PARABOLIC BUNDLES, PRODUCTS OF CONJUGACY CLASSES, AND GROMOV-WITTEN INVARIANTS

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ABSTRACT. The set of conjugacy classes appearing in a product of conjugacy classes in a compact, 1-connected Lie group K can be identified with a convex polytope in the Weyl alcove [34]. In this paper we identify linear inequalities defining this polytope. Each inequality corresponds to a non-vanishing Gromov-Witten invariant for a generalized flag variety G/P, where G is the complexification of K and P is a maximal parabolic subgroup. This generalizes the results for $SU(n)$ of Agnihotri and the second author [1] and Belkale [6] on the possible eigenvalues of a product of unitary matrices and quantum cohomology of Grassmannians.

1. STATEMENT OF RESULT

An old problem which goes back to Weyl is to determine the possible eigenvalues of a sum of traceless Hermitian matrices. According to a result of Klyachko [29], see also [6, 31], there is a finite set of homogeneous linear inequalities on the eigenvalues, each of which corresponds to a non-vanishing structure coefficient in the Schubert calculus of a Grassmannian. The same inequalities turn out to determine the non-vanishing of the Littlewood-Richardson numbers [30]. Berenstein-Sjamaar [7] and Leeb-Millson [35] generalize this result to arbitrary type as follows. Let $\mathfrak{k}$ be the Lie algebra of K, and T a maximal torus with Lie algebra $\mathfrak{t}$. The set of coadjoint orbits in $\mathfrak{k}^*$ is parametrized by a Weyl chamber $\mathfrak{t}^*_+$ in the fixed point set $\mathfrak{t}^*$ of the action of T on $\mathfrak{k}^*$. For any $\mu \in \mathfrak{t}^*_+$, we denote by $\mathcal{O}_\mu$ the corresponding coadjoint orbit $\mathcal{O}_\mu = K \cdot \mu$. For any $\mu_1, \ldots, \mu_{b-1} \in \mathfrak{t}^*_+$ the sum $\mathcal{O}_{\mu_1} + \ldots + \mathcal{O}_{\mu_b}$ is invariant under the coadjoint action and so decomposes into a union of coadjoint orbits $\mathcal{O}_{\mu_b}$. Which μ_b's occur is determined by a finite number of linear inequalities, each of which corresponds to a non-vanishing structure coefficient in the Schubert calculus for G/P. There are similar results for other symmetric spaces.

Biswas [10], Agnihotri-Woodward [1] and Belkale [6] generalize Klyachko's result to eigenvalues of products of special unitary matrices. For any special unitary matrix the logarithms $\lambda_1, \ldots, \lambda_r$ of the eigenvalues may be chosen so that

$$\lambda_1 + \ldots + \lambda_r = 0, \quad \lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_r \geq \lambda_r - 2\pi.$$

1991 *Mathematics Subject Classification.* 14L30,14L24,05E.

Key words and phrases. moment polytopes, parabolic bundles, Gromov-Witten invariants, quantum cohomology.

There are a finite set of linear inequalities on the λ_i's for a product, each of which corresponds to a non-vanishing structure coefficient of the *quantum* Schubert calculus of a Grassmannian.

In this paper we solve the multiplicative problem for arbitrary type; this includes as a special case some of the results of Berenstein-Sjamaar and Leeb-Millson. Let $\alpha_0 \in \mathfrak{t}^*$ denote the highest root. The set of conjugacy classes in K is parametrized by the Weyl alcove

$$\mathfrak{A} = \{\xi \in \mathfrak{t}_+, \ \alpha_0(\xi) \leq 1\}. \tag{1}$$

For any $\mu \in \mathfrak{A}$, we denote by $\mathcal{C}_\mu$ the conjugacy class of $\exp(\mu)$. For $\mu_1, \ldots, \mu_{b-1}$, the product $\mathcal{C}_{\mu_1} \cdot \mathcal{C}_{\mu_2} \ldots \cdot \mathcal{C}_{\mu_{b-1}}$ is invariant under conjugation; we wish to identify which conjugacy classes $\mathcal{C}_{\mu_b}$ appear in the product. We can write the problem in a more symmetric way as follows. Define

$$\Delta_b = \{(\mu_1, \ldots, \mu_b) \in \mathfrak{A}^b \mid \mathcal{C}_{\mu_1} \ \cdot \ldots \cdot \mathcal{C}_{\mu_b} \ni e\}$$

where e is the group unit. By [34, Corollary 4.13], Δ_b is a convex polytope of maximal dimension in $\mathfrak{A}^b$. We wish to find the defining inequalities for Δ_b.

This is equivalent to determining the existence of flat K-bundles on the punctured two-sphere with prescribed holonomies around the punctures. Let $x_1, \ldots, x_b$ be distinct points on an oriented surface X . For any markings $\mu_1, \ldots, \mu_b$ there exists a symplectic stratified space $\mathcal{R}_K(X; \mu_1, \ldots, \mu_b)$ whose points are the isomorphism classes of flat K-bundles on $X \backslash \{x_1, \ldots, x_b\}$ with holonomy around x_i in $\mathcal{C}_{\mu_i}$. Equivalently, $\mathcal{R}_K(X; \mu_1, \ldots, \mu_b)$ is the moduli space of representations of the fundamental group mapping a small loop around x_i to $\mathcal{C}_{\mu_i}$. In the case X has positive genus $\mathcal{R}_K(X; \mu_1, \ldots, \mu_b)$ is always non-empty. In the genus zero case we have

$$\Delta_b = \{(\mu_1, \ldots, \mu_b), \quad \mathcal{R}_K(X; \mu_1, \ldots, \mu_b) \neq \emptyset\}.$$

Fix a complex structure on X. The space $\mathcal{R}_K(X; \mu_1, \ldots, \mu_b)$ may be identified with the moduli space of semistable parabolic bundles on X, by a theorem of Mehta-Seshadri [33], Bhosle-Ramanathan [8], and the discussion in Section 4. Suppose that $n \in \mathbb{N}$ is such that $n\mu_i$ are all dominant weights. The determinant line bundle over $\mathcal{R}_K(X; \mu_1, \ldots, \mu_b)$ determines a projective embedding with Hilbert polynomial given by the dimension of the space of genus zero conformal blocks $\mathcal{H}(X; kn\mu_1, \ldots, kn\mu_b)$ at level k with markings $kn\mu_1, \ldots, kn\mu_b$ [36],[32],[44, Section 8]. Thus Δ_b is the set of b-tuples such that $\mathcal{H}(X; kn\mu_1, \ldots, kn\mu_b) \neq \{0\}$, for some k. This is the quantum generalization of the representation-theoretic version of Klyachko's problem, and another source of our interest in the problem.

The Schubert basis for $H^*(G/P)$ is given by the classes of closures of orbits of a Borel subgroup on G/P. Let B be the standard Borel subgroup whose Lie algebra contains the positive root spaces. Let $P \subset G$ denote a parabolic subgroup, corresponding to a subset Π_P of the simple roots Π. Let $W_P \subset W$ the subgroup of W generated by simple reflections for roots $\alpha \in \Pi_P$. For any $w \in W/W_P$, the *Schubert variety*

$$Y_w = \overline{BwP/P} \subset G/P$$

is a normal subvariety of G/P. The homology classes $[Y_w]$ form a basis for the homology $H^*(G/P)$; in this paper we use rational coefficients. Let $w_o \in W$ be the long element in the Weyl group. The class of $Y^w := Y_{w_o w}$ is Poincaré dual to $[Y_w]$. Its degree is $\deg[Y^w] = 2l_P(w)$, where $l_P(w)$ is the minimal length of a representative of w in W.

The small quantum cohomology $QH^*(G/P)$ is a deformation of the ordinary cohomology ring, defined by including contributions from higher degree rational curves in G/P. For simplicity, we discuss only the case that P is maximal. Let q be a formal variable. As a $\mathbb{Q}[q]$ module $QH^*(G/P)$ is freely generated by the ordinary $H^*(G/P)$. Fix $X = \mathbb{P}^1$ and choose distinct points $x_1, \ldots, x_b \in X$. For any holomorphic map $\varphi : X \to G/P$ the degree of φ is $\deg(\varphi) := \varphi_*[X] \in H_2(G/P) \cong \mathbb{Z}$. Let $g_i Y_{w_i}, i = 1, \ldots, b$ be general translates of the Schubert varieties Y_{w_i}. Define $n_d(w_1, \ldots, w_b)$ to be the number of holomorphic maps $\varphi : X \to G/P$ of degree d such that $\varphi(x_i) \in g_i Y_{w_i}$, if this number is finite, and zero otherwise. Define

$$[Y_{w_1}] \star \ldots \star [Y_{w_{b-1}}] = \sum_{d \in \mathbb{N},\ w_b \in W/W_P} n_d(w_1, \ldots, w_b) q^d [Y^{w_b}].$$

The resulting product is commutative, associative, and independent of the choice of $x_1, \ldots, x_b$ and general $g_1, \ldots, g_b$ [20, 21]. These Gromov-Witten invariants of G/P (as opposed to the invariants that appear in the large quantum cohomology) are computable in practice using formulas of Dale Peterson [37], whose proofs are given in [21] and [46]. An example, for the case G_2, is given at the end of the paper.

The non-vanishing Gromov-Witten invariants determine linear inequalities as follows. For any maximal parabolic subgroup P, let ω_P denote the fundamental weight that is invariant under W_P.

Theorem 1.1. *The polytope Δ_b is the set of points $(\mu_1, \ldots, \mu_b) \in \mathfrak{A}^b$ satisfying*

$$\sum_{i=1}^{b} (w_i \omega_P, \mu_i) \leq d$$

for all maximal parabolic subgroups $P \subset G$ and all $w_1, \ldots, w_b \in W/W_P$ and non-negative integers d such that the Gromov-Witten invariant $n_d(w_1, \ldots, w_b) = 1$.

A different approach to the problem, using the Hofer metric, is given in Entov [18].

There are several remaining open questions. We do not know which inequalities are independent. Also, are the inequalities necessary and sufficient conditions for the non-vanishing of the fusion coefficients, at least in the simply laced case? This is the quantum generalization of the saturation conjecture [30]. Also, there are similar polytopes for products of conjugacy classes in disconnected groups. It would be interesting to know if there is a twisted version of the small quantum cohomology, which computes the linear inequalities for these more general polytopes.

1.1. Index of notation.

K, G	simple 1-connected compact group, resp. complexification	p. 1
$T, \mathfrak{t}$	maximal torus, resp. Cartan subalgebra	p. 1
$\alpha_0, \mathfrak{t}_+, \mathfrak{A}$	highest root, resp. positive chamber, resp. alcove	p. 2
μ_j	marking in $\mathfrak{t}_+$ with $\alpha_0(\mu_j) < 1$	p. 5
B, P	Borel, resp. standard parabolic subgroup	p. 3
(P_1, P_2)	relative position of parabolics	p. 7
X	smooth curve $/\mathbb{C}$	p. 2
Y_w, C_w	Schubert variety, resp. Schubert cell	p. 3, p. 9
$n_d(w_1, \ldots, w_d)$	Gromov-Witten invariant	p. 3
$\mathcal{E} \to X$	holomorphic principal G-bundle	p. 4
$\varphi_j \in \mathcal{E}_{x_j}/P_j$	parabolic reduction of $\mathcal{E}$ at x_j	p. 5
$\sigma :\ X \to \mathcal{E}/P$	parabolic reduction	p. 7
$\pi :\ \tilde{X} \to X$	ramified cover	p. 8
$\tilde{x}_j, \tilde{U}_j$	ramification point of π, resp. neighborhood of $\tilde{x}_j$	p. 8
L, U	Levi, resp. unipotent subgroup	p. 9
$r :\ P \to L$	projection to L	p. 13
$\iota :\ L \to G$	inclusion of L	p. 13
Λ_P^*	weights of characters of P	p. 12
$\sigma_{\mathcal{E}}, \mu_{\mathcal{E}}$	canonical reduction, slope	p. 12
$\mathcal{U}_G(X)$	universal space for G-bundles	p. 19
$\mathcal{M}_G(X, x, \mu)$	moduli space of parabolic semistable G-bundles	p. 16
$\mathcal{R}_K(X, \mu)$	moduli space of flat K-bundles with fixed holonomy	p. 17
A, A_∞	a connection, resp. its Yang-Mills limit	p. 18

2. Parabolic G-bundles

In this section we develop the general theory of parabolic G-bundles. In particular we construct an equivalence with equivariant bundles for a finite group, canonical reductions, and coarse moduli spaces. Unfortunately, we could not understand the arguments in Bhosle-Ramanathan [8] which covers similar material so we chose to employ a different approach, basically switching the order of embedding G in $GL(n)$ and applying the equivalence with equivariant bundles. A different approach which is less useful for our purposes but works in any dimension is given by Balaji, Biswas, and Nagaraj [4].

2.1. **Definitions.** Let X be a complex manifold. A principal G-bundle over X is a complex manifold $\mathcal{E} \to X$ with a right action of G that is locally trivial. That is, any point in X is contained in a neighborhood U such that $\mathcal{E}|_U$ is G-equivariantly biholomorphic to $U \times G$. For X a scheme, principal G-bundles over X are required to be locally trivial in the étale topology. By the theorem of Drinfeld and Simpson [16] (another proof is given in [43]) any principal G-bundle over the product $X \times S$ of a smooth curve X with a scheme S is trivial locally in the product of the Zariski topology for X and the étale topology in S. The results of this section are mostly valid in both the analytic and algebraic categories.

The following definition of parabolic vector bundle is slightly more general than the original one given by Mehta and Seshadri. Let X be a curve with distinct marked points $x_1, \ldots, x_b$, and $E \to X$ a holomorphic vector bundle of rank r. A *parabolic structure* for E at x_i is a partial flag

$$E_{x_i}^1 \subset E_{x_i}^2 \subset \ldots \subset E_{x_i}^{l_i} = E_{x_i}$$

together with a set of *markings*

$$\mu_{i,1} \geq \mu_{i,2} \geq \ldots \geq \mu_{i,r}, \quad \mu_{i,1} - \mu_{i,r} < 1$$

corresponding to the type of the partial flag. That is, for all $j = 1, \ldots, r$,

$$\#\{\mu_{i,k},\ k \leq j\} = \#\{\dim(E_{x_i}^k) \leq j\}.$$

Note that Mehta-Seshadri require that the markings be non-negative, so that the $\mu_i = (\mu_{i,1}, \ldots, \mu_{i,r})$ lie in a fundamental domain for the affine Weyl group of $\mathfrak{gl}(r)$.

We define a parabolic $\mathrm{SL}(r)$-vector bundle to be a parabolic vector bundle $E \to X$ with degree zero and

$$\sum_{j=1}^{r} \mu_{i,j} = 0, \quad i = 1, \ldots, b.$$

Hence the markings μ_i lie in the Weyl alcove (1) for the Lie algebra $\mathfrak{sl}(r)$. There is an equivalent definition of parabolic structure in terms of the bundle $\mathcal{E}$ of frames for E. For any $\mu \in \mathfrak{t}_+$, there is a unique standard parabolic subgroup $P \subset \mathrm{SL}(r)$ such that the W_P is the stabilizer of μ in W. We say that P is the standard parabolic subgroup *corresponding to* μ. Let P_i denote the standard parabolic subgroup corresponding to the marking μ_i. The data of the filtration of E_{x_i} is equivalent to a reduction of $\mathcal{E}_{x_i}$ to the parabolic subgroup P_i. Explicitly, let φ_i denote the set of frames $\{v_1, \ldots, v_r\}$ for E_{x_i}, such that $v_l \in E_{x_i}^j$ for $l \leq \dim(E_{x_i}^j)$. Then P_i acts transitively on φ_i, that is, φ_i is a reduction of $\mathcal{E}_{x_i}$ to structure group P_i.

Let $\mathcal{E} \to X$ be a principal G-bundle.

Definition 2.1. A *parabolic structure* for $\mathcal{E}$ at x_i consists of

(a) a *marking* $\mu_i \in \mathfrak{A}$ with $\alpha_0(\mu_i) < 1$;
(b) a reduction $\varphi_i \in \mathcal{E}_{x_i}/P_i$, where P_i is the standard parabolic subgroup corresponding to μ_i.

A *parabolic bundle* on $(X, x_1, \ldots, x_b)$ is a bundle $\mathcal{E}$ with parabolic structures at $x_1, \ldots, x_b$. A *family of parabolic bundles* parametrized by a complex manifold S is a principal G-bundle over $X \times S$ with sections of $\mathcal{E}/P_i$ over $\{x_i\} \times S$ and markings μ_i. A morphism of bundles $\mathcal{E}_1 \to \mathcal{E}_2$ defines a morphism of parabolic bundles if the bundles have the same markings and parabolic reductions for $\mathcal{E}_1$ are mapped to parabolic reductions for $\mathcal{E}_2$.

We remark that one can replace the condition $\alpha_0(\mu_i) < 1$ with $\alpha_0(\mu_i) \leq 1$, by working with torsors (non-abelian cohomology classes) for a group sheaf which is locally a standard parabolic subgroup of the loop group; see [43] for definitions. However, we do not know any intrinsic formulation of the semistability condition in this language. In the case $G = \mathrm{SL}(r)$, all parabolic subgroups of the loop group are conjugated to subgroups of $G[[z]]$ by outer automorphisms. That is why this case does not need to be considered for moduli spaces of vector bundles.

The parabolic degree of a parabolic vector bundle E is defined by

$$\operatorname{pardeg}(E) := \deg(E) + \sum_{i=1}^{b} \sum_{j=1}^{r} \mu_{i,j}.$$

Here $\deg(E)$ denotes the first Chern class $c_1(E) \in H_2(X) \cong \mathbb{Z}$. The parabolic slope of E is

$$\mu(E) := \operatorname{pardeg}(E)/\operatorname{rank}(E).$$

The parabolic structure on E induces a parabolic structure on any holomorphic subbundle F. Define a flag in F_{x_i} by removing repeating terms from the sequence

$$F_{x_i} \cap E^1_{x_i} \subseteq F_{x_i} \cap E^2_{x_i} \ldots \subseteq F_{x_i} \cap E^{l_i}_{x_i} = F_{x_i}.$$

Define markings ν_i by $\nu_{i,j} = \mu_{i,k}$ where k is the smallest integer such that $F^j_{x_i} \subset E^k_{x_i}$. The parabolic bundle E is *semistable* if and only if the inequality

$$\mu(F) \leq \mu(E)$$

holds for all subbundles $F \subset E$.

In order to generalize these definitions to arbitrary type we give a definition of ordinary semistability using the frame bundle $\mathcal{E}$ for E. Let P_k denote the standard maximal parabolic subgroup of $GL(r)$ stabilizing a subspace of dimension k. Let $\sigma : X \to \mathcal{E}/P_k$ be the parabolic reduction with $\sigma(x)$ equal to the set of frames for E_x whose first k elements are in F_x. Let $\mathcal{E}(\omega_k)$ denote the line bundle $\mathcal{E} \times_{P_k} \mathbb{C}_{\omega_k}$ where $\mathbb{C}_{\omega_k}$ is the weight space for the k-th fundamental weight ω_k of $GL(r)$. A little yoga with the definition of Chern classes shows that

$$\deg(F) = \deg(\sigma^* \mathcal{E}(\omega_k)).$$

If E an SL-vector bundle, then E is ordinary semistable if and only if

$$\deg(\sigma^* \mathcal{E}(\omega_k)) \leq 0, \quad \forall \sigma : X \to \mathcal{E}/P_k$$

for all $k = 1, \ldots, r-1$.

Recall the definition of the Schubert cell decomposition of the Grassmannian. Let V be a vector space of dimension r, and

$$V^\bullet = \{V^1 \subset V^2 \subset \ldots V^r = V\}$$

a complete flag in V. Let $\mathrm{Gr}(k, V)$ denote the Grassmannian of k-planes in V. For each sequence of integers $J = \{j_1 < \ldots < j_k\}$ the Schubert variety corresponding to I is

$$Y_I = \{U \subset V, \quad \dim(U) \cap V^{j_m} \geq m, \quad m = 1, \ldots, k.\}$$

Let C_I denote the interior of Y_I, that is, $C_I = Y_I \backslash \cup Y_J$, where the union is over Y_J contained in Y_I. We say that U is in *relative position* I to $V^\bullet$ if U lies in Y_I.

Let $E \to X$ be a parabolic vector bundle, and $F \subset E$ a holomorphic sub-bundle. The marking for F_{x_i} is $(\mu_j, j \in J_i)$ where J_i is the relative position of F_{x_i} and $E^\bullet_{x_i}$. We can write this in the language of principal bundles as follows. The quotient $\mathcal{E}_{x_i}/P_i$ is isomorphic to the flag variety for E_{x_i} of type corresponding to μ_i, so the flag $E^\bullet_{x_i}$ defines a point $\varphi_i \in \mathcal{E}_{x_i}/P_i$. The quotient $\mathcal{E}_{x_i}/P$ is isomorphic to the Grassmannian $\mathrm{Gr}(k, E_{x_i})$, and any subspace $U_{x_i} \subset V$ defines a point $\sigma(x_i) \in \mathcal{E}_{x_i}/P_k$. The quotient of Weyl groups W/W_{P_k} maps bijectively to the set of elements of size k in $\{1, \ldots, r\}$, by

$$[w] \mapsto I([w]) := \{w(k+1), \ldots, w(r)\}.$$

We say that $\sigma(x_i)$ is in relative position $[w_i]$ to φ_i if U_{x_i} is in relative position $I([w_i])$ to $E^\bullet_{x_i}$. Hence E is parabolic semistable if and only if

$$\deg(\varphi^* \mathcal{E}(\omega_k)) + \sum_{i=1}^{b} (w_i \omega_k, \mu_i) \leq 0$$

for all $k = 1, \ldots, r-1$ and reductions $\varphi: \ X \to \mathcal{E}/P_k$, where $[w_i] \in W_{P_i} \backslash W / W_P$ is the relative position of $\sigma(x_i)$ and φ_i, and w_i is any representative of $[w_i]$ in W.

For arbitrary simple G and parabolic subgroups $P_1' = \mathrm{Ad}(g_1)P_1, P_2' = \mathrm{Ad}(g_2)P_2 \subset G$ given as conjugates of standard parabolics P_1 and P_2, define the *relative position* $(P_1', P_2') \in W_{P_1} \backslash W / W_{P_2}$ to be the image of (g_1, g_2) under the map

$$G \times G \to G \backslash (G \times G)/P_1 \times P_2 \cong P_1 \backslash G / P_2 \cong W_{P_1} \backslash W / W_{P_2}.$$

Note that $(P_2', P_1') = (P_1', P_2')^{-1}$ and $(P', P') = [1]$ for any parabolic subgroup P'.

Definition 2.2. A parabolic principal G-bundle $(\mathcal{E}, (\mu_1, \ldots, \mu_b), (\varphi_1, \ldots, \varphi_b))$ is *stable* (resp. *semistable*) if for any maximal parabolic subgroup P and reduction $\sigma: \ X \to \mathcal{E}/P$ we have

$$\deg(\sigma^* \mathcal{E}(\omega_P)) + \sum_{i=1}^{b} (w_i \omega_P, \mu_i) < 0 \quad (\text{resp.} \quad \leq 0 \ ) . \tag{2}$$

where $w_i \in W_{P_i} \backslash W / W_P$ is the relative position of $\sigma(x_i)$ and φ_i.

By $(w_i \omega_P, \mu_i)$ we mean $(\tilde{w}_i \omega_P, \mu_i)$, independent of the choice of representative $\tilde{w}_i$ of w_i. We call the left-hand-side of (2) the *parabolic degree* of σ.

If $G = \mathrm{SL}(r)$ and $\mathcal{E}$ is a parabolic principal G-bundle, then the parabolic structure induces on the associated vector bundle E the structure of a principal $\mathrm{SL}(r)$-vector bundle. For any smooth curve X with marked points $x_1, \ldots, x_b$, let $\mathrm{ParVect}_0(X; x_1, \ldots, x_b)$ denote the functor which assigns to any complex manifold S the set of isomorphism classes of families of parabolic $\mathrm{SL}(r)$-vector bundles on $(X; x_1, \ldots, x_b)$ parametrized by S. Let $\mathrm{ParBun}(X; x_1, \ldots, x_b; G)$ denote the functor that assigns to any complex manifold S the set of isomorphism classes of families of parabolic principal G-bundles on $(X; x_1, \ldots, x_b)$ parametrized by S. The map $E \mapsto \mathcal{E}$ defines an isomorphism of functors

$$\mathrm{ParVect}_0(X; x_1, \ldots, x_b) \to \mathrm{ParBun}(X; x_1, \ldots, x_b; \mathrm{SL}(r))$$

mapping families of semistable bundles to families of semistable bundles. There are similar statements in the algebraic category.

We warn the reader that an embedding $G \to H$ does not in general map the Weyl alcove for G into the Weyl alcove for H. This is the reason we could not understand the arguments in [8].

2.2. **Equivalence with equivariant bundles.** Parabolic principal bundles are equivalent to bundles equivariant for a finite group, just as in the vector bundle case.

Let Γ denote a finite group acting generically freely on a curve $\tilde{X}$, and let $X = \Gamma \backslash \tilde{X}$. Suppose that the projection $\pi : \tilde{X} \to X$ has ramification points $x_1, \ldots, x_b$. We denote the inverse image of x_i in $\tilde{X}$ by $\tilde{x}_i$. The stabilizer of $\tilde{x}_i$ under Γ is denoted $\Gamma_{\tilde{x}_i}$. We fix a generator $\gamma_{\tilde{x}_i}$ of $\Gamma_{\tilde{x}_i}$, so that its action in a neighborhood of $\tilde{x}_i$ is given by multiplication by a primitive root of unity.

Let $\tilde{E} \to \tilde{X}$ be a Γ-equivariant vector bundle. Define E to be the vector bundle whose sheaf of sections is sheaf of Γ-invariant sections of $\tilde{E}$. The parabolic structures are the filtrations of E_{x_i} induced by order of vanishing at the ramification points. The markings μ_i are the logarithms of the eigenvalues of the generator of Γ, acting on $\tilde{E}_{\tilde{x}_i}$, for any $\tilde{x}_i$ in the fiber over x_i. Let Vect_Γ denote the functor which assigns to any complex manifold S, the isomorphism classes of Γ-equivariant bundles $\tilde{E} \to \tilde{X}$. The map $\tilde{E} \mapsto E$ defines an isomorphism of functors, $\mathrm{Vect}_\Gamma(\tilde{X}) \to \mathrm{ParVect}(X; x_1, \ldots, x_b)$ [33, 22, 11, 9].

Let $\mathrm{Bun}_\Gamma(\tilde{X}; \tilde{x}; G)$ be the functor which assigns to any complex manifold S, the isomorphism classes of Γ-equivariant principal G-bundles $\tilde{\mathcal{E}} \to \tilde{X}$. We will sketch a proof of the following theorem:

Theorem 2.3. *There exists an isomorphism of functors*

$$\mathrm{Bun}_\Gamma(\tilde{X}; \tilde{x}; G) \to \mathrm{ParBun}(X; x; G)$$

mapping families of semistable bundles to semistable bundles, and a similar isomorphism in the algebraic category.

That is, there is a natural bijection between isomorphism classes of Γ-equivariant bundles (resp. semistable bundles) on $S \times \tilde{X}$, and isomorphism classes of parabolic bundles (resp. semistable bundles) on $S \times X$. Parabolic bundles with parabolic weights μ_i at the points x_i are mapped to Γ-equivariant bundles with action at $\tilde{x}_i$ in the conjugacy class fixed by μ_i.

Let $\tilde{\mathcal{E}} \to \tilde{X}$ be a Γ-equivariant principal G-bundle. We suppose for simplicity there is a single fixed point $\tilde{x} = \tilde{x}_j$ with marking $\mu = \mu_j$ and stabilizer $\Gamma_{\tilde{x}_j} = \Gamma$. Choose a neighborhood $\tilde{U} \to U$ with local coordinate z so that the projection is given by $z \mapsto z^N$, and the action of Γ by $z \mapsto \exp(2\pi i/N)z$. By the equivariant Oka principle of Heinzner and Kutzchebauch [26, Section 11], after shrinking $\tilde{U}$ we may assume that $\tilde{\mathcal{E}}$ is Γ-trivial over $\tilde{U}$. That is, there exists a Γ-equivariant biholomorphic map $\tau : \tilde{\mathcal{E}}|_{\tilde{U}} \to \tilde{U} \times G$ such that the action of Γ is given by $\gamma(z, g) = (\exp(2\pi i/N)z, \exp(\mu)g)$. Consider the one parameter subgroup,

$$\mathbb{C}^* \to G, \quad z \mapsto z^{N\mu/2\pi i} := \exp(\ln(z)N\mu/2\pi i).$$

Let $\tilde{\Sigma}^{-N\mu}$ denote the set of Γ-invariant meromorphic sections $s : \tilde{U} \to \tilde{\mathcal{E}}$ such that $s(z)z^{-N\mu/2\pi i}$ is regular on $\tilde{U}$. $\tilde{\Sigma}^{-N\mu}$ contains the section given locally by $s_0(z) = z^{N\mu/2\pi i}$.

We wish to show that there is a parabolic bundle $(\mathcal{E}, \varphi, \mu)$ isomorphic to $\Gamma\backslash\tilde{\mathcal{E}}$ over $\Gamma\backslash(\tilde{X}\backslash\tilde{x})$ such that $\tilde{\Sigma}^{-N\mu}$ is the set of sections of $\mathcal{E}$ over U. Form a bundle $\tilde{\mathcal{E}}^{-N\mu}$ by patching together $\tilde{\mathcal{E}}|_{\tilde{X}\backslash\{\tilde{x}\}}$ with $\tilde{U} \times G$, using the transition map $z^{-N\mu/2\pi i}$. The action of Γ extends to $\tilde{\mathcal{E}}^{-N\mu}$ and is trivial near x. Define $\mathcal{E} = \Gamma\backslash\tilde{\mathcal{E}}^{-N\mu}$. Since Γ acts trivially in the fiber at the ramification point, $\mathcal{E}$ is a principal G-bundle. Let $\varphi \in \mathcal{E}_{x_j}/P_j$ denote the parabolic reduction given as P_j in the trivialization at x_j. We leave it to the reader to check that the definition of $(\mathcal{E}, \varphi)$ is independent of the choices (this depends essentially on the assumption $\alpha_0(\mu) < 1$) and defines an isomorphism of functors.

In order to show that the isomorphism of functors preserves semistability, we show there is a one-to-one correspondence between parabolic reductions of $\tilde{\mathcal{E}}$ and $\mathcal{E}$, which maps the degree to a multiple of the parabolic degree. Let $\tilde{\mathcal{E}} \to \tilde{X}$ be a Γ-equivariant bundle, and $\mathcal{E} = \tilde{\mathcal{E}}^{-N\mu}/\Gamma$. Any parabolic reduction $\sigma : X \to \mathcal{E}/P$ induces a Γ-invariant parabolic reduction $\tilde{\sigma}$ of $\tilde{\mathcal{E}}$ and vice-versa, since G/P is complete. Fix a local trivialization $\tilde{U}_i \times G$ near $\tilde{x}_i$, so that the action of Γ is given by $\exp(\mu_i)$ on the fiber. The bundle $\mathcal{E}$ is formed by twisting by $z^{-N\mu_i/2\pi i}$ near $\tilde{x}_i$, and quotienting by Γ. Using this local trivialization, the fixed point set of Γ on $\tilde{\mathcal{E}}_{\tilde{x}_i}/P$ has components indexed by the double coset space of the Weyl group $W_{P_i}\backslash W/W_P$:

$$(\tilde{\mathcal{E}}_{\tilde{x}_i}/P)^\gamma \cong (G/P)^{\exp(\mu)} = \bigcup_{w \in W_{P_i}\backslash W/W_P} LwP \tag{3}$$

where L is the standard Levi subgroup of P.

Lemma 2.4. *$\tilde{\sigma}(\tilde{x}_i) \in LwP$, if and only if the relative position of $(\varphi_i, \sigma(x_i))$ is $[w]$.*

Proof. Let $O_w \subset G/P$ be the open cell containing $\mathrm{Ad}(w)P$, that is, $O_w = wB^-P$, where B^- is the Borel opposite to B. Let $C_w = BwP$ be the Schubert cell containing $Ad(w)P$. The set of elements $g \in G/P$ in relative position $[w]$ is C_w. Let $R_-(P)$ be the set of weights of $\mathfrak{g}/\mathfrak{p}$, i.e. roots of the negative unipotent complementary to P. Let $f : O_w \to \times\mathfrak{g}_\alpha$ be the T-equivariant isomorphism of the open cell O_w with the product of root spaces $\mathfrak{g}_\alpha$ for $\alpha \in wR_-(P)$. The image $f(C_w)$ is the product of $\mathfrak{g}_\alpha$ for $\alpha \in wR_-(P) \cap R_+(B)$. Therefore it suffices to show that each component $\tilde{\sigma}_\alpha^{-N\mu}$ is regular at $z = 0$ and $\tilde{\sigma}_\alpha^{-N\mu}(0) = 0$ unless $\alpha \in R_+(B)$. Since $\tilde{\sigma}$ is γ-invariant, $\tilde{\sigma}_\alpha(\gamma \cdot z) =$

$\exp(2\pi i(\mu,\alpha))\tilde{\sigma}_\alpha(z)$. Hence

$$\tilde{\sigma}_\alpha(z) = \sum_{j\geq 0,\ j\equiv_N (N\mu,\alpha)} c_j z^j.$$

Therefore

$$\sigma_\alpha^{-N\mu}(z) = \sigma_\alpha(z) z^{-(N\mu,\alpha)} = \sum_{j\geq 0,\ j\equiv_N (N\mu,\alpha)} c_j z^{j-(N\mu,\alpha)}.$$

It follows that $\sigma_\alpha^{-N\mu}(z) = 0$ at $z=0$ if $(\alpha,\mu)<0$, and is regular in any case. □

We compute the parabolic degree of σ as follows. In the local trivialization of $\tilde{\mathcal{E}}$ near $\tilde{x}_i$, the reduction $\tilde{\sigma}$ is given by $\tilde{\sigma}_i(z)P$ for some map $\tilde{\sigma}_i : \tilde{U}_i \to G$. By the previous paragraph we may assume $\tilde{\sigma}_i(0) = n_i$, for some representative n_i of w_i the relative position of $\sigma(x_i)$ and φ_i. By equivariant Oka [26] applied to the Γ-equivariant P-bundle corresponding to σ, we may assume that $\tilde{\sigma}_i(z) = n_i$ is constant. The bundle $(\tilde{\sigma}^{-N\mu})^*\tilde{\mathcal{E}}^{-N\mu}$ is formed by gluing $\tilde{\sigma}^*\tilde{\mathcal{E}}\backslash\bigcup\tilde{\sigma}^*\tilde{\mathcal{E}}_{x_i}$ with $\bigcup\tilde{\sigma}^*\tilde{\mathcal{E}}|_{\tilde{U}_i}$ using the maps $\operatorname{Ad}(n_i)z^{-N\mu_i/2\pi i} = z^{-Nw_i\mu_i/2\pi i}$. This implies that the gluing maps for $(\tilde{\sigma}^{-N\mu})^*\tilde{\mathcal{E}}^{-N\mu}(\omega_P)$ are $\chi_P(\operatorname{Ad}(n_i)z^{-N\mu_i/2\pi i}) = z^{-\omega_P(Nw_i\mu_i)}$. The degree of the line bundle is therefore

$$\deg((\tilde{\sigma}^{-N\mu})^*\tilde{\mathcal{E}}^{-N\mu}(\omega_P)) = \deg(\tilde{\sigma}^*\tilde{\mathcal{E}}(\omega_P)) - \sum_{i=1}^{b} N\omega_P(w_i\mu_i).$$

Since $\sigma = \Gamma\backslash\tilde{\sigma}^{N\mu}$, the degree of $\sigma^*\mathcal{E}(\omega_P)$ is $1/N$ times the degree of $(\tilde{\sigma}^{-N\mu})^*\tilde{\mathcal{E}}^{-N\mu}(\omega_P)$. Hence,

$$\deg(\sigma^*\mathcal{E}(\omega_P)) + \sum_{i=1}^{b}\omega_P(w_i\mu_i) = \frac{1}{N}\deg(\tilde{\sigma}^*\tilde{\mathcal{E}}(\omega_P)).$$

That is, the parabolic degree of σ is $\deg(\tilde{\sigma})/N$.

2.3. **Modifications for the algebraic case.** To prove the correspondence in the algebraic category one has to replace the equivariant Oka principle by a non-abelian cohomology argument, and the gluing by formal gluing.

Lemma 2.5. *Let $\tilde{\mathcal{E}} \to \tilde{X}\times S$ be a Γ-equivariant principal G-bundle. For any $s\in S$, there exists a neighborhood that is the product of an étale neighborhood in S and formal neighborhood of $\tilde{x}$ in $\tilde{X}$, such that the action of Γ on the restriction of $\tilde{\mathcal{E}}$ is of product form.*

Proof. Over the formal disk $D = \operatorname{Spec}(\mathbb{C}[[z]])$ at $\tilde{x}$, the bundle $\tilde{\mathcal{E}}$ is trivial and the action of Γ is given by $\gamma(z,\zeta) = (\gamma z, g(\gamma,z)\zeta)$ for some $g:\Gamma\to G[[z]]$. Since $\gamma^N = 1$ we have

$$g(\gamma,\gamma^{N-1}z)g(\gamma,\gamma^{N-2}z)\ldots g(\gamma,z) = e.$$

In particular, $g(\gamma,0)^N = e$. More generally, if $\tilde{\mathcal{E}}$ is a Γ-equivariant bundle over $D_R := \operatorname{Spec}(R[[z]])$, where R is any $\mathbb{C}$-algebra, then the action is given by an automorphism $g\in G(R[[z]])$. Let $\tilde{\mathcal{E}}$ be a Γ-equivariant bundle over D_R. We wish to show that there exists an automorphism $\tau \in G(R[[z]])$ which transforms the Γ-action on $\tilde{\mathcal{E}}|_{D_R} \cong D_R \times G$ to the product action, that is, $\tau(\gamma z)^{-1} g(\gamma, z)\tau(z) = g(\gamma, 0)$. Consider the element of $C^1(\Gamma, G(R[[z]]))$ defined by $\gamma \mapsto g(\gamma, z)$. Since

$$g(\gamma_1\gamma_2, z) = g(\gamma_1, \gamma_2 z)g(\gamma_2 z) = \gamma_2^* g(\gamma_1, z)g(\gamma_2, z)$$

$g(\cdot, z)$ is a cocycle in the cohomology of Γ with values in $G[[z]]$. Similarly $g(\cdot, 0) \in Z^1(\Gamma, G(R))$ which maps to $Z^1(\Gamma, G(R[[z]]))$. We claim there exists a 0-chain τ such that $(\delta\tau) : \ g(\cdot, z) \mapsto g(\cdot, 0)$. We construct τ order-by-order. Let $G_l = G(R[z]/z^l = 0)$. Let N_l be the kernel of the truncation map $G_{l+1} \to G_l$. The exact sequence of groups

$$1 \to N_l \to G_{l+1} \to G_l \to 1$$

induces an exact sequence of pointed sets in non-abelian cohomology (see e.g. [41, p. 49])

$$H^1(\Gamma, N_l) \to H^1(\Gamma, G_{l+1}) \to H^1(\Gamma, G_l).$$

Since N_l is a nilpotent, $H^1(\Gamma, N_l)$ is trivial, by induction on the length of the central series which reduces to the case that N_l is a Γ-module. Therefore, $H^1(\Gamma, G_{l+1})$ injects into $H^1(\Gamma, G_l)$ for all l. The complexes $C^0(\Gamma, G_l), C^1(\Gamma, G_l)$ satisfy the Mittag-Leffler condition: the image of $C^0(\Gamma, G_{l'})$ (resp. $C^1(\Gamma, G_{l'})$) in $C^0(\Gamma, G_l)$ resp. $C^1(\Gamma, G_l)$ stabilizes as $l' \to \infty$. Indeed, let $f_l : \ \mathrm{Spec}(R_l) \to G$. Extending f_l to a map $f_{l+1} : \ \mathrm{Spec}(R_{l+1}) \to G$ is equivalent to extending the map $f_0^{-1} f_{l+1}$; the latter extends because G is isomorphic to $\mathfrak{g}$ near the identity. Therefore, $G_{l+1} \to G_l$ is surjective, which implies the same result for the chain complexes. The Mittag-Leffler condition implies that

$$H^1(\Gamma, G[[z]]) = \lim_{l \to \infty} H^1(\Gamma, G_l)$$

(see [25, II.9.1] for the abelian case) and therefore also injects into $H^1(\Gamma, G)$. The claim follows since $g(\cdot, z)$ and $g(\cdot, 0)$ both map to $g(\cdot, 0)$ in $H^1(\Gamma, G(R))$. □

Recall the description of bundles on $\tilde{X}$ by formal gluing data [5], [32, Section 3]. For any algebra R, let $\tilde{X}_R := \tilde{X} \times \mathrm{Spec}(R)$. Let T denote the functor from algebras to sets which associates to R the set of isomorphism classes of triples $(\tilde{\mathcal{E}}, \rho, \sigma)$, where $\tilde{\mathcal{E}}$ is a G-bundle over $\tilde{X}_R$, ρ is a trivialization over $(\tilde{X}\backslash\{x\})_R$, and σ is a trivialization over the formal disk D_R. Then T is represented by $G(R((z)))$ [32, 3.8]. Choose a set of trivializations of $\tilde{\mathcal{E}}$ in formal neighborhoods of the form D_R as described above. Let $\tilde{\mathcal{E}}_R^{-N\mu}$ denote the bundle obtained from twisting by $z^{-N\mu/2\pi i} \in G(R((z)))$. The bundles $\tilde{\mathcal{E}}_R^{-N\mu}$ are canonically isomorphic away from $\tilde{x}$, the canonical isomorphisms extend to $\tilde{X}$, and the extension preserves the parabolic structures at the ramification points. By a simple case of étale descent, the bundles $\tilde{\mathcal{E}}_R^{-N\mu}$ patch together to a bundle $\tilde{\mathcal{E}}^{-N\mu} \to S \times \tilde{X}$. Since the gluing data for $\tilde{\mathcal{E}}^{-N\mu}$ are Γ-invariant, they define a G-bundle $\mathcal{E} \to S \times X$ with parabolic structure.

2.4. **Canonical reductions.** If a parabolic vector bundle E is unstable, the Harder-Narasimhan is a canonical sequence of sub-bundles violating the semistability condition. There is a unique sub-bundle $E_1 \subset E$ such that the slope $\mu(E_1)$ is maximal among all sub-bundles, and the rank of E_1 is maximal among sub-bundles with that slope. The Harder-Narasimhan filtration

$$E_\bullet = \{E_1 \subset \ldots \subset E_k = E\}$$

is defined inductively by $E_{i+1}/E_i = (E/E_i)_1$. It follows from the definition that the quotients E_{i+1}/E_i of the canonical filtration are semistable, the slopes $\mu_i = \mu(E_i/E_{i-1})$ decreasing, and $E_\bullet$ is the unique filtration with slopes μ_i and ranks $r_i = \dim(E_i)$.

Atiyah-Bott [3, Section 10] construct a canonical parabolic reduction $\sigma_E : X \to \mathcal{E}/P$ generalizing the Harder-Narasimhan filtration, using the adjoint bundle $\mathcal{E}(\mathfrak{g})$ of $\mathcal{E}$. The Harder-Narasimhan filtration

$$\ldots \mathcal{E}(\mathfrak{g})_{-1} \subset \mathcal{E}(\mathfrak{g})_0 \subset \ldots$$

of $\mathcal{E}(\mathfrak{g})$ has a term $\mathcal{E}(\mathfrak{g})_0$ such that $\mathcal{E}(\mathfrak{g})_0/\mathcal{E}(\mathfrak{g})_{-1}$ has degree zero, and $\mathcal{E}(\mathfrak{g})_0$ is a bundle of parabolic Lie algebras for some parabolic subgroup P. The infinitesimal action of $\mathcal{E}(\mathfrak{g})_0$ in each fiber of $\mathcal{E}/P$ has a unique fixed point, and so defines a reduction $\sigma_{\mathcal{E}} : X \to \mathcal{E}/P$.

The canonical reduction $\sigma_{\mathcal{E}}$ is functorial for homomorphisms $\phi : G \to G'$ such that the associated Lie algebra map $D\phi : \mathfrak{g} \to \mathfrak{g}'$ is injective. That is, for any principal G-bundle $\mathcal{E} \to X$, there is a parabolic subgroup P' of G' such that P is the inverse image of P' under ϕ and $\sigma_{\mathcal{E}}$ is the inverse image of $\sigma_{\phi_* E}$ under the map $\mathcal{E}/P \to \mathcal{E}'/P'$. This is because the image of $\mathcal{E}(\mathfrak{g})_0$ in $\phi_*\mathcal{E}(\mathfrak{g}')$ is contained in $\phi_*\mathcal{E}(\mathfrak{g}')_0$, for reasons of degree, and maximality of $\mathcal{E}(\mathfrak{g})_0$ among sub-bundles with the same degree implies that $\mathcal{E}(\mathfrak{g})_0$ contains the inverse image of $\phi_*\mathcal{E}(\mathfrak{g}')_0$. Hence $\mathfrak{p} = D\phi^{-1}(\mathfrak{p}')$.

Define a notion of slope for parabolic reductions as follows. Let Λ_P^* denote the abelian group of weights of characters of P, and Λ_P its dual. For a principal G-bundle $\mathcal{E} \to X$ and parabolic reduction $\sigma : X \to \mathcal{E}/P$, the *slope* $\mu(\sigma) \in \Lambda_P$ is given by

$$\mu(\sigma) : \lambda \mapsto \deg(\sigma^*\mathcal{E}(\lambda))$$

for $\lambda \in \Lambda_P^*$. The *type* of $\mathcal{E}$ is the slope $\mu(\sigma_{\mathcal{E}})$ of its canonical reduction. $\mu(\sigma_{\mathcal{E}})$ lies in the interior of the open face of $\mathfrak{t}_+$ corresponding to P.

Lemma 2.6. *The canonical reduction $\sigma_{\mathcal{E}}$ is the unique parabolic reduction with slope $\mu(\sigma_{\mathcal{E}})$.*

Proof. Consider an embedding $\phi : G \to \mathrm{Gl}(V)$, and let $\sigma_{\phi_* E} : X \to \phi_*\mathcal{E}/P'$ denote its canonical reduction. Let $\sigma : X \to \mathcal{E}/P$ be another reduction with slope $\mu(\sigma_{\mathcal{E}})$, and $\phi_*\sigma$ be the parabolic reduction of $\phi_*\mathcal{E}$ to P' induced by σ. Since $\deg((\phi_*\sigma)^*\phi_*\mathcal{E}(\lambda)) = \deg(\sigma^*\mathcal{E}(D\phi^*\lambda))$ for any weight $\lambda \in \Lambda_{P'}^*$, $\mu(\sigma_{\phi_* E}) = \mu(\phi_*\sigma)$. Since the Harder-Narasimhan filtration is the unique filtration of its slope, $\phi_*\sigma = \phi_*\sigma_{\mathcal{E}}$. This implies that $\sigma = \sigma_{\mathcal{E}}$. $\square$

Using the equivalence of parabolic bundles with equivariant bundles one can extend the theory of the canonical reduction to parabolic bundles. Let $\gamma : X \to X$ be an automorphism of the curve X. If $\mathcal{E}$ is an γ-equivariant bundle, then the canonical reduction is γ-invariant, since it is the unique reduction with its slope. Let Γ be a group of automorphisms of $\mathcal{E}$. We will call Γ-*stable* (resp. Γ-*semistable*) if

$$\deg(\sigma^*\mathcal{E}(\lambda)) \leq 0 \text{ (resp. } < 0 \text{)} \tag{4}$$

for all Γ-invariant parabolic reductions $\sigma : X \to \mathcal{E}/P$ and weights $\lambda \in \Lambda_P^*$. Since the canonical reduction is the unique reduction of its slope, a principal G-bundle is Γ-semistable if and only if it is ordinary semistable. On the other hand, Γ-stability, or Γ-irreducibility of $\mathcal{E}$ is not in general the same as ordinary stability or irreducibility. For any parabolic bundle $\mathcal{E} = (\mathcal{E}, \{(\varphi_i, \mu_i)\})$, let $\sigma_{\mathcal{E}}$ denote its canonical reduction, defined by the one-to-one correspondence between invariant parabolic reductions of $\tilde{\mathcal{E}}$ and parabolic reductions of $\mathcal{E}$. Define the slope of a parabolic reduction $\sigma : X \to \mathcal{E}/P$ by

$$\mu(\sigma) : \lambda \mapsto \deg \sigma^*\mathcal{E}(\lambda) + \sum \lambda(w_i \mu_i).$$

The *type* of $\mathcal{E}$ is the slope of $\sigma_{\mathcal{E}}$; by the discussion above $\sigma_{\mathcal{E}}$ is the unique reduction of this slope.

2.5. **Grade equivalence.** The rest of this section is included for the sake of completeness, and is not needed for the main result.

We extend Ramanathan's notion of grade equivalence to parabolic bundles. First, let $\mathcal{E} \to X$ be a G-bundle, and $\sigma : X \to E/P$ be a parabolic reduction. Let $r : P \to L$ the projection to a Levi subgroup $L \subset P$, and $\iota : L \to G$ the inclusion of L in G. The reduction σ is *admissible* if $\deg(\sigma^*\mathcal{E}(\lambda)) = 0$ for all weights λ. The equivalence relation on semistable bundles generated by

$$\mathcal{E} \sim \iota_* r_* \sigma^* \mathcal{E},$$

as σ ranges over all admissible reductions, is called *grade equivalence* [39, 40]. For any semistable bundle $\mathcal{E} \to X$, there is a semistable bundle $\mathrm{Gr}(\mathcal{E})$, unique up to isomorphism, defined by the condition that there is an admissible reduction $\sigma : X \to \mathcal{E}/P$ such that $r_*\sigma^*\mathcal{E}$ is stable and $\mathrm{Gr}(\mathcal{E}) \cong \iota_* r_* \sigma^* E$. The set of isomorphism classes of semistable G-bundles $\mathcal{E}$ such that $\mathcal{E} \cong \mathrm{Gr}(\mathcal{E})$ form a set of representatives for the equivalence classes of semistable G-bundles over X. That is, two bundles $\mathcal{E}_1, \mathcal{E}_2 \to X$ are grade equivalent, if and only if their grade bundles $\mathrm{Gr}(\mathcal{E}_1), \mathrm{Gr}(\mathcal{E}_2)$ are isomorphic.

For equivariant bundles we define grade equivalence to be the equivalence relation generated by $\mathcal{E} \sim \iota_* r_* \sigma^* \mathcal{E}$, where $\sigma : X \to \mathcal{E}/P$ is Γ-invariant.

Let $\mathcal{E} \to S \times X$ be a family of Γ-equivariant principal G-bundles, and $\mathcal{E}_0$ a bundle such that $\mathcal{E}_s$ is Γ-isomorphic to $\mathcal{E}_0$ for s varying in a dense open subset of S. The equivalence relation generated by $\mathcal{E}_s \to \mathcal{E}_0$ for any $s \in S$ is called S-equivalence. By [39, Proposition 3.24], this is the same as grade-equivalence.

To define grade equivalence for parabolic bundles, first let E be a parabolic vector bundle, with filtration $E_{x_i}^{\bullet}$. If $F \subset E$ is any sub-bundle, the filtrations $E_{x_i}^j$ induce filtrations on the fibers of the graded bundle $F \oplus E/F$ at the points x_i, and we say that $F \oplus E/F$ is parabolic grade equivalent to E.

This construction generalizes to arbitrary type as follows. let $(\mathcal{E}, \varphi_1, \ldots, \varphi_b, \mu_1, \ldots, \mu_b)$ be a parabolic bundle, $\sigma : X \to \mathcal{E}/P$ a parabolic reduction, and $r_* \sigma^* \mathcal{E}$ the associated L-bundle. Let $\mathrm{Aut}(\mathcal{E}_{x_i})$ denote the group of G-equivariant automorphisms of $\mathcal{E}_{x_i}$. $\mathrm{Aut}(\mathcal{E}_{x_i})$ is isomorphic to G, and the stabilizer $P' = \mathrm{Aut}(\mathcal{E}_{x_i})_{\sigma(x_i)}$ is isomorphic to P. Similarly, the stabilizer $P_i' = \mathrm{Aut}(\mathcal{E}_{x_i})_{\varphi_i}$ is isomorphic to P_i. In the vector bundle case, P_i' is the group of automorphisms preserving the filtration $E_{x_i}^{\bullet}$. The intersection $P' \cap P_i'$ is a subgroup isomorphic to $w_i P_i \cap P$, where w_i is the relative position of φ_i and $\sigma(x_i)$. Its image in $\mathrm{Aut}(r_* \sigma^* \mathcal{E}_{x_i})$ is a parabolic subgroup, isomorphic to $w_i P_i \cap L$. Therefore, it has a unique closed orbit in $r_* \sigma^* \mathcal{E}_{x_i}/(w_i P_i \cap L)$. Since $r_* \sigma^* \mathcal{E}_{x_i}/(w_i P_i \cap L)$ injects into $\iota_* r_* \sigma^* \mathcal{E}_{x_i}/P_i$, we get a reduction of $\iota_* r_* \sigma^* \mathcal{E}_{x_i}/P_i$ we denote by $\iota r \sigma \varphi_i$. Let *parabolic grade equivalence* be the equivalence relation generated by

$$(\mathcal{E}, \varphi_1, \ldots, \varphi_b, \mu_1, \ldots, \mu_b) \sim (\iota_* r_* \sigma^* \mathcal{E}, \iota r \sigma \varphi_1, \ldots, \iota r \sigma \varphi_b, \mu_1, \ldots, \mu_b).$$

We claim this equivalence relation corresponds to grade equivalence for equivariant bundles. Let $\tilde{\mathcal{E}} \to \tilde{X}$ be a Γ-equivariant bundle, and $(\mathcal{E}, \varphi_1, \ldots, \varphi_b, \mu_1, \ldots, \mu_b)$ the corresponding parabolic bundle. Let $\tilde{\sigma}$ be a Γ-invariant parabolic reduction to a parabolic subgroup P and σ the corresponding parabolic reduction of $\mathcal{E}$. Let $\tilde{U}_i \times G$ be a local trivialization near x_i, so that the action of Γ is $(z, g) \mapsto (\exp(2\pi i/N) z, \exp(\mu_i) g)$ and $\tilde{\sigma}(z) = w_i^{-1} P$, for some $w_i \in W_P \backslash W$. The local trivialization of $\tilde{\sigma}^* \tilde{\mathcal{E}}$ induces a local trivialization of $r_* \tilde{\sigma}^* \tilde{\mathcal{E}}$ near x_i. The action of Γ is given in this local trivialization by

$$(z, l) \mapsto (\exp(2\pi i/N) z, \exp(w_i \mu_i) l).$$

Let $\mu_{L,i}$ be the unique point in the positive chamber for L conjugate to $w_i \mu_i$. The parabolic bundle corresponding to $r_* \tilde{\sigma}^* \tilde{\mathcal{E}}$ is $(\mathcal{E}_L, \varphi_L, \mu_L)$ where $\mathcal{E}_L = (r_* \tilde{\sigma}^* \tilde{\mathcal{E}})^{-N\mu_L}/\Gamma$. Define $w_{L,i} \in W_L$ the Weyl group for L by $w_{L,i} \mu_{L,i} = w_i \mu_i$. By definition the gluing maps for $(r_* \tilde{\sigma}^* \tilde{\mathcal{E}})^{-N\mu_L}$ are given $w_{L,i} z^{N\mu_{L,i}/2\pi i}$. The gluing map for $r_*(\tilde{\sigma}^{-N\mu})^* \tilde{\mathcal{E}}^{-N\mu}$ is $z^{-N w_i \mu_i/2\pi i}$. Since $z^{-N w_i \mu_i/2\pi i} w_{L,i} z^{N\mu_{L,i}/2\pi i} = w_{L,i}$ is regular at $z = 0$, the bundles $(r_* \tilde{\sigma}^* \tilde{\mathcal{E}})^{-N\mu_L}$ and $r_*(\tilde{\sigma}^{-N\mu})^* \tilde{\mathcal{E}}^{-N\mu}$ are isomorphic. Therefore, their quotients by Γ are isomorphic. The parabolic structure for $(r_* \tilde{\sigma}^* \tilde{\mathcal{E}})^{-N\mu_L}$ at x_i is $r(P \cap w_i^{-1} P_i)$ in the trivialization near x_i. This completes the proof of the claim.

2.6. **Coarse moduli spaces.** Let $\overline{\mathrm{Bun}}^{\mathrm{ss}}(X)$ denote the functor which associates to any scheme S the set of grade equivalence classes of semistable algebraic principal G-bundles over $S \times X$. The main result of Ramanathan's thesis [40] (see also [19])) is the existence of an irreducible, normal projective variety $\mathcal{M}_G(X)$ and a morphism $\overline{\mathrm{Bun}}^{\mathrm{ss}}(X) \to \mathrm{Hom}(\cdot, \mathcal{M}_G(X))$ that is a coarse moduli space for $\overline{\mathrm{Bun}}^{\mathrm{ss}}(X)$. By definition, a *coarse moduli space* for a functor F is a scheme M and a morphism $\rho : F \to \mathrm{Hom}(\cdot, M)$ such that (i) ρ induces a bijection of points $\rho(*) : F(*) \to \mathrm{Hom}(*, M)$, where $* = \mathrm{Spec}(\mathbb{C})$, and (ii) for any scheme N and morphism $\chi: F \to \operatorname{Hom}(\cdot, N)$, there is a unique morphism $\phi: \operatorname{Hom}(\cdot, M) \to \operatorname{Hom}(\cdot, N)$ such that $\chi = \phi \circ \rho$. Usually, we omit the morphism ρ from the notation.

Let $\overline{\operatorname{Bun}}_{\Gamma}^{\mathrm{ss}}(\tilde{X})$ denote the functor that assigns to any scheme (or complex manifold, in the analytic category) S the set of grade-equivalence classes of Γ-equivariant bundles over $S \times \tilde{X}$.

Theorem 2.7. *There is a normal projective variety $\mathcal{M}_{G,\Gamma}(\tilde{X})$ that is a coarse moduli space for $\overline{\operatorname{Bun}}_{\Gamma}^{\mathrm{ss}}(\tilde{X})$.*

Sketch of Proof: We realize $\mathcal{M}_{G,\Gamma}(\tilde{X})$ as a subquotient of the moduli space of bundles with level structure. Recall that a *level structure* on $\mathcal{E}$ at a point $y \in \tilde{X}$ is a point e_y in the fiber $\mathcal{E}_y$. Bundles with level structure have no automorphisms, since the map $\operatorname{Aut}(\mathcal{E}) \to \operatorname{Aut}(\mathcal{E}_y)$ is injective. A morphism of bundles with level structure $(\mathcal{E}_1, e_{1,y}), (\mathcal{E}_2, e_{2,y})$ is a morphism $\varphi: \mathcal{E}_1 \to \mathcal{E}_2$ such that $\varphi(e_{1,y}) = e_{2,y}$. Let $\operatorname{Bun}(\tilde{X}; y_1, \ldots, y_m; G)$ denote the functor which associates to any scheme S the set of isomorphism classes of G-bundles over $S \times \tilde{X}$ with level structures at points $y_1, \ldots, y_m \in \tilde{X}$. Let $\operatorname{Bun}^{\mathrm{ss}}(\tilde{X}; y_1, \ldots, y_m; G)$ denote the open subfunctor defined by the condition that the underlying bundle is semistable. $\operatorname{Bun}^{\mathrm{ss}}(\tilde{X}, y_1, \ldots, y_m; G)$ is represented by a smooth quasi-projective moduli space $\mathcal{M}_G(\tilde{X}, y_1, \ldots, y_m)$, see [27]; for arbitrary G one needs the embedding arguments in [39, 4.8.1]. The right action of G on the fiber at each marked point induces an action of G^m on $\mathcal{M}_G(\tilde{X}, y_1, \ldots, y_m)$, with good quotient $\mathcal{M}_G(\tilde{X})$. Using Hilbert schemes as in [40, Section 5] one may construct a universal space $\mathcal{U}_G(\tilde{X}, y_1, \ldots, y_m)$ for G-bundles with level structure at $y_1, \ldots, y_m$, such that $\mathcal{U}_G(\tilde{X}, y_1, \ldots, y_m) \to \mathcal{M}_G(\tilde{X}, y_1, \ldots, y_m)$ is a good quotient.

Suppose that the set $\{y_1, \ldots, y_m\}$ is invariant under Γ, and the stabilizers Γ_{y_i} are trivial. An *equivariant bundle with level structure* is an equivariant bundle $\mathcal{E}$ with level structure $e_{y_1}, \ldots, e_{y_m}$ such that $\gamma(e_{y_i}) = e_{\gamma(y_i)}$. Let $\mathcal{M}_{G,\Gamma}(\tilde{X}, y_1, \ldots, y_m)$ denote the set of isomorphism classes of equivariant bundles with level structure whose underlying bundle is semistable. Since bundles with level structure have no automorphisms, forgetting the equivariant structure defines an injection

$$\mathcal{M}_{G,\Gamma}(\tilde{X}, y_1, \ldots, y_m) \to \mathcal{M}_G(\tilde{X}, y_1, \ldots, y_m).$$

The image is the fixed point set of the action of Γ, which is a non-singular quasi-projective variety. Let G_{Γ}^m denote the subgroup of G^m invariant under the action of Γ on G^m induced by the action of Γ on the set $\{y_1, \ldots, y_m\}$. Since the action of $G(y_1, \ldots, y_m)$ on $\mathcal{M}_G(\tilde{X}, y_1, \ldots, y_m)$ has a good quotient, the action of the subgroup G_{Γ}^m also has a good quotient. Therefore the action of G_{Γ}^m on the subvariety $\mathcal{M}_{G,\Gamma}(\tilde{X}, y_1, \ldots, y_m)$ also has a good quotient, which we denote $\mathcal{M}_{G,\Gamma}(\tilde{X})$. Since $\mathcal{M}_{G,\Gamma}(\tilde{X}, y_1, \ldots, y_m)$ is normal, so is $\mathcal{M}_{G,\Gamma}(\tilde{X})$. Quasiprojectivity of similar moduli spaces, in a more difficult situation, is shown in [27].

We will show that $\mathcal{M}_{G,\Gamma}(\tilde{X})$ is a coarse moduli space for the functor of equivalence classes of Γ-equivariant bundles. Let $\mathcal{E}$ be a Γ-equivariant semistable bundle over $S \times \tilde{X}$, and s any point in S. In a neighborhood S_1 of s, $\mathcal{E}$ admits equivariant level structures at $y_1, \ldots, y_m$ and defines $S_1 \to \mathcal{M}_{G,\Gamma}(\tilde{X}, y_1, \ldots, y_m)$. If $\mathcal{E}_s$ are equivariantly isomorphic for s in an open subset $S_0 \subset S$, then the image of $S_0 \cap S_1$ in $\mathcal{M}_{G,\Gamma}(\tilde{X}, y_1, \ldots, y_m)$ is contained in the closure of a single orbit. Conversely, if $[\mathcal{E}_0] \in \mathcal{M}_{G,\Gamma}(\tilde{X}, y_1, \ldots, y_m)$ lies in the closure of the orbit of $[\mathcal{E}_1] \in \mathcal{M}_{G,\Gamma}(\tilde{X}, y_1, \ldots, y_m)$, then forgetting the level structure shows that $\mathcal{E}_0$ and $\mathcal{E}_1$ are equivalent. Hence the points of $\mathcal{M}_{G,\Gamma}(\tilde{X})$ are equivalence classes of semistable bundles. For any family $\mathcal{E} \to S \times \tilde{X}$ of equivariant semistable G-bundles which admits equivariant level structure over $S_1 \subset S$, let $\varphi_{\mathcal{E},S_1} : S_1 \to \mathcal{M}_{G,\Gamma}$ denote the map induced by adding some level structure, $S_1 \to \mathcal{M}_{G,\Gamma}(\tilde{X}, y_1, \ldots, y_m)$, and then composing with the projection. It is clear that $\varphi_{\mathcal{E}}$ does not depend on the choice of level structure, so that $\varphi_{\mathcal{E},S_1}$ patches together to a map $\varphi_{\mathcal{E}}$, and $\mathcal{E} \mapsto \varphi_{\mathcal{E}}$ defines a morphism of functors

$$\rho_\Gamma : \overline{\mathrm{Bun}}_\Gamma^{\mathrm{ss}}(\tilde{X}; G) \to \mathcal{M}_{G,\Gamma}(\tilde{X}).$$

Part (ii) of the definition of the definition coarse moduli space follows from the properties of $\mathcal{U}_G(\tilde{X}, y_1, \ldots, y_m)$ as in [40, 4.5]. Projectivity of $\mathcal{M}_{G,\Gamma}(\tilde{X})$ follows from the correspondence theorem in Section 4, since the moduli space of flat bundles with fixed holonomy is compact.

Let $\mathcal{M}_G(X, x, \mu) := \mathcal{M}_G(X; x_1, \ldots, x_b, \mu_1, \ldots, \mu_b)$ be the moduli space of equivalence classes of parabolic G-bundles on $(X; x_1, \ldots, x_b)$ with markings $\mu_1, \ldots, \mu_b$. By the equivalence with equivariant bundles this is a normal projective variety and a coarse moduli space for the functor $\mathrm{ParBun}(X; x; \mu; G)$ of grade-equivalence classes of semistable parabolic bundles with markings $\mu_1, \ldots, \mu_b$.

3. Narasimhan-Seshadri correspondence for parabolic G-bundles

In this section we prove the correspondence between flat K-bundles and semistable holomorphic G-bundles, for markings μ_i satisfying $\alpha_0(\mu_i) < 1$. A related result for projective varieties X of any dimension is proved in [4]. A different approach to this correspondence in the case $SU(r)$ has been given by Simpson [42]; it would be interesting to know whether this can be generalized to arbitrary groups and markings.

The moduli space of flat K-bundles on a punctured surface can be constructed as in Atiyah-Bott as a symplectic quotient of the affine space of connections by the gauge group. Let X be a compact oriented surface with boundary ∂X. Since K is 1-connected, any principal K-bundle on X is trivial. Let

$$\mathcal{A}(X) := \Omega^1(X, \mathfrak{k}), \quad K(X) := \mathrm{Map}(X, K)$$

be the space of connections on $X \times K \to X$ and gauge group for $X \times K$. Choose an invariant inner product $\mathrm{Tr}(\ ,\ ) : \mathfrak{k} \times \mathfrak{k} \to \mathbb{R}$ on $\mathfrak{k}$. The affine space $\mathcal{A}(X)$ has a symplectic

form
$$a_1, a_2 \mapsto \int_X \mathrm{Tr}(a_1 \wedge a_2).$$
The action of $K(X)$ on $\mathcal{A}(X)$ is Hamiltonian, with moment map given by the curvature plus restriction to the boundary
$$\mathcal{A}(X) \mapsto \Omega^2(X, \mathfrak{k}) \oplus \Omega^1(\partial X, \mathfrak{k}), \quad A \mapsto (F_A, \iota_{\partial X}^* A).$$
The symplectic quotients of $\mathcal{A}(X)$ by $K(X)$ may be identified with moduli space of flat connections on $\mathcal{A}(X)$, with fixed holonomy around the boundary [34]. Let b denote the number of components of ∂X. The orbits of $K(X)$ on $\Omega^1(\partial X, \mathfrak{k})$ are parametrized by b-tuples $\mu = (\mu_1, \ldots, \mu_b) \in \mathfrak{A}^b$. Let
$$\mathrm{Hol}_i : \ \Omega^1(\partial X, \mathfrak{k}) \to K$$
denote the holonomy around the i-th boundary component. Then two connections $A_1, A_2 \in \Omega^1(\partial X, \mathfrak{k})$ are in the same orbit of $K(\partial X)$ if and only if $\mathrm{Hol}_i(A_1)$ is conjugate to $\mathrm{Hol}_i(A_2)$, for $i = 1, \ldots, b$. The symplectic quotient
$$\mathcal{R}_K(X, \mu_1, \ldots, \mu_b) = \mathcal{A}(X) /\!\!/_\mu K(X) = \{A \in \mathcal{A}(X), \ F_A = 0, \ \ \mathrm{Hol}_i(A) \in \mathcal{C}_{\mu_i}\}/K(X)$$
is the moduli space of flat connections on $X \times K$, with fixed holonomy. Up to symplectomorphism $\mathcal{R}_K(X, \mu) := \mathcal{R}_K(X, \mu_1, \ldots, \mu_b)$ does not depend on the choice of marked points x_i, which justifies dropping them from the notation.

The spaces $\mathcal{R}_K(X, \mu)$ may be identified with moduli spaces of representations of $\pi_1(X)$ in K. Any flat connection A determines a holonomy representation $\mathrm{Hol}(A) : \ \pi_1(X) \to K$. The i-th boundary component $(\partial X)_i$ determines a conjugacy class $[(\partial X)_i] \subset \pi_1(X)$. Two flat bundles are isomorphic if and only if their holonomy representations are conjugate by the action of K. Therefore, there is a bijection
$$\mathcal{R}_K(X, \mu) \to \{\rho \in \mathrm{Hom}(\pi_1(X), K), \ \ \rho([(\partial X)_i]) \subset \mathcal{C}_{\mu_i}\}/K. \tag{5}$$

Now suppose that X is a compact, oriented two-manifold without boundary, and $x_1, \ldots, x_b \in X$ distinct marked points. Then the moduli space of flat bundles on $X \backslash \{x_1, \ldots, x_b\}$ with holonomy around x_i in $\mathcal{C}_i$ is $\mathcal{R}_K(X', \mu_1, \ldots, \mu_b)$, where X' is the manifold obtained by removing a small open disk containing each marked point x_i. We denote this space by $\mathcal{R}_K(X, \mu)$. In the case $X = \mathbb{P}^1$, the fundamental group of $X \backslash \{x_1, \ldots, x_b\}$ is
$$\pi_1(X \backslash \{x_1, \ldots, x_b\}) =< c_1, \ldots, c_b > / \Pi_{i=1}^b c_i = 1.$$
By (5), the moduli space of flat bundles is given by
$$\mathcal{R}_K(\mathbb{P}^1; \mu_1, \ldots, \mu_b) = \{(k_1, \ldots, k_b) \in \mathcal{C}_{\mu_1} \times \ldots \times \mathcal{C}_{\mu_b} \mid \Pi_{i=1}^b k_i = e \ \}/K.$$

The moduli spaces on the punctured surface are homeomorphic to moduli spaces of Γ-invariant flat connections on a ramified cover $\tilde{X}$. Suppose Γ acts on $\tilde{\mathcal{E}} := \tilde{X} \times K$, so that the generator of $\Gamma_{\tilde{x}_i}$ acts on the fiber $\tilde{\mathcal{E}}_{\tilde{x}_i}$ by an element in the conjugacy class $\mathcal{C}_{\mu_i}$. Any invariant connection $\tilde{A}$ on $\tilde{X} \times K$ descends to a connection A on the quotient bundle

$(\tilde{\mathcal{E}}\backslash\{x_1,\dots,x_b\}\times K)/\Gamma$ with holonomy around x_i in $\mathcal{C}_{\mu_i}$. Let $\mathcal{R}_K^\Gamma(\tilde{X},\mu_1,\dots,\mu_b)$ denote the moduli space of Γ-equivariant flat bundles on $\tilde{X}$ up to Γ-equivariant isomorphism, such that the action of Γ on $\tilde{\mathcal{E}}_{x_i}$ is identified (up to conjugacy) with $\exp(\mu_i)$. If $\alpha_0(\mu_i)<1$ for $i=1,\dots,b$, the map $(\tilde{\mathcal{E}},\tilde{A})\mapsto(\mathcal{E},A)$ defines a bijection

$$\mathcal{R}_K^\Gamma(\tilde{X},\mu_1,\dots,\mu_b)\to\mathcal{R}_K(X,\mu_1,\dots,\mu_b). \tag{6}$$

Indeed, any Γ-equivariant isomorphism of flat bundles on $\tilde{X}$ induces an isomorphism of bundles on $X\backslash\{x_i\}$. Conversely, given a flat bundle on $X\backslash\{x_i\}$ one may pull-back to a flat bundle on $\tilde{X}\backslash\{\tilde{x}_i\}$. In polar coordinates r_i,θ_i near x_i the connection has the form $\mu_i d\theta_i$. It follows that one may glue in the trivial flat bundle using the gluing map $\exp(\theta_i\mu_i)$ to obtain a Γ-equivariant flat bundle on $\tilde{X}$. Any isomorphism of flat bundles on $X\backslash\{x_i\}$ lifts to an isomorphism of flat bundles on $\tilde{X}\backslash\{\tilde{x}_i\}$. In the local trivializations near the marked points x_i the isomorphism is given by a constant gauge transformation k in the centralizer of $\exp(\mu_i)$, which is equal to the stabilizer of μ_i since $\alpha_0(\mu_i)<1$. Therefore the isomorphism extends over the points $\tilde{x}_i$.

3.1. **The Yang-Mills heat flow.** According to Donaldson [15], the Narasimhan-Seshadri correspondence can be constructed by minimizing the Yang-Mills functional on the space of connections. Throughout this section we identify the space $\mathcal{A}(X)$ of connections on $X\times K$ with the space of holomorphic structures on $X\times G$. For any connection $A\in\mathcal{A}(X)$, let

$$d_A:\ \Omega^*(X,\mathfrak{k})\to\Omega^{*+1}(X,\mathfrak{k})$$

denote the corresponding covariant differentiation operator, and d_A^* its formal adjoint. The Yang-Mills functional is $A\mapsto\|F_A\|^2_{L^2}$. Let Θ be its contragradient, $\Theta(A)=-d_A^*F_A$. The connection A is *Yang-Mills* if $\Theta(A)=0$. The following summarizes results of Donaldson, Daskalopolous, and Råde for $\mathrm{Gl}(n)$, extended to arbitrary structure groups.

Theorem 3.1. (a) *For any $A_0\in\mathcal{A}(X)$, there exists a trajectory $A_t\in C^0([0,\infty),\mathcal{A}(X))$ satisfying $\frac{d}{dt}A_t=\Theta(A_t)$;*
(b) *A_t converges in the Sobolev space H^1 to a Yang-Mills connection A_∞;*
(c) *A_∞ is a flat connection if and only if A is semistable;*
(d) *the map $A\mapsto A_\infty$ defines a continuous retract of the space of semistable connections onto the space of flat connections, in the H^1-topology; and*
(e) *the map $[A]\mapsto[A_\infty]$ defines a homeomorphism $\mathcal{M}_G(X)\to\mathcal{R}_K(X)$.*

The results of [15, 12, 38] prove (a)–(e) for vector bundles. Fix an embedding $\phi:\ K\to U(n)$, and let the metric on $\mathfrak{k}$ be the pull-back of an invariant metric on $U(n)$. The Yang-Mills flow on $U(n)$-connections pulls back to the Yang-Mills flow on K-connections. This implies parts (a) and (b). Semistable holomorphic structures on $X\times G$ map to semistable holomorphic structures on $X\times\mathrm{Gl}(n)$, by functoriality of the canonical reduction. This implies (c) and (d). It remains to show (e).

We must show that two connections A_1,A_2 are grade equivalent if and only if the connections $A_{1,\infty},A_{2,\infty}$ are in the same $K(X)$-orbit. First, we show that S-equivalence is equivalent to topological S-equivalence, that is, S-equivalence where instead of algebraic or holomorphic families of connections we require only that the family be continuous, say in the Sobolev topology. By [39, 4.15.2], there exists a (non-singular, projective) universal space for semistable G-bundles on X, which we call $\mathcal{U}_G(X)$ (Ramanathan's R_3.) What we want to check is that $\mathcal{U}_G(X)$ has the universal property for *topological* families, at least locally. That is, a continuous family A_s of semistable G-bundles defines a continuous family B_s in $\mathcal{U}_G(X)$, in a neighborhood of any $s_0 \in S$. Choose an embedding $\iota: G \to \mathrm{Gl}(V)$ and a line bundle $L \to X$, such that any bundle $\iota_*(\mathcal{E}) \otimes L$ is generated by globally sections and the higher cohomology of $\iota_*(\mathcal{E}) \otimes L$ vanishes. A point in $\mathcal{U}_G(X)$ is a set of generating sections for $\iota_*(\mathcal{E}) \otimes L$, together with a G-structure on $\iota_*(\mathcal{E})$. Since higher cohomology vanishes, the global sections of $\iota_*(A_s) \otimes L$ form a topological vector bundle over the parameter space S. We can choose a continuous family of sections $f_1(s), \ldots, f_m(s)$ that generate $\iota_*(A_s)$ for any s in a neighborhood S_0 of s_0. Together with the G-structure on $\iota_*(A_s)$ these give the family B_s. Because $\mathcal{M}_G(X)$ is a good quotient of $\mathcal{U}_G(X)$, the family $[A_s]$ is a continuous path in $\mathcal{M}$. This shows that A_0 and A_s are S-equivalent, for any $s \in S$. In fact $\mathcal{M}_G(X)$ is a coarse moduli space in the topological category, that is, represents the functor from topological spaces to sets that assigns to any topological space S the set of continuous families $\mathcal{E}_s, s \in S$ of equivalence classes of semistable holomorphic G-bundles over X. This implies that the holomorphic bundles corresponding to $A_j, A_{j,\infty}$ are S-equivalent. Hence, if $A_{1,\infty}$ are isomorphic then A_1, A_2 are S-equivalent.

Conversely, suppose A_1, A_2 are semistable and S-equivalent. The grade bundles for A_1, A_2 are isomorphic, by [39, 3.12.1]. Also, the grade bundles for A_1, A_2 are isomorphic to the grade bundles of $A_{j,\infty}$, $j = 1, 2$, since these bundles are S-equivalent. Since $A_{i,\infty}$ is flat, it is its own grade bundle [39, 3.15]. Flat connections isomorphic by a complex gauge transformation are related by a unitary gauge transformation [15, Proposition 6.1.10]. Hence $A_{1,\infty}$ and $A_{2,\infty}$ are in the same $K(X)$-orbit, which completes the proof of (e).

3.2. **Narasimhan-Seshadri theorems for equivariant and parabolic bundles.** Let $\mathcal{R}_{K,\Gamma}(\tilde{X})$ denote the moduli space of Γ-equivariant flat bundles on $\tilde{X}$, up to equivariant isomorphism. Fix an action of Γ on $\tilde{X} \times K$. If $\tilde{A}$ is a Γ-invariant connection on $\tilde{X} \times K$, then the tangent vector $\Theta(\tilde{A})$ is also Γ-invariant. The Yang-Mills limit $\tilde{A}_\infty$ is therefore a Γ-invariant flat connection. If $\tilde{A}$ is semistable, then $\tilde{A}_\infty$ is flat, by 3.1 (c). The map

$$\mathcal{M}_{G,\Gamma}(\tilde{X}) \to \mathcal{R}_{K,\Gamma}(\tilde{X}), \quad [\tilde{A}] \mapsto [\tilde{A}_\infty] \tag{7}$$

is a bijection; the proof is essentially the same as in the non-equivariant case. This equivariant correspondence theorem implies a correspondence theorem for parabolic bundles. We need the following lemma on existence of finite ramified covers.

Lemma 3.2. [17, 5.2] *If N is odd or b is even, then there exists a $\mathbb{Z}_N$-cover $\pi: \tilde{X} \to X$ totally ramified at $x_1, \ldots, x_b$.*

Therefore, we can assume that $\tilde{X}$ exists, at least after adding a marked point with marking 0.

Theorem 3.3. *Let G be a connected simple, simply-connected Lie group with maximal compact subgroup K, and X a curve with distinct marked points $x_1, \dots, x_b$. Let $\mu_1, \dots, \mu_b$ be markings with $\alpha_0(\mu_i) < 1$. There is a homeomorphism*

$$\mathcal{M}_G(X; x_1, \dots, x_b, \mu_1, \dots, \mu_b) \to \mathcal{R}_K(X, \mu_1, \dots, \mu_b).$$

For rational markings, this follows from Theorem 2.3 and the bijections (7) and (6). We extend it to irrational markings by perturbation. We note that Simpson's method [42], see also [13] works just as well for the non-rational case, for $G = SL(r)$.

Theorem 3.4. *For any $(\mu_1, \dots, \mu_b) \in \mathfrak{A}^b$, there exists a rational affine subspace $C(\mu_1, \dots, \mu_b)$ such that for $(\mu_1', \dots, \mu_b')$ sufficiently close to $(\mu_1, \dots, \mu_b)$ in $C(\mu_1, \dots, \mu_b)$ there exist homeomorphisms*

$$\mathcal{M}_G(X; x_1, \dots, x_b, \mu_1, \dots, \mu_b) \to \mathcal{M}_G(X; x_1, \dots, x_b, \mu_1', \dots, \mu_b')$$

$$\mathcal{R}_K(X, \mu_1, \dots, \mu_b) \to \mathcal{R}_K(X, \mu_1', \dots, \mu_b'). \tag{8}$$

Proof. Suppose $(\mu_1, \dots, \mu_b)$ is contained in an open face $F(\mu_1, \dots, \mu_b) \subset \mathfrak{A}^b$. For each maximal parabolic P, there exist a finite set

$$S(\mu_1, \dots, \mu_b) = \{(d, w_1, \dots, w_b) \in \mathbb{N} \times (W/W_P)^b, \quad d + \sum (w_i \mu_i, \omega_P) = 0\}.$$

These inequalities define a rational affine subspace $C(\mu_1, \dots, \mu_b) \subset F(\mu_1, \dots, \mu_b)$. Let

$$m = \inf |d + \sum_{i=1}^{b} (w_i \mu_i, \omega_P)|, \quad (d, w_1, \dots, w_b) \notin S(\mu_1, \dots, \mu_b).$$

Since W/W_P is finite and $\mathfrak{A}$ is compact, m is non-zero. For $(\mu_1', \dots, \mu_b')$ sufficiently close to $(\mu_1, \dots, \mu_b)$ in $C(\mu_1, \dots, \mu_b)$ we have

$$\mathcal{M}_G(X; x_1, \dots, x_b, \mu_1, \dots, \mu_b) = \mathcal{M}_G(X; x_1, \dots, x_b, \mu_1', \dots, \mu_b')$$

since the semistability condition for the two sets of markings is the same.

To prove the bijection for flat K-bundles, consider the manifold

$$M = K^{2(g+b-1)} = \{(a_1, \dots, a_g, b_1, \dots, b_g, c_1, \dots, c_{b-1}, d_1, \dots, d_{b-1})\}$$

with action of $(k_1, \dots, k_b) \in K^b$ given by

$$a_i \mapsto \operatorname{Ad}(k_b) a_i, \ b_i \mapsto \operatorname{Ad}(k_b) b_i, \ c_i \mapsto k_b c_i k_i^{-1}, \quad d_i \mapsto \operatorname{Ad}(k_i) d_i$$

and group valued moment map (see [2]; one could use loop group actions here as well)

$$\Phi : \ M \to K^b, \ (a, b, c, d) \mapsto (d_1, \dots, d_{b-1}, (\Pi a_i b_i a_i^{-1} b_i^{-1} \Pi \operatorname{Ad}(c_i) d_i)^{-1}).$$

The moduli space of flat bundles is the symplectic quotient

$$\mathcal{R}_K(X, \mu_1, \dots, \mu_b) = \Phi^{-1}(\mu_1, \dots, \mu_b) / (K_{\mu_1} \times \dots \times K_{\mu_b}). \tag{9}$$

We claim that for $\nu \in TC(\mu_1, \ldots, \mu_b)$ and ϵ sufficiently small, there exists a homeomorphism

$$\mathcal{R}_K(X, \mu_1, \ldots, \mu_b) \to \mathcal{R}_K(X, \mu_1 + \epsilon\nu_1, \ldots, \mu_b + \epsilon\nu_b). \tag{10}$$

The quotient (9) can be taken in stages, first a quotient by $U(1)_\nu$ and then a quotient by $(K_{\mu_1} \times \ldots K_{\mu_b})/U(1)_\nu$. As in the Duistermaat-Heckman theorem, it suffices to show that the one-parameter subgroup $U(1)_\nu$ generated by $(\nu_1, \ldots, \nu_b)$ acts locally freely on $\Phi^{-1}(\mu_1, \ldots, \mu_b)$. Suppose $(a, b, c, d) \in \Phi^{-1}(\mu_1, \ldots, \mu_b)$ is fixed by $U(1)_\nu$. Then

$$a_i, b_i \in K_{\nu_b},\ d_i \in K_{\nu_i},\ \nu_i = \mathrm{Ad}(c_i)\nu_b.$$

Since $\nu_1, \ldots, \nu_b \in \mathfrak{t}$, we have $\nu_i = \mathrm{Ad}(w_i)\nu_b$ for some $w_1, \ldots, w_b \in W$ and c_i is a representative of w_i, up to multiplication by K_{ν_b}. We may assume $\nu_b \in \mathfrak{t}_+$. The fixed point set of $U(1)_\nu$ is

$$K_{\nu_b}^{2g} \times (K_{\nu_b} w_1 \times K_{\nu_i}) \times \ldots \times (K_{\nu_b} w_{b-1} \times K_{\nu_{b-1}}).$$

Its image under the moment map is equal to

$$\{(d_1, \ldots, d_b) \in K_{\nu_1} \times \ldots \times K_{\nu_b},\quad \prod_{i=1}^{b} \mathrm{Ad}(w_i) d_i \in [K_{\nu_b}, K_{\nu_b}]\}. \tag{11}$$

The stabilizer K_{ν_b} has roots α with $(\alpha, \nu_b) = 0$. Therefore, the Cartan $\mathfrak{t}_{\nu_b}$ of the semisimple part of K_{ν_b} is

$$\mathfrak{t}_{\nu_b} = \mathrm{span}\{\alpha,\ (\alpha, \nu_b) = 0)\} = \{\xi \in \mathfrak{t},\ (\omega_j, \xi) = 0,\ j = 1, \ldots, m\},$$

where $\omega_1, \ldots, \omega_m$ are the fundamental weights corresponding to simple roots $\alpha_1, \ldots, \alpha_m$ with $(\alpha_j, \nu_b) \neq 0$. Let us identify the Weyl alcove $\mathfrak{A}$ with a subset of K, using the exponential map. The torus $T_{\nu_b} \subset K_{\nu_b}$ intersects $\mathfrak{A}$ in the subset defined by the equations

$$T_{\nu_b} \cap \mathfrak{A} = \{\xi \in \mathfrak{A},\ (\omega_j, \xi) \in \mathbb{Z},\ j = 1, \ldots, m\}.$$

The intersection of (11) with $\mathfrak{A}^b$ is therefore

$$\{\xi \in \mathfrak{A}^b,\ (\omega_j, \sum_{i=1}^{b} w_i \xi_i) \in \mathbb{Z},\quad j = 1, \ldots, m\}.$$

If $\xi = \mu$ belongs to this set then so does ν, which implies

$$(\nu_b, \omega_j) = 0,\ j = 1, \ldots, m.$$

Hence ν_b is a combination of simple roots vanishing on ν_b which is a contradiction. □

Working on the analytic category we can define a canonical homeomorphism for non-rational markings

$$\mathcal{R}_K(X, \mu) \to \mathcal{M}_G(X, x, \mu) \tag{12}$$

as follows. Let $\tilde{X} \to X$ denote the ramified cover with covering group $\pi_1(X)$, given near the ramification points by the map $z \mapsto e^z$. Any flat bundle on $X \backslash \{x_1, \ldots, x_b\}$ with holonomies $\mu_1, \ldots, \mu_b$ defines a $\pi_1(X)$-equivariant bundle on $\tilde{X}$. The corresponding

$\pi_1(X)$-equivariant holomorphic G-bundle $\tilde{E} \to \tilde{X}$ defines a parabolic G-bundle $E \to X$. (One does not need equivariant Oka here, since the bundle is already trivialized near the ramification points.) The resulting map (12) is continuous and injective, since the argument that two flat bundles related by a complex gauge transformation are unitarily isomorphic [15, 6.1.10] does not use compactness of the curve. Now consider the map

$$\bigcup_{\nu \in U(\mu_1,\ldots,\mu_b)} \mathcal{R}_K(X;\nu_1,\ldots,\nu_b) \to \bigcup_{\nu \in U(\mu_1,\ldots,\mu_b)} \mathcal{M}_G(X;x_1,\ldots,x_b;\nu_1,\ldots,\nu_b)$$

where $U(\mu_1,\ldots,\mu_b)$ is a closed neighborhood of $(\mu_1,\ldots,\mu_b)$ in $C(\mu_1,\ldots,\mu_b)$ given by Theorem 3.4. Since this map is a homeomorphism for rational ν the image is dense. In fact since the domain is compact, the map is surjective. It follows that the map is a homeomorphism, since the domain is compact and the image is Hausdorff. This completes the proof of Theorem 3.3.

In this paper we do not deal with wall-crossing, that is, the change in $\mathcal{M}(X;x_1,\ldots,x_b,\mu_1,\ldots,\mu_b)$ as the markings $\mu_1,\ldots,\mu_b$ vary, see [14, 45].

4. Existence of parabolic bundles on the projective line

We now turn to the question of which moduli spaces of parabolic bundles are non-empty. We continue to identify the space of connections $\mathcal{A}(X)$ on $X \times K$ with the space of holomorphic structures on $X \times G$. Let $\mathcal{A}(X;x_1,\ldots,x_b)$ denote the set of holomorphic structures together with parabolic reductions at the marked points $x_1,\ldots,x_b$, and let $\mathcal{A}(X;x_1,\ldots,x_b)^{\mathrm{ss}}$ be the subset corresponding to parabolic semistable bundles with markings $\mu_1,\ldots,\mu_b$. The moduli space $\mathcal{M}_G(X,x,\mu)$ is the quotient of the $\mathcal{A}(X;x_1,\ldots,x_b)^{\mathrm{ss}}$ by grade equivalence. Let $f : \mathcal{A}(X;x_1,\ldots,x_b) \to \mathcal{A}(X)$ denote the map forgetting the reductions.

Proposition 4.1. *$\mathcal{A}(X;x_1,\ldots,x_b)^{\mathrm{ss}}$ is dense, if non-empty, in $\mathcal{A}(X;x_1,\ldots,x_b)$.*

Proof. First, let $\mathcal{E} \to X \times S$ be a family of G-bundles, with S connected. The set of s for which $\mathcal{E}_s$ is semistable is, if non-empty, a dense open subset of S by Ramanathan's [39, 5.8] or properties of the Shatz stratification [3]. Let $\tilde{\mathcal{E}} \to \tilde{X} \times S$ be a family of Γ-equivariant bundles. Since Γ-semistability is the same as ordinary semistability, the set of $s \in S$ for which $\tilde{\mathcal{E}}_s$ is semistable is dense and open if non-empty. By (2.3), the same holds for any family $\mathcal{E} \to (X;x_1,\ldots,x_b) \times S$ of parabolic bundles. Given any two points $(A_1,\varphi_1),(A_2,\varphi_2)$ in $\mathcal{A}(X;x_1,\ldots,x_b)$, let $\mathcal{E}$ be a family of parabolic bundles containing A_1 and A_2. For instance, the inverse image under f of the affine line through the points A_1, A_2 is a family parametrized by $S = \mathbb{C} \times (G/B)^b$. Since a general element of $\mathcal{E}$ is semistable, A_2 is contained in the closure of $\mathcal{A}(X;x_1,\ldots,x_b)^{\mathrm{ss}}$. □

Corollary 4.2. *For any markings $\mu_1,\ldots,\mu_b$, the moduli space $\mathcal{M}_G(X,x,\mu)$ is non-empty if and only if the general element of $\mathcal{M}_G(X,x,\mu)$ has a representative whose underlying principal bundle is ordinary semistable.*

Proof. The subset $\mathcal{A}(X)^{\mathrm{ss}}$ is open and dense, by [3]. Since the intersection of an open and dense set with a dense set is dense, the intersection $\mathcal{A}(X; x_1, \ldots, x_b)^{\mathrm{ss}} \cap \pi^{-1}(\mathcal{A}(X)^{\mathrm{ss}})$ is dense in $\mathcal{A}(X; x_1, \ldots, x_b)^{\mathrm{ss}}$. Since the image of a dense set under a surjective map is dense, the image of $\mathcal{A}(X; x_1, \ldots, x_b)^{\mathrm{ss}} \cap \pi^{-1}(\mathcal{A}(X)^{\mathrm{ss}})$ is dense in $\mathcal{M}_G(X, x, \mu)$. ☐

We warn the reader that it is *not* true that the grade bundle of a general element in the moduli space $\mathcal{M}_G(X, x, \mu)$ (that is, the holomorphic bundle corresponding to a general element in $\mathcal{R}_K(X, \mu)$) is ordinary semistable.

Proposition 4.3. (a) *If X has genus $g > 0$, then $\mathcal{M}_G(X, x, \mu)$ is non-empty.*

(b) *If X has genus $g = 0$, then $\mathcal{M}_G(X, x, \mu)$ is non-empty if and only if the trivial bundle $\mathcal{E} = X \times G$ with general parabolic structures at the marked points $x_1, \ldots, x_b$ is parabolic semistable.*

Proof. (a) follows from the holonomy description (5) and surjectivity of the commutator map $K \times K \to K$ [23]. (b) By Corollary 4.2, $\mathcal{M}_G(X, x, \mu)$ is non-empty if and only if a semistable bundle $\mathcal{E}$ with general parabolic structures φ_i is parabolic semistable. By the Birkhoff-Grothendieck theorem [24] any principle G-bundle admits a reduction of the structure group to $T_{\mathbb{C}}$. Since G is simple, $c_1(\mathcal{E}) = 0$. A principal $T_{\mathbb{C}}$-bundle $\mathcal{E}$ with $c_1(\mathcal{E}) = 0$ is semistable if and only if $\mathcal{E}$ is isomorphic to the trivial bundle, which completes the proof. ☐

The trivial bundle $X \times G$ with parabolic structures (φ_i, μ_i) is parabolic stable (resp. semistable) if and only if

$$\sum_{i=1}^{b} (w_i \omega_P, \mu_i) < d \quad (\text{resp.} \quad \leq d \ ) \tag{13}$$

for all maximal parabolics P and $([w_1], \ldots, [w_b]) \in W_i \backslash W / W_P$ such that there exists a reduction $\sigma : \ X \to G/P$ with $\deg(\sigma^* \mathcal{E}(\omega_P)) = d$ with $\sigma(x_i)$ in position w_i relative to φ_i for each $i = 1, \ldots, b$.

Remark 4.4. If $(\mu_1, \ldots, \mu_b)$ is sufficiently close to zero, then $\mathcal{M}_G(X, x, \mu)$ is isomorphic to the geometric invariant theory quotient $(G/B)^b /\!\!/ G$. More precisely, if

$$\sum_{i=1}^{b} (w_i \omega_P, \mu_i) < 1$$

for any maximal parabolic P and any $(w_1, \ldots, w_b)$, then any reduction $\sigma : \ X \to \mathcal{E}/P$ with $\deg \sigma^* \mathcal{E}(\omega_P) > 0$ violates semistability. Therefore, in this case $\mathcal{M}_G(X, x, \mu)$ consists entirely of parabolic bundles whose underlying bundle (forgetting the parabolic structure) is trivial. Restricting the condition (13) to constant reductions σ gives precisely the stability condition for an element of $(G/B)^b$ [7]. A symplectic argument for this fact is given in Jeffrey [28].

Since Δ_b is a polytope of maximal dimension, it suffices to consider the case that $\mu_1, \ldots, \mu_b$ are rational and lie in the interior of $\mathfrak{A}$. In this case, $P_i = B$ for all x_i. Let

$\pi : G \to G/B$ denote the projection. Suppose $\varphi_i = \pi(g_i)$ for some $g_1, \ldots, g_b \in G$. The element $\sigma(x_i) \in G/P$ lies in position w_i relative to $\varphi_i \in G/B$ only if $\sigma(x_i)$ lies in the Schubert cell $g_i C_{w_i}$. Therefore,

Proposition 4.5. *The polytope Δ_b is the set of points $(\mu_1, \ldots, \mu_b) \in \mathfrak{A}^b$ satisfying*

$$\sum_{i=1}^{b} (w_i \omega_P, \mu_i) \le d$$

for all maximal parabolics P and $(w_1, \ldots, w_b) \in (W/W_P)^b$ such that there exists a holomorphic curve $\sigma : \mathbb{P}^1 \to G/P$ with $\sigma(x_i) \in g_i C_{w_i}$, for general $g_i \in G$.

We call an inequality (13) essential if it actually defines a facet (codimension one face) of Δ_b. It remains to show that the essential inequalities are those corresponding to the structure coefficients $n_d(w_1, \ldots, w_b) = 1$. The argument is the same as that of Belkale [6] in the vector bundle case. Let $(P; w_1, \ldots, w_b; d)$ define an essential inequality, and let $(\mu_1, \ldots, \mu_b) \in \mathfrak{A}^b$ be a point which violates that inequality, and no others. Let $\mathcal{E}$ be a trivial G-bundle over $\mathbb{P}^1$, with generic parabolic structures φ_i and markings μ_i. Since $\mathcal{E}$ is unstable, the canonical parabolic reduction $\sigma_{\mathcal{E}}$ is non-trivial, and defines an inequality which is violated by $(\mu_1, \ldots, \mu_b)$. Since only one inequality is violated, $\sigma_{\mathcal{E}}$ must be a reduction to a maximal parabolic, and the corresponding inequality must be given by the data $(P; w_1, \ldots, w_b; d)$. The slope of $\sigma_{\mathcal{E}}$ is $-d + \sum_{i=1}^{b} (w_i \omega_P, \mu_i)$. Since the canonical reduction is the unique reduction of this slope, we must have $n_d(w_1, \ldots, w_b) = 1$. This completes the proof of Theorem 1.1.

5. The inequalities for type G_2

We compute the small quantum cohomology for the generalized flag varieties G/P with G of type G_2 and P maximal, using a variation on the quantum Chevalley formula of D. Peterson [37, 21].

5.1. The quantum Chevalley formula.

Let $c_1(G/P) \in H^2(G/P) \cong \mathbb{Z}$ be the first Chern class of G/P. For any root β, let h_β denote the corresponding co-root. Let α denote unique simple root, non-vanishing on ω_P.

Theorem 5.1. *For any $u \in W/W_P$ with minimal length representative $\tilde{u}$,*

$$[Y^{s_\alpha}] \star [Y^u] = \sum (h_\beta, \omega_P) [Y^{[\tilde{u} s_\beta]}] + \sum (h_\beta, \omega_P) q^{(h_\beta, \omega_P)} [Y^{[\tilde{u} s_\beta]}]$$

where the first sum is over positive roots β with $l_P([\tilde{u} s_\beta]) = l_P(u) + 1$ and the second is over positive roots β with $l_P([\tilde{u} s_\beta]) = l_P(u) + 1 - c_1(G/P)(h_\beta, \omega_P)$.

5.2. Small quantum cohomology for G_2/P, P maximal.

Let G be the group of type G_2, with simple roots α_1, α_2 and positive roots

$$\beta_1 = \alpha_1,\ \beta_2 = \alpha_2,\ \beta_3 = \alpha_1 + \alpha_2,\ \beta_4 = 3\alpha_1 + \alpha_2, \beta_5 = 2\alpha_1 + \alpha_2,\ \beta_6 = 3\alpha_1 + \alpha_2.$$

The highest root is β_4. We fix the inner product on $\mathfrak{t}^*$ so that $(\beta_4, \beta_4) = 2$ and use it to identify $\mathfrak{t}$ with $\mathfrak{t}^*$. The fundamental weights are $\omega_1 = \beta_5, \ \ \omega_2 = \beta_4$. The coroots and their pairings with the fundamental weights are

h_β	$h_\beta(\omega_1)$	$h_\beta(\omega_2)$
$h_{\beta_1} = 3\alpha_1$	1	0
$h_{\beta_2} = \alpha_2$	0	1
$h_{\beta_3} = 3\alpha_1 + 3\alpha_2$	1	3
$h_{\beta_4} = 3\alpha_1 + 2\alpha_2$	1	2
$h_{\beta_5} = 6\alpha_1 + 3\alpha_2$	2	3
$h_{\beta_6} = 3\alpha_1 + \alpha_2$	1	1

Let P_1, P_2 denote the corresponding maximal parabolics, so that the generalized flag varieties $G/P_j, j = 1, 2$ have dimension 10. The Weyl groups W_{P_1}, W_{P_2} are isomorphic to $\mathbb{Z}_2$. Therefore, the rational cohomology of G/P_i is generated by a single generator y_1 in degree 2, with the single relation $y_1^6 = 0$. The first Chern classes in $H^2(G/P_i) \cong \mathbb{Z}$ are

$$c_1(G/P_1) = 5, \quad c_1(G/P_2) = 3.$$

We denote by $y_i \in H_{10-2i}(G/P)$ the unique Schubert class of codegree $2i$, with $y_0 = 1$. The multiplication tables are given below. The second row in the table is given by Peterson's formula. Since the cohomology is generated by H^2, the remaining rows in the table may be computed recursively from the previous rows.

$QH^*(G/P_1)$	1	y_1	y_2	y_3	y_4	y_5
1	1	y_1	y_2	y_3	y_4	y_5
y_1		y_2	$2y_3$	y_4	$y_5 + q$	qy_1
y_2			$2y_4$	$y_5 + q$	$2qy_1$	qy_2
y_3				qy_1	qy_2	qy_3
y_4					$2qy_3$	qy_4
y_5						q^2

$QH^*(G/P_2)$	1	y_1	y_2	y_3	y_4	y_5
1	1	y_1	y_2	y_3	y_4	y_5
y_1		$3y_2$	$2y_3 + q$	$3y_4 + qy_1$	$y_5 + qy_2$	$qy_3 + 2q^2$
y_2			$2y_4 + qy_1$	$y_5 + 2qy_2$	$qy_3 + q^2$	$qy_4 + q^2y_1$
y_3				$2qy_3 + 2q^2$	$qy_4 + q^2y_1$	$2q^2y_2$
y_4					q^2y_2	q^2y_3
y_5						$2q^2y_4$

From the second row of the tables one may also compute the presentation of the quantum cohomology rings, in terms of the generator $y = y_1$. First, one obtains the following Giambelli-type expressions for the Schubert classes. In $QH^*(G/P_1)$, we have

$$y_2 = y_1^2, \ y_3 = y_1^3/2, \ y_4 = y_1^4/2, \ y_5 = y_1^5/2 - q$$

and in $QH^*(G/P_2)$

$$y_2 = y_1^2/3,\ y_3 = (y_1^3 - 3q)/6,\ y_4 = (y_1^4 - 9qy_1)/18,\ y_5 = (y_1^5 - 15qy_1^2)/18.$$

From the last entry in the first row in the tables one obtains

Proposition 5.2. *$QH^*(G/P_1)$ is generated by a single generator y of degree 2, with relation $y_1^6 = 4qy_1$. $QH^*(G/P_2)$ is generated by a single generator y of degree 2, with relation $y_1^6 = 18qy_1^3 + 9q^2$.*

In particular, both of these rings are semisimple at $q = 1$, since the relations have no multiple roots. Neither the classical integral or quantum rational cohomology of G/P_1 and G/P_2 is the same as that of complex projective space $\mathbb{C}P^5$, although the classical rational cohomology is the same.

5.3. **The inequalities.** From the tables, one may read off 33 classical and 40 quantum inequalities. Some of the quantum inequalities do not define facets; it would be interesting to determine which ones. For example, the last entry in the table for G/P_1 gives the inequality $(\omega_1, \mu_1 + \mu_2 + \mu_3) \leq 2$ which does not define a facet since $(\mu, \omega_1) \leq (\omega_1, \omega_1) = 2/3$ for any $\mu \in \mathfrak{A}$.

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