PARTLY-LOCAL DOMAIN-DEPENDENT ALMOST COMPLEX STRUCTURES

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Abstract. We fill a gap pointed out by Nick Sheridan in the proof of independence of genus zero Gromov-Witten invariants from the choice of divisor in the Cieliebak-Mohnke perturbation scheme [1].

Contents

1. Introduction

A convenient scheme to regularize moduli spaces of genus zero pseudoholomorphic maps was introduced by Cieliebak-Mohnke [1]. In this scheme one chooses a Donaldson hypersurface to stabilize the domains and then a generic domain-dependent almost complex structure to achieve regularity. The scheme leads to a definition of genus zero Gromov-Witten invariants over the rationals that counts some geometric objects, as opposed to the more abstract perturbation schemes in the Kuranishi or polyfold methods. The proof of independence of Gromov-Witten invariants from the choice of Donaldson hypersurfaces in [1, 8.18] depends on the construction of a parametrized moduli space for the following situation: Given a type Γ of stable marked curve let $\overline{\mathcal{U}}_\Gamma \to \overline{\mathcal{M}}_\Gamma$ denote the universal curve over the compactified moduli space $\overline{\mathcal{M}}_\Gamma$ of curves of type Γ. Let $\mathcal{J}_\tau(X,\omega)$ denote the space of ω-tamed almost complex structures on the given symplectic manifold (X,ω) with rational symplectic class $[\omega] \in H^2(X,\omega)$. A *domain-dependent almost complex structure* is a map

$$J_\Gamma : \overline{\mathcal{U}}_\Gamma \to J_\tau(X,\omega).$$

Associated to a coherent collection of sufficiently generic choices $\underline{J} = (J_\Gamma)$ is a Gromov-Witten pseudocycle $\overline{\mathcal{M}}_{0,n}(X,\beta) \subset X^n$ for each number of markings n and each class $\beta \in H_2(X)$.

Naturally one wishes to show that the resulting pseudocycle is independent, up to cobordism of pseudocycles, from the choice of Donaldson hypersurface. Suppose that $V', V'' \subset X$ are two Donaldson hypersurfaces and $J' = (J'_{\Gamma'}), J'' = (J''_{\Gamma''})$ are two collections of domain dependent almost complex structures depending on the intersection points with V' resp. V'', depending on some combinatorial type Γ' resp. Γ''. Consider the pullback

$$(f'')^* J'_{\Gamma'},\ (f')^* J''_{\Gamma''} : \overline{\mathcal{U}}_\Gamma \to \mathcal{J}(X, V', V'')$$

to a common universal curve $\overline{\mathcal{U}}_\Gamma$ for some type Γ recording both sets of markings (so that if Γ' resp. Γ'' has n' resp. n'' leaves then Γ has $n' + n''$ leaves.) One wishes to construct a homotopy between $(f'')^* J'_{\Gamma'},\ (f')^* J''_{\Gamma''}$ to construct a cobordism of the corresponding pseudocycles $\overline{\mathcal{M}}'_n(X, \beta)$ and $\overline{\mathcal{M}}''_n(X, \beta)$. Unfortunately, as pointed out by Nick Sheridan, the pullbacks $(f'')^* J'_{\Gamma'},\ (f')^* J''_{\Gamma''}$ do not satisfy the locality condition used to show compactness. That is, the restriction of the almost complex structures $(f')^* J''_{\Gamma''}$ (or $(f'')^* J'_{\Gamma'}$) to some irreducible component C_v of the domain curve C are not independent of markings on other components $C_{v'} \neq C_v$, because collapsed components C_v may map to non-special points $f'(C_v) = \{w\} \in f'(C)$ under the forgetful map f'.

In this note we modify the definition of the locality on the collapsed components so that one may homotope between the two domain-dependent almost complex structures without losing compactness. Instead of directly homotoping between the given pull-backs, one first homotopes each pullback to an almost complex structure that is equal to a base almost complex structure near any special point.

2. Partly local perturbations

We introduce the following notation for stable maps with two types of markings. Let Γ be a combinatorial type of genus zero stable curve with $n = n' + n''$ markings. Let V', V'' be Donaldson hypersurfaces in the symplectic manifold (X, ω), that is, symplectic hypersurfaces representing large multiples $k'[\omega]$ resp. $k''[\omega]$ of the symplectic class $[\omega] \in H^2(X, \mathbb{Q})$. Suppose V' and V'' intersect transversely. Let $\mathcal{J}(X, V', V'')$ be the space of ω-tamed almost complex structures on X thatmake V' and V'' almost complex. Let $\mathcal{J}^E(X, V', V'') \subset \mathcal{J}(X, V', V'')$ be some contractible subset of almost complex structures $J : TX \to TX$ preserving TV' and TV'' taming the symplectic form ω and so that any non-constant pseudoholomorphic J-holomorphic map $u : C \to X$ with some given energy bound $E(u) < E$ to X meets V', V'' each in at least three but finitely many distinct points $u^{-1}(V'), u^{-1}(V'')$ in the domain C as in [1, 8.18]. Let

$$J_{V',V''} \in \bigcap_E \mathcal{J}^E(X, V', V'')$$

be a base almost complex structure that satisfies these conditions without restriction on the energy of the map $u : C \to X$.

The universal curve breaks into irreducible components corresponding to the vertices of the combinatorial type. Let $\overline{\mathcal{U}}_\Gamma \to \overline{\mathcal{M}}_\Gamma$ be the closure of the universal curve of type Γ. For each vertex $v \in \mathrm{Vert}(\Gamma)$ let $\Gamma(v)$ denote the tree with the single vertex v and edges those of Γ meeting v. Let $\overline{\mathcal{U}}_{\Gamma,v} \subset \overline{\mathcal{U}}_\Gamma$ be the component corresponding to v, obtained by pulling back $\overline{\mathcal{U}}_{\Gamma(v)}$ so that $\mathcal{U}_\Gamma$ is obtained from the disjoint union of the curves $\mathcal{U}_{\Gamma,v} \to \mathcal{M}_\Gamma$ by identifying at nodes.

Cieliebak-Mohnke [1] requires that the almost complex structure is equal to the base almost complex structure near the nodes. This condition is not true for domain-dependent almost complex structures pulled back under forgetful maps, and so must be relaxed as follows. Recall that Knudsen's (genus zero) universal curve $\overline{\mathcal{U}}_\Gamma$ [3] is a smooth projective variety, and in particular a complex manifold. A *domain-dependent almost complex structure* for type Γ of stable genus zero curve is an almost complex structure

$$J_\Gamma : T(\overline{\mathcal{U}}_\Gamma \times X) \to T(\overline{\mathcal{U}}_\Gamma \times X)$$

that preserves the splitting of the tangent bundle $T(\overline{\mathcal{U}}_\Gamma \times X)$ into factors $T\overline{\mathcal{U}}_\Gamma \times TX$ and that is equal to the standard complex structure on the tangent space to the projective variety $\overline{\mathcal{U}}_\Gamma$, and gives rise to a map from $\overline{\mathcal{U}}_\Gamma$ to $\mathcal{J}(X, V', V'')$ with the same notation J_Γ. Let

$$\mathcal{J}^E_\Gamma(X, V', V'') \subset \mathrm{Map}(\overline{\mathcal{U}}_\Gamma, \mathcal{J}^E(X, V', V''))$$

denote the space of such maps taking values in $\mathcal{J}^E(X, V', V'')$. With this definition, the standard proof of Gromov convergence applies: Any sequence $u_\nu : C_\nu \to X$ of J_Γ-holomorphic maps with energy $E(u) < E$ may be viewed as a finite energy sequence of maps to $\overline{\mathcal{U}}_\Gamma \times X$. Therefore it has a subsequence with a Gromov limit $u : C \to X$ where the stabilization C^s of C is a fiber of $\overline{\mathcal{U}}_\Gamma$ and u is pseudoholomorphic for the pull-back of the restriction of J_Γ to C^s. If we restrict to sequences of maps $u_\nu : C_\nu \to X$ sending the markings to V' or V'' then in fact C^s is equal to C, since non-constant components of u with fewer than three markings are impossible.

We distinguish components of the curve that are collapsed under forgetting the first or second group of markings. Let

$$f' : \overline{\mathcal{U}}_\Gamma \to \overline{\mathcal{U}}_{\Gamma''}, \quad f'' : \overline{\mathcal{U}}_\Gamma \to \overline{\mathcal{U}}_{\Gamma'}$$

denote the forgetful maps forgetting the first n' resp. last n'' markings and stabilizing. Call a component of C *f'-unstable* if it is collapsed by f', and *f'-stable* otherwise, in which case it corresponds to a component of $f'(C)$. f''-unstable components are defined similarly.

Definition 2.1. (Local and partly local almost complex structures)

(a) A domain-dependent almost complex structure

$$J_\Gamma : \overline{\mathcal{U}}_\Gamma \to \mathcal{J}(X, V', V'')$$

is *local* if and only if for each $v \in \mathrm{Vert}(\Gamma)$ the restriction $J_\Gamma | \overline{\mathcal{U}}_{\Gamma,v}$ is local in the sense that $J_\Gamma | \overline{\mathcal{U}}_{\Gamma,v}$ is pulled back from some map $J_{\Gamma,v}$ defined on the universal curve $\overline{\mathcal{U}}_{\Gamma(v)}$ and equal to $J_{V',V''}$ near any special point of $\overline{\mathcal{U}}_{\Gamma,v}$.

(b) A domain-dependent almost complex structure

$$J_\Gamma : \overline{\mathcal{U}}_\Gamma \to \mathcal{J}(X, V', V'')$$

is f'-*local* if and only if
  (i) for each $v \in \mathrm{Vert}(\Gamma)$ such that $\mathcal{U}_{\Gamma,v}$ is f'-stable (that is, has sufficiently many V'' markings) then $J_\Gamma | \overline{\mathcal{U}}_{\Gamma,v}$ is local in the sense that $J_\Gamma | \overline{\mathcal{U}}_{\Gamma,v}$ is pulled back from some map $J_{\Gamma,v}$ defined on the universal curve $\overline{\mathcal{U}}_{\Gamma(v)}$ and equal to $J_{V,V'}$ near any point $z \in C$ mapping to a special point $f'(z)$ of $f'(C)$, and
  (ii) for each $v \in \mathrm{Vert}(\Gamma)$ such that $\mathcal{U}_{\Gamma,v}$ is f'-unstable (that is, does not have sufficiently many V'' markings) then $J_\Gamma | \overline{\mathcal{U}}_{\Gamma,v}$ is constant on each fiber of $\overline{\mathcal{U}}_{\Gamma,v}$.

The definition of f''-local is similar.

Remark 2.2. Note that f'-pullbacks $(f')^* J_{\Gamma''}$ are f'-local, and local almost complex structures are f'-local. The condition that an almost complex structure be f'-local is weaker than the condition that it be pulled back under f', because the restriction $J_\Gamma | \overline{\mathcal{U}}_{\Gamma,v}$ is allowed to depend on special points $z \in \overline{\mathcal{U}}_{\Gamma,v}$ that are forgotten under f'.

Remark 2.3. One can reformulate the f'-local condition as a pullback condition for a forgetful map that forgets almost the same markings as those forgotten by f'. Let C be a curve of type Γ. Let $C^{\mathrm{us}} \subset C$ be the locus collapsed by f'. For each connected component $C_i, i = 1, \ldots, k$ of C^{us} mapping to a marking of $f'(C)$ choose $j(i)$ so that $z_{j(i)} \in C_i$. Let $I^{\mathrm{us}} \subset \{1, \ldots, n\}$ denote the set of indices j of markings $z_j \in C^{\mathrm{us}}$ with $z_j \neq z_{j(i)}, \forall i$. Forgetting the markings with indices in I^{us} and collapsing defines a map $f : C \to f(C)$ such that any collapsed component of C maps to a special point of $f(C)$. Let Γ^f denote the combinatorial type of $f(C)$. Then J_Γ is f'-local if and only if $J_\Gamma = f^* J_{\Gamma^f}$ is pulled back from a local domain-dependent almost complex structure $J_{\Gamma^f} : \overline{\mathcal{U}}_{\Gamma^f} \to \mathcal{J}(X, V', V'')$. Indeed, the collapsed components under $C \to f(C)$ are the same as those of $f' : C \to f'(C)$ since adding a single marking z_i on the components that collapse C_v to markings $f(C_v) \subset f(C)$ does not stabilize C_v. So the pull-back condition $J_\Gamma = f^* J_{\Gamma^f}$ requires J_Γ to be constant on the components $C_v, \dim(f(C_v)) = 0$. On the other hand, any irreducible component of $f(C)$ is isomorphic, as a stable marked curve, to an irreducible component of C not collapsed under f'.

Remark 2.4. There also exist domain-dependent almost complex structures that are both f' and f''-local. Indeed suppose that C is a curve of type Γ, and $K \subset \{1, \ldots, n' + n''\}$ is the set of markings on components collapsed by f' or f''. Forgetting the markings $z_k, k \in K$ defines a forgetful map $f^{\mathrm{ss}} : C \to f^{\mathrm{ss}}(C)$, where $f^{\mathrm{ss}}(C)$ is of some (possibly empty) type Γ^{ss}. Let $J_{\Gamma^{\mathrm{ss}}} : \overline{\mathcal{U}}_{\Gamma^{\mathrm{ss}}} \to \mathcal{J}(X, V', V'')$ be a domain-dependent almost complex structure for type Γ^{ss}. Then $(f^{\mathrm{ss}})^* J_{\Gamma^{\mathrm{ss}}}$ is both f' and f''-local (taking the constant structure $J_{V',V''}$ if Γ^{ss} is empty.)

Lemma 2.5. *The space of f'-local resp. f''-local resp. f' and f''-local almost complex structures tamed by or compatible with the symplectic form ω is contractible. Any f'-local resp. f''-local resp. f' and f''-local $J_\Gamma | \partial \overline{\mathcal{U}}_\Gamma$ defined on the boundary $\partial \overline{\mathcal{U}}_\Gamma := \overline{\mathcal{U}}_\Gamma | \partial \overline{\mathcal{M}}_\Gamma$ extends to a f'-local resp. f''-local resp. f' and f''-local structure J_Γ over an open neighborhood of the boundary $\partial \overline{\mathcal{U}}_\Gamma$ in $\overline{\mathcal{U}}_\Gamma$.*

Proof. Contractibility follows from the contractibility of tamed or compatible almost complex structures. Since the space of f'-local tamed almost complex structures is contractible, it suffices to show the existence of an extension of J_Γ near any stratum $\overline{\mathcal{U}}_{\Gamma_1} \subset \overline{\mathcal{U}}_\Gamma$ and then patch together the extensions. Local domain-dependent almost complex structures J_Γ extend by a gluing construction in which open balls U_+, U_- around a node are replaced by a punctured ball $V \cong U_+^\times \cong U_-^\times$ on which the almost complex structure is equal to the base almost complex structure $J_{V',V''}$.

In the partly-local case recall from Remark 2.3 that J_Γ is the pull-back of a local almost complex structure J_{Γ^f} near any particular fiber of the universal curve. Define an extension of J_Γ near curves of type Γ_1 by first extending J_{Γ^f} and then pulling back. In more detail, let C be such a curve and let $C_1, \ldots, C_k$ denote the connected components of C collapsed by f' to a non-special point of $f'(C)$. Choose a marking $z_i \in C_i$ and let Γ^{s} resp. Γ_1^{s} denote the type obtained from Γ resp. Γ_1 by forgetting all markings on C_i except z_i, for each $i = 1, \ldots, k$. Consider the forgetful map

$$f : \overline{\mathcal{U}}_\Gamma \to \overline{\mathcal{U}}_{\Gamma^f}$$

that forgets all but the marking z_i on C_i. As discussed in Remark 2.3 J_{Γ_1} is the pullback of a complex structure

$$J_{\Gamma_1^f} : \overline{\mathcal{U}}_{\Gamma_1^f} \to \mathcal{J}(X, V', V'').$$

Since the complex structure $J_{\Gamma_1^f}$ is constant equal to the base almost complex structure $J_{V',V''}$ near the nodes (which must join non-collapsed components) $J_{\Gamma_1^f}$ naturally extends to a domain-dependent almost complex structure J_{Γ^f} on a neighborhood $\mathcal{N}_{\Gamma_1^f}$ of $\overline{\mathcal{U}}_{\Gamma_1^f}$ in $\overline{\mathcal{U}}_{\Gamma^f}$ by taking J_{Γ^f} to equal $J_{V',V''}$ near the nodes. Now take $J_\Gamma = f^* J_{\Gamma^f}$ to obtain an extension of J_Γ from $\mathcal{U}_{\Gamma_1}$ to a neighborhood $f^{-1}(\mathcal{N}_{\Gamma_1^f})$. The proof for f' local or f' and f''-local structures is similar. □

3. Transversality

We wish to inductively construct partly-local almost complex structures so that the moduli spaces of stable maps are pseudocycles. Recall that the combinatorial type of a stable map is obtained from the type of stable curve by decorating the vertices with homology classes; we also wish to record the intersection multiplicities with the Donaldson hypersurfaces. More precisely, a *type* of stable map u from C to (X, V', V'') consists of a type Γ the stable curve C (the graph with vertices corresponding to components and edges corresponding to markings and nodes) with the labelling of vertices $v \in \mathrm{Vert}(\Gamma)$ by homology class $d(v) = [u|C_v] \in H_2(X)$, labelling of the semi-infinite edges e by either V' or V'',[1] and by the intersection multiplicities $m'(e), m''(e)$ with V' and V'' (possibly zero if the corresponding marking does not map to V' or V''. A stable map is *adapted* of type Γ if each connected component of $u^{-1}(V')$ resp. $u^{-1}(V'')$ contains at least one marking z_e corresponding to an edge e labelled V' resp. V'', and each marking z_e maps to V' or V'' depending on its label. A stable map is *adapted* of type Γ if

(a) each connected component of $u^{-1}(V')$ resp. $u^{-1}(V'')$ contains at least one marking z_e corresponding to an edge e with labelling $m'(e) \geq 1$ resp. $m''(e) \geq 1$, and
(b) if $m'(e) \geq 1$ resp. $m''(e) \geq 1$, then the marking z_e is mapped to V' resp. V''.

By forgetting the extra data and stabilization one can associate to each type of stable maps to a type of stable curves. In notation we do not distinguish the two notions of types. Given a type of stable map Γ choose a domain-dependent almost complex structure J_Γ. Denote by $\mathcal{M}_\Gamma(X, J_\Gamma)$ the moduli space of adapted J_Γ-holomorphic stable maps $u : C \to X$ of type Γ, such that for each $v \in \mathrm{Vert}(\Gamma)$ with $d(v) \neq 0$, the image of u_v is not contained in $V' \cup V''$, and for each semi-infinite edge e attached to v, the local intersection number of u_v with V' resp. V'' at z_e is equal to $m'(e)$ resp. $m''(e)$. The moduli space $\mathcal{M}_\Gamma(X, J_\Gamma)$ is locally cut out by a smooth map of Banach manifolds: Given a local trivialization of the universal curve given by an subset $\mathcal{M}_\Gamma^i \subset \mathcal{M}_\Gamma$ and a trivialization $C \times \mathcal{M}_\Gamma^i \to \mathcal{U}_\Gamma^i = \mathcal{U}_\Gamma|_{\mathcal{M}_\Gamma^i}$, we consider the space of maps $\mathrm{Map}(C, X)_{k,p}$ of Sobolev class k, p for $p \geq 2$ satisfying the above constraints and k sufficiently large to the space of $0,1$-forms with values in TX given

[1] To obtain evaluation maps one should allow additional edges, but here we ignore evaluation maps

by the Cauchy-Riemann operator $\overline{\partial}_{J_\Gamma}$ associated to J_Γ. The linearization of this operator is denoted D_u (or D_{u,J_Γ} to emphasize dependence on J_Γ) and the map u is called *regular* if D_u is surjective. We call a type Γ of stable map $u : C \to X$ *crowded* if there is a maximal ghost subtree of the domain $C_1 \subset C$ with more than one marking $z_e \in C_1$ and *uncrowded* otherwise. It is not in general possible to achieve transversality for crowded types using the Cieliebak-Mohnke perturbation scheme.

Definition 3.1. We say a domain-dependent almost complex structure J_Γ is *regular* for a type of map Γ if

(a) if Γ is uncrowded then every element of the moduli space $\mathcal{M}_\Gamma(X, J_\Gamma)$ of adapted J_Γ-holomorphic maps is regular; and
(b) If Γ is crowded then there exists a regular J_{Γ^s} for some uncrowded type Γ^s obtained by forgetting all but one marking z_e on each maximal ghost component for curves of type Γ such that J_{Γ^s} is equal to J_Γ on all non-constant components, that is, all components of $\overline{\mathcal{U}}_\Gamma$ on which the maps $u : C \to X$ in $\mathcal{M}_\Gamma(X, J_\Gamma)$ are non-constant.

Recall the construction by Floer [2, Lemma 5.1] of a subspace of smooth functions with a separable Banach space structure. Let $\underline{\epsilon} = (\epsilon_\ell, \ell \in \mathbb{Z}_{\geq 0})$ be a sequence of constants converging to zero. Let $\mathcal{J}_\Gamma(X)_{\underline{\epsilon}}$ denote the space of domain-dependent almost complex structures of finite Floer norm as in [2, Section 5]. In particular, $\mathcal{J}_\Gamma(X)_{\underline{\epsilon}}$ allows variations with arbitrarily small support near any point.

Proposition 3.2. (a) *For a regular domain-dependent almost complex structure $J_{\Gamma''}$ the pull-back $(f')^* J_{\Gamma''}$ is regular, and similarly for the pull-back $(f'')^* J_{\Gamma'}$ for regular $J_{\Gamma'}$.*
(b) *Suppose that $J_\Gamma | \partial \overline{\mathcal{U}}_\Gamma$ is f'-local and is a regular domain-dependent almost complex structure defined on the boundary $\partial \overline{\mathcal{U}}_\Gamma \to \partial \overline{\mathcal{M}}_\Gamma$. The set of regular f'-local extensions is comeager, that is, is the intersection of countably many sets with dense interiors.*
(c) *Any parametrized-regular homotopy $J_{\Gamma,t} | \partial \overline{\mathcal{U}}_\Gamma$ between two regular f'-local domain-dependent almost complex structures $J_{\Gamma,0}, J_{\Gamma,1}$ on the boundary $\partial \overline{\mathcal{U}}_\Gamma$ may be extended to a parametrized-regular one-parameter family of f'-local structures $J_{\Gamma,t}$ equal to $J_{\Gamma,t}$ over $\overline{\mathcal{U}}_\Gamma$.*

Proof. Item (a) is immediate from the definition, since any variation of $J_{\Gamma''}$ induces a variation of $(f')^* J_{\Gamma''}$. (b) is an application of Sard-Smale applied to a universal moduli space. We sketch the proof which is analogous to that in Cieliebak-Mohnke [1, Chapter 5]. By Lemma 2.5, $J_\Gamma | \partial \overline{\mathcal{U}}_\Gamma$ has an extension over the interior. For transversality, first consider the case of an uncrowded type Γ of stable map. Choose open subsets $L_\Gamma, N_\Gamma \subset \overline{\mathcal{U}}_\Gamma$ of the boundary resp. markings and nodes, such that L_Γ is union of fibers of $\overline{\mathcal{U}}_\Gamma$ containing the restriction $\overline{\mathcal{U}}_\Gamma | \partial \mathcal{M}_\Gamma$ and N_Γ is sufficiently small so that the intersection of the complement of N_Γ with each component of each fiber of $\mathcal{U}_\Gamma$ not meeting L_Γ is non-empty. Let $\mathcal{M}_\Gamma^{\text{univ}}(X)$ denote the universal moduli space consisting of pairs (u, J_Γ), where $u : C \to X$ is a J_Γ-holomorphic map of some Sobolev class $W^{k,p}, kp \geq 3, p \geq 2$ on each component (with k sufficiently large so that the given vanishing order at the Donaldson hypersurfaces V', V'' is well-defined). Let $\mathcal{J}_\Gamma^E(X, N_\Gamma, S_\Gamma) \subset \mathcal{J}_\Gamma^E(X)$ denote the space of $J_\Gamma \in \mathcal{J}_\Gamma^E(X)_{\underline{\epsilon}}$ that are f'-local domain-dependent almost complex structures that agree with $J_{V',V''}$ on the neighborhood N_Γ of the nodes and markings $z \in \overline{\mathcal{U}}_\Gamma$ that map to special points $f'(z) \in \overline{\mathcal{U}}_{f'(\Gamma)}$ as in Definition 2.1, and equal to the given extension in the neighborhood L_Γ of the boundary, and constant on the components required by f'-locality in Definition 2.1. By elliptic regularity, $\mathcal{M}_\Gamma^{\text{univ}}(X)$ is independent of the choice of Sobolev exponents.

The universal moduli space is a smooth Banach manifold by an application of the implicit function theorem for Banach manifolds. Let $\mathcal{U}_\Gamma^i \to \mathcal{M}_\Gamma^i, i = 1, \ldots, m$ be a collection of open subsets of the universal curve $\mathcal{U}_\Gamma \to \mathcal{M}_\Gamma$ on which the universal curve is trivialized via diffeomorphisms $\mathcal{U}_\Gamma^i \to \mathcal{M}_\Gamma^i \times C$. The space of pairs $(u : C \to X, J_\Gamma)$ with $[C] \in \mathcal{M}_\Gamma^i$, u of type Γ of class $W^{k,p}$ on each component, and $J_\Gamma \in \mathcal{J}_\Gamma^E(X, N_\Gamma, S_\Gamma)$ is a smooth separable Banach manifold. Since we assume that J_Γ is regular on the boundary $\partial\mathcal{U}_\Gamma$, an argument using Gromov compactness shows that by choosing L_Γ sufficiently small we may assume that $\tilde{D}_{u,J_\Gamma}$ is surjective for $[C] \in L_\Gamma$. Let $\tilde{D}_{u,J_\Gamma}$ the linearization of $(u, J_\Gamma) \mapsto \overline{\partial}_{J_\Gamma} u$, and suppose that η lies in the cokernel of $\tilde{D}_{u,J_\Gamma}$. We have $D_u^* \eta^s = 0$ where D_u is the usual linearized Cauchy-Riemann operator [4, p. 258] for the map; in the case of vanishing constraints at the Donaldson hypersurfaces see Cieliebak-Mohnke [1, Lemma 6.6]. By variation of the almost complex structure J_Γ and unique continuation, η vanishes on any component on which u is non-constant. On the other hand, for any constant component $u_v : C_v \to X$, the linearized Cauchy-Riemann operator D_{u_v} on a trivial bundle $u_v^* TX$ is regular with kernel $\ker(D_{u_v})$ the space of constant maps $\xi : C_u \to (u_v)^* TX$. It follows by a standard inductive argument that the same holds true for a tree $C' = \cup_{v \in V} C_v, \mathrm{d}u|_{C'} = 0$ of constant pseudoholomorphic spheres so the element η vanishes on any component $C_v \subset C$ on which u is constant. It follows that $\mathcal{M}_\Gamma^{\text{univ},i}(X)$ is a smooth Banach manifold. For a comeager subset $\mathcal{J}_\Gamma^{\text{reg}}(X) \subset \mathcal{J}_\Gamma(X)$ of partly almost complex structures in the space above, the moduli spaces $\mathcal{M}_\Gamma^i(X) = \mathcal{M}_\Gamma(X)|_{\mathcal{M}_\Gamma^i}$ are transversally cut out for each $i = 1, \ldots, m$. The transition maps between the local trivializations $\mathcal{M}_\Gamma^i \cap \mathcal{M}_\Gamma^j \to \mathrm{Aut}(C)$ induce smooth maps $\mathcal{M}_\Gamma^i(X)|_{\mathcal{M}_\Gamma^i \cap \mathcal{M}_\Gamma^j} \to \mathcal{M}_\Gamma^j(X)|_{\mathcal{M}_\Gamma^i \cap \mathcal{M}_\Gamma^j}$ making $\mathcal{M}_\Gamma(X)$ into a smooth manifold.

Next consider a crowded type Γ. Let $f : \Gamma \to f(\Gamma)$ be a map forgetting all but one marking on each maximal ghost component $C' \subset C$ and stabilizing;

the multiplicity $m'(e), m''(e)$ at any marking z_e is the sum of the multiplicities of markings in its preimage $f^{-1}(z_e)$. Define J_{Γ^f} as follows.

(a) If $\mathcal{U}_{\Gamma^f,v} \cong \mathcal{U}_{\Gamma,v}$ let $J_{\Gamma^f}|\mathcal{U}_{\Gamma^f,v}$ be equal to $J_\Gamma|\mathcal{U}_{\Gamma,v}$.

(b) Otherwise let $J_{\Gamma^f} : \mathcal{U}_{\Gamma^f,v} \to \mathcal{J}^E(X, V', V'')$ be constant equal to $J_{V',V''}$.

The map J_{Γ^f} is continuous because any non-collapsed ghost component $C_v \subset C$ must connect at least two non-ghost components $C_{v_1}, C_{v_2} \subset C$ and the connecting points of the non-ghost components $f'(C_{v_1}), f'(C_{v_2})$ is a node of the curve $f'(f(C))$ of type $f'(\Gamma^f)$. For a comeager subset of J_Γ described above, the complex structures J_{Γ^f} are also regular by the argument for uncrowded types. Item (c) is a parametrized version of (b). □

Corollary 3.3. *There exists a regular homotopy $J_{\Gamma,t}, t \in [-1,1]$ between $(f'')^* J'_{\Gamma'}$ and $(f')^* J''_{\Gamma''}$ in the space of maps $\overline{\mathcal{U}}_\Gamma \to \mathcal{J}(X, V', V'')$ that are f'-local for $t \in [-1,0]$ and f''-local for $t \in [0,1]$*

Proof. Let $\underline{J} = (J_\Gamma)$ be a collection of regular domain-dependent almost complex structures that are both f' and f''-local, as in Remark 2.4. By part (b) above, for each type Γ there exists a regular homotopy from J_Γ to $(f')^* J_{\Gamma''}$ resp. $(f'')^* J_{\Gamma'}$ extending given homotopies on the boundary. The existence of a regular homotopy now follows by induction. □

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