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GEOMETRIC REALIZATIONS OF THE MULTIPLIHEDRON

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Abstract. We realize Stasheff's multiplihedron geometrically as the moduli space of stable quilted disks. This generalizes the geometric realization of the associahedron as the moduli space of stable disks. We show that this moduli space is the non-negative real part of a complex moduli space of *stable scaled marked curves*.

1. Introduction

The Stasheff polytopes, also known as *associahedra*, have had many incarnations since their original appearance in Stasheff's work on homotopy associativity [12]. A particular realization of the associahedra as the compactified moduli space of nodal disks with markings is described by Fukaya and Oh [4]. The natural cell decomposition arising from this compactification is dual to the cell decomposition arising from the compactification of a space of metric trees studied by Boardman and Vogt [1]. In this paper we describe analogous constructions for a related family of polytopes J_n, called the *multiplihedra*, which appeared in [12] when defining A_∞ maps between A_∞ spaces, see also Iwase and Mimura [6]. The multiplihedra have a realization as metric trees with levels as found in [1], which in a certain sense dualizes the CW structure in Stasheff. We consider a moduli space $M_{n,1}$ of marked *quilted* disks, which are disks with $n+1$ marked points $z_0, \ldots, z_n$ on the boundary, and an interior circle passing through the marked point z_0. This moduli space has a compactification $\overline{M}_{n,1}$ by allowing nodal disks as in the definition of the moduli space of stable marked disks. Our first main result is

Theorem 1.1. *The moduli space of stable n-marked quilted disk $\overline{M}_{n,1}$ is isomorphic as a CW-complex to the multiplihedron J_n.*

Another geometric realization of the multiplihedron, which gives a different CW structure, appears in Fukaya-Oh-Ohta-Ono [3]. The authors of [3] denote them by $\overline{M}_n^w$ for $k = 1, 2, \ldots$ and use them to define A_∞ maps. The geometric description of $\overline{M}_n^w$ is similar to the space of quilted disks, in that it is a moduli space of stable marked nodal disks with some additional structure. The main difference is that their complex is has the structure of a manifold with corners, whereas the moduli space of quilted disks has real toric singularities on its boundary.

Using our geometric realization, we introduce a natural complexification of the multiplihedron. The moduli space of quilted disks $M_{n,1}$ can also be naturally identified with the moduli space of n points on the real line modulo translation only. As such, it sits inside the moduli space $M_{n,1}(\mathbb{C})$ of n points on the complex plane modulo translation. A natural compactification $\overline{M}_{n,1}(\mathbb{C})$ of this space was constructed in Ziltener's thesis [13], as the moduli space of symplectic vortices on the affine line with trivial target. Our second main result concerns the structure of Ziltener's compactification $\overline{M}_{n,1}(\mathbb{C})$, and its relationship with the multiplihedron:

Theorem 1.2. *The moduli space of stable scaled marked curves $\overline{M}_{n,1}(\mathbb{C})$ admits the structure of a complex projective variety with toric singularities that contains the multiplihedron $\overline{M}_{n,1}$ as a fundamental domain of the action of the symmetric group S_n on its real locus.*

This result is analogous to that for the Grothendieck-Knudsen moduli space of genus zero marked stable curves, which contains the associahedron as a fundamental domain for the action of the symmetric group on its real locus. In [9] this moduli space is used to define a notion of *morphism of cohomological field theories.*

2. Background on associahedra

Let $n > 2$ be an integer. The n-th associahedron K_n is a CW-complex of dimension $n-2$ whose vertices correspond to the possible ways of parenthesizing n variables $x_1, \ldots, x_n$. Each facet of K_n is the image of an embedding

$$\phi_{i,e} : K_i \times K_e \to K_n, \quad i + e = n + 1 \tag{1}$$

corresponding to the expression $x_1 \ldots x_{i-1}(x_i \ldots x_{i+e})x_{i+e+1} \ldots x_n$. The associahedra have geometric realizations as moduli spaces of genus zero nodal disks with markings:

Definition 2.1. A *marked nodal disk* consists of a collection of disks, a collection of *nodal points*, and a collection of *markings* $(z_1, \ldots, z_n)$ disjoint from the nodes, in clockwise order around the boundary, see [4]. The *combinatorial type* of the nodal disk is the ribbon tree obtained by replacing each disk with a vertex, each nodal point with a finite edge between the vertices corresponding to the two disk components, and each marking with a semi-infinite edge. A marked nodal disk is *stable* if each disk component contains at least three nodes or markings. A *morphism* between nodal disks is a collection of holomorphic isomorphisms between the disk components, preserving the singularities and markings.

Any combinatorial type has a distinguished edge defined by the component containing the zeroth marking z_0. Thus the combinatorial type of a nodal disk with markings is a rooted tree. Let $M_{n,T}$ denote the set of isomorphism classes of stable nodal marked disks of combinatorial type T, and $\overline{M}_n = \bigcup_T M_{n,T}$. $\overline{M}_n$ can be identified with a part of the real locus of the Grothendieck-Knudsen moduli space $\overline{M}_{n+1}(\mathbb{C})$ of stable genus zero marked complex curves. The topology on $\overline{M}_{n+1}(\mathbb{C})$ has an explicit description in terms of cross-ratios [8, Appendix D], hence so does the topology on $\overline{M}_n$. The cross-ratio of four distinct points $w_1, w_2, w_3, w_4 \in \mathbb{C}$ is

$$\rho_4(w_1, w_2, w_3, w_4) = \frac{(w_2 - w_3)(w_4 - w_1)}{(w_1 - w_2)(w_3 - w_4)}$$

and represents the image of w_4 under the fractional linear transformation that sends w_1 to 0, w_2 to 1, and w_3 to ∞. ρ_4 is invariant under the action of $SL(2, \mathbb{C})$ on $\mathbb{C}$ by fractional linear transformations. By identifying $\mathbb{P}^1(\mathbb{C}) \to \mathbb{C} \cup \{\infty\}$ and using invariance we obtain an extension of ρ_4 to $\mathbb{P}^1(\mathbb{C})$, that is, a map

$$\rho_4 : \{(w_1, w_2, w_3, w_4) \in (\mathbb{P}^1(\mathbb{C}))^4, \quad i \neq j \implies w_i \neq w_j\} \to \mathbb{C} - \{0\}.$$

ρ_4 naturally extends to the geometric invariant theory quotient

$$(\mathbb{P}^1(\mathbb{C}))^4 /\!\!/ SL(2,\mathbb{C}) = \{(w_1,w_2,w_3,w_4),\ \text{no more than two points equal}\}/SL(2,\mathbb{C})$$

by setting

$$\rho_4(w_1,w_2,w_3,w_4) = \left\{ \begin{array}{lll} 0 & \text{if} & w_2 = w_3 \text{ or } w_1 = w_4 \\ 1 & \text{if} & w_1 = w_3 \text{ or } w_2 = w_4 \\ \infty & \text{if} & w_1 = w_2 \text{ or } w_3 = w_4 \end{array} \right\} \tag{2}$$

and defines an isomorphism from $(\mathbb{P}^1(\mathbb{C}))^4 /\!\!/ SL(2,\mathbb{C})$ to $\mathbb{P}^1(\mathbb{C})$. Let $\mathbb{R}^4_+ \subset \mathbb{R}^4$ denote the subset of distinct points $(w_1,w_2,w_3,w_4) \in \mathbb{R}^4$ in cyclic order. The restriction of ρ_4 to $\mathbb{R}^4_+$ takes values in $(-\infty,0)$ and is invariant under the action of $SL(2,\mathbb{R})$ by fractional linear transformations. Hence it descends to a map $(\mathbb{R}^4)_+/SL(2,\mathbb{R}) \to (-\infty,0)$. Let D denote the unit disk, and identify $D \setminus \{-1\}$ with the half plane $\mathbb{H}$ by $z \mapsto 1/(z+1)$. Using invariance one constructs an extension $\rho_4 : (\partial D)^4_+/SL(2,\mathbb{R}) = M_4 \to (-\infty,0)$ where $(\partial D)^4_+$ is the set of distinct points on ∂D in counterclockwise cyclic order. ρ_4 admits an extension to $\overline{M}_4$ via (2) and so defines an isomorphism $\rho_4 : \overline{M}_4 \to [-\infty,0]$. For any distinct indices i,j,k,l the cross-ratio ρ_{ijkl} is the function

$$\rho_{ijkl} : M_n \to \mathbb{R}, \quad [w_0,\dots,w_n] \mapsto \rho_4(w_i,w_j,w_k,w_l).$$

Extend ρ_{ijkl} to $\overline{M}_n$ as follows. Let $T(ijkl) \subset T$ be the subtree whose ending edges are the semi-infinite edges i,j,k,l. The subtree $T(ijkl)$ is one of the three types in Figure 1. In the first

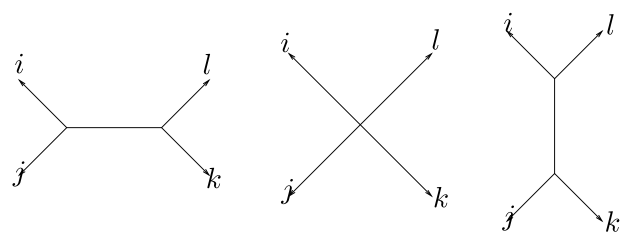

FIGURE 1. Cross-ratios by combinatorial type: for the first type, $\rho_{ijkl}(S) = -\infty$, for the second type $\rho_{ijkl}(S) \in (-\infty,0)$, and for the third type $\rho_{ijkl}(S) = 0$.

resp. third case, we define $\rho_{ijkl}(S) = -\infty$ resp. 0. In the second case, let $\overline{w}_i, \overline{w}_j, \overline{w}_k, \overline{w}_l$ be the points on the component where the four branches meet and define $\rho_{ijkl}(S) = \rho(\overline{w}_i, \overline{w}_j, \overline{w}_k, \overline{w}_l)$. Properties of ρ_{ijkl} that follow from elementary facts about cross-ratios [8, Appendix D] are

Proposition 2.2. (a) **(Invariance):** *For all marked nodal disks S, and for all $\phi \in SL(2,\mathbb{R})$, $\rho_{ijkl}(\phi(S)) = \rho_{ijkl}(S)$.*

(b) **(Symmetry):** $\rho_{jikl} = \rho_{ijlk} = 1 - \rho_{ijkl}$, *and* $\rho_{ikjl} = \rho_{ijkl}(\rho_{ijkl} - 1)$.

(c) **(Normalization):** $\rho_{ijkl} = \begin{cases} \infty, & \textit{if } i = j \textit{ or } k = l, \\ 1, & \textit{if } i = k \textit{ or } j = l, \\ 0, & \textit{if } i = l \textit{ or } j = k. \end{cases}$

(d) **(Recursion):** *As long as the set $\{1,\infty,\rho_{ijkl},\rho_{ijkm}\}$ contains three distinct numbers, then $\rho_{jklm} = \frac{\rho_{ijkm}-1}{\rho_{ijkm}-\rho_{ijkl}}$ for any five pairwise distinct integers $i,j,k,l,m \in \{0,1,\dots,d\}$.*

The collection of functions $\rho_{ijkl},\ i<j<k<l$ defines a map of sets

$$\rho_n : \overline{M}_n \mapsto [-\infty,0]^N, \qquad N = \binom{d+1}{4}, \tag{3}$$

which is the restriction of the corresponding map $\rho_n : \overline{M}_{n+1}(\mathbb{C}) \to (\mathbb{P}^1(\mathbb{C}))^N$ defined by cross-ratios.

Theorem 2.3 (Theorem D.4.5, [8]). *The map $\rho_n : \overline{M}_{n+1}(\mathbb{C}) \to (\mathbb{P}^1(\mathbb{C}))^N$ is injective, and its image is closed.*

Corollary 2.4. *The map $\rho_n : \overline{M}_n \mapsto [-\infty,0]^N$ is an embedding, and its image is closed.*

The topology on $\overline{M}_n$ is defined by pulling back the topology on $[-\infty,0]^N$. With respect to this topology, $\overline{M}_n$ is compact and Hausdorff. Explicit coordinate charts which give $\overline{M}_n$ the structure of a $(n-2)$ dimensional manifold-with-corners can be defined with cross-ratios. There is a canonical partial order on the combinatorial types, and we write $T_0 \leq T_1$ to mean that T_1 is obtained from T_0 by contracting a subset of finite edges of T_0, in other words, there is a surjective morphism of trees from T_0 to T_1. Let

$$\overline{M}_{n,\leq T_1} := \bigcup_{T_0 \leq T_1} M_{n,T_0} \subset \overline{M}_n.$$

Definition 2.5. A *cross-ratio chart* for a combinatorial type T is a map

$$\psi_T : \overline{M}_{n,\leq T} \to (0,\infty)^{n-2-|E|} \times [0,\infty)^{|E|}$$

where $|E|$ is the number of interior edges of T, given by

(a) $n-2-|E|$ coordinates taking values in $(0,\infty)$, obtained by choosing $m-3$ coordinates $-\rho_{ijkl}$ for each disk component with m marked or singular points,
(b) $|E|$ coordinates with values in $[0,\infty)$, obtained by choosing a coordinate $-\rho_{ijkl}=0$ for each internal edge such that a $\rho_{ijkl}=0$ for any combinatorial type modeled on that edge.

Theorem 2.6 (Theorem D.5.1 in [8]). *For any combinatorial type T, suppose that $n-2$ cross-ratios have been chosen as prescribed by (a), (b) above. Then, in the open set $\overline{M}_{n,\leq T}$, all cross-ratios are smooth functions of those chosen. Hence $\overline{M}_n$ is a smooth manifold-with-corners of real dimension $n-2$.*

The associahedra have another geometric realization as *metric trees*, introduced in Boardman-Vogt [1]. Here we follow the presentation in [4].

Definition 2.7. A *rooted metric ribbon tree* consists of

(a) a finite tree $T=(V(T),\overline{E}(T))$ where $\overline{E}(T)$ is the union of a set $E(T)$ of *finite edges* incident to two vertices and a set $E_\infty(T)=\{e_0,\ldots,e_n\}$ of *semi-infinite edges*, each of which is incident to a single vertex;
(b) a cyclic ordering on the edges $\{e \in \overline{E}(T), v \in e\}$ at each vertex $v \in V(T)$;
(c) a distinguished edge $e_0 \in E_\infty(T)$, called the *root*; the other semi-infinite edges are called *leaves*.

(d) a *metric* $\lambda : E(T) \to (0, \infty)$

A tree is *stable* if each vertex has valence at least 3.

Given a rooted ribbon tree T we denote by $\mathrm{W}_{n,T}$ the set of all metrics $\lambda : E(T) \to \mathbb{R}_+$. The space of all stable rooted metric ribbon trees with n leaves is denoted

$$\mathrm{W}_n = \bigcup_T \mathrm{W}_{n,T} .$$

There is a natural topology on W_n, which allows the collapse of edges whose lengths approach zero in a sequence. The closure of $\mathrm{W}_{n,T}$ in W_n is given by

$$\overline{\mathrm{W}}_T = \bigcup_{T' \leq T} \mathrm{W}_{n,T'} .$$

Each cell $\mathrm{W}_{n,T}$ is compactified by allowing the edge lengths to be infinite. We denote the induced compactification of W_n by $\overline{\mathrm{W}}_n$. The following theorem is well-known:

Theorem 2.8. *There exists a homeomorphism $\Theta : \overline{\mathrm{W}}_n \to \overline{M}_n$ such that for any combinatorial type T, $\Theta(\overline{\mathrm{W}}_{n,T})$ intersects $M_{n,T}$ in a single point.*

In other words, the realization as metric trees is dual, in a CW-sense, to the realization as marked disks. We give a proof of the corresponding statement for the multiplihedra in the next section.

3. The multiplihedra

Stasheff [11] introduced a family of CW-complexes called the *multiplihedra*, which play the same role for maps of loop spaces as the associahedra play in the recognition principle for loop spaces. The n-th multiplihedron J_n is a complex of dimension $n-1$ whose vertices correspond to ways of bracketing n variables $x_1, \ldots, x_n$ and applying an operation, say f. The multiplihedron J_3 is the hexagon shown in Figure 2.

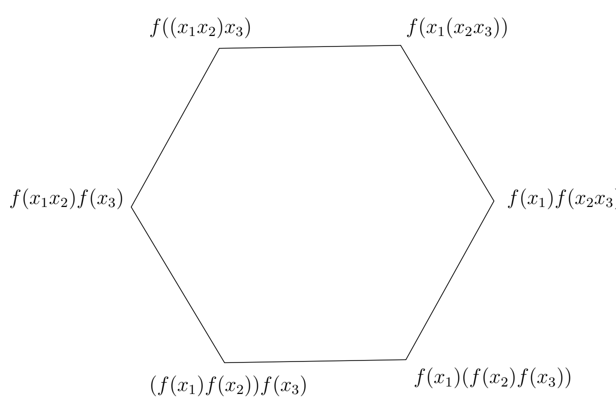

Figure 2. Vertices of J_3

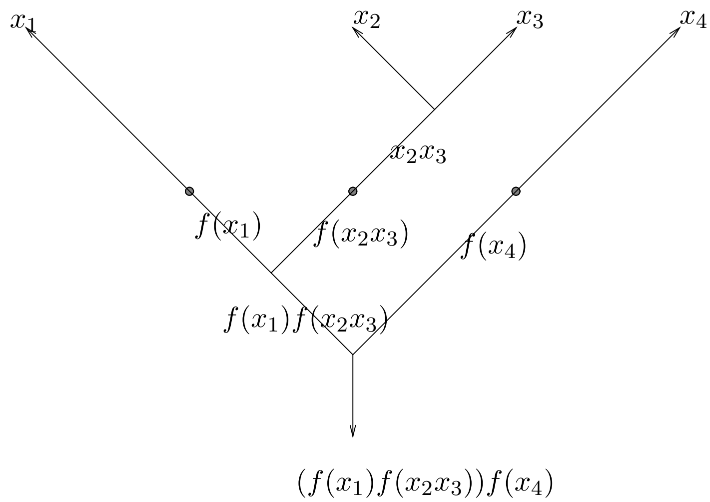

FIGURE 3. Tree for $(f(x_1)f(x_2x_3))f(x_4)$

The facets of J_n are of two types. First, there are the images of the inclusions

$$J_{i_1} \times \ldots \times J_{i_j} \times K_j \to J_n$$

for partitions $i_1 + \ldots + i_j = n$, and secondly the images of the inclusions

$$J_{n-e+1} \times K_e \to J_n$$

for $2 \leq e \leq n$. One constructs the multiplihedron inductively starting from setting J_2 and K_3 equal to closed intervals.

Each vertex corresponds to a rooted tree with two types of vertices, the first a trivalent vertex corresponding to a bracketing of two variables and the second a bivalent vertex corresponding to an application of f, see Figure 3. Dualizing the rooted tree gives a triangulation of the $n+1$-gon together with a partition of the two-cells into two types, depending on whether they occur before or after a bivalent vertex in a path from the root, see Figure 4. The edges of J_n

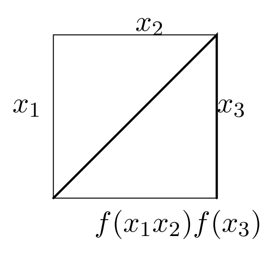

FIGURE 4. Triangulation corresponding to $f(x_1x_2)f(x_3)$

are of two types: (a) A change in bracketing $\ldots x_{i-1}(x_ix_{i+1})\ldots \mapsto (x_{i-1}x_i)x_{i+1}$ or vice-versa; (b) A move of the form $\ldots f(x_ix_{i+1})\ldots \mapsto f(x_i)f(x_{i+1})\ldots$ or vice versa, which corresponds to moving one of the bivalent vertices past a trivalent vertex, after which it becomes a pair of bivalent vertices, or vice-versa; see Figure 5.

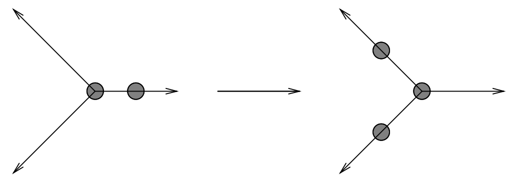

FIGURE 5. Splitting of bivalent vertices

4. Quilted disks

Definition 4.1. A *quilted disk* is a closed disk $D \subset \mathbb{C}$ together with a circle $C \subset D$ (the *seam* of the quilt) tangent to a unique point in the boundary. Thus C divides the interior of D into two components. Given quilted disks (D_0, C_0) and (D_1, C_1), a *isomorphism* from (D_0, C_0) to (D_1, C_1) is a holomorphic isomorphism $D_0 \to D_1$ mapping C_0 to C_1. Any quilted disk is isomorphic to the pair (D, C) where D is the unit disk in the complex plane and C the circle of radius $1/2$ passing through 1 and 0. Thus the automorphism group of (D, C) is canonically isomorphic to the group $T \subset SL(2, R)$ of translations by real numbers.

Let $n \geq 2$ be an integer. A *quilted disk with* $n+1$ *markings on the boundary* consists of a disk $D \subset \mathbb{C}$ (which we may take to be the unit disk), distinct points $z_0, \ldots, z_n \in \partial D$ and a circle $C \subset D$ tangent to z_0, of radius between 0 and 1. A *morphism of quilted disks* from $(D_0, C_0; z_0, \ldots, z_n) \to (D_1, C_0; w_0, \ldots, w_n)$ is a holomorphic isomorphism $D_0 \to D_1$ mapping C_0 to C_1 and z_j to w_j for $j = 0, \ldots, n$.

Let $M_{n,1}$ be the set of isomorphism classes of $n+1$-marked quilted disks. We compactify $M_{n,1}$ by allowing nodal quilted disks whose combinatorial type is described as follows.

Definition 4.2. A *colored, rooted ribbon tree* is a ribbon tree $T = (\overline{E}(T), V(T))$ together with a distinguished subset $V_{col}(T) \subset V(T)$ of *colored vertices*, such that in any non-self-crossing path from a leaf e_i to the root e_0, exactly one vertex is a colored vertex.

Definition 4.3. A *nodal* $(d+1)$*-quilted disk* S is a collection of quilted and unquilted marked disks, identified at pairs of points on the boundary. The combinatorial type of S is a colored rooted ribbon tree T, where the colored vertices represent quilted disks, and the remaining vertices represent unquilted disks. A nodal quilted disk is *stable* if and only if

(a) Each quilted disk component contains at least 2 singular or marked points;
(b) Each unquilted disk component contains at least 3 singular or marked points.

Thus the automorphism group of any disk component of a stable disk is trivial, and from this one may derive that the automorphism group of any stable $n+1$-marked nodal quilted disk is also trivial.

The appearance of the two kinds of disks can be explained in the language of bubbling as in [8]. Suppose that S_α is a sequence of quilted disks. We identify the complement of z_0 with the upper half-space $\mathbb{H}$, so that the circle C_α becomes a horizontal line $L_\alpha \subset \mathbb{H}$. After a sequence of automorphisms φ_α, we may assume that $z_{1,\alpha} - z_{n,\alpha}$ is constant. If the line L_α approaches the real axis, or two points $z_{i,\alpha}, z_{j,\alpha}$ converge then we re-scale so that the distances remain finite and encode the limit of the re-scaled data as a bubble. There are three-different types of bubbles: either $L_\alpha \to \partial\mathbb{H}$ in the limit, in which case we say that the resulting bubble is unquilted, or L_α approaches a fixed line L_∞, in which case the bubble is a quilted disk, or L_α goes to ∞, in which case the bubble is also unquilted. Thus the limiting sequence is a bubble tree, whose bubbles are of the types discussed above.

Let $\overline{M}_{n,1}$ denote the set of isomorphism classes of stable $n+1$-marked nodal quilted disks. For example, $\overline{M}_{3,1}$ is a hexagon.

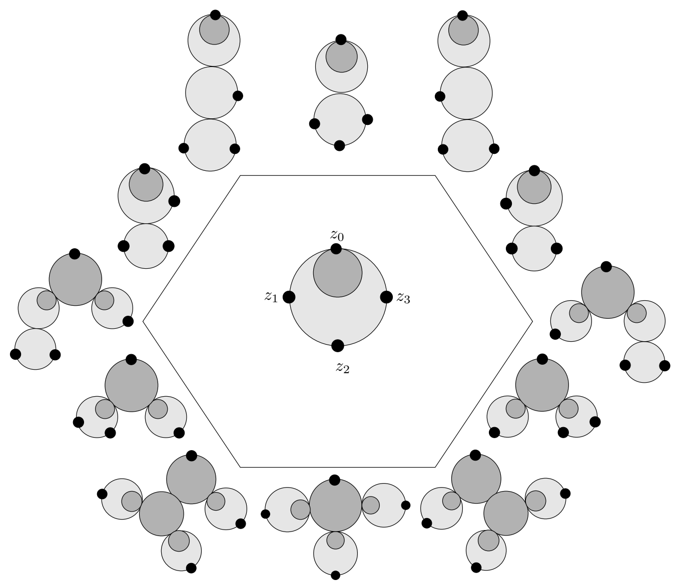

FIGURE 6. $\overline{M}_{3,1}$

5. THE CANONICAL EMBEDDING

$\overline{M}_{n,1}$ admits a canonical embedding into a product of closed intervals via a natural generalization of cross-ratios. Let D denote the unit disk, C a circle in D passing through a unique point z_0 and $z_1, z_2 \in D$ points in D such that z_0, z_1, z_2 are distinct. Let w be a point in C not equal to z_0. Define $\rho_{3,1}(D, C, z_1, z_2) = \mathrm{Im}(\rho(z_0, z_1, z_2, w))$, the imaginary part of $\rho(z_0, z_1, z_2, w)$. $\rho_{3,1}$ is independent of the choice of w and invariant under the group of automorphisms of the disk and so defines a map $\rho_{3,1} : M_{3,1} \to (0, \infty)$. We extend $\rho_{3,1}$ to $\overline{M}_{3,1}$ by setting $\rho_{3,1}(S) = 0$ if S is the 3-marked quilted nodal disk with three components, and $\rho_{3,1}(S) = \infty$ if S is the 3-marked nodal disk with two components. Thus $\rho_{3,1}$ extends to a bijection

$$\rho_{3,1} : \overline{M}_{3,1} \to [0, \infty].$$

More generally, given $n \geq 3$ and a pair i, j of distinct, non-zero vertices, let T_{ij} denote the minimal connected subtree of T containing the semi-infinite edges corresponding to z_i, z_j, z_0.

There are three possibilities for T_{ij}, depending on whether the quilted vertex appears closer or further away than the trivalent vertex from z_0, or equals the trivalent vertex. In the first, resp.

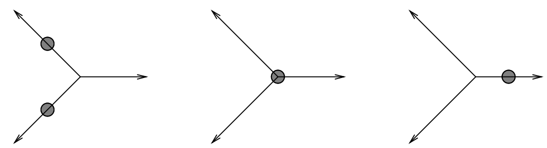

FIGURE 7. Tree types for J_3

third case define $\rho_{ij}(S) = 0$ resp ∞. In the second case let (D, C) denote the disk component corresponding to the trivalent vertex, $w_i, w_j \in \partial D$ the points corresponding to the images in ∂D of the marked points z_i, z_j, and define

$$\rho_{ij}(S) = \rho_{3,1}(D, C, w_i, w_j).$$

The ρ_{ij} have properties very similar to the ρ_{ijkl}:

Proposition 5.1. *For all quilted disks S,*

(a) **(Invariance):** *for all $\phi \in SL(2,\mathbb{R})$, $\rho_{ij}(\phi(S)) = \rho_{ij}(S)$.*
(b) **(Symmetry):** $\rho_{ij}(S) = -\rho_{ji}(S)$.
(c) **(Normalization):** $\rho_{ij}(S) = \begin{cases} \infty, & \text{if } i \neq j \text{ and } L = \mathbb{R} + i\infty, \\ 0, & \text{if } i \neq j \text{ and } L = \mathbb{R} + i0. \end{cases}$
(d) **(Recursion):** $\rho_{ik}(S) = \frac{\rho_{ij}(S)}{\rho_{jk}(S)}$
(e) **(Relations):** $\rho_{jk} = \frac{\rho_{ij}}{-\rho_{ijk0}}, \quad \rho_{ik} = \frac{\rho_{ij}}{1-\rho_{ijk0}}$.

By the invariance property, ρ_{ij} descends to a map

$$\overline{M}_{n,1} \to [0, \infty].$$

In addition, for any four distinct indices i, j, k, l we have the cross-ratio $\rho_{ijkl} : \overline{M}_{n,1} \to [0, \infty]$ defined in the previous section, obtained by treating the quilted disk component as an ordinary component.

Theorem 5.2. *The map*

$$\rho_{n,1} : \overline{M}_{n,1} \to [-\infty, 0]^N \times [0, \infty]^{n(n-1)/2}, \quad N = \binom{d+1}{4}$$

obtained from all the cross-ratios is injective, and its image is closed.

Proof. The proof is similar that of Theorem 2.3. In fact, it is a corollary of Theorem 10.3 in Section 10, which deals with a complex space $\overline{M}_{n,1}(\mathbb{C})$ in which $\overline{M}_{n,1}$ sits as a part of the real locus. □

We define the topology on $\overline{M}_{n,1}$ by pulling back the topology on the codomain. Since the codomain is Hausdorff and compact,

Corollary 5.3. *$\overline{M}_{n,1}$ is Hausdorff and compact.*

Remark 5.4. The maps $\overline{M}_{n,1} \to \overline{M}_4$, $\overline{M}_{n,1} \to \overline{M}_{3,1}$ are special cases of forgetful morphisms: For any subset $I \subset \{0, \ldots, n\}$ of size k we have a map $\overline{M}_{n,1} \mapsto \overline{M}_k$ obtained by forgetting the position of the circle and collapsing all unstable components. Similarly, for any subset $J \subset \{1, \ldots, n\}$ of size l we have a map $\overline{M}_{n,1} \mapsto \overline{M}_{l,1}$ obtained by forgetting the positions of $z_j, j \notin J$ and collapsing all unstable disk components. The topology on $\overline{M}_{n,1}$ is the minimal topology such that all forgetful morphisms are continuous and the topology on $\overline{M}_{3,1} \cong [0, \infty]$, $\overline{M}_4 \cong [0, \infty]$ is induced by the cross-ratio.

The full collection of cross-ratios contains a large amount of redundant information. In the remainder of this section we discuss certain "minimal sets" of cross-ratios, to be used later. Let T be a combinatorial type in $\overline{M}_{n,1}$.

Definition 5.5. A *cross-ratio chart* associated to T is a map $\psi_T : \overline{M}_{n,1,\leq T} \to (0, \infty)^p \times [0, \infty)^q$ for some $p, q \geq 0$ given by

(a) p coordinates taking values in $(0, \infty)$, obtained by taking $m - 3$ coordinates of the form $-\rho_{abcd}$ or ρ_{ab} for each disk component that has m special features, where a special feature is either a marked point, a nodal point, or an inner circle of radius $0 < r < 1$;
(b) q coordinates taking values in $[0, \infty)$, obtained by choosing (i) a coordinate $-\rho_{abcd}$ for each finite edge in T, such that a combinatorial type has that edge if and only if $\rho_{abcd} = 0$; (ii) a coordinate ρ_{ab} for each finite edge in T that is incident to a *bivalent* colored vertex from above, such that $\rho_{ab} = 0$ for every combinatorial type modeled on that edge; (iii) a coordinate $1/\rho_{ab}$ for each finite edge in T that is incident to a bivalent colored vertex from below, such that $1/\rho_{ab} = 0$ for every combinatorial type modeled on that edge.

Proposition 5.6. *Let ψ_T be as above. On $\overline{M}_{n,1,\leq T}$ all cross-ratios ρ_{ijkl} and ρ_{ij} are compositions of smooth functions with ψ_T.*

Proof. First we prove that all cross-ratios of the form ρ_{ijkl} are smooth functions of those in the chart associated to T. Let T' be the combinatorial type in $\overline{M}_n$ obtained by forgetting colored vertices. Taking all cross-ratios of the form ρ_{ijkl} in the chart associated to T is almost a chart for T' in the sense of Definition 2.5, the only chart coordinates that might be missing correspond to edges whose pre-image in T had a bivalent colored vertex. For each bivalent vertex in T, we can assume that the lower edge has coordinate $\rho_{i,j} = \infty$ and the upper edge is either $\rho_{jk} = 0$ or $\rho_{hi} = 0$. Assuming the first case, relation (e) holds and $\rho_{ijk0} = -\rho_{ij}/\rho_{jk}$, which expresses ρ_{ijk0} as a smooth function of the chart coordinates, and ρ_{ijk0} is a valid chart coordinate for the edge in T'. The other case is very similar, by relation (e), $\rho_{hij0} = -\rho_{hi}/\rho_{ij}$, which expresses ρ_{hij0} as a smooth function of the chart coordinates, and ρ_{hij0} is a chart coordinate for the corresponding edge in T'. Thus we get a chart for T', so by Theorem 2.6 all cross-ratios of the form ρ_{abcd} are smooth functions of these coordinates. Finally, all cross-ratios ρ_{ab} are smooth functions of the cross-ratios ρ_{ij} in the chart and the appropriate ρ_{ijk0}'s, by **(Relations)**. $\square$

6. Local structure

In general, $\overline{M}_{n,1}$ is not CW-isomorphic to a manifold with corners, but rather has more complicated singularities that we now describe. Quilted disks in the interior $M_{n,1}$ can be identified with configurations of n distinct points $-\infty < z_1 < z_2 < \ldots < z_n < \infty$ in $\mathbb{R} \subset \mathbb{C}$, together with a horizontal line L in $\mathbb{H}$. Isomorphisms are transformations of the form $z \mapsto az+b$ for $a, b \in \mathbb{R}$ such that $a > 0$, i.e. dilation and translation. For such configurations define coordinates $(x_1, x_2, \ldots, x_n, y)$ by $x_i = z_{i+1} - z_i$, and $y = \mathrm{dist}(L, \mathbb{R})$. A transformation $z \mapsto az+b$ for $a, b \in \mathbb{R}$ sends $(x_1, x_2, \ldots, x_{n-1}, y) \mapsto (ax_1, ax_2, \ldots, ax_{n-1}, ay)$, so $(x_1 : x_2 : \ldots : x_{n-1} : y)$ are projective coordinates on $M_{n,1}$.

Let T be a maximal colored rooted ribbon tree, hence its colored vertices are bivalent, and all other vertices trivalent. We construct a *simple ratio chart*

$$\phi_T : \overline{M}_{n,1,\leq T} \to \mathrm{Hom}(\mathrm{E}(T), \mathbb{R}_{\geq 0}), \quad [S] \mapsto (\phi_{T,e}(S))_{e \in \mathrm{E}(T)} \tag{4}$$

as follows. Let $[S] \in \overline{M}_{n,1,\leq T}$. For each $1 \leq i \leq n-1$, there is a unique vertex of T at which the path from the leaf i and the leaf $i+1$ back to the root intersect; we label this vertex v_i. Every trivalent vertex of T can be labeled this way, so all remaining vertices are colored. The interior edges of T are of two types: edges that connect two vertices v_i and v_j, and edges that connect a vertex v_i to a colored vertex. Suppose that $e \in \mathrm{E}(T)$ connects the vertex v_i with the vertex v_j, with v_j closer to the root (i.e., v_j is above v_i). The vertex v_i labels the unique component of the nodal disk S on which the markings corresponding to the leaves z_j, z_{j+1} and z_0 are distinct. On this component, choose a parametrization that sends z_0 to ∞, then label the edge between v_i and v_j with $\phi_{T,e} = (z_{i+1} - z_i)/(z_{j+1} - z_j) = x_i/x_j$. Note that the label $\phi_{T,e}$ is therefore independent of the choice of parametrization. If the edge e connects the vertex v_i with a colored vertex immediately above it, choose the unique component at which z_{i+1} is distinct from z_i, and label e with $\phi_{T,e} = x_i/y$; if the edge e connects the vertex v_i with a colored vertex immediately below it, label e with the value $\phi_{T,e} = y/x_i$. We claim that the map ϕ_T is defined for all $[S] \in \overline{M}_{n,1,\leq T}$ with all the ratios $\phi_{T,e}(S)$ landing in $[0, \infty)$, and with the property that $\phi_{T,e}(S) = 0$ if and only if the combinatorial type of S has the edge e. To see this, recall that if $[S] \in \overline{M}_{n,1,\leq T}$, its combinatorial type T_S must be obtained from T by contracting a subset of edges (that is, $T_S \leq T$). In particular, every edge in T_S corresponds to a unique edge in T. Hence, if $e \in \mathrm{E}(T)$ connects a vertex v_i with a vertex v_j above it, then in T_S either $v_i = v_j$ if e is contracted, or the edge e remains. If $v_i = v_j$ in T_S, it implies that the disk component of S on which $z_j \neq z_{j+1}$ is also the disk component on which $z_i \neq z_{i+1}$, hence $x_i > 0$ and $x_j > 0$ and the ratio $\phi_{T,e}(S) = x_i/x_j > 0$. If the edge e remains in T_S, it means that with respect to the markings on the disk component where $z_j \neq z_{j+1}$, we have $z_i = z_{i+1}$ and so $\phi_{T,e}(S) = 0/x_j = 0$.

Now if $e \in \mathrm{E}(T)$ connects vertex v_i with a colored vertex above it, then in T_S either v_i becomes a colored vertex, or e is an edge. In the first case, it implies that the unique component where $z_i \neq z_{i+1}$ is a quilted component, so parametrizing the component such that $z_0 = \infty$ the inner circle is a line of height $y > 0$, and $\phi_{T,e}(S) = x_i/y > 0$. In the second case, the unique component where $z_i \neq z_{i+1}$ is unquilted and corresponds to having the line at height $y = \infty$,

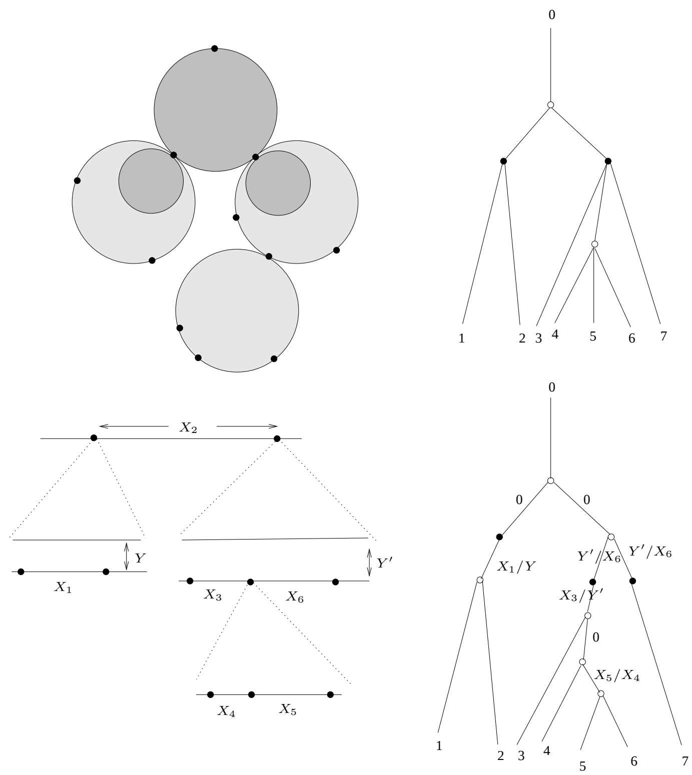

FIGURE 8. Identifying a nodal quilted disk in $\overline{M}_{7,1,\leq T}$ with a balanced labelling of T, using simple ratios.

so $\phi_{T,e}(S) = x_i/\infty = 0$. The case of a colored vertex below v_i is similar. This completes the construction of ϕ_T.

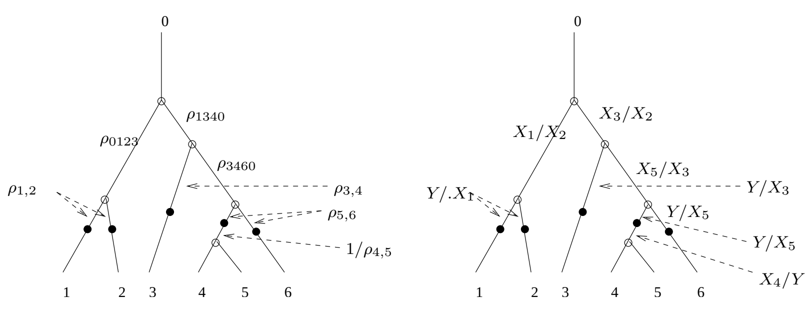

FIGURE 9. A cross-ratio chart and a simple-ratio chart for the same maximal colored tree.

Definition 6.1. Let T be a colored tree. A labelling $\varphi : \mathrm{E}(T) \to \mathbb{R}_{\geq 0}$ is *balanced* if it satisfies the following condition: denote by $V_-(T)$ the set of vertices on the root side of the colored vertices, that is, connected to the root by a path not crossing a colored vertex. For each vertex $v_0 \in V(T)$ and any colored vertex v connected by a path of edges not crossing the root, let

$\pi(v_0, v)$ denote the product of the values of φ along the unique path of edges from v_0 to v. Then φ is balanced if $\pi(v_0, v)$ is independent of the choice of colored vertex v. Let $X(T)$ denote the set of balanced labellings:

$$X(T) := \{\varphi : \mathrm{E}(T) \to \mathbb{R}_{\geq 0} \,|\, \forall v_0 \in V_-(T),\ \pi(v_0, v) \text{ is independent of } v \in V_{\mathrm{col}}(T)\}.$$

We denote by $G(T) \subset X(T)$ the subset of non-zero labellings.

Proposition 6.2. *Let T be a maximal colored tree. Then ϕ_T is a homeomorphism from $\overline{M}_{n,1,\leq T}$ onto $X(T)$, mapping $M_{n,1}$ onto $G(T)$.*

Thus in particular the simple-ratios and cross-ratios define the same topology on $\overline{M}_{n,1,\leq T}$.

Proof. It follows from the definition that ϕ_T takes values in balanced labellings, with products y/x_i where i is the top vertex. The construction of ϕ_T also makes it clear how to construct a pointed nodal quilted disk from a balanced labeling of T, showing that ϕ_T is onto $X(T)$. To make the relationship between the coordinates in the balanced labeling and the cross-ratio coordinates in a chart for $\overline{M}_{n,1,\leq T}$ explicit, let $\rho = (\rho_e)_{e\in \mathrm{E}(T)}$ be the cross-ratios in a chart covering $\overline{M}_{n,1,\leq T}$. Without loss of generality, assume that all chart cross-ratios of the form ρ_{ijkl} are either of the form $\rho_e = \rho_{ijk0}$, or $\rho_e = \rho_{0ijk}$, so that on $\overline{M}_{n,1,\leq T}$ they take values in $(-\infty, 0]$, and such that for $[S] \in \overline{M}_{n,1,\leq T}$, $\rho_e(S) = 0$ implies that the combinatorial type of S has the edge e. Let $\zeta = (\phi_{T,e})_{e\in \mathrm{E}(T)}$ denote the simple ratios in a balanced labelling of T. We claim that $\rho_e = \zeta_e f(\zeta)$ for every edge $e \in \mathrm{E}(T)$, where $f(\zeta)$ is a smooth function on the interior of $\overline{M}_{n,1,\leq T}$ which is continuous up to the boundary, and $f(\zeta) \neq 0$ on $\overline{M}_{n,1,\leq T}$. In particular, $\rho_e = 0 \iff \zeta_e = 0$. First we prove it for the cross-ratios ρ_{ijk0} in the chart. By symmetry it suffices to consider the edge pictured in Figure 10, where an edge e joins vertices v_r and v_s, with v_r above v_s, so $\phi_{T,e} = x_s/x_r$, and a chart cross-ratio for e is ρ_{ijk0}. Parametrizing so that

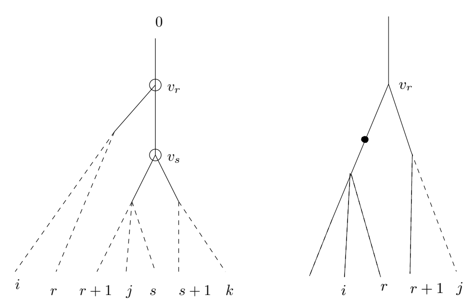

FIGURE 10. Comparing cross-ratios with simple ratios.

$z_0 = \infty$,

$$\rho_{ijk0} \;=\; -\frac{z_j - z_k}{z_j - z_i} = -\frac{x_s}{x_r}\left(\frac{x_j/x_s + x_{j+1}/x_s + \ldots + 1 + \ldots + x_{k-1}/x_s}{x_i/x_r + \ldots + 1 + \ldots + x_{j-1}/x_r}\right) = \phi_{T,e} f(\zeta).$$

The ratios in the bracketed function are products of ratios labeling edges below v_r and v_s, and the presence of the 1's means that bracketed function is smooth and never zero for all positive non-zero ratios and continuous as the labels in the chart go to 0, so $\rho_{ijk0} = 0$ if and only if $\phi_{T,e} = x_s/x_r = 0$. Now we prove it for a cross-ratio ρ_{ij} in the chart. Parametrizing so that $z_0 = \infty$ and using y to denote the height of the line with respect to this parametrization, consider an edge such as the one pictured in Figure 10, where the cross-ratio labelling e in a cross-ratio chart is ρ_{ij}, and the simple ratio $\phi_{T,e} = y/x_r$. Then

$$\rho_{ij} = \frac{y}{z_j - z_i} = \frac{y}{x_r}\left(\frac{1}{\frac{x_i}{x_r} + \ldots + 1 + \ldots + \frac{x_{j-1}}{x_r}}\right) = \phi_{T,e} f(\zeta)$$

where the ratios appearing in the big bracket are products of simple ratios labelling edges below v_r. The function $f(z)$ is smooth and never 0 for all positive non-zero ratios and it is continuous as ratios $\phi_{T,e} \to 0$. Moreover $\rho_{ij} = 0$ if and only if $y/x_r = 0$. The case of a colored vertex above a regular vertex is very similar so we omit it. This proves that the transition from a simple ratio chart to a cross ratios chart is a smooth change of coordinates on $\overline{M}_{n,1,\leq T}$. □

One sees from this description that $\overline{M}_{n,1}$ is *not* a manifold-with-corners. We say that a point $[S] \in \overline{M}_{n,1}$ is a *singularity* if $\overline{M}_{n,1}$ is not CW-isomorphic to a manifold with corners near $[S]$.

Example 6.3. The first singular point occurs for $n = 4$. The expression $(f(x_1)f(x_2))(f(x_3)f(x_4))$ is adjacent to the expressions $f(x_1x_2)(f(x_3)f(x_4))$, $(f(x_1)f(x_2))f(x_3x_4)$, $f(x_1)(f(x_2)(f(x_3)f(x_4)))$, $((f(x_1)f(x_2))f(x_3))f(x_4)$ and hence there are four edges coming out of the corresponding vertex. On the other hand, the dimension of $M_{4,1}$ is 3, see Figure 11. Thus $M_{4,1}$ cannot be a manifold with corners (and therefore, not a simplicial polytope.)

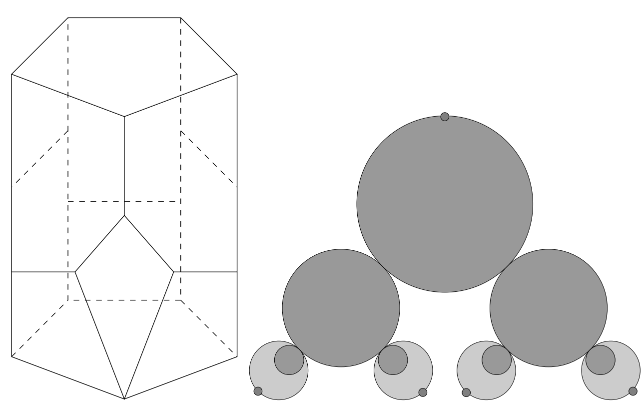

FIGURE 11. $\overline{M}_{4,1}$, sometimes called the "Chinese lantern". The singular point on the boundary, which has 4 edges coming out of it, corresponds to the nodal quilted disk at right.

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Lemma 6.4. *Any morphism of trees $f : T_0 \to T_1$ induces a morphism of balanced labellings $X(f) : X(T_0) \to X(T_1)$.*

Proof. Let $f : T_0 \to T_1$ be a morphism of trees. Given a balanced labelling $\varphi^m : \mathrm{E}(T_0) \to \mathbb{R}_{\geq 0}$ we obtain a balanced labelling $\varphi_1 : \mathrm{E}(T_1) \to \mathbb{R}_{\geq 0}$ by setting $\varphi_1(e_1) = \prod \varphi_0(e_0)$ where the product is over edges e_0 above e_1 that are collapsed under f. One sees easily that the resulting labelling of T_1 is balanced. $\square$

Corollary 6.5. *Let T be a colored tree. There exists a CW-isomorphism of $M_{n,1,T} \times X(T)$ onto a neighborhood of $M_{n,1,T}$ in $\overline{M}_{n,1}$.*

Proof. Let T^m be a maximal tree such that there exists a morphism of trees $f : T^m \to T$. For each vertex $v \in V(T)$, let $T_v^m \subset T$ be the subtree whose vertices map to v. Given a labelling $\varphi_m \in X(T^m)$, we obtain by restriction labellings $\varphi_v \in X(T_v^m)$ for each $v \in V(T)$, and from Lemma 6.4 a labelling $\varphi \in X(T)$. Thus we obtain a map

$$X(T^m) \to \left(\prod_{v \in V(T)} X(T_v^m) \right) \times X(T).$$

It is straightforward to verify that this map induces an isomorphism of $\{\varphi \in X(T^m) | \varphi(e) \neq 0 \forall e \in \mathrm{E}(T)\}$ onto $(\prod_{v \in V(T)} X(T_v^m)^*) \times X(T)$. The former is the image of $\overline{M}_{n,1,\leq T}$ under ϕ_T^{-1}. Since each tree T_v^m is maximal, Proposition 6.2 gives an isomorphism of $\overline{M}_{n,1,\leq T}$ onto $M_{n,1,T} \times X(T)$. $\square$

For later use, we describe subsets of the edges whose labels determine all others.

Definition 6.6. Let T be a maximal colored tree. Let e be an interior edge of T that is incident to a pair of trivalent vertices. Contraction of e produces a 4-valent vertex, and we say that the tree obtained by a *flop* of e is that which corresponds to the alternative resolution of the 4-valent vertex. A *fusion* move through an interior vertex v_i is one by which two colored vertices immediately below v_i become a single colored vertex immediately above v_i; we call the reverse move a *splitting* move. We say that two maximal colored trees T and T' differ by a *basic move* if they differ by a single flop, fusion, or splitting move.

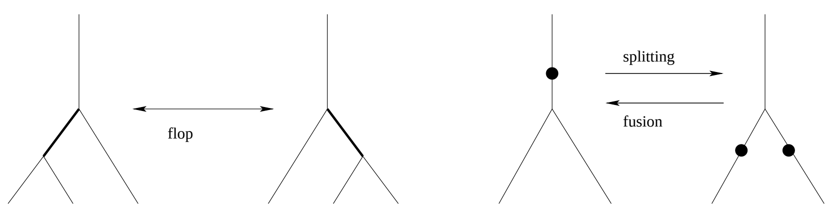

FIGURE 12. Basic moves on edges in a colored tree.

Any maximal colored tree can be obtained from a fixed maximal colored tree by a sequence of basic moves. Let T be a maximal colored tree. The simple ratios chart covering the open set $\overline{M}_{n,1,\leq T}$ assigns a simple ratio coordinate to each interior edge of T; however, on a given chart, some of those ratios may be functions of other ratios in the chart. There are six possibilities for an interior edge e of T, pictured in Figure 13. Note that the edges in the top four cases have a basic move associated to them, but not the two lower cases.

Lemma 6.7. *All simple ratios in ϕ_T are determined by the simple ratios labeling edges which have an associated basic move.*

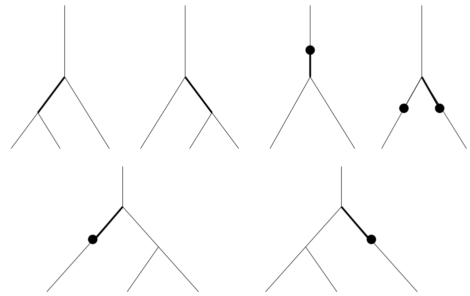

FIGURE 13. The top 4 types of edge have associated basic moves; the lower two types do not.

Proof. The simple ratios for the other edges are redundant: if an edge e doesn't have an associated basic move, it must be one of the lower two types in Figure 13. Let v be the vertex directly above e. Observe that there must exist a path from v to another colored vertex below it, such that every edge in the path is one of the top four types in Figure 13. Thus the simple ratios labeling the edges in that path appear in the reduced chart, and the relations imply that the product of the simple ratios in that path is equal to the simple ratio labeling e. □

Definition 6.8. A *reduced simple ratio chart* is given by restricting ϕ_T to the edges that have an associated basic move; we call these edges *reduced chart edges*.

Example 6.9. In the example of Figure 9, the reduced simple ratio chart consists of $y/x_1, x_1/x_2, x_3/x_2, x_5/x_3, y/x_5$ The simple ratio y/x_3 is redundant.

7. Colored metric ribbon trees

The multiplihedra have another geometric realization as *colored metric ribbon trees*. Colored trees were introduced in Boardman-Vogt [1], although their construction does not have the relations described below.

Definition 7.1. A *colored rooted metric ribbon tree* is a colored rooted ribbon tree with a metric $\lambda : E(T) \to (0, \infty)$ such that the sum of the edge lengths in any non-self-crossing path from a colored vertex $v \in V_{\text{col}}(T)$ back to the root is independent of $v \in V_{\text{col}}(T)$. A colored rooted metric ribbon tree is *stable* if each colored resp. non-colored vertex has valency at least 2 resp. 3.

Example 7.2. For the tree in Figure 14, an edge length map is subject to the relations $\lambda_1 + \lambda_2 + \lambda_3 = \lambda_1 + \lambda_2 + \lambda_4 = \lambda_1 + \lambda_2 + \lambda_5 = \lambda_1 + \lambda_6 = \lambda_7$.

For each stable colored tree T, we denote by $\mathrm{W}_{n,1,T}$ the set of all maps λ colored metric trees with underlying colored tree λ and by W_n the union

$$\mathrm{W}_{n,1} = \bigcup_T \mathrm{W}_{n,1,T}\,.$$

We define a topology on $\mathrm{W}_{n,1,T}$ as follows. Assume that a sequence $T_i = (T, \{\lambda_i\}_{i \in \mathbb{N}})$ of metric trees converges for each interior edge e to a non-negative real number. In other words,

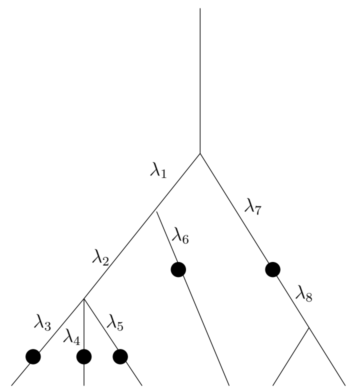

FIGURE 14. A colored ribbon tree. The relations on $\lambda_1, \ldots, \lambda_8$ imply that $\lambda_3 = \lambda_4 = \lambda_5, \lambda_3 + \lambda_2 = \lambda_6$, and $\lambda_6 + \lambda_1 = \lambda_7$.

$\lambda_i(e) \to \lambda_\infty(e) \in [0, \infty)$ for every $e \in E(T)$. We say that the corresponding colored metric trees T_i converge to T_∞ if

(a) T_∞ is the tree obtained from T by collapsing edges e for which $\lambda_\infty(e) := \lim_{i\to\infty} \lambda_i(e) = 0$. This defines a surjective morphism of colored rooted ribbon trees, $f : T \to T_\infty$.
(b) $V_{\mathrm{col}}(T_\infty) = f(V_{\mathrm{col}}(T))$
(c) $\lambda_\infty(e) = \lim_{i\to\infty} \lambda_i(f^{-1}(e))$, if this limit is non-zero.

Proposition 7.3. $\mathrm{W}_{n,1,T}$ *is a polyhedral cone in* $\mathbb{R}^n$*, where* $n = |E(T)| - |V_{\mathrm{col}}(T)| + 1$.

Proof. There is an $\mathbb{R}_+$ action on $\mathrm{W}_{n,1,T}$, given by $(\delta \cdot \lambda)(e) := \delta\lambda(e)$, so it is clearly a cone. The dimension follows from the fact that there are $|E(T)|$ variables and $|V_{\mathrm{col}}(T)| - 1$ relations. The polyhedral structure can be seen by writing $|V_{\mathrm{col}}(T)| - 1$ variables as linear combinations of n independent variables. Then the condition that all $\lambda(e) \geq 0$ means that $\mathrm{W}_{n,T}$ is an intersection of half-spaces. □

Example 7.4. In the example of Figure 14, $|E(T)| = 8$, and $|V_{\mathrm{col}}(T)| = 5$. We can choose independent variables to be $\lambda_1, \lambda_2, \lambda_3, \lambda_8$, and express the remaining variables as $\lambda_4 = \lambda_3,\ \lambda_5 = \lambda_3,\ \lambda_6 = \lambda_2 + \lambda_3,\ \lambda_7 = \lambda_1 + \lambda_2 + \lambda_3$. Thus the space of admissible edge lengths is parametrized by points in the polyhedral cone that is the intersection of $\mathbb{R}^4_+$ (for the independent variables being non-negative) with the half-spaces $\lambda_4 \geq 0, \lambda_5 \geq 0, \lambda_6 \geq 0$ and $\lambda_7 \geq 0$.

Exponentiating the labellings of the edges gives a map

$$\Theta : \mathrm{W}_{n,1} \to M_{n,1}, \quad (\Theta(\lambda)(e) = e^{-\lambda(e)}$$

Since $\lambda \geq 0$, the image of a cone $\mathrm{W}_{n,1,T}$ is identified directly with the subset of $\overline{M}_{n,1,\leq T}$ consisting of balanced labellings with $\phi_{T,e} \in (0, 1]$ for every $e \in \mathrm{E}(T)$.

Example 7.5. Consider the case $n = 3$, where we have fixed the parametrization of the elements of $M_{n,1}$ so that the interior circle is identified with a line of height L in half-space. Let $x = z_2 - z_1$ and $y = z_3 - z_2$. The images of $\mathrm{W}_{3,1,T}$ subdivide $\mathbb{R}^2_{>0}$ into 6 regions, see Figure 15, each of which corresponds to a cone in $\mathbb{R}^2$ via the homeomorphism $(x, y) \mapsto (\log x, \log y)$.

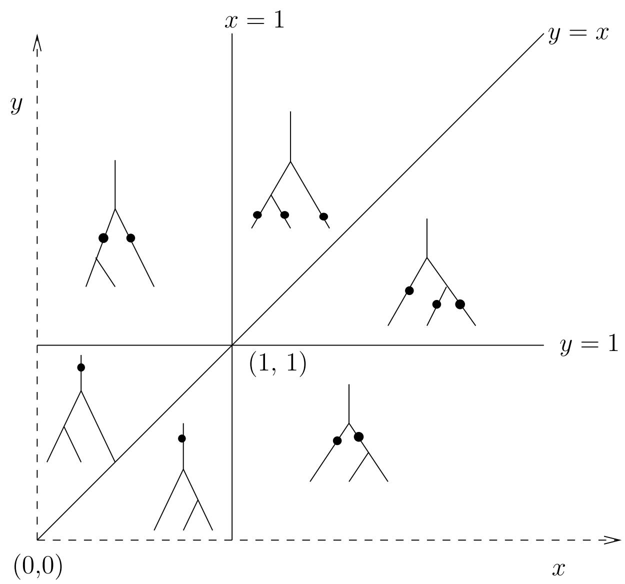

FIGURE 15. The image of the cones of $\mathrm{W}_{3,1}$ in the moduli space $M_{3,1}$.

There is a natural compactification $\overline{\mathrm{W}}_{n,1}$ of $\mathrm{W}_{n,1}$ by allowing edges to have length ∞. The map Θ extends to the compactifications by taking limits in appropriate charts.

Theorem 7.6. *The map $\Theta : \overline{\mathrm{W}}_{n,1} \to \overline{M}_{n,1}$ is a homeomorphism, with the property that for any combinatorial type T, $\Theta(\overline{\mathrm{W}}_{n,1,T})$ intersects $M_{n,T,1}$ in a single point.*

This is the colored analog of Theorem 2.8.

Proof. As $\lambda(e) \to \infty$, the identification $\lambda(e) \to \phi_{T,e} = e^{-\lambda}$ implies $\phi_{T,e} \to 0$. Thus the image of a compactified cone $\overline{\mathrm{W}}_{n,1,T}$ is identified with the subset of $\overline{M}_{n,1,\leq T}$ consisting of balanced labellings with $\phi_{T,e} \in [0,1]$ for every $e \in E(T)$. □

8. TORIC VARIETIES AND MOMENT POLYTOPES.

In this section we show

Theorem 8.1. *$\overline{M}_{n,1}$ is homeomorphic to the non-negative part of an embedded toric variety V in $\mathbb{P}^k(\mathbb{C})$, where k is the number of maximal colored trees with n leaves.*

In particular, $\overline{M}_{n,1}$ is isomorphic as a CW-complex to a convex polytope; this reproduces the result of Forcey [2]. Using this we prove the main Theorem 1.1. First we define the toric variety V. Recall from Section 6 that a point in $M_{n,1}$ can be identified with a projective coordinate $\underline{x} = (x_1 : x_2 : \ldots : x_{n-1} : y)$, by parametrizing such that $z_0 = \infty$, and setting $x_i = z_{i+1} - z_i$ and y to be the height of the line. Let T be a maximal colored tree. Adapting the algorithm of Forcey in [2], we associate a *weight vector* $\mu_T \in \mathbb{Z}^n$ to the tree T as follows. A pair of adjacent leaves in T, say i and $i+1$, determines a unique vertex in T, which we label v_i. Let a_i be the number of leaves on the left side of v_i, and let b_i be the number of leaves on the right side of v_i. Let

$$\delta_i = \begin{cases} 0 & \text{if } v_i \text{ is below the level of the colored vertices, and} \\ 1 & \text{if } v_i \text{ is above the colored vertices.} \end{cases}$$

Set

$$\mu_T := (a_1 b_1(1+\delta_1), \ldots, a_i b_i (1+\delta_i), \ldots, a_{n-1}b_{n-1}(1+\delta_{n-1}), -\sum_i \delta_i a_i b_i).$$

Example 8.2. The tree in Figure 16 has weight vector $(2, 16, 6, 1, 4, -14)$, and monomial $x_1^2 x_2^{16} x_3^6 x_4 x_5^4 y^{-14}$.

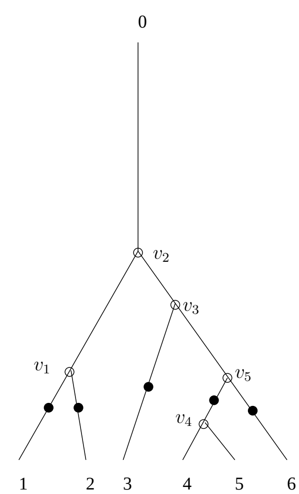

FIGURE 16. A maximal colored tree, whose weight vector is $(2, 16, 6, 1, 4, -14)$.

Fix some ordering of the k maximal colored trees with n leaves, $T_1, \ldots, T_k$. The projective toric variety $V \subset \mathbb{P}^{k-1}(\mathbb{C})$ is the closure of the image of the embedding

$$(x_1 : \ldots : x_{n-1} : y) \mapsto (\underline{x}^{\mu_{T_1}} : \ldots : \underline{x}^{\mu_{T_k}}). \tag{5}$$

The entries in the weight vectors always sum to $n(n-1)/2$, so the map is well-defined on the homogeneous coordinates.

Lemma 8.3. *Suppose that two maximal colored trees T and T' differ by a single basic move involving an edge $e \in \mathrm{E}(T)$. Let $\phi_{T,e}$ denote the simple ratio labeling the edge e in the chart determined by T. Then*

$$\frac{\underline{x}^{\mu_{T'}}}{\underline{x}^{\mu_T}} = \phi_{T,e}^m$$

for an integer $m > 0$. In general, for two maximal trees T and T',

$$\frac{\underline{x}^{\mu_{T'}}}{\underline{x}^{\mu_T}} = \phi_{T,e_1}^{m_1} \phi_{T,e_2}^{m_2} \cdots \phi_{T,e_r}^{m_r}$$

for some edges $e_1, \ldots, e_r$ of T and positive integers $m_1, \ldots, m_r$.

Proof. First let us consider the case of a single flop. Without loss of generality consider the situation in Figure 17. Say T is on the left, and T' is on the right, and the affected edges are in bold. The weight vectors μ_T and $\mu_{T'}$ are the same in all entries except entries i and j, where

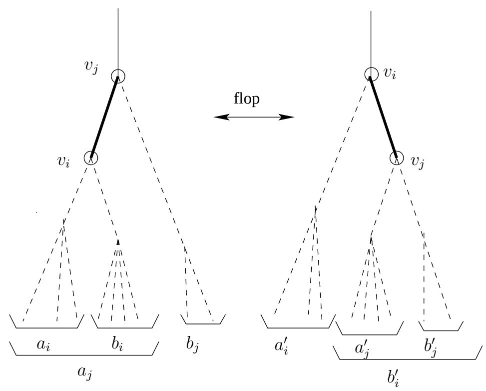

FIGURE 17. The effect of a flop on weight vectors.

$$(\mu_T)_i = a_i b_i, \quad (\mu_T)_j = a_j b_j = (a_i + b_i) b_j, \quad (\mu_{T'})_i = a_i(b_i + b_j), \quad (\mu_{T'})_j = b_i b_j.$$

Therefore,

$$\frac{\underline{x}^{\mu_{T'}}}{\underline{x}^{\mu_T}} = \frac{x_i^{a_i(b_i+b_j)} x_j^{b_i b_j}}{x_i^{a_i b_i} x_j^{(a_i+b_i)b_j}} = \frac{x_i^{a_i b_j}}{x_j^{a_i b_j}} = \left(\frac{x_i}{x_j}\right)^{a_i b_j}$$

and observe that $\phi_{T,e} = x_i/x_j$ is the ratio labeling that edge of T, and $a_i b_j \geq 1$.

For the other kinds of basic move it suffices to consider fusion, in which a pair of colored vertices are below v_i in T, and above v_i in T'. The weight vectors of T and T' are identical in all entries except for the i-th entry, which corresponds to the exponent of x_i, and the $n+1$-th entry, which corresponds to the exponent of y:

$$(\mu_T)_i = 2a_i b_i, \quad \mu_{T'} = a_i b_i, \quad (\mu_{T'})_{n+1} - (\mu_T)_{n+1} = -(0) - (-a_i b_i)$$

Therefore

$$\frac{\underline{x}^{\mu_{T'}}}{\mu_T} = \frac{x_i^{a_i b_i} y^{-0}}{x_i^{2a_i b_i} y^{-a_i b_i}} = \frac{y^{a_i b_i}}{x_i^{a_i b_i}} = \left(\frac{y}{x_i}\right)^{a_i b_i}$$

where $\phi_{T,e} = y/x_i$ is the ratio labeling the two edges below v_i of T, and $a_i b_j \geq 1$.

The vertices are partially ordered by their positions in the tree; the effect of basic moves on the partial ordering are individual changes $(v_i \leq v_j) \leftrightarrow (v_j \leq v_i)$, or $(v_{\mathrm{col}} \leq v_i) \leftrightarrow (v_i \leq v_{\mathrm{col}})$, between adjacent vertices. In general, every maximal tree is obtained from a *fixed* tree T by a sequence of independent basic moves – by independent we just mean that each one involves a different pair of vertices. We prove the general case by induction on the number of independent basic moves needed to get from a fixed maximal tree T, to any other maximal tree T'. Having proved the base case, now consider a tree T' obtained after a sequence of $k+1$ flops. Write $\widetilde{T}$ for a tree which is k independent moves away from T and one move away from T'. Suppose that the the final move between $\widetilde{T}$ and T' is described by the $(v_i \leq v_j) \to (v_j \leq v_i)$. By the inductive hypothesis and the base step,

$$\frac{\underline{x}^{\mu_{T'}}}{\underline{x}^{\mu_T}} = \frac{\underline{x}^{\mu_{T'}}}{\underline{x}^{\mu_{\widetilde{T}}}} \frac{\underline{x}^{\mu_{\widetilde{T}}}}{\underline{x}^{\mu_T}} = \left(\frac{x_i}{x_j}\right)^m \phi_{T,e_1}^{m_1} \phi_{T,e_2}^{m_2} \dots \phi_{T,e_r}^{m_r}$$

for some positive integers $m_1, \dots, m_r$ and m, and some edges $e_1, \dots, e_r$ of T. Since none of the previous flops involved the pair v_i and v_j, the partial order in the original tree T must have also had $v_i \leq v_j$, although they were possibly not adjacent in T. In any case, the ratio x_i/x_j is a product of the ratios in the chart ϕ_T labeling the edges from v_i to v_j. The case where the final move is one of $(v_{\mathrm{col}}, v_i) \leftrightarrow (v_i, v_{\mathrm{col}})$ is similarly straightforward. This completes the inductive step. □

Proof of Theorem 8.1. We use Lemma 8.3 to identify the simple ratios in a reduced chart for each maximal colored tree T_i, with the non-negative part of the affine slice $V \cap \mathbb{A}_i$. Consider T_1. The affine piece $V \cap \mathbb{A}_1$ consists of all points

$$\left(1 : \frac{\underline{x}^{\mu_{T_2}}}{\underline{x}^{\mu_{T_1}}} : \dots : \frac{\underline{x}^{\mu_{T_k}}}{\underline{x}^{\mu_{T_1}}}\right)$$

where the entries may be 0. Now let $\phi_{T_1,e_1}, \dots, \phi_{T_1,e_l}$ be the simple ratio coordinates in a reduced chart for the open set $M_{n,1,\leq T_1} \subset \overline{M}_{n,1}$ (Definition 6.8). By construction, the edges $e_1, \dots, e_l$ of T_1 have associated basic moves. Thus they determine a set $T_1(e_1), \dots, T_1(e_l)$ of maximal colored trees, where each $T_1(e_i)$ is obtained from T_1 by the basic move associated to the edge e_i. Without loss of generality, assume that the l maximal trees $T_2, \dots, T_{l+1}$ are respectively $T_1(e_1), \dots, T_1(e_l)$. By Lemma 8.3, we identify the non-negative part of $V \cap \mathbb{A}_1$ with the chart ϕ_{T_1} by the map

$$\overline{M}_{n,1,T_1} \longrightarrow V \cap \mathbb{A}_1 \quad (\phi_{T_1,e_1}, \dots, \phi_{T_1,e_l}) \mapsto (1 : \phi_{T_1,e_1}^{m_1} : \phi_{T_1,e_2}^{m_2} : \dots : \phi_{T_1,e_l}^{m_l} : * : \dots : *)$$

where $m_1, \dots, m_l$ are positive integers depending on the combinatorics of T_1, and the entries labeled $*$ are higher products of $\phi_{T_1,e_1}, \dots, \phi_{T_1,e_l} l$. This map is well-defined, one-to-one and onto for ϕ_{T_1,e_i} and $\phi_{T_1,e_i}^{m_i}$ which are all in the non-negative range $[0, \infty)$. □

Corollary 8.4. *$\overline{M}_{n,1}$ is CW-isomorphic to the convex hull of the weight vectors in $\mathbb{R}^n$, and thus CW-isomorphic to a $(n-1)$-dimensional polytope.*

Proof. The non-negative part of a projective toric variety constructed with weight vectors is homeomorphic, via the moment map, to the convex hull of the weight vectors (see, for example, [5], [10]). □

Proof of Theorem 1.1. By induction on n: The one-dimensional spaces $\overline{M}_{2,1}, \overline{M}_3, J_2$ and K_3 are all compact and connected, and so CW-isomorphic. It suffices, therefore, to show that $\overline{M}_{n,1}$ is the cone on its boundary. This is true since it is homeomorphic to a convex polytope. □

9. Stable weighted disks

Fukaya, Oh, Ohta, and Ono [3] introduced another geometric realization of the multiplihedron, although the CW-structure is slightly different. A *weighted stable* $n+1$*-marked disk* consists of

(a) a stable nodal disk $(\underline{\Sigma} = (\Sigma_1, \ldots, \Sigma_m), \underline{z} = (z_0, \ldots, z_n))$
(b) for each component $\Sigma_1, \ldots, \Sigma_m$ of $\underline{\Sigma}$, a *weight* $x_i \in [0,1]$

with the following property: if Σ_i is further away from Σ_j from the *root marking* z_0 then $x_i \leq x_j$. An *isomorphism* of rooted stable disks is an isomorphism of stable disks intertwining with the weights. Let $\overline{M}^w_{n+1}$ denote the moduli space of stable weighted marked disks, equipped with the natural extension of the Gromov topology in which a sequence $(\underline{\Sigma}_i, \underline{z}_i, \underline{\lambda}_i)$ converges to $(\underline{\Sigma}, \underline{z}, \underline{\lambda})$ if $(\underline{\Sigma}_i, \underline{z}_i)$ Gromov converges to $(\underline{\Sigma}, \underline{z})$ and the weights on the limit curve are pulled back from those on $\underline{\Sigma}$ via the morphism of trees appearing in the limit.

For example, $\overline{M}^w_3$ is an interval; $\overline{M}^w_4$ is a hexagon consisting of a square and two triangles, joined along two edges, see Figure 18. Each triangle is defined by the inequality $0 \leq x_2 \leq x_1 \leq 1$. The moduli space $\overline{M}^w_5$ has 23 cells of dimension 2 on the boundary (2 projecting onto 2-cell of $\overline{M}_5$, 10 projecting onto 1-cells of $\overline{M}_5$, and 11 projecting onto vertices of $\overline{M}_5$.) On the other hand, the multiplihedron $\overline{M}_{4,1}$ has 13 cells of dimension 2 on the boundary, see Figure 11.

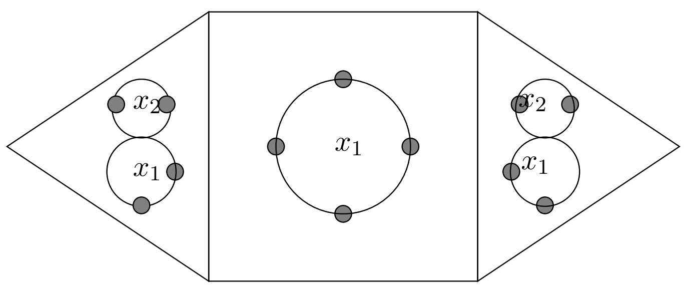

Figure 18. Moduli of weighted 4-marked disks

10. Stable scaled affine lines.

In this section we re-interpret the moduli space of quilted disks as a *moduli space of stable scaled lines*. This construction has the advantage that it works for any field. Working over $k = \mathbb{C}$ gives a moduli space introduced Ziltener's study [13] of gauged pseudoholomorphic maps from the complex plane; we show it is a projective variety with toric singularities.

Definition 10.1. Let k be a field. A *scaled marked line* is a datum $(\mathbb{A}, \underline{z}, \phi)$, where $\mathbb{A}$ is an affine line over k, $\underline{z} = (z_1, \ldots, z_n) \in \mathbb{A}$ are distinct points, and $\phi \in \Omega^1(\mathbb{A}, k)^k$ is a translationally invariant area form. An *isomorphism of scaled marked lines* is an isomorphism $\psi : \mathbb{A} \to \mathbb{A}$ that intertwines the area forms and markings. A scaled line is *stable* if the automorphism group is finite, that is, has at least one marking. Denote by $M_{n,1}(k)$ the corresponding moduli space of stable, scaled marked lines.

In the case $k = \mathbb{R}$, $M_{n,1}(\mathbb{R})$ has as a component (given by requiring that the markings appear in order) the moduli space $M_{n,1}$ of the previous section, through identifying $(z_1, \dots, z_n, \phi)$ with $(z_0 = \infty, z_1, \dots, z_n, L_\phi)$, where $L_\phi \subset \mathbb{A}$ is a line of height $1/\phi(1)$. $M_{n,1}(k)$ has a natural compactification, obtained by allowing the points to come together and the volume form ϕ to scale. For any nodal curve $\overline{C}$ with markings $z_0, \dots, z_n$, and component $\overline{C}_\alpha$ of $\overline{C}$, we write $z_{\alpha i}$ for the special point in $\overline{C}_\alpha$ that is either the marking z_i, or the node closest to z_i.

Definition 10.2. A *(genus zero) nodal scaled marked line* is a datum $(\overline{C}, z, \phi)$, where $\overline{C}$ is a (genus zero) projective nodal curve, $\underline{z} = (z_0, \dots, z_n)$ is a collection of markings disjoint from the nodes, and for each component $\overline{C}_\alpha$ of $\overline{C}$, the affine line $C_\alpha := \overline{C}_\alpha \setminus \{z_{\alpha 0}\}$ is equipped with a (possibly zero or infinite) translationally invariant volume form $\phi_i \in \Omega^1(C_\alpha, k)^k$. We call a volume form ϕ_i *degenerate* if it is zero or infinite. An automorphism of a stable nodal scaled curve is an automorphism of the nodal curve preserving the volume forms and the markings. A nodal scaled marked curve is *stable* if it has finite automorphism group, or equivalently, if each component with non-degenerate (resp. degenerate) volume form has at least two (resp. three) special points.

The affine structure on C_α is unique up to dilation, so that $\Omega^1(C_\alpha, k)^k$ is well-defined. The *combinatorial type* of a nodal scaled marked affine line is a rooted colored tree: Vertices represent components of the nodal curve, edges represent nodes, labeled semi-infinite edges represent the markings, with the root always labelled by z_0. Every path from a leaf back to the root must pass through exactly one colored vertex.

Now we specialize to the case $k = \mathbb{C}$. $M_{n,1}(\mathbb{C})$ contains as a subspace those scaled marked curves such that all markings lie on the projective real line, $\mathbb{R}P := \mathbb{R} \cup \{\infty\}$; these are naturally identified with marked disks. More accurately, $M_{n,1}(\mathbb{C})$ admits an antiholomorphic involution induced by the antiholomorphic involution of $\mathbb{P}^1(\mathbb{C})$. The involution extends to an antiholomorphic involution of $\overline{M}_{n,1}(\mathbb{C})$. The multiplihedron $\overline{M}_{n,1}$ can be identified with the subset of the fixed point set such that the points are in the required order.

We introduce coordinates on $M_{n,1}(\mathbb{C})$ in the same way as we did for $M_{n,1}$. Define two types of coordinates, of the form ρ_{ijkl} where i, j, k, l are distinct indices in $0, 1, \dots, n$, and of the form ρ_{ij}, where i, j are distinct indices in $1, \dots, n$. The ρ_{ijkl} are defined as before, and the ρ_{ij} are defined as follows: given a representative $(z_1, z_2, \dots, z_n, \phi)$,

$$\rho_{ij}([z_1, \dots, z_n, \phi]) := (\phi(1)(z_j - z_i))^{-1}.$$

The coordinates extend to the compactification $\overline{M}_{n,1}(\mathbb{C})$. For the coordinates ρ_{ijkl}, we evaluate the cross-ratio at a component $\overline{C}_\alpha$ in the bubble tree for which at least three of $z_{\alpha i}, z_{\alpha j}, z_{\alpha k}$ and $z_{\alpha l}$ are distinct, normalizing by

$$\rho_{ijkl} = \begin{cases} \infty, & \text{if} \quad z_{\alpha i} = z_{\alpha j} \quad \text{or} \quad z_{\alpha k} = z_{\alpha l}, \\ 1, & \text{if} \quad z_{\alpha i} = z_{\alpha k} \quad \text{or} \quad z_{\alpha j} = z_{\alpha l}, \\ 0, & \text{if} \quad z_{\alpha i} = z_{\alpha l} \quad \text{or} \quad z_{\alpha j} = z_{\alpha k}. \end{cases} \tag{6}$$

For the coordinates ρ_{ij}, we evaluate them at the unique component $\overline{C}_\alpha$ at which $z_{\alpha 0}, z_{\alpha i}$ and $z_{\alpha j}$ are distinct, normalizing by

$$\rho_{ij} = \begin{cases} 0, & \text{if } C_\alpha \text{ has infinite scaling,} \\ \infty, & \text{if } C_\alpha \text{ has zero scaling.} \end{cases}$$

The same arguments as in the real case show that the product of forgetful morphisms defines an embedding

$$\rho_{n,1} : \overline{M}_{n,1} \to (\mathbb{P}^1(\mathbb{C}))^{(n+1)n(n-1)(n-2)/4!+n(n-1)/2}.$$

The coordinates ρ_{ijkl} and ρ_{ij} also satisfy recursion relations

$$\rho_{jklm} = \frac{\rho_{ijkm} - 1}{\rho_{ijkm} - \rho_{ijkl}}, \quad \rho_{jk} = \frac{\rho_{ij}}{\rho_{ijk0}}.$$

Let $\overline{A}_{n,1}(\mathbb{C})$ denote the closure of the algebraic variety defined by the two types of cross ratio coordinate and the relations (10).

Theorem 10.3. *The map $\rho_{n,1} : \overline{M}_{n,1}(\mathbb{C}) \to \overline{A}_{n,1}(\mathbb{C})$ is a bijection.*

The proof of the bijection is an extension of the corresponding result for genus zero stable nodal $(n+1)$-pointed curves, $\overline{M}_{n+1}(\mathbb{C})$. In this case the cross-ratios ρ_{ijkl} satisfy (10). Let $\overline{A}_n(\mathbb{C})$ denote the closure of the algebraic variety defined by the cross-ratio coordinates and the relation (10). The image of the canonical embedding $\rho_n(\overline{M}_{n+1}(\mathbb{C}))$ with cross-ratios is contained in $\overline{A}_n(\mathbb{C})$, the map $\rho_n : \overline{M}_{n+1}(\mathbb{C}) \to \overline{A}_n(\mathbb{C})$ is a bijection [8, Theorem D.4.5].

Proof of Theorem 10.3. First we show that $\rho_{n,1}$ is injective. Given a nodal stable scaled marked line $(\overline{C}, \underline{z}, \phi)$, by construction the combinatorial type uniquely determines which cross-ratios ρ_{ijkl} are $0, 1$ or ∞, and which cross-ratios ρ_{ij} are 0 and ∞. In addition, the isomorphism class of each component of $\overline{C}$ is determined by the cross-ratios ρ_{ijkl} with values in $\mathbb{P}^1(\mathbb{C}) \setminus \{0, 1, \infty\}$ and ρ_{ij} with values in $\mathbb{P}^1(\mathbb{C}) \setminus \{0, \infty\}$, so the map $\rho_{n,1}$ is injective. To show that $\rho_{n,1}$ is surjective, let $\rho \in \overline{A}_{n,1}(\mathbb{C})$. By the result for stable curves [8, D.4.5], there is a unique stable, nodal $(n+1)$-marked genus zero curve of combinatorial type given by a rooted tree T (the root corresponds to the marking z_0), which realizes the cross-ratios of the form ρ_{ijkl}.

Lemma 10.4. *Let $\alpha \in V(T)$, and suppose that for some $1 \le i < j \le n$, $z_{\alpha_i}, z_{\alpha_j}, z_{\alpha_0}$ are distinct at α.*

(a) *If $\rho_{ij} = 0$, then for every vertex $\beta \in V(T)$ in a path from α to the root (including α itself), $\rho_{kl} = 0$ for every distinct triple $z_{\beta_k}, z_{\beta_l}, z_{\beta_0}$.*
(b) *If $\rho_{ij} = \infty$, then for every vertex $\beta \in V(T)$ in a path from α away from the root (including α itself), $\rho_{kl} = \infty$ for every distinct triple $z_{\beta_k}, z_{\beta_l}, z_{\beta_0}$.*
(c) *If $0 < |\rho_{ij}| < \infty$, then*
   (i) *for every other distinct triple $z_{\alpha_k}, z_{\alpha_l}, z_{\alpha_0}$ on α, $\rho_{kl} \notin \{0, \infty\}$;*
   (ii) *for every vertex $\beta \in V(T)$ adjacent to α towards the root, and every distinct triple $z_{\beta_k}, z_{\beta_l}, z_{\beta_0}$, $\rho_{kl} = 0$;*
   (iii) *for every vertex $\beta \in V(T)$ adjacent to α away from the root, and every distinct triple $z_{\beta_k}, z_{\beta_l}, z_{\beta_0}$, $\rho_{kl} = \infty$.*

Proof. (a) First, we show that $\rho_{kl} = 0$ for every k, l such that $z_{\alpha_k}, z_{\alpha_l}, z_{\alpha_0}$ are distinct. Without loss of generality suppose that z_{α_k} is distinct from z_{α_j} and z_{α_0}. Then $\rho_{ijk0} \neq 0$ and so by (10), $\rho_{jk} = \rho_{ij}/\rho_{ijk0} = 0$. Now without loss of generality suppose that z_{α_l} is distinct from z_{α_k} and z_{α_0}. Then $\rho_{jkl0} \neq 0$ so by (10) $\rho_{kl} = \rho_{jk}/\rho_{jkl0} = 0/\rho_{jkl0} = 0$. Now consider the vertex β that is immediately adjacent to α in the direction of the root. Without loss of generality, suppose that z_{β_k} is distinct from $z_{\beta_i} = z_{\beta_j}$. The combinatorics of T at α and β imply that $\rho_{ijk0} = \infty$, hence by (10), $\rho_{jk} = \frac{\rho_{ij}}{\rho_{ijk0}} = 0/\infty = 0$. Applying the first argument that $\rho_{kl} = 0$ for all k, l with $z_{\beta_k}, z_{\beta_l}, z_{\beta_0}$ distinct. The result holds by remaining vertices in the path from α by induction. (b) First, we show that $\rho_{kl} = \infty$ for every k, l such that $z_{\alpha_k}, z_{\alpha_l}, z_{\alpha_0}$ are distinct. Without loss of generality suppose that z_{α_k} is distinct from z_{α_j} and z_{α_0}. Then $\rho_{ijk0} \neq \infty$, hence by (10) $\rho_{jk} = \rho_{ij}/\rho_{ijk0} = \infty/\rho_{ijk0} = \infty$. Now without loss of generality suppose that z_{α_l} is distinct from z_{α_k} and z_{α_0}. Then $\rho_{jkl0} \neq \infty$ hence by (10) $\rho_{kl} = \rho_{jk}/\rho_{jkl0} = \infty/\rho_{jkl0} = \infty$. Now consider a vertex β that is immediately adjacent to α away from the root. It is now enough to show that $\rho_{mn} = \infty$ for some m, n such that z_{β_m}, z_{β_n} and z_{β_0} are distinct. Pick k and l such that $z_{\alpha_k}, z_{\alpha_l}, z_{\alpha_0}$ are distinct (hence by the previous argument $\rho_{kl} = \infty$), and such that β is adjacent to α through a node that identifies z_{α_l} with z_{β_0}. Now let z_{β_m} be distinct from z_{β_l} and z_{β_0}. Then $\rho_{klm0} \neq \infty$, so by (10), $\rho_{lm} = \rho_{kl}/\rho_{klm0} = \infty/\rho_{klm0} = \infty$.

(c) *Proof of (i):* If z_{α_k} is distinct from $z_{\alpha_i}, z_{\alpha_j}$ and z_{α_0}, then $\rho_{ijk0} \notin \{0, 1, \infty\}$ so $\rho_{jk} = \rho_{ij}/\rho_{ijk0}$ hence $0 < |\rho_{jk}| < \infty$. Repeating this argument implies that $0 < |\rho_{kl}| < \infty$ for any k and l such that $z_{\alpha_k}, z_{\alpha_l}, z_{\alpha_0}$ are distinct. *Proof of (ii):* In light of (a) and the proof of (c)(i), it is enough to prove that for any k such that z_{β_j}, z_{β_k} and z_{β_0} are distinct, then $\rho_{jk} = 0$. Note that since β is closer to the root than α, $z_{\beta_i} = z_{\beta_j}$. Hence, $\rho_{ijk0} = \infty$, and by (10), $\rho_{jk} = \rho_{ij}/\rho_{ijk0} = 0$. *Proof of (iii):* In light of (b) and (c)(i), it is enough to prove the following case: if α is incident to β in such a way that $z_{\beta_i} = z_{\beta_0}$ is distinct from z_{β_j} and z_{β_k}, then $\rho_{jk} = \infty$. In this case, $\rho_{ijk0} = 0$, so (10) implies $\rho_{jk} = \rho_{ij}/\rho_{ijk0} = \infty$.

□

By Lemma 10.4, the vertices of the tree T can be partitioned into subsets for which the cross-ratios ρ_{ij} defined on them are 0, ∞, or finite non-zero. Let

$$\begin{aligned} V_0 &:= \{\alpha \in V(T) | z_{\alpha_i}, z_{\alpha_j}, z_{\alpha_0} \text{ are distinct and } \rho_{ij} = 0\}, \\ V_f &:= \{\alpha \in V(T) | z_{\alpha_i}, z_{\alpha_j}, z_{\alpha_0} \text{ are distinct and } 0 < |\rho_{ij}| < \infty\}, \\ V_\infty &:= \{\alpha \in V(T) | z_{\alpha_i}, z_{\alpha_j}, z_{\alpha_0} \text{ are distinct and } \rho_{ij} = \infty\}. \end{aligned}$$

If V_0 is empty, turn the marked point z_0 into a nodal point and attach it to the nodal point ζ of a scaled curve $(\overline{C}', z_0, \zeta, \phi)$. If V_0 is non-empty, by Lemma 10.4 it must be a connected sub-tree which includes the component containing the root z_0. If a marked point z_i, $i = 1, \ldots, n$ is on a component labeled by $\alpha \in V_0$, turn the marked point z_i into a nodal point z_{α_i} and attach it to the nodal point ζ of a scaled curve $(\overline{C}, \zeta, z_i, v)$. If $\alpha \in V_f$ then by Lemma 10.4 it is attached by a node to V_0. Suppose that $z_{\alpha_i}, z_{\alpha_j}$ and z_{α_0} are distinct and $0 < |\rho_{ij}| < \infty$. Identify this sphere component and its markings with a stable marked curve with the same markings, and a volume form ϕ determined by parametrizing $z_{\alpha_0} = \infty, z_{\alpha_i} = 0, z_{\alpha_j} = 1$ and putting $1/\phi(1) = \rho_{ij}$. Finally, suppose that $\alpha \in V_\infty$ is connected by a nodal point z_{α_0} to a nodal point

z_{β_i} of V_0. Then insert a stable marked curve $(\overline{C}, \zeta_0, \zeta_1, \phi)$ such that the node identifications are ζ_0 with z_{β_i}, and ζ_1 with z_{α_0}.

At the end of this process one obtains a stable nodal, marked scaled curve in $\overline{M}_{n,1}(\mathbb{C})$, whose combinatorial type is a colored tree refining the tree T, and whose image under the cross-ratio embedding is the same as the original point $\rho \in \overline{A}_{n,1}(\mathbb{C})$. $\square$

Given the combinatorial type of a stable scaled curve, one can choose a local chart of cross-ratios according to the same prescription as given in the real case.

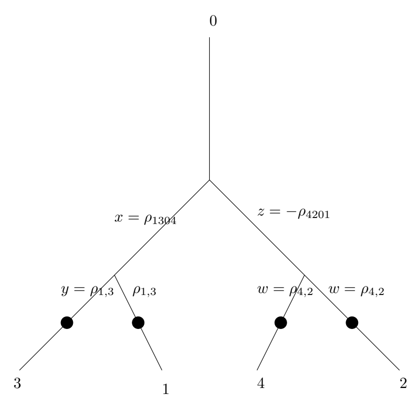

FIGURE 19. A cross-ratio chart in $\overline{M}_{4,1}(\mathbb{C})$. The relation is $xy = zw$, which gives a toric singularity.

The subset $G(T) = X(T) \cap \mathrm{Hom}(\mathrm{E}(T), \mathbb{C}^*)$ of points with non-zero labels is the kernel of the homomorphism $\mathrm{Hom}(\mathrm{E}(T), \mathbb{C}^*) \to \mathrm{Hom}(\mathrm{Vert}^-(T), \mathbb{C}^*)$ given by taking the product of labels from the given vertex to the colored vertex above it, and is therefore an algebraic torus. The torus $G(T)$ acts on $X(T)$ by multiplication with a dense orbit. Choose a planar structure on T. Let

$$\phi_T : \overline{M}_{n,1,\leq T}(\mathbb{C}) \to X(T)$$

denote the map given by the simple ratios in Definition (4) (now allowed to be complex). After re-labelling it suffices to consider the case that the ordering is the standard ordering. We denote by $X^*(T)$ the Zariski open subset of $X(T)$ defined by the equations $1 + x_{i+1}/x_i + \ldots + x_j/x_i = (z_i - z_j)/(z_i - z_{i+1}) = 0$, for $1 \leq i < j \leq n$.

Theorem 10.5. *ϕ_T is an isomorphism of $\overline{M}_{n,1,\leq T}(\mathbb{C})$ onto $X^*(T)$.*

Proof. Let T be a maximal colored tree and consider the map $\overline{M}_{n,1,\leq T} \to X(T)$ given by the simple ratios. The same argument as in the real case shows that any $\lambda \in X(T)$ is in the image of some quilted disk unless at some stage the reconstruction procedure assigns the same position to two markings i, j in different branches; in this case we have $\lambda \in X^*(T)$. The set of exceptional points in $X(T)$ is an affine subvariety of $X(T)$ disjoint from $0 \in X(T)$ hence the Theorem. $\square$

Corollary 10.6. *Let T be a colored tree. There exists an isomorphism of a Zariski open neighborhood of $M_{n,1,T}(\mathbb{C}) \times \{0\}$ in $M_{n,1,T}(\mathbb{C}) \times X(T)$ with a Zariski open neighborhood of $M_{n,1,T}(\mathbb{C})$ in $\overline{M}_{n,1}(\mathbb{C})$. Thus $\overline{M}_{n,1}(\mathbb{C})$ is a projective variety with at most toric singularities.*

The proof is similar to the real case in Corollary 6.5 and left to the reader. This completes the proof of Theorem 1.2 in the introduction.

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