *collapses edges* if Υ is injective except for one colored vertex of Γ' whose inverse image in $\mathrm{Vert}(\Gamma)$ is a collection of colored vertices in Γ and a single vertex in $\mathrm{Vert}^\infty(\Gamma)$, joined to each of the colored vertices by a single edge. Note that one cannot write such a morphism as a composition of morphisms each collapsing a single edge, since there is no way to assign the coloring of vertices of the resulting graph which results in a colored forest.

(c) (Cutting an edge without relations) $\Upsilon : \Gamma \to \Gamma'$ *cuts an edge* of Γ if the vertices are the same, but Γ' has one fewer edge than Γ and the edge does not lie between the colored vertices and the root edge.

(d) (Cutting edges with relations) $\Upsilon : \Gamma \to \Gamma'$ *cuts edges with relations* of Γ if the vertices are the same, but Γ' has fewer edges than Γ, with each removed edge lying on a path between the root edge and the colored vertices, and each path passing through a unique such edge. See Figure 9.

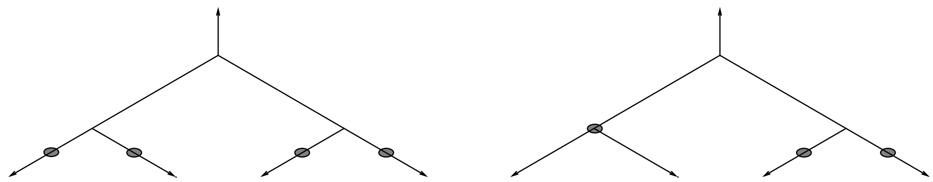


FIGURE 8. Collapsing edges with relations

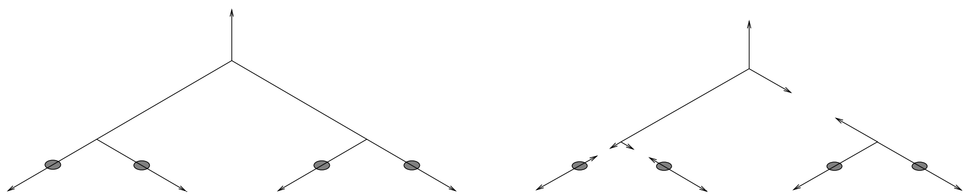


FIGURE 9. Cutting edges with relations

(e) (Forgetting tails) $\Upsilon : \Gamma \to \Gamma'$ *forgets a tail* (semiinfinite edge) and any vertices that become unstable, recursively starting from the semiinfinite edges furthest away from the root edge.

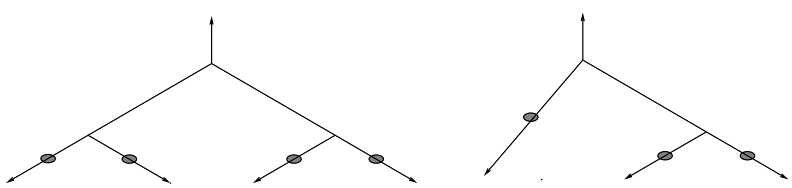


FIGURE 10. Forgetting a tail and collapsing

Remark 2.26. (More explanation on forgetting tails) Forgetting a tail leaves possibly only the vertex adjacent to the tail unstable, if it is colored with a single other edge adjacent, or non-colored with two other adjacent edges. In the first case, removing that vertex and the other adjacent edge still leaves a non-colored vertex which may be unstable, since it has one fewer edge. If unstable, removing this vertex and identifying the other two edges gives a stable colored tree. In the second case, removing the vertex gives a stable colored tree.

Proposition 2.27. (Morphisms of moduli spaces induced by morphisms of colored forests) To any morphism Υ of colored forests $\Gamma \to \Gamma'$ one can associate a morphism $\overline{M}_n(\Upsilon) : \overline{M}_{n,1,\Gamma}(\mathbb{A}) \to \overline{M}_{n,1,\Gamma'}(\mathbb{A})$ as follows.

(a) (Collapsing edges without relations) Any morphism $\Upsilon : \Gamma \to \Gamma'$ collapsing an edge induces an inclusion $\overline{M}(\Upsilon) : \overline{M}_{n,1,\Gamma}(\mathbb{A}) \to \overline{M}_{n,1,\Gamma'}(\mathbb{A})$.
(b) (Collapsing edges with relations) Any morphism $\Upsilon : \Gamma \to \Gamma'$ collapsing edges with relations induces an inclusion $\overline{M}(\Upsilon) : \overline{M}_{n,1,\Gamma}(\mathbb{A}) \to \overline{M}_{n,1,\Gamma'}(\mathbb{A})$.

(c) (Cutting an edge or edges with relations) Any morphism $\Upsilon : \Gamma \to \Gamma'$ cutting an edge or edges with relations of Γ induces a homeomorphism from $\overline{M}_{n,1,\Gamma}(\mathbb{A})$ to $\overline{M}_{n,1,\Gamma'}(\mathbb{A})$ by identifying the markings corresponding to the additional semiinfinite edges.

(d) (Forgetting tails) Any morphism $\Upsilon : \Gamma \to \Gamma'$ *forgetting a tail* induces a map $\overline{M}(\Upsilon) : \overline{M}_{n,1,\Gamma}(\mathbb{A}) \to \overline{M}_{n,1,\Gamma'}(\mathbb{A})$ which forgets the corresponding marking and collapses any unstable components recursively starting with the semiinfinite edges corresponding to the finite markings.

Proof. The existence of these maps is immediate from the definitions except for the existence of the morphism for forgetting tails, which requires an inductive argument collapsing the unstable components. For this note that forgetting a tail leaves possibly only the component containing the corresponding marking unstable, if it had either non-degenerate scaling and two special points, or degenerate scaling and three special points. Removing that component, and if the second possibility holds, replacing the node with the remaining marking or identifying the two remaining nodes, produces a new curve with one fewer irreducible component. In the second case, the resulting curve is automatically stable. In the first case, the adjacent irreducible component has one fewer special point, and so now may be unstable. If so, removing that component, and either (i) identifying the nodes, if the two special points were nodes, or (ii) changing the node to a marking, if the two special points were a node and a marking, produces a stable scaled affine curve. We give a stronger, algebraic version of the forgetful morphism in [52, Example 4.2].

Lemma 2.28. *For each Γ, the boundary of $\overline{M}_{n,1,\Gamma}(\mathbb{A})$ consists of those moduli spaces $M_{n,1,\Gamma'}(\mathbb{A})$ such that Γ is obtained from Γ' by collapsing an edge or edges with relations. Furthermore, each $\overline{M}_{n,1,\Gamma}(\mathbb{A})$ is a product of the moduli spaces $\overline{M}_{n_j,1}(\mathbb{A})$ and $\overline{M}_{0,n_j}$ corresponding to the vertices of Γ, where n_j are the valences.*

Ma'u-Woodward [36] shows that the compactification $\overline{M}_{n,1}(\mathbb{A})$ has the structure of a projective variety, locally isomorphic to a toric variety. The local structure of $\overline{M}_{n,1}(\mathbb{A})$ near the stratum $M_{n,1,\Gamma}(\mathbb{A})$ of nodal lines with combinatorial type Γ may be described as follows.

Definition 2.29. (Balanced labellings) For any colored tree Γ, a labelling

$$\gamma : \mathrm{Edge}_{<\infty}(\Gamma) \to \mathbb{C}$$

is *balanced* iff

$$\prod_{e \in P_{vw}} \gamma(e)^{\pm} = 1 \tag{7}$$

where v, w range over elements of $\mathrm{Vert}^1(\Gamma)$ and P_{vw} is the unique non-self-crossing path from v to w, and in the product the sign is positive if e is pointing towards the root edge marked z_0 and negative otherwise.

Let $Z_\Gamma \subset \mathrm{Map}(\mathrm{Edge}_{<\infty}(\Gamma), \mathbb{C})$ denote the space of balanced labellings. An element of Z_Γ is called a tuple of *gluing parameters*.

Example 2.30. (A singularity in the moduli space) For the tree Γ in Figure 11, with large dots indicating vertices in $\mathrm{Vert}^1(\Gamma)$, the relations are $\gamma_3 = \gamma_4, \gamma_1\gamma_3 = \gamma_2\gamma_5, \gamma_5 = \gamma_6$. The corresponding toric variety Z_Γ corresponds to a 3-dimensional cone with 4 extremal rays, and so the moduli space has a singularity at the vertex.

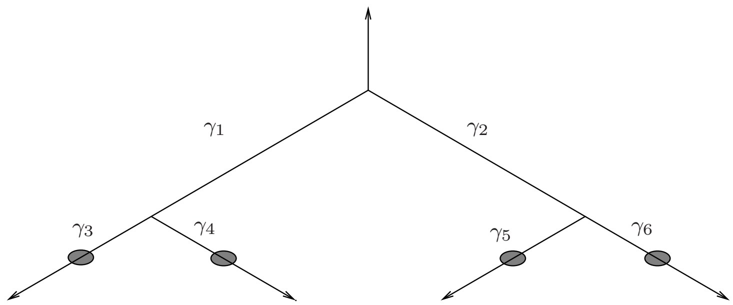


FIGURE 11. An example of a colored tree

Proposition 2.31. [36] There exists an open neighborhood of $M_{n,1,\Gamma}(\mathbb{A})$ in $\overline{M}_{n,1}(\mathbb{A})$ isomorphic to an open neighborhood of $0 \times M_{n,1,\Gamma}(\mathbb{A})$ in $Z_\Gamma \times M_{n,1,\Gamma}(\mathbb{A})$.

Proof. The construction is a version of the construction of the universal deformation of a genus zero nodal curve [2, p. 184] in which small balls around the nodes are removed and the components glued together via maps $z \mapsto \gamma/z$. The scaling is determined by the product of the gluing parameters from the root component to the irreducible components with finite scaling, independent of the choice of irreducible component with finite scaling by the balanced condition (7).

Remark 2.32. (Codimension formula) The codimension of a stratum $\overline{M}_{n,1}(\mathbb{A})$ corresponding to a colored tree Γ is *not* the number of finite edges, but rather

$$\mathrm{codim}(M_{n,1,\Gamma}(\mathbb{A})) = \#\,\mathrm{Edge}_{<\infty}(\Gamma) + 1 - \#\,\mathrm{Vert}^1(\Gamma)$$

where the extra summand $1 - \mathrm{Vert}^1(\Gamma)$ corresponds to the minus the number of relations on the gluing parameters $(\gamma(e))_{e\in\mathrm{Edge}_{<\infty}(\Gamma)} \in Z_\Gamma$.

Proposition 2.33. The boundary of $\overline{M}_{n,1}(\mathbb{A})$ consists of the following subsets (which will be *divisors* with respect to the algebraic structure on $\overline{M}_{n,1}(\mathbb{A})$ introduced later):

(a) (Bubbling points) For any $I \subset \{1,\ldots,n\}$ of order at least two the subset

$$\iota_I : D_I \to \overline{M}_{n,1}(\mathbb{A})$$

corresponding to the formation of a single bubble containing the markings I, with an isomorphism

$$D_I \to \overline{M}_{0,|I|+1} \times \overline{M}_{n-|I|+1,1}(\mathbb{A}). \tag{8}$$

(b) (Blowing up scaling) For any $r > 0$ and unordered partition $[I_1, \ldots, I_r]$, $I_1 \cup \ldots \cup I_r = \{1, \ldots n\}$ of order at least two, with each I_j non-empty, a subset $D_{[I_1,\ldots,I_r]}$ corresponding to the formation of r bubbles with markings $I_1, \ldots, I_r$, attached to a remaining component with infinite scaling. The map forgetting all but the infinitely-scaled locus induces an isomorphism [2]

$$D_{[I_1,\ldots,I_r]} \to \overline{M}_{r,1}(\mathbb{A}) \times \left(\prod_{i=1}^{r} \overline{M}_{|I_i|,1}(\mathbb{A}) \right). \tag{9}$$

Remark 2.34. (a) The inclusions of these subspaces give the collection of spaces $\overline{M}_{n,1}(\mathbb{A})$ the structure of an algebra over the operad associated to the notion of *homotopy morphism* of operads in [35]. However, we will not use or need this language and will not discuss it further.

(b) (Relation to the multiplihedon) The moduli space $\overline{M}_{n,1}(\mathbb{A})$ has a "positive real locus" that appears in Stasheff's description of A_∞ morphisms [46]. Namely, taking a real structure on $\mathbb{A}$, the anti-holomorphic involution on $\mathbb{A}$ induces an anti-holomorphic involution of $\overline{M}_{n,1}(\mathbb{A})$. We denote by $\overline{M}_{n,1}(\mathbb{A})^{\mathbb{R}}$ the fixed point locus, in which all markings are on the real line. The symmetric group S_n acts on $\overline{M}_{n,1}(\mathbb{A})$, and restricts to an action on $\overline{M}_{n,1}(\mathbb{A})^{\mathbb{R}}$ with fundamental domain given as the closure of the subset $M_{n,1}(\mathbb{A})^{\mathbb{R},+}$ where $z_1 < z_2 < \ldots < z_n$, homeomorphic to Stasheff's multiplihedron Mult_n [36]. An A_∞ morphism of A_∞ spaces X, Y consists of a sequence of maps

$$X^n \times \mathrm{Mult}_n \to Y, n \geq 0$$

satisfying a suitable splitting axiom on the boundary.

The splitting axiom for morphisms of CohFT algebras is defined via divisors on $\overline{M}_{n,1}(\mathbb{A})$. Because the singularities of the toric variety Z_Γ occur in complex codimension at least three, $\overline{M}_{n,1}(\mathbb{A})$ has a unique homology class of top dimension. In particular, each of the boundary divisors above has a well-defined homology class in $\overline{M}_{n,1}(\mathbb{A})$. However, $\overline{M}_{n,1}(\mathbb{A})$ is not smooth (and not a rational homology manifold) and not every boundary stratum has a dual class. That is, given a divisor

$$D = \sum_I n_I D_I + \sum_{r,[I_1,\ldots,I_r]} n_{[I_1,\ldots,I_r]} D_{[I_1,\ldots,I_r]} \tag{10}$$

there may or may not exist a class $\gamma \in H^2(\overline{M}_{n,1}(\mathbb{A}))$ that satisfies

$$\langle \beta, [D] \rangle = \langle \beta \cup \gamma, [\overline{M}_{n,1}(\mathbb{A})] \rangle.$$

This requires the restriction to combinations of boundary divisors that have dual classes in the following definition.

Let $(V, (\mu_V^n)_{n\geq 2})$ and $(W, (\mu_W^n)_{n\geq 2})$ be CohFT algebras.

[2] Equation (9) is corrected from the published version.

Definition 2.35. A *morphism of CohFT algebras* from V to W is a collection of S_n-invariant (with Koszul signs) multilinear maps

$$\phi^n : V^n \times H(\overline{M}_{n,1}(\mathbb{A})) \to W, \quad n \geq 0$$

such that for any divisor D of the form (10) with dual class $\gamma \in H^2(\overline{M}_{n,1}(\mathbb{A}))$ we have

$$\phi^n(\alpha, \beta \cup \gamma) = \sum_I n_I \phi^{n-|I|+1}(\mu_V^{|I|}(\alpha_i, i \in I; \cdot), \alpha_j, j \notin I; \cdot)(\iota_I^* \beta) + \sum_{r,[I_1,\ldots,I_r]} n_{[I_1,\ldots,I_r]} \mu_W^r(\phi^{I_1}(\alpha_i, i \in I_1; \cdot), \ldots, \phi^{I_r}(\alpha_i, i \in I_r; \cdot); \cdot)(\iota^*_{[I_1,\ldots,I_r]}\beta), \tag{11}$$

where the sum is over unordered partitions $[I_1, \ldots, I_r]$ of $\{1, \ldots, n\}$ with some I_j possibly empty, $\cdot$ indicates insertion of the Künneth components of $\iota_I^*\beta$, $\iota^*_{[I_1,\ldots,I_r]}\beta$, using the homeomorphisms (8), (9), the sum on the right-hand-side is assumed finite, and by convention if $n = 0$ we replace $H(\overline{M}_{n,1}(\mathbb{A}))$ with Λ (since in this case $\overline{M}_{n,1}(\mathbb{A})$ is empty). A morphism of CohFT algebras ϕ is *flat* resp. *curved* if ϕ^0 is zero resp. non-zero.

Remark 2.36.
(a) In our examples, $\Lambda = \cup_{a\in\mathbb{R}}\Lambda_{\geq a}$ will be a filtered ring, and $V = \cup_{a\in\mathbb{R}} V_{\geq a}$, $W = \cup_{a\in\mathbb{R}} W_{\geq a}$ filtered Λ-modules. We say that a *morphism of filtered CohFT algebras* is defined as above but where the right-hand-side of (11) is finite modulo $W_{\geq a}$ for any $a \in \mathbb{R}$.
(b) See Nguyen-Woodward-Ziltener [41] for a description of the space of Cartier divisors in $\overline{M}_{n,1}(\mathbb{A})$, that is, a description of which combinations of codimension two strata have dual classes.
(c) The definition of flat morphism of CohFT algebras (which has nothing to do with flat morphism of rings etc.) is analogous to the definition of flat morphism of A_∞ algebras in [16]. That is, ϕ^0 is analogous to the *curvature* of an A_∞ morphism.
(d) The divisors $D_{\{1\},\{2\}}, D_{\{1,2\}} \subset \overline{M}_{2,1} \cong \mathbb{P}$ are points (the limiting points in Figures 5, 6) and so have the same homology class. Using the splitting axiom (11) this implies that if ϕ^0 vanishes then ϕ^1 is a homomorphism from (V, μ_V^2) to (W, μ_W^2). This is an analog of the fact that a flat A_∞ morphism induces an algebra homomorphism of cohomology groups.
(e) For simplicity we will consider here only the even case, that is, V is a usual vector space and there are no signs.

Now we discuss the connection of morphisms of CohFT algebras with Frobenius manifolds, or rather, the underlying family of algebras:

Definition 2.37. Let V, W be vector spaces equipped with associative products $\star_v : T_vV^2 \to T_vV, \star_w : T_wW^2 \to T_wW$ varying smoothly in v, w. A *$\star$-morphism* from V to W is an analytic map $\phi : V \to W$ whose derivative $D_v\phi$ is a morphism of algebras from T_vV to $T_{\phi(v)}W$ for all $v \in V$.

In particular, if $\phi : V \to W$ is a $\star$-morphism with Taylor coefficients ϕ^n then $D_0\phi = \phi^1$ is an algebra homomorphism from T_0V to $T_{\phi^0(1)}W$.

Proposition 2.38. Any morphism of CohFT algebras $(\phi^n)_{n\geq 0}$ from V to W defines a formal $\star$-morphism from V to W via the formula

$$\phi : V \to W, \quad v \mapsto \sum_{n\geq 0} \frac{1}{n!}\phi^n(v,\dots,v).$$

ϕ arises from a flat morphism of CohFT algebras iff $\phi(0) = 0$.

Proof. For convenience, we reproduce the argument from [41, Proposition 2.43]. Consider the relation $[D_{\{1,2\}}] = [D_{\{1\},\{2\}}] \in H^2(\overline{M}_{2,1}(\mathbb{A}))$ from Remark 2.36 (d). Its pull-back under the morphism $\overline{M}_{n,1}(\mathbb{A}) \to \overline{M}_{2,1}(\mathbb{A})$ forgetting all but the first two markings is the relation

$$\sum_{r,[I_1,\dots,I_r]} [D_{[I_1,I_2,\dots,I_r]}] = \sum_I [D_I] \tag{12}$$

where the first sum is over partitions $I_1,\dots,I_r$ with $1 \in I_1, 2 \in I_2$, and the second is over subsets $I \subset \{1,\dots,n\}$ with $\{1,2\} \subset I$. Indeed, the pull-back of the coordinate on $\overline{M}_{2,1}(\mathbb{A})$ to $\overline{M}_{n,1}(\mathbb{A})$ is equal to any of the (equal) gluing parameters at the nodes at a generic point in any divisor $D_{[I_1,I_2,\dots,I_r]}$ appearing on the left-hand-side, see Ma'u-Woodward [36], and so has a zero of order one at that divisor. On the other hand, the pull-back is the inverse of the gluing parameter at a generic point in any divisor D_I appearing on the right-hand side, and so has a pole of order one at that divisor. The splitting axiom implies for each $a, b \in T_vV$

$$\begin{aligned}
D_v\phi(a \star_v b) &= \sum_{n,i} \frac{1}{(i-2)!(n-i)!}\phi^{n-i+1}\big(\mu_V^i(a,b,v,\dots,v;1),v,\dots,v;1\big) \\
&= \sum_{n,I}((n-2)!)^{-1}\phi^{n-|I|+1}\big(\mu_V^{|I|}(a,b,v,\dots,v;1),v,\dots,v;1\big) \\
&= \sum_{I_1\ni 1, I_2\ni 2, I_3,\dots,I_r} ((n-2)!\#\{j \mid I_j = \emptyset\}!)^{-1}\mu_W^r\big(\phi^{|I_1|}(a,v,\dots,v;1), \\
&\quad \phi^{|I_2|}(b,v,\dots,v;1),\phi^{|I_3|}(v,\dots,v;1),\dots,\phi^{|I_r|}(v,\dots,v;1);1\big) \\
&= \sum_{i_1,i_2\geq 1, i_3\dots,i_r\geq 0} \frac{1}{(i_1-1)!(i_2-1)!i_3!\cdots i_r!(r-2)!}\mu_W^r\big(\phi^{i_1}(a,v,\dots,v;1), \\
&\quad \phi^{i_2}(b,v,\dots,v;1),\phi^{i_3}(v,\dots,v;1),\dots,\phi^{i_r}(v,\dots,v;1);1\big) \\
&= \sum_r \frac{1}{(r-2)!}\mu_W^r\big(D_v\phi(a),D_v\phi(b),\phi(v),\dots,\phi(v);1\big) \\
&= D_v\phi(a) \star_{\phi(v)} D_v\phi(b)
\end{aligned}$$

where the right-hand-side is defined formally, that is, via Taylor series. By definition $\phi(0) = 0$ iff ϕ^0 vanishes iff $(\phi^n)_{n\geq 0}$ is flat.

2.4. Compositions of morphisms and traces

Morphisms and traces on a CohFT algebra admit a notion of composition, which generalizes the usual homotopy notions of composition in the A_∞ setting. The definition of 2-morphism of a composition of a trace with a morphism depends on a moduli space of *scaled parametrized curves* which combines features of the complexified multiplihedron and cyclohedron. Let $M_{n,1}(C)$ denote the space of n-marked 1-scaled curves with underlying curve C; we do not quotient by automorphisms of C. The space $M_{n,1}(C)$ admits a compactification $\overline{M}_{n,1}(C)$ by allowing *stable scaled curves* allowing bubbles with zero area form or allowing the area form on C to degenerate to zero and *affine scaled curves* to develop as bubbles. Recall that any nodal map $u : \hat{C} \to C$ of class $[C]$ has a *relative dualizing sheaf* given as the tensor product of the dualizing sheaf $T^\vee \hat{C}$ for $\hat{C}$ and the inverse of the pull-back of the cotangent bundle $T^\vee C$ to C:

$$T_u^\vee := T^\vee \hat{C} \otimes (u^* T^\vee C)^{-1};$$

(More detail is given below in [52, Example 4.2].) We denote by $\mathbb{P}(T_u^\vee \oplus \mathbb{C})$ the associated bundle with projective line fibers.

Definition 2.39. (Nodal Scaled Marked Curves) Let $u : \hat{C} \to C$ be a map of class $[C]$. A *scaling form* is a section $\lambda : \hat{C} \to \mathbb{P}(T_u^\vee \oplus \mathbb{C})$ such that on any connected component $\hat{C}'$ of $\hat{C} - C_0$ (that is, bubble tree attached to the principal component) the pair $(\hat{C}', \lambda | \hat{C}')$ is an affine scaled curve (if the scaling λ is infinite on C_0) or has zero scaling, otherwise. A *nodal scaled curve* parametrized by C is a map $\hat{C} \to C$ equipped with a scaling form. An *isomorphism* of nodal scaled parametrized curves $(\hat{C}_j, u_j, \omega_j), j = 0, 1$, is an isomorphism $\psi : \hat{C}_0 \to \hat{C}_1$ such that

(a) (Scalings are intertwined) $\psi^* \lambda_1 = \lambda_0$.
(b) (Markings are intertwined) $\psi(z_{0,i}) = z_{1,i}, i = 1, \ldots, n$;
(c) (Parametrization is intertwined) $\psi \circ u_0 = u_1$.

A nodal scaled parametrized curve is *stable* iff it has no infinitesimal automorphisms, that is, each irreducible non-principal component has at least three special points or a non-degenerate scaling and two special points. The *combinatorial type* of a nodal scaled parametrized curve is a colored tree $\Gamma = (\mathrm{Vert}(\Gamma), \mathrm{Edge}(\Gamma))$ with finite resp. semiinfinite edges $\mathrm{Edge}_{<\infty}(\Gamma)$ resp. $\mathrm{Edge}_\infty(\Gamma)$ obtained by replacing every irreducible component by a vertex and every node or marking with an edge, equipped with a *root vertex* $v_0 \in \mathrm{Vert}(\Gamma)$ corresponding to the principal component and a partition

$$\mathrm{Vert}(\Gamma) = \mathrm{Vert}^0(\Gamma) \cup \mathrm{Vert}^1(\Gamma) \cup \mathrm{Vert}^\infty(\Gamma)$$

corresponding to the irreducible components with zero resp. finite resp. infinite scalings. A rooted colored tree is *stable* if it corresponds to a stable scaled curve, that is, every non-root vertex in $\mathrm{Vert}^0(\Gamma)$ or $\mathrm{Vert}^\infty(\Gamma)$ resp. $\mathrm{Vert}^1(\Gamma)$ has at least 3 resp. 2 incident edges.

In other words, a stable scaled curve is a copy of the curve C with a section of the trivial bundle (necessarily constant) and a collection of stable curves attached (if the scaling on the principal component is zero or finite) or a curve with infinite scaling and a collection of stable scaled affine lines attached (if the scaling on the principal component is infinite).

Let $\overline{M}_{n,1}(C)$ denote the moduli space of isomorphism classes of n-marked, scaled curves with principal component C, and $M_{n,1,\Gamma}(C)$ the subset of combinatorial type Γ so that

$$\overline{M}_{n,1}(C) = \cup_\Gamma M_{n,1,\Gamma}(C).$$

The topology on $\overline{M}_{n,1}(C)$ is similar to that for $\overline{M}_{n,1}(\mathbb{A})$ and is compact and Hausdorff. The local structure of $\overline{M}_{n,1}(C)$ near the stratum $M_{n,1,\Gamma}(C)$ of nodal lines with combinatorial type Γ may be described as follows. As in (7), for any colored rooted tree Γ, a labelling

$$\gamma : \mathrm{Edge}_{<\infty}(\Gamma) \to \mathbb{C}$$

is *balanced* iff

$$\prod_{e \in P_{vw}} \gamma(e)^{\pm 1} = 1 \tag{13}$$

where v, w range over elements of $\mathrm{Vert}^1(\Gamma)$ and P_{vw} is the unique non-self-crossing path from v to w, and the sign in the exponent is positive if e is pointing towards the root vertex and negative otherwise. Let Z_Γ denote the set of balanced labellings. As in [36] for each rooted tree Γ there exists a tubular neighborhood of the form

$$M_{n,1,\Gamma}(C) \times Z_\Gamma \to \overline{M}_{n,1}(C)$$

given by removing small neighborhoods of the nodes and gluing together using identifications depending on the gluing parameters.

Example 2.40. Suppose that $\hat{C}$ consists of a principal component $C_0 \cong C$ with infinite scaling, two other components with infinite scaling (dark shading), four components with finite scaling (medium shading), and two components with zero scaling (light shading) as shown in Figure 12. The relations on the gluing parameters $\gamma_1 = \gamma_2, \gamma_3 = \gamma_4, \gamma_1\gamma_5 = \gamma_3\gamma_6$ imply that the curve obtained with gluing with non-zero gluing parameters is equipped with the area form $\gamma_1\gamma_5\omega_C = \gamma_2\gamma_5\omega_C = \gamma_3\gamma_6\omega_C = \gamma_4\gamma_6\omega_C$.

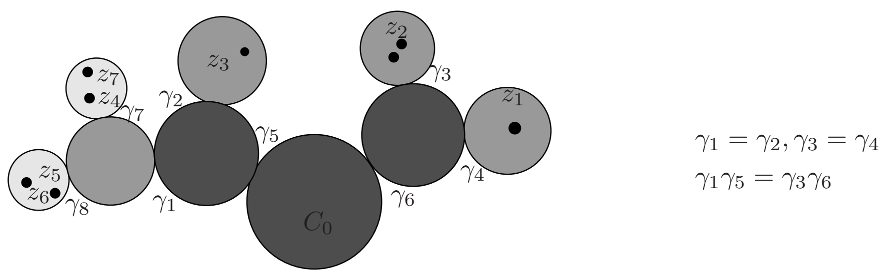


FIGURE 12. Gluing relations on a marked scaled curve

Proposition 2.41. The boundary of $\overline{M}_{n,1}(C)$ is the union of the following sets (which will be *divisors* once the algebraic structure on $\overline{M}_{n,1}(C)$ is introduced)

(a) (Bubbling points) For any subset $I \subset \{1, \ldots, n\}$ of order at least two we have a subspace
$$\iota_I : D_I \to \overline{M}_{n,1}(C)$$
and an isomorphism
$$\varphi_I : D_I \to \overline{M}_{0,|I|+1} \times \overline{M}_{n-|I|+1,1}(C)$$
corresponding to the formation of a bubble with markings $z_i, i \in I$ with zero scaling.

(b) (Blowing up scaling) For any unordered partition $[I_1, \ldots, I_r]$ of $\{1, \ldots, n\}$ we have a subspace
$$\iota_{[I_1,\ldots,I_r]} : D_{[I_1,\ldots,I_r]} \to \overline{M}_{n,1}(C)$$
and an fibration forgetting all but the infinitely-scaled component
$$D_{[I_1,\ldots,I_r]} \to \overline{M}_r(C)$$
whose fiber isomorphic to $\prod_{j=1}^r \overline{M}_{|I_j|,1}(\mathbb{A})$ describes bubbles with non-degenerate area form containing the markings.[3]

(c) (Fixing a scaling) For any $\rho \in \mathbb{C}$, in particular for $\rho = 0$, there is an inclusion
$$\iota_\rho : \overline{M}_n(C) \to \overline{M}_{n,1}(C)$$
by choosing any scaling $\rho\omega_C$.

Proof. The description of boundary subspaces is immediate from the tubular neighborhood description of each stratum and a dimension count, which is the same as in Remark 2.32.

The adiabatic limit Theorem 1.5 will be deduced from the following divisor class relation:

Proposition 2.42. (The basic divisor class relation in the moduli of stable scaled curves) The homology class of $\iota_\rho(\overline{M}_n(C))$ is equal to that of the union of classes of $D_{[I_1,\ldots,I_r]}$ over unordered partitions.

Proof. The equivalence in homology induced by the map $\rho : \overline{M}_{n,1}(C) \to \mathbb{P}$ equates the homology classes of $\rho^{-1}(0) \cong \overline{M}_n(C)$ with $\rho^{-1}(\infty) = \cup_{r,[I_1,\ldots,I_r]} D_{[I_1,\ldots,I_r]}$, as one can check that the multiplicity of each divisor on the right hand side is 1, using the fact that linear equivalence of divisors implies homology equivalence.

Suppose that V, W are (even, genus zero) CohFT algebras, with structure maps
$$\mu_V^n : V^n \times H(\overline{M}_{0,n+1}) \to V, \quad \mu_W^n : W^n \times H(\overline{M}_{0,n+1}) \to W.$$
Let $\phi^n : V^n \times H(\overline{M}_{n,1}(\mathbb{A})) \to V$ be a morphism of CohFT algebras, and τ_V, τ_W traces on V, W respectively.

[3] This sentence corrected from the published version.

Definition 2.43. (2-morphisms for compositions of traces and morphisms) A 2-*morphism from $\phi \circ \tau_W$ to τ_V* is a collection of maps

$$\psi : V^n \times H(\overline{M}_{n,1}(C)) \to W, \quad n \geq 0$$

such that

(a) (Fixing Scaling) if $\gamma \in H^2(\overline{M}_{n,1}(C))$ is the dual class to $\iota_\rho(\overline{M}_n(C))$ then

$$\psi^n(\alpha_1, \ldots, \alpha_n; \beta \cup \gamma) = \tau_V^n(\alpha_1, \ldots, \alpha_n; \iota_\rho^* \beta);$$

(b) (Bubbling points) if $\gamma_I \subset H^2(\overline{M}_{n,1}(C))$ is the dual class to the divisor D_I corresponding to bubbling off markings $z_i, i \in I$, then

$$\psi^n(\alpha_1, \ldots, \alpha_n; \beta \cup \gamma_I) = \psi^{n-|I|+1}(\alpha_j, j \notin I, \mu^{|I|}(\alpha_i, i \in I, \cdot); \cdot)(\iota_{I,1}^* \beta);$$

(c) (Blowing up scaling) if $D = \sum n_{[I_1,\ldots,I_r]} D_{[I_1,\ldots,I_r]}$ is a boundary divisor with dual class γ then

$$\begin{aligned}\psi^n(\alpha_1, \ldots, \alpha_n; \beta \cup \gamma) = \sum_{[I_1,\ldots,I_r]} n_{[I_1,\ldots,I_r]} \tau_W^r(\phi^{|I_1|}(\alpha_i, i \in I_1; \cdot), \ldots, \\ \phi^{|I_r|}(\alpha_i, i \in I_r; \cdot); \cdot)(\iota_{[I_1,\ldots,I_r]}^* \beta). \quad (14)\end{aligned}$$

We write $\tau_V \cong_\psi \tau_W \circ \phi$.

Lemma 2.44. *If $\tau_V \cong_\psi \tau_W \circ \phi$ then*

$$\tau_V(\alpha_1, \ldots, \alpha_n; \iota_\rho^* \beta) = \sum_{r,[I_1,\ldots,I_r]} \tau_W^r(\phi^{|I_1|}(\alpha_i, i \in I_1; \cdot), \ldots, \phi^{|I_r|}(\alpha_i, i \in I_r; \cdot); \cdot)(\iota_{[I_1,\ldots,I_r]}^* \beta)$$

Proof. The proof follows by combining (a) and (c) in the definition. These are applied to using the equality of the homology class $[D_\rho]$ of the divisor D_ρ corresponding to fixed scaling and the divisor at infinite scaling $\sum [D_{[I_1,\ldots,I_r]}]$.

3. Symplectic vortices

In physics, a *vortex* refers to a stable solution of classical field equations which has finite energy in two spatial dimensions, see for example Preskill [43] and, for a more mathematical treatment, Jaffe-Taubes [25] who classified vortices for scalar fields. In mathematics, vortices often refer to pairs of a connection and section of a line bundle satisfying an equation involving the curvature and a quadratic function of the section, see for example Bradlow [6]. Symplectic vortices are vortices in which the "field" takes values in a symplectic manifold with Hamiltonian group action. In this section we review the symplectic approach to gauged Gromov-Witten invariants, also known as symplectic vortex invariants or Hamiltonian gauged Gromov-Witten invariants, as introduced by Mundet and Salamon, see [8], [40]. If the moduli spaces of symplectic vortices are smooth, then integration over them defines the required invariants and the proofs of the Theorems 1.3, 1.5 are immediate. Of course, the moduli spaces are not smooth, or of expected dimension in general, and to define the needed virtual fundamental cycles we pass to algebraic geometry, starting in the following section.

3.1. Gauged holomorphic maps

In this section we review the construction of the moduli space of symplectic vortices as the symplectic quotient of the action of the group of gauge transformations on the space of gauged holomorphic maps; this generalizes the construction of the space of flat connections as the symplectic quotient of the action of the group of gauge transformations on the space of connections on a bundle. Unfortunately since the space of gauged holomorphic maps is in general singular, this construction is rather formal and serves only for motivation for what follows.

We begin with notation for equivariant cohomology. Let K be a compact group with Lie algebra $\mathfrak{k}$. We assume that $\mathfrak{k}$ is equipped with a K-invariant metric, inducing an identification $\mathfrak{k}^\vee \to \mathfrak{k}$. We denote by $EK \to BK$ a universal K-bundle, unique up to homotopy equivalence. For any K-space X, we denote by $X_K := X \times_K EK$ the homotopy quotient and by $H^K(X) := H(X_K)$ the equivariant cohomology.

Next we introduce connections and curvature. Let $P \to C$ be a principal K-bundle over a compact surface C, and $\psi : C \to BK$ a classifying map for P. Denote by

$$\Omega(P, \mathfrak{k})^K = \{\theta \in \Omega(P, \mathfrak{k}) \mid (k^{-1})^*\theta = \mathrm{Ad}(k)\theta\}$$

the space of K-invariant forms and by

$$\mathcal{A}(P) = \{\theta \in \Omega^1(P, \mathfrak{k})^K \mid \theta_{\pi(p)}(\xi_P(p)) = \xi, \forall \xi \in \mathfrak{k}, p \in P\}$$

the space of (principal) connections on P, where $\xi_P(p) = \frac{d}{dt}|_{t=0} p \exp(t\xi)$ is the generating vector field at p. The space $\mathcal{A}(P)$ is an affine space with a free transitive action of the space $\Omega^1(C, P(\mathfrak{k}))$ of one-forms with values in the *adjoint bundle* $P(\mathfrak{k}) = P \times_K \mathfrak{k}$. Let

$$\mathcal{A}(P) \to \Omega^2(C, P(\mathfrak{k})), \quad A \mapsto F_A$$

denote the curvature map.

For Hamiltonian actions we introduce a notation of a gauged map, which will mean a pair of a connection and a section of the associated bundle. Let X be a Hamiltonian K-manifold with symplectic form ω and moment map $\Phi : X \to \mathfrak{k}^\vee$. By our convention this means that for all $\xi \in \mathfrak{k}$, we have $\omega(\xi_X, \cdot) = -\mathrm{d}(\Phi, \xi)$ where $\xi_X(x) = \frac{d}{dt}|_{t=0} \exp(-t\xi)x$ is the generating vector field for ξ. Recall that $P(X)$ is the associated X-bundle. Sections $u : C \to P(X)$ are in one-to-one correspondence with lifts u_K of ψ to X_K. Given a section $u : C \to P(X)$, the homology class $[u]$ is defined to be the homology class $[u] := u_{K,*}[C] \in H_2^K(X, \mathbb{Z})$. The equivariant symplectic form $\omega_K \in \Omega_K^2(X)$ pulls back to $\Omega_K^2(P \times X)$ and descends to a closed, fiber-wise symplectic two-form $\omega_{P(X),A} \in \Omega^2(P(X))$ depending on the choice of connection A. Its cohomology class $[\omega_{P(X)}] \in H^2(P(X))$ is independent of the choice of connection. The *equivariant symplectic area* of u is

$$D(u) = ([u], [\omega_K]) = ([C], u^*[\omega_{P(X)}])$$

where $(\ ,\ )$ denotes the pairing between homology and cohomology. More concretely, we have $D(u) = \int_C u^*\omega_{P(X),A}$ independently of the connection A. We denote by $P(\Phi) : P(X) \to P(\mathfrak{k}^\vee) \cong P(\mathfrak{k})$ the map induced by Φ. A *gauged map* from C to X is a datum (P, A, u) where $A \in \mathcal{A}(P)$ and $u : C \to P(X)$ is a section. Given a metric on C, we denote by

$$* : \Omega^\bullet(C) \to \Omega^{2-\bullet}(C)$$

the associated Hodge star. The *energy* of a gauged map (A, u) is given by

$$E(A, u) = \frac{1}{2} \int_C * \left(\|\mathrm{d}_A u\|^2 + \|F_A\|^2 + \|u^* P(\Phi)\|^2 \right). \tag{15}$$

Finally we introduce gauged holomorphic maps and symplectic vortices. Suppose the C is a complex curve. Denote by $\mathcal{J}(X)$ the space of almost complex structures on X compatible with ω. The action of K induces an action on $\mathcal{J}(X)$, and we denote by $\mathcal{J}(X)^K$ the invariant subspace. Any connection $A \in \mathcal{A}(P)$ induces a map of spaces of almost complex structures

$$\mathcal{J}(X)^K \to \mathcal{J}(P(X)), \quad J \mapsto J_A$$

by combining the almost complex structure on X and C using the splitting defined by the connection. Let $\Gamma(C, P(X))$ denote the space of sections of $P(X)$. We denote by

$$\overline{\partial}_A : \Gamma(C, P(X)) \to \bigcup_{u \in \Gamma(C, P(X))} \Omega^{0,1}(C, u^* T^{\mathrm{vert}} P(X))$$

the Cauchy-Riemann operator defined by J_A.

Definition 3.1. (Gauged holomorphic maps) Suppose that X is a Hamiltonian K-manifold equipped with an invariant almost complex structure J, C is a complex curve, and $P \to C$ is a principal K-bundle. A *gauged holomorphic map* for P is a pair (A, u) satisfying $\overline{\partial}_A u = 0$.

Let $\mathcal{H}(P, X)$ be the space of gauged holomorphic maps for P:

$$\mathcal{H}(P, X) = \{(A, u) \in \mathcal{A}(P) \times \Gamma(C, P(X)), \quad \overline{\partial}_A u = 0\}.$$

The energy and symplectic area are related by a generalization of the familiar energy-area relation for pseudoholomorphic maps in [37]:

Proposition 3.2. (Energy-area relation, [8, Proposition 3.1]) Suppose that X is a Hamiltonian K-manifold equipped with an invariant almost complex structure J, C is a compact complex curve, and $P \to C$ is a principal K-bundle. Let (A, u) be a gauged map from C to X with bundle P. Let ω_C be the area form determined by a choice of metric on C. The energy and equivariant symplectic area are related by

$$E(A, u) = D(u) + \int_C * \|\overline{\partial}_A u\|^2 + * \frac{1}{2} \|F_A + u^* P(\Phi) \omega_C\|^2. \tag{16}$$

Definition 3.3. A gauged map $(A, u) \in \mathcal{H}(P, X)$ is a *symplectic vortex* if it satisfies the

$$\text{(Vortex Equation)} \quad F_{A,u} := F_A + u^* P(\Phi)\omega_C = 0.$$

An *isomorphism* of symplectic vortices $(A_j, u_j), j = 0, 1$ with bundle P is a gauge transformation $\phi : P \to P$ such that $\phi^* A_1 = A_0$ and $\phi(X) \circ u_0 = u_1$, where $\phi(X) : P(X) \to P(X)$ is the fiber-bundle-automorphism induced by ϕ. An *n-marked* symplectic vortex is a vortex (A, u) together with an n-tuple $\underline{z} = (z_1, \ldots, z_n)$ of distinct points in C. A *framed vortex* is a collection $(A, u, \underline{z}, \underline{\phi})$, where $(A, u, \underline{z})$ is a marked vortex and $\underline{\phi} = (\phi_1, \ldots, \phi_n)$ is an n-tuple where each $\phi_j : P_{z_j} \to K$ is a K-equivariant isomorphism, that is, a trivialization of the fiber.

Let $M_n(P, X)$ denote the moduli space of isomorphism classes of n-marked vortices and $M_n^K(C, X)$ the union over isomorphism classes of bundles $P \to C$. The moduli space $M_n^K(C, X)$ is homeomorphic to the product $M^K(C, X) \times M_n(C)$ where $M_n(C)$ denotes the configuration space of n-tuples of distinct points in C.

3.2. Nodal symplectic vortices

The bubbling phenomenon for holomorphic curves also occurs in the case of symplectic vortices and prevents compactness of the moduli spaces. However, once one incorporates bubbles and fixes the homology class the moduli spaces become compact, as we now explain following Salamon et al [8] who proved compactness of the moduli space of vortices of fixed homology class in the case that X has no holomorphic spheres, and Mundet [40] and Ott [42] who compactify the moduli space of vortices by allowing bubbling in the fibers of $P(X)$.

Definition 3.4. (Nodal vortices) Suppose that X is a Hamiltonian K-manifold equipped with an invariant almost complex structure J and C is a connected smooth projective curve. A *nodal gauged n-marked map* from C to X with underlying bundle P consists of a datum $(P, A, \hat{C}, u, \underline{z})$ where

(a) (Bundle and Connection) $P \to C$ is a K-bundle and A is a connection on P;
(b) (Nodal section) $u : \hat{C} \to P(X)$ is a map from nodal curve $\hat{C}$ to $P(X)$ of base degree one, that is, $\pi \circ u : \hat{C} \to C$ is a map of class $[C]$;
(c) (Markings) an n-tuple $\underline{z} = (z_1, \ldots, z_n) \in \hat{C}^n$ of distinct, smooth points of $\hat{C}$.

An *isomorphism* of nodal gauged maps $(\hat{C}', A', u', \underline{z}'), (\hat{C}'', A'', u'', \underline{z}'')$ with bundles P', P'' consists of

(a) (Domain automorphism) an automorphism of the domain $\psi : \hat{C}' \to \hat{C}''$, inducing the identity on C, and
(b) (Bundle automorphism) a bundle isomorphism $\phi : P' \to P''$, inducing the identity on C,

intertwining the connections and maps and exchanging the markings, that is,

(a) (Isomorphism of connections) $\phi^* A'' = A'$,
(b) (Isomorphism of maps) $u'' = \phi(X) \circ u' \circ \psi^{-1}$ where $\phi(X) : P(X) \to P'(X)$ is the map induced by ϕ and
(c) (Isomorphism of markings) $\psi(z_i') = z_i'', i = 1, \ldots, n$.

A *nodal vortex* is a nodal gauged map such that the principal component is a vortex and there are no automorphisms with trivial bundle automorphism (gauge transformation). A nodal vortex is *stable* if the map $u : \hat{C} \to P(X)$ is a stable map from the marked curve $(\hat{C}, \underline{z})$ and the pair (A, u) has finite automorphism group under the action of gauge transformations. Note that there is no condition on the number of special points on the principal component. In particular, constant gauged maps with no markings can be stable. For any map $u : \hat{C} \to P(X)$, the *homology class* $[u] \in H_2^K(X, \mathbb{Z})$ of u is the push-forward of $[\hat{C}]$ under a map $u_K : \hat{C} \to X_K$ obtained from a classifying map for P.

The following extends the notion of convergence to the case of nodal marked vortices.

Definition 3.5. (Convergence of nodal vortices) Suppose that X is a Hamiltonian K-manifold equipped with an invariant almost complex structure J and C is a connected smooth projective curve. Suppose that $(P_\nu, A_\nu, \hat{C}_\nu, u_\nu, \underline{z}_\nu)$ is a sequence of marked nodal vortices on C with values in X, and $(P, A, \hat{C}, u, \underline{z})$ is a nodal vortex with values in X. We say that $[(P_\nu, A_\nu, \hat{C}_\nu, A_\nu, u_\nu, \underline{z}_\nu)]$ *converges* to $[(P, A, \hat{C}, A, \underline{z})]$ if after a sequence of bundle isomorphism $\phi_\nu : P_\nu \to P$ the connections A_ν converge to A_∞ in the C^0 topology and $[(\hat{C}_\nu, u_\nu, \underline{z}_\nu)]$ Gromov converges to $[(\hat{C}, u, \underline{z})]$.

See Ott [42] for more details on the definition of convergence. The definition of convergence implies in particular that the curvature F_{A_ν} converges to F_{A_∞} in L^2, but A_ν is not required to (and does not) converge to A uniformly in all derivatives. Recall from [8] a condition which guarantees compactness of moduli spaces of symplectic vortices with non-compact target:

Definition 3.6. A Hamiltonian K-manifold X with moment map $\Phi : X \to \mathfrak{k}^\vee$ is *convex at infinity* if there exists $f \in C^\infty(X)^G$ such that

$$\text{(Convexity)} \quad \langle \nabla_v \nabla f(x), v \rangle + \langle \nabla_{Jv} \nabla f(x), Jv \rangle \geq 0, \quad \mathrm{d}f(x) J_x \Phi(x)_X(x) \geq 0 \quad (17)$$

for every $x \in X$ and $v \in T_x X$ outside of a compact subset of X.

For example, if X is a vector space with a linear Hamiltonian action of K with *proper* moment map Φ then X is convex. The following is proved by Mundet [40] and Ott [42].

Theorem 3.7. (Sequential compactness for vortices with compact domain) *Suppose that X is Hamiltonian K-manifold equipped with a proper moment map convex at infinity and an invariant almost complex structure J, C is a connected smooth compact complex curve. Any sequence of nodal symplectic vortices with bounded energy has a convergent subsequence.*

Convergence for sequences with bounded first derivative is proved as follows: Suppose that (A_ν, u_ν) is a sequence of symplectic vortices on a bundle P with smooth domain with the property that $c_\nu := \sup \| d_{A_\nu} u_\nu \| = \| \mathrm{d}_{A_\nu} u_\nu(z_\nu) \|$ is bounded. By Uhlenbeck compactness, after gauge transformation, we may assume that A_ν converges weakly in the Sobolev $W^{1,p}$ topology and strongly in C^0

to a limit A_∞. After putting A_ν in Coulomb gauge with respect to A_∞, elliptic regularity for vortices (see [8, p.20]) implies that (u_ν, A_ν) has a C^l-convergent subsequence of any l.

More generally in the case of unbounded first derivative one has a bubbling analysis similar to that for pseudoholomorphic curves [8], [42]: If $\|\mathrm{d}_{A_\nu} u_\nu(z_\nu)\| \to \infty$ and $z_\nu \to z$ we say that z is a *bubble point* for the sequence (A_ν, u_ν).

Theorem 3.8. (Bubbling analysis for vortices with compact domain) *Let X be a Hamiltonian K-manifold with proper moment map convex at infinity.*

(a) (Removal of Singularities) *[42, Theorem 1.1] Any finite energy vortex (A, u) on the punctured disk $D - \{0\}$ extends to a vortex on D.*

(b) (Energy Quantization) *[42, Lemma 4.2] There exists an $E_0 > 0$ such that for any bubble point of a sequence (A_ν, u_ν), one has*

$$\lim_{\epsilon \to 0} \lim_{\nu \to \infty} E(A_\nu, u_\nu | B_\epsilon(z)) > E_0.$$

(c) (Annulus Lemma) *[42, Section 5.2] There exists a constant $\epsilon > 0$ such that if (A, u) is a vortex on $\mathcal{A}(r, R) := B_R(0) - B_r(0)$ then for every $\mu < 1$ there exist constants $R_0, \delta_0, C > 0$ such that if $R > R_0$ and $E(A, u) < \delta_0$, then for $\log(2) \le T \le \log(\sqrt{R/r})$ the restriction of (A, u) to $\mathcal{A}(e^T r, e^{-T} R)$ satisfies*

$$E((A, u)|_{\mathcal{A}(e^T r, e^{-T} R)}) < Ce^{-2\mu T} E(A, u).$$

(d) (Mean Value Inequality) *[42, Corollary 2.2] Let $r > 0$ and let $\lambda(s, t)\mathrm{d}s\mathrm{d}t$ be an area form on the ball $B_r(0)$ of radius r around 0 in $\mathbb{C}$. There exist constants $C, \delta > 0$ such that if (A, u) is a vortex on $B_r(0)$ with $E(A, u|B_r(0)) < \delta$ then*

$$(1/2)\|\mathrm{d}_A u(0)\|^2 + \|\Phi(u(0))\|^2 < (C/r^2) E((A, u)|_{B_r(0)}).$$

We remark that an examination of the proof in [42] shows that the constants C, δ depend only on bounds on λ and its first and second derivatives.

Remark 3.9. Given these ingredients the proof of compactness goes as follows. For each *bubbling sequence* z_ν with $\|\mathrm{d}_{A_\nu} u_\nu(z_\nu)\| \to \infty$ one constructs by *soft rescaling* a sequence of $J_{A_\nu^1}$-holomorphic maps u_ν^1 on balls of increasing radius, with the property that the limit A_ν^1 is zero and u_ν^1 is an ordinary holomorphic map from $\mathbb{C}$ to X. Note that one does not have C^2 convergence of the sequence $J_{A_\nu^1}$; however, [42, Appendix] shows that C^0 convergence of a sequence of almost complex structures is sufficient as long as one has a version of the mean value inequality, which in this case follows from the vortex equations. The limiting configuration is then constructed by induction.

Denote by $\overline{M}_n^K(C, X)$ the set of isomorphism classes of nodal vortices with connected domain. The *combinatorial type* of a nodal vortex is a rooted tree Γ with root vertex corresponding to the principal component, equipped with a labelling of vertices by elements of $H_2^K(X, \mathbb{Z})$. For any tree Γ and homology class

$d \in H_2^K(X,\mathbb{Z})$ we denote by $M_{n,\Gamma}^K(C,X,d)$ the space of isomorphism classes of vortices of combinatorial type Γ, so that

$$\overline{M}_n^K(C,X) = \bigcup_\Gamma M_{n,\Gamma}^K(C,X,d)$$

as Γ ranges over connected combinatorial types. We say that a subset S of $\overline{M}_n^K(C,X)$ is *closed* if any convergent sequence in S has limit point in S, and *open* if its complement is closed. The open sets form a topology for which any convergent sequence is convergent, and any convergent sequence has a unique limit, by arguments similar to [37, Lemma 5.6.5]. Namely local "distance functions" can be defined by combining the distance functions of [37] with the L^2-metric on the space of connections. In the case $n = 0$, given a constant $\epsilon > 0$ and stable vortex $(P, A_0, \hat{C}_0, u_0)$ and another stable vortex $(P, A_1, \hat{C}_1, u_1)$ with the same underlying bundle P define

$$\begin{aligned}\operatorname{dist}_\epsilon([(P, A_0, \hat{C}_0, u_0)], [(P, A_1, \hat{C}_1, u_1)]) = \\ \inf_{k \in \mathcal{K}(P)} \|k \cdot A_1 - A_0\|_{L^2} + \operatorname{dist}^0_\epsilon([(\hat{C}_0, u_0)], [(\hat{C}_1, k \cdot u_1)])\end{aligned} \tag{18}$$

where $\operatorname{dist}^0_\epsilon$ is the distance on isomorphism classes of stable maps defined on [37, p. 134], but using the Yang-Mills-Higgs energy on a small ball around each node. It follows from the results in Ott [42] and elliptic regularity for vortices that for ϵ sufficiently small, a sequence $[(P, A_\nu, \hat{C}_\nu, u_\nu)]$ converges to $[(P, A_0, \hat{C}_0, u_0)]$ if and only if

$$\operatorname{dist}_\epsilon([(P, A_0, \hat{C}_0, u_0)], [(P, A_\nu, \hat{C}_\nu, u_\nu)]) \to 0$$

and the remainder of the proof is similar to that in [37]. The sequential compactness Theorem 3.7 implies:

Theorem 3.10. (Properness of the moduli space of symplectic vortices) *Suppose that X is a Hamiltonian K-manifold equipped with a proper moment map convex at infinity and an invariant almost complex structure J and C is a connected smooth projective curve. Then $\overline{M}_n^K(C,X)$ is Hausdorff and the energy map $E : \overline{M}_n^K(C,X) \to [0,\infty)$ is proper.*

3.3. Large area limit

In the large area limit the vortices are related to holomorphic maps to the (possibly orbifold) symplectic quotient, as pointed out by Gaio-Salamon [18]. The limiting process involves various kinds of bubbling which one hopes to incorporate into a description of the relationship.

First recall the notion of symplectic quotient of X by K as introduced by Mayer and Marsden-Weinstein: Suppose that X is a Hamiltonian K-manifold equipped with a proper moment map $\Phi : X \to \mathfrak{k}^\vee$. Let

$$X /\!\!/ K := \Phi^{-1}(0)/K$$

denote the symplectic quotient. If K acts freely on $\Phi^{-1}(0)$, then $X/\!\!/K$ has the structure of a smooth manifold of dimension $\dim(X) - 2\dim(K)$. The quotient

$X/\!\!/K$ has a unique symplectic form ω_0 satisfying $i^*\omega = p^*\omega_0$, where $i : \Phi^{-1}(0) \to X$ and $p : \Phi^{-1}(0) \to X/\!\!/K$ are the inclusion and projection respectively. Any invariant almost complex structure J on X induces an almost complex structure on $X/\!\!/K$.

If $X \subset \mathbb{P}(V)$ is a smooth projectively embedded variety in a G-representation V then $X/\!\!/K$ is canonically homeomorphic to the geometric invariant theory quotient $X/\!\!/G$ introduced by Mumford, by a theorem of Kempf-Ness. The latter is defined as the quotient of the *semistable locus*

$$X^{\mathrm{ss}} = \{x \in X | \exists k \in \mathbb{Z}_+, s \in H^0(X, \mathcal{O}_X(k))^G, s(x) \neq 0\}$$

where $\mathcal{O}_X(k)$ is the k-th tensor product of the hyperplane bundle on X. The quotient $X/\!\!/G$ is the quotient of X^{ss} by the *orbit-equivalence relation*

$$x_1 \sim x_2 \iff \overline{Gx_1} \cap \overline{Gx_2} \cap X^{\mathrm{ss}} \neq \emptyset.$$

A point $x \in X$ is *stable* if Gx is closed in X^{ss} and the stabilizer G_x is finite. If stable=semistable in X then the points of $X/\!\!/G$ are the orbits of G in X^{ss}, that is, two orbits are equivalent iff they are equal.

Let C be a connected smooth projective curve, and suppose that $X/\!\!/K$ is a locally free quotient.

Definition 3.11. A gauged holomorphic map (A, u) is an *infinite-area vortex* if it satisfies $u^*P(\Phi) = 0$.

Let $M^K(C, X)_\infty$ denote the set of gauge-equivalence classes of infinite-area vortices, and $M(C, X/\!\!/K)$ the set of holomorphic maps from C to $X/\!\!/K$.

Proposition 3.12. Suppose that K acts locally freely on $\Phi^{-1}(0)$. Then there is a bijection from $M^K(C, X)_\infty$ to $M(C, X/\!\!/K)$.

Proof. See Gaio-Salamon [18, Section 2]. Given an infinite-area vortex (A, u), let $\overline{u} : C \to X/\!\!/K$ denote the composition of u with the quotient map $\Phi^{-1}(0) \to X/\!\!/K$. The equation $\overline{\partial}_A u = 0$ implies $\overline{\partial}\overline{u} = 0$ (since $TX \to T(X/\!\!/K)$ is holomorphic) and so $\overline{u} \in M(C, X/\!\!/K)$. Conversely, given $\overline{u} : C \to X/\!\!/K$ let P be the pull-back of $\Phi^{-1}(0) \to X/\!\!/K$, equipped with the connection given by $JTX|\Phi^{-1}(0) \cap TX|\Phi^{-1}(0) \cong \pi^*T(X/\!\!/K)$. The equivariant map $P \to \Phi^{-1}(0)$ defines a section $u : C \to P \times_K X$. The equation $\overline{\partial}\overline{u} = 0$ implies $\overline{\partial}_A u = 0$, since J_A agrees with $J_{X/\!\!/K}$ on $\pi^*T(X/\!\!/K)$. If (A, u) is constructed in this way from $\overline{u}$ then the corresponding map to $X/\!\!/K$ is $\overline{u}$. To see that the map $(A, u) \mapsto \overline{u}$ is injective, suppose that (A, u) and (A', u') define the same holomorphic map to $X/\!\!/K$. This means that there is a gauge transformation k so that $ku = u'$. the equations $\overline{\partial}_A u = 0 = \overline{\partial}_{A'} u'$ and local freeness of the action imply that $kA = A'$.

Gaio-Salamon [18] studied under what conditions a sequence of symplectic vortices defined with respect to an area form $\rho_\nu \omega_C$ with $\rho_\nu \to \infty$ converge to a solution of the limiting equations:

Proposition 3.13. (Sequential compactness for bounded first derivative, [18]) Suppose that X is a Hamiltonian K-manifold with proper moment map convex at infinity equipped with an invariant almost complex structure J and C is a connected smooth projective curve. Suppose that (A_ν, u_ν) is a sequence of symplectic vortices for $\rho_\nu \omega_C$ with the property that $\sup \| \mathrm{d}_{A_\nu} u_\nu \|$ is bounded. Then there exists a subsequence and a sequence of gauge transformations k_ν such that $k_\nu(A_\nu, u_\nu)$ converges to an infinite-area vortex (A_∞, u_∞) uniformly in all derivatives.

Without a bound on the first derivative, various kinds of bubbling occur.

Proposition 3.14. (Bubble zoology for the infinite area limit) Suppose that X is a Hamiltonian K-manifold with a proper moment map convex at infinity and an invariant almost complex structure J and C is a connected smooth projective curve. Suppose that (A_ν, u_ν) is a sequence of vortices for $\rho_\nu \omega_C, \rho_\nu \to \infty$ with

$$c_\nu := \sup \| \mathrm{d}_{A_\nu} u_\nu \| = \| \mathrm{d}_{A_\nu} u_\nu(z_\nu) \|, \quad \epsilon_\nu := \rho_\nu / c_\nu.$$

Consider the *rescaled pair* $\phi_\nu^*(A_\nu, u_\nu)$ on $B_{c_\nu}(0)$ where $\phi_\nu(z) = z_\nu + z/c_\nu$. Noting that $\phi_\nu^*(A_\nu, u_\nu)$ has $\sup \| \mathrm{d}_{\phi_\nu^* A_\nu} u_\nu \| = 1$ bounded. Then after passing to a subsequence one of the following possibilities occurs:

(a) (Sphere Bubble in $X /\!\!/ K$) If $\lim_{\nu \to \infty} \epsilon_\nu = \infty$, then $\phi_\nu^*(A_\nu, u_\nu)$ converges to a solution to a solution to 3.11, that is, is equivalent to a holomorphic map to $\mathbb{C} \to X /\!\!/ K$.

(b) (Affine vortex) If $\lim_{\nu \to \infty} \epsilon_\nu \in (0, \infty)$, then $\phi_\nu^*(A_\nu, u_\nu)$ converges to a vortex on the affine line $\mathbb{A}$, with respect to the Euclidean area form $\omega_{\mathbb{A}} = \frac{i}{2} \mathrm{d}z \wedge \mathrm{d}\overline{z}$.

(c) (Sphere Bubble in X) If $\lim_{\nu \to \infty} \epsilon_\nu = 0$, then $\phi_\nu^*(A_\nu, u_\nu)$ converges to a vortex with zero vortex parameter, that is, a holomorphic map $\mathbb{C} \to X$.

The bubble trees that occur are described further in the following section.

3.4. Affine symplectic vortices

In this section we further study vortices on the complex affine line $\mathbb{A}$, which arose in the Gaio-Salamon study [18] of the large area limit. Ziltener [54] studied the bubbles that arise in more detail.

Definition 3.15. (Affine vortices) Suppose that X is a compact Hamiltonian K-manifold equipped with an invariant almost complex structure J. An *n-marked affine symplectic vortex* with target X is a datum $(A, u, \underline{z})$, where $A \in \Omega^1(\mathbb{A}, \mathfrak{g})$ is a connection on the trivial bundle $P := \mathbb{A} \times K \to \mathbb{A}$, $u : \mathbb{A} \to X$ is a J_A-holomorphic map, $\underline{z} = (z_0, \ldots, z_n) \in \mathbb{A}^n$ is an n-tuple of distinct points, and

$$\text{(Affine Vortex Equation)} \quad F_A + u^* \Phi \, \omega_{\mathbb{A}} = 0.$$

An *isomorphism* of scaled vortices (A_0, u_0) to (A_1, u_1) is translation ϕ of $\mathbb{A}$ and a gauge transformation $k : \mathbb{A} \to K$ satisfying $k^* \phi^* (A_1, u_1) = (A_0, u_0)$.

Let $M_{n,1}^K(\mathbb{A}, X)$ denote the moduli space of isomorphism classes of n-marked affine vortices. It has a natural compactification by isomorphism classes of *nodal* *affine vortices* described as follows. Let C be an $(n+1)$-marked connected genus zero nodal curve with irreducible components $C_1, \ldots, C_k$. For each irreducible component C_i not containing z_0 there is a unique node $\hat{w}_i$ which disconnects C_i from the marking z_0; we denote by $C_i^\circ = C_i - \{\hat{w}_i\}$ the complement. In case C_i contains z_0 we denote $C_i^\circ = C_i - \{z_0\}$. Each affine curve C_i° is isomorphic to $\mathbb{C}$, uniquely up to translation and dilation. Thus C_i° admits a unique equivalence class of Kähler forms, equal to $\omega_{\mathbb{A}} = \mathrm{d}z \wedge \mathrm{d}\overline{z}(i/2)$ up to scalar multiplication.

Definition 3.16. (Nodal affine vortices) Suppose that X is a compact Hamiltonian K-manifold equipped with an invariant almost complex structure J. An *n-marked nodal affine symplectic vortex* with target X is a datum $(C, P, A, u, \omega, \underline{z})$ consisting of a connected $(n+1)$-marked nodal curve C together with a principal K-bundle $P \to C$, a (possibly infinite or zero) two form $\omega : C \to \mathbb{P}(\Lambda^2 T^\vee_{\mathbb{R}} C \oplus \mathbb{R})$, a connection A_i on each $P|C_i$, a section $u : C \to P(X)$, and markings $\underline{z} = (z_1, \ldots, z_n)$, such that each irreducible component C_i of C is one of the following three types:

(a) (Zero Scaling) Components with zero two-form, equipped with a trivial bundle $P|C_i$ and a pseudoholomorphic map $u_i : C_i \to X$.
(b) (Finite, non-zero scaling) Components with non-zero, finite area form $\omega|C_i \in \Omega^2(C_i^\circ)$ equal to a non-zero multiple of $\omega_{\mathbb{A}}$ on $C_i^\circ \cong \mathbb{A}$ and an affine vortex (A_i, u_i) on C_i°.
(c) (Infinite scaling) Components C_i with infinite two-form $\omega|C_i$, equipped with holomorphic sections $u|C_i : C_i \to P(X)$ mapping to the zero level set $P(\Phi^{-1}(0))$, and so defining a holomorphic map $\overline{u}_i : C_i \to X/\!\!/K$.

This datum should satisfy the following conditions

(a) (Monotonicity) For every non-self-crossing path from a marking $z_i, i > 0$ to the marking z_0, the path crosses exactly one irreducible component with finite, non-zero area form, and all irreducible components before resp. after that irreducible component have zero resp. infinite area form.
(b) (Continuity) If C_i meets C_j at a node represented by a pair $(w_{ij}, w_{ji}) \in C_i \times C_j$ then $u_i(w_{ij}) = u_j(w_{ji})$.
(c) (Stability) If C_i is an irreducible component on which the two-form is zero or infinity resp. finite and non-zero and u_i is constant resp. u_i is covariant constant and A_i is flat then C_i contains at least two resp. three special points.

Remark 3.17. An irreducible component C_i is a *ghost component* if it satisfies one of the hypotheses requiring at least three special points or non-degenerate scalings; that is, $\overline{u}_i$ is constant or $A|C_i$ is flat and $u|C_i$ is covariant constant. The stability condition can then be reformulated as the condition that any ghost component has at least three special points (nodes or markings) or two special points and a non-zero, finite area form. Either of these conditions is equivalent to the absence of non-trivial infinitesimal automorphisms: infinitesimal automorphisms arising from gauge transformations are impossible because of the local freeness assumption for the action on the zero level set of the moment map.

There is a natural notion of *convergence* of affine nodal vortices which generalizes convergence with fixed area form in Definition 3.5.

Definition 3.18. (Convergence of nodal affine vortices) A sequence of isomorphism classes of nodal marked affine symplectic vortices $[(C_\nu, P_\nu, A_\nu, u_\nu, \omega_\nu, \underline{z}_\nu)]$ *converges* to a nodal marked affine vortex $[(C, P, A, u, \omega, \underline{z})]$ iff there exists for each irreducible component C_j of C, a sequence $U_{j,\nu} \subset C_j$ of increasing open neighborhoods and for each ν, a holomorphic embedding $\phi_{j,\nu} : U_{j,\nu} \to C_\nu$, an isomorphism $\psi_{j,\nu} : \phi_{j,\nu}^* P_\nu \to P$ such that if $\psi_{j,\nu}(X) : \phi_{j,\nu}^* P_\nu(X) \to P(X)$ denotes the associated maps of fiber bundles then

(a) (Open neighborhoods cover) the union of $\phi_{j,\nu}(U_{j,\nu})$ is C_ν;
(b) (Marked curves converge) If $z_i \in C_j$ then the limit of $\phi_{j,\nu}^{-1}(z_{\nu,i})$ is defined and equal to z_i. Furthermore, if $i \neq j$ then the image of $\phi_{j,\nu}^{-1} \circ \phi_{i,\nu}$ converges to the node of C_i connecting to C_j;
(c) (Connections converge) $\psi_{j,\nu}^* A_\nu$ converges to $A|C_j$ uniformly in all derivatives in any compact subset of any $U_{j,\nu'}$;
(d) (Sections converge) $\psi_{j,\nu}(X)^* u_\nu$ converges to $u|C_j$ uniformly in all derivatives in any compact subset of any $U_{j,\nu}$;
(e) (Scalings converge) $\phi_{j,\nu}^* \omega$ converges to $\omega|C_j$ in C^0 on any compact subset of any $U_{j,\nu'}$;
(f) (Energies converge) $\lim_{\nu\to\infty} E(C_\nu, P_\nu, A_\nu, u_\nu, \omega_\nu, \underline{z}_\nu)) = E(C, P, A, u, \omega, \underline{z})$.

Proposition 3.19. (Sequential compactness for affine vortices) Suppose that X is a Hamiltonian K-manifold with a proper moment map convex at infinity and an invariant almost complex structure J. Any sequence of isomorphism of classes of stable marked affine vortices with bounded energy has a convergent subsequence.

The proof of the theorem above combines results of Ziltener [54], [55] who discusses bubbling of affine vortices and bubbles in $X/\!\!/K$, and Ott [42], who treats bubbling in X. Namely:

Theorem 3.20. (Bubbling analysis for affine vortices) *Suppose that X is a Hamiltonian K-manifold with proper moment map convex at infinity equipped with an invariant almost complex structure J.*

(a) (Energy quantization) *[54, Lemma D.1] There exists a constant $E_0 > 0$ such that any non-trivial symplectic vortex (A, u) on $\mathbb{C}$ satisfies $E(A, u) > E_0$.*
(b) (Annulus Lemma) *[54, Lemma 4.11] For every compact subset $X_0 \subset X$ and every number $r_0 > 0$ there are constants $E_1 > 0, a > 0$ and $c_1 > 0$ such that the following holds. Assume that $r_0 \le r < R \le \infty$ and (A, u) is a vortex on the annulus $A(r, R) = B_R(0) - B_r(0)$ such that $u(z) \in X_0$ for every $z \in A(r, R)$, and suppose that $E((A, u)|A(r, R)) \le E_1$. Then for every $\rho \ge 2$ we have*

$$E((A, u)|_{A(\rho r, \rho^{-1} R)}) \le c_1 E(A, u)\rho^{-a}.$$

(c) (Mean value inequality) *Let $X_0 \subset X$ be a compact subset. Then there exists a constant $E_0 > 0$ such that for every $z_0 \in \mathbb{C}$, $r > 0$ and every symplectic vortex (A, u) satisfying $u(B_r(z_0)) \subset X_0$ and $E((A, u)|B_r(z_0)) \le E_0$, the energy density $e_{A,u}$ given by the integrand in (15) satisfies the estimate*

$$e_{A,u}(z_0) := \frac{1}{2}\|\mathrm{d}_A u(z_0)\|^2 + \|\Phi(u(z_0))\|^2 \le (8/\pi r^2) E((A, u)|B_r(z_0)).$$

(d) (Removal of singularities) *[54, Proposition D.6] Let* (A,u) *be a finite energy vortex on* $\mathbb{C}$*. The map* $Ku : \mathbb{C} \to X/K$ *extends continuously to a map* $\mathbb{P} \to X/K$*, such that* $Ku(\infty) \in X/\!\!/K$*. Furthermore,*

   i. there are constants $E > 0, C > 0$ *and* $\delta > 0$ *such that the following holds. For every vortex* (A,u) *on* $\mathbb{C}$ *and every* $R \geq 1$ *such that* $E(w, \mathbb{C}\backslash B_R) < E$ *and every* $z \in \mathbb{C}\backslash B_{2R}$ *we have* $e_{A,u}(z) \leq CR^{\delta}|z|^{-2-\delta}$*;*

   ii. there exist a number $\delta > 0$ *such that for* $2 \leq p < 4/(2-\delta)$*, then*

$$x_0 := \lim_{r\to\infty} u(r,0)$$

   exists, and there exists a map $k_0 \in W^{1,p}([0,2\pi], G)$ *such that if* $A_\theta(r)$ *denotes the restriction of the connection* A *in radial gauge to the circle* $\{|z| = e^r\} \cong S^1$*, then*

$$\lim_{r\to\infty} \max_{\theta\in S^1} \mathrm{d}(u(re^{i\theta}), k_0(\theta)x_0) = 0.$$

$$\sup_{r\geq 0} \|\partial_\theta k_0 k_0^{-1} + A_\theta(r)\|_{L^p(S^1)} e^{(-1+2/p+\delta/2)r} < \infty.$$

   Necessarily x_0 *is fixed by* $k_{2\pi}$*, which since* K *is compact and acts locally freely on* $\Phi^{-1}(0)$*, is finite order.*

Remark 3.21. The compactness argument for some of the moduli spaces below uses a slightly more general argument than that stated in Ziltener [55] and Ott [42]. Namely, multiples of the standard area form on the projective line gives rise to area forms in local coordinates

$$\lambda(s,t)\ \mathrm{d}s \wedge \mathrm{d}t = (1 + \epsilon(s^2+t^2))^{-2}\ \mathrm{d}s \wedge \mathrm{d}t \tag{19}$$

for $\epsilon > 0$. We will need the (Annulus Lemma) (b) above and the associated exponential decay of the derivative of a vortex for this area form. To see that the result still holds, we remark that Ziltener's results on the invariant symplectic action in [56] hold for any area form $\lambda \mathrm{d}s \wedge \mathrm{d}t$ satisfying the following property: Let

$$m = \inf\{\|\xi_X(x)\| \mid x \in X,\ \|\xi\| = 1\}$$

denote the minimal length of generating vector fields for Lie algebra vectors of length one. We say that $\lambda(s,t)\mathrm{d}s\mathrm{d}t$ is *admissible* for some constant $a > 0$ iff

$$\lambda \geq \frac{2\pi}{am}, \quad \sup |d(\lambda^{-1})|^2 + |\Delta(\lambda^{-2})| < 2m^2 \tag{20}$$

see Ziltener [56, Equation 1.7], in particular, for the area form (19). The mean value inequality in Ott [42, Section 2] holds uniformly in ϵ for the family of area forms (19). Thus the inequality in the (Annulus Lemma) implies an inequality for the covariant derivative of the map away from the ends.

In the case that K acts freely on $\Phi^{-1}(0)$, any finite energy vortex (A,u) on $\mathbb{C}$ has a well-defined *evaluation at infinity* $\mathrm{ev}_\infty(A,u) \in X/\!\!/K$, given by the limit of

$u(s+it)$ along any ray $s+it = re^{i\theta}, r \to \infty$. More generally in the locally free case, $\mathrm{ev}_\infty(A,u) = [k_0(2\pi), x_0]$ lies in the *inertia orbifold*

$$I_{X/\!\!/K} \in \{(k,x) \in K \times \Phi^{-1}(0) \mid kx = x\}/K$$

which, for example, appeared in Kawasaki [26]. The orbifold case was not treated in Ziltener [54]; however, the proof is almost the same as the manifold case.

Remark 3.22. (Failure of removal of singularities) The area form $\omega_{\mathbb{A}}$ has a pole at infinity and so one cannot expect an extension of (A,u) to the projective line to satisfy the vortex equations. Instead, taking P to be a orbi-bundle on $\mathbb{P}^1$ given by gluing in the trivial bundle using transition map z^λ, the pair (A,u) extends to a gauged holomorphic map on $\mathbb{P}$ mapping ∞ to $\Phi^{-1}(0)$ [50]. However, the extension will not satisfy any vortex equation on the entire projective line $\mathbb{P}$.

Let $\overline{M}^K_{n,1}(\mathbb{A},X)$ resp. $\overline{M}^{K,\mathrm{fr}}_{n,1}(\mathbb{A},X)$ denote the moduli space of isomorphism classes of nodal scaled affine vortices to X, resp. the moduli space of isomorphism classes of framed nodal scaled affine vortices to X. $\overline{M}^{\mathrm{fr}}_{n,1}(\mathbb{A},X)$ admits an evaluation maps at the markings, and, as explained in [54] an additional evaluation map at infinity to $I_{X/\!\!/K}$:

$$\mathrm{ev}^{\mathrm{fr}} \times \mathrm{ev}_\infty : \overline{M}^{K,\mathrm{fr}}_{n,1}(\mathbb{A},X) \to X^n \times I_{X/\!\!/K}.$$

If K^n acts freely, combining this map with a classifying map gives a map

$$\mathrm{ev} \times \mathrm{ev}_\infty : \overline{M}^{K}_{n,1}(\mathbb{A},X) \to X^n_K \times I_{X/\!\!/K}.$$

If K^n only acts locally freely, then the map above exists as a morphism of stacks, or after passing to a classifying space for the groupoid $\overline{M}^K_{n,1}(\mathbb{A},X)$, which has the same rational cohomology as $\overline{M}^K_{n,1}(\mathbb{A},X)$.

Remark 3.23. The case that X and K are trivial gives the complexified multiplihedron $\overline{M}_{n,1}$ constructed in Section 2.3. Indeed, each irreducible component C_i with finite or non-zero scaling is equipped with a isomorphism with the affine line, unique up to translation, and therefore a scaling λ_i. Thus the underlying curve of any affine vortex is automatically a (possibly unstable) scaled affine curve in the sense of Section 2.3.

Let $M^K_{n,1}(\mathbb{A},X)$ denote the moduli space of isomorphism classes of finite energy n-marked vortices on $\mathbb{A}$ (the additional marking at infinity) with values in X. By combining the sequential compactness theorem 3.19 with local distance functions as in (18), one has:

Theorem 3.24. (Properness of the moduli space of affine vortices) *Suppose that X is a Hamiltonian K-manifold with proper moment map convex at infinity, equipped with an invariant compatible almost complex structure. $\overline{M}^K_{n,1}(\mathbb{A},X)$ is Hausdorff and the energy map $E : \overline{M}^K_{n,1}(\mathbb{A},X) \to [0,\infty)$ is proper.*

3.5. Stable maps to orbifolds

In order to understand bubbling in the infinite area limit we briefly review the definition of a stable maps for orbifold targets, studied in Chen-Ruan [7], which appear as bubbles in the definition of certain gauged maps. The corresponding algebraic theory by Abramovich-Graber-Vistoli [1] will be reviewed in [52, Section 5]. Recall that an *orbifold structure* on a topological space C is a proper étale groupoid together with a homeomorphism of the space of objects to C. An *nodal orbifold structure* on a nodal complex curve C is an orbifold structure on its normalization, such that the automorphism group of any nodal point is independent of the choice of irreducible component containing it. A nodal orbifold structure is a *twisting* if each point with non-trivial automorphism is either a node or marking, and for each node, the orbifold charts satisfy the following *balanced condition*: the chart on one side of the node is of the form U/μ_r for some r where μ_r acts on $U \subset C$ a neighborhood of 0, while the orbifold structure on the other side is U/μ_r with the conjugate action by $\exp(-2\pi i/r)$.

Let Y be a compact symplectic orbifold equipped with a compatible almost complex structure J. A *stable map* to Y is a complex nodal curve C equipped with a twisting and a representable pseudoholomorphic orbifold map $u : C \to Y$. Representability means that the map u is smooth after passing to étale cover in Y: If $z \in C$ and $u(z) \in Y$ has automorphism group $\operatorname{Aut}(u(z))$, then there exists an orbifold chart $U/\operatorname{Aut}(u(z))$ for Y near $u(z)$ so that the groupoid fiber product $u^{-1}(U/\operatorname{Aut}(z)) \times_{U/\operatorname{Aut}(u(z))} U$ is an orbifold chart for z and u has a smooth local lift $\tilde{u}$ to U given by projection on the second factor. In particular, $\operatorname{Aut}(z)$ injects into $\operatorname{Aut}(u(z))$. The notion of Gromov convergence of twisted pseudoholomorphic maps which generalizes Gromov convergence of stable pseudoholomorphic maps. Let $\overline{M}_{g,n}(Y)$ denote the space of isomorphism classes of connected, genus g, n-marked stable maps to Y.

Theorem 3.25. (Properness of moduli spaces of stable maps to orbifolds) *Let Y be a compact symplectic orbifold equipped with a compatible almost complex structure. $\overline{M}_{g,n}(Y)$ is Hausdorff and the energy map $E : \overline{M}_{g,n}(Y) \to [0,\infty)$ is proper.*

A proof of this theorem is sketched by Chen-Ruan [7], who list the necessary changes in the proof of the compactness of the moduli space of stable maps for manifold targets.

3.6. Vortices with varying scaling

The adiabatic limit Theorem 1.5 will be proved by studying a moduli space of curves with *varying scaling* which interpolates between the moduli space of vortices for a fixed area form ω_C and the infinite area limit. More precisely, Theorem 1.5 will follow from a divisor class relation in the source moduli space constructed in Section 2.4, just as associativity of the quantum product follows from a divisor class relation in the moduli space of stable curves.

First we construct the objects that appear in the infinite area limit. Let X be a Hamiltonian K-manifold with proper moment map, so that $X /\!\!/ K$ is a locally free quotient and so an orbifold.

Definition 3.26. (Nodal infinite area vortices) An n-marked *nodal infinite area vortex* consists of the following datum:

(a) (Stable map to the quotient) a stable r-marked pseudoholomorphic map $u : C_0 \to C \times X/\!\!/K$ of class $([C], d_0)$ for some $d_0 \in H_2(X/\!\!/K)$;
(b) (Affine vortex bubbles) for each marking $z_j \in C_0$ a stable i_j-marked affine vortex $(C_j, A_j, u_j, \underline{z}_j)$ with markings $\underline{z}_j = (z_{j,1}, \ldots, z_{j,i_j})$ and orbifold structure of C_j at the point at infinity $z_{j,0}$ matching the orbifold structure of C_0 at $z_{0,j} \in C_0$, so that the union $C_0 \cup C_1 \cup \ldots \cup C_r$ is a balanced orbifold curve;
(c) (Matching condition) the value $u_i(z_{i,0}) = p_2(u_i(z_{i,0}))$ in $X/\!\!/K$, where $p_2 : C \times X/\!\!/K \to X/\!\!/K$ is the projection on the second factor.

An *isomorphism* of nodal infinite area vortices is a combination of an automorphism of the underlying curves intertwining the scalings and a bundle isomorphism on the affine vortex bubbles, intertwining the markings, scalings, connections and maps.

Remark 3.27. We denote by C the nodal curve obtained by gluing together the curves C_0 and C_i at $(z_i, z_{0,i})$; the matching condition means that the orbifold structures glue together to a twisting of C. Removal of singularities for vortices in Remark 3.22 implies that orbi-bundles P_j defined by the limit of the connections at infinity glue together with the bundle $P_0 \to C_0$ given by pull-back of $\Phi^{-1}(0) \to X/\!\!/K$, to give a bundle $P \to C$. The matching condition then implies that the section u_0 over C_0 and u_j over C_j glue together to a section of $P(X)$ over C with fairly weak regularity properties at the nodes with infinite scaling, so that u takes values in the zero level set $\Phi^{-1}(0)$ on the subset of C with infinite scaling. An isomorphism of nodal symplectic vortices is then an automorphism of the domain intertwining the connections (singular at the nodes with infinite scaling), maps, scalings, and markings.

Let $\overline{M}_n^K(C, X)_\infty$ denote the moduli space of isomorphism classes of nodal infinite-area vortices. From the description above, $\overline{M}_n^K(C, X)_\infty$ is the union of fiber products

$$\overline{M}_r^K(C, X)_\infty \times_{(I_{X/\!\!/K})^r} \prod_{j=1}^r \overline{M}_{|I_j|,1}^K(\mathbb{A}, X)$$

over unordered partitions $[I_1, \ldots, I_r]$ of $\{1, \ldots, n\}$.

We combine the moduli spaces above into a moduli space of vortices with varying area form. Let $\omega_C \in \Omega^2(C, \mathbb{R})$ be an area form.

Definition 3.28. (a) (Two-form corresponding to a scaling) Any scaled curve $(v : \hat{C} \to C, \lambda : \hat{C} \to \mathbb{P}(T_v^\vee \oplus \mathbb{C}))$ in the sense of Definition 2.39 defines a volume form $\omega_C(v, \lambda) \in \Omega^2(\hat{C}, \mathbb{R} \cup \{\infty\})$ as follows.

i. If λ is finite on the principal component, then $\omega_C(v, \lambda)$ is $|\lambda|^2$ times the form ω_C on the principal component, and on the bubble component $\omega_C(v, \lambda)$ vanishes.
ii. If λ is infinite on the principal component, then $\omega_C(v, \lambda)$ is equal to $\lambda \wedge \overline{\lambda}$ on each bubble tree.

(b) (Scaled vortices) A *marked scaled vortex* with domain C and target X consists of a marked scaled curve $(v : \hat{C} \to C, \lambda : \hat{C} \to \mathbb{P}(T_v^\vee \oplus \mathbb{C}))$ together with a vortex on $\hat{C}$ corresponding to the area form $\omega_C(v, \lambda)$. That is,
   i. If λ is finite on the principal component, then a vortex v_0 on the principal component corresponding to the area form $\omega_C(v,\lambda)|C_0$ and a collection of sphere bubbles $v_i : C_i \to X$ for each component $C_i \subset \hat{C}, i \neq 0$;
   ii. If λ is infinite on the principal component, then a holomorphic map $u_0 : C_0 \to X/\!\!/K$ on the principal component C_0 and a collection of nodal affine vortices on the bubble trees attached to C_0;

   Both should satisfy natural matching conditions at the nodes. A marked scaled vortex is *polystable* if each non-principal component with finite, non-zero scaling (resp. zero or infinite scaling) has at least 2 (resp. 3) special points. The notion of *isomorphism* of marked scaled vortices, using gauge transformations and automorphisms of the domain, is left to the reader. A marked scaled vortex is *stable* if it is polystable and has no infinitesimal automorphisms.

We denote by $\overline{M}^K_{n,1}(C, X)$ the moduli space of isomorphism classes of marked scaled vortices, and by $\overline{M}^K_{n,1}(C, X, d)$ the component with homology class $d \in H_2^K(X, \mathbb{Z})$. The notion of convergence of vortices extends naturally to the case of varying scaling, and defines a topology on $\overline{M}^K_{n,1}(C, X)$.

Theorem 3.29. (Properness of the moduli space of scaled symplectic vortices) *Let X be a Hamiltonian K-manifold with proper moment map convex at infinity equipped with a compatible almost complex structure, such that $X/\!\!/K$ is a locally free quotient. Then $\overline{M}^K_{n,1}(C, X)$ is Hausdorff and the energy map $E : \overline{M}^K_{n,1}(C, X) \to [0, \infty)$ is proper.*

The proof depends on an extension of the results of Ziltener [55], [56] and Ott [42] to the case of vortices with varying scaling. We need the following extension of [56, Theorem 1.3]:

Lemma 3.30. *For every constant $\epsilon > 0$, there exist constants $C, \delta > 0$ such that if (A, u) is a vortex on the cylinder $\Sigma = [-S, S] \times S^1$ with respect to an admissible area form as in (20) with respect to some constant a with $E(A, u) < \delta$, then the energy density $e_{A,u}$ satisfying for $(s, t) \in [-S + 1, S - 1] \times S^1$*

$$e_{A,u}(s, t) < CE(A, u)\lambda^{-2}e^{(-2\pi/a+\epsilon)s}.$$

Proof. We assume that $(A, u), \epsilon, a$ are as in the statement of the Lemma. As in Ziltener [56, p.17], the assumptions imply that for δ sufficiently small, the invariant symplectic action $\mathcal{A}$ in [56] is well-defined and satisfies the energy-action inequality

$$E((A, u)|_{[s,s']\times S^1}) = -\mathcal{A}((A, u)(s', \cdot)) + \mathcal{A}((A, u)(s, \cdot)).$$

Then as in [55, p. 46],

$$\frac{d}{ds}E((A, u)|_{[S+s,S-s]\times S^1}) \leq -(2\pi/a - \epsilon)E((A, u)|_{[S+s,S-s]\times S^1})$$

which implies the necessary exponential decay for the energy. The exponential decay for the energy density follows from the point-wise estimate in [56, Lemma 3.3]: If (A, u) has sufficiently small energy on a ball of radius 1 around (s, t) contained in $[-S, S] \times S^1$ then the energy density satisfies

$$e_{A,u}(s,t) \leq \frac{32}{\pi} E((A,u)_{B_1(s,t)}).$$

Note that the statement of Lemma [56, Lemma 3.3] is only for a fixed vortex on a half-cylinder, on a sufficiently small neighborhood of infinity, but an examination of the short proof shows that it suffices to have sufficiently small energy on the ball.

Proof of Theorem 3.29. We claim sequential compactness, that is, that any sequence of vortices $(\hat{C}_\nu, A_\nu, u_\nu)$ with varying scaling λ_ν with bounded energy has a convergent subsequence. For sequences with scaling bounded away from infinity, this follows from Ott [42]. For scalings λ_ν approaching infinity, one obtains convergence in the absence of bubbling by the arguments of Gaio-Salamon [18], see Theorem 3.13. That reference also shows that bubbles in $X /\!\!/ K$ or X or affine vortices satisfy energy quantization. Hence bubbling (blow-up of the first derivative of the section) happens only at finitely many points. It follows that after passing to a subsequence, $(\hat{C}_\nu, A_\nu, u_\nu)$ converges on the complement of a finite subset of the principal component, an infinite area vortex. By Proposition 3.12 such an infinite area vortex corresponds to a pseudoholomorphic map to $X /\!\!/ K$. By removal of singularities, one obtains a stable map to $X /\!\!/ K$. The annulus lemma in parts (c,d) of Theorem 3.20, and the extension described in Lemma 3.30, implies that bubbles in X connect with the affine vortices. The standard soft rescaling argument (see for example [55, Section 2.6]) shows the sequential compactness statement. Let the closed sets be those for which any convergent sequence has a limit. That this defines a topology, and that this topology is Hausdorff, uses the construction of local distance functions as in (18) which will be left to the reader.

References

[1] D. Abramovich, T. Graber, and A. Vistoli. Gromov-Witten theory of Deligne-Mumford stacks. *Amer. J. Math.*, 130(5):1337–1398, 2008.

[2] E. Arbarello, M. Cornalba, and P. A. Griffiths. *Geometry of algebraic curves. Volume II*, volume 268 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer, Heidelberg, 2011. With a contribution by Joseph Daniel Harris.

[3] K. Behrend and B. Fantechi. The intrinsic normal cone. *Invent. Math.*, 128(1):45–88, 1997.

[4] K. Behrend and Yu. Manin. Stacks of stable maps and Gromov-Witten invariants. *Duke Math. J.*, 85(1):1–60, 1996.

[5] A. Bertram, I. Ciocan-Fontanine, and B. Kim. Gromov-Witten invariants for abelian and nonabelian quotients. *J. Algebraic Geom.*, 17(2):275–294, 2008.

[6] S. B. Bradlow. Vortices in holomorphic line bundles over closed Kähler manifolds. *Comm. Math. Phys.*, 135(1):1–17, 1990.

[7] W. Chen and Y. Ruan. Orbifold Gromov-Witten theory. In *Orbifolds in mathematics and physics (Madison, WI, 2001)*, volume 310 of *Contemp. Math.*, pages 25–85. Amer. Math. Soc., Providence, RI, 2002.

[8] K. Cieliebak, A. Rita Gaio, I. Mundet i Riera, and D. A. Salamon. The symplectic vortex equations and invariants of Hamiltonian group actions. *J. Symplectic Geom.*, 1(3):543–645, 2002.

[9] I. Ciocan-Fontanine and B. Kim. Big I-functions. arxiv:1401.7417.

[10] I. Ciocan-Fontanine and B. Kim. Wall-crossing in genus zero quasimap theory and mirror maps. *Algebr. Geom.*, 1:400–448, 2014.

[11] I. Ciocan-Fontanine and B. Kim. Higher genus quasimap wall-crossing for semi-positive targets. arxiv:1308.6377.

[12] I. Ciocan-Fontanine, B. Kim, and D. Maulik. Stable quasimaps to GIT quotients. *J. Geom. Phys.*, 75:17–47, 2014.

[13] T. Coates, Y.-P. Lee, A. Corti, and H. H. Tseng. The quantum orbifold cohomology of weighted projective spaces. *Acta Math.*, 202(2):139–193, 2009.

[14] T. Coates, A. Corti, H. Iritani, Hsian-Hua Tseng. A Mirror Theorem for Toric Stacks. arXiv:1310.4163.

[15] P. Deligne and D. Mumford. The irreducibility of the space of curves of given genus. *Inst. Hautes Études Sci. Publ. Math.*, (36):75–109, 1969.

[16] K. Fukaya, Y.-G. Oh, H. Ohta, and K. Ono. *Lagrangian intersection Floer theory: anomaly and obstruction.*, volume 46 of *AMS/IP Studies in Advanced Mathematics*. American Mathematical Society, Providence, RI, 2009.

[17] W. Fulton and R. MacPherson. A compactification of configuration spaces. *Ann. of Math. (2)*, 139(1):183–225, 1994.

[18] A. R. P. Gaio and D. A. Salamon. Gromov-Witten invariants of symplectic quotients and adiabatic limits. *J. Symplectic Geom.*, 3(1):55–159, 2005.

[19] E. Getzler and M. M. Kapranov. Modular operads. *Compositio Math.*, 110(1):65–126, 1998.

[20] A. B. Givental. Equivariant Gromov-Witten invariants. *Internat. Math. Res. Notices*, (13):613–663, 1996.

[21] E. Gonzalez and C. Woodward. Quantum Witten localization and abelianization for qde solutions. arxiv:0811.3358.

[22] V. Guillemin and S. Sternberg. Geometric quantization and multiplicities of group representations. *Invent. Math.*, 67:515–538, 1982.

[23] K. Hori and C. Vafa. Mirror symmetry. arxiv:hep-th/0002222.

[24] H. Iritani. Quantum D-modules and generalized mirror transformations. *Topology*, 47(4):225–276, 2008.

[25] A. Jaffe and C. Taubes. *Vortices and monopoles*, volume 2 of *Progress in Physics*. Birkhäuser Boston, Mass., 1980. Structure of static gauge theories.

[26] T. Kawasaki. The Riemann-Roch theorem for complex V-manifolds. *Osaka J. Math.*, 16:151–159, 1979.

[27] F. C. Kirwan. *Cohomology of Quotients in Symplectic and Algebraic Geometry*, volume 31 of *Mathematical Notes*. Princeton Univ. Press, Princeton, 1984.

[28] F. F. Knudsen. The projectivity of the moduli space of stable curves. II. The stacks $M_{g,n}$. *Math. Scand.*, 52(2):161–199, 1983.

[29] S.-C. Lau, N. C. Leung, and B. Wu. Mirror maps equal SYZ maps for toric Calabi-Yau surfaces. *Bull. Lond. Math. Soc.* 44:255–270, 2012.

[30] J. Li and G. Tian. Virtual moduli cycles and Gromov-Witten invariants of general symplectic manifolds. In *Topics in symplectic 4-manifolds (Irvine, CA, 1996)*, First Int. Press Lect. Ser., I, pages 47–83. Internat. Press, Cambridge, MA, 1998.

[31] B. H. Lian, K. Liu, and S.-T. Yau. Mirror principle. I. *Asian J. Math.*, 1(4):729–763, 1997.

[32] Y. I. Manin. *Frobenius manifolds, quantum cohomology, and moduli spaces*, volume 47 of *American Mathematical Society Colloquium Publications*. American Mathematical Society, Providence, RI, 1999.

[33] M. Markl. Simplex, associahedron, and cyclohedron. In *Higher homotopy structures in topology and mathematical physics (Poughkeepsie, NY, 1996)*, volume 227 of *Contemp. Math.*, pages 235–265. Amer. Math. Soc., Providence, RI, 1999.

[34] M. Markl. Free loop spaces and cyclohedra. In *Proceedings of the 22nd Winter School "Geometry and Physics" (Srní, 2002)*, number 71, pages 151–157, 2003.

[35] M. Markl, S. Shnider, and J. Stasheff. *Operads in algebra, topology and physics*, volume 96 of *Mathematical Surveys and Monographs*. American Mathematical Society, Providence, RI, 2002.

[36] S. Ma'u and C. Woodward. Geometric realizations of the multiplihedra. *Compos. Math.*, 146(4):1002–1028, 2010.

[37] D. McDuff and D. Salamon. *J-holomorphic curves and symplectic topology*, volume 52 of *American Mathematical Society Colloquium Publications*. American Mathematical Society, Providence, RI, 2004.

[38] D. Mumford. Boundary points of moduli schemes. In *Lecture notes prepared in connection with the seminars held at the Summer Institute on Algebraic Geometry, Whitney Estate, Woods Hole, Massachusetts, July 6 – July 31, 1964*. woodshole.pdf.

[39] I. Mundet i Riera. A Hitchin-Kobayashi correspondence for Kähler fibrations. *J. Reine Angew. Math.*, 528:41–80, 2000.

[40] I. Mundet i Riera. Hamiltonian Gromov-Witten invariants. *Topology*, 42(3):525–553, 2003.

[41] K. Nguyen, C. Woodward, and F. Ziltener. Morphisms of cohomological field theories and quantization of the Kirwan map. In *Symplectic, Poisson, and Noncommutative Geometry*, Cambridge University Press, pages 131–170, 2014.

[42] A. Ott. Removal of singularities and Gromov compactness for symplectic vortices. *J. Symplectic Geom.* 12:257–311, 2014.

[43] J. Preskill. Vortices and monopoeles. In *Architecture of Fundamental Interactions at Short Distances (1987), P. Ramond and R. Stora, editors.*

[44] A. Schmitt. A universal construction for moduli spaces of decorated vector bundles over curves. *Transform. Groups*, 9(2):167–209, 2004.

[45] A. H. W. Schmitt. *Geometric invariant theory and decorated principal bundles*. Zurich Lectures in Advanced Mathematics. European Mathematical Society (EMS), Zürich, 2008.

[46] J. Stasheff. *H-spaces from a homotopy point of view.* Lecture Notes in Mathematics, Vol. 161. Springer-Verlag, Berlin, 1970.

[47] C. Teleman. Topological field theories in 2 dimensions. In *European Congress of Mathematics*, pages 197–210. Eur. Math. Soc., Zürich, 2010.

[48] S. Venugopalan. Vortices on surfaces with cylindrical ends. arxiv:1201.1933.

[49] S. Venugopalan. Yang-Mills heat flow on gauged holomorphic maps. arXiv:1201.1933.

[50] S. Venugopalan and C. Woodward. Classification of affine vortices. arXiv:1301.7052.

[51] E. Witten. Phases of $N = 2$ theories in two dimensions. *Nuclear Phys. B*, 403(1-2):159–222, 1993.

[52] C. Woodward. Quantum Kirwan morphism and Gromov-Witten invariants of quotients II. preprint.

[53] C. Woodward. Quantum Kirwan morphism and Gromov-Witten invariants of quotients III. preprint.

[54] F. Ziltener. Symplectic vortices on the complex plane. ETH Zurich 2006 Ph.D. thesis.

[55] F. Ziltener. *A Quantum Kirwan Map: Bubbling and Fredholm Theory for Symplectic Vortices over the Plane*, volume 230 of *Mem. Amer. Math. Soc.* no. 1082, 2014.

[56] F. Ziltener. The invariant symplectic action and decay for vortices. *J. Symplectic Geom.*, 7(3):357–376, 2009.