Fulton volume

Bill's birthday

Edited by
Bill's friends

Contents

Contributors

Eduardo González
Chris T. Woodward

Department of Mathematics
University of Massachusetts Boston
100 William T. Morrissey Boulevard
Boston, MA 02125
eduardo@math.umb.edu
Mathematics-Hill Center
Rutgers University, 110 Frelinghuysen Road
Piscataway, NJ 08854-8019
ctw@math.rutgers.edu

1

Quantum Kirwan for quantum K-theory

E. González and C. Woodward

Abstract

For G a complex reductive group and X a smooth projective or convex quasi-projective polarized G-variety we construct a formal map in quantum K-theory

$$\kappa_X^G : QK_G^0(X) \to QK^0(X/\!\!/G)$$

from the equivariant quantum K-theory $QK_G^0(X)$ to the quantum K-theory of the geometric invariant theory quotient $X/\!\!/G$, assuming the quotient $X/\!\!/G$ is a smooth Deligne-Mumford stack with projective coarse moduli space. As an example, we give a presentation of the (possibly bulk-shifted) quantum K-theory of any smooth proper toric Deligne-Mumford stack with projective coarse moduli space, generalizing the presentation for quantum K-theory of projective spaces due to Buch-Mihalcea [7] and (implicitly) of Givental-Tonita [18]. We also provide a wall-crossing formula for the K-theoretic gauged potential

$$\tau_X^G : QK_G^0(X) \to \Lambda_X^G$$

under variation of geometric invariant theory quotient, a proof of the invariance of τ_X^G under (strong) crepant transformation assumptions, and a proof of the abelian non-abelian correspondence relating τ_X^G and τ_X^T for $T \subset G$ a maximal torus.

Partially supported by NSF grants DMS-1510518 and DMS-1711070

1.1 Introduction

We aim to describe the behavior of quantum K-theory under the operation of geometric invariant theory quotient. Let X be a smooth projective G-variety with polarization $\mathcal{L} \to X$. The geometric invariant theory (git) quotient $X /\!\!/ G$ is a proper smooth Deligne-Mumford stack with projective coarse moduli space. Let $K_G^0(X)$ denote the even topological or algebraic G-equivariant K-cohomology of X. The *Kirwan map* is the ring homomorphism in K-theory ("K-theoretic reduction")

$$\kappa_X^G : K_G^0(X) \to K^0(X /\!\!/ G), \quad [E] \mapsto [(E|X^{\mathrm{ss}})/G] \tag{1.1}$$

obtained by restricting a vector bundle E to the semistable locus X^{ss} and passing to the stack quotient. If X is projective then Kirwan showed that (1.1) is surjective in rational cohomology [39]. The analogous results in K-theory hold by work of Harada-Landweber [30, Theorem 3.1] and Halpern-Leistner [27, Corollary 1.2.3]. The Kirwan map can often be used to compute the K-theory of a git quotient. In particular, the Kirwan map allows a simple presentation of the K-theory of a smooth projective toric variety, equivalent to the presentation given in Vezzosi-Vistoli [55, Section 6.2].

A quantum deformation of the K-theory ring was introduced by Givental [17] and Y.P. Lee [41]. In this deformation, the tensor product of vector bundles is replaced by a certain push-pull over the moduli space of stable maps. The virtual fundamental class in cohomology is replaced by a *virtual structure sheaf* introduced in [41], and integrals over the moduli space of stable maps in K-theory are called *K-theoretic Gromov-Witten invariants*. For many purposes quantum K-theory is expected to be more natural than the quantum cohomology ring, which can be obtained as a limit; see for example [43]. In particular, the K-theoretic Gromov-Witten invariants are integers. Computations in quantum K-theory have been rather rare; even the quantum K-theory of projective space seems to have been computed only recently by Buch-Mihalcea [7].

We develop a quantum version of Kirwan's map in K-theory. As applications, we give presentations of the quantum K-theory of toric varieties, generalizing a computation of Buch-Mihalcea [7] in the case of projective spaces. Let

$$\Lambda_X^G \subset \mathrm{Map}(H_2^G(X, \mathbb{Q}), \mathbb{Q})$$

denote the equivariant Novikov ring associated to the polarization, with q^d denoting for q a formal variable the delta function at $d \in H_2^G(X, \mathbb{Q})$.

The quantum K-theory product is defined by Givental-Lee [17, 41] as a pull-push over moduli spaces of stable maps; in order to make the products finite we define the equivariant quantum K-theory as the completion of $K_G^0 \otimes \Lambda_X^G$ with respect to the ideal $I_G^X(c)$ of elements $E \in K_G^0 \otimes \Lambda_X^G$ with $\mathrm{val}_q(E) > c$:

$$QK_G^0(X) = \lim_{n \leftarrow} (K_G^0(X) \otimes \Lambda_X^G)/I_G^X(c)^n. \tag{1.2}$$

The main result is the following:

Theorem 1.1. *Let G be a complex reductive group and X be a smooth polarized projective (or convex quasiprojective) G-variety with locally free git quotient $X/\!\!/G$. There exists a canonical* Kirwan map *in quantum K-theory*

$$\kappa_X^G : QK_G^0(X) \to QK^0(X/\!\!/G)$$

with the property that the linearization $D_\alpha \kappa_X^G$ is a homomorphism:

$$D_\alpha \kappa_X^G(\beta \star \gamma) = D_\alpha \kappa_X^G(\beta) \star D_\alpha \kappa_X^G(\gamma).$$

If $X/\!\!/G$ is a free quotient then κ_X^G is surjective.[1]

The convexity assumption is satisfied if, for example, the variety is a vector space with a torus action such that all weights are properly contained in a half-space. For example, for toric varieties that may be realized as git quotients we explicitly compute the kernel of the map to obtain a presentation of the orbifold quantum K-theory at a point determined by the presentation, generalizing the presentation of ordinary K-theory of non-singular toric varieties due to Vezzosi-Vistoli [55, Section 6.2] and the case of quantum K-theory of projective spaces due to Buch-Mihalcea [7]: Let G be a complex torus and X a vector space with weight spaces $X_1, \ldots, X_k$ and weights $\mu_1, \ldots, \mu_k$ define the *completed equivariant quantum K-theory* $\widehat{QK}{}_G^0(X)$ to be the ring with generators $X_1^{\pm 1}, \ldots, X_k^{\pm 1}$ formally completed by the ideal generated by $(1 - X_j^{-1}), j = 1, \ldots, k$.[2] The K-theoretic Batyrev (or quantum K-theoretic Stanley-Reisner) ideal is the ideal $QKSR_X^G$ generated by the relations

$$\prod_{(\mu_j, d) \geq 0} (1 - X_j^{-1})^{\mu_j(d)} = q^d \prod_{(\mu_j, d) < 0} (1 - X_j^{-1})^{-\mu_j(d)}. \tag{1.3}$$

[1] Computations suggest that the map κ_X^G might be surjective even for locally free quotients of proper free actions.

[2] In the case of orbifold quotients, a more complicated formal completion is necessary, see Definition 1.2.

Theorem 1.2. *Suppose that G is a torus with Lie algebra $\mathfrak{g}$, that X is a G-vector space with weights $\mu_1, \dots, \mu_k \in \mathfrak{g}_{\mathbb{R}}^{\vee}$ contained in an open half-space in $\mathfrak{g}_{\mathbb{R}}^{\vee}$, and that X is equipped with a polarization so that $X /\!\!/ G$ is a non-singular proper toric Deligne-Mumford stack with projective coarse moduli space. Let $T = (\mathbb{C}^\times)^k / G$ denote the residual torus. Then the quantum K-theory ring $QK^0(X/\!\!/G)$ with bulk deformation $\kappa_X^G(0)$ is isomorphic to the quotient $\widehat{QK_G^0(X)}/QKSR_X^G$.*

Example 1.3. (Weighted projective spaces) Suppose that $G = \mathbb{C}^\times$ acts on $X = \mathbb{C}^k$ with weights $\mu_1, \dots, \mu_k \in \mathbb{Z}$, so that $X/\!\!/G$ is the weighted projective space $\mathbb{P}(\mu_1, \dots, \mu_k)$. Then the T-equivariant quantum K-theory of $X/\!\!/G$ has canonical presentation with generators and a single relation (in this case the formal completion is not necessary):

$$QK^0(X/\!\!/G) \cong \frac{\Lambda_X^G[X_1^{\pm 1}, \dots, X_k^{\pm 1}]}{\left\langle \prod_j (1 - X_j^{-1})^{\mu_j(d)} - q \right\rangle}. \tag{1.4}$$

In this case, the bulk deformation $\kappa_X^G(0)$ turns out to vanish, see Lemma 1.6 below, and X_j is the class of the line bundle associated to the weight μ_j.

Example 1.4. ($B\mathbb{Z}_2$) This is a sub-example of the previous Example 1.3; we include it to emphasize the importance of working over the equivariant Novikov ring. Suppose that $G = \mathbb{C}^\times$ acts on $X = \mathbb{C}$ with weight two. Then the quantum K-theory of $X/\!\!/G = B\mathbb{Z}_2$ has generators $X^{\pm 1}$ with single relation

$$QK(B\mathbb{Z}_2) = \frac{\mathbb{Z}[X^{\pm 1}, q]}{\left\langle (1 - X^{-1})^2 - q \right\rangle}.$$

On the other hand, without the equivariant Novikov ring $K(B\mathbb{Z}_2; \mathbb{Z})$ is simply the group ring on $\mathbb{Z}_2$ via the identification of representations with their characters. Let

$$\delta_{\pm 1} \in K(B\mathbb{Z}_2; \mathbb{Z})$$

be the delta functions at the group elements $\pm 1 \in \mathbb{Z}_2$. Proposition 1.1 below shows that $(1 - X^{-1})$ maps to $\sqrt{q}\delta_{-1}$ under $D_0\kappa_X^G$. This implies the relation

$$(\sqrt{q}\delta_{-1})^2 - q = q\delta_{(-1)(-1)} - q = 0$$

since δ_1 is the identity. This matches with the relation

$$\delta_{-1}^2 = \delta_1$$

in $K(B\mathbb{Z}_2)$, the group algebra of $\mathbb{Z}_2$ since the product is given by convolution.

Since the relations are essentially the same as those in quantum cohomology, one obtains a generalization of the isomorphism between K-theory and cohomology of toric varieties induced by identifying cohomological and K-theoretic first Chern classes of divisors in Vezzosi-Vistoli [55, Section 6.2]: The quantum K-theory ring at bulk deformation $\kappa_X^G(0) \in QK^0(X /\!\!/ G)$ is canonically isomorphic to the quantum cohomology ring $QH(X /\!\!/ G)$ at bulk deformation $\kappa_X^{G,\mathrm{coh}}(0) \in QH(X /\!\!/ G)$ (where $\kappa_X^{G,\mathrm{coh}} : QH_G(X) \to QH(X /\!\!/ G)$ is the cohomological quantum Kirwan map) via a map defined on generators by

$$QK^0(X /\!\!/ G) \to QH(X /\!\!/ G), \quad D_\alpha \kappa_X^G(X_j - 1) \mapsto D_\alpha \kappa_X^{G,\mathrm{coh}}(c_1(X_j)).$$

In particular, the quantum K-theory of toric Deligne-Mumford stacks is generically semisimple.

We thank Ming Zhang for helpful comments and an anonymous referee for pointing out an important omission in the orbifold case.

1.2 Equivariant quantum K-theory

We recall the following basics of equivariant quantum K-theory, following Buch-Mihalcea [7] and Iritani-Milanov-Tonita [33]. Let X be a G-variety. The *equivariant K-homology group* $K_0^G(X)$ is the Grothendieck group of coherent G-sheaves on X, that is, the free Abelian group generated by isomorphism classes of G-equivariant coherent sheaves modulo relations whenever there exists an equivariant exact sequence: For G-equivariant sheaves $\mathcal{F}, \mathcal{F}', \mathcal{F}''$ we have an implication

$$0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0 \quad \implies \quad [\mathcal{F}] = [\mathcal{F}'] + [\mathcal{F}''].$$

The K-homology $K_G^0(X)$ is naturally a module over the *equivariant K-cohomology ring* $K_G^0(X)$ of G-equivariant vector bundles on X. Both the multiplicative structure of $K_G^0(X)$ and the module structure of $K_0^G(X)$ are given by tensor products. If X is non-singular then the map from $K_G^0(X)$ to $K_0^G(X)$ that sends a vector bundle to its sheaf of sections is an isomorphism.

Equivariant K-theory has the following functoriality properties. Given a G-equivariant morphism $f : X \to Y$ between varieties X, Y there is a ring homomorphism

$$f^* : K^0_G(Y) \to K^0_G(X), \quad [E] \mapsto [f^* E]$$

given by pull-back of bundles. If f is proper then there is a pushforward map

$$f_* : K_0^G(X) \to K_0^G(Y), \quad f_*[\mathcal{F}] \mapsto \sum_{i \geq 0} (-1)^i [R^i f_* \mathcal{F}].$$

This map is a homomorphism of $K^0_G(Y)$ modules by the projection formula.

The quantum product in K-theory is defined by incorporating contributions from moduli spaces of stable maps. Let X be a smooth projective G-variety. For integers $g, n \geq 0$ and a class $d \in H_2(X, \mathbb{Z})$ let

$$\overline{\mathcal{M}}_{g,n}(X,d) = \left\{ (u : C \to X, \underline{z} \in C^n) \;\middle|\; \begin{array}{l} \# \operatorname{Aut}(u, \underline{z}) < \infty, \\ g(C) = g, \ u_*[C] = d \end{array} \right\}$$

denote the moduli stack of stable maps to X with n markings, genus g, and homology class d. The evaluation maps are denoted

$$\operatorname{ev} = (\operatorname{ev}_1, \ldots, \operatorname{ev}_n) : \overline{\mathcal{M}}_{g,n}(X,d) \to X^n.$$

Recall that a *perfect obstruction theory* $E^\bullet$ *admitting a global resolution by vector bundles* on a stack $\mathcal{M}$ is a pair (E, ϕ) consisting of an object of the bounded derived category of coherent sheaves on $\mathcal{M}$ that can be presented as a two term complex

$$E = [E^{-1} \to E^0] \in D^{[-1,0]}(\operatorname{Coh}(\mathcal{M}))$$

of vector bundles, together with a morphism

$$\phi : E \to L_{\mathcal{M}}, \quad h^0(\phi) \text{ iso}, \quad h^{-1}(\phi) \text{ epi}$$

to the ($L^{\geq -1}$ truncation of the) cotangent complex of $L_{\mathcal{M}}$, satisfying that (c.f. [4], [21]) $h^0(\phi)$ is an isomorphism and $h^{-1}(\phi)$ is an epimorphism. The perfect obstruction theory defines the *virtual tangent bundle*

$$T^{\text{vir}}_{\mathcal{M}} = \operatorname{Def} - \operatorname{Obs} \in K^0(\mathcal{M}), \quad \operatorname{Def} := [(E^0)^\vee], \quad \operatorname{Obs} := [(E^{-1})^\vee]$$

as in [10, p. 21]. That is, the virtual tangent bundle is the K-theoretic difference of the deformation space and the obstruction space. The virtual normal cone $C \hookrightarrow E_1 = (E^{-1})^\vee$ induces the *virtual structure sheaf* [41] as the derived tensor product

$$\mathcal{O}^{\text{vir}}_{\mathcal{M}} := \mathcal{O}_{\mathcal{M}} \bigotimes^{L}_{\mathcal{O}_{E_1}} \mathcal{O}_C \in \operatorname{Ob}(D \operatorname{Coh}(\mathcal{M})),$$

whose class in $K(\mathcal{M})$ is

$$[\mathcal{O}_{\mathcal{M}}^{\text{vir}}] = \sum_{i=0}^{\infty} (-1)^i \operatorname{Tor}^i_{\mathcal{O}_{E_1}}(\mathcal{O}_{\mathcal{M}}, \mathcal{O}_C) \in K(\mathcal{M}).$$

For any class $\alpha \in K(\mathcal{M})$ we define the *virtual Euler characteristic*

$$\chi^{\text{vir}}(\mathcal{M}; \alpha) = \chi(\mathcal{M}; \alpha \otimes \mathcal{O}_{\mathcal{M}}^{\text{vir}}) \in \mathbb{Z}$$

the Euler characteristic after twisting by $\mathcal{O}_{\mathcal{M}}^{\text{vir}}$. These constructions admit equivariant generalizations, so that for any genus $g \geq 0$ and markings $n \geq 0$ with $\mathcal{M} = \overline{\mathcal{M}}_{g,n}(X, d)$ the *virtual structure sheaf* $\mathcal{O}_{\mathcal{M}}^{\text{vir}}$ is an object in the G-equivariant bounded derived category for $\overline{\mathcal{M}}_{g,n}(X, d)$ introduced by Y.P. Lee [41]. It defines a class $[\mathcal{O}^{\text{vir}}]$ in the equivariant K-theory of $\overline{\mathcal{M}}_{g,n}(X, d)$. For classes $\alpha_1, \ldots, \alpha_n \in K_G^0(X)$ and a class $\beta \in K(\overline{\mathcal{M}}_{g,n})$ define the equivariant K-theoretic Gromov-Witten invariants

$$\langle \alpha_1, \ldots, \alpha_n; \beta \rangle_{g,n,d} := \chi_G(\operatorname{ev}_1^* \alpha_1 \otimes \ldots \otimes \operatorname{ev}_n^* \alpha_n \otimes f^* \beta \otimes [\mathcal{O}^{\text{vir}}]), \quad (1.5)$$

where χ_G is the equivariant Euler characteristic. In fact, because X is smooth, one may replace the algebraic K-cohomology group above by the topological equivariant K-cohomology, that is, the Grothendieck group of equivariant complex vector bundles on X, as in [41, Section 4]. In this case the equivariant Euler characteristic is then replaced by a proper push-forward in topological K-theory. Define *descendant invariants* by

$$\langle \alpha_1 L^{d_1}, \ldots, \alpha_n L^{d_n}; \beta \rangle_{g,n,d} \in K_G^0(\text{pt})$$

defined by insertion d_i cotangent lines L at the i-th marking z_i.

The K-theoretic Gromov-Witten invariants can be organized into a potential as follows. Let $\Lambda_X \subset \operatorname{Map}(H_2(X), \mathbb{Q})$ denote the *Novikov ring* associated to the ample line bundle $\mathcal{L} \to X$. The elements of the Novikov ring are formal combinations

$$\Lambda_X = \left\{ \sum_{d \in H_2(X)} c_d q^d \right\}$$

such that for any $E > 0$ the number of coefficients c_d with $(d, c_1(\mathcal{L})) < E$ is finite. Define the *equivariant quantum K-theory* as the completion (1.2). The K-theoretic *genus zero Gromov-Witten potential* with insertions is the formal function which we write informally

$$\mu_X : QK_G^0(X) \to QK_G^0(\text{pt}), \quad \alpha \mapsto \sum_{d \in H_2(X)} \sum_{n \geq 0} \langle \alpha, \ldots, \alpha; 1 \rangle_{0,n,d} \frac{q^d}{n!}; \quad (1.6)$$

what this means is that each Taylor coefficient of μ_X is well-defined. In the following expressions involving μ_X will be understood in this sense. For any element $\sigma \in QK_G^0(\mathrm{pt})$ we denote by $\partial_\sigma \mu_X$ the differentiation of μ_X in the direction of σ. The *quantum K-theory pairing* at $\alpha \in QK_G^0(X)$ is for $\sigma, \gamma \in QK_G^0(X)$

$$B_\alpha(\sigma, \gamma) = \partial_1 \partial_\sigma \partial_\gamma \mu_X(\alpha) \in QK_G^0(\mathrm{pt}) \tag{1.7}$$

where the identity in $QK_G^0(X)$ is the structure sheaf $\mathcal{O}_X$. This recovers the usual pairing $\chi(\sigma \otimes \gamma)$ when $\alpha = 0, q = 0$. Note that the corresponding pairing in quantum cohomology is the classical pairing; the existence of quantum corrections in the K-theoretic pairing is due to a modification in the contraction axioms [41, Section 3.7], this is an important new feature of quantum K-theory. The *quantum K-theory product* on $QK_G^0(X)$ with bulk deformation α is the formal product defined by

$$B_\alpha(\sigma \star_\alpha \gamma, \kappa) = \partial_\sigma \partial_\gamma \partial_\kappa \mu_X(\alpha).$$

As in the product in quantum cohomology, for each choice of $\alpha \in QK_G^0(X)$ we obtain a formal Frobenius algebra structure on $QK_G^0(X)$, by an argument of Givental [17]. Notably, the product does not satisfy a divisor axiom. However, quantum K-theory has better properties in other respects. For example, the small quantum K-theory (product at $\alpha = 0$) is defined over the integers, since the virtual Euler characteristics are virtual representations.

Later we will need a slight reformulation of the quantum-corrected inner product in (1.7). Let ev_{n+1}^d denote the restriction of the evaluation map

$$\mathrm{ev}_{n+1} : \overline{\mathcal{M}}_{0,n+1}(X) \to X$$

to $\overline{\mathcal{M}}_{0,n+1}(X, d)$. Define the (formal) *Maurer-Cartan map*

$$\begin{aligned} \mathcal{MC}_X^G &: QK_G^0(X) \to QK_G^0(X) \\ \alpha &\mapsto \sum_{n \geq 1, d \in H_2(X)} \mathrm{ev}_{n+2,*}^d \left(\mathrm{ev}_1^* \alpha \otimes \ldots \otimes \mathrm{ev}_n^* \alpha \otimes ev_{n+1}^* 1 \right) \frac{q^d}{n!} \end{aligned} \tag{1.8}$$

where the push-forward is defined using the virtual structure sheaf. Then if $\underline{B}$ denotes the classical Mukai pairing we have

$$B_\alpha(\sigma, \gamma) = \underline{B}(D_\alpha \mathcal{MC}_X^G(\sigma), \gamma), \quad \underline{B}^{-1} B_\alpha = D_\alpha \mathcal{MC}_X^G,$$

where $D_\alpha \mathcal{MC}_X^G$ denotes the linearization of $\mathcal{MC}_X^G$ at α

$$\sigma \mapsto \sum_{n\geq 1, d\in H_2(X)} \mathrm{ev}^d_{n+2,*}\left(\mathrm{ev}_1^*\,\sigma \otimes \mathrm{ev}_2^*\,\alpha \otimes \ldots \otimes \mathrm{ev}_n^*\,\alpha \otimes \mathrm{ev}_{n+1}^*\,1\right)\frac{q^d}{n!}.$$

One can also consider twisted K-theoretic Gromov-Witten invariants as in Tonita [54]: Let

$$\begin{array}{ccc} \overline{\mathcal{C}}_{g,n}(X,d) & \xrightarrow{\ e\ } & X \\ \downarrow p & & \\ \overline{\mathcal{M}}_{g,n}(X,d). & & \end{array}$$

denote the universal curve. If $E \to X$ is a G-equivariant vector bundle then the index class is defined by

$$\mathrm{Ind}(E) := [Rp_* e^* E] \in K^0(\overline{\mathcal{M}}_{g,n}(X,d)).$$

Its Euler class $[\mathrm{Eul}(\mathrm{Ind}(E))]$ is well-defined in $K^0_{\mathbb{C}^\times}(\overline{\mathcal{M}}_{g,n}(X,d))$ after localizing the equivariant parameter for the action of $\mathbb{C}^\times$ by scalar multiplication on the fibers of E at roots of unity. The genus $g=0$ sum

$$\begin{aligned} &\varphi^E_{n,d} : K^0_{G\times\mathbb{C}^\times}(X)^{\otimes n} \to K^{0,\mathrm{loc}}_{G\times\mathbb{C}^\times}(\mathrm{pt}), \\ &(\alpha_1,\ldots,\alpha_n) \mapsto \chi_{G\times\mathbb{C}^\times}(\mathrm{ev}_1^*\,\alpha_1 \otimes \ldots \otimes \mathrm{ev}_n^*\,\alpha_n \otimes [\mathcal{O}^{\mathrm{vir}}] \otimes [\mathrm{Eul}(\mathrm{Ind}(E))]) \end{aligned} \tag{1.9}$$

produces the *E-twisted quantum K-theory*, again a Frobenius manifold.

An analog of the quantum connection in quantum K-theory was introduced by Givental [17]:With $m(\alpha)(\cdot) = \alpha \star \cdot$ quantum multiplication define a connection

$$\nabla^q_\alpha = (1-z)\partial_\alpha + m(\alpha) \in \mathrm{End}(QK^0_{\mathbb{C}^\times}(X)).$$

By Givental [17] the quantum connection is flat. One of the goals of this paper is to give (somewhat non-explicit) formulas for its fundamental solutions.

1.3 Quantum K-theoretic Kirwan map

In this section we extend the definition of the quantum Kirwan map, defined in [57] to K-theory. Let G be a complex reductive group as in the introduction and let X be a smooth polarized projective G-variety with G-polarization, that is, ample G-line bundle, $\mathcal{L} \to X$. Suppose that

G acts with finite stabilizers on the semistable locus, defined as the locus of points with non-vanishing invariant sections of some positive power of the polarization:

$$X^{\mathrm{ss}} = X^{\mathrm{ss}}(\mathcal{L}) = \{x \in X \mid \exists k > 0, s \in H^0(X, \mathcal{L}^{\otimes k})^G,\ s(x) \neq 0\} \subset X.$$

Equivalently, suppose that every orbit $Gx \subseteq X^{\mathrm{ss}}$ for $x \in X^{\mathrm{ss}}$ is closed. Denote the stack-theoretic quotient

$$X/\!\!/G := X^{\mathrm{ss}}/G$$

which is necessarily a smooth proper Deligne-Mumford stack with projective coarse moduli space. By definition X/G is a category whose objects

$$\mathrm{Ob}(X/G) = \{(P \to C, u : P \to X)\}$$

are pairs consisting of principal G-bundles P over some base C and equivariant maps $u : P \to X$, and whose morphisms are the natural commutative diagrams.

1.3.1 Affine gauged maps

The quantum Kirwan map is defined by push-forward (in cohomology by integration) over moduli spaces of *affine gauged maps*.

Definition 1.1. (Affine gauged maps) An affine gauged map to X/G is a datum $(C, \underline{z}, \lambda, u : C \to X/G)$ consisting of

- (Curve) a possibly-nodal projective curve $p : C \to S$ of arithmetic genus 0 over an algebraic space S;
- (Markings) sections $\underline{z} = (z_0, \ldots, z_n : S \to C)$ disjoint from each other and the nodes;
- (One-form) a section $\lambda : C \to \mathbb{P}(\omega_{C/S} \oplus \mathbb{C})$ of the projective dualizing sheaf $\omega_{C/S}$;
- (Map) a map $u : C \to X/G$ to the quotient stack X/G;

satisfying the following conditions:

- (Scalings at markings) $\lambda(z_0) = \infty$ and $\lambda(z_i)$ is finite for $i = 1, \ldots, n$;
- (Monotonicity) on any component $C_v \subset C$ on which $\lambda|C_v$ is non-constant, $\lambda|C_v$ has a single double pole, at the node $w \in C_v$ closest to z_0;
- (Map stability for infinity scaling) u takes values in the semistable locus $X/\!\!/G$ on $\lambda^{-1}(\infty)$;

- (Bundle triviality for zero scaling) The bundle $u^*(X \to X/G)$ is trivializable on $\lambda^{-1}(0)$, or equivalently, u lifts to a map to X on $\lambda^{-1}(0)$.

The monotonicity assumption gives an affine structure near the double pole, thus the term *affine*. An affine gauged map given by a datum $(u : C \to X/G, z_0, \ldots, z_n, \lambda)$ is *stable* if any component C_v of C on which u is trivializable has at least three special points, if $\lambda|C_v$ is zero or infinite, or two special points, if $\lambda|C_v$ is finite and non-zero. In the case that $X/\!\!/G$ is only locally free, that is, has some finite but non-trivial stabilizers we also allow orbifold twistings at the nodes of C where $\lambda = \infty$, as in orbifold quantum cohomology. The *homology class* of an affine gauged map is the class $u_*[C] \in H_2^G(X, \mathbb{Q})$.

We introduce the following notation for moduli stacks. Let $\overline{\mathcal{M}}_n^G(\mathbb{A}, X)$ be the moduli stack of stable affine gauged maps to X and $\overline{\mathcal{M}}_n^G(\mathbb{A}, X, d)$ the locus of homology class d. Each $\overline{\mathcal{M}}_n^G(\mathbb{A}, X, d)$ is a proper Deligne-Mumford stack equipped with a perfect obstruction theory. The relative perfect obstruction theory on $\overline{\mathcal{M}}_n^G(\mathbb{A}, X)$ has complex dual to $Rp_* e^* T_{X/G}$, where p, e are maps from the universal curve $\overline{\mathcal{C}}_n^G(\mathbb{A}, X)$ as in the diagram

$$\begin{array}{ccc} \overline{\mathcal{C}}_n^G(\mathbb{A}, X) & \xrightarrow{e} & X/G \\ \Big\downarrow p & & \\ \overline{\mathcal{M}}_n^G(\mathbb{A}, X). & & \end{array}$$

As in the construction of Y.P. Lee [41], the perfect obstruction theory determines a virtual structure sheaf $\mathcal{O}_{\mathcal{M}}^{\mathrm{vir}}$ in the bounded derived category of coherent sheaves on $\overline{\mathcal{M}}_n^G(\mathbb{A}, X)$. It defines a class $[\mathcal{O}_{\mathcal{M}}^{\mathrm{vir}}]$ in the rational K-theory of $\overline{\mathcal{M}}_n^G(\mathbb{A}, X)$. Let $\overline{I}_{X/\!\!/G}$ denote the rigidified inertia stack of $X/\!\!/G$ and

$$\mathrm{ev} = (\mathrm{ev}_0, \mathrm{ev}_1, \ldots, \mathrm{ev}_n) : \overline{\mathcal{M}}_n^G(\mathbb{A}, X) \to \overline{I}_{X/\!\!/G} \times (X/G)^n$$

denote the evaluation maps at $z_0, \ldots, z_n$. If X is smooth projective then the moduli stack $\overline{\mathcal{M}}_n^G(\mathbb{A}, X)$ is proper. Properness also holds in certain other situations, such as if X is a vector space, G is a torus, and the weights of G are contained in an open half-space in $\mathfrak{g}^\vee$. For details on the proof of properness we refer the reader to [26].

The moduli stack of affine gauged maps also admits a forgetful map to a stack of domain curves. Denote by $\overline{\mathcal{M}}_n(\mathbb{A})$ the moduli stack $\overline{\mathcal{M}}_n^G(\mathbb{A}, X)$

in the case that X and G are points, which we call the stack of *affine scaled curves*. There is a forgetful morphism

$$f : \overline{\mathcal{M}}_n^G(\mathbb{A}, X) \to \overline{\mathcal{M}}_n(\mathbb{A})$$

defined by forgetting the morphism to X/G and collapsing all components that become unstable.

Example 1.2. The moduli stack $\overline{\mathcal{M}}_2(\mathbb{A})$ of twice-marked affine scaled curves is isomorphic to the projective line via the map

$$\overline{\mathcal{M}}_2(\mathbb{A}) \to \mathbb{P}^1, \quad (C, z_0, z_1, z_2, \lambda) \mapsto \int_{z_1}^{z_2} \lambda;$$

more precisely, the map above identifies the locus of affine scaled curves with irreducible domain with $\mathbb{C}^\times$. The compactification adds the two distinguished divisors

$$D_{\{1,2\}}, D_{\{1\},\{2\}} \subset \overline{\mathcal{M}}_2(\mathbb{A})$$

corresponding to loci where the markings $z_1, z_2 \in C$ are on the same component $C_v \subseteq C, v \in \mathrm{Vert}(\Gamma)$ with zero scaling $\lambda | C_v = 0$ resp. different components $C_{v_1}, C_{v_2} \subset C, v_1 \neq v_2 \in \mathrm{Vert}(\Gamma)$.

1.3.2 Affine gauged invariants and quantum Kirwan map

K-theoretic affine gauged Gromov-Witten invariants are defined as virtual Euler characteristics over the moduli stack of affine gauged maps. We introduce an equivariant version of the Novikov ring. Denote the equivariant polarization $\mathcal{L} \to X$. We denote the equivariant Novikov ring

$$\Lambda_X^G = \left\{ f(q) = \sum_{d \in H_2^G(X,\mathbb{Q})} c_d q^d \;\middle|\; \forall E, \# \mathrm{Supp}^E(f) < \infty \right\} \tag{1.10}$$

where

$$\mathrm{Supp}^E(f) = \left\{ d \in H_2^G(X) \;\middle|\; c_d \neq 0, \langle d, c_1^G(\mathcal{L}) \rangle < E \right\}.$$

The ring Λ_X^G depends on the (class of the) polarisation $\mathcal{L}$, but we omit $\mathcal{L}$ from the notation when it is clear which polarisation we are using.

Redefine

$$QK^0_G(X) = \lim_{n\leftarrow} K^0_G(X) \otimes \Lambda^G_X / I^G_X(c)^n,$$
$$QK^0(X/\!\!/G) = \lim_{n\leftarrow} K^0(X/\!\!/G) \otimes \Lambda^G_X / I^G_X(c)^n;$$

in other words, from now on we work over the Novikov ring Λ^G_X. Let ev^d_0 denote the restriction of

$$\mathrm{ev}_0 : \overline{\mathcal{M}}^G_n(\mathbb{A}, X) \to \overline{I}_{X/\!\!/G}$$

to $\overline{\mathcal{M}}^G_n(\mathbb{A}, X, d)$. Recall the formal map $\mathcal{MC}_{X/\!\!/G}$ from $QK(X/\!\!/G)$ to $QK(X/\!\!/G)$ from (1.8). The map $\mathcal{MC}_{X/\!\!/G}$ is formally invertible near 0. Its linearization at 0 is the identity modulo higher order terms, involving positive powers of q, hence it has a formal inverse $\mathcal{MC}^{-1}_{X/\!\!/G}$ with $\mathcal{MC}^{-1}_{X/\!\!/G}(0) = 0$.

Definition 1.3. The *quantum Kirwan map* in quantum K-theory with insertions $\beta_n \in K(\overline{\mathcal{M}}_{0,n})$ is the formal map

$$\begin{aligned} &\kappa^G_X : QK^0_G(X) \to QK^0(X/\!\!/G), \\ &\alpha \mapsto \mathcal{MC}^{-1}_{X/\!\!/G} \sum_{d \in H_2(X/\!\!/G,\mathbb{Q}), n \geq 0} \frac{q^d}{n!} \mathrm{ev}^d_{0,*}(\mathrm{ev}^*_1 \alpha \otimes \ldots \otimes \ldots \mathrm{ev}^*_n \alpha \otimes f^* \beta_n). \end{aligned} \tag{1.11}$$

The *linearized quantum Kirwan map* is obtained from the linearization of κ^G_X and correction terms arising from the quantum corrections in the inner product:

$$\begin{aligned} &D_\alpha \kappa^G_X : QK^0_G(X) \to QK^0(X/\!\!/G), \\ &\sigma \mapsto (D_{\kappa^G_X(\alpha)} \mathcal{MC}_{X/\!\!/G})^{-1} \sum_{d,n} \frac{q^d}{(n-1)!} \mathrm{ev}^d_{0,*}(\mathrm{ev}^*_1 \sigma \otimes \mathrm{ev}^*_2 \alpha \otimes \ldots \mathrm{ev}^*_n \alpha). \end{aligned} \tag{1.12}$$

Theorem 1.4. *Each linearization of κ^G_X is a $\star$-homomorphism:*

$$D_\alpha \kappa^G_X(\sigma \star_\alpha \gamma) = D_\alpha \kappa^G_X(\sigma) \star_{\kappa^G_X(\alpha)} D_\alpha \kappa^G_X(\gamma)$$

for any $\alpha \in QK^0_G(X)$.

Proof The proof is a consequence of an equivalence of divisor classes. As in the proof of associativity of quantum K-theory by Givental [17], consider the forgetful map

$$f_2 : \overline{\mathcal{M}}^G_n(\mathbb{A}, X) \to \overline{\mathcal{M}}_2(\mathbb{A}) \cong \mathbb{P}^1$$

forgetting all but the first and second markings and scaling. The inverse image $f_2^{-1}(\infty)$ consists of configurations

$$(u : C \to X/G, \lambda, \underline{z}) \in \mathrm{Ob}(\overline{\mathcal{M}}_n^G(\mathbb{A}, X))$$

where the first two incoming markings $z_1, z_2 \in C$ are on different components of the domain C is a union of boundary divisors:

$$f_2^{-1}(\infty) = \bigcup_\Gamma \overline{\mathcal{M}}_\Gamma^G(\mathbb{A}, X)$$

where Γ ranges over combinatorial type of colored tree with r colored vertices $v_1, \ldots, v_r \in \mathrm{Vert}(\Gamma)$ and one non-colored vertex $v_0 \in \mathrm{Vert}(\Gamma)$, with the edge labelled 1 attached to the first vertex v_1 and the edge labelled 2 attached to the second colored vertex v_2. We digress briefly to recall that if

$$D = \cup_{i=1}^n D_i$$

is a divisor with normal crossing singularities on a variety Y then the class of the structure sheaf $\mathcal{O}_D$ is

$$[\mathcal{O}_D] = \sum_{I \subset \{1, \ldots n\}} (-1)^{|I|} [\mathcal{O}_{D_I}] \in K(Y), \quad D_I = \bigcap_{i \in I} D_i.$$

The corresponding property for virtual structure sheaves in $\overline{\mathcal{M}}_{g,n}(X)$ is proved by Y.P. Lee [41, Proposition 11]. The intersection of any two strata $\overline{\mathcal{M}}_{\Gamma_1}^G(\mathbb{A}, X)$, $\overline{\mathcal{M}}_{\Gamma_2}^G(\mathbb{A}, X)$ of codimension one is a stratum $\overline{\mathcal{M}}_{\Gamma_3}^G(\mathbb{A}, X)$ of lower codimension; there is an exact sequence of sheaves whose i-th term is the union of structure sheaves of strata of codimension i. Thus the structure sheaf of $f_2^{-1}(\infty)$ is identified in K-theory with the alternating sum of structure sheaves

$$[\mathcal{O}_{f_2^{-1}(\infty)}] = \sum_{k_1, k_2 \geq 0} (-1)^{k_1 + k_2} \sum_{\Gamma \in \mathcal{T}_\infty(k_1, k_2)} [\mathcal{O}_{\overline{\mathcal{M}}_\Gamma^G(\mathbb{A}, X, d)}]$$

where $\mathcal{T}_\infty(k_1, k_2)$ is the set of combinatorial types of affine scaled gauged maps with k_1, k_2 rational curves connecting the components containing z_1, z_2, with finite and non-zero scaling, and the component with infinite scaling containing z_0. On the other hand, the structure sheaf of $f_2^{-1}(0)$ is the alternating sum of structure sheaves

$$[\mathcal{O}_{f_2^{-1}(0)}] = \sum_{k \geq 0} (-1)^k \sum_{\Gamma \in \mathcal{T}_0(k)} [\mathcal{O}_{\overline{\mathcal{M}}_\Gamma^G(\mathbb{A}, X, d)}]$$

where $\mathcal{T}_0(k)$ is the set of combinatorial types with z_1, z_2 on one component, z_0 on another, and these two components related by a chain of

k rational curves. The contribution of $\mathcal{T}_\infty(k)$ to the push-forward over $f_2^{-1}(0)$ is

$$(D_\alpha \mathcal{MC}_{X/\!\!/G} - I)^k \sum_{d \in H_2^G(X,\mathbb{Q}), n \geq 1} \frac{q^d}{(n-1)!} \operatorname{ev}^d_{0,*}(\operatorname{ev}_1^* \sigma \otimes \operatorname{ev}_2^* \alpha \otimes \ldots \operatorname{ev}_n^* \alpha).$$

The inverse of the linearization of the map $\mathcal{MC}_{X/\!\!/G}$,

$$D_\alpha \mathcal{MC}_{X/\!\!/G}^{-1} = (I + (D_\alpha \mathcal{MC}_{X/\!\!/G} - I))^{-1} = \sum_{k \geq 0} (-1)^k (D_\alpha \mathcal{MC}_{X/\!\!/G} - I)^k. \tag{1.13}$$

Putting everything together and using (1.12) gives

$$(D_{\kappa_X^G(\alpha)} \mathcal{MC}_{X/\!\!/G})(D_\alpha \kappa_X^G)(\sigma \star_\alpha \gamma) = (D_{\kappa_X^G(\alpha)} \mathcal{MC}_{X/\!\!/G})(((D_\alpha \kappa_X^G)\sigma) \star_{\kappa_X^G(\alpha)} ((D_\alpha \kappa_X^G)\gamma)).$$

Since $D_{\kappa_X^G(\alpha)} \mathcal{MC}_{X/\!\!/G}$ is invertible, this implies the result. □

Remark. (Inductive definition) For classes

$$\aleph_0 \in QK^0(X/\!\!/G), \quad \aleph_1, \ldots, \aleph_n \in QK^0_G(X)$$

denote by

$$m_{n,d}(\aleph_0, \aleph_1, \ldots, \aleph_n) := \chi(\overline{\mathcal{M}}_n^G(\mathbb{A}, X, d), \operatorname{ev}_0^* \aleph_0 \otimes \operatorname{ev}_1^* \aleph_1 \otimes \ldots \operatorname{ev}_n^* \aleph_n \otimes [\mathcal{O}^{\mathrm{vir}}]) \in \mathbb{Z} \tag{1.14}$$

the virtual Euler characteristic. For classes

$$\aleph_0 \in QK^0(X/\!\!/G), \quad \alpha, \aleph_1, \ldots, \aleph_k \in QK^0_G(X)$$

define

$$m^\alpha_{k,d}(\aleph_0, \ldots, \aleph_k) = \sum_{n \geq 0} \frac{1}{n!} m_{k+n,d}(\aleph_0, \ldots, \aleph_k, \alpha, \ldots, \alpha)$$

and similarly define $\varphi_{n,d}^{\kappa_X^G(\alpha)}$ by summing over all possible numbers of insertions of $\kappa_X^G(\alpha)$. Expanding the definition of the inner product we have (in topological K-theory)

$$\underline{B}(D_\alpha \kappa_X^G(\sigma), \gamma) = \sum_{d_0, \ldots, d_r, \aleph_1, \ldots, \aleph_r} (-1)^r q^d m^\alpha_{2,d_0}(\sigma, \aleph_1^\vee) \left(\prod_{i=1}^{r-1} \varphi_{2,d_i}^{\kappa_X^G(\alpha)}(\aleph_i, \aleph_{i+1}^\vee) \right) \varphi_{2,d_r}^{\kappa_X^G(\alpha)}(\aleph_r, \gamma) \tag{1.15}$$

where the sum is over all sequences of non-negative classes $(d_0,\ldots,d_r)$ such that $\sum d_i = d$ and $d_i > 0$ for $i > 0$ and $\aleph_1,\ldots,\aleph_r$ range over a basis for $QK(X/\!\!/G)$. Equivalently, $D_\alpha \kappa_X^G(\sigma)$ can be defined by the inductive formula

$$\underline{B}(D_\alpha \kappa_X^G(\sigma),\gamma) = \sum_{d>0} q^d m_{2,d}^{\alpha}(\sigma,\gamma) - \sum_{\aleph,e>0} q^e \underline{B}(D_\alpha\kappa_X^G(\sigma),\aleph)\varphi_{2,e}^{\kappa_X^G(\alpha)}(\aleph^\vee,\gamma). \quad (1.16)$$

Definition 1.5. The *canonical bulk deformation* of $QK(X/\!\!/G)$ is the value $\kappa_X^G(0) \in QK(X/\!\!/G)$ of κ_X^G at 0.

Note that κ_X^G depends on the choice of presentation of $X/\!\!/G$ as a git quotient. We give the following criterion for the canonical bulk deformation to vanish.

Lemma 1.6. *Suppose that the coarse moduli spaces of $\overline{\mathcal{M}}_0^G(\mathbb{A},X,d)$ and $\overline{\mathcal{M}}_{0,n}(X/\!\!/G,d)$ are smooth, rational and connected with virtual fundamental sheaf equal to the usual structure sheaf. Suppose further that $H_2^G(X) \cong H_2(X/\!\!/G)$ with $\overline{\mathcal{M}}_0^G(\mathbb{A},X,d)$ non-empty iff $\overline{\mathcal{M}}_{0,n}(X/\!\!/G,d)$ is non-empty. Then $\kappa_X^G(0) = 0$.*

Proof Under the conditions in the Lemma, a result of Buch-Mihalcea [7, Theorem 3.1] implies that the push-forward of the structure sheaf over any moduli space of maps in the Lemma under any evaluation map is the structure sheaf of the target:

$$\mathrm{ev}_{0,*}^d[\mathcal{O}_{\overline{\mathcal{M}}_n^G(\mathbb{A},X,d)}] = [\mathcal{O}_{X/\!\!/G}], \quad \mathrm{ev}_{1,*}^d[\mathcal{O}_{\overline{\mathcal{M}}_{0,n}(X/\!\!/G,d)}] = [\mathcal{O}_{X/\!\!/G}], \qquad \forall d \in H_2^G(X) \cong H_2(X/\!\!/G), n \geq 0. \quad (1.17)$$

Thus

$$\sum_d q^d\, \mathrm{ev}_{0,*}^d[\mathcal{O}_{\overline{\mathcal{M}}_n^G(\mathbb{A},X,d)}] = \left(\sum_d q^d\right)[\mathcal{O}_{X/\!\!/G}],$$
$$\sum_{d,n} q^d\, \mathrm{ev}_{1,*}^d[\mathcal{O}_{\overline{\mathcal{M}}_{0,1}(X,d)}] = \left(\sum_d q^d\right)[\mathcal{O}_{X/\!\!/G}]$$

where both sums are over d such that $\overline{\mathcal{M}}_n^G(\mathbb{A},X,d), \overline{\mathcal{M}}_{0,n}(X/\!\!/G,d)$ are non-empty. It follows that

$$\mathcal{MC}_{X/\!\!/G}(0) = \left(\sum_d q^d\right)[\mathcal{O}_{X/\!\!/G}], \quad \kappa_X^G(0) = 0$$

as claimed. □

Under the conditions of the lemma, one obtains a map of *small* quantum K-rings given by the linearized quantum Kirwan map

$$D_0\kappa_X^G : QK_G^0(X) \to QK^0(X/\!\!/G).$$

1.4 Quantum K-theory of toric quotients

In this section we use the K-theoretic quantum Kirwan map to give a presentation of the quantum K-theory of any smooth proper toric Deligne-Mumford stack with projective coarse moduli space. The case of projective spaces was treated in Buch-Mihalcea [7] and Iritani-Milanov-Tonita [33], and general toric varieties were treated in Givental [18]. The toric stacks we consider are obtained as git quotients for actions of tori on vector spaces. Let G be a torus with Lie algebra $\mathfrak{g}$. Let X be a vector space with a representation of G such that the weights

$$\mu_i \in \mathfrak{g}_{\mathbb{R}}^\vee, i = 1, \dots, k$$

of the action are contained in the interior of a half-space in $\mathfrak{g}_{\mathbb{R}}^\vee$. For generic polarization, the git quotient $X/\!\!/G$ is smooth. We assume that $X/\!\!/G$ is non-empty, and for simplicity that the generic stabilizer is trivial. We have

$$QK_G^0(X) \cong QK_G^0(\mathrm{pt}) = R(G)$$

where $R(G)$ denotes the Grothendieck group of finite-dimensional representations of G. Denote by $X_k \subset X$ the representation given by the k-th weight space. For any class $d \in H_2^G(X) \cong \mathfrak{g}_{\mathbb{Z}}$, define elements of $QK_G^0(X)$ by

$$\zeta_+(d) = \prod_{\mu_j(d) \geq 0} (1 - X_j^{-1})^{\mu_j(d)}, \quad \zeta_-(d) = q^d \prod_{\mu_j(d) \leq 0} (1 - X_j^{-1})^{-\mu_j(d)}.$$

Define the d-th *K-theoretic Batyrev element*

$$\zeta(d) = \zeta_+(d) - \zeta_-(d) \in QK_G^0(\mathrm{pt}) \cong QK_G^0(X). \tag{1.18}$$

Proposition 1.1. *The kernel of $D_0\kappa_X^G$ contains the elements $\zeta(d), d \in H_2^G(X)$.*

Proof An argument using divided difference operators is given later in Example 1.6; here we give a geometric proof. The target X itself defines an element of $K_G^0(X)$ via pull-back under $X \to \mathrm{pt}$. The pull-back $[\mathrm{ev}_j^* X]$ is a class in $K(\overline{\mathcal{M}}_1^G(\mathbb{A}, X))$ for $j = 1, \ldots, n$. Define sections

$$\sigma_{i,j} : \overline{\mathcal{M}}_n^G(\mathbb{A}, X) \to \mathrm{ev}_j^* X, \quad i \geq 0 \tag{1.19}$$

by composing the map $\overline{\mathcal{M}}_1^G(\mathbb{A}, X) \to \mathrm{ev}_1^* X$ taking the i-th derivative of the map $u : C \to X/G$ at the marking z_j with the forgetful morphism $\overline{\mathcal{M}}_n^G(\mathbb{A}, X) \to \overline{\mathcal{M}}_1^G(\mathbb{A}, X)$. More precisely, suppose that $u : C \to X/G$ is given by a bundle $P \to C$ and a section $v : C \to P \times_G X$. In a local trivialization near z_j the section is given by a map $v : \mathbb{C} \to X$. Furthermore, the scaling λ on C necessarily pulls back to a non-zero scaling on $\mathbb{C}$, since there are no components of C with zero scaling. (There are no holomorphic curves in X, hence all curves with zero scaling are constant, and there is only one marking, hence no constant components with three special points and zero scaling.) Choose a coordinate z so that $\lambda = \mathrm{d}z$. Let $\sigma_{i,j}([u]) \in \mathrm{ev}_1^* X_j$ denote the i-th derivative of v at z_j with respect to the coordinate z.

We apply these canonical sections to the following Euler class computation. Each factor X_j defines a corresponding class $[X_j]$ in $K_G^0(X)$; we often omit the square brackets to simplify notation. Define bundles $E_\pm \to \overline{\mathcal{M}}_1^G(\mathbb{A}, X)$

$$E_\pm := \bigoplus_{\pm \mu_j(d) \geq 0} \mathrm{ev}_1^* X_j^{\oplus \mu_j(d)}.$$

The Euler class of $E_\pm$ is

$$\mathrm{Eul}(E)_\pm = \bigotimes_{\pm \mu_j(d) \geq 0} (1 - \mathrm{ev}_1^* X_j)^{\otimes \mu_j(d)} \in K(\overline{\mathcal{M}}_1^G(\mathbb{A}, X)).$$

Given a section σ_j of $\mathrm{ev}_1^* X_j$ transverse to the zero section, there is a canonical isomorphism of the structure sheaf $\mathcal{O}_{\sigma_j^{-1}(0)}$ of $\sigma_j^{-1}(0)$ with

$$[\mathrm{Eul}(\mathrm{ev}_1^* X_j^\vee)] = [1 - \mathrm{ev}_1^* X_j^\vee] \in K(\overline{\mathcal{M}}_1^G(\mathbb{A}, X, d')).$$

This isomorphism is defined by the exact sequence

$$0 \to \mathrm{ev}_1^* X_j^\vee \to \mathcal{O} \to \mathcal{O}_{\sigma_j^{-1}(0)} \to 0.$$

Extending this by direct sums, any section $\sigma : \overline{\mathcal{M}}_1^G(\mathbb{A}, X, d') \to E_\pm$ transverse to the zero section defines an equality

$$[\mathcal{O}_{\sigma^{-1}(0)}] = [\mathrm{Eul}(E)_\pm^\vee] \in K(\overline{\mathcal{M}}_1^G(\mathbb{A}, X), d').$$

In particular, let σ denote the section of E_+ given by the derivatives

$$\sigma_{i,j}, i = 1, \ldots, d_j := \min(\mu_j(d), \mu_j(d'))$$

defined in (1.19). We construct a diagram

$$\begin{array}{ccc} \sigma^{-1}(0) & \xrightarrow{\iota} & \overline{\mathcal{M}}_1^G(\mathbb{A}, X, d') \\ \downarrow \delta & & \\ .\overline{\mathcal{M}}_1^G(\mathbb{A}, X, d' - d) & & \end{array}$$

as follows. The map ι is the inclusion. To construct δ, note that $\sigma^{-1}(0) \subset \overline{\mathcal{M}}_1^G(\mathbb{A}, X)$ consists of maps u whose j-th component u_j vanishes to order d_j at the marking z_1. Therefore, for any $[u] \in \sigma^{-1}(0)$ define a map of degree $d' - d$ by dividing by the j-th component of $u : C \to X/G$ on the component of C containing z_1 by $(z - z_1)^{d_j}$ on the component containing z_1, to obtain a map denoted $(z - z_1)^{-d}u$. The other components of C all map to $X/\!\!/G$, and the action of $(z - z_1)^{-d}$ on the other components does not change the isomorphism class of u. It follows that there is a canonical map

$$\delta : \sigma^{-1}(0) \to \overline{\mathcal{M}}_1^G(\mathbb{A}, X, d' - d), \quad [u] \mapsto [u/(z - z_1)^d]. \tag{1.20}$$

The normal bundle to δ has Euler class the product of factors $(1 - X_j^{-1})^{\min(\mu_j(-d), \mu_j(d'-d))}$ over j such that $\mu_j(-d) \geq 0$.

The remaining factors are explained by the difference in obstruction theories. We denote by $p^{d'}$ the restriction of the projection p to maps of homology class d'. To compute the difference in classes we note that δ lifts to an inclusion of universal curves and (if $e^{d'}, e^{d'-d}$ denote the universal evaluation maps)

$$\begin{aligned} \iota^*[Rp_*^{d'} e^* T_{X/G}] - \delta^*[Rp_*^{d'-d} e^* T_{X/G}] &= \iota^*[Rp_*^{d'} \bigoplus_j (\mathcal{O}_{z_1}(\mu_j(d'))] \\ &\quad - \delta^*[Rp^{d'-d} \bigoplus_j \mathcal{O}_{z_1}(\mu_j(d' - d))] \\ &= \iota^*[E_+] - \rho^*[E_-]. \end{aligned}$$

Hence for any class $\alpha_0 \in K(X/\!\!/G)$ we obtain

$$\begin{aligned} &\chi^{\mathrm{vir}}(\overline{\mathcal{M}}_1^G(\mathbb{A}, X, d'), \mathrm{ev}_0^* \alpha_0 \otimes \mathrm{ev}^* \zeta_+(d)) \\ &\qquad = \chi^{\mathrm{vir}}(\overline{\mathcal{M}}_1^G(\mathbb{A}, X, d' - d), \mathrm{ev}_0^* \alpha_0 \otimes \mathrm{ev}^* \zeta_-(d)). \end{aligned} \tag{1.21}$$

That is,

$$m_{1,d'}(\alpha_0, \zeta_+(d)) = m_{1,d'-d}(\alpha_0, \zeta_-(d)).$$

By definition of the quantum Kirwan map this implies

$$D_0\kappa_X^G(\zeta_+(d)) = q^d D_0\kappa_X^G(\zeta_-(d)).$$ □

We wish to show that the elements in the lemma above generate, in a suitable sense, the kernel of the K-theoretic quantum Kirwan map. Define the *quantum K-theoretic Stanley-Reisner ideal* $QKSR_X^G$ to be the ideal in $QK_G^0(X)$ spanned by the K-theoretic Batyrev elements $\zeta(d), d \in H_2(X,\mathbb{Z})$ of (1.18). In general there are additional elements in the kernel of the linearized quantum Kirwan map. In order to remove these one must pass to a formal completion.

Definition 1.2. a. In the case that G acts freely on the semistable locus in X, for $l = (l_1, \dots, l_k) \in \mathbb{Z}^k_{\geq 0}$ define a filtration

$$QK_G^0(X)^{\geq l} := \prod_{j=1}^k (1 - X_j^{-1})^{l_j} QK_G^0(X) \subset QK_G^0(X).$$

b. More generally suppose that G acts on the semistable locus in X with finite stabilizers $G_x, x \in X^{\text{ss}}$ and let

$$\mathcal{F}(X) = \bigcup_{j=1}^k \mathcal{F}_j(X) \subset \mathbb{C}$$

denote the set of roots of unity

$$\mathcal{F}_j(X) := \{ g^{\mu_j} = \exp(2\pi i \mu_j(\xi)) \in \mathbb{C}^\times \mid g = \exp(\xi) \in G_x, x \in X^{\text{ss}} \}$$

representing the roots of unity for the action of $g \in G_x, x \in X^{\text{ss}}$ on X_j. Define

$$QK_G^0(X)^{\geq l} := \prod_{j=1}^k \prod_{\zeta \in \mathcal{F}_j(X)} (1 - \zeta X_j^{-1})^{l_j}.$$

Let $\widehat{QK}_G(X)$ denote the completion with respect to this filtration

$$\widehat{QK}_G(X) = \lim_{\leftarrow l} QK_G^0(X)/QK_G^0(X)^{\geq l}.$$

Lemma 1.3. *For any d, n the K-theoretic Gromov-Witten invariants in the definition of κ_X^G vanish for $\alpha_0, \dots, \alpha_n \in QK_G^0(X)^{\geq l}$ for $l_1 + \ldots + l_k$ sufficiently large.*

Proof We apply Tonita's virtual Riemann-Roch theorem [53]: If $\mathcal{M} = \overline{\mathcal{M}}_n^G(\mathbb{A}, X, d)$ embeds in a smooth global quotient stack and $I_{\mathcal{M}}$ its inertia stack then for any vector bundle V

$$\chi^{\mathrm{vir}}(\mathcal{M}, V) = \int_{[I_{\mathcal{M}}]^{\mathrm{vir}}} m^{-1} \operatorname{Ch}(V \otimes \mathcal{O}_{I_{\mathcal{M}}}^{\mathrm{vir}}) \operatorname{Td}(TI_{\mathcal{M}}) \operatorname{Ch}(\operatorname{Eul}(\nu_{\mathcal{M}}^\vee))$$

where $\nu_{\mathcal{M}}$ is the normal bundle of $I_{\mathcal{M}} \to \mathcal{M}$ and $m : I_{\mathcal{M}} \to \mathbb{Z}$ is the order of stabilizer function. The pullback $\mathrm{ev}_j^*(1 - \zeta X_j^{-1})$ restricts on a component of the inertia stack corresponding to an element $h \in G$ to the K-theory class $1 - \chi_j(h)^{-1} X_j^{-1}$, where χ_j is the character of X_j. The Chern character of each $1 - \chi_j(h)^{-1} \zeta X_j^{-1}$ has degree at least two for $\zeta = \chi_j(h)$. In this case if the sum $l_j, j \in I(h)$ of weights for which $\chi_j(h)$ is trivial is larger than the virtual dimension for the component of $I\mathcal{M}$ corresponding to h, the virtual integral over $I\mathcal{M}$ (which embeds in a smooth global quotient stack, see [57, Part 3, Proposition 9.14] and [24, Proposition 3.1 part (d)]) vanishes. □

Lemma 1.3 implies that the quantum Kirwan map admits a natural extension to the formal completion:

$$\widehat{\kappa_X^G} : \widehat{QK}_G(X) \to QK(X/\!\!/G).$$

Theorem 1.4. *(Theorem 1.2 from the Introduction.) The completed quantum Stanley-Reisner ideal is the kernel of the linearized quantum Kirwan map $D_0\kappa_X^G$: We have an exact sequence*

$$0 \to \widehat{QKSR_X^G} \to \widehat{QK_G^0(X)} \to QK(X/\!\!/G) \to 0.$$

Proof In [23, Theorem 2.6] we proved a version of Kirwan surjectivity for the cohomological quantum Kirwan map. The arguments given there hold equally well in rational topological K-theory as in cohomology. We address first the surjectivity of the right arrow in the sequence. By [23, Proposition 2.9] for d such that $\mu_j(d) > 0$ for all j we have

$$D_0\kappa_X^G \left(\prod_i^k (1 - X_j^{-1})^{s\lceil \mu_i(d) \rceil} \right) = q^d [\mathcal{O}_{I_{X/\!\!/G}(\exp(d))}] + h.o.t. \qquad (1.22)$$

where $h.o.t.$ denotes terms higher order in q. Since divisor intersections $[D_I] = \cap_{i \in I} [D_i]$ generate the cohomology $H(I_{X/\!\!/G})$ of any $I_{X/\!\!/G}$, the classes of their structure sheaves $[\mathcal{O}_{D_i}]$ generate the rational K-theory $K(I_{X/\!\!/G})$. It follows that $D_0\widehat{\kappa_X^G}$ is surjective.

To show exactness of the sequence, it suffices to show the equality of dimensions

$$\dim(\widehat{QK}_G(X)/QKSR_X^G) = \dim(QK(X/\!\!/G)).$$

We recall the argument in the case that the generic stabilizer is trivial. Let $T = (\mathbb{C}^\times)^k/G$ denote the residual torus acting on $X/\!\!/G$. The moment polytope of $X/\!\!/G$ may be written

$$\Delta_{X/\!\!/G} = \{\mu \in \mathfrak{t}^\vee \mid (\mu, \nu_j) \geq c_j, j = 1, \ldots, k\}$$

where ν_j are normal vectors to the facets of $\Delta_{X/\!\!/G}$, determined by the image of the standard basis vectors in $\mathbb{R}^k$ in $\mathfrak{t}$ under the quotient map, and c_j are constants determined by the equivariant polarization on X.

The quantum cohomology may be identified with the Jacobian ring of a *Givental potential* defined on the dual torus. Let $\Lambda = \exp^{-1}(1) \subset \mathfrak{t}$ denote the coweight lattice, and $\Lambda^\vee \subset \mathfrak{t}^\vee$ the weight lattice. The dual torus and *Givental potential* are

$$T^\vee = \mathfrak{t}^\vee/\Lambda^\vee, \quad W : T^\vee \to \mathbb{C}[q, q^{-1}], \quad y \mapsto \sum_{j=1}^{k} q^{c_j} y^{\nu_j}.$$

The quotient of $QK_G(X)$ by the Batyrev ideal maps to the ring $\mathrm{Jac}(W)$ of functions on $\mathrm{Crit}(W)$ by $(1 - X_j^{-1}) \mapsto y^{\nu_j}$. Denote by $\widehat{\mathrm{Crit}(W)}$ the intersection of $\mathrm{Crit}(W)$ with a product U of formal disks around $q = 0, X_j \in \mathcal{F}_j(X)$:

$$\widehat{\mathrm{Crit}(W)} = \mathrm{Crit}(W) \cap U.$$

Under the identification of the Jacobian ring, $\widehat{\mathrm{Crit}(W)}$ is the scheme of critical points $y(q)$ such that each y^{ν_j} approaches an element of $\mathcal{F}_j(X)$ as $q \to 0$. This definition differs from that in [23] in that we allow in theory critical points that converge to non-trivial roots of unity $\mathcal{F}_j(X)$ in the limit $q \to 0$.

To see that this gives the same definition as in [23] we must show that there are no families of critical points that converge to non-zero values of y^{ν_j} as $q \to 0$. Let $y(q)$ be such a family. Necessarily the q-valuation $\mathrm{val}_q(y(q))$ of $y(q)$ lies in some face of the moment polytope. Since the moment polytope is simplicial, the normal vectors ν_j of facets containing $\mathrm{val}_q(y(q))$ cannot be linearly dependent. Taking such ν_j as part of a basis for the Lie algebra of the torus one may write $W(y) = y_1 + \ldots + y_k + h.o.t$ and taking partials with respect to $y_1, \ldots, y_k$ shows that these variables must vanish. Thus $y(q)$ converges to zero as $q \to 0$.

The dimension of $\widehat{\mathrm{Crit}(W)}$ can be computed using the toric minimal model program [23, Lemma 4.15]: under the toric minimal model program each flip changes the dimension of $\widehat{\mathrm{Crit}(W)}$ in the same as way as the dimension of $QK(X/\!\!/G)$. .

Similarly for a Mori fibration one has a product formula representing $\dim(QK(X/\!\!/G))$ as the product of dimension of the base and fiber, and similarly for the Jacobian ring. It follows that $\dim(\widehat{\mathrm{Crit}(W)})$ is equal to $\dim(\widehat{QK}_G(X)/QKSR_X^G) = \dim(QK(X/\!\!/G))$. □

Remark. a. The presentation above specializes to Vezzosi-Vistoli presentation [55, Theorem 6.4] by setting $q = 1$ in the case of smooth projective toric varieties. See Borisov-Horja [6] for the case of smooth Deligne-Mumford stacks.

b. The presentation above restricts Buch-Mihalcea presentation [7] in the case of projective spaces. In the case of projective (or more generally weighted projective spaces realized as quotients of a vector space by a $\mathbb{C}^\times$ action) we have $m_{0,d}(\alpha_0) = 1$ for any $d > 0$. This implies $\kappa_X^G(0) = 0$, by the arguments in Buch-Mihalcea [7]: The moduli stack is non-singular, has rational singularities, the evaluation map

$$\mathrm{ev}_0(d) : \overline{\mathcal{M}}_0^G(\mathbb{A}, X, d) \to X^{\exp(d)}/\!\!/G$$

is surjective and has irreducible and rational fibers. By [7, Theorem 3.1], the push-forward of the structure sheaf $\mathcal{O}_{\overline{\mathcal{M}}_0^G(\mathbb{A},X,d)}$ is a multiple of the structure sheaf on $X^{\exp(d)}/\!\!/G$. It follows from the inductive formula (1.16) for $\kappa_X^G(0)$ that $\kappa_X^G(0)$ is the structure sheaf on $I_{X/\!\!/G}$.

c. The linearized quantum Kirwan map has no quantum corrections in the case of circle group actions on vector spaces with positive weights. To see this, note that for $d > 0$ the push-forward of $\mathrm{ev}_1^*(1 - X_j)$ is the pushforward of the structure sheaf $\mathcal{O}_{\sigma_j^{-1}(0)}$ to $X/\!\!/G$. By the argument in the previous item, this push-forward is equal to $[\mathcal{O}_{X/\!\!/G}]$. For similar reasons, for $d > 0$ we have

$$m_{2,d}(\mathrm{ev}_1^*(1 - X_j), [\mathcal{O}_{\mathrm{pt}}]) = 1.$$

The formula (1.16) then implies that $D_0\kappa_X^G(1 - X_j)$ has no quantum corrections, hence neither does $D_0\kappa_X^G(X_j)$. It follows that in the presentation (1.4) the class X_j may be taken to be the line bundle associated to the weight μ_j on the weighted projective space $X/\!\!/G = \mathbb{P}(\mu_1, \ldots, \mu_k)$. It seems to us at the moment that even in the case of Fano toric stacks one might have $\kappa_X^G(0) \neq 0$ and so the bulk deformation above may be non-trivial.

1.5 K-theoretic gauged Gromov-Witten invariants

In this section we define gauged K-theoretic Gromov-Witten invariants by K-theoretic integration over moduli stacks of Mundet-semistable maps to the quotient stack, and prove an adiabatic limit Theorem 1.4 relating the invariants.

1.5.1 The K-theoretic gauged potential

In our terminology, a gauged Gromov-Witten invariant is an integral over gauged maps, by which we mean maps to the quotient stack. Let C be a smooth projective curve.

Definition 1.1. (Gauged maps) A gauged map from C to X/G consists of

- (Curve) a nodal projective curve $\hat{C} \to S$ over an algebraic space S;
- (Markings) sections $z_0, \ldots, z_n : S \to \hat{C}$ disjoint from each other and the nodes;
- (One-form) a stable map $\hat{C} \to C$ of homology class $[C]$;
- (Map) a map $\hat{C} \to X/G$ to the quotient stack X/G, corresponding to a bundle $P \to \hat{C}$ and section $u : \hat{C} \to P(X) := (P \times X)/G$ which we required to be pulled back from a map $C \to BG$.

A gauged map is *stable* it satisfies a slope condition introduced by Mundet [46] which combines the slope conditions in Hilbert-Mumford and Ramanathan for G-actions and principal G-bundles respectively. Given a gauged map

$$(u : \hat{C} \to C \times X/G, z_0, \ldots, z_n)$$

let

$$\sigma : C \to P/R$$

be a parabolic reduction of P to a parabolic subgroup $R \subset P$. Let

$$\lambda \in \mathfrak{l}(P)^\vee$$

be a central weight of the Levi subgroup $L(P)$ of R. By twisting the bundle and section by the one-parameter subgroup $z^\lambda, z \in \mathbb{C}$ we obtain a family of gauged maps

$$u_\lambda : \hat{C} \times \mathbb{C}^\times \to C \times X/G.$$

By Gromov compactness the limit $z \to 0$ gives rise to an *associated graded* gauged map

$$u_\infty : \hat{C}_\infty \to X/G$$

equipped with a canonical infinitesimal automorphism

$$\lambda_\infty : \hat{C}_\infty \to \operatorname{aut}(P_\infty)$$

where P_∞ is the G-bundle corresponding to u_∞. The automorphism naturally acts on the determinant line bundle $\det \operatorname{aut}(P_\infty)$ as well as on the line bundle induced by the linearization $u_\infty^*(P_\infty(\mathcal{L}) \to P(X))$. The action on the first line bundle is given by a *Ramanathan weight* while the second is the *Hilbert-Mumford weight*

$$\lambda_\infty \cdot \delta_\infty = i\mu_\lambda^R(u)\delta_\infty \quad \lambda_\infty \cdot \tilde{u}_\infty = i\mu_\lambda^{HM}(u)\tilde{u}_\infty$$

for points δ_∞ resp. $\tilde{u}_\infty$ in the fiber of $\det \operatorname{aut}(P_\infty)$ resp. $u_\infty^*(P_\infty(\mathcal{L}) \to P(X))$. Let $\rho > 0$ be a real number. The *Mundet weight* is combination of the Ramanathan and Hilbert-Mumford weights with *vortex parameter* $\rho \in \mathbb{R}_{>0}$

$$\mu^M(\sigma, \lambda) = \rho\mu^R(\sigma, \lambda) + \mu^{HM}(\sigma, \lambda). \tag{1.23}$$

Definition 1.2. A gauged map $u : \hat{C} \to C \times X/G$ is *Mundet semistable* if

a. $\mu^M(\sigma, \lambda) \leq 0$ for all pairs (σ, λ) and
b. each component C_j of $\hat{C}$ on which u is trivializable (as a bundle with section) has at least three special points $z_i \in C_j$.

The map u is *stable* if, in addition, there are only finitely many automorphisms in $\operatorname{Aut}(u)$.

Gauged K-theoretic Gromov-Witten invariants are defined as virtual Euler characteristics over moduli stacks of Mundet-semistable gauged maps. Denote by $\overline{\mathcal{M}}^G(C, X)$ the moduli stack of Mundet semistable gauged maps. Assume that the semistable locus is equal to the stable locus, in which case $\overline{\mathcal{M}}^G(C, X)$ is a Deligne-Mumford stack with a perfect obstruction theory, proper for fixed numerical invariants [57]. Restriction to the sections defines an evaluation map

$$\operatorname{ev} : \overline{\mathcal{M}}^G(C, X) \to (X/G)^n.$$

Forgetting the map and stabilizing defines a morphism

$$f : \overline{\mathcal{M}}^G(C, X) \to \overline{\mathcal{M}}_n(C), \quad (C, u) \mapsto C^{\mathrm{st}}$$

where $\overline{\mathcal{M}}_n(C)$ is the moduli stack of stable maps to C of class $[C]$. For classes $\alpha_1, \ldots, \alpha_n \in K^0_G(X)$, $\beta_n \in \overline{\mathcal{M}}_n(C)$ and $d \in H_2^G(X)$ we denote by

$$\tau^G_{X,n,d}(C, \alpha_1, \ldots, \alpha_n; \beta) := \\ \chi^{\mathrm{vir}}(\overline{\mathcal{M}}^G_n(C, X, d), \mathrm{ev}_0^* \alpha_0 \otimes \mathrm{ev}_1^* \alpha_1 \otimes \ldots \mathrm{ev}_n^* \alpha_n) \otimes f^* \beta_n \in \mathbb{Z} \quad (1.24)$$

the virtual Euler characteristic. Define the *gauged K-theoretic Gromov-Witten potential* as the formal sum

$$\tau^G_X : QK^0_G(X) \times K(\overline{\mathcal{M}}_n(C)) \to \Lambda^G_X, \\ (\alpha, \sigma) \mapsto \sum_{n \geq 0, d \in H_2^G(X,\mathbb{Z})} \frac{q^d}{n!} \tau^G_{X,n,d}(C, \alpha, \ldots, \alpha; \beta_n). \quad (1.25)$$

In the case of domain the projective line the gauged potential can be further localized as follows. Let $C = \mathbb{P}^1$ be equipped with the standard $\mathbb{C}^\times$ action with fixed points $0, \infty \in \mathbb{P}^1$. Denote by z the equivariant parameter corresponding to the $\mathbb{C}^\times$-action. As in [57, Section 9] let $\overline{\mathcal{M}}^G_{n+1}(\mathbb{C}_+, X, d)^{\mathbb{C}^\times}$ denote the stack of $n+1$-marked Mundet-semistable gauged maps $P \to \mathbb{P}^1, u : \mathbb{P}^1 \to P(X)$ with the following properties: the data (P, u) are fixed up to automorphism by the $\mathbb{C}^\times$ action, and with one marking at 0 and the remaining markings mapping to components attached to 0, and the pair (P, u) is trivializable in a neighborhood of $\infty \in \mathbb{P}^1$. The moduli space $\overline{\mathcal{M}}^G_{n+1}(\mathbb{C}_-, X, d)^{\mathbb{C}^\times}$ is defined similarly but replacing 0 with ∞ and vice-versa. It is an observation of Givental (in a more restrictive setting) that the $\mathbb{C}^\times$-fixed locus in $\overline{\mathcal{M}}^G(\mathbb{P}^1, X, d)$ for sufficiently small vortex parameter ρ is naturally a union of fiber products

$$\overline{\mathcal{M}}^G_{n_- + n_+}(\mathbb{P}^1, X)^{\mathbb{C}^\times} = \\ \bigcup_{\substack{n_- + n_+ = n \\ d_- + d_+ = d}} \overline{\mathcal{M}}^G_{n_-+1}(\mathbb{C}_-, X, d_-)^{\mathbb{C}^\times} \times_{\overline{I}_{X /\!\!/ G}} \overline{\mathcal{M}}^G_{n_++1}(\mathbb{C}_+, X, d_+)^{\mathbb{C}^\times} \quad (1.26)$$

Indeed the bundle $P \to \mathbb{P}^1$ is given via the clutching construction by a transition map corresponding to an element $d \in \mathfrak{g}$ and the map u on $\mathbb{C}_\pm$ is given by an orbit of the one-parameter subgroup $u(z) = \exp(zd)x$ generated by d, for some $x \in X$.

Integration over the factors in this fiber product define *localized gauged graph potentials* $\tau^G_{X,\pm}$ as follows. The stack $\mathcal{M}^G(\mathbb{C}_\pm, X, d)^{\mathbb{C}^\times}$ has a natural equivariant perfect obstruction theory, as a fixed point stack in

$\mathcal{M}^G(\mathbb{P}^1, X, d)$. The perfect obstruction theory for $\mathcal{M}^G(\mathbb{C}_\pm, X, d)$ on the fixed locus splits in the *fixed* and *moving* parts. A perfect obstruction theory for $\mathcal{M}^G(\mathbb{C}_\pm, X, d)^{\mathbb{C}^\times}$ can be taken to be the fixed part. Let $N_\pm$ denote the virtual normal complex of $\mathcal{M}^G(\mathbb{C}_\pm, X, d)^{\mathbb{C}^\times}$ in $\mathcal{M}^G(\mathbb{P}^1, X, d)$. Define

$$\tau^G_{X,\pm} : QK^0_G(X) \to QK(X/\!\!/G)[z^{\pm 1}, z^{\mp 1}]],$$
$$\alpha \mapsto \sum_{n\geq 0, d\in H_2^G(X,\mathbb{Q})} \frac{q^d}{n!} \operatorname{ev}^d_{\infty,*} \frac{\operatorname{ev}^* \alpha^{\otimes n}}{\operatorname{Eul}(N^\vee_\pm)}.$$

Example 1.3. The gauged graph potentials for toric quotients are q-hypergeometric functions described in Givental-Lee [19]. Let G be a torus acting on a vector space X is a vector space with weights $\mu_1, \ldots, \mu_k$ and weight spaces $X_1, \ldots, X_k$ with free quotient $X/\!\!/G$. For any given class $\phi \in H_2^G(X, \mathbb{Z}) \cong \mathfrak{g}_\mathbb{Z}$, we have (omitting the classical map $K^G(X) \to K(X/\!\!/G)$ from the notation)

$$\tau^G_{X,-}(0) = \sum_{d\in H_2^G(X)} q^d \frac{\prod_{j=1}^k \prod_{m=-\infty}^0 (1 - z^m X_j^{-1})}{\prod_{j=1}^k \prod_{m=-\infty}^{\mu_j(d)} (1 - z^m X_j^{-1})}. \tag{1.27}$$

Note that the terms with $X/\!\!/_d G = \emptyset$ contribute zero in the above sum since in this case the factor in the numerator $\prod_{\mu_j(d)<0}(1 - X_j^{-1})$ vanishes.

Arbitrary values of the gauged potential can be computed as follows, using a result of Y.P. Lee [42] on Euler characteristics on the moduli spaces of genus zero marked curves. Since there are no non-constant holomorphic spheres in X, the evaluation maps $\operatorname{ev}_1, \ldots, \operatorname{ev}_n$ are equal on $\overline{\mathcal{M}}_n^G(\mathbb{C}_\pm, X)^{\mathbb{C}^\times}$. Let

$$L_i \to \overline{\mathcal{M}}_n^G(\mathbb{C}_\pm, X), \quad (L_i)_{u:C\to X/G,\lambda,\underline{z}} = T^\vee_{z_i} C$$

denote the cotangent line at the i-th marked point. We compute the push-pull as follows: On the component of $\overline{\mathcal{M}}_n^G(\mathbb{C}_\pm, X)^{\mathbb{C}^\times}$ corresponding to maps of degree d the pushforward is given by

$$\operatorname{ev}^d_{\infty,*} \frac{\operatorname{ev}^* \alpha^n}{\mp(1 - z^{\pm 1})(1 - z^{\pm 1} L_{n+1})} = \frac{\Psi_d(\alpha)^{\otimes n}}{(1 - z^{\pm 1})^2} \chi\left(\overline{\mathcal{M}}_{0,n+1}, \sum_d (L_{n+1} z^{\pm 1})^d\right) \tag{1.28}$$

where

$$(\Psi_d \alpha)(g) = \alpha(g z^d).$$

The integral (1.28) can be computed using a result of Y.P. Lee [42, Equation (3)] on Euler characteristics over $\overline{\mathcal{M}}_{0,n+1}$ (note the shift by 1 in the variable n to relate to Lee's conventions):

$$\chi\left(\overline{\mathcal{M}}_{0,n+1}, \sum_d ((z^{\pm 1} L_{n+1})^{\otimes d})\right) = (1 - z^{\pm})^{n-1}.$$

This implies that for $\alpha \in K^0_G(X)$

$$\tau^G_{X,-}(\alpha) = \sum_{d \in H^G_2(X)} q^d \exp\left(\frac{\Psi_d(\alpha)}{1 - z^{-1}}\right) \frac{\prod_{j=1}^k \prod_{m=-\infty}^0 (1 - X_j^{-1} z^m)}{\prod_{j=1}^k \prod_{m=-\infty}^{\mu_j(d)} (1 - X_j^{-1} z^m)}. \tag{1.29}$$

This is a version of Givental's K-theoretic *I-function*, see Givental-Lee [19] and Taipale [50].

1.5.2 The adiabatic limit theorem

In the limit that the linearization tends to infinity, the gauged Gromov-Witten invariants are related to the Gromov-Witten invariants of the quotient in K-theory. Let $\mathbb{C}^\times$ act on $\mathcal{M}^G_X(\mathbb{A}, X)$ via the weight 1 action on $\mathbb{A}$. The quantum Kirwan map then has a natural $\mathbb{C}^\times$-equivariant extension

$$\kappa^G_X : K^0_G(X) \to K^0_{\mathbb{C}^\times}(X/\!\!/G)$$

defined by push-forward using the $\mathbb{C}^\times$-equivariant virtual fundamental sheaf. The following is a K-theoretic version of a result of Gaio-Salamon [15]:

Theorem 1.4 (Adiabatic Limit Theorem)**.** *Suppose that C is a smooth projective curve and X a polarized projective G-variety such that stable=semistable for gauged maps from C to X of sufficiently small ρ. Then*

$$\tau_{X/\!\!/G} \circ \kappa^G_X = \lim_{\rho \to 0} \tau^G_X : QK^0_G(X) \to \Lambda^G_X \tag{1.30}$$

in the following sense: For a class $\beta \in K(\overline{\mathcal{M}}_{n,1}(C))$ let

$$\sum_{k=1}^l \beta^k_\infty \otimes \beta^k_1 \otimes \ldots \beta^k_r, \quad \beta_0$$

be its pullbacks to

$$K\left(\overline{\mathcal{M}}_r(C) \times \prod_{j=1}^r \overline{\mathcal{M}}_{|I_j|}(\mathbb{C})\right), \quad \textit{resp. } K(\overline{\mathcal{M}}_n(C))$$

respectively. Then

$$\sum_{I_1 \cup \ldots \cup I_r = \{1,\ldots,n\}} \sum_{k=1}^{l} \tau^r_{X/\!\!/G}(\alpha, \beta^k_\infty) \circ \kappa_X^{G,|I_j|}(\alpha, \beta^k_j) = \lim_{\rho \to 0} \tau_X^{G,n}(\alpha, \beta_0).$$

Similarly for the localized graph potentials (without insertions of classes on the source moduli spaces)

$$\tau_{X/\!\!/G,\pm} \circ \kappa_X^G = \tau^G_{X,\pm} : QK^0_G(X) \to QK^0_{\mathbb{C}^\times}(X/\!\!/G).$$

In other words, the diagram

$$\begin{array}{ccc} QK^0_G(X) & \xrightarrow{\kappa_X^G} & QK^0_{\mathbb{C}^\times}(X/\!\!/G) \\ & \searrow^{\tau_X^G} \quad \swarrow_{\tau_{X/\!\!/G}} & \\ & \Lambda_X^G & \end{array} \tag{1.31}$$

commutes in the limit $\rho \to 0$.

Sketch of Proof The proof is similar to that in the cohomology case in [57]: the proof only used an equivalence of divisor classes in the moduli stacks of scaled gauged maps. Let $\overline{\mathcal{M}}_{n,1}(\mathbb{P}^1)$ denote the moduli space *scaled* maps to $\mathbb{P}^1$, that is, the space of maps $\phi : C \to \mathbb{P}^1$ of class $[\mathbb{P}^1]$ equipped with sections λ of the projectivized relative dualizing sheaf from [57]; this means that some component C_0 mapped isomorphically onto $\mathbb{P}^1$ while the remaining components maps to points; either $\lambda|C_0$ is finite, in which case the remaining components $C_v \subset C$ have $\lambda|C_v = 0$, or $\lambda|C_0$ is infinite in which case there are a collection of bubble trees $C_1, \ldots, C_k \subset C$ attached to C_0 of the form described in 1.1. In particular, if there are no markings $n = 0$ then there are no bubble components and there exists an isomorphism

$$\overline{\mathcal{M}}_{0,1}(\mathbb{P}^1) \cong \mathbb{P}$$

corresponding to the choice of section of the projectivized trivial sheaf. Denote by

$$f_{0,0} : \overline{\mathcal{M}}_{0,n}(\mathbb{P}^1) \to \overline{\mathcal{M}}_{0,0}(\mathbb{P}^1) \cong \mathbb{P} \tag{1.32}$$

the forgetful morphism forgetting the markings $z_1, \ldots, z_n$ but remembering the scaling λ. We have the relation

$$[\mathcal{O}_{\cup_{\mathcal{P}} D_{\mathcal{P}}}] = [\mathcal{O}_{f_{0,0}^{-1}(0)}] \in K(\overline{\mathcal{M}}_n(\mathbb{P}^1))$$

where

$$D_{\mathcal{P}} \cong \overline{\mathcal{M}}_r(\mathbb{P}^1) \times \prod_{i=1}^{r} \overline{\mathcal{M}}_{i_j}(\mathbb{A})$$

is a divisor corresponding to the unordered partition

$$\mathcal{P} = \{\mathcal{P}_1, \ldots, \mathcal{P}_r\}, \quad |\mathcal{P}_j| = I_j$$

of the markings $\mathcal{P}$ in groups of size $i_1, \ldots, i_r$. The class $\cup_{\mathcal{P}} D_{\mathcal{P}}$ is locally the union of prime divisors in a toric variety and the standard resolution gives the equality in K-theory

$$[\mathcal{O}_{\cup_{\mathcal{P}} D_{\mathcal{P}}}] = \sum_{\mathcal{P}_1, \ldots, \mathcal{P}_k} (-1)^{k-1} [\mathcal{O}_{D_{\mathcal{P}_1} \cap \ldots \cap D_{\mathcal{P}_k}}].$$

Two divisors $D_{\mathcal{P}_1}, \ldots, D_{\mathcal{P}_k}$ intersect if and only the partitions $\mathcal{P}_1, \ldots, \mathcal{P}_k$ have a common refinement (take a curve $(C, \underline{z})$ in the intersection and the partition determined by the equivalence class given by two markings are equivalent if they lie on the same irreducible component) and any type $\mathcal{M}_\Gamma(\mathbb{P}^1)$ in the intersection corresponds to some common refinement $\mathcal{P}$. Thus in the case of a non-empty intersection $D_{\mathcal{P}_1} \cap \ldots \cap D_{\mathcal{P}_k}$ we may assume that each $\mathcal{P}$ refines each $\mathcal{P}_j$.

Define a moduli space of maps with scaling as follows. If $(C, \lambda, \underline{z})$ is a scaled curve and $u : C \to X/G$ a morphism we say that the data $(C, \lambda, \underline{z})$ is stable if either $\lambda | C_0$ is finite and is Mundet semistable or $\lambda | C_0$ is infinite and each bubble tree is stable in the sense of 1.1. By [26], the moduli stack of scaled gauged maps $\overline{\mathcal{M}}^G_{n,1}(\mathbb{P}^1, X)$ is proper with a perfect obstruction theory. Virtual Euler characteristics over $\overline{\mathcal{M}}^G_{n,1}(\mathbb{P}^1, X)$ define invariants

$$K^0_G(X) \otimes K(\overline{\mathcal{M}}_{n,1}(\mathbb{P}^1)) \to \mathbb{Z}$$

with the property that $\alpha \otimes [\mathcal{O}_{\mathrm{pt}}]$ maps to $\tau^X_G(\alpha)$. We have a natural forgetful morphism

$$f : \overline{\mathcal{M}}^G_{n,1}(\mathbb{P}^1, X) \to \overline{\mathcal{M}}_{0,1}(\mathbb{P}^1) \cong \mathbb{P}$$

and the inverse image of ∞ is the union of divisor classes corresponding to partitions according to which markings lie on which bubble tree.

The contributions of intersections of divisors correspond to the terms in the Taylor expansion of the inverse of the Maurer-Cartan map, using the *tree inversion formula* of Bass-Connell-Wright [3, Theorem 4.1]. Consider the Taylor expansion

$$\mathcal{MC}(\alpha) = \mathrm{Id} + \sum_{k_1,\dots,k_r \geq 0} \frac{\mathcal{MC}_{(i_1,\dots,i_r;)} \alpha_1^{i_1} \dots \alpha_r^{i_r}}{i_1! \dots i_r!},$$
$$\mathcal{MC}_{(i_1,\dots,i_r)} \in QK(X /\!\!/ G). \quad (1.33)$$

The tree formula for the formal inverse $\mathcal{MC}^{-1}$ of $\mathcal{MC}$ reads

$$\mathcal{MC}^{-1} = \sum_{\Gamma} |\operatorname{Aut}(\Gamma)|^{-1} \prod_{v \in \operatorname{Vert}(\Gamma)} (-\mathcal{MC}_v)$$

where the sum over $\{1, \dots, \dim(QH(X /\!\!/ G))\}$-labelled trees Γ,

$$\mathcal{MC}_v(\sigma_1, \dots, \sigma_{k(v)}) = \sum_{i_1,\dots,i_{k(v)}} \mathcal{MC}_{i_1(v),\dots,i_r(v)}(\sigma_1, \dots, \sigma_r)$$

is the $|v|$-th Taylor coefficient in $\mathcal{MC} - \mathrm{Id}$ for the labels incoming the vertex v considered as a symmetric polynomial in the entries; and composition is taken on the tensor algebra using the tree structure on Γ. For example, for the tree corresponding to the bracketing (12)3 the contribution of $\mathcal{MC}_2$ to $\mathcal{MC}^{-1}$ is $(-\mathcal{MC}_2)(-\mathcal{MC}_2 \otimes \mathrm{Id})$. See Wright [58] for the extension of [3] to power series, and also Kapranov-Ginzburg [16, Theorem 3.3.9]. The argument for the localized gauged potential is similar, by taking fixed point components for the $\mathbb{C}^\times$-action on $\overline{\mathcal{M}}^G(\mathbb{P}^1, X)$. □

1.5.3 Divided difference operators

A result of Givental and Tonita [20] shows the existence of a difference module structure on quantum K-theory for arbitrary target which gives rise to relations in the quantum K-theory. We follow the treatment in [33, Section 2.5, esp. 2.10-2.11]. Let

$$\lambda_1, \dots, \lambda_k \in K(X)$$

be classes of nef line bundles corresponding to a basis of $H^2(X, \mathbb{Z})$ and $m(\lambda_i)$ the corresponding endomorphisms of $QK(X)$ given by multiplication. Define endomorphisms

$$\mathcal{E}_i = \tau(m(\lambda_i)^{-1} z^{q_i \partial_{q_i}} \tau^{-1}) \in \operatorname{End}(QK^0_{\mathbb{C}^\times}(X))$$

where τ is a fundamental solution to $\nabla^q_\alpha \tau = 0$. Define

$$\mathcal{E}_{i,\mathrm{com}} = \mathcal{E}_i|_{z=1} \in \operatorname{End}(QK(X)).$$

Then any difference operator annihilating the J-function defines a relation in the quantum K-theory: Following [33, Remark 2.11] define

$$\tilde{\tau}^G_{X,\pm} = \left(\prod_{i=1}^r \lambda_i^{-\ln(q_i)/\ln(z)} \right) \tau^G_{X,\pm}.$$

The results in [33, Section 2] give relations corresponding to operators annihilating the fundamental solution. Working with topological K-theory we define coordinates $t_1, \ldots, t_r$ by

$$\aleph = t_1 \aleph_1 + \ldots t_r \aleph_r, \quad \aleph_1, \ldots, \aleph_r \text{ a basis of } QK(X /\!\!/ G).$$

Theorem 1.5. *For any divided-difference operator*

$$\square \in \mathbb{Q}[z, z^{-1}][[q, t]] \langle z^{q_i \partial_{q_i}}, (1 - z^{-1}) \partial_\alpha, \alpha \in QH(X /\!\!/ G) \rangle$$

(where angle brackets denote the sub-ring of differential operators generated by these symbols) we have

$$\square(z, q, t, z^{q_i \partial_{q_i}}, (1 - z^{-1}) \partial_\alpha) \tilde{\tau}^G_{X,-} = 0 \implies \square(z, q, t, \mathcal{E}_i, \nabla^z_\alpha) 1 = 0$$

and setting $z = 1$ gives rise to the relation involving quantum multiplication $m(\alpha)$ by α

$$\square(1, q, t, \mathcal{E}_{i,\mathrm{com}}, m(\alpha)) = 0$$

in the quantum K-theory $QK(X /\!\!/ G)$.

Example 1.6. (Toric varieties) In the toric case, we recover some of the relations on quantum K-theory from Theorem 1.2 as follows. The operators

$$\mathcal{D}_{i,k} := 1 - X_i (z^{q_i \partial_{q_i}} + z^{-k} (1 - z^{-1}) \partial_i)$$

satisfy

$$\left(\prod_{\mu_i(d) \geq 0} \prod_{k=0}^{\mu_i(d)-1} \mathcal{D}_{i,k} - q^d \prod_{\mu_i(d) < 0} \prod_{i=1}^{-\mu_i(d)} \mathcal{D}_{i,k} \right) \hat{\tau}^G_{X,-} = 0. \tag{1.34}$$

Using the expression (1.29) and $\Psi_d(X_i) = X_i z^{\mu_i(d)}$ we have

$$\begin{aligned} &(1 - z) \partial_i \tau^G_{X,-} \\ &= \sum_{d \in H_2^G(X)} q^d X_i z^{\mu_i(d)} \exp\left(\frac{\Psi_d(\alpha)}{1 - z^{-1}} \right) \frac{\prod_{j=1}^k \prod_{m=-\infty}^0 (1 - X_j^{-1} z^m)}{\prod_{j=1}^k \prod_{m=-\infty}^{\mu_j(d)} (1 - X_j^{-1} z^m)}. \end{aligned} \tag{1.35}$$

Hence

$$\prod_{\mu_i(d)\geq 0} \prod_{k=0}^{\mu_i(d)-1} \mathcal{D}_{i,k}\tau^G_{X,-}(\alpha,q,z)$$
$$= \sum_{d\in H_2^G(X)} q^d \prod_{k=0}^{\mu_i(d)-1}(1-X_i z^{\mu_i(d)-k}) \exp\left(\frac{\Psi_d(\alpha)}{1-z^{-1}}\right)$$
$$\frac{\prod_{j=1}^k \prod_{m=-\infty}^0 (1-X_j^{-1}z^m)}{\prod_{j=1}^k \prod_{m=-\infty}^{\mu_j(d)}(1-X_j^{-1}z^m)}. \quad (1.36)$$

This equals

$$q^d \prod_{\mu_i(d)<0} \prod_{k=1}^{-\mu_i(d)} \mathcal{D}_{i,k}\tau^G_{X,-}$$

hence the relation (1.34). By 1.5 one obtains quantum Stanley-Reisner relations as in (1.3).

1.6 Wall-crossing in K-theory

In this section we recall results on the dependence of the K-theoretic invariants of the quotient on the choice of polarization due to Kalkman [36], in the case of cohomology, and Metzler [45], in the case of K-theory.

1.6.1 The master space and its fixed point loci.

The geometric invariant theory quotients for two different polarizations may be written as quotients by a circle group action on a *master space*. Let $\mathcal{L}_\pm \to X$ two G-polarizations of X. Define

$$\mathcal{L}_t := \mathcal{L}_-^{(1-t)/2} \otimes \mathcal{L}_+^{(1+t)/2}, \quad t\in(-1,1)\cap\mathbb{Q} \quad (1.37)$$

the one-parameter family of rational polarizations given by interpolation.

Definition 1.1. The wall-crossing datum $(\mathcal{L}_-,\mathcal{L}_+)$ is *simple* if the following conditions are satisfied:

a. The only *singular value* is $t=0$, this is,

$$X^{\mathrm{ss}}(\pm) := X^{\mathrm{ss}}(\mathcal{L}_t), \quad 0 < \pm t \leq 1$$

is constant. We assume that stable equals semi-stable for $t\neq 0$, that is, there are no positive-dimensional stabilizer subgroups of points in the semi-stable locus.

b. The *strictly semistable set* $X^{ss}(\mathcal{L}_0) \setminus (X^{ss}(+) \cup X^{ss}(-))$ is connected.
c. The only infinite stabiliser subgroup is a circle group:

$$G_x \cong \mathbb{C}^\times, \quad \forall x \in X^{ss}(\mathcal{L}_0) \setminus (X^{ss}(+) \cup X^{ss}(-)), \quad \dim(G_x) > 0.$$

Given a simple wall-crossing, choose a one-parameter subgroup $\lambda : \mathbb{C}^\times \to G$, a connected component Z_λ of the semi-stable fixed points $X^{ss}(\mathcal{L}_0)^\lambda$.

Definition 1.2. The *master space* introduced Dolgachev-Hu [13] and Thaddeus [51] is defined as follows. The projectivization $\mathbb{P}(\mathcal{L}_- \oplus \mathcal{L}_+) \to X$ of the direct sum $\mathcal{L}_- \oplus \mathcal{L}_+ \to X$ is itself a G-variety, and has a natural polarization given by the relative hyperplane bundle $\mathcal{O}_{\mathbb{P}(\mathcal{L}_- \oplus \mathcal{L}_+)}(1)$ of the fibers, with stalks

$$\mathcal{O}_{\mathbb{P}(\mathcal{L}_- \oplus \mathcal{L}_+)}(1)_{[l_-, l_+]} = \operatorname{span}(l_- + l_+)^\vee. \tag{1.38}$$

Let $\pi : \mathbb{P}(\mathcal{L}_- \oplus \mathcal{L}_+) \to X$ denote the projection. The group $\mathbb{C}^\times$ acts on $\mathbb{P}(\mathcal{L}_- \oplus \mathcal{L}_+)$ by rotating the fiber $w[l_-, l_+] = [l_-, wl_+]$. The space of sections of $\mathcal{O}_{\mathbb{P}(\mathcal{L}_- \oplus \mathcal{L}_+)}(k)$ has a decomposition under the natural $\mathbb{C}^\times$-action with eigenspaces given by the sections of

$$\pi^* \mathcal{L}_-^{k_-} \otimes \pi^* \mathcal{L}_+^{k_+}, \quad k_- + k_+ = k, k_\pm \geq 0.$$

The G-semistable locus in $\mathbb{P}(\mathcal{L}_- \oplus \mathcal{L}_+)$ is the union of loci of invariant eigensections and so the union of $\pi^{-1}(X^{ss,t})$ where $X^{ss,t} \subset X$ is the semistable locus for

$$\mathcal{L}_t = \mathcal{L}_-^{(1-t)/2} \otimes \mathcal{L}_+^{(1+t)/2}, t \in [-1, 1].$$

Let

$$\tilde{X} := \mathbb{P}(\mathcal{L}_- \oplus \mathcal{L}_+) /\!\!/ G$$

denote the geometric invariant theory quotient, by which we mean the stack-theoretic quotient of the semistable locus. The assumption on the action of the stabilizers implies that the action of G on the semistable locus in $\mathbb{P}(\mathcal{L}_- \oplus \mathcal{L}_+)$ is locally free, so that stable=semistable for $\mathbb{P}(\mathcal{L}_- \oplus \mathcal{L}_+)$. It follows that $\tilde{X}$ is a proper smooth Deligne-Mumford stack. The quotient $\tilde{X}$ contains the quotients of $\mathbb{P}(\mathcal{L}_\pm) \cong X$ with respect to the polarizations $\mathcal{L}_\pm$, that is, $X /\!\!/_\pm G$. Moreover the quotient $\tilde{X}^{ss}(t)/\mathbb{C}^\times$ with respect to the $\mathbb{C}^\times$-linearisation $\mathcal{O}(t)$ is isomorphic to $X^{ss}(\mathcal{L}_t)/G$. This ends the definition.

Next we recall from [24] the fixed point sets for the circle action on the master space.

Lemma 1.3. *The fixed point set* $\tilde{X}^{\mathbb{C}^\times}$ *is the union of the quotients* $X/\!\!/_{\pm}G$ *and the locus in* $X^{\mathbb{C}^\times,\mathrm{ss}}$ *in* $X^{\mathbb{C}^\times}$ *that is semistable for some* $t \in (-1,1)$*. The normal bundle of* $X^{\mathbb{C}^\times,\mathrm{ss}}$ *in* $\tilde{X}$ *is isomorphic to the normal bundle in* X*, while the normal bundles of* $X/\!\!/_{\pm}G$ *are isomorphic to* $\mathcal{L}_{\pm}^{\pm 1}$*.*

Proof Any fixed point is a pair $[l_-, l_+]$ with a positive dimensional stabiliser under the action of $\mathbb{C}^\times$. Necessarily the points with $l_- = 0, l_+ = 0$ are fixed and they correspond to the quotients $X/\!\!/_{\pm}G$. However there are other fixed points when l_-, l_+ are both non-zero when the projection to X is fixed by a one-parameter subgroup, as we now explain. For any $\zeta \in \mathfrak{g}$, we denote by $G_\zeta \subset G$ the stabiliser of the line $\mathbb{C}\zeta$ under the adjoint action of G. If $[l] \in \tilde{X}^{\mathbb{C}^\times}$, with $l \in \mathbb{P}(\mathcal{L}_- \oplus \mathcal{L}_+)$ then $[l] \in \tilde{X}^\xi$, where ξ is a generator of the Lie algebra of $\mathbb{C}^\times$ and $\tilde{X}^\xi$ is the zero set of the vector field $\xi_{\tilde{X}}$ generated by ξ. Since $\tilde{X}$ is the quotient of $\mathbb{P}(\mathcal{L}_- \oplus \mathcal{L}_+)$ by G, if ξ_L denotes the vector field on $\mathbb{P}(\mathcal{L}_- \oplus \mathcal{L}_+)$ generated by ξ then $\xi_L(l) = \zeta_L(l)$ for some $\zeta \in \mathfrak{g}$. Since G acts locally freely ζ must be unique. Integrating gives $z \cdot l = z^\zeta l$ for all $z \in \mathbb{C}^\times$. By uniqueness of ζ, this holds for every point in the component of $\tilde{X}^\xi$ containing $[l]$. Thus for any fixed point $\tilde{x} \in \tilde{X}^{\mathbb{C}^\times}$ with $\tilde{x} = [l]$ for some $l \in \mathbb{P}(\mathcal{L}_- \oplus \mathcal{L}_+)$, there exists $\zeta \in \mathfrak{g}$ such that

$$\forall z \in \mathbb{C}^\times, \quad z \cdot l = z^\zeta l,$$

where $z \mapsto z^\zeta$ is the one-parameter subgroup generated by ζ. By the definition of semistability, the argument above and our wall-crossing assumption, any fixed point $\tilde{x} \in \tilde{X}^{\mathbb{C}^\times}$ is in the fibre over $x \in X$ that is 0-semistable and has stabilizer given by the one-parameter subgroup generated by $\zeta \in \mathfrak{g}$, that is, the weight of the one-parameter subgroup generated by ζ on $(\mathcal{L}_t)_x$ vanishes:

$$z^\zeta l = l, \quad \forall l \in (\mathcal{L}_t)_x.$$

Denote by $X^\zeta/\!\!/_0(G_\zeta/\mathbb{C}^\times_\zeta)$ the quotient $[X^{\mathrm{ss}}(\mathcal{L}_0)^\zeta/(G_\zeta/\mathbb{C}^\times_\zeta)]$. Conversely, taking any lift gives a morphism

$$\iota_\zeta : X^\zeta/\!\!/_0(G_\zeta/\mathbb{C}^\times_\zeta) \to \tilde{X}^{\mathbb{C}^\times}.$$

The argument above shows that any (l_-, l_+) with both non-zero is in the image of some ι_ζ. The normal bundle $\nu_{\tilde{X}^{\mathbb{C}^\times}}$ of $\tilde{X}^{\mathbb{C}^\times}$ in $\tilde{X}$ restricted to the image of ι_ζ is isomorphic to the quotient of the normal bundle of $G \times_{G_\zeta} \mathbb{P}(\mathcal{L}_- \oplus \mathcal{L}_+)^{\xi+\zeta}$ by G, which in turn is isomorphic via projection to the quotient of the normal bundle of $\mathbb{P}(\mathcal{L}_- \oplus \mathcal{L}_+)^{\xi+\zeta}$ by $\mathfrak{g}/\mathfrak{g}_\zeta$, and then by the induced action of the smaller group $G_\zeta/\mathbb{C}_\zeta^\times$. $G_\zeta/\mathbb{C}_\zeta^\times$ acts canonically on the normal bundle $\nu_{X^\zeta}/(\mathfrak{g}/\mathfrak{g}_\zeta)$ and induces a bundle $(\nu_{X^\zeta}/(\mathfrak{g}/\mathfrak{g}_\zeta))/\!\!/(G_\zeta/\mathbb{C}_\zeta^\times)$ over $X^\zeta/\!\!/_0(G_\zeta/C_\zeta^\times)$. The pull-back of the normal bundle $\iota_\zeta^*(\nu_{\tilde{X}^{\mathbb{C}^\times}})$ is canonically isomorphic to the quotient of $\nu_{X^\zeta}/(\mathfrak{g}/\mathfrak{g}_\zeta)$ by $G_\zeta/\mathbb{C}_\zeta^\times$, by an isomorphism that intertwines the action of $\mathbb{C}_\zeta^\times$ on $(\nu_{X^\zeta}/(\mathfrak{g}/\mathfrak{g}_\zeta))/\!\!/(G_\zeta/\mathbb{C}_\zeta^\times)$ with the action of $\mathbb{C}^\times$ on $\nu_{\tilde{X}^{\mathbb{C}^\times}}$. The final claim is left to the reader. □

Definition 1.4. We introduce the following notation. Denote by $X^{\zeta,0} \subset X^\zeta$ the fixed point loci $X^{\mathrm{ss}}(\mathcal{L}_0)^\zeta$ that are 0-semistable. Denote by $\nu_{X^{\zeta,0}}$ the $\mathbb{C}_\zeta^\times$-equivariant normal bundle of $X^{\zeta,0}$ modulo $\mathfrak{g}/\mathfrak{g}_\zeta$, quotiented by $G_\zeta/\mathbb{C}_\zeta^\times$.

A natural collection of K-theory classes on the master space is obtained by pull back. The projection $\mathbb{P}(\mathcal{L}_- \oplus \mathcal{L}_+) \to X$ is G-equivariant and $\mathbb{C}^\times$-invariant by construction. Consider the canonical map

$$\delta : K_G^0(X) \to K_{\mathbb{C}^\times}^0(\tilde{X}). \tag{1.39}$$

obtained by composition of the natural pull-back

$$K_G^0(X) \to K_{G\times\mathbb{C}^\times}^0(\mathbb{P}(\mathcal{L}_- \oplus \mathcal{L}_+))$$

with the Kirwan map

$$K_{G\times\mathbb{C}^\times}^0(\mathbb{P}(\mathcal{L}_- \oplus \mathcal{L}_+)) \to K_{\mathbb{C}^\times}(\mathbb{P}(\mathcal{L}_- \oplus \mathcal{L}_+)/\!\!/G) = K_{\mathbb{C}^\times}^0(\tilde{X}).$$

The composition of δ with the Kirwan map

$$\tilde{\kappa}_{X,t}^{\mathbb{C}^\times} : K_{\mathbb{C}^\times}^0(\tilde{X}) \to K(\tilde{X}/\!\!/_t\mathbb{C}^\times) = K(X/\!\!/_t G)$$

agrees with the pull-back to the $\mathcal{L}_t$-semistable locus $X^{\mathrm{ss}}(\mathcal{L}_t)$. It follows that

$$\tilde{\kappa}_{X,t}^{\mathbb{C}^\times} \circ \delta = \kappa_{X,t}^G : K_G^0(X) \to K(X/\!\!/_t G),$$

is the Kirwan map for the geometric invariant theory quotient of X with respect to the polarisation $\mathcal{L}_t$. In particular, $\delta(\alpha) \in K_{\mathbb{C}^\times}^0(\tilde{X})$ restricts to $\kappa_{X,\pm}^G \alpha$ on the two distinguished fixed point components $X/\!\!/_\pm G \subset \tilde{X}^{\mathbb{C}^\times}$.

The restrictions of these classes to fixed point sets are described as follows. After passing to a finite cover, we may assume that G_ζ splits as the product $G_\zeta/\mathbb{C}_\zeta^\times \times \mathbb{C}_\zeta^\times$. For any $\alpha \in K_G^0(X)$, the pull-back of $\tilde{\kappa}_{X,0}^{\mathbb{C}^\times}|_{\tilde{X}^{\mathbb{C}^\times}}(\alpha)$ under ι_ζ is equal to the image of α under the restriction map $K_G^0(X) \to K_{\mathbb{C}_\zeta^\times}(X^\zeta/\!\!/_0(G_\zeta/\mathbb{C}_\zeta^\times))$. Indeed $TX|_{X^\zeta}$ is the quotient of $T\mathbb{P}(\mathcal{L}_- \oplus \mathcal{L}_+)|_{\pi^{-1}(X^\zeta)}$ by $\mathbb{C}^\times$. Hence the action of G_ζ on $T\mathbb{P}(\mathcal{L}_- \oplus \mathcal{L}_+)|_{\pi^{-1}(X^\zeta)}$ induces an action of $G_\zeta/\mathbb{C}^\times$ on $TX|_{X^\zeta}$. After identifying $\mathbb{C}^\times \cong \mathbb{C}^\times_\zeta$, we have that $\iota_\zeta \circ \tilde{\kappa}^{\mathbb{C}^\times}_{X,0}$ is the pullback to $X^\zeta /\!\!/_0 (G_\zeta/\mathbb{C}^\times_\zeta)$. For $\alpha \in K^0_G(X)$ we will denote by

$$\alpha_0 \in K_G(X^\zeta /\!\!/_0 (G_\zeta/\mathbb{C}^\times_\zeta))$$

its restriction.

1.6.2 The Atiyah-Segal localization formula

In this section we review the Atiyah-Segal localization formula in equivariant K-theory [2], which expresses Euler characteristics as a sum over fixed point loci. Recall the definition of the Euler class in K-theory. Suppose Y is a smooth variety. For a vector bundle $E \to Y$ and a formal variable q denote the graded exterior power by

$$\bigwedge\nolimits_q E = \sum_{k=0}^{\infty} q^k \bigwedge\nolimits^k E \in K^0(Y)[q]. \tag{1.40}$$

The K-theoretic Euler class of E is defined by

$$\mathrm{Eul}(E) = \bigwedge\nolimits_{-1} E^\vee = 1 - E^\vee + \bigwedge\nolimits^2 E^\vee - \cdots \in K^0(Y).$$

Suppose now that $T = \mathbb{C}^\times$ acts trivially on Y, and that $a = \mathbb{C}_{(1)}$ is the weight 1 one-dimensional representation. Thus the equivariant K-theory of Y is given by

$$K^0_T(Y) = K^0(Y) \otimes K^0_T(\mathrm{pt}) = K^0(Y) \otimes \mathbb{Z}[z, z^{-1}].$$

If E is a coherent T-equivariant sheaf, its decomposition into isotypical components will be denoted

$$E = \bigoplus_{i=1}^{k} z^{\mu_i} E_i \tag{1.41}$$

where $\mu_i \in \mathbb{Z}$ is the weight of the action on E_i. The K-theoretical equivariant Euler class of E is given by

$$\mathrm{Eul}_T(E) = \prod_i \mathrm{Eul}(z^{\mu_i} E_i) \in K_T(Y).$$

The localization formula involves an integral over fixed point components with insertion of the inverted Euler class. Suppose that $T = \mathbb{C}^\times$ acts on X (non-trivially) and with fixed point set X^T. The previous paragraph discussion applies to the T-equivariant normal bundle ν_{X^T}, that is, the Euler class

$$\mathrm{Eul}_T(\nu_{X^T}) \in K_T(X^T) = K(X^T)[z, z^{-1}]$$

has been formally inverted through localisation. Denote by $K_T^{\mathrm{loc}}(X)$, the equivariant K-theory ring localized at the ideal of $R(T)$ generated by the Euler classes of representations $\mathbb{C}_{\mu_i}$ with weights $\mu_i, i = 1, \ldots, k$ (or more alternatively, one could localize at the roots of unity in the equivariant parameter).[3] The *K-theoretic localisation formula* of Atiyah-Segal [2], see also [12, Chapter 5]) states that in $K_T^{\mathrm{loc}}(X)$

$$[\mathcal{O}_X] = [\iota_* \left(\mathcal{O}_{X^T} \otimes \mathrm{Eul}_T(\nu_{X^T}^\vee)^{-1}\right)] = \left[\sum_{F \to X^T} \mathcal{O}_F \otimes \mathrm{Eul}_T(\nu_F^\vee)^{-1}\right] \tag{1.42}$$

for the inclusion $\iota : X^T \to X$. Here the sum runs over all fixed components F of X^T. In terms of Euler characteristics we have

$$\chi(X; \mathcal{F}) = \sum_{F \to X^T} \chi(F; \mathcal{F} \otimes \mathrm{Eul}_T(\nu_F^\vee)^{-1}) \in R(T).$$

The Atiyah-Segal localization formula implies a wall-crossing formula for K-theoretic integrals under variation of geometric invariant theory quotient due to Metzler [45]. Metzler's formula uses the following notion of *residue*, related to a power series expansion in a localized ring. For any T-equivariant locally free sheaf $\mathcal{F}$ of rank r on a variety Y with trivial T-action the class

$$\mathcal{F}_0 = \mathcal{F} - r\mathcal{O}_Y \in K_T(Y)$$

is nilpotent (c.f. [12, Prop. 5.9.5]). By taking exterior powers with notation as in (1.40) we have

$$\bigwedge_q \mathcal{F} = \bigwedge_q (\mathcal{F}_0 + r\mathcal{O}_Y) = (1+q)^r \bigwedge_{\frac{q}{1+q}} \mathcal{F}_0 \in K_T(Y)[q].$$

Using this formula on the weight μ_i bundle E_i gives

$$\mathrm{Eul}_T(E_i) = (1 - z^{-\mu_i})^{r_i}(1 + N_i)$$

where $r_i = \mathrm{rank}\, E_i$, and $N_i \in K(Y) \otimes \mathbb{Z}[z, z^{-1}]$ is a combination of nilpotent elements in $K(Y)$ whose coefficients are monomials of the form

[3] In the orbifold case, one must localize at the roots of unity that appear in the denominators in the Riemann-Roch formula as in Tonita [53].

$\frac{z^{-\mu_i}}{1-z^{-\mu_i}}$. Thus

$$\mathrm{Eul}_T(E) = \prod_i (1 - z^{-\mu_i})^{r_i}(1 + N_i), \tag{1.43}$$

where

$$N_i = \sum N_{k,i} s_k(z) \in K(Y) \otimes \mathbb{Z}[[z, z^{-1}]]. \tag{1.44}$$

The equation (1.44) is a finite sum where $N_{k,i}$ is a nilpotent element in $K(Y)$ and $s_k(z)$ is a rational function in z that has no pole at $z = 0$ nor $z = \infty$. Thus $\mathrm{Eul}_T(E)^{-1}$ has only poles at roots of unity. In particular the Euler classes can formally be inverted, since the leading term is invertible.

Example 1.5. We give the following example of formal power series associated to inverted Euler classes. Let $L \to Y$ be a T-equivariant line bundle of weight $\mu = \pm 1$. Then

$$\mathrm{Eul}_T(L) = 1 - z^{\mp 1} L^\vee \in K_T(Y).$$

This expression can be expanded, using the nilpotent element $L_0 = L^\vee - 1$, as

$$1 - z^{\mp 1} L^\vee = 1 - z^{\mp 1}(1 + L_0) = (1 - z^{\mp 1}) \left(1 + \frac{z^{\mp 1} L_0}{1 - z^{\mp 1}} \right).$$

This ends the example.

The residue of a K-theoretic class is a difference between the residues of the characters at zero and infinity. For any class $\alpha \in K_T^{\mathrm{loc}}(Y)$ there exist unique expansions

$$\alpha = \sum_{n \geq 0} \alpha_{0,n} z^n \in K(Y)[z^{-1}, z]], \quad \alpha_{0,n} \in K(Y)$$

and

$$\alpha = \sum_{n \geq 0} \alpha_{\infty,n} z^{-n} \in K(Y)[[z^{-1}, z], \quad \alpha_{\infty,n} \in K(Y).$$

The *residue* is the map

$$\mathrm{Resid} : K_T^{\mathrm{loc}}(Y) \to K(Y); \;\; \alpha \mapsto \alpha_{\infty,0} - \alpha_{0,0} \tag{1.45}$$

assigning the difference of the constant coefficients in the power series expansions above.

Lemma 1.6. *Let $\alpha \in K_T^{\mathrm{loc}}(Y)$ be a class in the image of the inclusion $K_T(Y) \to K_T^{\mathrm{loc}}(Y)$. Then* $\mathrm{Resid}\, \alpha = 0$.

Proof In this case the coefficients $\alpha_{0,i} = \alpha_{\infty,i}$ are the coefficients of z^i in α, hence in particular $\alpha_{0,0} = \alpha_{\infty,0}$. □

Example 1.7. Let Y be a point and $E = \mathbb{C}_{(1)}$ the representation with weight one. The inverted Euler class of E is defined formally by the associated series

$$\mathrm{Eul}(\mathbb{C}_{(1)})^{-1} = (1 - z^{-1})^{-1} = 1 + z^{-1} + z^{-2} + \dots$$

$$\mathrm{Eul}(\mathbb{C}_{(1)})^{-1} = (1 - z^{-1})^{-1} = \frac{-1}{z^{-1}(1-z)} = -z - z^2 - z^3 + \dots .$$

It follows that the residue of the inverted Euler class is

$$\mathrm{Resid}(\mathrm{Eul}(\mathbb{C}_{(1)})) = 1 - 0 = 1 \in K(\mathrm{pt}) \cong \mathbb{Z}.$$

Similarly

$$\mathrm{Resid}(\mathrm{Eul}(\mathbb{C}_{(-1)})) = 0 - 1 = -1 \in K(\mathrm{pt}) \cong \mathbb{Z}.$$

More generally for any line bundle L of weight ± 1, by Example 1.5 we have

$$\mathrm{Resid}\,\mathrm{Eul}(L)^{-1} = \mathrm{Resid}\left(\frac{1}{1 - L^\vee}\right) = \pm 1. \tag{1.46}$$

Example 1.8. Suppose that T acts trivially on Y and that $E \to Y$ is a bundle with isotypical decomposition as in (1.41). Let

$$S_+, S_- \subset \{1, \dots, k\}$$

denote the index sets for positive and negative weights μ_i respectively. Using (1.43), we have

$$\mathrm{Eul}_T(E)^{-1} = \left(\prod_{i \in S_+} (1 - z^{-\mu_i})^{r_i} (1 + E_{+,i}) \prod_{i \in S_-} (1 - z^{-\mu_i})^{r_i} (1 + E_{-,i}) \right)^{-1}$$

where both $E_\pm \in K_T^{\mathrm{loc}}(X)$ are as in Equation (1.44). To compute the residue, we may ignore the N_i terms. In this case the lowest order terms in the power series expansions in both z and z^{-1} have degree given by $\prod_{i \in S_+} \mu_i r_i$ and $\prod_{i \in S_-} (-\mu_i) r_i$ respectively. If both $S_\pm$ are non-empty, then $\mathrm{Resid}\,\mathrm{Eul}_T(E)^{-1} = 0$.

In general one can make different choices of residues, and the choice depends on the problem at hand, see [45, Section 4] for other examples.

1.6.3 The Kalkman-Metzler formula

The formulas of Kalkman and Metzler describe the wall-crossing of K-theoretic integrals under variation of geometric invariant theory quotient. We follow the same notations as in Section 1.6.1. Consider family of polarisations $\mathcal{L}_t \to X$ with a simple wall crossing at the singular time $t = 0$ with an associated master space $\tilde{X}$. Let

$$X^{\zeta,0} = X^{\mathrm{ss}}(\mathcal{L}_0) \cap X^{\mathbb{C}^\times}$$

be the fixed point component of the one-parameter subgroup generated by ζ involved in the wall-crossing.

Definition 1.9. For an element $\zeta \in \mathfrak{g}$, define the *fixed point contribution*

$$\tau_{X,\zeta,0} : K^0_G(X) \to \mathbb{Z},$$
$$\alpha \mapsto \chi\left(X^{\zeta,0} /\!\!/ (G_\zeta / \mathbb{C}^\times_\zeta);\ \mathrm{Resid}\left(\frac{\alpha_0}{\mathrm{Eul}_{\mathbb{C}^\times_\zeta}(\nu_{X^{\zeta,0}})} \right)\right). \quad (1.47)$$

Here α_0 is the image of α under the map $K^0_G(X) \to K^0_{\mathbb{C}^\times_\zeta}(X^{\zeta,0})$.

A version of the following formula was proved by Metzler [45, Theorem 1.1] in the differential geometric setting of manifolds with circle actions.

Theorem 1.10 (Kalkman-Metzler wall-crossing)**.** *Let X be a smooth G-variety (projective or affine) and suppose that $\mathcal{L}_\pm \to X$ are polarizations inducing a simple wall-crossing. Then*

$$\tau_{X/\!\!/_+ G} \kappa^G_{X,+} - \tau_{X/\!\!/_- G} \kappa^G_{X,-} = \tau_{X,\zeta,0}. \quad (1.48)$$

Example 1.11. Our first example concerns the passage from projective space to the empty set by variation of git quotient. Consider $G = \mathbb{C}^\times$ acting on $X = \mathbb{C}^{n+1}$ diagonally with weight one. A polarisation is given by a choice of another character $\ell \in \mathbb{Z}$. For $\ell < 0$ the quotient is empty and for $\ell > 0$, $X/\!\!/G = \mathbb{P}^n$. Let $\mathbb{C}_{(k)}$ be the weight $k \geq 0$ representation. The sheaf $\mathcal{F} = \mathcal{O}_X \otimes \mathbb{C}_{(k)}$ descends to $\mathcal{O}(k) \to \mathbb{P}^n$. The fixed point set X^G is the single point $0 \in X$ whose normal bundle is identified with $\mathbb{C}^{n+1}$ itself. The inverted Euler class is $(1 - z^{-1})^{-(n+1)}$ and can be identified with the class of the symmetric algebra $\mathrm{Sym}(z^{-1}\mathbb{C}^{n+1})$. The fixed point contribution of the wall crossing formula (1.48) equals

$$\mathcal{O}(k)|_0 \otimes \mathrm{Sym}(z^{-1}\mathbb{C}^{n+1}) = z^k \,\mathrm{Sym}(z^{-1}\mathbb{C}^{n+1}).$$

The residue, corresponding to the z^0 coefficient is the degree k part

$\mathrm{Sym}^k(\mathbb{C}^{n+1})$. The wall crossing formula now reads

$$\chi(\mathbb{P}^n, \mathcal{O}(k)) = \chi(X^G, \mathrm{Sym}^k(\mathbb{C}^{n+1})) = \dim \mathrm{Sym}^k(\mathbb{C}^{n+1}) = \binom{n+k}{k}.$$

Similarly, if one considers $\mathcal{O}(-k)$ and the identity

$$\mathrm{Sym}\, V = (-1)^{n+1} \det V^\vee \otimes \mathrm{Sym}\, V^\vee,$$

we get that the wall crossing contribution is

$$\mathrm{Resid}\, z^{-k} z^{n+1}\, \mathrm{Sym}(\mathbb{C}^{n+1}).$$

This is the degree $k-n-1$ part $\mathrm{Sym}^{k-n-1}(\mathbb{C}^{n+1})$, hence

$$\chi(\mathbb{P}^n, \mathcal{O}(-k)) = (-1)^n \chi(X^G, \mathrm{Sym}^{k-n-1}(\mathbb{C}^{n+1})) = (-1)^n \binom{k-1}{k-n-1}.$$

Example 1.12. Our second example concerns the Cremona transformation. Let $X = (\mathbb{P}^1)^3$ with polarization

$$\mathcal{L} = \mathcal{O}_{\mathbb{P}^1}(c_1) \boxtimes \mathcal{O}_{\mathbb{P}^1}(c_2) \boxtimes \mathcal{O}_{\mathbb{P}^1}(c_3), \quad c_1 < c_2 < c_3.$$

Consider $G = \mathbb{C}^\times$ acting on each factor $\mathbb{P}^1$ by $t[z_0, z_1] = [z_0, tz_1]$ and the action on $\mathcal{O}_{\mathbb{P}^1}(n)$ so that the weights at the fibres over the fixed points $[1,0]$, $[0,1]$ are $n/2, -n/2$. We consider the family of polarizations $\mathcal{L}_t = \mathcal{L} \otimes \mathbb{C}_t$ obtained by shifting $\mathcal{L}$ by a trivial line bundle weight weight t. The singular values t are given by the weights $(\pm c_1 \pm c_2 \pm c_3)/2$ of the action on the fibre $\mathcal{L}_t|_p$ at the fixed points. Hence, there are eight chambers for t on which the geometric invariant theory quotient $X/\!\!/_t G$ is constant. Two have empty git quotients. In the first and last chamber, we have $X/\!\!/_t G \cong \mathbb{P}(\mathbb{C}^3)$ resp. $\mathbb{P}((\mathbb{C}^3)^\vee)$, while the six intermediate wall-crossings induce three blow-ups and three blow-downs involved in the classic Cremona transformation of $\mathbb{P}^2$.

We consider how the Euler characteristic of the structure sheaf changes under wall-crossing. Consider the structure sheaf $\mathcal{O}_X$, whose restriction to any of the chambers is again the respective structure sheaf $\mathcal{O}_t$. We now describe the change in the Euler characteristic $\chi(X/\!\!/_t G, \mathcal{O}_t)$ as we vary t under the wall-crossing from the chamber $t < -c_1 - c_2 - c_3$ to the first non-empty chamber $t \in (c_1 - c_2 - c_3, -c_1 + c_2 - c_3)$, corresponding to the wall defined by the fixed point $x = ([1,0],[1,0],[1,0]) \in X$. Since all weights of the action on the tangent bundle at this fixed point $x \in X$ are 1, we have that the Euler class at this fixed point is $(1 - z^{-1})^3$. The wall crossing formula yields

$$\chi(\mathbb{P}^2, \mathcal{O}) = 0 + \chi\left(x, \mathrm{Resid}\, \frac{\mathcal{O}|_x}{(1 - z^{-1})^3}\right) = 1$$

as expected.

The next wall-crossing, corresponds to the blow-up $\mathrm{Bl}(\mathbb{P}^2)$ of $\mathbb{P}^2$ at a fixed point. The normal bundle at this point has weights $-1, 1, 1$ and thus the wall crossing formula yields

$$\begin{aligned}\chi(\mathrm{Bl}(\mathbb{P}^2), \mathcal{O}) &= \chi(\mathbb{P}^2, \mathcal{O}) + \chi\left(x, \mathrm{Resid}\, \frac{\mathcal{O}|_x}{(1-z)(1-z^{-1})^2}\right) \\ &= \chi(\mathbb{P}^2, \mathcal{O}) + 0 = 1\end{aligned}$$

since the residue is zero each time there are contributions with both positive and negative weights (since the numerator is trivial) which is consistent, since the Euler characteristic $\chi(X, \mathcal{O}_X)$ is a birational invariant. In fact the invariance of the Euler characteristic $\chi(X/\!\!/G, \mathcal{O}_{X/\!\!/G})$ under git birational transformation follows from Example 1.8.

Proof of Theorem 1.10 Consider the master space $\tilde{X}$ associated to the wall-crossing. Let $\alpha \in K_G^0(X)$, and consider its image $\delta(\alpha) \in K^0_{\mathbb{C}^\times}(\tilde{X})$ of (1.39). Using K-theoretic localization (1.42) on the $\mathbb{C}^\times$-space $\tilde{X}$, and the identification of the fixed point components and normal bundles with those in the ambient space we obtain the relation

$$\delta(\alpha) = \iota_* \frac{\kappa^G_{X,-}(\alpha)}{\mathrm{Eul}_{\mathbb{C}^\times}(\nu_-)} + \iota_* \frac{\kappa^G_{X,+}(\alpha)}{\mathrm{Eul}_{\mathbb{C}^\times}(\nu_+)} + \iota_* \frac{\alpha_0}{\mathrm{Eul}_{\mathbb{C}^\times_\zeta}(\nu_{X^{\zeta,0}})},$$

where $\iota : F \to \tilde{X}^{\mathbb{C}^\times}$ is the inclusion of (each) fixed point components. By applying residues and Euler characteristics

$$\begin{aligned}\chi\left(\tilde{X}; \mathrm{Resid}\,\delta(\alpha)\right) &= \chi\left(X/\!\!/_-G; \mathrm{Resid}\, \frac{\kappa^G_{X,-}(\alpha)}{\mathrm{Eul}_{\mathbb{C}^\times}(\nu_-)}\right) \\ &+ \chi\left(X/\!\!/_+G; \mathrm{Resid}\, \frac{\kappa^G_{X,+}(\alpha)}{\mathrm{Eul}_{\mathbb{C}^\times}(\nu_+)}\right) + \chi\left(X^{\zeta,0}; \mathrm{Resid}\, \frac{\alpha_0}{\mathrm{Eul}_{\mathbb{C}^\times}(\nu_{X^{\zeta,0}})}\right).\end{aligned} \tag{1.49}$$

The left hand side of (1.49) is zero by the definition of residue (c.f. Remark 1.6) . The group $\mathbb{C}^\times$ acts on $\nu_\pm$ with weights ∓ 1 on the normal bundles $\nu_\pm \to X/\!\!/_\pm G$ of $X/\!\!/_\pm G$ in $\tilde{X}$ respectively, since they are canonically identified with $(\mathcal{L}_+ \otimes \mathcal{L}_-)^{\pm 1}$. Hence one obtains

$$0 = -\tau_{X/\!\!/_+G}\kappa^G_{X,+}\alpha + \tau_{X/\!\!/_-G}\kappa^G_{X,-}\alpha + \chi\left(X^{\zeta,0};\ \mathrm{Resid}\, \frac{\alpha_0}{\mathrm{Eul}_{\mathbb{C}^\times_\zeta}(\nu_{X^{\zeta,0}})}\right)$$

as claimed. □

We remark that Kalkman-Metzler wall-crossing formula Theorem 1.10 can be generalised to more complicated wall-crossing, such as when there are multiple singular times and fixed components. The details are left to the reader.

1.6.4 The virtual wall-crossing formula

A virtual version of the Kalkman-Metzler formula follows from a virtual version of localization in K-theory proved by Halpern-Leistner [28, Theorem 5.7]. Let $\mathcal{X}$ be a Deligne-Mumford G-stack with coarse moduli space X. Let $\mathcal{L}_\pm \to \mathcal{X}$ be G-polarizations of $\mathcal{X}$. That is, $\mathcal{L}_\pm$ are G-line bundles which are ample on the coarse moduli space X. Let

$$\mathrm{Pic}_{\mathbb{Q}}(\mathcal{X}) = \mathrm{Pic}(\mathcal{X}) \otimes_{\mathbb{Z}} \mathbb{Q}$$

denote the rational Picard group and

$$\mathcal{L}_t := \mathcal{L}_-^{(1-t)/2} \otimes \mathcal{L}_+^{(1+t)/2} \in \mathrm{Pic}_{\mathbb{Q}}(\mathcal{X}), \quad t \in (-1,1) \cap \mathbb{Q}. \tag{1.50}$$

The family (1.50) is a one-parameter family of rational polarizations given by interpolation. We will also assume, for simplicity that there is a simple wall-crossing:

Assumption 1.13. *There is only one* singular value $t = 0$*, such that the semistable loci*

$$\mathcal{X}^{\mathrm{ss}}(+) := \mathcal{X}^{\mathrm{ss}}(\mathcal{L}_t), \quad \mathcal{X}^{\mathrm{ss}}(-) := \mathcal{X}^{\mathrm{ss}}(\mathcal{L}_t)$$

are constant for $\pm 1 \leq t < 0$. We assume that stable equals semi-stable for $t \neq 0$. Moreover $\mathcal{X}^{\mathrm{ss}}(\mathcal{L}_0) \setminus (\mathcal{X}^{\mathrm{ss}}(+) \cup \mathcal{X}^{\mathrm{ss}}(-))$ is connected. Assume that the stabiliser G_x for $x \in \mathcal{X}^{\mathrm{ss}}(\mathcal{L}_0) \setminus (\mathcal{X}^{\mathrm{ss}}(+) \cup \mathcal{X}^{\mathrm{ss}}(-))$ is isomorphic to $\mathbb{C}^\times$.

Consider the case that the given stack has an equivariant perfect obstruction theory. The G-action on $\mathcal{X}$ induces a canonical morphism

$$a^\vee : L_{\mathcal{X}} \to \underline{\mathfrak{g}}^\vee \subset L_{\mathcal{X} \times G} = L_{\mathcal{X}} \times \underline{\mathfrak{g}}^\vee$$

that we call the *infinitesimal action*. Composing with the morphism $E \to L_{\mathcal{X}}$ gives a natural lift $\tilde{a}^\vee : E \to \underline{\mathfrak{g}}^\vee$. Let $\mathcal{X}^{\mathrm{ss}}$ be the semistable locus for some polarization and assume stable=semistable, that is all the stabilisers are finite. Let $E^{\mathrm{ss}}, L_{\mathcal{X}^{\mathrm{ss}}}$ etc. be the restrictions to the semistable locus.

Lemma 1.14. *The perfect obstruction theory $E^{\mathrm{ss}} \to L_{\mathcal{X}^{\mathrm{ss}}} := L_{\mathcal{X}}|_{\mathcal{X}^{\mathrm{ss}}}$ descends to a perfect obstruction theory on the quotient $\mathcal{X}^{\mathrm{ss}}/G$.*

Proof From the fibration $\pi : \mathcal{X}^{\mathrm{ss}} \to \mathcal{X}^{\mathrm{ss}}/G$ one obtains an exact triangle of cotangent complexes

$$L_{\mathcal{X}^{\mathrm{ss}}/G} \to L_{\mathcal{X}^{\mathrm{ss}}} \xrightarrow{a^\vee} \underline{\mathfrak{g}}^\vee \to L_{\mathcal{X}^{\mathrm{ss}}/G}[1], \tag{1.51}$$

thus we can consider $L_{\mathcal{X}^{\mathrm{ss}}} \to \underline{\mathfrak{g}}^\vee$ as the cotangent complex of $\mathcal{X}^{\mathrm{ss}}/G$. Let $\mathrm{Cone}(\tilde{a}^\vee)$ denote the mapping cone of $\tilde{a}^\vee$. Then the exact triangle

$$\mathrm{Cone}(\tilde{a}^\vee) \to E^{\mathrm{ss}} \xrightarrow{\tilde{a}^\vee} \underline{\mathfrak{g}}^\vee \to \mathrm{Cone}(\tilde{a}^\vee)[1]$$

admits a morphism to (1.51), in particular making $\mathrm{Cone}(\tilde{a}^\vee) \to L_{\mathcal{X}^{\mathrm{ss}}/G}$ into an obstruction theory with amplitude in $[-1, 1]$. By the assumption on the finite stabilizers, this obstruction theory is perfect. □

The perfect obstruction theories on the stack induce perfect obstruction theories on the fixed point components. Assume again that $\mathcal{X}$ is equipped with a G action. For any $\zeta \in \mathfrak{g}$, consider the fixed point stacks $\mathcal{X}^\zeta$. The restriction of the perfect obstruction theory $E^\bullet|_{\mathcal{X}^\zeta}$ decomposes as

$$E^\bullet|_{\mathcal{X}^\zeta} = E^{\bullet,\mathrm{mov}} + E^{\bullet,\mathrm{fix}}$$

where $E^{\bullet,\mathrm{mov}}$ is the *moving part* and $E^{\bullet,\mathrm{fix}}$ the *fixed part*. By results of [21], $E^{\bullet,\mathrm{fix}}$ yields an equivariant perfect obstruction theory for $\mathcal{X}^\zeta$, which is compatible with that on $\mathcal{X}$. Denote by

$$a_\zeta^\vee : L_{\mathcal{X}/G_\zeta} \to (\mathfrak{g}_\zeta/\mathbb{C}\zeta)^\vee$$

the map given by the infinitesimal action of $G_\zeta/\mathbb{C}_\zeta^\times$ on $\mathcal{X}/G_\zeta$. Denote by

$$\nu_{\mathcal{X}^\zeta} \in \mathrm{Ob}(D^b\,\mathrm{Coh}(\mathcal{X}^\zeta/G_\zeta))$$

the *moving part* of the mapping cone $\mathrm{Cone}(a_\zeta)$; this is the conormal complex for the embedding $\mathcal{X}^\zeta/G_\zeta \to \mathcal{X}/G_\zeta$ except for the factor $\mathbb{C}\zeta$ of automorphisms of the fixed point set. Denote by $\mathcal{X}^{\zeta,0}$ the locus of $\mathcal{X}^\zeta$ which is $\mathcal{L}_0$-semistable and by $\nu_{\mathcal{X}^{\zeta,0}}$ the restriction of $\nu_{\mathcal{X}^\zeta}$ to $\mathcal{X}^{\zeta,0}$.

A virtual version of localization in K-theory for stacks is proved in Halpern-Leistner [28, Theorem 5.7]. A special case is the following: Assume that a torus T acts on $\mathcal{X}$ (e.g. the action of $\mathbb{C}^\times$ on a master space for wall-crossing) and let $\mathcal{X}^T$ denote the fixed locus. The virtual tangent space restricted to $\mathcal{X}^T$ decomposes into

$$\mathrm{Def} - \mathrm{Obs}\,|_{\mathcal{X}^T} = \mathrm{Def}^{\mathrm{fix}} - \mathrm{Obs}^{\mathrm{fix}} + \mathrm{Def}^{\mathrm{mov}} - \mathrm{Obs}^{\mathrm{mov}}$$

its fixed and moving parts consisting of T-modules with trivial and nontrivial weights. The K-class $\nu_{\mathcal{X}^T} = (\mathrm{Def}^{\mathrm{mov}} - \mathrm{Obs}^{\mathrm{mov}})$ is the virtual *normal bundle*. In the case $T = \mathbb{C}^\times$ this class splits into classes $\nu_{\mathcal{X}^T,\pm}$ corresponding to negative and positive weights and the inverted Euler class of $\nu_{\mathcal{X}^T}$ maybe defined as the tensor product

$$\operatorname{Eul}(\nu_{\mathcal{X}^T})^{-1} = \operatorname{Eul}(\nu_{\mathcal{X}^T,+})^{-1} \operatorname{Eul}(\nu_{\mathcal{X}^T,-})^{-1}$$

where as in [28, Footnote 24]

$$\begin{aligned} \operatorname{Eul}(\nu_{\mathcal{X}^T,-})^{-1} &= \operatorname{Sym}(\nu_{\mathcal{X}^T,-}) \\ \operatorname{Eul}(\nu_{\mathcal{X}^T,+})^{-1} &= \operatorname{Sym}(\nu^\vee_{\mathcal{X}^T,+}) \otimes \det(\nu_{\mathcal{X}^T,+})[-\operatorname{rank}(\nu_{\mathcal{X}^T,+})] \end{aligned}$$

is an infinite sum of bundles such that each weight component is finite; this suffices for the finiteness of the formulas below. The residue of such classes is defined as before in (1.45), by taking the difference in expansion as formal power series in z and z^{-1}.

Theorem 1.15. (Virtual localisation) *[28, 5.6,5.7] Suppose that $\mathcal{X}$ is the a proper global quotient of a quasiprojective scheme by a reductive group equipped with an equivariant $T = \mathbb{C}^\times$-action, and E is a T-equivariant sheaf on $\mathcal{X}$. Then the Euler characteristic of E is computed by*

$$\begin{aligned} \chi^{\text{vir}}(\mathcal{X}, E) &= \chi^{\text{vir}}(\mathcal{X}^T, E \otimes \operatorname{Eul}_T(\nu_{\mathcal{X}_T})^{-1}) \\ &= \sum_{F \subset \mathcal{X}^T} \chi^{\text{vir}}(F, E \otimes \operatorname{Eul}_T(\nu_F)^{-1}) \end{aligned}$$

for the inclusion $\iota : \mathcal{X}^T \to \mathcal{X}$. Here the sum runs over all fixed components F of $\mathcal{X}^T$ and the Euler class $\operatorname{Eul}_T(\nu_{\mathcal{X}_T})$ *are defined as in 1.6.2, and it is invertible in localised equivariant K-theory.*

Proof This is stated for quasi-smooth schemes Y in [28, Theorem 5.7]. The statement for quasi-smooth global quotient stacks $\mathcal{X} = Y/G$ by reductive groups G follows from the identification of the Euler characteristic $\chi(\mathcal{X}, E/G)$ on $\mathcal{X}$ with the invariant part of the Euler characteristic $\chi(Y, E)^G$ upstairs, where the Bialynicki-Birula decomposition satisfies the requirements of the stratification by [29, Section 1.4.1]. □

The formula (1.52) implies a formula for Euler characteristics as a sum over fixed point components, in which the fixed point contributions are of the following form. Let $\operatorname{Eul}_{\mathbb{C}^\times_\zeta}(\nu_{\mathcal{X}^{\zeta,0}})$ be the equivariant Euler class of the normal bundle in $K^0_{\mathbb{C}^\times_\zeta}(\mathcal{X}^{\zeta,0})$, Let $\tau_{\mathcal{X},\zeta,0}$ denote the equivariant virtual Euler characteristic twisted by $\operatorname{Eul}^{-1}_{\mathbb{C}^\times_\zeta}(\nu_{\mathcal{X}^{\zeta,0}})$ combined with the residue

$$\tau_{\mathcal{X},\zeta,0} : K^0_{\mathbb{C}^\times_\zeta}(\mathcal{X}^{\zeta,0}) \to \mathbb{Z}; \quad \sigma \mapsto \chi^{\mathrm{vir}}_{\mathbb{C}^\times_\zeta}\left(\mathcal{X}^{\zeta,0}; \mathrm{Resid}\,\frac{\sigma}{\mathrm{Eul}_{\mathbb{C}^\times_\zeta}(\nu_{\mathcal{X}^{\zeta,0}})}\right).$$

Let $\tau_{\mathcal{X}/\!\!/_\pm G}$ denote the virtual Euler characteristic

$$\tau_{\mathcal{X}/\!\!/_\pm G} : K(\mathcal{X}/\!\!/_\pm G) \to \mathbb{Z}, \quad \alpha \mapsto \chi^{\mathrm{vir}}(\mathcal{X}/\!\!/_\pm G; \alpha),$$

on the quotients as before. We have the following virtual Kalkman-Metzler wall-crossing formula:

Theorem 1.16. *Let $\mathcal{X}$ be a proper Deligne-Mumford global-quotient G-stack equipped with a G-equivariant perfect obstruction theory which admits a global resolution by vector bundles. Let $\mathcal{L}_\pm \to \mathcal{X}$ be G-line bundles whose associated wall-crossing is simple. Then*

$$\tau_{\mathcal{X}/\!\!/_+ G}\ \kappa^G_{\mathcal{X},+} - \tau_{\mathcal{X}/\!\!/_- G}\ \kappa^G_{\mathcal{X},-} = \tau_{\mathcal{X},\zeta,0} \tag{1.52}$$

Example 1.17. (Wall-crossing over a nodal fixed point) Suppose that $\mathcal{X} = \mathbb{P}^1 \cup \mathbb{P}^1$ is a nodal projective line with a single node $x_0 \in \mathcal{X}$, equipped with the standard $G = \mathbb{C}^\times$ action on each component, so that the weights of the action on the tangent spaces at the node are ± 1. Equip $\mathcal{X}$ with a polarization so that the weights are ± 1 at the smooth fixed points, and 0 at the nodal point $x_0 = \mathcal{X}^{\zeta,0}$. Then $\mathcal{X}/\!\!/_t G$ is a point for $t \in (-1,1)$, and is singular for $t = 0$. Since $\mathcal{X}$ is a complete intersection, $\mathcal{X}$ it has a perfect obstruction theory [4, Example before Remark 5.4] and the virtual wall-crossing formula of Theorem 1.16 applies. We examine the wall-crossing for the class $\mathcal{O}_\mathcal{X}$ at the singular value $t = 0$. The virtual normal complex at the nodal point is the quotient of $\mathbb{C}_1 \oplus \mathbb{C}_{-1}$, the sum of one-dimensional representations with weights $1, -1$, modulo their tensor product $\mathbb{C}_1 \otimes \mathbb{C}_{-1}$, which has weight $1 - 1 = 0$. Hence the normal complex has inverted Euler class

$$\mathrm{Eul}_{\mathbb{C}^\times}(\nu_{\mathcal{X}^{\zeta,0}})^{-1} = \frac{1}{(1-z^{-1})(z-1)}$$

whose residue is zero by Example 1.8. The Euler characteristics on the left and right hand sides are 1 while the wall-crossing term is

$$\begin{aligned} 1 - 1 &= \tau_{\mathcal{X}/\!\!/_+ G}\ \kappa^G_{\mathcal{X},+} - \tau_{\mathcal{X}/\!\!/_- G}\ \kappa^G_{\mathcal{X},-} \\ &= \tau_{\mathcal{X},\zeta,0} = \chi^{\mathrm{vir}}\left(x_0, \mathrm{Eul}_{\mathbb{C}^\times}(\nu_{\mathcal{X}^{\zeta,0}})^{-1}\right) = 0 \end{aligned}$$

compatible with the wall-crossing formula.

The proof of Theorem 1.16 uses the construction of a master space for this set up. However, the same construction as before with small modifications applies.

Lemma 1.18. *There exists a proper Deligne-Mumford $\mathbb{C}^\times$-stack $\tilde{\mathcal{X}}$ equipped with a line bundle ample for the coarse moduli space whose git quotients $\tilde{\mathcal{X}}/\!\!/_t\mathbb{C}^\times$ are isomorphic to those $\mathcal{X}/\!\!/_t G$ of $\mathcal{X}$ by the action of G with respect to the polarization $\mathcal{L}_t$ and whose fixed point set $\tilde{\mathcal{X}}^{\mathbb{C}^\times}$ is given by the union*

$$\tilde{\mathcal{X}}^{\mathbb{C}^\times} = (\mathcal{X}/\!\!/_- G) \cup (\mathcal{X}/\!\!/_+ G) \cup \iota_\zeta(\mathcal{X}^\zeta/\!\!/_0(G_\zeta/\mathbb{C}^\times_\zeta)) \tag{1.53}$$

where ι_ζ is the natural map to $\tilde{\mathcal{X}}$ as before. Furthermore, $\tilde{\mathcal{X}}$ has a perfect obstruction theory admitting a global resolution by vector bundles with the property that the virtual normal complex of $\mathcal{X}^\zeta/\!\!/_0(G_\zeta/\mathbb{C}^\times_\zeta)$ is isomorphic to the image of $\nu_{\mathcal{X}^\zeta}/(\mathfrak{g}/\mathbb{C}\zeta)$ under the quotient map $\mathcal{X}^\zeta \to \mathcal{X}^\zeta/\!\!/(G_\zeta/\mathbb{C}^\times_\zeta)$, by an isomorphism that intertwines the action of $\mathbb{C}^\times_\zeta$ on $(\nu_{\mathcal{X}^\zeta}/(\mathfrak{g}/\mathbb{C}\zeta))/\!\!/(G_\zeta/\mathbb{C}^\times_\zeta)$ with the action of $\mathbb{C}^\times$ on $\nu_{\tilde{\mathcal{X}}^{\mathbb{C}^\times}}$.

Proof The construction of the master space is the same as in the smooth case, that is, the master space is the stack-theoretic quotient $\tilde{\mathcal{X}} = \mathbb{P}(\mathcal{L}_- \oplus \mathcal{L}_+)/\!\!/G$. The action of G on the semistable locus in $\mathbb{P}(\mathcal{L}_- \oplus \mathcal{L}_+)$ is locally free by assumption. It follows that $\tilde{\mathcal{X}}$ is a proper Deligne-Mumford stack, and by Lemma 1.14 has a perfect obstruction theory induced from the natural obstruction theory on $\mathbb{P}(\mathcal{L}_- \oplus \mathcal{L}_+)$ given by considering it as a bundle over $\mathcal{X}$. The quotient $\tilde{\mathcal{X}}$ is such that $\tilde{\mathcal{X}}/\!\!/_t\mathbb{C}^\times$ is isomorphic to $\mathcal{X}/\!\!/_t G$ for $t \neq 0$. In fact it contains the quotients of $\mathbb{P}(\mathcal{L}_\pm) \cong \mathcal{X}$ with respect to the polarizations $\mathcal{L}_\pm$, that is, $\mathcal{X}/\!\!/_\pm G$.

The same argument as before describes the fixed point loci: they correspond to fixed point loci in $\mathbb{P}(\mathcal{L}_- \oplus \mathcal{L}_+)$ for one-parameter subgroups of $\mathbb{C}^\times \times G$. Given such a locus $\mathbb{P}(\mathcal{L}_- \oplus \mathcal{L}_+)^{\xi+\zeta}$, the pull-back of the virtual normal complex is by definition the moving part of $\operatorname{Cone}(\tilde{a}^\vee_{\mathbb{P}(\mathcal{L}_- \oplus \mathcal{L}_+)})$, where

$$\tilde{a}^\vee_{\mathbb{P}(\mathcal{L}_- \oplus \mathcal{L}_+)} : E_{\mathbb{P}(\mathcal{L}_- \oplus \mathcal{L}_+)} \to \underline{\mathfrak{g}}^\vee_\zeta$$

is the lift of the infinitesimal action of G_ζ. Consider the fibration $\pi : \mathbb{P}(\mathcal{L}_- \oplus \mathcal{L}_+) \to \mathcal{X}$. By definition $E_{\mathbb{P}(\mathcal{L}_- \oplus \mathcal{L}_+)}$ fits into an exact triangle

$$E_{\mathcal{X}} \to E_{\mathbb{P}(\mathcal{L}_- \oplus \mathcal{L}_+)} \to L_\pi \to E_{\mathcal{X}}[1].$$

Over the complement $\mathbb{P}(\mathcal{L}_- \oplus \mathcal{L}_+) \setminus (D_0 \cup D_\infty) \subset \mathbb{P}(\mathcal{L}_- \oplus \mathcal{L}_+)$ of the sections at zero and infinity we may identify $L_\pi \cong \underline{\mathbb{C}}$ using the $\mathbb{C}^\times$-action on the fibers, by the assumption on the weights of the $\mathbb{C}_\zeta$ action on the fiber. Thus the projection to $\mathcal{X}$ identifies

$$\operatorname{Cone}(\tilde{a}^\vee_{\mathbb{P}(\mathcal{L}_-\oplus\mathcal{L}_+)}|_{\mathbb{P}(\mathcal{L}_-\oplus\mathcal{L}_+)^{\xi+\zeta}}) \to \pi^* \operatorname{Cone}(\tilde{a}^\vee_{\mathcal{X}^\zeta})$$

where

$$\tilde{a}^\vee_{\mathcal{X}^\zeta} : E_{\mathcal{X}}|\mathcal{X}^\zeta \to (\underline{\mathfrak{g}_\zeta/\mathbb{C}\zeta})^\vee$$

is the lift of the infinitesimal action of $\mathfrak{g}_\zeta/\mathbb{C}\zeta$. Now the virtual normal complex is by definition the $\mathbb{C}^\times$-moving part of the perfect obstruction theory; the Lemma follows. □

Proof of Theorem 1.16 For any equivariant class $\alpha \in K^0_G(\mathcal{X})$, its pullback to $\mathbb{P}(\mathcal{L}_- \oplus \mathcal{L}_+)$ descends to a class $\tilde{\alpha} \in K^0_G(\tilde{\mathcal{X}})$ whose restriction to $\mathcal{X}/\!\!/_\pm G$ is $\kappa^G_{X,\pm}(\alpha)$, and whose pullback under $X^{\zeta,0} \to \tilde{X}^{\mathbb{C}^\times}$ is $\iota_{X^{\zeta,0}}\alpha$. By virtual localisation (1.52) and the description of the fixed set, the normal bundles in Lemma 1.18

$$0 = \chi^{\mathrm{vir}}(\tilde{\mathcal{X}}; \operatorname{Resid}\delta(\alpha)) =$$
$$\chi^{\mathrm{vir}}\left(\mathcal{X}/\!\!/_- G; \operatorname{Resid}\frac{\kappa^G_{\mathcal{X},-}(\alpha)}{\operatorname{Eul}_{\mathbb{C}^\times}(\nu_-)}\right) + \chi^{\mathrm{vir}}\left(\mathcal{X}/\!\!/_+ G; \operatorname{Resid}\frac{\kappa^G_{\mathcal{X},+}(\alpha)}{\operatorname{Eul}_{\mathbb{C}^\times}(\nu_+)}\right) +$$
$$\chi^{\mathrm{vir}}\left(\mathcal{X}^{\zeta,0}/(G_\zeta/\mathbb{C}^\times); \operatorname{Resid}\frac{\iota^*_{\mathcal{X}^{\zeta,0}}\alpha}{\operatorname{Eul}_{\mathbb{C}^\times_\zeta}(\nu_{\mathcal{X}^{\zeta,0}})}\right). \quad (1.54)$$

As in (1.7) we have

$$\operatorname{Resid}\chi(\mathcal{X}/\!\!/_\mp G, \frac{\kappa^G_{\mathcal{X},\mp}(\alpha)}{\operatorname{Eul}_{\mathbb{C}^\times}(\nu_\mp)} = \pm\chi(\mathcal{X}/\!\!/_\mp G, \kappa^G_{\mathcal{X},\mp}(\alpha).)$$

Indeed, by definition the normal bundle ν_F at $\mathcal{X}/\!\!/_\mp G$ has virtual dimension one with positive resp. negative weight, and the inverted Euler class $\operatorname{Eul}(\nu)^{-1}$ is the symmetric product $\operatorname{Sym}(\nu)$ for the Bialynicki-Birula decomposition for positive weight, or $\operatorname{Sym}(\nu^\vee)\det(\nu^\vee)$ for the decomposition with negative weight. [4] For $\chi(\mathcal{X}/\!\!/_+ G)$ the invariant part of the first is the trivial line bundle, while for the second the invariant part vanishes since the weights are positive so the difference in (1.45) of $\frac{\kappa^G_{\mathcal{X},\mp}(\alpha)}{\operatorname{Eul}_{\mathbb{C}^\times}(\nu_\mp)}$ is

[4] That these classes both agree with the inverted Euler class in localized K-theory follows by inspection from Riemann-Roch [53]. In fact, the agreement with the inverted Euler class is not necessary and one may take the difference in the Halpern-Leistner version of virtual localization for the Bialynicki-Birula decomposition and its opposite as the definition of residue.

$\kappa^G_{\mathcal{X},\mp}(\alpha)$. For $\chi(\mathcal{X}/\!\!/_-G)$, the weight of ν is negative and $\mathrm{Sym}(\nu)$ appears in the Bialynicki-Birula decomposition for negative weight. Thus the residue is $\kappa^G_{\mathcal{X},\mp}(\alpha)$, which completes the proof. □

1.7 Wall-crossing in quantum K-theory

The main result in this section, Theorem 1.5 below, relates the quantum K-theory pairings on both sides of a wall-crossing. Let $X/\!\!/_\pm G$ denote the associated quotient stacks $[X^{\mathrm{ss}}(\pm)/G]$ at times $t = \pm 1$ and let

$$\kappa^G_{X,\pm} : QK^0_G(X, \mathcal{L}_\pm) \to QK^0_{\mathbb{C}^\times}(X/\!\!/_\pm G)$$

denote the associated quantum Kirwan maps. Consider the graph potentials

$$\tau_{X/\!\!/_\pm G} : QK^0_{\mathbb{C}^\times}(X/\!\!/_\pm G) \to \Lambda^G_{X,\mathcal{L}_\pm}.$$

Denote by

$$QK^{0,\mathrm{fin}}_G(X) \subset QK^0_G(X, \mathcal{L}_-) \cap QK^0_G(X, \mathcal{L}_+)$$

the subset of classes of sums lying in both completions; for example, any finite sum lies in this intersection. We want to establish a formula for the difference

$$\tau_{X/\!\!/_+ G}\ \kappa^G_{X,+} - \tau_{X/\!\!/_- G}\ \kappa^G_{X,-} : QK^{0,\mathrm{fin}}_G(X) \to \Lambda^G_X$$

as a sum of fixed point contributions given by gauged Gromov-Witten invariants with smaller structure groups $G_\lambda/\mathbb{C}^\times_\lambda$.

1.7.1 Master space for gauged maps and wall-crossing

We recall from [24, Proposition 3.1] the construction of master space whose quotients are the moduli stacks of Mundet stable gauged maps.

Proposition 1.1 (Existence of a master space)**.** *Under assumptions of simple wall-crossing 1.13, for each equivariant degree $d \in H^G_2(X)$ there exists a proper Deligne-Mumford $\mathbb{C}^\times$-stack $\overline{\mathcal{M}}^G_n(C, X, \mathcal{L}_-, \mathcal{L}_+, d)$ with the following properties:*

a. The stack $\overline{\mathcal{M}}^G_n(C, X, \mathcal{L}_-, \mathcal{L}_+, d)$ has a $\mathbb{C}^\times$-equivariant perfect obstruction theory, relative over the moduli stack $\overline{\mathfrak{M}}_n(C)$ of prestable maps to C of class $[C]$.

b. *the git quotients of* $\overline{\mathcal{M}}_n^G(C, X, \mathcal{L}_-, \mathcal{L}_+)$ *by the* $\mathbb{C}^\times$*-action are isomorphic to the moduli stacks* $\overline{\mathcal{M}}_n^G(C, X, \mathcal{L}_-^{(1-t)/2} \otimes \mathcal{L}_+^{(1+t)/2})$ *for parameter* $t \in (-1, 1)$*;*

c. *the* $\mathbb{C}^\times$*-fixed substack includes* $\overline{\mathcal{M}}_n^G(C, X, \mathcal{L}_-, d)$ *and* $\overline{\mathcal{M}}_n^G(C, X, \mathcal{L}_+, d)$*; the other fixed point components are isomorphic to substacks of* reducible *elements of* $\overline{\mathcal{M}}_n^G(C, X, \mathcal{L}_-^{(1-t)/2} \otimes \mathcal{L}_+^{(1+t)/2})$ *for* $t \in (-1, 1)$ *consisting of gauged maps with* $\mathbb{C}^\times$*-automorphisms.*

d. $\overline{\mathcal{M}}_n^G(C, X, \mathcal{L}_-, \mathcal{L}_+, d)$ *admits an embedding in a non-singular Deligne-Mumford stack.*

For any fixed point component $F \subset \overline{\mathcal{M}}_n^G(C, X, \mathcal{L}_-, \mathcal{L}_+, d)$ denote by ν_F the normal complex, that is, the $\mathbb{C}^\times$-moving part of the perfect obstruction theory of Proposition 1.1 part (a). The following is a direct application of virtual wall crossing applied to the master space. Denote the evaluation map

$$\mathrm{ev} : \overline{\mathcal{M}}_n^G(C, X, \mathcal{L}_-, \mathcal{L}_+, d) \to (X/G)^n.$$

Proposition 1.2. *For any class* $\alpha \in K_G^0(X)^n$

$$\chi^{\mathrm{vir}}\left(\overline{\mathcal{M}}_n^G(C, X, \mathcal{L}_+, d); \mathrm{ev}^* \alpha\right) - \chi^{\mathrm{vir}}\left(\overline{\mathcal{M}}_n^G(C, X, \mathcal{L}_-, d); \mathrm{ev}^* \alpha\right)$$
$$= \sum_F \chi^{\mathrm{vir}}\left(F; \mathrm{Resid}\, \frac{\iota_F^* \mathrm{ev}^* \alpha}{\mathrm{Eul}_{\mathbb{C}^\times}(\nu_F)}\right) \quad (1.55)$$

where F *ranges over the fixed point components of* $\mathbb{C}^\times$ *on the moduli* $\overline{\mathcal{M}}_n^G(C, X, \mathcal{L}_-, \mathcal{L}_+, d)$ *that are not equal to* $\overline{\mathcal{M}}_n^G(C, X, \mathcal{L}_\pm, d)$*.*

1.7.2 Reducible gauged maps

We analyze further the fixed point contributions in (1.55), which come from reducible gauged maps. Let X be a smooth projective G-variety. Let $Z \subset G$ a central subgroup. For any principal G-bundle $P \to C$, the right action of Z on P induces an action on the associated bundle $P(X)$, and so on the space of sections of $P(X)$. The fixed point set of Z on $P(X)$ is $P(X)^Z = P(X^Z)$, the associated bundle with fiber the fixed point set $X^Z \subset X$. The action of Z on the space of sections of $P(X)$ preserves Mundet semistability, since the parabolic reductions are invariant under the action and the Mundet weights are preserved. This induces an action of Z on $\overline{\mathcal{M}}_n^G(C, X)$.

Proposition 1.3. *Let $Z \subset G$ be a central subgroup. The fixed point locus for the action of Z on $\overline{\mathcal{M}}_n^G(C, X)$ is the substack whose objects are tuples*

$$(p : P \to C, u : \hat{C} \to P(X), \underline{z})$$

such that

a. u takes values in $P(X^Z)$ on the principal component C_0;
b. for any bubble component $C_i \subset \hat{C}$ mapping to a point in C, u maps C_i to a one-dimensional orbit of Z on $P(X)$; and
c. any node or marking of $\hat{C}$ maps to the fixed point set $P(X^Z)$.

We introduce notation for the various substacks of reducible maps. Let $\zeta \in \mathfrak{g}$ generate a one-parameter subgroup $\mathbb{C}_\zeta^\times \subset G$. Recall that G_ζ denotes the centralizer in G and so it contains $\mathbb{C}_\zeta^\times$ as a central subgroup. Let

$$\overline{\mathcal{M}}_n^{G_\zeta}(C, X, \mathcal{L}_t, \zeta, d)$$

denote the stack of $\mathcal{L}_t$-Mundet-semistable morphisms from C to X/G_ζ that are $\mathbb{C}_\zeta^\times$-fixed and take values in X^ζ on the principal component and in X/G_ζ on the bubbles. Because these gauged maps correspond to the smaller group G_ζ, we call them as *reducible* gauged maps.

Each component of reducibles has an equivariant perfect obstruction theory. Recall that the obstruction theory for the moduli of gauged maps $\overline{\mathcal{M}}_n^G(C, X, d)$ is given by the complex $Rp_* e^* T_{X/G}^\vee$, which is relative with respect to $\overline{\mathfrak{M}}_n(C)$. The moduli $\overline{\mathcal{M}}_n^{G_\zeta}(C, X, \mathcal{L}_0, \zeta, d)$ is an Artin stack, and if every automorphism group is finite modulo $\mathbb{C}_\zeta^\times$, it is a proper Deligne-Mumford stack with a $\mathbb{C}^\times$-equivariant relatively perfect obstruction theory over $\overline{\mathfrak{M}}_n(C)$. This follows from the fact that the relative perfect obstruction theory for $\overline{\mathcal{M}}_n^{G_\zeta}(C, X, \mathcal{L}_t, \zeta, d)$ is pulled back from that on the $\mathbb{C}^\times$-fixed point set in the master space $\overline{\mathcal{M}}_n^G(C, X, \mathcal{L}_-, \mathcal{L}_+, d)^{\mathbb{C}^\times}$. This coincides with the $\mathbb{C}_\zeta^\times$-equivariant obstruction theory on the stack $\overline{\mathcal{M}}_n^{G_\zeta}(C, X, \mathcal{L}_0, \zeta, d)$ whose relative part is the *cone* with target the trivial bundle $\underline{\mathbb{C}}_\zeta^\vee$ with fiber the Lie algebra $\mathbb{C}_\zeta$ of $\mathbb{C}_\zeta^\times$

$$\operatorname{Cone}(Rp_* e^* T_{X/G}^\vee \to \mathbb{C}_\zeta^\vee)$$

given by the infinitesimal action of $\mathbb{C}_\zeta^\times$. The complex $Rp_* e^* T_{X/G}^\vee$ itself is not perfect because of the $\mathbb{C}_\zeta^\times$-automorphisms; taking the cone has the effect of cancelling this additional automorphisms. Denote by ν_0 the virtual (co)normal complex of the morphism

$$\overline{\mathcal{M}}_n^{G_\zeta}(C, X, \mathcal{L}_t, \zeta) \to \overline{\mathcal{M}}_n^G(C, X, \mathcal{L}_-, \mathcal{L}_+),$$

and as before, denote by

$$\mathrm{Eul}_{\mathbb{C}^\times}(\nu_0) \in K(\overline{\mathcal{M}}_n^G(C, X, \mathcal{L}_0, \zeta))$$

its Euler class.

Virtual Euler characteristics over the reducible gauged maps gives rise to the fixed point contributions in the wall-crossing formula: these are *ζ-fixed K-theoretic gauged Gromov-Witten invariants.* The ζ-fixed K-theoretic gauged Gromov-Witten invariants that appear in the wall-crossing formula involve further twists by the inverse of Euler classes of the virtual normal complex $\mathrm{Eul}_{\mathbb{C}^\times}(\nu_0)^{-1}$. Before we made this explicit, we need to allow a slightly larger coefficient ring. Denote by

$$\tilde{\Lambda}_X^G := \mathrm{Map}(H_2^G(X, \mathbb{Z}), \mathbb{Q})$$

the space of $\mathbb{Q}$-valued functions on $H_2^G(X, \mathbb{Z})$ (cf. Equation (1.10)). Note that $\tilde{\Lambda}_X^G$ has no ring structure extending that on Λ_X^G. The space $\tilde{\Lambda}_X^G$ can be viewed as the space of distributions in the quantum parameter q, and we use it as a master space interpolating Novikov parameters for the quotients with respect to $\mathcal{L}_t$ as t varies. Let $\tilde{\Lambda}_{X,\mathrm{fin}}^G$ denote the subspace of finite sums.

Definition 1.4. Let $X, G, \mathcal{L}_\pm, \zeta \in \mathfrak{g}$ as above, such that there is simple wall-crossing at the unique singular time $t = 0$ and such that $X^{\zeta,0}$ is non-empty. The *fixed point potential* associated to this data is the map

$$\begin{aligned} &\tau_{X,\zeta,0} : QK_G^{0,\mathrm{fin}}(X) \to \tilde{\Lambda}_X^G \\ &\alpha \mapsto \sum_{d \in H_2^G(X,\mathbb{Z})} \sum_{n \geq 0} \chi\left(\overline{\mathcal{M}}_n^G(C, X, \mathcal{L}_0, \zeta, d); \mathrm{Resid}\, \frac{\mathrm{ev}^*(\alpha, \ldots, \alpha)}{\mathrm{Eul}_{\mathbb{C}^\times}(\nu_0)}\right) \frac{q^d}{n!}, \end{aligned} \tag{1.56}$$

for $\alpha \in K_G^0(X)$, extended to $QK_G^{0,\mathrm{fin}}(X)$ by linearity. Here we omit the restriction map $K_G^{0,\mathrm{fin}}(X) \to K_{G_\zeta}^0(X)$ to simplify notation.

Remark. The fixed point potential $\tau_{X,\zeta,0}$ takes values in $\tilde{\Lambda}_X^G$ rather than in $\Lambda_X^G(\mathcal{L}_0)$ because Gromov compactness fails for gauged maps in the case that a central subgroup $\mathbb{C}_\zeta^\times$ acts trivially. Indeed, in this case, the energy $\langle d, c_1(L) \rangle$ of a gauged map of class d does not determine the isomorphism class of the bundle, since twisting by a character of $\mathbb{C}_\zeta^\times$ does not change the energy.

1.7.3 The wall-crossing formula

We may now prove the quantum version of the Kalkman-Metzler formula.

Theorem 1.5 (Wall-crossing for gauged potentials)**.** *Let X be a smooth G-variety. Suppose that $\mathcal{L}_\pm \to X$ are polarizations such that there is simple wall-crossing. Then the gauged Gromov-Witten potentials are related by*

$$\tau^G_{X,+} - \tau^G_{X,-} = \tau_{X,\zeta,0} \tag{1.57}$$

where the same Mundet semistability parameter should be used to define the potentials on both sides of the equation.

Proof of Theorem 1.5 The statement follows from virtual Kalkman-Metzler formula 1.16 applied to the master space $\overline{\mathcal{M}}^G_n(C, X, \mathcal{L}_-, \mathcal{L}_+)$ and the identification of fixed point contributions as reducible gauged maps described in sections 1.7.1,1.7.2. □

Combining Theorem 1.5 with the adiabatic limit (1.30) yields:

Theorem 1.6 (Quantum Kalkman-Metzler formula)**.** *Suppose that X is equipped with polarizations $\mathcal{L}_\pm$ so that the wall crossing is simple (the only singular polarisation is $\mathcal{L}_0$). Then the Gromov-Witten invariants of $X/\!\!/_\pm G$ are related by twisted gauged Gromov-Witten invariants with smaller subgroup $G_\zeta \subset G$*

$$\tau_{X/\!\!/_+ G}\ \kappa^G_{X,+} - \tau_{X/\!\!/_- G}\ \kappa^G_{X,-} = \lim_{\rho \to 0} \tau_{X,\zeta,0}. \tag{1.58}$$

In other words, failure of the following square

$$\begin{array}{ccccc} QK^0_G(X, \mathcal{L}_-) & \longleftarrow & QK^{0,\mathrm{fin}}_G(X) & \longrightarrow & QK^0_G(X, \mathcal{L}_+) \\ \kappa^G_{X,-}\downarrow & & & & \downarrow\kappa^G_{X,+} \\ QK(X/\!\!/_- G) & & & & QK(X/\!\!/_+ G) \\ \tau_{X/\!\!/_- G}\downarrow & & & & \downarrow\tau_{X/\!\!/_+ G} \\ \Lambda^G_{X,\mathcal{L}_-} & \longrightarrow & \tilde{\Lambda}^G_X & \longleftarrow & \Lambda^G_{X,\mathcal{L}_+} \end{array} \tag{1.59}$$

to commute is measured by an explicit sum of wall-crossing terms given by the contribution of the fixed gauged potential. We remark that if the wall-crossing is not simple, the contributions on the right-hand side of the wall-crossing formula might come from several singular values $t \in (-1, 1)$ as the polarisations $\mathcal{L}_t$ varies; however a simple modification of the argument above proves it as well.

1.8 Crepant wall-crossing

In this section we use the quantum Kalkman-Metzler formula to prove a version of the crepant transformation conjecture for K-theoretic Gromov-Witten invariants, under some rather strong assumptions on the weights involved in the wall-crossing. We assume the following symmetry condition on the weights involved in the wall-crossing. Suppose we have a birational transformation of git type

$$\phi : X/\!\!/_{-}G \dashrightarrow X/\!\!/_{+}G$$

defined by a simple wall-crossing induced by two polarisations $\mathcal{L}_+, \mathcal{L}_-$ as in the previous sections. Suppose that for $\zeta \in \mathfrak{g}$, the fixed point component $X^{\zeta,0}$ is the one contributing to the wall-crossing term, and let $\nu_{X^{\zeta,0}} \to X^{\zeta,0}$ be its normal bundle in X. Let

$$\nu_{X^{\zeta,0}} = \bigoplus_j \nu_j$$

be the isotypical decomposition so that $\mathbb{C}^\times_\zeta$ acts on ν_j with weight μ_j. Note as before that all the $\mu_j \neq 0$. Let $r_j = \operatorname{rank} \nu_j$.

Definition 1.1. The birational transformation $\phi : X/\!\!/_{-}G \to X/\!\!/_{+}G$ is *simply crepant* if the set of weights μ_i of the normal bundle of $X^{\zeta,0}$ in X is invariant under multiplication by -1, that is, whenever μ_j is a weight with multiplicity r_j then so is $-\mu_j$ with the same multiplicity.

If the wall-crossing is not simple, it is simply crepant if the condition in 1.1 holds for all fixed point components contributing to the wall-crossing terms.

We show invariance for the gauged potentials under crepant wall-crossing if a certain symmetrised version of the Euler characteristics are used. Let T be a torus acting on a Deligne-Mumford stack $\mathcal{X}$, endowed with a perfect obstruction theory. Suppose $x \in \mathcal{X}^T$ is an isolated fixed point. Locally the virtual tangent space

$$T_x^{\text{vir}} := \operatorname{Def}_x - \operatorname{Obs}_x .$$

can be decomposed as

$$T_x^{\text{vir}} = \bigoplus_i \mathbb{C}_{a_i} - \bigoplus_j \mathbb{C}_{b_j}$$

where a_i, b_j are the weights of the deformation and obstruction spaces respectively. Define

$$\widehat{\mathcal{O}}_{\mathcal{X}}^{\text{vir}} := \mathcal{O}_{\mathcal{X},x}^{\text{vir}} \otimes (K_{\mathcal{X}}^{\text{vir}})^{1/2}, \quad K_{\mathcal{X}}^{\text{vir}} := (\det T_{\mathcal{X}}^{\text{vir}})^{-1}$$

where a square root can be defined in rational K-theory via the Chern character [14]. The resulting K-theoretic Gromov-Witten invariants obtained by replacing the virtual structure sheaf by this shift quantum K-theory at level $-1/2$ in the language of Ruan-Zhang [47]. At an isolated fixed point x we have

$$\widehat{\mathcal{O}}_x^{\mathrm{vir}} := \frac{\prod_j (b_j^{1/2} - b_j^{-1/2})}{\prod_i (a_i^{1/2} - a_i^{-1/2})}$$

where $a_i^{1/2}, b_j^{1/2}$ are formal, since they represent weights only after passing to a cover $\hat{T} \to T$.

The virtual localization formula may be re-written in terms of the shifted structure sheaves. Let $\hat{A}(\cdot)$ be the denominator of the A-hat genus, mapping $R(T)$ to the space of functions defined on some cover

$$\widehat{A}(a_1 + a_2) = \widehat{A}(a_1)\widehat{A}(a_2); \quad \widehat{A}(a) = \frac{1}{a^{1/2} - a^{-1/2}}.$$

where a is a weight (representation) of T. Define an extension to $\mathcal{F} \in K_T^0(X)$ by

$$\widehat{A}(\mathcal{F}) = \prod_j \widehat{A}(y_j),$$

where the product runs over the equivariant Chern roots $y_j \in K_T^0(X)$ of $\mathcal{F}$. Then localization (1.52) becomes

$$\widehat{\mathcal{O}}_{\mathcal{X}}^{\mathrm{vir}} = \iota_*(\widehat{A}(T_{\mathcal{X}^T}^{\mathrm{vir}})\ \widehat{\mathcal{O}}_{\mathcal{X}^T}^{\mathrm{vir}}). \tag{1.60}$$

This can be made more explicit as follows. For each component $F \subset \mathcal{X}^T$ we have a decomposition

$$T_{\mathcal{X}}^{\mathrm{vir}}|_F = T_F^{\mathrm{vir}} + \nu_F$$

and therefore

$$(K_{\mathcal{X}}|_F)^{1/2} = K_F^{1/2} (\det \nu_F)^{-1/2}.$$

It follows that

$$\frac{\mathcal{O}_F \otimes (K_{\mathcal{X}}^{\mathrm{vir}}|_F)^{1/2}}{\mathrm{Eul}_T(\nu_F)} = \mathcal{O}_F \otimes (K_F^{\mathrm{vir}})^{1/2} \otimes \frac{(\det \nu_F)^{-1/2}}{\mathrm{Eul}_T(\nu_F)}. \tag{1.61}$$

By considering the decomposition

$$\nu_F = \bigoplus_i z^{\mu_i} \nu_{F,i}, \tag{1.62}$$

in isotypical components, we have

$$\mathrm{Eul}_T(\nu_F) = \prod_i \mathrm{Eul}_T(z^{\mu_i}\nu_{F,i}) = \prod_{i,j}(1 - z^{-\mu_i}x_{i,j}^{-1})$$

where $x_{i,j}$ are the Chern roots of $\nu_{F,i}$. Since $(\det \nu_F)^{1/2} = \prod_{i,j}(z^{\mu_i}x_{ij})^{1/2}$ we have

$$\mathrm{Eul}_T(\nu_F)^{-1}(\det \nu_F)^{-1/2} = \prod_{i,j} \widehat{A}(z^{\mu_i}x_{ij})^{-1} = \widehat{A}(\nu_F)^{-1}. \tag{1.63}$$

For our arguments below, we need to discuss the asymptotic behaviour of $\widehat{A}(\nu_F)$. Consider the decomposition of ν_F as in (1.62) and the Euler class expansion (1.43) for each of its isotypical components. Thus

$$\widehat{A}(\nu_F) = \frac{(\det \nu_F)^{-1/2}}{\prod_i(1 - z^{-\mu_i})^{r_i}(1 + N_i)}$$

with $N_i \in K(F) \otimes K_T^{\mathrm{loc}}(\mathrm{pt})$ as in (1.44). Therefore

$$\widehat{A}(\nu_F) = \prod_i \left(\frac{z^{-\mu_i/2}}{1 - z^{-\mu_i}}\right)^{r_i} \cdot O(z) = \prod_i \widehat{A}(\mathbb{C}_{\mu_i})^{r_i} \cdot O(z), \tag{1.64}$$

where $\mathbb{C}_{\mu_i}$ is the representation with weight μ_i and $O(z)$ is a term that converges to zero as $z^{\pm 1} \to 0$.

Symmetrised wall-crossing.

We can define symmetric versions of the gauged K-theoretic Gromov-Witten potentials previously studied by considering Euler characteristics with respect to $\widehat{\mathcal{O}}^{\mathrm{vir}}$. In the following, we add a hat to any expression whose definition now uses $\widehat{\mathcal{O}}^{\mathrm{vir}}$ rather than $\mathcal{O}^{\mathrm{vir}}$. The proof of the quantum Kalkman formula in Theorem 1.6 relied on virtual localisation. If instead we use localisation for the symmetrised virtual structure sheaf we obtain the following:

$$\widehat{\tau}_{X/\!\!/_+ G}\, \widehat{\kappa}^G_{X,+} - \widehat{\tau}_{X/\!\!/_- G}\, \widehat{\kappa}^G_{X,-} = \lim_{\rho \to 0} \widehat{\tau}_{X,\zeta,0}. \tag{1.65}$$

The symmetrised virtual structure sheaves satisfy good properties under the action of the Picard stack on the locus of reducible maps. Let

$$\mathrm{Pic}(C) := \mathrm{Hom}(C, B\mathbb{C}^\times)$$

denote the Picard stack of line bundles on C. The Lie algebra $\mathfrak{g}_\zeta$ has a distinguished factor generated by ζ, and using an invariant metric the weight lattice of $\mathfrak{g}_\zeta$ has a distinguished factor $\mathbb{Z}$ given by its intersection with the Lie algebra of $\mathbb{C}^\times_\zeta$. After passing to a finite cover, we may assume that $G_\zeta \cong (G_\zeta/\mathbb{C}^\times_\zeta) \times \mathbb{C}^\times_\zeta$. The Picard stack $\mathrm{Pic}(C)$ acts on the moduli stack of reducible gauged maps $\overline{\mathcal{M}}_n^{G_\zeta}(C, X, \mathcal{L}_0, \zeta)$ as follows. Recall that a reducible gauged map $(P, \hat{C}, u)$, where $P \to C$ is a G-bundle and $u : \hat{C} \to P(X)$ is ζ-fixed. The restriction of u to the principal component of C maps into the fixed point locus X^ζ. For Q an object of $\mathrm{Pic}(C)$ and $(P, \hat{C}, u)$ an object of $\overline{\mathcal{M}}_n^{G_\zeta}(C, X, \mathcal{L}_0, \zeta)$ define

$$Q(P, \hat{C}, u) := (P \times_{\mathbb{C}^\times_\zeta} Q, \hat{C}, v) \tag{1.66}$$

where the section v is defined as follows: We have an identification of bundles $(P \times_{\mathbb{C}^\times_\zeta} Q)(X^\zeta) \cong P(X^\zeta)$ since the action of $\mathbb{C}^\times_\zeta$ on X^ζ is trivial. Hence the principal component u_0, which is a section of $P(X^\zeta)$ induces the corresponding section v_0 of $(P \times_{\mathbb{C}^\times_\zeta} Q)(X^\zeta)$. Each bubble component of u maps into a fiber of $P(X)$, canonically identified with X up to the action of G_ζ. So u induces the corresponding bubble map of v into a fiber of $(P \times_{\mathbb{C}^\times_\zeta} Q)(X)$, well-defined up to isomorphism. Note that if the degree of $(P, \hat{C}, u)$ is d the degree of $Q(P, \hat{C}, u)$ is $d + c_1(Q)$.

The Picard action preserves semistable loci in the large area limit. Indeed, because the Mundet weights $\mu_M(\sigma, \lambda)$ approach the Hilbert-Mumford weight $\mu_{HM}(\sigma, \lambda)$ as $\rho \to 0$, the limiting Mundet weight is unchanged by the shift by Q in the limit $\rho \to 0$ and so Mundet semistability is preserved. Thus for ρ^{-1} sufficiently large the action of an object Q of $\mathrm{Pic}(C)$ induces an isomorphism

$$\mathcal{S}^\delta : \overline{\mathcal{M}}_n^{G_\zeta}(C, X, \mathcal{L}_0, \zeta, d) \to \overline{\mathcal{M}}_n^{G_\zeta}(C, X, \mathcal{L}_0, \zeta, d + \delta) \tag{1.67}$$

where $\delta = c_1(Q)$.

Lemma 1.2. (Action of the Picard stack on fixed loci) *The action of Pic(C) in (1.67) induces isomorphisms of the relative obstruction theories on $\overline{\mathcal{M}}_n^{G_\zeta}(C, X, \mathcal{L}_0, \zeta, d)$ preserving the restriction of symmetrised virtual structure sheaves $\hat{\mathcal{O}}^{\mathrm{vir}}_{\overline{\mathcal{M}}_n(C,X,\mathcal{L}_0)}$, and preserving the class* $\mathrm{ev}^* \alpha$ *for any $\alpha \in K^0_G(X)^n$.*

Proof The action of $\mathrm{Pic}(C)$ lifts to the universal curves, denoted by the same notation. Since the relative part of the obstruction theory on $\overline{\mathcal{M}}_n^{G_\zeta}(C, X, \mathcal{L}_0, \zeta, d)$ is the $\mathbb{C}^\times_\zeta$-invariant part of $Rp_* e^* T^\vee_{X/G_\zeta}$ up to the factor $\mathbb{C}\zeta$, the isomorphism preserves the relative obstruction theories and so the virtual structure sheaves. (Note that on the principal component, the invariant part is $Rp_* e^* T^\vee_{X^\zeta/G_\zeta}$ which is unchanged by the tensor product by $\mathbb{C}^\times_\zeta$-bundles, while on the bubble components $Rp_* e^* T^\vee_{X/G}$

is unchanged by the tensor product by Q.) Since the evaluation map is unchanged by pull-back by $\mathcal{S}^\delta$ (up to isomorphism given by twisting by Q), the class $\mathrm{ev}^*\alpha$ is preserved. □

Theorem 1.3 (Wall-crossing for crepant birational transformations of git type)**.** *Suppose that $X, G, \mathcal{L}_\pm$ define a simple wall-crossing, and C has genus zero. If the wall-crossings is simply crepant then*

$$\widehat{\tau}_{X/\!\!/_- G} \circ \widehat{\kappa}^G_{X,-} \underset{a.e.}{=} \widehat{\tau}_{X/\!\!/_+ G} \circ \widehat{\kappa}^G_{X,+}$$

almost everywhere (a.e.) in the quantum parameter q.

The following remark explains precisely in what sense *a.e.* is used in Theorem 1.3.

Remark. In the Schwartz theory of distributions (Hörmander [32]) denote by $\mathcal{T}(S^1)$ the space of *tempered distributions*. Fourier transform identifies $\mathcal{T}(S^1)$ with the space of functions on $\mathbb{Z}$ with polynomial growth. The variable q is a coordinate on the punctured plane $\mathbb{C}^\times$ and any formal power series in q, q^{-1} defines a distribution on S^1, which is tempered if the coefficient of q^d has polynomial growth in d. In particular the series $\sum_{d\in\mathbb{Z}} q^d$ is the delta function at $q = 1$, and its Fourier transform is the constant function with value 1. Any distribution of the form $\sum_{d\in\mathbb{Z}} f(d)q^d$, for $f(d)$ polynomial, is a sum of derivatives of the delta function (since Fourier transform takes multiplication to differentiation) and so is almost everywhere zero.

We study the dependence of the fixed point contributions $\tau_{X,\zeta,d,0}$ with respect to the Picard action defined in (1.66). Suppose that Q is a $\mathbb{C}^\times_\zeta$-bundle of first Chern class the generator of $H^2(C)$, after the identification $\mathbb{C}^\times_\zeta \to \mathbb{C}^\times$. Denote the corresponding class in $H_2^{G_\zeta}(X^\zeta)$ by δ. Consider the action of the $\mathbb{Z}$-subgroup $\mathbb{Z}_Q \subset \mathrm{Pic}(C)$ generated by Q. The contribution of any component $\overline{\mathcal{M}}_n^{G_\zeta}(C, X, \mathcal{L}_0, \zeta, d)$ of class $d \in H_2^G(X)$ differs from that from the component induced by acting by $Q^{\otimes r}, r \in \mathbb{Z}_Q$, of class $d + r\delta$, by the ratio of symmetrised Euler classes of the moving parts of the virtual normal complexes

$$\widehat{A}((Rp_*e^*T^\vee_{X/G})^+)\widehat{A}(\mathcal{S}^{r\delta,*}(Rp_*e^*T^\vee_{X/G})^+)^{-1} \tag{1.68}$$

As before, denote by ν_i be the subbundle of $\nu_{X^\zeta,0}$ of weight μ_i.

Lemma 1.4. *The $\widehat{A}$ classes relate by*

$$\widehat{A}((Rp_*e^*T^\vee_{X/G})^+) = \widehat{A}(\mathcal{S}^{r\delta,*}(Rp_*e^*T^\vee_{X/G})^+)\left(\prod_i \widehat{A}\,(\nu_i)^{\mu_i}\right)^r.$$

Proof The Grothendieck-Riemann-Roch allows a computation of the Chern characters of the (representable) push-forwards. Consider the isotypical decomposition into $\mathbb{C}^\times$-bundles of the normal bundle to the fixed component $X^{\zeta,0}$ in X

$$\nu_{X^{\zeta,0}} = \bigoplus_{i=1}^{k} \nu_i$$

where $\mathbb{C}^\times$ acts on ν_i with non-zero weight $\mu_i \in \mathbb{Z}$. By the discussion above $e^* T^\vee_{X/G}$ is canonically isomorphic to $\mathcal{S}^{r\delta}(e^* T^\vee_{X/G})$ on the bubble components, since the G-bundles are trivial on those components. Because the pull-back complexes are isomorphic on the bubble components, the difference $(e^* T^\vee_{X/G})^+ - \mathcal{S}^{r\delta,*}(e^* T^\vee_{X/G})^+$ is the pullback of the difference of the restrictions to the principal part of the universal curve, that is, the projection on the second factor

$$p_0 : C \times \overline{\mathcal{M}}_n^{G_\zeta}(C, X, \mathcal{L}_0, \zeta) \to \overline{\mathcal{M}}_n^{G_\zeta}(C, X, \mathcal{L}_0, \zeta).$$

These restrictions are given by

$$\begin{aligned} (e^* T_{X/G})^{+,\mathrm{prin}} &\cong \bigoplus_i e^* \nu_{X^{\zeta,t},i} \\ \mathcal{S}^{r\delta,*}(e^* T_{X/G})^{+,\mathrm{prin}} &\cong \bigoplus_i e^* \nu_{X^{\zeta,t},i} \otimes (e_C^* Q \times_{\mathbb{C}^\times_\zeta} \mathbb{C}_{r\mu_i}) \end{aligned}$$

where e_C is the map from the universal curve to C. The projection p_0 is a representable morphism of stacks given as global quotients. Let

$$\sigma : \overline{\mathcal{M}}_n^{G_\zeta}(C, X, \mathcal{L}_0, \zeta) \to C \times \overline{\mathcal{M}}_n^{G_\zeta}(C, X, \mathcal{L}_0, \zeta)$$

be a constant section of p_0, so that $c_1(\sigma^* e^* \nu_{X^{\zeta,0},i})$ is the "horizontal" part of $c_1(\nu_{X^{\zeta,0},i})$. By Grothendieck-Riemann-Roch for such stacks [52],

[14]

$$\begin{aligned}\mathrm{Td}_{\mathcal{M}} \quad & \mathrm{Ch}(\mathcal{S}^{r\delta,*}\,\mathrm{Ind}(T_{X/G})^+) = p_{0,*}(\mathrm{Td}_{C\times\mathcal{M}}\,\mathrm{Ch}(\mathcal{S}^{r\delta,*}T_{X/G})^+) \\ &= (1-g) + \mathrm{Td}_{\mathcal{M}}\, p_{0,*}\,\mathrm{Ch}(\mathcal{S}^{r\delta,*}T_{X/G})^+) \\ &= (1-g) + \mathrm{Td}_{\mathcal{M}}\, p_{0,*}\sum_i \mathrm{Ch}(e^*\nu_{X^{\zeta,t},i})\,\mathrm{Ch}((e_C^* Q \times_{\mathbb{C}_\zeta^\times} \mathbb{C}_{r\mu_i})) \\ &= (1-g) + \mathrm{Td}_{\mathcal{M}}\, p_{0,*}\sum_i \mathrm{Ch}(e^*\nu_{X^{\zeta,t},i})(1 + r\mu_i\omega_C) \\ &= p_{0,*}(\mathrm{Td}_{C\times\mathcal{M}}\,\mathrm{Ch}\left(\mathrm{Ind}(T_{X/G})^+ \oplus \bigoplus_i (\sigma^* e^* \nu_{X^{\zeta,t},i})^{\oplus r\mu_i}\right) \\ &= \mathrm{Td}_{\mathcal{M}}\,\mathrm{Ch}\left(\mathrm{Ind}(T_{X/G})^+ \oplus \bigoplus_i (\sigma^* e^* \nu_{X^{\zeta,0},i})^{\oplus r\mu_i}\right).\end{aligned}$$

Hence

$$\mathrm{Ch}(\mathcal{S}^{r\delta,*}\,\mathrm{Ind}(T_{X/G})^+) = \mathrm{Ch}\left(\mathrm{Ind}(T_{X/G})^+ \oplus \bigoplus_{i=1}^m (\sigma^* e^* \nu_{X^{\zeta,0},i})^{\oplus r\mu_i}\right) \tag{1.69}$$

The equality of Chern characters above implies an isomorphism in rational topological K-theory. The difference in Euler classes (1.68) is therefore given by the Euler class of the last summand in (1.69)

$$\begin{aligned}\frac{\widehat{A}((Rp_* e^* T^\vee_{X/G})^+)}{\widehat{A}(\mathcal{S}^{r\delta,*}(Rp_* e^* T^\vee_{X/G})^+)} &= \widehat{A}\left(\bigoplus_i (\sigma^* e^*(\nu_i))^{\oplus \mu_i r}\right) \\ &= \left(\prod_i \widehat{A}\,(\nu_i)^{\mu_i}\right)^r. \quad \square\end{aligned}$$

Proof of Theorem 1.3 Using the expansion of Euler classes as in (1.64) and (1.43), we have that by setting $r_i = \mathrm{rank}\,\nu_i$ (on each component of the inertia stack, in the orbifold case)

$$\prod_i \widehat{A}\,(\nu_i)^{-\mu_i} = \prod_i (\zeta_i^{1/2} z^{\mu_i/2} - \zeta_i^{-1/2} z^{-\mu_i/2})^{-\mu_i r_i}(1+N)$$

where ζ_i are the roots of unity appearing in orbifold Riemann-Roch [53] and N is nilpotent. Let S_-, S_+ denote the indices for which μ_i is negative and respectively positive. Define

$$\Delta(z) := \frac{\prod_{j\in S_-}(\zeta_j^{1/2} z^{\mu_j/2} - \zeta_j^{-1/2} z^{-\mu_j/2})^{-\mu_j r_j}}{\prod_{i\in S_+}(\zeta_i^{1/2} z^{\mu_i/2} - \zeta_i^{-1/2} z^{-\mu_i/2})^{\mu_i r_i}} \tag{1.70}$$

We can rewrite the difference

$$\left(\prod_i \widehat{A}\,(\nu_i)^{-\mu_i}\right)^r = (\Delta(z)(1+N))^r$$

By the crepant wall-crossing assumption 1.1, the function $\Delta(z)$ is a constant, denoted Δ. Summing the terms from $\mathcal{S}^{r\delta}, r \in \mathbb{Z}$ we obtain that the wall-crossing contribution is

$$\sum_{r\in\mathbb{Z}} q^{d+\delta r}\widehat{\tau}_{X,\zeta,d+\delta r,0}(\alpha) = \sum_{r\in\mathbb{Z}} q^{d+r\delta}.\chi_0(r) \tag{1.71}$$

where $\chi_0(r)$ is polynomial in r, since each N is nilpotent and the binomial coefficients from the expansion of $(1+N)^r$ are polynomial. Now

$$\sum_{r\in\mathbb{Z}} q^{\delta r} \in \mathcal{T}(S^1)$$

is a delta function and it vanishes almost everywhere in q^δ, see Remark 1.8. Since $\chi_0(r)$ is polynomial, (1.71) is the derivative of a delta function. Since

$$\widehat{\kappa}_X^{G,+}\widehat{\tau}_{X/\!/_+G} - \widehat{\kappa}_X^{G,-}\widehat{\tau}_{X/\!/_-G}$$

is a sum of wall-crossing terms of the type in (1.71), this completes the proof of Theorem 1.3. □

1.9 Abelianization

In this section we compare the K-theoretic Gromov-Witten invariants of a git quotient with the quotient by a maximal torus, along the lines of the case of quantum cohomology investigated by Bertram-Ciocan-Fontanine-Kim [5] and Ciocan-Fontanine-Kim-Sabbah [8]. The analogous question for K-theoretic I-functions of git quotients was already considered in Taipale [50] as well as Wen [56] and, around the same time as the first draft of this paper, Jockers, Mayr, Ninad, and Tabler [35].

Recall that the equivariant cohomology may be identified with the Weyl-invariant equivariant cohomology for the action of a maximal torus. We assume that G is a connected complex reductive group. Choose a maximal torus T and W its Weyl group. By a theorem of Harada-Landweber-Sjamaar [31, Theorem 4.9(ii), Section 6] if X is either a smooth projective G-variety or a G-vector space then restriction from the action of G to the action of the torus T defines an isomorphism onto the space of W-invariants

$$\mathrm{Restr}_T^G : K_G^0(X) \cong K_T^0(X)^W$$

for either the topological or algebraic K-cohomology. Given a polarisation $\mathcal{L} \to X$ of the G action, consider the naturally induced T polarisation on X so that

$$X^{\mathrm{ss},G}(\mathcal{L}) \subset X^{\mathrm{ss},T}(\mathcal{L}).$$

We assume from now in this section that $QK_G^0(X)$ denotes the algebraic equivariant quantum K-cohomology. We relate the K-theoretic potentials of the two geometric invariant theory quotients $X/\!\!/G$, and the *abelian* quotient $X/\!\!/T$. Let $\nu_{\mathfrak{g}/\mathfrak{t}}$ denote the bundle over $X/\!\!/T$ induced from the trivial bundle over X with fibre $\mathfrak{g}/\mathfrak{t}$. Consider the graph potential

$$\tau_{X/\!\!/T} : QK_T^0(X) \to \Lambda_X^T$$

twisted by the Euler class of the index bundle associated to $\mathfrak{g}/\mathfrak{t}$:

$$\tau_{X/\!\!/T}(\alpha, q) := \sum_{n \geq 0} \sum_{d \in H_2^G(X,\mathbb{Q})} \chi^{\mathrm{vir}} \left(\overline{\mathcal{M}}_n(C, X/\!\!/T, d); \mathrm{ev}^* \alpha^n \, \mathrm{Eul}(\mathrm{Ind}(\mathfrak{g}/\mathfrak{t})) \right) \frac{q^d}{n!}. \quad (1.72)$$

Similarly T-gauged potential τ_X^T and the Kirwan map $\kappa_{X,T}$ will from now on denote the maps with the Euler twist above. The natural map $H_2^T(X,\mathbb{Q}) \to H_2^G(X,\mathbb{Q})$ induces a map of Novikov rings

$$\pi_T^G : \Lambda_X^T \to \Lambda_X^G, \sum_{d \in H_2^T(X)} c_q q^d \mapsto \sum_{d \in H_2^G(X)} c_q q^{\pi(d)}.$$

By abuse of notation, denote again by

$$\mathrm{Restr} : QK_G^0(X) \to QK_T^0(X)$$

the map induced by the restriction map above and the inclusion of the Weyl invariants $\Lambda_X^G \cong (\Lambda_X^T)^W \subset \Lambda_X^T$. As in Martin [44] the restriction map

$$\mathrm{Restr}_G^T : K(X/\!\!/T)^W \to K(X/\!\!/G)$$

is surjective and has kernel is the annihilator of $\mathrm{Eul}(\mathfrak{g}/\mathfrak{t})$, the set of classes that vanish when multiplied by $\mathrm{Eul}(\mathfrak{g}/\mathfrak{t})$. This map naturally extends to a map

$$\mathrm{Restr}_G^T : QK(X/\!\!/T)^W \to QK(X/\!\!/G)$$

by a similar definition on the twisted sectors. On the main sector Restr_G^T is given by restriction of a class

$$\alpha \in K(X /\!\!/ T) = K(X^{\mathrm{ss}}(T)/T)$$

to $X^{\mathrm{ss}}(G)/T$ then followed by the identification of the Weyl invariant part with $K(X /\!\!/ G)$ [44, Theorem A]. With these notations we have the following result.

Theorem 1.1. (Quantum Martin formula in quantum K-theory) *Let C be a smooth connected projective genus 0 curve, X a smooth projective or convex quasiprojective G-variety, and suppose that stable=semistable for T and G actions on X. The following equality holds on the topological quantum K-theory $QK_G^0(X)$:*

$$\tau_{X /\!\!/ G} \circ \kappa_{X,G} = |W|^{-1} \pi_T^G \circ \tau_{X /\!\!/ T}^{\mathfrak{g}/\mathfrak{t}} \circ \kappa_{X,T}^{\mathfrak{g}/\mathfrak{t}} \circ \mathrm{Restr}_T^G .$$

Similarly for (J-functions) localised graph potentials

$$\begin{aligned} \tau_{X /\!\!/ G,-} &: QK_G^0(X) \to QK(X /\!\!/ G)[z, z^{-1}]] \\ \tau_{X /\!\!/ T,-} &: QK_T^0(X) \to QK(X /\!\!/ T)[z, z^{-1}]] \end{aligned}$$

we have

$$\begin{aligned} \tau_{X /\!\!/ G,-} \circ \kappa_{X,G} &= \tau_{X,-}^G \\ &= \mathrm{Restr}_G^T \circ \mathrm{Eul}(\mathfrak{g}/\mathfrak{t})^{-1} \tau_{X,-}^{T,\mathfrak{g}/\mathfrak{t}} \circ \mathrm{Restr}_T^G \qquad (1.73) \\ &= \mathrm{Restr}_G^T \circ \mathrm{Eul}(\mathfrak{g}/\mathfrak{t})^{-1} \tau_{X /\!\!/ T,-}^{\mathfrak{g}/\mathfrak{t}} \circ \kappa_{X,T}^{\mathfrak{g}/\mathfrak{t}} \circ \mathrm{Restr}_T^G . \end{aligned}$$

In particular commutativity of the following diagram holds:

$$\begin{array}{ccc} QK_G^0(X) & \xrightarrow{\mathrm{Restr}_T^G} & QK_T^0(X)^W \\ \downarrow{\scriptstyle \kappa_{X,G}} & & \downarrow{\scriptstyle \kappa_{X,T}} \\ QK(X /\!\!/ G) & & QK(X /\!\!/ T) \\ \downarrow{\scriptstyle \tau_{X /\!\!/ G}} & & \downarrow{\scriptstyle |W|^{-1} \tau_{X /\!\!/ T}} \\ \Lambda_X^G & \xleftarrow[\pi_T^G]{} & \Lambda_X^T \end{array}$$

Sketch of proof The argument is the same as that for cohomology in [25, Section 4]. In the case of projective target X, one can vary the vortex parameter $\rho \in \mathbb{R}_{>0}$ until one reaches the small-area chamber in which every bundle $P \to C$ appearing in the vortex moduli space is trivial (in genus zero). Indeed, for ρ^{-1} sufficiently large the Mundet weight is dominated by the Ramanathan weight, and this forces the bundle to be semistable of vanishing Chern class and so trivial. It follows that both $\overline{\mathcal{M}}^G(C,X,d)$ and $\overline{\mathcal{M}}^T(C,X,d)$ are quotients of open loci in the moduli stacks of stable maps $\overline{\mathcal{M}}_{0,n}(X,d)$ by G resp. T. In the small-area chamber abelianization holds for the localized potentials $\tau_{X,G}$ and $\tau_{X,T}$ by virtual non-abelian localization [28]: For sufficiently positive equivariant vector bundles V over $\overline{\mathcal{M}}_{0,n}(X,d)$ denote by $V/\!\!/G$ the restriction to $\overline{\mathcal{M}}^G(C,X,d)$. Then

$$\begin{aligned}\chi(\overline{\mathcal{M}}^G(C,X,d), V/\!\!/G) &= \chi(\overline{\mathcal{M}}_{0,n}(X,d), V)^G \\ &= |W|^{-1}\chi(\overline{\mathcal{M}}_{0,n}(X,d), V\otimes \mathrm{Eul}(\mathfrak{g}/\mathfrak{t}))^T \\ &= |W|^{-1}\chi(\overline{\mathcal{M}}^T(C,X,d), (V\otimes \mathrm{Eul}(\mathfrak{g}/\mathfrak{t}))/\!\!/T)\end{aligned}$$

where the second equality holds by the Weyl character formula. If the stabilizers are at most one-dimensional then the wall-crossing formula of [24] implies that the variation in the gauged Gromov-Witten invariants τ_X^G with respect to the vortex parameter ρ is given by wall-crossing terms $\tau_{X,\zeta}^{G_\zeta}$ involving smaller-dimensional structure group given by the centralizers $G_\zeta, \zeta \in \mathfrak{g}$ of one-parameter subgroups generated by ζ: For any singular value ρ_0 and $\rho_\pm = \rho_0 \pm \epsilon$ for ϵ small we have

$$\tau_{X,d}^G(\alpha,\rho_+) - \tau_{X,d}^G(\alpha,\rho_-) = \sum_\zeta \tau_{X,\zeta}^{G_\zeta}(\alpha,\rho_0) \tag{1.74}$$

where $\tau_{X,\zeta}^{G_\zeta}$ is a moduli stack of ρ_0-semistable gauged maps fixed by the one-parameter subgroup generated by ζ as in Section 1.7. After possibly adding a parabolic structure the stabilizers of gauged maps that are ρ_0-semistable are one-dimensional and so the wall-crossing formula (1.74) holds. Furthermore, the fixed point components have structure group that reduces to $G_\zeta/\mathbb{C}_\zeta^\times$, which as such that objects in the fixed point components have trivial stabilizer. By induction we may assume that the abelianization formula holds for structure groups $G_\zeta/\mathbb{C}_\zeta^\times$ of lower dimension, and in particular for the invariants $\tau_{X,\zeta}^{G_\zeta}(\alpha,\rho_0)$. The result for other chambers $\rho \in (\rho_i,\rho_{i+1})$ holds by the wall-crossing formula Theorem 1.5 since, by the inductive hypothesis, the wall-crossing terms are equal. The conclusion for the git quotients then follows from the adiabatic limit theorem 1.4.

In the case of quasiprojective X we assume that G has a central factor $\mathbb{C}^\times \subset G$ and the moment map $\Phi : X \to \mathbb{R}$ for this factor on X is bounded from below. Then a similar wall-crossing argument obtained by varying the polarization $\lambda(t) \in H_G^2(X)$ from $\lambda(0) = \omega$ to to a chamber corresponding to an equivariant symplectic class $\lambda \in H_G^2(X)$ where

$X/\!\!/_{\lambda(1)}G$ is empty, produces the same result [25]. Indeed the moduli space of gauged maps $\overline{\mathcal{M}}_n^G(\mathbb{P}^1, X)$ for the polarization $\lambda(1)$ is also empty, since for ρ small elements of $\overline{\mathcal{M}}_n^G(\mathbb{P}^1, X)$ must be generically semistable. On the other hand, the wall-crossing terms correspond to integrals over gauged maps whose structure group G_ζ is the centralizer of some one-parameter subgroup generated by a rational element $\zeta \in \mathfrak{t}$. By induction on the dimension of G_ζ we may assume that the wall-crossing terms for $\overline{\mathcal{M}}_n^G(\mathbb{P}^1, X)$ and $\overline{\mathcal{M}}_n^T(\mathbb{P}^1, X)$ are equal and we obtain the abelianization formula by induction.

For the localized potentials the same wall-crossing argument applied to the $\mathbb{C}^\times$-fixed point components $\overline{\mathcal{M}}_n^G(\mathbb{P}^1, X)^{\mathbb{C}^\times}$ and $\overline{\mathcal{M}}_n^T(\mathbb{P}^1, X)^{\mathbb{C}^\times}$ produces the abelianization formula

$$\begin{aligned}\chi(\overline{\mathcal{M}}_n^G(\mathbb{P}^1, X)^{\mathbb{C}^\times}, V \otimes \operatorname{Eul}(\nu_G)^{-1}) \\ = |W|^{-1}\chi(\overline{\mathcal{M}}_n^T(\mathbb{P}^1, X)^{\mathbb{C}^\times}, V \otimes \operatorname{Eul}(\operatorname{Ind}(\mathfrak{g}/\mathfrak{t}))) \quad (1.75)\end{aligned}$$

for any equivariant K-class V, where ν_G, ν_T are the normal bundles for the inclusion of the fixed point sets of the action of $\mathbb{C}^\times$. This formula holds as well after restricting to fixed point components with markings $z_1, \ldots, z_n$ mapping to $0 \in \mathbb{P}^1$ and z_{n+1} mapping to ∞, and taking V to be a class of the form $\operatorname{ev}_1^* \alpha \otimes \ldots \operatorname{ev}_n^* \alpha \otimes \operatorname{ev}_{n+1}^* \alpha_0$. One obtains the formula (with superscript class denoting the classical Kirwan map)

$$\begin{aligned}\langle \tau_{X,-}^G(\alpha), \kappa_{X,G}^{\text{class}}(\alpha_0)\rangle &= |W|^{-1}\langle \tau_{X,-}^{T,\mathfrak{g}/\mathfrak{t}}(\operatorname{Restr}_T^G \alpha), \kappa_{X,T}^{\text{class}}(\alpha_0)\rangle \\ &= \langle \operatorname{Restr}_G^T \operatorname{Eul}(\mathfrak{g}/\mathfrak{t})^{-1} \tau_{X,-}^{T,\mathfrak{g}/\mathfrak{t}}(\operatorname{Restr}_T^G \alpha), \kappa_{X,G}^{\text{class}}(\alpha_0)\rangle\end{aligned}$$

(the second by Martin's formula [44]) from which the localized abelianization formula (1.73) follows. □

Example 1.2. The fundamental solution in quantum K-theory for the Grassmannian $G(r, n)$ is studied in Taipale [50, Theorem 1], Wen [56], and Jockers, Mayr, Ninad, and Tabler [35]. Let

$$X = \operatorname{Hom}(\mathbb{C}^r, \mathbb{C}^n), \quad G = GL(r).$$

The group G acts on X by composition on the right: $gx = x \circ g$. Choose a polarization $\mathcal{L} = X \times \mathbb{C}$ corresponding a positive central character of G. The semistable locus is then

$$X^{\text{ss}} = \{x \in X \mid \operatorname{rank}(x) = r\}$$

and the git quotient

$$X/\!\!/G \cong G(r, n).$$

The torus $T = (\mathbb{C}^\times)^r$ is a maximal torus of G and the git quotient by the maximal torus is

$$X/\!\!/T \cong (\mathbb{P}^{n-1})^r.$$

We claim that the localized gauged potential $\tau^G_{X,-}$ is the restriction of the $\operatorname{Ind}(\mathfrak{g}/\mathfrak{t})$-twisted potential $\tau^{T,\mathfrak{g}/\mathfrak{t}}_{X,-}$ given by

$$\tau^{T,\mathfrak{g}/\mathfrak{t}}_{X,-}(\alpha,q,z) = \sum_d q^d \tau^{T,\mathfrak{g}/\mathfrak{t}}_{X,-,d}(\alpha,q,z) \prod_{1\le i<j\le r}(1 - X_i X_j^\vee) X^{2\rho}$$

$$\tau^{T,\mathfrak{g}/\mathfrak{t}}_{X,-,d}(\alpha,q,z) := \sum_{d_1+\ldots+d_r=d} \exp\left(\frac{\Psi_d(\alpha)}{1-z^{-1}}\right)(-1)^{d(r-1)} z^{\langle d+\rho,d+\rho\rangle - \langle\rho,\rho\rangle} \left(\frac{\prod_{i<j}(1 - X_i X_j^\vee z^{d_i-d_j})}{\prod_{i=1}^r \prod_{l=1}^{d_i}(1 - X_i z^l)^n}\right). \quad (1.76)$$

Without the factors $1 - X_i X_j^\vee z^{d_i-d_j}$ and $1 - X_i X_j^\vee$ the expression (1.76) in the Lemma would be the formula (1.29) for $\tau^T_{X,-}$ discussed previously in the toric setting. The additional factors are given by the Euler class of the index bundle

$$\operatorname{Eul}(\operatorname{Ind}(\mathfrak{g}/\mathfrak{t})) = \frac{\prod_{i<j}\prod_{k=0}^{d_j-d_i}(1 - X_i X_j^\vee z^k)}{\prod_{i<j}\prod_{k=1}^{d_j-d_i-1}(1 - X_j X_i^\vee z^k)} \quad (1.77)$$

$$= \frac{\prod_{i<j}\prod_{k=0}^{d_j-d_i}(1 - X_i X_j^\vee z^{-k})}{\prod_{i<j}(-1)^{d_j-d_i-2}\prod_{k=1}^{d_j-d_i-1} z^{-2k} X_j X_i^\vee (1 - X_i X_j^\vee z^k)}$$

$$= (-1)^{d(r-1)} z^{\sum_{i<j} -2(d_j-d_i)(d_j-d_i-1)/2} \prod_{i<j} X^{2\rho}(1 - X_i X_j^\vee)(1 - X_i X_j^\vee z^{d_j-d_i}). \quad (1.78)$$

$$= (-1)^{d(r-1)} z^{\langle d,d\rangle + 2\langle d,\rho\rangle} \prod_{i<j} X^{2\rho}(1 - X_i X_j^\vee)(1 - X_i X_j^\vee z^{d_j-d_i}) \quad (1.79)$$

$$= (-1)^{d(r-1)} z^{\langle d+\rho,d+\rho\rangle - \langle\rho,\rho\rangle} \prod_{i<j} X^{2\rho}(1 - X_i X_j^\vee)(1 - X_i X_j^\vee z^{d_j-d_i}) \quad (1.80)$$

where $\langle\cdot,\cdot\rangle$ is the Killing form. Note the missing factor of $X^{2\rho}$ in [50, (20)]; this factor re-appears in [50, (31)] but without the powers of z.

As pointed out to us by M. Zhang, the additional factors arising from the ρ-shift in (1.77) vanish when one uses the level $-1/2$-theory introduced by Ruan-Zhang [47]. It would be interesting to know how the relations depend on the level structure, and whether at level $-1/2$ the relations can be found using the difference module structure in (1.5); see Jockers, Mayr, Ninad, and Tabler [35] for further developments.

References

[1] S. Agnihotri. Quantum Cohomology and the Verlinde Algebra. PhD thesis, Oxford University, 1995.

[2] Atiyah, M. F.; Segal, G. B. The index of elliptic operators. II. *Ann. of Math.* (2) 87 1968 531–545.

[3] H. Bass, E. H. Connell, and D. Wright. The Jacobian conjecture: Reduction of degree and formal expansion of the inverse. *Bull. Amer. Math. Soc. (N.S.)* Volume 7, Number 2 (1982), 287–330.

[4] K. Behrend and B. Fantechi. The intrinsic normal cone. *Invent. Math.*, 128(1):45–88, 1997.

[5] A. Bertram, I. Ciocan-Fontanine, and B. Kim. Gromov-Witten invariants for abelian and nonabelian quotients. *J. Algebraic Geom.*, 17(2):275–294, 2008.

[6] L. A. Borisov and R. P. Horja. On the K-theory of smooth toric DM stacks. In *Snowbird lectures on string geometry*, volume 401 of *Contemp. Math.*, pages 21–42. Amer. Math. Soc., Providence, RI, 2006.

[7] A. S. Buch and L. C. Mihalcea. Quantum K-theory of Grassmannians. *Duke Math. J.*, 156(3):501–538, 2011.

[8] I. Ciocan-Fontanine, B. Kim, and C. Sabbah. The abelian/nonabelian correspondence and Frobenius manifolds. *Invent. Math.*, 171(2):301–343, 2008.

[9] H.L. Chang, Y.H. Kiem and J. Li Torus localization formulas for cosection localized virtual cycles. arXiv:1502.00078.

[10] T.H. Coates. Riemann-Roch theorems in Gromov-Witten theory. Thesis (Ph.D.)–University of California, Berkeley. 2003.

[11] J.-L. Colliot-Thélène and J.-J. Sansuc. Fibrès quadratiques et composantes connexes rèelles. *Math. Ann.*, 244(2):105–134, 1979.

[12] N. Chriss and V. Ginzburg. Representation theory and complex geometry. Reprint of the 1997 edition. Modern Birkhäuser Classics. Birkhäuser Boston, Inc., Boston, MA, 2010.

[13] I. V. Dolgachev and Y. Hu. Variation of geometric invariant theory quotients. *Inst. Hautes Études Sci. Publ. Math.*, (87):5–56, 1998. With an appendix by Nicolas Ressayre.

[14] D. Edidin. Riemann-Roch for Deligne-Mumford stacks. *A celebration of algebraic geometry*, Clay Math. Proc. (18): 241–266, 2013. arXiv:1205.4742.

[15] A. R. P. Gaio and D. A. Salamon. Gromov-Witten invariants of symplectic quotients and adiabatic limits. *J. Symplectic Geom.*, 3(1):55–159, 2005.

[16] V. Ginzburg and M. Kapranov. Koszul duality for operads. *Duke Math. J.* 76 (1994), no. 1, 203–272.

[17] A. Givental. On the WDVV equation in quantum K-theory. *Michigan Math. J.*, 48:295–304, 2000. Dedicated to William Fulton on the occasion of his 60th birthday.

[18] A. Givental. Permutation-equivariant quantum K-theory V. Toric q-hypergeometric functions. arXiv:1509.03903.

[19] A. Givental and Y.-P. Lee. Quantum K-theory on flag manifolds, finite-difference Toda lattices and quantum groups. *Invent. Math.*, 151(1):193–219, 2003.

[20] A. Givental and V. Tonita. The Hirzebruch–Riemann–Roch theorem in true genus-0 quantum K-theory. arXiv:1106.3136.

[21] T. Graber and R. Pandharipande. Localization of virtual classes. *Invent. Math.*, 135(2):487–518, 1999.

[22] R. Goldin, M. Harada, T. S. Holm, and T. Kimura. The full orbifold K-theory of abelian symplectic quotients. *J. K-Theory*, 8(2):339–362, 2011.

[23] E. Gonzalez and C. Woodward. Quantum cohomology and minimal model programs. *Adv. Math.* 353:591–646, 2019. arXiv:1207.3253.

[24] E. Gonzalez and C. Woodward. A wall-crossing formula for Gromov-Witten invariants under variation of git quotient 1208.1727.

[25] E. Gonzalez and C. Woodward. Quantum Witten localization and abelianization for qde solutions. arXiv:0811.3358.

[26] E. González, P. Solis and C. Woodward. Properness for scaled gauged maps Journal of Algebra, 490, 104-157. 2017

[27] D. Halpern-Leistner. The derived category of a GIT quotient. *J. Amer. Math. Soc.* 28: 871–912, 2015. arXiv:1203.0276.

[28] D. Halpern-Leistner. Θ-stratifications, Θ-reductive stacks, and applications. Algebraic geometry: Salt Lake City 2015. Proc. Sympos. Pure Math. 97, 349–379. Amer. Math. Soc., Providence, RI. 2018.

[29] D. Halpern-Leistner. On the structure of instability in moduli theory, arXiv: 1411.0627.

[30] Megumi Harada and Gregory D. Landweber. Surjectivity for Hamiltonian G-spaces in K-theory. *Trans. Amer. Math. Soc.*, 359(12):6001–6025 (electronic), 2007.

[31] M. Harada, G. D. Landweber, and R. Sjamaar. Divided differences and the Weyl character formula in equivariant K-theory. *Math. Res. Lett.*, 17(3):507–527, 2010.

[32] L. Hörmander. *The Analysis of Linear Partial Differential Operators I*, volume 256 of *Grundlehren der mathematischen Wissenschaften*. Springer-Verlag, Berlin-Heidelberg-New York, second edition, 1990.

[33] H. Iritani, T. Milanov, and V. Tonita. Reconstruction and Convergence in Quantum K-Theory via Difference Equations. arXiv:1309.3750.

[34] H. Iritani. An integral structure in quantum cohomology and mirror symmetry for toric orbifolds. *Adv. Math.*, 222(3):1016–1079, 2009.

[35] H. Jockers, P. Mayr, U. Ninad, A. Tabler. Wilson loop algebras and quantum K-theory for Grassmannians. . *J. High Energ. Phys.* 36, 2020. arXiv:1911.13286.

[36] J. Kalkman. Cohomology rings of symplectic quotients. *J. Reine Angew. Math.*, 485:37–52, 1995.

[37] Y. Kawamata. D-equivalence and K-equivalence. *J. Differential Geom.*, 61(1):147–171, 2002.

[38] Y. -H. Kiem and J. Li. A wall crossing formula of Donaldson-Thomas invariants without Chern-Simons functional. *Asian J. Math.* 17:63–94, 2013. arXiv:0905.4770.

[39] F. C. Kirwan. *Cohomology of Quotients in Symplectic and Algebraic Geometry*, volume 31 of *Mathematical Notes.* Princeton Univ. Press, Princeton, 1984.

[40] M. Kontsevich. Enumeration of rational curves via torus actions. In *The moduli space of curves (Texel Island, 1994)*, pages 335–368. Birkhäuser Boston, Boston, MA, 1995.

[41] Y.-P. Lee. Quantum K-theory. I. Foundations. *Duke Math. J.*, 121(3):389–424, 2004.

[42] Y.-P. Lee. A formula for Euler characteristics of tautological line bundles on the Deligne-Mumford moduli spaces. *Int. Math. Res. Not.*, (8):393–400, 1997.

[43] A. Okounkov. Lectures on K-theoretic computations in enumerative geometry. Geometry of moduli spaces and representation theory, 251–380, IAS/Park City Math. Ser., 24, Amer. Math. Soc., Providence, RI, 2017.

[44] S. Martin. Symplectic quotients by a nonabelian group and by its maximal torus. arXiv:math.SG/0001002.

[45] D. Metzler. Cohomological localization for manifolds with boundary *Int. Math. Res. Not.*, (24):1239–1274, 2002.

[46] I. Mundet i Riera. A Hitchin-Kobayashi correspondence for Kähler fibrations. *J. Reine Angew. Math.*, 528:41–80, 2000.

[47] Y. Ruan and M. Zhang. The level structure in quantum K-theory and mock theta functions. arXiv:1804.06552.

[48] Y. Ruan and M. Zhang. Verlinde/Grassmannian Correspondence and Rank 2 δ-wall-crossing. arXiv:1811.01377.

[49] P. Solis A complete degeneration of the moduli of G-bundles on a curve. arXiv:1311.6847.

[50] K. Taipale. K-theoretic J-functions of type A flag varieties. *Int. Math. Res. Not. IMRN*, (16):3647–3677, 2013.

[51] M. Thaddeus. Geometric invariant theory and flips. *J. Amer. Math. Soc.*, 9(3):691–723, 1996.

[52] B. Toen. Théorèmes de Riemann-Roch pour les champs de Deligne-Mumford. *K-Theory*, 18(1):33–76, 1999.

[53] V. Tonita. A virtual Kawasaki-Riemann-Roch formula. *Pacific J. Math.* 268: 249–255, 2014. arXiv:1110.3916.

[54] V. Tonita. Twisted K-theoretic Gromov-Witten invariants. *Math. Ann.* 372:489–526, 2018.

[55] G. Vezzosi and A. Vistoli. Higher algebraic K-theory for actions of diagonalizable groups. *Invent. Math.*, 153(1):1–44, 2003.

[56] Y. Wen. K-Theoretic I-function of $V//_\theta\mathbf{G}$ and Application. arXiv: 1906.00775.

[57] C. T. Woodward. Quantum Kirwan morphism and Gromov-Witten invariants of quotients I, II, III. Transformation Groups 20 (2015) 507–556, 881–920, 21 (2016) 1–39. arXiv:1204.1765. arXiv:1408.5864. arXiv:1408.5869.

[58] D.Wright. The tree formulas for reversion of power series. Journal of Pure and Applied Algebra 57 (1989) 191–211.