ORIENTATIONS FOR PSEUDOHOLOMORPHIC QUILTS

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ABSTRACT. We construct coherent orientations on moduli spaces of pseudoholomorphic quilts and determine the effect of various gluing operations on the orientations. We also investigate the behavior of the orientations under composition of Lagrangian correspondences.

CONTENTS

1. INTRODUCTION

In previous work on quilted Floer cohomology [24], [25], [29] we associated to a sequence of Lagrangian correspondences between symplectic manifolds satisfying certain conditions a *quilted Floer cohomology group*. The boundary operator in quilted Floer theory is defined by a signed count of isolated *quilted pseudoholomorphic cylinders* consisting of collections of pseudoholomorphic strips with Lagrangian seam conditions, analogous to the way that Morse homology is defined by a signed count of gradient trajectories. In the case of Morse homology the signs are derived from orientations on the spaces of Morse trajectories induced by choices of orientations on stable manifolds for each critical point and an overall orientation on the manifold. In this paper we construct coherent orientations on moduli spaces of pseudoholomorphic quilts by auxiliary choices similar to those in the Morse case.

The main result, for a single quilted domain, is the following: Let $\underline{S}$ be a *quilted surface*. Such a surface is obtained from a collection of *patches* $(S_p, j_p),\ p \in \mathcal{P}$, complex surfaces with strip-like ends, by gluing along boundary components. Let $\underline{M}$ be a collection of *symplectic labels* for the patches

$$\underline{M} = (M_p,\ p \in \mathcal{P})$$

assigning to each patch S_p a symplectic manifold M_p, equipped with compatible almost structures $J_p : TM_p \to TM_p$. Let $\underline{L}$ be a collection of *Lagrangian seam and boundary*

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conditions: for each seam or boundary component $\sigma \subset \underline{S}_{p_-} \cap \underline{S}_{p_+}$ a Lagrangian seam or boundary condition

$$\underline{L} = (L_\sigma \subset M_{p_-}^- \times M_{p_+})$$

where $M_{p_\pm}$ is a point if σ represents a boundary component. Suppose that $\underline{L}$ that the components of $\underline{L}$ are equipped with relative spin structures. Let $\underline{x}$ denote a collection of generalized intersection points for the quilted ends and $\mathcal{M}(\underline{M}, \underline{L}, \underline{x})$ the space of pseudo-holomorphic quilts

$$\underline{u} = (u_p : S_p \to M_p, \quad J_p \mathrm{d}u_p = \mathrm{d}u_p j_p \quad p \in \mathcal{P}) \tag{1}$$

with domain $\underline{S}$, targets $\underline{M}$, Lagrangian seam and boundary conditions $\underline{L}$, and limits $\underline{x}$. We suppose that almost complex structures have been chosen so that the moduli space $\mathcal{M}(\underline{M}, \underline{L}, \underline{x})$ is regular; that is, is cut out transversally by the Cauchy-Riemann equation so that the tangent space at $u \in \mathcal{M}(\underline{M}, \underline{L}, \underline{x})$ is the kernel of a surjective Fredholm operator denoted D_u:

$$T_u\mathcal{M}(\underline{M}, \underline{L}, \underline{x}) \cong \ker(D_u).$$

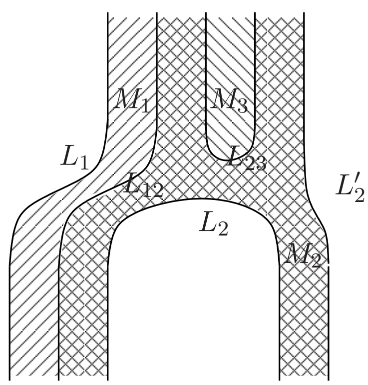

FIGURE 1. Lagrangian boundary conditions for a quilt

Our goal is to orient the moduli space. That is, we wish to provide the top exterior power of the tangent space, isomorphic to the determinant line of the Fredholm operator, with a system of non-zero elements

$$o_u \in \Lambda^{\mathrm{top}}(T_u\mathcal{M}(\underline{M}, \underline{L}, \underline{x})) \cong \det(D_u).$$

We then investigate the signs of various gluing operations. The main result is:

Theorem 1.0.1. *Suppose that $\mathcal{M}(\underline{M}, \underline{L}, \underline{x})$ is regular and $\underline{L}$ are relatively spin as above. Then $\mathcal{M}(\underline{M}, \underline{L}, \underline{x})$ admits a canonical orientation. The operations of gluing along strip-like ends, gluing at boundary nodes, and composition of seam conditions act on determinant lines by universal signs (to be specified).*

In particular, under suitable monotonicity assumptions we obtain versions of Lagrangian Floer cohomology defined over the integers and the Fukaya category, described in Sections 5.2 and 5.3 respectively. A family version is given in Theorem 5.1.6 below.

The orientations are constructed as follows. For each generator of the cochain complex we fix an orientation on a certain determinant line associated to a once-marked disk with a path of Lagrangian subspaces along the boundary. We then show that the orientations constructed in this way have good gluing properties: the quilted Floer boundary operator squares to zero, and the composition theorem of [29] holds over the integers. The construction is a generalization of the work of several authors already in the literature. For periodic Floer trajectories the construction is given in Floer-Hofer [6], while for pseudoholomorphic disks the construction is outlined in Fukaya-Oh-Ohta-Ono [8], and generalized in Ekholm-Etnyre-Sullivan [5]. In the setting of Fukaya categories orientations are constructed in Seidel's book [19]. Most of the proofs in this paper are slight modifications of proofs in one of these sources, and so the paper should be considered largely expository.

The present paper is an updated and more detailed version of a paper the authors have circulated since 2007. The authors have unreconciled differences over the exposition in the paper, and explain their points of view at https://math.berkeley.edu/~katrin/wwpapers/ resp. http://christwoodwardmath.blogspot.com/. The publication in the current form is the result of a mediation.

2. Orientations for Cauchy-Riemann operators

In this section we construct orientations for determinant lines of Cauchy-Riemann operators. Most of this material is standard: Knudsen-Mumford [14] study determinant lines of complexes with a view towards orienting moduli spaces of curves; Quillen [17] introduced determinant lines of Cauchy-Riemann operators; Segal gave a construction of a determinant line bundle over the space of Fredholm operators published eventually in [18, Appendix D]. This construction is described in more detail in Huang [9, Appendix D], see also Freed [7]; Determinant lines for families of Cauchy-Riemann operators occurring in symplectic geometry are studied in McDuff-Salamon [13, Appendix A.2], Seidel [19, Section 11], and Solomon [22] sometimes with different conventions.

2.1. Determinant lines.

We begin with a review of the construction and properties of the determinant line bundle over the space of Fredholm operators. Let V, W be real Banach spaces. Let $\mathrm{Fred}(V, W)$ be the space of Fredholm operators $D : V \to W$, that is, operators with finite dimensional kernel and cokernel

$$\ker(D) = \{v \in V \mid D(v) = 0\}, \quad \mathrm{coker}(D) = W/\{D(v) \mid v \in V\}.$$

The image of a Fredholm operator is necessarily closed.

Definition 2.1.1. (a) (Indices) The *index* of a Fredholm operator $D : V \to W$ is the integer

$$\mathrm{Ind}(D) = \dim(\ker(D)) - \dim(\mathrm{coker}(D)).$$

(b) (Determinant lines) The *determinant line* of a Fredholm operator $D : V \to W$ is the one-dimensional vector space

$$\det(D) = \Lambda^{\max}(\mathrm{coker}(D)^\vee) \otimes \Lambda^{\max}(\ker(D))$$

where $\mathrm{coker}(D)^\vee := \mathrm{Hom}(\mathrm{coker}(D), \mathbb{R})$ is the dual of $\mathrm{coker}(D)$ and $\Lambda^{\max}$ denotes the top exterior power.

Remark 2.1.2. (a) (Behavior under direct sums) Let V_j, W_j be real Banach spaces for $j = 1,2$ and $D_j : V_j \to W_j$ Fredholm operators. Denote by

$$D_1 \oplus D_2 : V_1 \oplus V_2 \to W_1 \oplus W_2$$

the direct sum of the operators D_1 and D_2. Equality of indices

$$\mathrm{Ind}(D_1 \oplus D_2) = \mathrm{Ind}(D_1) + \mathrm{Ind}(D_2)$$

holds and there is a canonical isomorphism of determinant lines

$$\det(D_1 \oplus D_2) \to \det(D_1) \otimes \det(D_2). \tag{2}$$

Explicitly let

$$\{v_{k,i}, i = 1, \ldots, a_k := \dim(\ker(D_k))\} \subset \ker(D_k)$$

$$\{w^\vee_{k,i}, i = 1, \ldots, b_k := \dim(\mathrm{coker}(D_k))\} \subset \mathrm{coker}(D_k)^\vee$$

be bases for $k = 1,2$. The isomorphism (2) is defined by

$$\begin{aligned} &\left(\wedge_{i=1}^{b_2} w^\vee_{2,i} \wedge \wedge_{i=1}^{b_1} w^\vee_{1,i}\right) \otimes (\wedge_{i=1}^{a_1} v_{1,i} \wedge \wedge_{i=1}^{a_2} v_{2,i}) \\ &\qquad \mapsto (-1)^{\dim(\mathrm{coker}(D_2))\,\mathrm{Ind}(D_1)} \left(\wedge_{i=1}^{b_1} w^\vee_{1,i} \otimes \wedge_{i=1}^{a_1} v_{1,i}\right) \otimes \left(\wedge_{i=1}^{b_2} w^\vee_{2,i} \otimes \wedge_{i=1}^{a_2} v_{2,i}\right). \end{aligned} \tag{3}$$

The isomorphism (2) is associative and graded commutative in the following sense: The composition

$$\det(D_2) \otimes \det(D_1) \to \det(D_2 \oplus D_1) \to \det(D_1 \oplus D_2) \to \det(D_1) \otimes \det(D_2), \tag{4}$$

where the middle map is induced by exchange of summands, agrees with the map

$$\det(D_2) \otimes \det(D_1) \to \det(D_1) \otimes \det(D_2)$$

induced by exchange of factors by a sign $(-1)^{\mathrm{Ind}(D_1)\,\mathrm{Ind}(D_2)}$.

(b) (Determinant lines in finite dimensions) Let $D : V \to W$ be a linear operator on finite dimensional spaces V, W. There is a canonical isomorphism to the determinant of the trivial operator from V to W,

$$t_D : \det(D) \to \det(0) = \Lambda^{\max}(W^\vee) \otimes \Lambda^{\max}(V). \tag{5}$$

To define the map (5) explicitly, choose bases

$$\{e_1, \ldots, e_n\} \subset V, \quad \{f_1, \ldots, f_m\} \subset W$$

so that

$$D(e_j) = f_j,\ j = 1, \ldots, k, \quad D(e_j) = 0,\ j = k+1, \ldots, n.$$

Let $f_1^\vee, \ldots, f_m^\vee$ be the basis for $W^\vee$ dual to $f_1, \ldots, f_m \in W$. Define

$$t_D((f_n^\vee \wedge \ldots \wedge f_{k+1}^\vee) \otimes (e_{k+1} \wedge \ldots \wedge e_m)) := (f_n^\vee \wedge \ldots \wedge f_1^\vee) \otimes (e_1 \wedge \ldots \wedge e_m).$$

Note that t_D is independent of the choice of bases e_i, f_j.

Remark 2.1.3. (Determinant line bundles) For real Banach spaces V, W let

$$\det(V,W) \to \mathrm{Fred}(V,W), \quad \det(V,W)_D := \det(D)$$

be the *determinant line bundle* whose fiber $\det(V,W)_D$ over D is the one-dimensional vector space $\det(D)$. The line bundle $\det(V,W)$ has a topological structure uniquely determined by the following conditions; see for example Zinger [30]:

(a) for finite dimensional V, W, the trivialization (5) is continuous;
(b) the isomorphism for a direct sum in (2) defines a continuous isomorphism from $\det(V_1,W_1) \otimes \det(V_2,W_2)$ to the pullback of $\det(V_1 \oplus V_1, W_1 \oplus W_2)$ under

$$\mathrm{Fred}(V_1,W_1) \times \mathrm{Fred}(V_2,W_2) \to \mathrm{Fred}(V_1 \oplus V_1, W_1 \oplus W_2);$$

(c) on the locus of surjective operators $\mathrm{Fred}^{\mathrm{sur}}(V,W) \subset \mathrm{Fred}(V,W)$, the determinant line $\det(V,W)$ is isomorphic to the top exterior power of the bundle given by the kernel via the canonical isomorphism

$$\det(D) \cong \Lambda^{\max}(\ker(D)), \quad D \in \mathrm{Fred}^{\mathrm{sur}}(V,W).$$

Different conventions give rise to different topologies on the space of determinant lines; the resulting determinant line bundles are isomorphic topologically, but via non-obvious isomorphisms.

The construction of determinant lines works in families: For a topological space X consider Fredholm morphisms $\tilde{D} : \tilde{V} \to \tilde{W}$ of Banach vector bundles $\tilde{V} \to X$, $\tilde{W} \to X$. The determinant line bundle of $\tilde{D}$ is a line bundle over X

$$\det(\tilde{D}) \to X, \quad \det(\tilde{D})_x := \det(\tilde{D} | \tilde{V}_x \to \tilde{W}_x)$$

with fibers $\det(\tilde{D})_x$ the determinant lines of the restriction of $\tilde{D}$ to fibers. In particular any homotopy of Fredholm operators $\tilde{D} = (D_t)_{t \in [0,1]}$ induces a determinant line bundle $\det(\tilde{D})$ over $X = [0,1]$. Trivializing $\det(\tilde{D})$ induces an isomorphism of determinant lines $\det(D_0) \to \det(D_1)$. We refer to this throughout the text as an *isomorphism of determinant lines induced by a deformation of operators.*

2.2. **Orientations for Fredholm operators.** By definition an orientation for a Fredholm operator is an orientation of the corresponding determinant line. There are natural constructions of orientations on duals and sums of Fredholm operators.

Definition 2.2.1. (Orientations for Fredholm operators)

(a) Let V be a finite dimensional real vector space, and $\Lambda^{\max}(V)$ its top exterior power. An *orientation* for V is a component of $\Lambda^{\max} V \setminus \{0\}$, that is, a non-vanishing element of $\Lambda^{\max} V$ up to homotopy. Denote by

$$\mathrm{Or}(V) := (\Lambda^{\max} \setminus \{0\}) / \mathbb{R}_{>0}$$

the space of orientations.
(b) An *oriented vector space* is a pair (V, o) of a vector space V and an orientation $o \in \mathrm{Or}(V)$. Given an oriented vector space (V, o) of dimension n, we say that a basis $e_1, \ldots, e_n$ of V is *oriented* if

$$o = \mathbb{R}_{>0}(e_1 \wedge \ldots \wedge e_n) \in \mathrm{Or}(V)$$

defines the orientation o on V.

(c) Let V and W be finite dimensional vector spaces. A linear isomorphism $T : V \to W$ induces a map on orientations

$$\mathrm{Or}(T) : \mathrm{Or}(V) \to \mathrm{Or}(W).$$

If V, W are oriented then the map T is orientation preserving resp. reversing if $\mathrm{Or}(T)$ is orientation preserving resp. reversing.

(d) An *orientation* of a Fredholm operator $D : V \to W$ between real Banach spaces V, W is an orientation of the one-dimensional vector space given by its determinant line $\det(D)$.

Remark 2.2.2. (a) (Orientations on duals) An orientation for a finite dimensional vector space V induces an orientation for the dual $V^\vee$. Explicitly, let $e_1, \ldots, e_n, n = \dim(V)$ be an oriented basis for V and $e_1^\vee, \ldots, e_n^\vee$ the dual basis for $V^\vee$. Give $V^\vee$ the orientation defined by

(6) $$o_{V^\vee} := [e_n^\vee \wedge \ldots \wedge e_1^\vee \in \Lambda^{\max}(V^\vee)].$$

Note the reverse order. Identify V with $V^\vee$ by an inner product $B : V \times V \to \mathbb{R}$:

$$L : V \to V^\vee, \quad v \mapsto B(v, \cdot).$$

The orientation on V relates to the pull-back orientation on $V^\vee$ by

$$L^* o_{V^\vee} = (-1)^{\dim(V)(\dim(V)-1)/2} o_V.$$

This convention is opposite to the convention of [5].

(b) (Orientations on direct sums) Orientations on finite dimensional vector spaces V, W induce an orientation on the direct sum $V \oplus W$ as follows. Let

$$\{e_1, \ldots, e_n\} \subset V, \quad \{f_1, \ldots, f_m\} \subset W$$

be oriented bases. Define on the sum $V \oplus W$ on the orientation given by

$$e_1 \wedge \ldots \wedge e_n \wedge f_1 \wedge \ldots \wedge f_m \in \Lambda^{\max}(V \oplus W).$$

The isomorphism $i : V \oplus W \to W \oplus V$ given by transposition acts on orientations

$$o_{V \oplus W} = (-1)^{\dim(V)\dim(W)} i^* o_{W \oplus V}.$$

(c) (Orientation for the identity) For finite-dimensional V, W, orientations on V and W induce an orientation o_0 on $\det(0)$. By (5), o_0 induces an orientation o_D on $\det(D)$. By convention (6) o_D is compatible with the canonical orientation on $\det(\mathrm{Id}) \cong \mathbb{R}$ for the identity operator $D = \mathrm{Id}$ if $V = W$.

(d) (Orientation double cover) For real Banach spaces V, W let

$$\mathrm{Fred}^+(V, W) = \{(D, o) \mid D : V \to W \text{ Fredholm}, \ o \in \mathrm{Or}(D) := \det(D)^\times / \mathbb{R}_{>} 0\}$$

denote the space of Fredholm operators equipped with orientations of their determinant bundles $\det(D)$. Thus

$$\mathrm{Fred}^+(V, W) \to \mathrm{Fred}(V, W), \quad (D, o) \mapsto D$$

is a double cover. The pull-back of the determinant line bundle to $\mathrm{Fred}^+(V, W)$ is automatically orientable.

Example 2.2.3. (Orientations induced by difference maps) The following example of orientations for difference maps will be used later. Consider the map

$$D: \mathbb{R} \oplus \mathbb{R} \to \mathbb{R}, \quad (x_1, x_2) \mapsto x_1 - x_2.$$

The kernel and cokernel of D are

$$\ker(D) = \{(x, x) \mid x \in \mathbb{R}\}, \quad \operatorname{coker}(D) = \{0\}.$$

Choose standard bases for $\mathbb{R} \oplus \mathbb{R}, \mathbb{R}$:

$$\{e_1 = (1, 0), e_2 = (0, 1)\} \subset \mathbb{R}^2, \quad \{f = 1\} \subset \mathbb{R}$$

The isomorphism (5) identifies

$$e_1 + e_2 \mapsto 2f^\vee \wedge (e_1 - e_2) \wedge (e_1 + e_2)$$

and so induces the standard orientation on the diagonal $\ker(D)$. On the other hand, consider the map

$$D^-: \mathbb{R} \oplus \mathbb{R} \to \mathbb{R}, \quad (x_1, x_2) \mapsto x_2 - x_1.$$

The isomorphism (5) in this case identifies

$$e_1 + e_2 \mapsto -2f^\vee \wedge (e_1 - e_2) \wedge (e_1 + e_2)$$

and so one obtains the opposite orientation on $\ker(D^-)$ from that on $\ker(D)$.

2.3. Cauchy-Riemann operators.

Our terminology for Cauchy-Riemann operators follows that of McDuff-Salamon [13]; in particular, the Cauchy-Riemann operators arising in pseudoholomorphic curve theory are *real* Cauchy-Riemann operators in the sense that the zero-th order term is not complex-linear.

2.3.1. *Cauchy-Riemann operators on surfaces with boundary.*

Let S be a compact surface with boundary.

Definition 2.3.1. (a) (Bundles with boundary condition) A *bundle with boundary condition* for S is a complex vector bundle E (given as a real vector bundle $E \to S$ together with an operator $J_E : E \to E, J_E^2 = -\operatorname{id}$) over S with a maximally totally real subbundle $F \subset E|_{\partial S}$; that is,

$$F \cap J_E(F) = \{0\}, \quad \operatorname{rank}_{\mathbb{R}}(F) = \operatorname{rank}_{\mathbb{R}}(E)/2 = \operatorname{rank}_{\mathbb{C}}(E).$$

For each component $\partial S_b \subset \partial S$ we denote by F_b the restriction of F to ∂S_b. Given $E = (E, J_E)$ we denote by $E^- = (E, -J_E)$ over S the bundle obtained by reversing the complex structure. Denote by F^- the bundle F considered as a totally real sub-bundle of E^-.

(b) (Forms with boundary condition) Let $\Omega^k(E)$ denote the space of k-forms with values in E for integers $k \geq 0$. For $k = 0$ let

$$\Omega^0(E, F) = \{\xi \in \Omega^0(E) \mid \xi|_{\partial S} \in \Omega^0(\partial S, F)\}$$

denote the space of sections of E with boundary values in F.

(c) (Dolbeault forms) Suppose that S is equipped with a complex structure. Let $\Omega^{k,l}(E)$ denote the forms of type k, l for integers k, l with values in E. Thus

$$\Omega^j(E) = \bigoplus_{k+l=j} \Omega^{k,l}(E).$$

Definition 2.3.2. (Cauchy-Riemann operators)

(a) An operator $D_{E,F} : \Omega^0(E,F) \to \Omega^{0,1}(E)$ is a *Cauchy-Riemann operator* if it is complex linear and satisfies the Leibniz rule

$$D_{E,F}(f\xi) = fD_{E,F}(\xi) + (\overline{\partial} f)(\xi), \quad \forall f \in C^\infty(S,\mathbb{C}),\ \xi \in \Omega^0(E,F).$$

(b) A *real Cauchy-Riemann operator* is the sum of a Cauchy-Riemann operator with a zeroth order term taking values in $\mathrm{End}_{\mathbb{R}}(E)$.

(c) Consider trivial bundles with rank n for some integer $n \geq 0$ given by

$$E = S \times \mathbb{C}^n, \quad F = \partial S \times \mathbb{R}^n.$$

The *trivial Cauchy-Riemann operator* is the operator $D_{E,F}$ defined by

$$D_{E,F}(f \otimes \xi) = (\overline{\partial} f) \otimes \xi.$$

(d) (Adjoint Cauchy-Riemann operator) Let $D_{E,F}$ denote a real Cauchy-Riemann operator acting on sections of E with boundary values in F. The cokernel of $D_{E,F}$ can be identified with the kernel of the adjoint $D^*_{E,F}$. The operator $D^*_{E,F}$ is a real Cauchy-Riemann operator acting on sections of

$$(E \otimes (TS))^* = \mathrm{Hom}(E^- \otimes TS^-, \mathbb{C}).$$

The sections are required to have boundary values in the subbundle $(F \otimes T(\partial S))^{\mathrm{ann}}$, the real sub-bundle of $E^* \otimes (TS)^*$ whose evaluations on $F \otimes T(\partial S)$ vanish.

Remark 2.3.3. The set of all Cauchy-Riemann operators is an affine space modelled on $\Omega^{0,1}(S, \mathrm{End}(E))$ in the sense of $D^0_{E,F}$ and $D^1_{E,F}$ are two such operators then

$$(D^0_{E,F} - D^1_{E,F})\sigma = \alpha \wedge \sigma, \forall \sigma \in \Omega^0(E,F), \quad \text{for some } \alpha \in \Omega^{0,1}(S, \mathrm{End}(E)).$$

The set of all real Cauchy-Riemann operators forms an affine space modelled on $\Omega^{0,1}(S) \otimes_{\mathbb{R}} \mathrm{End}_{\mathbb{R}}(E)$. In particular both spaces are contractible.

The Riemann-Roch theorem generalizes to Cauchy-Riemann operators on compact surfaces with boundary as follows; see for example [13, Appendix].

Definition 2.3.4. (Euler characteristic and Maslov index) For any compact surface with boundary S denote by

$$H^j(S) = \ker(\mathrm{d}^j)/\mathrm{im}(\mathrm{d}^{j+1}), \quad \mathrm{d}^j : \Omega^j(S) \to \Omega^{j+1}(S)$$

the j-th de Rham cohomology of S for integers $j \geq 0$. Let

$$\chi(S) = \dim H^0(S) - \dim H^1(S) + \dim H^2(S)$$

denote the Euler characteristic of S. Let $I(E,F) \in \mathbb{Z}$ be the Maslov index of the pair (E,F), as in [13, Appendix]. For S without boundary, the index $I(E,F)$ is twice the Chern number,

$$I(E,\emptyset) = \int_S c_1(E).$$

On the other hand, for E trivial, F is the sum of the winding numbers of the boundary conditions around the boundary components, considered as paths in the Grassmannian of totally real subspaces.

Proposition 2.3.5. (Riemann-Roch for surfaces with boundary [13, Appendix]) *For any Cauchy-Riemann operator $D_{E,F}$ on a surface with boundary S,*

$$\mathrm{Ind}(D_{E,F}) = \mathrm{rank}_{\mathbb{R}}(F)\chi(S) + I(E,F). \tag{7}$$

2.3.2. *Cauchy-Riemann operators on surfaces with strip-like ends.*

Definition 2.3.6.

(a) A *surface with strip-like ends* consists of the following data:

(i) a compact surface $\overline{S}$ with boundary

$$\partial\overline{S} = C_1 \sqcup \ldots \sqcup C_m$$

and $d_n \geq 0$ distinct points

$$z_{n,1}, \ldots, z_{n,d_n} \in C_n$$

in cyclic order on each boundary circle $C_n \cong S^1$. We will use the indices on C_n modulo d_n. The index set for the marked points is denoted

$$\mathcal{E} = \mathcal{E}(S) := \big\{ e = (n,l) \,\big|\, n \in \{1,\ldots,m\}, l \in \{1,\ldots,d_n\} \big\};$$

We use the notation $e \pm 1 := (n, l \pm 1)$ for the cyclically adjacent index to $e = (n,l)$. Denote by

$$I_e := I_{n,l} \subset C_n$$

the component of ∂S between $z_e = z_{n,l}$ and $z_{e+1} = z_{n,l+1}$. However, ∂S may also have compact components $I = C_n \cong S^1$.

(ii) a complex structure j_S on $S := \overline{S} \setminus \{z_e \,|\, e \in \mathcal{E}\}$;

(iii) a set of *strip-like ends* for S, that is a set of embeddings with disjoint images

$$\epsilon_e : \mathbb{R}^{\pm} \times [0,\delta_e] \to S$$

for all $e \in \mathcal{E}$ such that

$$\begin{aligned} \epsilon_e(\mathbb{R}^{\pm} \times \{0,\delta_e\}) &\subset \partial S \\ \lim_{s\to\pm\infty} (\epsilon_e(s,t)) &= z_e \\ \epsilon_e^* j_S &= j_0 \end{aligned}$$

where j_0 is the canonical complex structure on the half-strip $\mathbb{R}^{\pm} \times [0,\delta_e]$ of width[1] $\delta_e > 0$. Denote the set of incoming resp. outgoing ends

$$\mathcal{E}_{\pm} := \mathcal{E}_{\pm}(S) := \{\epsilon_e : \mathbb{R}^{\mp} \times [0,\delta_e] \to S\}.$$

(iv) An ordering of the set of (compact) boundary components of $\overline{S}$ and orderings

$$\mathcal{E}_- = (e_1^-, \ldots, e_{N_-}^-), \quad \mathcal{E}_+ = (e_1^+, \ldots, e_{N_+}^+)$$

of the sets of incoming and outgoing ends. Here $e_i^{\pm} = (n_i^{\pm}, l_i^{\pm})$ denotes the incoming or outgoing end at $z_{e_i^{\pm}}$.

[1] Note that here, by a conformal change of coordinates, we can always assume the width to be $\delta_e = 1$. The freedom of widths will only become relevant in the definition of quilted surfaces with strip-like ends.

(b) Let S be a surface with strip-like ends, and E, F a pair of vector bundles as in Definition 4.1.1 of [26]. The bundle E admits a trivialization with fiber E_e over each strip like end e, and $F \subset E|\partial S$ is a totally real sub-bundle constant on the strip-like ends with fibers $F_e = (F_{e,+}, F_{e,-})$. A real Cauchy-Riemann operator $D_{E,F}$ for (E, F) is *asymptotically constant* if the following condition is satisfied: on each strip-like end $e \in \mathcal{E}(S)$ there exists a time-dependent operator

$$\mathcal{H}_e : [0,1] \to \mathrm{End}_{\mathbb{R}}(E_e)$$

such that the operator D_{E^ρ,F^ρ} on sections $(\epsilon_e)_*\xi, \xi : \mathbb{R}^\pm \times [0,1] \to E_e$ has asymptotic limit given by the following operator:

$$\tfrac{1}{2}\big(\mathrm{d}\xi + i_{E_e} \circ \mathrm{d}\xi \circ j\big) + \big((\mathcal{H}_e\xi)\mathrm{d}s - (i_{E_e} \circ \mathcal{H}_e\xi)\mathrm{d}t\big) \tag{8}$$

where i_{E_e} and j denote the complex structures on E_e and $\mathbb{R}^\pm \times [0,1]$ respectively, and d is the trivial connection on the trivial bundle E_e over $\mathbb{R}^\pm \times [0,1]$. More precisely, the difference between $\epsilon_e^*\big(D_{E,F}(\epsilon_e)_*\xi\big)$ and (8) is a zero-th order operator that approaches 0 uniformly in all derivatives in t as $s \to \infty$.

(c) An asymptotically constant Cauchy-Riemann operator $D_{E,F}$ is *non-degenerate* if the operator

$$\partial_t + \mathcal{H}_e : \Omega^0([0,1]; E_e, F_e) \to \Omega^0([0,1]; E_e)$$

has trivial kernel. Any non-degenerate, asymptotically constant operator $D_{E,F}$ is Fredholm on suitable Sobolev completions; see for example Lockhart-McOwen [10] for the case of surfaces with cylindrical ends.

Remark 2.3.7. (Non-degeneracy of the trivial operator) Suppose that E, F are trivial and for each end $e \in \mathcal{E}(S)$ the subspaces F_{b_0}, F_{b_1} for the components b_0, b_1 of ∂S meeting e are transversal, that is, $F_{b_0} \cap F_{b_1} = \{0\}$. Then the trivial Cauchy-Riemann operator $D_{E,F}$ is non-degenerate:

$$\ker(\partial_t) = F_{b_0} \cap F_{b_1} = \{0\}.$$

Furthermore if in addition the surface is a strip then the kernel and cokernel are trivial:

$$S \cong \mathbb{R} \times [0,1] \quad \Longrightarrow \quad \ker(D_{E,F}) = \{0\}, \ \ \mathrm{coker}(D_{E,F}) = \{0\}.$$

2.3.3. *Cauchy-Riemann operators on nodal surfaces.*

Definition 2.3.8. (Cauchy-Riemann operators on nodal surfaces)

(a) A *nodal surface S (with boundary and strip-like ends)* consists of

  (i) A surface with strip-like ends S^ρ (here the superscript ρ is used to indicate the surface "with nodes resolved") with boundary ∂S^ρ,

  (ii) An collection of *interior nodes*: pairs

$$Z = \{\{z_1^-, z_1^+\}, \ldots, \{z_r^-, z_r^+\}\}$$

  of distinct interior points of S^ρ;

  (iii) A collection of *boundary nodes*: ordered pairs

$$W = \{(w_1^-, w_1^+), \ldots, (w_s^-, w_s^+)\}$$

  of distinct boundary points of S^ρ; and

  (iv) An *ordering* of the set $\{I_i\} \cup \{w_j\} \cup \{e_k\}$ of boundary components I_i, boundary nodes w_j, and strip-like ends e_k.

Note that S^ρ is the normalization (resolution of singularities) of S.

(b) A *complex vector bundle* $E \to S$ on a nodal surface with boundary consists of
   (i) a complex vector bundle $E^\rho \to S^\rho$;
   (ii) isomorphisms $E^\rho_{z_i^+} \to E^\rho_{z_i^-}$ and $E^\rho_{w_i^+} \to E^\rho_{w_i^-}$ for each interior node $z_i^\pm$ and boundary node $w_i^\pm$; and
   (iii) a trivialization $E^\rho|_{\mathrm{im}\,\epsilon_e} \cong E_e \times (\mathbb{R}^\pm \times [0,1])$ for each strip-like end $e \in \mathcal{E}(S^\rho)$.

(c) A *totally real boundary condition* F for $E \to S$ is a totally real subbundle $F^\rho \subset E^\rho|_{\partial S}$ such that:
   (i) The identifications of the fibers at the boundary nodes induce isomorphisms $F^\rho_{w_i^+} \to F^\rho_{w_i^-}$;
   (ii) F^ρ is maximally totally real, that is $\mathrm{rank}_{\mathbb{R}}(F^\rho) = \mathrm{rank}_{\mathbb{C}}(E^\rho)$;
   (iii) In the trivialization over each strip-like end $e \in \mathcal{E}(S^\rho)$, the subspaces

$$F^\rho_{\epsilon_e(s,0)} = F_{e,0} \subset E_e, \quad F^\rho_{\epsilon_e(s,1)} = F_{e,1} \subset E_e$$

   are constant along $s \in \mathbb{R}^\pm$. These subspaces form a transverse pair $F_{e,0} \oplus F_{e,1} = E_e$.

(d) Let $E \to S$ be a complex vector bundle on a nodal surface S with totally real boundary condition F. A *real Cauchy-Riemann operator* $D_{E,F}$ for (S,E,F) is an operator

$$D_{E,F} : \Omega^0(E,F) \to \Omega^{0,1}(E,F), \quad \sigma \mapsto D_{E^\rho,F^\rho}\sigma$$

   defined in terms of a real Cauchy-Riemann operator D_{E^ρ,F^ρ} on S^ρ with values in E^ρ and boundary conditions in F^ρ. Here we set $\Omega^{0,1}(E,F) := \Omega^{0,1}(E^\rho,F^\rho)$ and define $\Omega^0(E,F) \subset \Omega^0(E^\rho,F^\rho)$ as the kernel of the surjective map

$$\delta : \quad \Omega^0(E^\rho,F^\rho) \longrightarrow \bigoplus_i E^\rho_{z_i^+} \oplus \bigoplus_j F^\rho_{w_j^+} \tag{9}$$

$$\sigma \longmapsto \bigoplus_i (\sigma(z_i^+) - \sigma(z_i^-)) \oplus \bigoplus_j (\sigma(w_j^+) - \sigma(w_j^-)). \tag{10}$$

(e) The family versions of the above definitions are as follows. A *family of nodal surfaces* $S \to B$ is a smooth family $S^\rho \to B$ of complex surfaces (compact, possibly with boundary) over a smooth, open base B, together with nodes $Z, W \subset (S^\rho)^2$ varying smoothly over B. A *family of complex vector bundles* $E \to S$ is a complex vector bundle $E^\rho \to S^\rho$, together with smoothly varying identifications of the fibers at the nodes and constant trivializations on the strip-like ends. A *family of totally real boundary conditions* $F \to \partial S$ consists of a totally real boundary condition $F^\rho \to \partial S^\rho$ that is constant in the trivializations on the strip-like ends. A family of real Cauchy-Riemann operators $D_{E,F}$ for the families $(S,E,F) \to B$ is a family of real Cauchy-Riemann operators D_b for (S_b, E_b, F_b), varying smoothly with $b \in B$.

Remark 2.3.9. (Unreduced and reduced Cauchy-Riemann operators) The determinant line $\det(D_{E,F})$ for a Cauchy-Riemann operator $D_{E,F}$ over a nodal surface S is isomorphic to the determinant $\det(D_{E^\rho,F^\rho})$ for the corresponding operator over the smooth surface S^ρ

with resolved nodes by the following construction: Consider the "unreduced" operator

$$D_{E,F}^{\mathrm{unred}} : \Omega^0(E^\rho, F^\rho) \to \bigoplus_i E^\rho_{z_i^+} \oplus \bigoplus_j F^\rho_{w_j^+} \oplus \Omega^{0,1}(E^\rho, F^\rho), \quad \sigma \mapsto (\delta(\sigma), D_{E^\rho,F^\rho}\sigma)$$

where δ is the operator of (9). The kernel and cokernel are canonically isomorphic to those of $D_{E,F}$. The isomorphisms define an isomorphism of determinant lines

$$\det(D_{E,F}) \to \det(D_{E,F}^{\mathrm{unred}}). \tag{11}$$

From this we construct the "reduced operator"

$$D_{E,F}^{\mathrm{red}} : \ker(D_{E^\rho,F^\rho}) \to \bigoplus_i E^\rho_{z_i^+} \oplus \bigoplus_j F^\rho_{w_j^+} \oplus \operatorname{coker}(D_{E^\rho,F^\rho}), \quad \sigma \mapsto (\delta(\sigma), 0). \tag{12}$$

The kernel and cokernel of $D_{E,F}^{\mathrm{red}}$ are canonically isomorphic to those of $D_{E,F}^{\mathrm{unred}}$. The isomorphisms define an isomorphism of determinant lines

$$\det(D_{E,F}^{\mathrm{unred}}) \to \det(D_{E,F}^{\mathrm{red}}). \tag{13}$$

Since the domain and codomain of $D_{E,F}^{\mathrm{red}}$ are finite dimensional, we have by (5) a canonical isomorphism

$$\det(D_{E,F}^{\mathrm{red}}) \to \Lambda^{\max}\Big(\bigoplus_i E^\rho_{z_i^+} \oplus \bigoplus_j F^\rho_{w_j^+}\Big)^\vee \otimes \det(D_{E^\rho,F^\rho}). \tag{14}$$

Hence orientations on D_{E^ρ,F^ρ} and the fibers $E^\rho_{z_i^+}, F^\rho_{w_j^+}$ induce an orientation on $D_{E,F}$. A similar isomorphism holds when a surface S and bundles E, F are obtained from another nodal surface $\hat{S}$ and bundles $\hat{E}, \hat{F}$ by resolving some subset of the nodes of $\hat{S}$. That is, $\hat{S}$ is obtained by removing some subset of the sets of interior and boundary nodes Z, W so that some nodal points of $\hat{S}$ are replaced by pairs of points in S, and $\hat{E}, \hat{F}$ are the bundles obtained by pullback under $\hat{S} \to S$.

In the case that the ordering of the boundary nodes and components is such that the boundary nodes are ordered first (w_i appears before p_j, for each i, j) we take the orientation from the previous paragraph to be the orientation of $D_{E,F}$. In general, an orientation of $D_{E,F}$ is defined by the orientation from the previous paragraph times the sign arising from permuting the determinant lines $\Lambda^{\max}(F_{w_i})$ of the boundary nodes so that they appear before the determinant lines for the boundary components.

Example 2.3.10. (Orientation for the trivial bundle over a nodal disk) Continuing Example 2.2.3, suppose that S is a nodal surface consisting of two disks joined with a single boundary node (w_-, w_+). Also suppose that the ordering of the disks inducing the ordering of (w_-, w_+). Equip S with the trivial bundles E, F. Then the reduced operator is

$$D_{E,F}^{\mathrm{red}} : (x_1, x_2) \mapsto x_1 - x_2.$$

Thus the reduced operator has kernel equal to the diagonal

$$\ker(D_{E,F}^{\mathrm{red}}) = \{(x, x) \mid x \in F\} \subset F \oplus F.$$

By 2.2.3, the determinant line $\det(D_{E,F}^{\mathrm{red}})$ inherits the orientation of $\det(F)$ times $(-1)^{\mathrm{rank}(F)}$. If the ordering of the boundary components and boundary node is (first component, boundary node, second component), then the orientation induced on $D_{E,F}$ is the standard one.

Remark 2.3.11. The class of Cauchy-Riemann operators is closed under the following operations:

(a) (Conjugates) Let (E, F) be a bundle with boundary condition over S. Let E^- the complex conjugate of E, and F^- the subspace F considered as a totally real subspace of F. Let S^- denote the surface S with complex structure $\overline{j} = -j$. Given a Cauchy-Riemann operator $D_{E,F}$ the first order part of $D_{E,F}$ is complex linear with respect to the dual complex structures $-J, -j$ and defines a Cauchy-Riemann operator D_{E^-,F^-} on the dual (E^-, F^-).

(b) (Direct Sums) Let $(E_k, F_k), k = 0, 1$ be bundles with real boundary conditions over a surface S, and

$$(E, F) = (E_0, F_0) \oplus (E_1, F_1).$$

Let $D_{E_k,F_k}, k = 0, 1$ are Cauchy-Riemann operators for the components. The direct sum

$$D_{E,F} = D_{E_0,F_0} \oplus D_{E_1,F_1}$$

is a Cauchy-Riemann operator for the direct sum.

(c) (Disjoint Unions) Let (E_k, F_k) denoted bundles with totally real boundary condition over surfaces S_k for $k = 0, 1$. Then

$$(E, F) = (E_0, F_0) \sqcup (E_1, F_1)$$

is a bundle with totally real boundary condition over $S = S_0 \sqcup S_1$. Then the space of forms $\Omega^0(E, F)$ is naturally isomorphic to the direct sum of the $\Omega^0(E_k, F_k)$. If $D_{E_k,F_k}, k = 0, 1$ are Cauchy-Riemann operators for the components then the direct sum $D_{E,F} = D_{E_0,F_0} \oplus D_{E_1,F_1}$ is a Cauchy-Riemann operator for the disjoint union.

Now we turn to quilted surfaces. We could allow nodes in the following definition, but have no need for nodal quilted surfaces and so that extension is left to the interested reader.

Definition 2.3.12. (Quilted surfaces) A *quilted surface* $\underline{S}$ with strip-like ends consists of the following data:

(a) A collection of *patches* $(S_p)_{p\in\mathcal{P}}$ indexed by a set $\mathcal{P}$, so that each patch S_p is a surface with strip-like ends. Each S_p carries a complex structures j_p and has strip-like ends $(\epsilon_{p,e})_{e\in\mathcal{E}(S_p)}$ of widths $\delta_{p,e} > 0$. Each end has limit equal to a marked point

$$\lim_{s\to\pm\infty} \epsilon_{p,e}(s, t) =: z_{p,e} \in \partial\overline{S}_p.$$

Denote by $I_{p,e} \subset \partial S_p$ the noncompact boundary component between $z_{p,e-1}$ and $z_{p,e}$.

(b) A collection of *seams* $\mathcal{S}$. Each seam $\sigma \in \mathcal{S}$ is a pairwise disjoint 2-element subset of the set of patches and boundary components:

$$\sigma \subset \bigcup_{p\in\mathcal{P}} \{p\} \times \pi_0(\partial S_p).$$

We write

$$\sigma = \{(p_-(\sigma), I_{\sigma,-}), (p_+(\sigma), I_{\sigma,+})\}$$

recording the patches and components of the boundary that are identified. For each $\sigma \in \mathcal{S}$, a diffeomorphism of boundary components

$$\varphi_\sigma : \partial S_{p_-(\sigma)} \ni I_{\sigma,-} \xrightarrow{\sim} I_{\sigma,+} \subset \partial S_{p_+(\sigma)}$$

is given and supposed to satisfy the conditions:

(i) *real analytic:* Every point $z \in I_\sigma$ has an open neighborhood $\mathcal{U} \subset S_{p_-(\sigma)}$ on one side of the seam such that $\varphi_\sigma|_{\mathcal{U}\cap I_\sigma}$ extends to an antiholomorphic embedding on the other side:

$$\psi_z : \mathcal{U} \to S_{p_+(\sigma)}, \quad \psi_z^* j_{p_+(\sigma)} = -j_{p_-(\sigma)}.$$

In particular, this condition forces φ_σ to reverse the orientation on the boundary components.

(ii) *compatible with strip-like ends :* Let I_σ (and hence I'_σ) be noncompact, i.e. lie between marked points, $I_\sigma = I_{p_\sigma,e_\sigma}$ and $I'_\sigma = I_{p'_\sigma,e'_\sigma}$. We require that φ_σ matches up the end e_σ with $e'_\sigma - 1$ and the end $e_\sigma - 1$ with e'_σ. That is, $\epsilon^{-1}_{p'_\sigma,e'_\sigma} \circ \varphi_\sigma \circ \epsilon_{p_\sigma,e_\sigma-1}$ maps $(s, \delta_{p_\sigma,e_\sigma-1}) \mapsto (s, 0)$ if both ends are incoming, or it maps $(s, 0) \mapsto (s, \delta_{p'_\sigma,e'_\sigma})$ if both ends are outgoing. We disallow matching of an incoming with an outgoing end, and the condition on the other pair of ends is analogous.

Given a quilted surface with strip-like ends $\underline{S}$ as above:

(a) The *true boundary components* $I_b \subset \partial S_{p_b}, b \in \mathcal{B}$ are those that are not identified with another boundary component of $\underline{S}$. Let $\mathcal{B}$ denote the set of true boundary components, and for each $b \in \mathcal{B}$ let p_b denote the patch and I_b the component.

(b) The *quilted ends*

$$\underline{e} \in \mathcal{E}(\underline{S}) = \mathcal{E}_-(\underline{S}) \sqcup \mathcal{E}_+(\underline{S})$$

consist of a maximal sequence

$$\underline{e} = (p_i, e_i)_{i=1,\ldots,n_{\underline{e}}}$$

of ends of patches. The boundaries of each end of each patch are identified

$$\epsilon_{p_i,e_i}(\cdot, \delta_{p_i,e_i}) \cong \epsilon_{p_{i+1},e_{i+1}}(\cdot, 0)$$

via some seam ϕ_{σ_i}. The end sequence could be cyclic, i.e. with an additional identification $\epsilon_{p_n,e_n}(\cdot, \delta_{p_n,e_n}) \cong \epsilon_{p_1,e_1}(\cdot, 0)$ via some seam ϕ_{σ_n}. Otherwise the end sequence is noncyclic, i.e. $\epsilon_{p_1,e_1}(\cdot, 0) \in I_{b_0}$ and $\epsilon_{p_n,e_n}(\cdot, \delta_{p_n,e_n}) \in I_{b_n}$ take values in some true boundary components $b_0, b_n \in \mathcal{B}$.

(c) The ends ϵ_{p_i,e_i} of patches in a quilted end $\underline{e}$ are either all incoming, $e_i \in \mathcal{E}_-(S_{p_i})$, in which case we call the quilted end *incoming*, $\underline{e} \in \mathcal{E}_-(\underline{S})$, or they are all outgoing, $e_i \in \mathcal{E}_+(S_{p_i})$, in which case we call the quilted end *outgoing*, $\underline{e} \in \mathcal{E}_+(\underline{S})$.

We assume, as part of the definition, that orderings of the patches and of the boundary components of each $\overline{S}_k$, orderings $\mathcal{E}_\pm(\underline{S}) = (\underline{e}^\pm_1, \ldots, \underline{e}^\pm_{N_\pm(\underline{S})})$ of the quilted ends are given.

Definition 2.3.13. (Cauchy-Riemann operators for quilted surfaces with strip-like ends)

(a) (Boundary and seam conditions) A collection of *bundles with totally real boundary and seam conditions* is a pair $(\underline{E}, \underline{F}) \to \underline{S}$ consisting of a family of complex vector bundles over the components $\underline{E}$ together with totally real subbundles $\underline{F}$ over the boundary components and seams. That is, for each seam σ, the corresponding component F_σ is a totally real subspace of the restriction of components of $\underline{E}$:

$$F_\sigma \subset E^-_{p_+(\sigma)}|(\partial S_{p_+(\sigma)})_\sigma \times E_{p_-(\sigma)}|(\partial S_{p_-(\sigma)})_\sigma$$

of the bundles $E_{p_\pm(\sigma)}$ on the patches $S_{p_\pm(\sigma)}$ adjacent to σ.

(b) (Quilted Cauchy-Riemann operators) A *quilted Cauchy-Riemann operator* for $(\underline{E}, \underline{F})$ is a collection of Cauchy-Riemann operators

$$D_{\underline{E},\underline{F}} = (D_p, p \in \mathcal{P})$$

on the patches $S_p, p \in \mathcal{P}$, acting on the space of sections with the given boundary and seam conditions.

2.4. **Gluing of Cauchy-Riemann operators.** Nodal surfaces with strip-like ends can be glued along the ends, or at interior or boundary nodes. In this section we explain the corresponding gluing operators on Cauchy-Riemann operators. First we explain the behavior of determinant lines under gluing of strip-like ends.

Definition 2.4.1. (Gluable ends) Let S be a surface with strip-like ends. Let $E \to S$ be a complex vector bundle and $F \to \partial S$ a totally real boundary condition. Let $D_{E,F}$ be a real Cauchy-Riemann operator. Let $e_+ \in \mathcal{E}_+(S)$ and $e_- \in \mathcal{E}_-(S)$ be an outgoing resp. incoming end. Suppose a complex isomorphism is given that maps the totally real boundary conditions on the ends:

$$E_{e_+} \to E_{e_-}, \quad F_{e_+,k} \mapsto F_{e_-,k}, \quad k \in \{0,1\} \tag{15}$$

We say that the ends $e_\pm$ are *gluable* if the asymptotic limits (8) of $D_{E,F}$ on the ends $e_\pm$ are equal, after the identification of fibers (15).

Definition 2.4.2. (Glued surface and Cauchy-Riemann operator) Let S be a surface with gluable ends $e_\pm$ equipped with bundles E, F and a gluable Cauchy-Riemann operator $D_{E,F}$.

(a) Let $\tilde{S} = \#_{e_+}^{e_-}(S)$ be the *glued surface* formed by gluing the ends of S. That is, consider a pair of ends

$$\epsilon_{e_+}(\mathbb{R}^+ \times [0,1]) \cup \epsilon_{e_-}(\mathbb{R}^- \times [0,1]) \subset S.$$

Replacing these ends by a strip $[-\tau, \tau] \times [0,1]$ depending on a gluing parameter $\tau > 0$, where $\{\pm\tau\} \times [0,1]$ is identified with $\epsilon_{e_\mp}(\{0\} \times [0,1])$, gives a surface $\tilde{S}$ with two fewer ends, after a choice of a new ordering on the boundary components and strip-like ends. See also Section 4.1 of [26] and Definition 4.1.1 of [26].

(b) Let $\tilde{E}, \tilde{F}$ be the complex vector bundle and totally real boundary condition over $\tilde{S}$ that arise from gluing E, F via the isomorphism $E_{e_+} \cong E_{e_-}$ on the middle strip. Let $\rho_\pm$ be cutoff functions on the strip-like ends with $\rho_+ + \rho_- = 1$. Given a section $\tilde{\sigma}$ of $\tilde{E}$ define a section σ of E by

$$\sigma = \tilde{\sigma} \text{ on } \tilde{S} \backslash \epsilon_\pm(\pm(0,\infty) \times [0,1]), \quad \sigma = \rho_\pm \tilde{\sigma} \text{ on } \epsilon_\pm(\pm(0,\infty) \times [0,1]).$$

Given $D_{E,F}$ define a *glued real Cauchy-Riemann operator* $D_{\tilde{E},\tilde{F}}$ for $(\tilde{S}, \tilde{E}, \tilde{F})$ by defining $D_{\tilde{E},\tilde{F}} \tilde{\sigma}$ to be the section of E obtained from $D_{E,F}\sigma$ by adding together the forms on the strip-like ends:

$$D_{\tilde{E},\tilde{F}} \tilde{\sigma} = \pi_* D_{E,F} \sigma \tag{16}$$

where

$$\pi : S \backslash \epsilon_\pm(\pm(0,\infty) \times [0,1]) \to \tilde{S}$$

is the gluing map, and π_* is integration over the fibers

$$\pi_*\eta(z) = D_{z_-}\pi_*\eta(z_-) + D_{z_+}\pi_*\eta(z_+), \quad \pi^{-1}(z) = (z_-, z_+).$$

Proposition 2.4.3. (Identification of indices and determinant lines under gluing strip-like ends) *Suppose that $D_{\tilde{E},\tilde{F}}$ is obtained from $D_{E,F}$ by gluing strip-like ends. Then there is an equality of indices* $\mathrm{Ind}(D_{E,F}) = \mathrm{Ind}(D_{\tilde{E},\tilde{F}})$ *and a canonical isomorphism of determinant lines*

$$\det(D_{E,F}) \to \det(D_{\tilde{E},\tilde{F}}). \tag{17}$$

Proof. For simplicity we assume that the surface is unquilted. For sufficiently large τ there exist isomorphisms

$$\ker(D_{E,F}) \xrightarrow{\sim} \ker(D_{\tilde{E},\tilde{F}}), \quad \mathrm{coker}(D_{E,F}) \xrightarrow{\sim} \mathrm{coker}(D_{\tilde{E},\tilde{F}})$$

defined as follows. Given a section ξ in the kernel of $D_{E,F}$, one may use cutoff functions on $[-\tau,\tau]$ to glue it together to a section $\tilde{\xi} = \#_\tau\xi$ of $\tilde{E} \to \tilde{S}$ with boundary conditions in $\tilde{F}$. Explicitly

$$\tilde{\xi} = \xi \text{ on } \tilde{S}\backslash\epsilon_\pm(\pm(0,\infty)\times[0,1]), \quad \tilde{\xi} = \rho_\pm\xi \text{ on } \epsilon_\pm(\pm(0,\infty)\times[0,1]).$$

Then $\tilde{\xi}$ is an approximate zero of $D_{\tilde{E},\tilde{F}}$. Gluing followed by orthogonal projection onto the kernel of $D_{\tilde{E},\tilde{F}}$ defines, for τ sufficiently large, the isomorphism, see [11, Section 5.3] for details of the analysis. The construction for the cokernels follows by identifying the cokernels of $D_{E,F}$ and $D_{\tilde{E},\tilde{F}}$ with the kernels of their adjoints. Gluing of Cauchy-Riemann operators on quilted surfaces along quilted ends is similar. □

Next we describe the behavior of determinant lines under deformation of nodes. The story here is analogous to the one in algebraic geometry, where one has a long exact sequence in homology induced from the short exact sequence of sheaves induced by the normalization.

Definition 2.4.4. (Deformation of a node) Consider an interior node of S represented by a pair $z^\pm \in S^\rho$, and $\tau \in \mathbb{R}_{>0} + [0,1]i$.

(a) (Deformed surface) Let $\tilde{S}$ be the (possibly still nodal) *deformed surface* with strip-like ends obtained by deforming the node. Thus $\tilde{S}$ is the surface obtained gluing punctured disks around $z^\pm$ using the map $z \mapsto \exp(2\pi\tau)/z$. Denote by

$$s + it = \ln(z)/\pi - \tau$$

the coordinates on the cylindrical neck $[-|\tau|,|\tau|]\times S^1$. In the case of a boundary node, we require that the gluing parameter τ is real and glue together half-disks by $z \mapsto \exp(2\pi\tau)/z$ and identify the neck with $[-\tau,\tau]\times[0,1]$ with coordinates $s+it$. See Figure 2, in which the glued disks/neck regions are shaded.

(b) (Deformed vector bundles and Cauchy-Riemann operators) Let $\tilde{E},\tilde{F}$ denote the vector bundles over $\tilde{S},\partial\tilde{S}$ obtained by gluing in the trivial bundles

$$(E_z, F_z) = (E^\tau_{z^-}, F^\tau_{z^-}) = (E^\tau_{z^+}, F^\tau_{z^+})$$

in the fixed trivialization over the (half)disks around $z^\pm$. Using cutoff functions, one constructs from $D_{E,F}$ a family of real Cauchy-Riemann operators

$$D_{\tilde{E},\tilde{F}} : \Omega^0(\tilde{E},\tilde{F}) \to \Omega^{0,1}(\tilde{E})$$

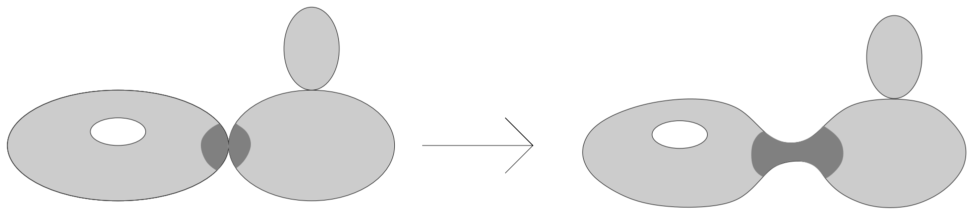

FIGURE 2. Deformation of a boundary node

for $(\tilde{S}, \tilde{E}, \tilde{F})$ similar to the construction of (16). Each operator $D_{\tilde{E},\tilde{F}}$ in the family is equal to $D_{E,F}$ away from the gluing region and approaches the trivial operator on the neck in the limit $\tau \to \infty$.

Note that the conformal structure of $\tilde{S}$ depends on the value of the gluing parameter τ, as well as the choices of local coordinates $\mathbb{R}^\pm \times S^1$ or $\mathbb{R}^\pm \times [0,1]$ on punctured neighborhoods of $z^\pm$. In addition, to obtain a surface with strip-like ends in our sense one has to choose a new ordering on the nodes and possibly the boundary components of $\tilde{S}$.

The following gluing result is the basic result used in the identification of determinant lines of the deformed Cauchy-Riemann operator with the determinant line of the original. The result is a slight modification of [5, Lemma 3.1]. We suppose that the node is on the boundary; the interior case is similar. We also assume for simplicity that $\tilde{S}$ is smooth.

Theorem 2.4.5. (Long exact sequence in homology for surfaces with boundary and strip-like ends) *Let $\tilde{S}, \tilde{E}, \tilde{F}$ be a deformation of a node from S, E, F obtained from the resolved surface and bundles S^ρ, E^ρ, F^ρ by identifying small balls around the node. For sufficiently large values of the gluing parameter τ there is an exact sequence*

$$0 \to \ker(D_{\tilde{E},\tilde{F}}) \xrightarrow{\iota} \ker(D_{E^\rho,F^\rho}) \xrightarrow{D^{\mathrm{red}}_{\tilde{E},\tilde{F}}} F_z \oplus \operatorname{coker}(D_{E^\rho,F^\rho}) \to \operatorname{coker}(D_{\tilde{E},\tilde{F}}) \to 0 \tag{18}$$

such that in the limit $\tau \to \infty$, the middle map $D^{\mathrm{red}}_{\tilde{E},\tilde{F}}$ converges to $D^{\mathrm{red}}_{E,F}$ from (12).

Remark 2.4.6. (Relation to the long exact sequence for a normalization) The sequence (18) is a real version of a standard long exact sequence in algebraic geometry. Namely suppose $\mathbb{E}$ is a vector bundle on a nodal curve S with a node at z, and $\pi : S^\rho \to S$ is the normalization of S at z. There is a short exact sequence of sheaves given by

$$0 \to \mathbb{E} \to \pi_*\pi^*\mathbb{E} \to \iota_{z,*}\iota_z^*\mathbb{E} \to 0$$

where abusing notation $\iota_{z,*}\iota_z^*\mathbb{E}$ denotes the skyscraper sheaf with fiber $\mathbb{E}_z$. The short exact sequence induces a long exact sequence of cohomology groups, described in terms of linearized Cauchy-Riemann operators as

$$0 \to \ker(D_{\mathbb{E}}) \to \ker(D_{\mathbb{E}^\rho}) \to \mathbb{E}_z \to \operatorname{coker}(D_{\mathbb{E}}) \to \operatorname{coker}(D_{\mathbb{E}^\rho}) \to 0.$$

See for example [3, (1)].

Proof of Theorem 2.4.5. First we construct suitable Sobolev spaces on the glued surface $\tilde{S}$, depending on the gluing parameter τ. We will require a nested sequence of cutoff functions on the neck region of $\tilde{S}$ for which we introduce the following notation. For each integer

$n \in [2,6]$ let β_n be a family of smooth functions on $\tilde{S}$ depending on τ with the following properties:

(a) β_n is supported on the part of the neck parametrized by $[-n\tau/7, n\tau/7] \times [0,1]$;
(b) β_n is equal to 1 on $[-(n-1)\tau/7, (n-1)\tau/7] \times [0,1]$, and takes values in $[0,1]$ elsewhere;
(c) β_n has first derivative bounded by C/τ for some constant $C > 0$.

For $\delta \in (-1,0)$ consider the weight function

$$\zeta_\tau \in C^\infty(\tilde{S}), \quad \zeta_\tau = (1-\beta_6) + \beta_6(e^{\delta(s-\tau)} + e^{-\delta(s+\tau)}). \tag{19}$$

The second term is well-defined since β_6 is supported on the neck. Let $W^{1,2}_\delta(\tilde{E},\tilde{F})$ be the Sobolev space with weight function ζ_τ. This is the space of $W^{1,2}_{\mathrm{loc}}$ functions with finite weighted norm

$$\|\xi\|_{W^{1,2}_\delta(\tilde{E},\tilde{F})} = \|\zeta_\tau \xi\|_{W^{1,2}(\tilde{E},\tilde{F})}.$$

The Cauchy-Riemann operator $D_{\tilde{E},\tilde{F}}$ is Fredholm on this Sobolev space by standard results.

With these Sobolev spaces defined we study the kernel and the cokernel of the linearized Cauchy-Riemann operator on the glued surface. We have an exact sequence of Banach spaces defined by the linearized

$$0 \to \ker(D_{\tilde{E},\tilde{F}}) \to \Omega^0(\tilde{E},\tilde{F}) \to \Omega^{0,1}(\tilde{E}) \to \operatorname{coker}(D_{\tilde{E},\tilde{F}}) \to 0 \tag{20}$$

where the middle terms are the $W^{1,2}_\delta$ resp. $W^{0,2}_\delta$ spaces defined above. We wish to show that (20) is equivalent, modulo stabilization, to an exact sequence of finite-dimensional spaces. More precisely, we show that the middle map in (20) is uniformly bounded from below on a space isomorphic to the complement of the kernel of the Cauchy-Riemann operator for the normalization.

Next we identify the kernel of the operator on the normalization with a subspace of sections on the deformed bundle. For any $\xi \in \Omega^0(E,F)$ we denote by $(1-\beta_2)\xi \in \Omega^0(\tilde{E},\tilde{F})$ the section obtained by multiplying by the cutoff function $(1-\beta_2)$ and using the identification of $\tilde{E}$ and E away from the neck. The map

$$\ker(D_{E^\rho,F^\rho}) \to (1-\beta_2)\ker(D_{E^\rho,F^\rho}), \quad \xi \mapsto (1-\beta_2)\xi \tag{21}$$

is an isomorphism, since by analytic continuation no element is supported on the neck. We identify $\ker(D_{E^\rho,F^\rho})$ with its image under the map (21).

The first map in the four-term sequence (18) is now defined by orthogonal projection. Namely let V_τ denote the $W^{0,2}_\delta$-orthogonal complement of $\ker(D_{E^\rho,F^\rho})$. Define the first map in (18) to be the composition

$$\ker(D_{\tilde{E},\tilde{F}}) \to \ker(D_{E^\rho,F^\rho}) \subset \Omega^0(\tilde{E},\tilde{F})$$

of inclusion and projection along V_τ.

To find the second map in the four-term sequence (18), we find an approximate description of the image of V_τ under the linearized operator $D_{\tilde{E},\tilde{F}}$ for the glued surface.

Claim: For $\delta \in (-1,0)$, the restriction of $D_{\tilde{E},\tilde{F}}$ to V_τ is uniformly right invertible, that is, there exist constants C and τ_0 such that for $\tau > \tau_0$,

$$C\|\xi\|_{W^{1,2}_\delta} \le \|D_{\tilde{E},\tilde{F}}\xi\|_{L^2_\delta}, \ \forall \xi \in V_\tau. \tag{22}$$

Suppose otherwise. There exists a sequence $\tau(\alpha) \to \infty, \xi(\alpha) \in V_{\tau(\alpha)}$ with

$$\|\xi(\alpha)\|_{W^{1,2}_\delta} = 1, \quad \lim_{\alpha \to \infty} \|D_{\tilde{E},\tilde{F}} \xi(\alpha)\|_{L^2_\delta} = 0. \tag{23}$$

Denote by S° resp. E°, F° the surface with strip-like ends obtained by removing the node z, resp. the fibers E_z, F_z. Let

$$S^\nu \cong [-\tau, \tau] \times [0,1], \quad E^\nu = E|S^\nu, \quad F^\nu = F^\nu|S^\nu$$

be the neck and bundles restricted to the neck. We split $\xi(\alpha)$ into sections supported away from and on the neck, and apply elliptic estimates for S°, S^ν to obtain a contradiction. First note that the kernel of D_{E^ρ,F^ρ} may be identified with the kernel of D_{E°,F° for any Sobolev weight $\delta \in (0,-1)$. Indeed, we may identify S locally with the half-space $\mathcal{H}$. We assume that our Sobolev spaces on S use a measure that is locally the pull-back of the standard measure on $\mathcal{H}$. The conformal transformation $(s,t) \mapsto \exp(-s - i\pi t)$ maps the infinite strip $\mathbb{R} \times [0,1]$ to $\mathcal{H}$. The pull-back of the canonical measure on $\mathcal{H}$ is $\pi e^{-2s} \mathrm{d}s\mathrm{d}t$. With our conventions, this pullback is the measure with Sobolev weight $\delta = -1$. Thus pullback gives an identification of the kernel

$$W^{1,2}_{-1}(E^\circ, F^\circ) \supset \ker(D_{E^\circ,F^\circ}) \to \ker(D_{E^\rho,F^\rho}) \subset W^{1,2}(E^\rho, F^\rho).$$

Elliptic regularity gives an identification with the kernel of D_{E^ρ,F^ρ} on $W^{k,2}(E^\rho, F^\rho)$ for any $k \geq 1$. The operator D_{E°,F° is Fredholm for weights δ not in the spectrum $\mathbb{Z}$ of the limiting operator on the strip-like ends, see e.g. [10]. The kernel $\ker(D_{E^\circ,F^\circ})$ is unchanged by any non-negative perturbation of Sobolev weight not passing through the spectrum of the limiting operator:

$$([\delta_1, \delta_2] \cap \operatorname{Spec}(\partial_t + \mathcal{H}_e) = \emptyset, \quad \forall e \in \mathcal{E}) \implies (\ker(D_{E^\circ,F^\circ})_{W^{1,2}_{\delta_1}} = \ker(D_{E^\circ,F^\circ})_{W^{1,2}_{\delta_2}}).$$

Hence

$$\ker(D_{E^\circ,F^\circ})_{W^{1,2}_{-1}} = \ker(D_{E^\circ,F^\circ})_{W^{1,2}_\delta}, \quad \forall \delta \in (-1,0).$$

Let β_2 denote the cutoff function introduced at the beginning of the proof. Since the operator $D_{\tilde{E},\tilde{F}}$ is equal to D_{E°,F° away from the neck and approaches D_{E^ν,F^ν} on the neck, we have for some constant $C > 0$ independent of α,

$$\begin{aligned} \|\xi(\alpha)\|_{\tilde{E}} &\leq C\|(1-\beta_2)\xi(\alpha)\|_{E^\circ} + C\|\beta_2\xi(\alpha)\|_{E^\nu} \\ &\leq C\|D_{E^\circ,F^\circ}(1-\beta_2)\xi(\alpha)\|_{E^\circ} + C\|\operatorname{proj}_{\ker(D_{E^\circ,F^\circ})}(1-\beta_2)\xi(\alpha)\|_{E^\circ} \\ &\quad + C\|D_{E^\nu,F^\nu}\beta_2\xi(\alpha)\|_{E^\nu} \\ &\to 0. \end{aligned}$$

This is a contradiction. The first inequality follows from comparability of the norms on E°, E^ν, and E, the second inequality combines the elliptic estimates for (E°, F°) and (E^ν, F^ν). The last limit uses the bound on the derivative of β_2 and the fact that $D_{\tilde{E},\tilde{F}}\xi(\alpha) \to 0$. This proves the claim.

We continue with the construction of the second map in the four-term sequence. Identify $\operatorname{coker}(D_{E^\rho,F^\rho})$ with the $W^{1,2}_\delta$-perpendicular of $\operatorname{im}(D_{E^\rho,F^\rho})$. Also identify E and E° away from the neck. Let β_4 denote the cutoff function introduced above, and define an injection for τ sufficiently large

$$\operatorname{coker}(D_{E^\rho,F^\rho}) \to \Omega^{0,1}(\tilde{E},\tilde{F}), \quad \xi \mapsto \beta_4\xi;$$

let $\beta_4 \operatorname{coker}(D_{E^\rho,F^\rho})$ denote its image. Let $\underline{F_z}$ the subspace of $\Omega^{0,1}(\tilde{E},\tilde{F})_{L^2_\delta}$ consisting of one-forms equal on the neck to $f(\mathrm{d}s - i\mathrm{d}t)$ for some $f \in F_z$. By multiplying by β_4 gives a finite-dimensional subspace of $\Omega^{0,1}(\tilde{E},\tilde{F})_{L^2_\delta}$, isomorphic to F_z by evaluation at a point z_{mid} at the mid-point of the neck:

$$\underline{F_z} \cong F_z, \quad \xi \mapsto \xi(z_{\mathrm{mid}}).$$

For τ sufficiently large, the sum $(1-\beta_4)\operatorname{coker}(D_{E^\rho,F^\rho}) + \beta_4\underline{F_z}$ is direct, since the intersection is trivial. Let

$$U_\tau := ((1-\beta_4)\operatorname{coker}(D_{E^\rho,F^\rho}) + \beta_4\underline{F_z})^\perp \subset \Omega^{0,1}(\tilde{E},\tilde{F})$$

denote the $W^{1,2}_\delta$-perpendicular. Let

$$\pi_\tau : \Omega^{0,1}(\tilde{E},\tilde{F}) \to U_\tau$$

denote the projection.

Claim: The operator $\pi_\tau \circ D_{\tilde{E},\tilde{F}} : V_\tau \to U_\tau$ is an isomorphism with uniformly bounded right inverse, for τ sufficiently large.

Suppose otherwise. Then there is a sequence

$$\tau(\alpha) \to \infty, \quad \xi(\alpha) \in V_{\tau(\alpha)}, \quad \zeta(\alpha) \in U_{\tau(\alpha)}$$

with

$$\|\xi(\alpha)\|_{V_{\tau(\alpha)}} = \|\zeta(\alpha)\|_{U_{\tau(\alpha)}} = 1, \quad (D_{\tilde{E}_{\tau(\alpha)},\tilde{F}_{\tau(\alpha)}}\xi(\alpha), \zeta(\alpha)) \to 0. \tag{24}$$

The pairing of $D_{\tilde{E}_{\tau(\alpha)},\tilde{F}_{\tau(\alpha)}}\xi(\alpha)$ with any sequence of elements

$$(1-\beta_4)\zeta(\alpha) + \beta_4\zeta(\alpha)' \in (1-\beta_4)\operatorname{coker}(D_{E^\rho,F^\rho}) + \beta_4\underline{F_z}$$

of norm one approaches zero since the cut-off functions are slowly varying. This convergence implies $\|D_{\tilde{E}_{\tau(\alpha)},\tilde{F}_{\tau(\alpha)}}\xi(\alpha)\| \to 0$ which contradicts (22). The claim follows.

The discussion above shows that we have splittings

$$\Omega^0(\tilde{E},\tilde{F}) \cong V_\tau \oplus \ker(D_{E^\rho,F^\rho}), \quad \Omega^{0,1}(\tilde{E}) \cong U_\tau \oplus F_z \oplus \operatorname{coker}(D_{E^\rho,F^\rho}). \tag{25}$$

By (20) and (25) for τ sufficiently large there is an exact sequence

$$0 \to \ker(D_{\tilde{E},\tilde{F}}) \to V_\tau \oplus \ker(D_{E^\rho,F^\rho}) \to U_\tau \oplus F_z \oplus \operatorname{coker}(D_{E^\rho,F^\rho}) \to \operatorname{coker}(D_{\tilde{E},\tilde{F}}) \to 0. \tag{26}$$

We obtain from this sequence of Banach spaces an exact sequence of finite-dimensional spaces as follows. By the Riemann-Roch theorem for surfaces with boundary (7), the index of the deformed Cauchy-Riemann operator is

$$\operatorname{Ind}(D_{\tilde{E},\tilde{F}}) = \operatorname{Ind}(D_{E^\rho,F^\rho}) - \dim(F_z) \tag{27}$$

since the Euler characteristic of the glued surface is one less than the resolved surface. The identity (27) implies that the restriction of $\pi_\tau \circ D_{\tilde{E},\tilde{F}}$ to V_τ is an isomorphism onto U_τ.

Let $\tilde{D}_{ij}, i, j = 1, 2$ denote the components of $D_{\tilde{E},\tilde{F}}$ with respect to the splittings (25). The kernel of $D_{\tilde{E},\tilde{F}}$ consists of pairs (ξ_1, ξ_2) such that

$$\xi_1 = -\tilde{D}_{11}^{-1}\tilde{D}_{12}\xi_2, \quad (-\tilde{D}_{21}\tilde{D}_{11}^{-1}\tilde{D}_{12} + \tilde{D}_{22})\xi_2 = 0.$$

Define

$$D^{\mathrm{red}}_{\tilde{E},\tilde{F}} := -\tilde{D}_{21}\tilde{D}_{11}^{-1}\tilde{D}_{12} + \tilde{D}_{22}.$$

We have an identification

$$\ker(D_{\tilde{E},\tilde{F}}) \to \ker(D^{\mathrm{red}}_{\tilde{E},\tilde{F}}), \quad \xi_2 \mapsto (-\tilde{D}_{11}^{-1}\tilde{D}_{12}\xi_2, \xi_2).$$

The image of $D_{\tilde{E},\tilde{F}}$ consists of pairs (η_1, η_2) such that $\eta_2 - D_{21}D_{11}^{-1}\eta_1$ lies in the image of $D^{\mathrm{red}}_{\tilde{E},\tilde{F}}$. The inclusion of $F_z \oplus \mathrm{coker}(D_{E^\rho,F^\rho})$ into $U_\tau \oplus F_z \oplus \mathrm{coker}(D_{E^\rho,F^\rho})$ induces an identification of cokernels of $D_{\tilde{E},\tilde{F}}$ and $D^{\mathrm{red}}_{\tilde{E},\tilde{F}}$. Applying this identification to (26) gives the desired exact sequence.

To compute the limit of the middle operator in the limit of large gluing parameter, note that the component of $D^{\mathrm{red}}_{\tilde{E},\tilde{F}}$ in F_z is given asymptotically by projecting $D_{\tilde{E},\tilde{F}}((1-\beta_2)\xi)$ onto $\beta_4\underline{F}_z$. We have

$$D_{\tilde{E},\tilde{F}}((1-\beta_2)\xi) \to -(\partial_s\beta_2)\xi(\mathrm{d}s + i\mathrm{d}t).$$

Pairing with $f \in \underline{F}_z$ gives the difference of evaluation maps $\xi(z_+) - \xi(z_-)$ paired with f. It follows that the limit is

$$\lim_{\tau\to\infty} D^{\mathrm{red}}_{\tilde{E},\tilde{F}}\xi = (\xi(z_+) - \xi(z_-), 0) = D^{\mathrm{red}}_{E,F}\xi.$$

□

Corollary 2.4.7. (Isomorphism of determinant lines induced by deformations of nodes) *Let $D_{\tilde{E},\tilde{F}}$ be the operator obtained from a Cauchy-Riemann operator $D_{E,F}$ by deforming a node. There is a canonical up to deformation* gluing isomorphism $\det(D_{E,F}) \to \det(D_{\tilde{E},\tilde{F}})$.

Proof. The existence of the exact sequence is equivalent to the existence of isomorphisms

$$\ker(D^{\mathrm{red}}_{\tilde{E},\tilde{F}}) \to \ker(D_{\tilde{E},\tilde{F}}), \quad \mathrm{coker}(D^{\mathrm{red}}_{\tilde{E},\tilde{F}}) \to \mathrm{coker}(D_{\tilde{E},\tilde{F}}). \tag{28}$$

These induce an isomorphism of determinant lines

$$\det(D_{\tilde{E},\tilde{F}}) \to \det(D^{\mathrm{red}}_{\tilde{E},\tilde{F}}). \tag{29}$$

The homotopy of Theorem 2.4.5 induces an isomorphism of determinant lines $\det(D^{\mathrm{red}}_{E,F}) \to \det(D^{\mathrm{red}}_{\tilde{E},\tilde{F}})$. Combining this with (29), (13), and (11) proves the corollary. □

Next we show that the gluing maps of Proposition 2.4.3 and Corollary 2.4.7 satisfy an *associativity* property:

Proposition 2.4.8. (Associativity of gluing) *Let S be a nodal surface with strip-like ends and $\tilde{S}$ the surface obtained by one of the following:*

(a) *deforming two nodes $\underline{w}_0, \underline{w}_1$, or*
(b) *deforming one node $\underline{w}$ and gluing two strip-like ends e_-, e_+, or*
(c) *gluing two pairs of strip-like ends $e_{0,\pm}, e_{1,\pm}$.*

Suppose that $D_{\tilde{E},\tilde{F}}$ is obtained from $D_{E,F}$ by deforming the nodes. Then the resulting gluing isomorphisms $\det(D_{E,F}) \to \det(D_{\tilde{E},\tilde{F}})$ are independent of the order of deformation/gluing.

Proof. We consider only the case of two boundary nodes z, z'; the cases of interior nodes, strip-like ends, and mixed cases are similar but easier. We claim that if δ denotes the deformation of z and δ' the deformation of z' then the diagram

$$\begin{array}{ccc} \det(D_{E,F}) & \longrightarrow & \det(D_{E,F^\delta}) \\ \downarrow & & \downarrow \\ \det(D_{E,F^{\delta'}}) & \longrightarrow & \det(D_{E,F^{\delta,\delta'}}) \end{array} \tag{30}$$

commutes. The proof is a minor modification of e.g. [5, Lemma 3.5]. Simultaneous deformation of the two nodes leads to an exact sequence

$$0 \to \ker(D_{E,F^{\delta,\delta'}}) \to \ker(D_{E,F^{\rho,\rho'}}) \to F_z \oplus F_{z'} \oplus \operatorname{coker}(D_{E,F^{\rho,\rho'}}) \to \operatorname{coker}(D_{E,F^{\delta,\delta'}}) \to 0. \tag{31}$$

Together with the identification of $D_{E,F}$ with the reduced operator in (12), this induces an isomorphism

$$\det(D_{E,F}) \to \det(D_{E,F^{\delta,\delta'}}). \tag{32}$$

We claim that this isomorphism is equal to the isomorphism given by going either way around the square (30). To prove the claim consider the diagram

$$\begin{array}{ccccccc} \ker(D_{E,F^{\delta,\rho'}}) & \xrightarrow{\mathrm{Id}} & \ker(D_{E,F^{\delta,\rho'}}) & \xrightarrow{D^{\mathrm{red}}_{E^\delta,F^\delta}} & F_z \oplus \operatorname{coker}(D_{E,F^{\delta,\rho'}}) & \xrightarrow{\mathrm{Id}} & F_z \oplus \operatorname{coker}(D_{E,F^{\delta,\rho'}}) \\ \uparrow & & \downarrow & & \uparrow & & \downarrow \\ \ker(D_{E,F^{\delta,\delta'}}) & \longrightarrow & \ker(D_{E,F^{\rho,\rho'}}) & \xrightarrow{D^{\mathrm{red}}_{E,F}} & F_z \oplus F_{z'} \oplus \operatorname{coker}(D_{E,F^{\rho,\rho'}}) & \longrightarrow & \operatorname{coker}(D_{E,F^{\delta,\delta'}}) \end{array} .$$

For fixed gluing parameters τ, τ' the diagram commutes up to a small error term which is irrelevant for the purposes of orientations. By approximate commutativity of the diagram the composition of the top and right maps in (30) is equal up to homotopy to (32). A similar argument shows the same for the composition of the two maps on the other side of (30). This completes the proof. □

The existence of the gluing isomorphisms of determinant lines can be phrased in the following more conceptual way, following the discussion in [12]:

Definition 2.4.9. (Decomposed spaces) Let $\mathcal{G}$ be a partially ordered set with partial order $\leq$. Let B be a Hausdorff paracompact space. A *$\mathcal{G}$-decomposition* of B is a locally finite collection of disjoint locally closed subspaces $B_\Gamma, \Gamma \in \mathcal{G}$ each equipped with a smooth manifold structure of constant dimension $\dim(B_\Gamma)$, such that

$$B = \bigcup_{\Gamma \in \mathcal{G}} B_\Gamma$$

and

$$B_\Gamma \cap \overline{B_{\Gamma'}} \neq \emptyset \iff B_\Gamma \subset \overline{B_{\Gamma'}} \iff \Gamma \leq \Gamma'.$$

The *dimension* of a $\mathcal{G}$-decomposed space B is

$$\dim B = \sup_{\Gamma \in \mathcal{G}} \dim(B_\Gamma).$$

The *stratified boundary* $\partial_s B$ resp. *stratified interior* $\mathrm{int}_s\, B$ of a $\mathcal{G}$-decomposed space B is the union of pieces

$$\partial_s B = \bigcup_{\dim(B_\Gamma) < \dim(B)} B_\Gamma, \quad \mathrm{int}_s\, B = \bigcup_{\dim(B_\Gamma) = \dim(B)} B_\Gamma.$$

An *isomorphism* of $\mathcal{G}$-decomposed spaces B_0, B_1 is a homeomorphism $B_0 \to B_1$ that restricts to a diffeomorphism on each piece $B_{0,\Gamma}$.

Example 2.4.10. (Cone construction) Let B is a $\mathcal{G}$-decomposed space. The *cone* on B

$$CB := (B \times [0,\infty)) \,/\, \big((r,0) \sim (r',0), r, r' \in B\big)$$

has a natural $\mathcal{G}$-decomposition with

$$(CB)_\Gamma = C(B_\Gamma), \quad \dim(CB) = \dim(B) + 1.$$

More generally, if B is a $\mathcal{G}$-decomposed space equipped with a locally trivial map π to a manifold A, the *cone bundle* on B is the union of cones on the fibers, that is,

$$C_A B := (B \times [0,\infty)) \,/\, \big((r,0) \sim (r',0), \pi(r) = \pi(r') \in B\big),$$

is again a $\mathcal{G}$-decomposed space with dimension $\dim(C_A B) = \dim(B) + 1$.

Definition 2.4.11. (a) (Stratified spaces) A decomposition $B = \cup_{\Gamma \in \mathcal{G}} B_\Gamma$ of a space B is a *stratification* if the pieces B_Γ fit together in a nice way: Given a point r in a piece B_Γ there exists an open neighborhood U of r in B, an open ball V around r in B_Γ, a stratified space L (the *link* of the stratum) and an isomorphism of decomposed spaces $\phi : V \times CL \to U$ that preserves the decompositions. That is, ϕ restricts to a diffeomorphism $\phi_{\Gamma'}$ from each piece $(V \times CL)_{\Gamma'}$ of $V \times CL$ to a piece $U \cap B_{\Gamma'}$. A *stratified space* is a space equipped with a stratification.

(b) (Families of quilted surface) Let $B = \cup_{\Gamma \in \mathcal{G}} B_\Gamma$ be a stratified space. A *family of quilted surfaces with strip like ends* over B is a stratified space $\mathcal{S} = \cup_{\Gamma \in \mathcal{G}} \mathcal{S}_\Gamma$ equipped with a stratification-preserving map to B such that each $\mathcal{S}_\Gamma \to B_\Gamma$ is a smooth family of quilted surfaces with fixed type, and furthermore local neighborhoods of $\mathcal{S}_\Gamma$ in $\mathcal{S}$ are given by the gluing construction of Definition 2.4.4: there exists
   (i) a neighborhood U_Γ of $\mathcal{S}_\Gamma$,
   (ii) a projection $\pi_\Gamma : U_\Gamma \to B_\Gamma$, and
   (iii) a map $\delta_\Gamma : U_\Gamma \to (\mathbb{R}_{\geq 0})^m \times \mathbb{C}^n$

   such that if $r \in B_\Gamma$ then $\underline{S}_r$ is obtained from gluing $\underline{S}_{\pi_\Gamma(r)}$ with gluing parameters $\delta_\Gamma(r)$.

(c) (Families of bundles) A *family of complex bundles with totally real boundary and seam conditions* is a collection $(\underline{E}, \underline{F}) = (\underline{E}_b, \underline{F}_b)_{b \in B}$ of complex bundles with totally real boundary and seam conditions, such that for each $b \in B$, the nearby bundles are given by the gluing construction of Definition 2.4.4.

In other words, for a family of quilted surfaces with strip-like ends, degeneration as one moves to a boundary stratum is given by neck-stretching.

Proposition 2.4.12. (Orientation double cover of a family with nodal degeneration) *Let $\underline{S}_b, \underline{E}_b, \underline{F}_b, b \in B$ be a family of complex vector bundles with totally real boundary conditions on quilted surfaces with strip-like ends over a stratified space B. Then the collection of determinant lines* $\det(D_{\underline{E},\underline{F},b}), b \in B$ *has the structure of a topological line bundle over B.*

Proof. Proposition 2.4.8 shows that the isomorphisms of Corollary 2.4.7 define local trivializations of

$$\mathrm{Or}(D_{\underline{E},\underline{F}}) := \bigcup_{b \in B} \mathrm{Or}(D_{\underline{E}_b,\underline{F}_b}), \quad \mathrm{Or}(D_{\underline{E}_b,\underline{F}_b}) = \det(D_{\underline{E},\underline{F},b})^\times / \mathbb{R}_{>0}. \tag{33}$$

Since each fiber has two components, the bundle (33) is a double cover of B. The determinant line is the associated line bundle to the double cover and so inherits a topological structure. □

3. Relative non-abelian cohomology

The construction of orientations for pseudoholomorphic maps with Lagrangian boundary conditions depends on the existence of a structure on the Lagrangians called a *relative spin structure* as introduced by Fukaya-Oh-Ohta-Ono [8]. In this section, we give a description of these groups in somewhat greater generality. In the latter case the discussion is equivalent to the one introduced in [8], but avoids triangulations. A more general notion of relative pin structures that does not require orientability of the Lagrangians is developed in Solomon [22].

3.1. Principal bundles and non-abelian cohomology.

Definition 3.1.1. (a) (First non-abelian cohomology) Let G be a Lie group and M a smooth manifold. Let $\mathcal{U} = \{U_i, i \in I\}$ be an open cover of M. For integers $j \geq 0$ let

$$C^j(\mathcal{U}, G) = (g_{i_0,\ldots,i_j} : U_{i_0} \cap \ldots \cap U_{i_j} \to G)_{i_0,\ldots,i_j}$$

be the space of cochains of degree j and ∂ the coboundary operator defined by

$$\partial : \; C^j(\mathcal{U}, G) \to C^{j+1}(\mathcal{U}, G), \; (\partial g)_{i_0,\ldots,i_{j+1}} = \prod_{k=0}^{j+1} g^{(-1)^k}_{i_0,\ldots,\widehat{i_k},\ldots,i_{j+1}}.$$

The groups $C^j(\mathcal{U}, G)$ form a complex in the following sense. Consider the space of *one-cycles*

$$Z^1(\mathcal{U}, G) := \ker(\partial|_{C^1} : C^1(\mathcal{U}, G) \to C^2(\mathcal{U}, G)).$$

Then $C^0(\mathcal{U}, G)$ acts on the left on $Z^1(\mathcal{U}, G)$ by the formula

$$(h, g) \mapsto hg, \quad (hg)_{i_0,i_1} := h_{i_0} g_{i_0,i_1} h_{i_1}^{-1}.$$

The zeroth and first non-abelian cohomology groups are

$$H^0(\mathcal{U}, G) := Z^0(\mathcal{U}, G), \quad H^1(\mathcal{U}, G) := C^0(\mathcal{U}, G) \backslash Z^1(\mathcal{U}, G).$$

Any refinement $\mathcal{V} \to \mathcal{U}$ induces maps $H^1(\mathcal{U}, G) \to H^1(\mathcal{V}, G)$ for $j = 0, 1$. Denote by

$$H^k(M, G) = \varprojlim_{\mathcal{U}} H^k(\mathcal{U}, G), \quad k = 0, 1$$

the limit over refinements. For G abelian, all cohomology groups $H^j(M, G), j = 0, 1, 2, \ldots$ are well-defined in a similar way.

(b) (Long exact sequence) If $A \subset G$ is an abelian subgroup then there is a long exact sequence of pointed sets

$$\ldots H^0(M, G/A) \to H^1(M, A) \to H^1(M, G) \to H^1(M, G/A) \to H^2(M, A). \tag{34}$$

That is, $H^1(M, A)$ acts transitively on the kernel of $H^1(M, G) \to H^1(M, G/A)$, and the set-theoretic kernel of the connecting homomorphism $H^1(M, G/A) \to H^2(M, A)$ is equal to the image of $H^1(M, G)$.

(c) (Characteristic class) The image of a class in $H^1(M, G/A)$ under the connecting homomorphism $c : H^1(M, G/A) \to H^2(M, A)$ in (34) is called the *characteristic class*.

(d) (Chern class) As an example of the previous item, consider the exact sequence

$$1 \to \mathbb{Z} \to \mathbb{R} \to S^1 \to 1$$

where 1 denotes the trivial group. In this case, there exists an isomorphism with the Picard group

$$H^1(M, S^1) \to \mathrm{Pic}(M)$$

of isomorphism classes of line bundles. The characteristic class map $H^1(M, S^1) \to H^2(M, \mathbb{Z})$ is equivalent to the first Chern class

$$c_1 : \mathrm{Pic}(M) \to H^2(M, \mathbb{Z}).$$

(e) (Relative cohomology for a group homomorphism) Let $\mathcal{U}$ be an open cover of M as above, $A \subset G$ an abelian subgroup and $\phi : G \to G/A$ the projection, the last map in the exact sequence

$$1 \to A \to G \to G/A \to 1.$$

Let $z \in Z^1(\mathcal{U}, G/A)$ be a cocycle. A *G-structure* on z is a cocycle $a \in C^1(\mathcal{U}, G)$ with $\phi_*(a) = z$; an isomorphism from a to a' is an element $b \in C^0(\mathcal{U}, A)$. Let $H^1(\mathcal{U}, G, z)$ denote the set of isomorphism classes of Čech G-structures and

$$H^1(M, G, z) = \lim_{\to} H^1(\mathcal{U}, G, Z)$$

the direct limit over open covers. The obstruction to admitting a G-structure is the characteristic class in (34).

(f) (Relative cohomology for a map) Let G be an abelian Lie group and $f : M \to N$ a smooth map of manifolds. Suppose the open cover $\mathcal{U}$ on M is a refinement of the pull-back $f^*\mathcal{V}$ of the open cover $\mathcal{V}$ on N. Let $\psi : \mathcal{U} \to f^*\mathcal{V}$ be a morphism of open covers; that is, for each $U \in \mathcal{U}$ an element $\psi(U) \in \mathcal{V}$ such that $f(U) \subset \psi(U)$. Pull-back gives a morphism of cochain groups

$$\psi^* : C^j(\mathcal{V}, G) \to C^j(\mathcal{U}, G).$$

For non-negative integers j define

$$C^j(\psi, G) := C^j(\mathcal{U}, G) \times C^{j+1}(\mathcal{V}, G), \quad \partial(a,b) = ((\partial a)\cdot(\psi^* b)^{(-1)^j}, \partial b).$$

The space $C^{j-1}(\psi, G)$ acts on the space of cocycles $Z^j(\psi, G)$. Let

$$H^1(\psi, G) := Z^1(\psi, G)\backslash C^0(\psi, G).$$

Let

$$H^1(f, G) = \lim_{\rightarrow} H^1(\psi, G)$$

denote the limit over morphisms of open covers ψ; standard arguments (see e.g. [20, Theorem 2.4.1]) show that $H^1(f, G) = H^1(\psi, G)$ where ψ is any morphism of good covers.

(g) (Relative cohomology for a map of manifolds and a homomorphism of groups) Let $f : M \to N$ be a smooth map of smooth manifolds, and $\mathcal{U}, \mathcal{V}$ open covers. A *morphism of open covers* $\psi : \mathcal{U} \to \mathcal{V}$ is an assignment of an element $\psi(U) \in \mathcal{V}$ for every $U \in \mathcal{U}$ such that $f(U) \subset \psi(U)$. Let $A \subset G$ a closed central subgroup of a Lie group G and $\phi : G \to G/A$ the projection. Let

$$z \in C^1(\mathcal{U}, G/A), \quad \partial z = 0 \in C^2(\mathcal{U}, G/A)$$

be a cocycle. A *Čech relative G-structure* on z is a cocycle

$$(a,b) \in Z^1(\psi, G), \quad \phi_*(a,b) = (z, 0).$$

An isomorphism from (a,b) to (a',b') is an element $h \in C^0(f, A)$ with $h(a,b) = (a',b')$. Denote by $H^1(\psi, G, z)$ the space of isomorphism classes of relative G-structures and

$$H^1(f, G, z) = \lim_{\rightarrow} H^1(\psi, G, z)$$

the inverse limit over morphisms of open covers.

The above constructions in Čech cohomology can be connected to principal bundles as follows.

Definition 3.1.2. (a) (Principal bundles) A principal G-bundle over a smooth manifold M consists of a smooth right G-manifold P together with a projection $\pi : P \to M$ such that G acts freely transitively on the fibers of π and is locally trivial in the following sense: for any $m \in M$, there exists an open neighborhood U of m and a G-equivariant diffeomorphism

$$\tau : \pi^{-1}(U) \to U \times G, \quad \pi_1 \circ \tau = \pi$$

where $\pi_1 : U \times G \to U$ is projection on the first factor. An *isomorphism* of G-bundles P_1, P_2 is a G-equivariant diffeomorphism from P_1 to P_2 that induces the identity on M. Let

$$\mathrm{Prin}(M, G) = \{P \to M\}/\sim$$

denote the set of isomorphism classes of G-bundles over M. Then $\mathrm{Prin}(M, G)$ is canonically in bijection with $H^1(M, G)$ via the map given by gluing:

$$[\psi_{ij} \in C^1(\mathcal{U}, G)] \mapsto \sqcup_{U_i \in \mathcal{U}} (U_i \times G)/(u_i, p) \sim (u_j, \psi_{ij}(u)p).$$

(b) (Relative G-structures) A *relative G-structure* on a G/A-bundle $Q \to M$ trivial on a open cover $\mathcal{U}$ relative to a map $f : M \to N$ is given by a morphism of open covers $\psi : \mathcal{U} \to \mathcal{V}$ and a relative G-structure $(a_1, a_2) \in C^1(\psi, G)$ on a cocycle $z \in C^1(\mathcal{U}, G/A)$ representing Q. The class

$$b(Q) = [a_2] \in H^2(N, A)$$

is the *background class* of the relative G-structure on Q. An *isomorphism* of relative G-structures $(a_1, a_2), (a_1', a_2')$ is a zero cycle

$$w \in C^0(\mathcal{U}, A), \quad w(a_1, a_2) = (a_1', a_2').$$

(c) (Relative Spin-structures) Recall that for $r = 2$ resp. $r > 2$ the special orthogonal group $SO(r)$ has fundamental group $\mathbb{Z}$ resp. $\mathbb{Z}_2$. The spin group $\mathrm{Spin}(r)$ is the canonical double cover of $SO(r)$ and its universal cover for $r > 2$:

$$1 \to \mathbb{Z} \text{ resp. } \mathbb{Z}_2 \to \mathrm{Spin}(r) \to SO(r) \to 1.$$

A *relative spin structure* on an $SO(r)$-bundle $Q \to M$ relative to a morphism of covers ψ is a relative $\mathrm{Spin}(r)$-structure on a cocycle representing Q. For $f : M \to N$ a smooth map, let

$$H^1(f, \mathrm{Spin}(r), E) = \{(a_1, a_2) \in C^1(f, \mathrm{Spin}(r)) \mid (a_1, a_2) \text{ represents } E\}/\sim$$

denote the set of isomorphism classes of relative spin structures on E.

Remark 3.1.3. (Relative spin structures as relative trivializations of the second Stiefel-Whitney class) In concrete terms, a relative spin structure is a lift of the transition maps $\psi_{ij} : U_i \cap U_j \to SO(r)$ of the bundle Q to a collection

$$\hat{\psi}_{ij} : U_i \cap U_j \to \mathrm{Spin}(r)$$

of transition maps with values in $\mathrm{Spin}(r)$ that satisfy the cocycle condition up to the boundary of a $\mathbb{Z}_2$-valued cochain pulled back from a two-chain $\delta_{ijk} \in C^2(N, \mathbb{Z}_2)$ on N:

$$\hat{\psi}_{jk} \hat{\psi}_{ik}^{-1} \hat{\psi}_{ij} = f^* \delta_{ijk}.$$

Since

$$[\hat{\psi}_{jk} \hat{\psi}_{ik}^{-1} \hat{\psi}_{ij}] = w_2(E) \in H^2(M, \mathbb{Z}_2)$$

is the second Stiefel-Whitney class of E, this description shows that there is a natural bijection from $H^1(f, \mathrm{Spin}(r), E)$ to the set of isomorphism classes of trivializations of the image of the second Stiefel-Whitney class $w_2(E)$ in $C^2(f, \mathbb{Z}_2)$.

There is a topology definition of relative G-structures in terms of classifying spaces.

Definition 3.1.4. (a) (Homotopy G-structures) Let $EG \to BG$ be a universal G-bundle for G, and $[M, BG]$ the set of homotopy classes of continuous maps to BG, canonically in bijection with $\mathrm{Prin}(M, G)$. Let $A \subset G$ be an abelian subgroup. Let ϕ be a map from M to $B(G/A)$. A *homotopy G-structure* on ϕ is a lift to BG.

(b) (Homotopy relative G-structures) Let $A \subset G$ be an abelian subgroup and $f : M \to N$ a smooth map. Since A is abelian, we have a natural A-bundle defined by

$$EA \times_A EA \to BA \times BA.$$

The corresponding classifying map

$$m : BA \times BA \to BA, \quad m^* EA \cong (EA \times_A EA)$$

gives BA the structure of an H-space. Let $B^2 A$ denote the classifying space of BA. Consider a G/A-bundle $Q \to M$ with a classifying map $\phi : M \to B(G/A)$. A *homotopy relative G-structure* on ϕ is a homotopy class of a pair

$$\beta : N \to B^2 A, \quad \alpha : M \to f^* \beta^* E(BA)) \times_{BA} BG$$

where α is a section of the BG-bundle associated to the BA-bundle pulled back from β, such that the associated section of the trivial $B(G/A)$-bundle is the given classifying map ϕ for Q.

The space of homotopy relative G-structures is in one-to-one correspondence with Čech relative versions; this seems to be a special case of [2, Theorem 1]. See also Shahbahzi [20] for a discussion of relative gerbes in the abelian case. The following proposition connects the definitions above with that of Fukaya et al [8]:

Proposition 3.1.5. (a) *Suppose that $Q \to M$ is a G/A-bundle and $R \to N$ a G/A-bundle with characteristic class $c(Q) = f^* c(R)$. There is a one-to-one correspondence between equivalence classes of relative G-structures on Q and equivalence classes of $G \times_A G$-structures on $Q \oplus f^* R$.*

(b) *Suppose that $Q \to M$ and $R \to N$ are Euclidean vector bundles over manifolds M, N and $f : M \to N$ is a smooth map with $w_2(Q) = f^* w_2(R)$. There is a one-to-one correspondence between isomorphism classes of relative spin structures on Q and isomorphism classes of spin structures on $Q \oplus f^* R$.*

Proof. For the first statement, suppose that $(a, b) \in C^1(M, G) \times C^2(N, A)$ is a relative G-structure on Q. Let $c \in C^1(N, G/A)$ be a cocycle representing R, mapping to $b \in C^2(N, A)$ under the coboundary map. By definition c has a lift

$$d \in C^1(N, G), \quad \partial d = b + \partial e \text{ for some } e \in C^1(M, A).$$

The cochain

$$(a, f^* d - e) \in C^1(M, G) \times C^1(M, G) \cong C^1(M, G \times G)$$

has boundary

$$\partial(a, f^* d - e) = (-f^* b, f^* b) \in C^2(M, A \times A).$$

The image of $(a, f^* d - e)$ in $C^1(M, G/A \times G/A)$ represents $Q \oplus f^* R$. Conversely, suppose $Q \oplus f^* R$ is equipped with a $G \times_A G$-structure. Any lift of the transition maps is of the form

$$(a, f^* d) \in C^1(M, G \times G), \quad \partial a = -\partial f^* d.$$

Thus the pair $(a, f^* d)$ defines a relative G-structure on Q with $b = f^* d$. This proves the first part of the statement of the Proposition. The second statement is the special case $G = \mathrm{Spin}(r), A = \mathbb{Z}_2$. □

The usual operations of duals, direct sums, and tensor products extend to the relative spin case: In addition, there is also a canonical relative spin structure on the "double" of any oriented vector bundle:

Proposition 3.1.6. (Relative spin structures on direct sums and tensor products)

(a) *Let $f : M \to N$ be a smooth map and $E_1, E_2 \to M$ and oriented Euclidean vector bundles equipped with relative spin structures for the map f. There are canonical relative spin structures on $E_1 \oplus E_2$ and $E_1 \otimes E_2$ for the map f.*
(b) *Let $E \to M$ be an oriented vector bundle. The direct sum $E \oplus E$ has a canonical spin structure.*
(c) *Let $f : M \to N$ be a smooth map, $E \to M$ an oriented vector bundle and $F \to N$ an oriented vector bundle such that $f^*F \cong G \oplus G$ for some oriented vector bundle $G \to N$. There is a bijection between relative spin structures on E for the map f with background class b and relative spin structures on E for the map f with background class $b + w_2(F)$.*

Proof. (a) Let r_1, r_2 denote the ranks of E_1, E_2. The claim on the tensor product and direct sum follows from the existence of the group homomorphisms:

$$\mathrm{Spin}(r_1) \times \mathrm{Spin}(r_2) \to \mathrm{Spin}(r_1 + r_2), \quad \mathrm{Spin}(r_1) \times \mathrm{Spin}(r_2) \to \mathrm{Spin}(r_1 r_2).$$

(b) The claim on the self-sum follows from the fact that, if $r \geq 1$ denotes the rank of the bundle E, then the diagonal homomorphism $SO(r) \to SO(2r)$ induces the trivial map on fundamental groups and so lifts to $\mathrm{Spin}(2r)$. (c) The third item follows by combining the first two: by (b), $f^*F \cong G \oplus G$ has a canonical spin structure. By (a), relative spin structures on E with background class b are in one-to-one correspondence with relative spin structures on E with background class $b + w_2(F)$. □

The relevance of relative spin structures in Floer theory is provided by the following proposition. In particular the proposition implies that relative spin structures for boundaries of surfaces give stable trivializations:

Proposition 3.1.7. (Stable trivializations via relative spin structures) *Suppose that S is a compact, oriented surface with boundary ∂S, and $Q \to \partial S$ is an $SO(r)$-bundle. There is a canonical bijection between the set of isomorphism classes of relative spin structures on Q for the inclusion $\partial S \to S$ and the set of homotopy classes of stable trivializations of Q.*

Proof. We first show that any relative spin structure induces a stable trivialization. Let $f : \partial S \to S$ be the inclusion of the boundary. Since S is two-dimensional, any cohomology class $w \in H^2(S, \mathbb{Z}_2)$ is the second Stiefel-Whitney class of some oriented bundle:

$$\exists R \to S, \quad w = w_2(R).$$

Indeed, the third Postnikov truncation of BSO is the Eilenberg-Maclane space $K(\mathbb{Z}_2, 2)$. From Proposition 3.1.5 (or the homotopy definition) we obtain a bundle $R \to S$ together with a spin structure on $Q \oplus f^*R$. We may assume that ∂S is non-empty, since otherwise the statement is vacuous. Thus S is homotopy equivalent to a bouquet of circles:

$$S \cong S^1 \vee \ldots \vee S^1.$$

Since $\pi_2(S)$ is trivial, the bundle $R \to S$ is trivial. The relative spin structure gives a stable trivialization of Q. If S is a disk, then the trivialization of R (and therefore also the stable trivialization of S) is unique up to homotopy. In general, two stable trivializations differ by a map $S \to SO(r)$ for some r sufficiently large. Since

$$[S, SO(r)] \cong [S, (SO(r))_2] \cong H^1(S, \mathbb{Z}_2)$$

(where $(SO(r))_2$ is the Postnikov truncation) there is no longer a distinguished stable trivialization. However, the image of $H^1(S,\mathbb{Z}_2) \to H^1(\partial S,\mathbb{Z}_2)$ is trivial. This implies that f^*R has a distinguished trivialization. Hence Q has a distinguished stable trivialization. Conversely, any stable trivialization of Q induces a relative spin structure (by taking R to be the trivial bundle). These two constructions are inverses of each other by construction and this gives the bijection. □

The following lemma will be used later to show that quilted Floer cohomology is unaffected, in a certain sense, by "insertion of a diagonal seam", see Proposition 5.4.2 below. We consider the following situation. Let $(S,\partial S)$ be a surface with boundary and let $f:\partial S \to S$ denote the inclusion of the boundary as above, and $Q \to \partial S, R \to S$ vector bundles. Suppose that $Q \oplus f^*R$ is equipped with a spin structure, giving rise to a relative spin structure σ_1 on Q with background class $w_2(R)$ and a stable trivialization of Q. We consider the effect of "shifting" the relative spin structure on Q by adding another bundle whose restriction to the boundary is the complexification of a real bundle.

Lemma 3.1.8. *Let Q,R,σ_1 be as above. Suppose that $U \to S, V \to \partial S$ are bundles equipped with an isomorphism*

$$U|_{\partial S} \to V \oplus V.$$

Let σ_2 be the induced relative spin structure on Q with background class $w_2(R)\oplus w_2(U)$ as in Proposition 3.1.6. Then the stable trivializations of Q induced by σ_1,σ_2 are equivalent resp. opposite if the Maslov index of (U,V) is equal to 0 resp. 2 mod 4.

Proof. The pullback $f^*U = V \oplus V$ is stably trivialized on the boundary ∂S using the triviality of the diagonal map $\pi_1(SO(n)) \to \pi_1(SO(2n))$. On the other hand, the identification of stable trivializations of $Q \oplus f^*R$ and $Q \oplus f^*R \oplus f^*U$ uses the trivialization of U over the disk. The two trivialization differ exactly if the Maslov index of the pair (U,V) is equal to 2 mod 4. □

4. Orientations for families of operators

In this section we define orientations for quilted Cauchy-Riemann operators from an orientation and relative spin structure on the totally real boundary condition, and investigate their behavior under gluing. These results are generalizations of results from Fukaya et al [8], Seidel [19], and Solomon [22]. The constructions of this section are for families of quilted maps over smooth bases. That is, while the determinant line bundle exists for a family of quilts with varying type over a stratified base, our purpose here is to construct trivializations for families of a fixed type. We then compute the gluing signs relating these trivializations for different types.

4.1. **Construction of orientations for surfaces without strip-like ends.** First we construct orientations on the determinant lines arising from Cauchy-Riemann operators on surfaces without strip-like ends with relative spin structures on the boundary conditions:

Proposition 4.1.1. (Orientations via relative spin structures) *Suppose that $S \to B$ is a family of nodal surfaces without strip-like ends, $(E,F) \to B$ is a family of complex vector bundles $E \to S$ with oriented totally real boundary conditions $F \subset E|_{\partial S}$, and $D_{E,F}$ is a* *family of real Cauchy-Riemann operators for* (S, E, F)*. A relative spin structure for the bundle* $F \to \partial S$*, if it exists, defines an orientation*

$$o_{E,F} : B \to \mathrm{Or}(D_{E,F}) = \det(D_{E,F})^{\times}/\mathbb{R}_{>0}$$

for the determinant line bundle $\det(D_{E,F}) \to B$*.*

Here B is a smooth open base, so $\partial S = \bigcup_{b \in B} \partial S_b$ is a bundle over B whose fibers are the boundaries of the fibers of S.

Proof of Proposition 4.1.1. Step 1: Orientations for families of smooth, closed surfaces: Suppose that $S \to B$ and (E, F) are as in the statement of the Proposition and S has empty boundary. Consider a family

$$D_E = (D_{E,b})_{b \in B}$$

of real Cauchy-Riemann operators acting on sections of a family of complex vector bundles $E = (E_b \to S)_{b \in B}$. Since the space of real Cauchy-Riemann operators is an affine space containing the complex linear Cauchy-Riemann operators, there exists a homotopy from

$$(D_{E,b})_{b \in B} \sim (D'_{E,b})_{b \in B}$$

where $D'_{E,b}$ is a family of complex linear operators. The complex structure on the kernels and cokernels of D'_E induce orientations for D'_E. These pull back under the isomorphism of determinant lines $\det(D_E) \to \det(D'_E)$ to orientations of D_E. Any two homotopies are related by a homotopy of homotopies, since the spaces of real Cauchy-Riemann operators and complex-linear Cauchy-Riemann operators are contractible. So the orientation on $\det(D_E)$ is independent of the choice of D'_E and homotopy to D'_E.

Step 2: Orientations for smooth, compact surfaces with boundary: Suppose that the base of the family $B = \{\mathrm{pt}\}$ is a point, S is a smooth, compact surface with boundary, and E, F are as above. The relative spin structure on F gives a homotopy class of stable trivializations of $F \to \partial S$ by Proposition 3.1.7. We first fix a stable trivialization of F and construct an orientation for $D_{E,F}$; later we will show that the orientation depends only on the homotopy class of stable trivializations. The real Cauchy-Riemann operator $D_{E,F}$ acts on sections of $E \to S$ with totally real boundary conditions $F \subset E|_{\partial S}$. We may assume, after adding a trivial bundle, that F is trivialized. The trivialization $F \cong \mathbb{R}^n \times \partial S$ induces a trivialization

$$E|_{\partial S} = F \oplus iF \cong \mathbb{C}^n \times \partial S$$

which extends to a neighborhood $U \subset S$ of ∂S.

Deform the surface to a nodal surface by pinching off a disk for each boundary component, as follows. Choose a tubular neighborhood of the boundary

$$U = \sqcup_i U_i \subset S$$

equal to a disjoint union of annuli

$$U_i \cong [-1, 1] \times S^1, \quad \partial S \cap U_i \cong \{1\} \times S^1.$$

Replacing U_i with complex annuli of increasing radius produces a family of surfaces. The limit is the nodal surface

$$\hat{S} = S / \left(U_i \mapsto (D_i^- \sqcup D_i^+)/(z_i^- \sim z_i^+), i = 1, \ldots, n \right)$$

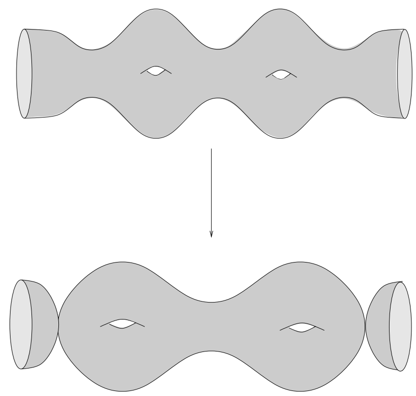

Figure 3. Pinching off a set of disks

obtained by replacing U_i with two disks $D_i^- \sqcup D_i^+$ glued at an interior node

$$\{z_i^-, z_i^+\}, \ z_i^- = 0 \in D_i^-, \ z_i^+ = 0 \in D_i^+.$$

Here D_i^+ is the unit disk with standard complex structure j_{std} and boundary ∂D_i^+ identified with $\{1\}\times S^1 \subset \partial U_i$, whereas D_i^- is the unit disk with complex structure $-j_{\mathrm{std}}$ and boundary ∂D_i^- identified with $\{-1\} \times S^1 \subset \partial U_i$. So the nodal surface $\hat{S}$ has resolution $\hat{S}^\rho = \hat{S}_{\mathrm{c}} \sqcup \hat{S}_{\mathrm{d}}$, consisting of a closed surface and a union of disks

$$\hat{S}_{\mathrm{c}} = (S \setminus U) \cup \sqcup_i D_i^-, \quad \hat{S}_{\mathrm{d}} = \sqcup_i D_i^+.$$

The identifications needed to produce $\hat{S}$ from $\hat{S}^\rho$ are a collection of interior nodes

$$Z = \{\{z_i^-, z_i^+\}\}, \quad z_i^- \in \hat{S}_{\mathrm{c}}, \quad z_i^+ \in \hat{S}_{\mathrm{d}},$$

see Figure 3.

The pinching construction extends to define a complex vector bundle and totally real boundary condition as follows: Let $\hat{E}_{\mathrm{c}} \to \hat{S}_{\mathrm{c}}$ be the complex vector bundle defined by gluing together $E|_{S\setminus U}$ (which is trivialized $\cong \mathbb{C}^n \times \sqcup_i \partial D_i^-$ on the boundary) with the trivial bundle on $\sqcup_i D_i^-$. Consider the trivial bundles

$$\hat{E}_{\mathrm{d}} = \mathbb{C}^n \times \sqcup_i D_i^+, \quad \hat{E}_{\mathrm{d}}|_{\partial \hat{S}_{\mathrm{d}}} \supset \hat{F}^\rho = \mathbb{R}^n \times \sqcup_i \partial D_i^+ \cong F.$$

Then $\hat{E} \to \hat{S}$ is obtained from $\hat{E}^\rho := \hat{E}_{\mathrm{c}} \sqcup \hat{E}_{\mathrm{d}} \to \hat{S}^\rho$ and identification at the nodes Z. Similarly the boundary condition $\hat{F} = \hat{F}_{\mathrm{d}}$ is induced from $\hat{F}^\rho$. Conversely, (S, E, F) is obtained from $(\hat{S}, \hat{E}, \hat{F})$ by gluing at the interior nodes.

We use the canonical identification of determinant lines produced by the homotopy above to produce an orientation on the determinant line of the original surface. By 2.4.7 the pinching of bundles induces an isomorphism of determinant lines

$$\det(D_{\hat{E},\hat{F}}) \to \det(D_{E,F}).$$

Combining with (11), (13), and (14) we obtain an isomorphism

$$\det(D_{E,F}) \to \Lambda^{\max}\Big(\bigoplus_i \hat{E}^\rho_{z_i^+}\Big)^\vee \otimes \det(D_{\hat{E}^\rho,\hat{F}^\rho}). \tag{35}$$

Here the first factor is oriented by the complex structure on $\hat{E}^\rho_{z_i^+}$. The second factor decomposes into

$$\det(D_{\hat{E}^\rho,\hat{F}^\rho}) = \det(D_{\hat{E}_c} \oplus D_{\hat{E}_d,\hat{F}_d}) \cong \det(D_{\hat{E}_c}) \otimes \det(D_{\hat{E}_d,\hat{F}_d}).$$

The operator $D_{\hat{E}_c}$ has an orientation given by the previous step, since $\hat{S}_c$ is smooth and closed. On the other hand, by construction the operator $D_{\hat{E}_d,\hat{F}_d}$ is the direct sum of real Cauchy-Riemann operators on the disk. After a homotopy, the operators on the disks are the standard Cauchy-Riemann operators which are surjective. Their kernel is isomorphic to a sum of fibers via evaluation at points $\underline{s} = (s_i \in \partial D_i^+ \subset \partial S)$ on the boundary:

$$\ker D_{\hat{E}_d,\hat{F}_d} \to \oplus_i F_{z_i}, \quad \underline{\xi} \mapsto \underline{\xi}(\underline{s}).$$

The orientation of the boundary condition F (induced by the trivialization) thus defines an orientation on $D_{\hat{E}_d,\hat{F}_d}$. The orientation on $D_{E,F}$ is induced from the isomorphism (35).

We claim that the orientation is independent of the auxiliary choices: the trivialization of F, the extension of the induced trivialization of E to the neighborhood U, and the choice of coordinates on U. Any two choices of extensions and coordinates on U are homotopic. Any two trivializations of $F \to \partial S$ differ by a map

$$\partial S \to SO(\mathrm{rank}(F)).$$

Hence there are two trivializations up to homotopy for each boundary component if $\mathrm{rank}(F) > 2$, infinitely many if $\mathrm{rank}(F) = 2$, and a unique trivialization if $\mathrm{rank}(F) = 1$. So there are two stable homotopy classes of stable trivializations of F, for any rank. Consider two choices of extensions and coordinates, and a homotopic pair of trivializations

$$\tau_\delta : F \to \partial S \times \mathbb{R}^k$$

of F. The homotopies τ_δ gives continuous families of nodal surfaces and bundles $\hat{S}_\delta, \hat{E}_\delta, \hat{F}_\delta$, Cauchy-Riemann operators $D_{\hat{E}_\delta,\hat{F}_\delta}$, and isomorphisms

$$\det(D_{E,F}) \to \det(D_{\hat{E}_\delta,\hat{F}_\delta}), \quad \delta \in [0,1].$$

The construction fixes an orientation for each $\det(D_{\hat{E}_\delta,\hat{F}_\delta})$ from the orientations for the nodal fibers $(\hat{E}^\rho_\delta)_{z^i_+(\delta)}$, the operators $D_{(\hat{E}_c)_\delta}$ on complex bundles over closed surfaces, and the operators $D_{(\hat{E}_d)_\delta,(\hat{F}_d)_\delta}$ on trivial bundles over disks. Each of these orientations is continuous in families. Hence the orientations on $\det(D_{\hat{E}_\delta,\hat{F}_\delta})$ vary continuously in δ. It follows that the map $\det(D_{\hat{E}_0,\hat{F}_0}) \to \det(D_{\hat{E}_1,\hat{F}_1})$ induced by the homotopy of operators $(D_{\hat{E}_t,\hat{F}_t})_{\delta\in[0,1]}$ preserves the given orientations. The composition of this map with $\det(D_{E,F}) \to \det(D_{\hat{E}_0,\hat{F}_0})$ is homotopic to $\det(D_{E,F}) \to \det(D_{\hat{E}_1,\hat{F}_1})$. Hence the two isomorphisms induce the same orientation on $\det(D_{E,F})$.

Finally we show that trivializations of the boundary condition that are homotopic after stabilization also define the same orientation on the determinant line of the Cauchy-Riemann operator. For $\mathrm{rank}(F) > 2$ there is nothing to show, since the trivializations are homotopic iff they are stably homotopic. Let F_τ be the trivial $\mathbb{R}^k$-bundle over ∂S, and E_τ the trivial $\mathbb{C}^k$-bundle over S. Consider two trivializations

$$\tau_i : F \to \mathbb{R}^k \times \partial S, \quad i \in \{0,1\}$$

of F such that the induced trivializations of

$$\widehat{\tau}_i : F_\tau \oplus F \to \mathbb{R}^{2k} \times \partial S$$

are homotopic. By the previous discussion the trivializations $\widehat{\tau}_i$ define the same orientation $o_{E_\tau \oplus E, F_\tau \oplus F}$ for

$$D_{E_\tau \oplus E, F_\tau \oplus F} := D_{E_\tau, F_\tau} \oplus D_{E,F},$$

where D_{E_τ, F_τ} is the standard Cauchy-Riemann operator. On the other hand, applying the direct sum isomorphism (2) provides an orientation o_{E_τ, F_τ} of

$$\det(D_{E_\tau, F_\tau}) \otimes \det(D_{E,F}) \cong \det(D_{E_\tau \oplus E, F_\tau \oplus F}).$$

The orientation $o_{E,F,i}$ induced by τ_i for $i \in \{0,1\}$ is related to $o_{E_\tau \oplus E, F_\tau \oplus F}$ by a universal sign that only depends on the combinatorics of the surface and the ranks of the bundles. Hence $o_{E,F,0} = o_{E,F,1}$ as claimed.

Step 3: Orientations for families of smooth, compact surfaces with boundary: We now consider the case of families. Let $\det(D_{E,F})_b, b \in B$ be a family of Cauchy-Riemann operators. It suffices to show that the orientations constructed above vary continuously in B. For this it suffices to consider family $S \to B$ of smooth surfaces with B contractible. A trivialization of $F \to \partial S$ induces a trivialization of E near the boundary ∂S:

$$\tau : E|_{\partial S} \cong F \oplus F \to \mathbb{R}^{2k} \times \partial S.$$

Deforming the conformal structure on $S \to B$ as in the previous step produces a family of nodal surfaces $\hat{S} \to B$. The family $\hat{S}$ consists of a family of disks $\hat{S}_\mathrm{d} \to B$, a family of closed surfaces $\hat{S}_\mathrm{c} \to B$ (obtained by gluing a disk bundle to ∂S), and identifications of $\hat{S}_\mathrm{d}$ and $\hat{S}_\mathrm{c}$ at families of interior nodes. This deformation provides an isomorphism of determinant line bundles

$$\det(D_{E,F}) \to \det(D_{\hat{E},\hat{F}}) \to \Lambda^{\max}\Big(\bigoplus_i \hat{E}^\rho_{z_i^+}\Big)^\vee \otimes \det(D_{\hat{E}_\mathrm{c}}) \otimes \det(D_{\hat{E}_\mathrm{d},\hat{F}_\mathrm{d}}).$$

This isomorphism defines the orientation on $\det(D_{E,F})$ by pullback from the right hand side. To see that these orientations vary continuously, note that the orientation on the first factor is induced from the complex structure on $\hat{E}^\rho_{z_i^+} \to B$. On the second factor the orientation is given by the previous construction for families of closed surfaces. The third factor is isomorphic (using a homotopy to the standard Cauchy-Riemann operator on disks) to $\Lambda^{\max}(\oplus_i F_{z_i})$ for a smooth family of boundary points $z_i \subset \partial S$ in each connected component. These fibers of $F \to \partial S$ are oriented by assumption, inducing a continuous orientation on $\det(D_{\hat{E}_\mathrm{d},\hat{F}_\mathrm{d}})$ and hence on $\det(D_{E,F})$.

Step 4: General definition of orientations: Finally, we consider a general family of nodal (but compact) surfaces. Let $S \to B$ be such a family equipped with families of complex vector bundles $E \to S$ and totally real boundary conditions F, and a family of real Cauchy-Riemann operators $D_{E,F}$. By assumption the family of operators $D_{E,F}$ is produced from identifications of families of nodes from a family of real Cauchy-Riemann operators D_{E^ρ,F^ρ} for families of bundles $E^\rho \to S^\rho$ and $F^\rho \to \partial S^\rho$ over the family of smooth resolutions

$S^\rho \to B$. We fix a trivialization of F, that is a trivialization of $F^\rho \to \partial S^\rho$ that is compatible with the identifications at nodes. From (11), (13), and (14) we have a bundle isomorphism

$$\det(D_{E,F}) \to \Lambda^{\max}\Big(\bigoplus_i E^\rho_{z_i^+} \oplus \bigoplus_j F^\rho_{w_j^+}\Big)^\vee \otimes \det(D_{E^\rho,F^\rho}) \tag{36}$$

Here an orientation on D_{E^ρ,F^ρ} is defined by the previous step, the complex fibers of E are naturally oriented, and the fibers of F are oriented by assumption. Hence this isomorphism defines orientations on $D_{E,F}$. □

Remark 4.1.2. (Orientation of the trivial operator) Suppose that S is a disk, (E,F) are trivial and $D_{E,F}$ is a trivial Cauchy-Riemann operator. In this case the constructed orientation on $\det(D_{E,F})$ is isomorphic to the given orientation on $\Lambda^{\max}(F)$, via the identification $\ker(D_{E,F}) \to F_z$ for any point $z \in \partial S$. Indeed, in this case the Maslov index is already zero.

We now investigate the behavior of orientations with respect to basic operations:

Remark 4.1.3. (a) (Conjugates) Let (E,F) a bundle with totally real boundary condition, and suppose that F is equipped with a relative spin structure. Let E^- the complex conjugate of E, and F^- the subspace F considered as a totally real subspace of F. Let S^- denote the surface S with complex structure $\overline{j} = -j$. Given a Cauchy-Riemann operator $D_{E,F}$ let D_{E^-,F^-} denote the same operator on the conjugate spaces (minus complex structures), as Section 2.3. The kernel and cokernel of $D_{E,F}$ are canonically identified with those of D_{E^-,F^-} as real vector spaces, hence $\det(D_{E,F})$ is canonically identified with that of $\det(D_{E^-,F^-})$. However, the complex structures on the kernel and cokernel of $D_{E,F}$ are reversed. It follows that the orientations are related by

$$o_{E^-,F^-} = (-1)^{(\mathrm{Ind}(D_{E,F})-\mathrm{rank}(F))/2} o_{E,F}. \tag{37}$$

(b) (Direct Sums) Let $(E_j,F_j), j = 0,1$ be bundles with real boundary conditions over a closed surface with boundary S, and $(E,F) = (E_0,F_0)\oplus(E_1,F_1)$. The isomorphisms

$$\ker D_{E_0,F_0} \oplus \ker D_{E_1,F_1} \to \ker D_{E,F}, \quad \mathrm{coker}\, D_{E_0,F_0} \oplus \mathrm{coker}\, D_{E_1,F_1} \to \mathrm{coker}\, D_{E,F}$$

induce an isomorphism

$$p : \det(D_{E_0,F_0}) \otimes \det(D_{E_1,F_1}) \to \det(D_{E,F}). \tag{38}$$

By definition the isomorphism (2) is continuous in families. For each $j = 0,1$ the orientation o_{E_j,F_j} is defined via an isomorphism

$$\det(D_{E_j,F_j}) \to \det(D_{E'_j}) \otimes \Lambda^{\max}(E'_{j,z})^{-1} \otimes \Lambda^{\max}(F_{j,w})$$

where z is the node in the deformed surface and w a point in the boundary of the deformed surface. Since the operators $D_{E'_j}$ have complex linear kernel and cokernel, their indices are even dimensional. Similarly the fiber at the node has even dimension. It follows that the $\Lambda^{\max}(F_{j,w})$ factor commutes with the other factors, and

$$o_{E,F} = p(o_{E_0,F_0} \otimes o_{E_1,F_1})$$

is the image of $o_{E_0,F_0} \otimes o_{E_1,F_1}$ under (38):

(c) (Disjoint Unions) Let (E_j, F_j) denoted bundles with totally real boundary condition over surfaces S_j for $j = 0, 1$. Then

$$(E, F) = (E_0, F_0) \sqcup (E_1, F_1)$$

is a bundle with totally real boundary condition over $S = S_0 \cup S_1$. We take $o_{E,F}$ to be the image of $o_{E_0,F_0} \otimes o_{E_1,F_1}$ under the canonical isomorphism, by a special case of the previous paragraph.

4.2. **Construction of orientations for surfaces with strip-like ends.** As in the case of Morse theory, in order to construct orientations we must make auxiliary choices. In our case, these auxiliary choices involve a once-punctured disk S_1 with a complex structure such that a neighborhood of the puncture corresponds to an incoming strip-like end. We identify its boundary $\partial S_1 \cong \mathbb{R}$, preserving the orientation.

Definition 4.2.1. (a) (End Datum) An *end datum* is a tuple $(E, F_-, F_+, \mathcal{H})$ consisting of

(i) a finite-dimensional complex vector space E,
(ii) a pair (F_-, F_+) of transverse, oriented, totally real subspaces of half-dimension, equipped with spin structures, and
(iii) $\mathcal{H}$ a normal form for a Cauchy-Riemann operator on the strip as in (8).

(b) (Orientation for an end datum) An *orientation* for an end datum $(E, F_-, F_+, \mathcal{H})$ consists of

(i) a smooth path

$$\Gamma : \mathbb{R} \to \mathrm{Real}(E), \quad \Gamma(\pm\infty) = F_\pm$$

of totally real subspaces connecting $F_\pm$. We view Γ as a totally real boundary condition $\Gamma \subset E \times \partial S_1$ for the trivial bundle $E \times S_1$;
(ii) a real Cauchy-Riemann operator

$$D_\Gamma : \Omega^0(E, \Gamma) \to \Omega^{0,1}(E)$$

on S_1 for sections with values in the trivial bundle E and boundary values in Γ, with asymptotic limit $\lim_{s\to\infty} \epsilon^* D_\Gamma$ given by $\mathcal{H}$ in the sense of (8);
(iii) an orientation for D_Γ;
(iv) a spin structure on Γ, extending the given spin structures on F_-, F_+.

Remark 4.2.2. (a) (Conjugates) Let $(E, F_-, F_+, \mathcal{H})$ be an end datum equipped with an orientation. The *dual* end datum $(E^-, F_+^-, F_-^-, \mathcal{H}^-)$ has a canonical orientation given by the same path on ∂S_1 with the complex structure (hence direction on the boundary) of S_1 reversed, the same Cauchy-Riemann operator D_Γ, (now complex linear with respect to the reversed complex structures on the domain and codomain), the given orientation on D_Γ, and the given spin structure on Γ.

(b) (Direct Sums) Let $(E_j, F_{j,0}, F_{j,1}, \mathcal{H}_j)$ be end data equipped with orientations for $j \in \{0, 1\}$. The direct sum

$$E = E_0 \oplus E_1, \quad F_k = F_{0,k} \oplus F_{1,k}, \quad \mathcal{H} = \mathcal{H}_0 \oplus \mathcal{H}_1$$

has an orientation given by the direct sum of the paths and Cauchy-Riemann operators, the orientation given by the direct sum isomorphism (2), and the direct sum spin structure.

Definition 4.2.3. (End orientations) Let S be a surface with strip-like ends and $\overline{S}$ the surface obtained by adding the points at infinity:

$$\overline{S} = S \cup \{z_e, e \in \mathcal{E}(S)\}.$$

Let $E \to S$ be a complex vector bundle and $F \subset E|_{\partial S}$ totally real boundary conditions. For each end $e \in \mathcal{E}(S)$, the corresponding point at infinity is denoted $z_e \in \overline{S}$, and the two real boundary conditions meeting it are denoted F_e, F_{e-1}. By assumption, these are constant transverse subspaces, so that $F_{e,0} \oplus F_{e,1} = E_e$ over the strip-like end. We suppose we have chosen relative spin structures on the fibers $F_{e,0}, F_{e,1}$ at infinity, and there are asymptotic limits $\mathcal{H} = (\mathcal{H}_e, e \in \mathcal{E}(S))$. Let S be a surface with strip-like ends S equipped with a bundle with totally real boundary condition (E, F) and a spin structure $\mathcal{H}$ on F. An *orientation* for this datum is an orientation for each end

$$(\Gamma_e, D_e, o_e, \mathrm{Spin}(\Gamma_e)), \quad e \in \mathcal{E}(S).$$

Orientations for the ends and a relative spin structure suffice to induce orientations on the determinant line of a family of Cauchy-Riemann operators:

Proposition 4.2.4. (Orientations for determinant line bundles via relative spin structures and end orientations) *Let $S \to B$ be a family of nodal surfaces with strip-like ends, and $E \to B$ a family of complex vector bundles with totally real boundary conditions $F \to B$. A choice of a relative spin structure on F, if it exists, and orientations for the ends of $(S, E, F, \mathcal{H})$ induce an orientation of the determinant line bundle* $\det(D_{E,F}) \to B$.

Proof. First consider the case that the boundary of S is connected. A deformation of S is obtained by "bubbling off" disks with strip-like ends for each strip-like end, see Figure 4 below, using the path of totally real subspaces specified by the end datum. Using the orientations for the ends and the behavior of determinant lines under deformation, this reduces the construction of orientations to the case without strip-like ends. Namely, on each strip-like end e consider the deformation $F_{e,\pm,\delta}$ of the boundary conditions $F_{e,\pm}$ on a neighborhood of infinity to the boundary condition formed by concatenating the restriction of Γ with Γ^{-1}:

$$F_{e,\pm,0} = F_{e,\pm}, \quad F_{e,\pm,1} = \Gamma \# \Gamma^{-1}.$$

The sub-bundle $\Gamma \# \Gamma^{-1}$ has a canonical deformation to the boundary condition with constant value $\Gamma(\infty)$. The resulting boundary value problem is obtained by deformation of the nodes of a nodal surface S with vector bundles

$$(E, F) = \coprod_e (E_e, F_{e,0} \sqcup F_{e,1}) \# (E^c, F^c)$$

obtained by gluing together the problems $(E_e, F_{e,0} \sqcup F_{e,1})$ and a problem (E^c, F^c) on a closed (possibly nodal) surface with bundles obtained by gluing $(D_e, E_e, F_{e,0}, F_{e,1})$ onto (S, E, F) for each end e. The procedure is illustrated in Figure 4. The nodal surface $\hat{S}$ has a canonical order of patches given by taking the ordering of the additional patches and the boundary nodes so that the original component $\overline{S}$ is ordered first, and the nodes $\underline{w}_e, e \in \mathcal{E}(S)$ are ordered in the ordering of the strip-like ends $e \in \mathcal{E}(S)$. Let $(\hat{E}, \hat{F})$ denote the vector bundles on the nodal surface. The equation (17) gives an isomorphism of determinant lines

$$\det(D) \to \det(\hat{D}). \tag{39}$$

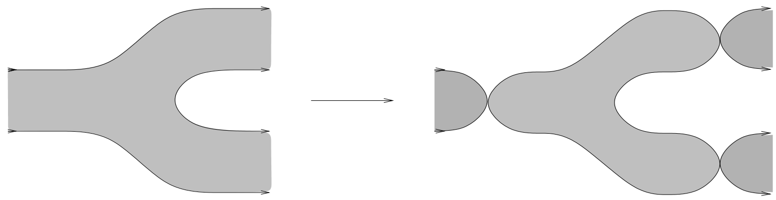

FIGURE 4. Bubbling off the strip-like ends

The equation (14) gives an identification
(40)

$$\det(\hat{D}) \to \bigotimes_{e\in\mathcal{E}_-} \big(\det(D_e^-) \otimes \Lambda^{\max}(\Gamma_e(0)^\vee)\big) \otimes \det(D^c) \otimes \left(\bigotimes_{e\in\mathcal{E}_+} \Lambda^{\max}(\Gamma_e(0)^\vee)\right) \otimes \left(\bigotimes_{e\in\mathcal{E}_+} \det(D_e^+)\right)$$

where $\Gamma_e(0)$ is the fiber given by evaluating the corresponding path Γ_e at 0, and the order of the two products over $\mathcal{E}_-$ is *reversed*. This choice of order means that when gluing, we obtain a product over ends of an incoming end, and outgoing end, and a determinant line of a dualized fiber

$$\det(D_e^+) \otimes \det(D_e^-) \otimes \Lambda^{\max}(\Gamma_e(0)^\vee).$$

Each of these is canonically trivial by the construction of end orientations, leaving a product of determinant lines of dualized fibers $\Lambda^{\max}(\Gamma_e(0)^\vee)$ and the determinant lines $\det(D^c)$ for the surfaces without strip-like ends. The relative spin structure on F and the bundles Γ_e define a relative spin structure on F^c. This construction gives an orientation on the corresponding index by Proposition 4.1.1.

In the case that S is disconnected, we define the orientation on S as a product of expressions in the right hand side of (40) for each component S_i. Since in our applications, all but at most one of these expressions $\det(D_{E_i,F_i})$ corresponds to an even Fredholm index, so the determinant lines $\det(D_{E_i,F_i})$ commute and the particular details of the ordering are irrelevant. □

We investigate the behavior of the constructed orientations under various operations. The behavior of orientations under duals and direct sum is the same as that in the case of no strip-like ends.

Remark 4.2.5. (a) (Conjugates) Suppose that S is equipped with a bundle with totally real boundary conditions (E, F), a spin structure $\mathcal{H}$ on F, and orientations for the ends. Let S^-, E^-, F^- denote the complex-conjugate surface and bundles, and suppose these have been equipped with the orientations given by the dual construction in Remark 2.2.2. Let

$$D_{E^-,F^-} : \Omega^0(E^-, F^-) \to \Omega^{0,1}(E^-)$$

be the dual Cauchy-Riemann operator as in 4.1.3 (a). The determinant lines $D_{E,F}, D_{E^-,F^-}$ are oriented by (40) using gluing on the orientations for the ends and bubbling off the boundary components on disks with Maslov index zero. Complex conjugation acts on the resulting products (40) of determinant lines

$$\bigotimes_{e\in\mathcal{E}_-}\big(\det(D_e^-)\otimes\Lambda^{\max}(\Gamma_e(0)^\vee)\big)\otimes\det(D^c)\otimes\left(\bigotimes_{e\in\mathcal{E}_+}\Lambda^{\max}(\Gamma_e(0)^\vee)\right)\otimes\left(\bigotimes_{e\in\mathcal{E}_+}\det(D_e^+)\right)$$

preserving the orientation on the determinant lines of the disks and the orientations on the ends. However it reverses the complex structure on the bundles (E^c, F^c) on the closed surfaces and interior nodes. Thus the total sign change is

$$\det(D_{E^-,F^-})\cong\det(D_{E,F})^{(-1)^{(\mathrm{Ind}(D^c)+\#\pi_0(\partial S)\operatorname{rank}_{\mathbb{C}}(E))/2}}.$$

(b) (Disjoint union) Suppose that $S = S_1 \cup S_2$, and S_j has $d_j^\pm$ incoming resp. outgoing ends for $j = 1, 2$. Consider the identification

$$\det(D_{E_0,F_0})\otimes\det(D_{E_1,F_1})\to\det(D_{E,F}).$$

The difference between the orientations is given by

$$(-1)^{\operatorname{rank}(F)(\#\pi_0(\partial S_2)+d_2^+)(\sum_{e\in\mathcal{E}_{-,1}}(\operatorname{rank}(E)/2+\mathrm{Ind}(D_e)))}$$

from the reordering of the determinant lines of the ends.

(c) (Re-ordering components or ends)

(i) Suppose that S' is a nodal surface obtained by re-ordering a boundary node:

$$S' = S/((w_+, w_-)\mapsto(w_-, w_+)).$$

Let $D'_{E,F}$ be the Cauchy-Riemann operator obtained from $D_{E,F}$. The isomorphism $\det(D_{E,F})\to\det(D'_{E,F})$ of determinant lines induced by the isomorphism of kernel and cokernel acts on orientations by $(-1)^{\operatorname{rank}(F)}$.

(ii) Suppose that S' is a nodal surface obtained by transposing two patches S_i, S_j of S. The isomorphism $\det(D_{E,F})\to\det(D'_{E,F})$ of determinant lines induced by the isomorphism of kernel and cokernel acts on orientations by $(-1)^{\mathrm{Ind}(D_{E_i,F_i})\,\mathrm{Ind}(D_{E_j,F_j})}$.

(iii) Suppose that S' is a nodal surface obtained by re-ordering the boundary components (resp. boundary nodes) by a permutation σ of the set of nodes

$$\sigma:\{w_1,\dots,w_m\}\to\{w_1,\dots,w_m\}.$$

The isomorphism $\det(D_{E,F})\to\det(D'_{E,F})$ of determinant lines induced by the isomorphism of kernel and cokernel acts on orientations by $\det(\sigma)^{\operatorname{rank}(F)}$.

(iv) Suppose that S' is a nodal surface obtained by transposing a pair e, e' of consecutive strip-like ends. The isomorphism $\det(D_{E,F})\to\det(D'_{E,F})$ of determinant lines induced by the isomorphism of kernel and cokernel acts on orientations by $(-1)^{\mathrm{Ind}(D_e)\,\mathrm{Ind}(D_{e'})}$.

These follows from the behavior of determinant lines under permutations (63), the behavior of the isomorphism with the trivial determinant (5), and the definition of the orientation on nodal surfaces (14).

4.3. **Effect of gluing on orientations.** We have already introduced in Section 2.3 three types of gluing for Cauchy-Riemann operators: gluing along strip-like ends, gluing at an interior node, and gluing at a boundary node. Let S be a nodal surface with strip-like ends, and $\tilde{S}$ a nodal surface obtained by either deforming away a boundary node, deforming away an interior node, or gluing two strip-like ends. Let E be a complex vector bundle with totally real boundary condition F, and $\tilde{E}, \tilde{F}$ the vector bundles on $\tilde{F}$ obtained by gluing. Similarly let $D_{E,F}$ be a real Cauchy-Riemann operator with non-degenerate limits that are equal along the strip-like ends $e_{\pm}$, and $D_{\tilde{E},\tilde{F}}$ an operator obtained by gluing the ends e_+, e_-.

Definition 4.3.1. (Compatibility of end orientations) Let S be a surface with strip-like ends $\mathcal{E} = \mathcal{E}_- \cup \mathcal{E}_+$ and $o_e, e \in \mathcal{E}(S)$ a set of orientations on the ends $D_{e^\pm}$. Let $D_{e^\pm}$ denote Cauchy-Riemann operators on the caps $S_{e^\pm}$ added to the outgoing and incoming ends in (39). Gluing together the caps $S_{e^\pm}$ produces a surface

$$\tilde{S}_e = S_{e^-} \# S_{e_+}$$

diffeomorphic to the disk with zero Maslov index. By the previous constructions the Cauchy-Riemann operator $\tilde{D}_e$ on $\tilde{S}_e$ obtained by gluing from D_e is equipped with an orientation. We say that the orientations $o_e, e \in \mathcal{E}_\pm$ are *compatible* if the gluing isomorphism

$$\det(D_{e^-}) \otimes \det(D_{e^+}) \to \det(\tilde{D}_e) \tag{41}$$

is orientation preserving.

Compatibly chosen orientations are compatible with the basic operations on Cauchy-Riemann operators:

Remark 4.3.2. (a) (Conjugates) Suppose that E, F is a pair of bundles for a surface with strip like ends S, and E^-, F^- denote the conjugate bundles over the conjugate surface S^-. Suppose that a set of orientations $o_e, e \in \mathcal{E}(S)$ for the operators $D_{e^\pm}$ have been chosen compatibly. Then the isomorphisms $\det(D_{e^\pm}) \to \det(D^-_{e^\pm})$ induce a collection of orientations

$$o_e^- \in \det(D^-_{e^\pm})$$

for E^-, F^-. Since complex conjugation induces an isomorphism of orientations for disks with Maslov index zero, the orientations o_e^- are also compatible.

(b) (Direct Sums) Suppose that E_j, F_j are bundles over a surface with strip-like ends S. Let $o_{e,j}$ be a set of orientations for $D_{e^\pm,j}$ the operators for E_j, F_j at end $e \in \mathcal{E}(S)$. Let E, F denote the direct sum bundles and $D_{e^\pm}$ the direct sum Cauchy-Riemann operators. Let $o_{e^\pm}$ denote the orientations induced by the isomorphism of determinant lines

$$\det(D_{e^\pm,0}) \otimes \det(D_{e^\pm,1}) \to \det(D_{e^\pm}).$$

The gluing isomorphism $\det(D_{e^-}) \otimes \det(D_{e^+}) \to \det(\tilde{D})$ is orientation preserving hence the orientations o_e are compatible.

In the following, in the case of gluing boundary nodes or strip-like ends we assume that the boundary components joined by the gluing are adjacent in ordering; then we give the boundary components of $\overline{S}$ the ordering obtained by inserting the new boundary component(s) in place of the old in the ordered sequence.

Theorem 4.3.3. (Behavior of orientations under gluing) *Suppose that a surface with strip-like ends $\overline{S}$ is obtained from S by gluing. The isomorphism of determinant lines*

$$\mathcal{G}_{E,F} : \det(D_{\overline{E},\overline{F}}) \to \det(D_{E,F})$$

from (17) *has the following signs*

$$\mathrm{Or}(\mathcal{G}_{E,F}) : \mathrm{Or}(\det(D_{\overline{E},\overline{F}})) \to \mathrm{Or}(\det(D_{E,F}))$$

in the respective cases below with respect to the constructed orientations $o_{\overline{E},\overline{F}}, o_{E,F}$.

(a) (Gluing at interior nodes)

$$(\mathrm{Or}(\mathcal{G}_{E,F}))(o_{\overline{E},\overline{F}}) = (o_{E,F}).$$

(b) (Gluing at a boundary node for a nodal surface with a single node (w_+, w_-) joining two distinct boundary components adjacent in ordering)

$$(\mathrm{Or}(\mathcal{G}_{E,F}))(o_{\overline{E},\overline{F}}) = (o_{E,F})(\pm 1)^{\mathrm{rank}(F)},$$

with positive sign if and only if the ordering of w_-, w_+ agrees with the ordering of the boundary components for the pre-glued surface; that is, the component $(\partial S)_-$ containing w_- is ordered before $(\partial S)_+$ if and only if w_- is ordered before w_+, and both boundary components $(\partial S)_\pm$ are ordered before the node $w_\pm$.

(c) (Gluing at a boundary node for a nodal surface with a single node (w_+, w_-) joining a single boundary component)

$$(\mathrm{Or}(\mathcal{G}_{E,F}))(o_{\overline{E},\overline{F}}) = (o_{E,F})(\pm 1)^{\mathrm{rank}(F)},$$

with positive sign if and only if the ordering of the boundary components of the glued surface has the boundary component corresponding to the segment from w_- to w_+ ordered first;

(d) (Gluing of strip-like ends of distinct components S_-, S_+ such that $S_\pm$ have connected boundary, the end e_+ is the last outgoing end of S_- and the end e_- is the first incoming end of S_+, and the ordering of the ends on the glued surface is induced by the ordering of ends on S_-, S_+.)

$$(\mathrm{Or}(\mathcal{G}_{E,F}))(o_{\overline{E},\overline{F}}) = (o_{E,F})(\pm 1)^{\mathrm{rank}(F)} \heartsuit \diamondsuit$$

with positive sign in (± 1) if and only if the ordering of e_-, e_+ is (e_-, e_+), with

$$\heartsuit := (-1)^{(\sum_{e \in \mathcal{E}_-(S_+) - \{e_-\}} \mathrm{rank}(F) - \mathrm{Ind}(D_e))(\sum_{f \in \mathcal{E}_+(S_-) - \{e_+\}} \mathrm{rank}(F) - \mathrm{Ind}(D_f))}$$

times

$$\diamondsuit := (-1)^{(\sum_{f \in \mathcal{E}_+(S_+)} \mathrm{rank}(F))(\sum_{f \in \mathcal{E}_+(S_-) - \{e_+\}} \mathrm{rank}(F) - \mathrm{Ind}(D_f))};$$

In particular, for one outgoing end or one incoming end and ordering (e_-, e_+), the gluing sign is positive.

Proof. Case (a), Interior Gluing: In the case of interior gluing we deduce preservation of orientations from complex linearity. Let S^ρ denote the resolution of S, and E^ρ, F^ρ the corresponding vector bundles. First we assume that S^ρ has empty boundary. Consider a deformation

$$D_{E^\rho,F^\rho,\delta} : \Omega^0(E,F) \to \Omega^{0,1}(E,F), \quad D_{E^\rho,F^\rho,0} = D_{E^\rho,F^\rho}, \quad D_{E^\rho,F^\rho,1} \text{ complex linear}$$

of D_{E^ρ,F^ρ} to a complex-linear operator. The gluing isomorphism 2.4.7 induces an identification of determinant lines for each bundle in the homotopy. Since the identification of determinant lines for the complex-linear operators is complex linear, the identification of determinant lines is orientation-preserving, for each bundle in the homotopy.

Next suppose the boundary is non-empty. A deformation of (E^ρ, F^ρ) to the connect sum of a problem on a closed surface $\hat{S}_\mathrm{c}$, glued to a trivial problem on a union of disks $\hat{S}_\mathrm{d}$, induces a corresponding deformation for the glued problem $(\tilde{E}, \tilde{F})$. Compatibility of orientations for gluing closed surfaces implies that the gluing map is orientation preserving.

Cases (b,c), Boundary Gluing: We reduce to the case of gluing disks of index zero by the following argument. Suppose that (w^+, w^-) is a boundary node of S, and $\tilde{S}, \tilde{E}, \tilde{F}$ a surface and bundles obtained by deforming the node. Consider the diagram of indices shown in Figure 5; for self-gluing of a disk, see also Figure 6. In the diagram ϕ_1, ϕ_2 are the gluing maps for the determinant lines for $\hat{S}_\mathrm{c} \cup \hat{S}_\mathrm{d}$ to those of S and $\hat{S}_\mathrm{c} \cup \hat{S}_\mathrm{d}$ to S, and are orientation preserving by definition. The surface $S_\times$ at bottom left is obtained as follows. First, glue at the boundary. Second, degenerate the circles used for the degeneration of S. The gluing map ϕ_3 for $S_{\times,+} \cup S_{\times,-}$ to $S_\times$ is orientation preserving by definition. The map δ represents gluing of a collection of disks equipped with trivialized boundary condition, while β, ψ_2 represent gluing at an interior node. The corresponding isomorphism of determinant lines are orientation preserving by the previous section. Both the lower square and the upper left triangle in the diagram commute by associativity of gluing Proposition 2.4.8. Therefore, the map ψ_1 representing gluing of determinant lines from $\tilde{S}$ to S, induces the same sign on orientations as δ.

We next determine the sign for boundary gluing of disks of index zero, Suppose that S is obtained from a pair S^ρ of disks by joining them at a boundary node $w^\pm$. After deformation we may assume that the Cauchy-Riemann operator D_{E^ρ,F^ρ} is the trivial operator. Then the kernel is isomorphic to $F_{w^\pm} \oplus F_{w^\mp}$ (via the two evaluation maps on the boundary) and the cokernel vanishes. The reduced operator of (12) is

$$D^{\mathrm{red}}_{E,F} : F_{w^\pm} \oplus F_{w^\mp} \to F_w, \quad (f_\pm, f_\mp) \mapsto f_+ - f_-. \tag{42}$$

The ordering of the factors is determined by the ordering of the boundary components of S. By (5) and (29) the induced map $\det(D^{\mathrm{red}}_{E,F}) \to \det(D_{\tilde{E},\tilde{F}})$ changes the defined orientations by a sign

$$\epsilon(S, \tilde{S}, \mathrm{rank}(F)) = (\pm 1)^{\mathrm{rank}(F)}$$

depending on whether the ordering of the pair $w^\pm$ agrees with the ordering of the boundary components of S^ρ. See Example 2.3.10.

Next we consider the case of a single disk joined to itself by a boundary node and the Cauchy-Riemann operator is the trivial one. Thus the boundary component is split into two, as in Figure 6. On the normalization S^ρ of the nodal disk S the kernel $\ker(D_{E^\rho,F^\rho})$ is isomorphic to $F_{w^\pm}$ via evaluation at a boundary point and trivial cokernel. The reduced operator of (12) is

$$D^{\mathrm{red}}_{E,F} : F_{w^\pm} \to F_{w^\pm}, \quad f \mapsto 0. \tag{43}$$

The kernel is isomorphic to $F_{w^\pm}$ and the cokernel is isomorphic to $F_{w^\pm}$. The deformed surface $\tilde{S}$ is an annulus, equipped with trivial bundles $\tilde{E}, \tilde{F}$. The orientation for $D_{\tilde{E},\tilde{F}}$ is

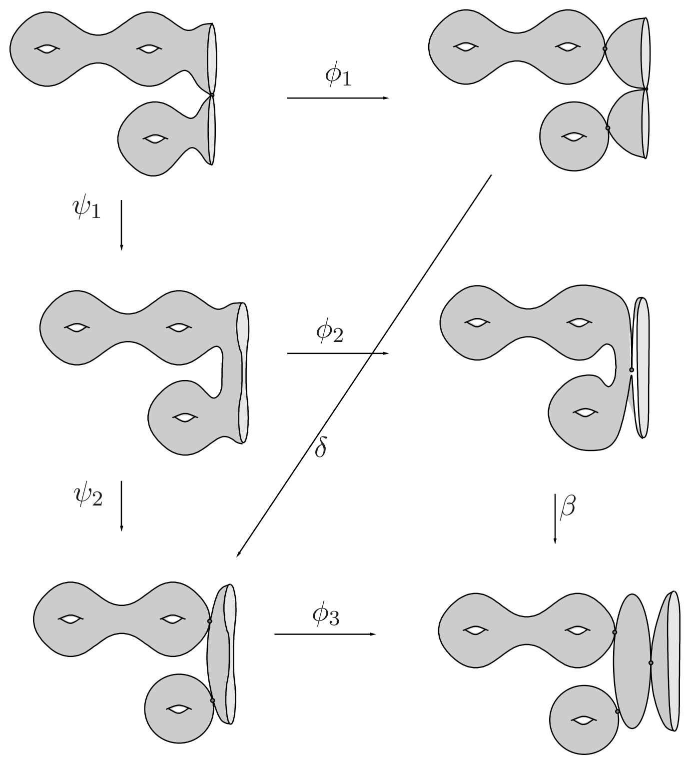

FIGURE 5. Gluing at a boundary point

induced from pinching off a pair of disks so that S_1 is obtained by joining two disks and a sphere at interior points. A choice of ordering of boundary components on $\tilde{S}$ induces an ordering of the nodes z_-, z_+ of S_1. On the normalization S_1^ρ the reduced operator can be identified with

$$D^{\mathrm{red}}_{E_1,F_1} : F_{z^-} \oplus E_{z^\pm} \oplus F_{z^+} \to E_{z^-} \oplus E_{z^+}, \quad (f_-, e, f_+) \mapsto (f_- - e, f_+ - e). \tag{44}$$

The kernel is isomorphic to $F_{z^\pm}$, via evaluation at any boundary point. On the other hand, the cokernel is isomorphic to $E_{z^\pm}/F_{z^\pm} = iF_{z^\pm}$, via projection onto the second factor of the codomain.

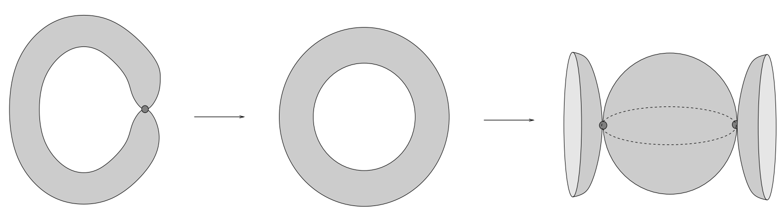

FIGURE 6. Gluing a disk to itself

We compare the orientations coming from the two degenerations of the annulus above. Let $S_0 = S, S_1$ be the nodal surfaces obtained by stretching the two different directions.

We compare the identifications of the kernel and cokernels

$$\ker(D_{E_0,F_0}) \cong \ker(D_{E_1,F_1}), \quad \mathrm{coker}(D_{E_0,F_0}) \cong \mathrm{coker}(D_{E_1,F_1}).$$

In the first case, the reduced operator in (43) is the trivial operator on the space of sections with values in F. The kernel and cokernel are

$$\ker(D_{E_0,F_0}) \cong \mathrm{coker}(D_{E_0,F_0}) \cong F_{w^\pm}.$$

via isomorphisms given by evaluation at a boundary point resp. evaluation at a boundary point on the strip-like neck, see Figure 7. For the surface S_1 the reduced operator in (44) has kernel and cokernel

$$\ker(D^{\mathrm{red}}_{E_1,F_1}) \cong F_{w^\pm} \quad \mathrm{coker}(D^{\mathrm{red}}_{E_1,F_1}) \cong iF_{w^\pm}. \tag{45}$$

The isomorphisms in (45) are given by evaluation at a point on one of the cylindrical necks for S_1. By construction, the bundle $\tilde{E}$ is trivial. Choose a homotopy between the two

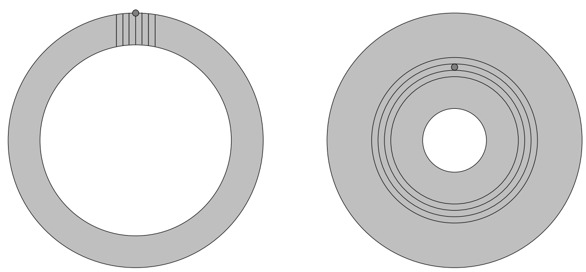

Figure 7. Two kinds of neck

conformal structures on the annulus:

$$j_t \in \mathcal{J}(S), t \in [0,1], \quad (S, j_\delta) \cong S^\delta, \quad (S, j_{1-\delta}) \cong S^\delta_1.$$

Taking the trivial bundle (E_t, F_t) over the homotopy gives a family of trivial operators $D_{E_t,F_t})$ each with kernel and cokernel isomorphic to $F_{w^\pm}$. We can also deform the evaluation maps to all lie on the boundary of S, without changing the induced orientations. It remains to compare the orientations of

$$F_{w^\pm} \cong \mathrm{coker}(D_{E_0,F_0}) \cong \mathrm{coker}(D_{E_1,F_1}) \cong iF_{w^\pm}$$

given by (18). In each case, $F_{w^\pm}$ is identified with the cokernel via wedge product with a one-form supported on the neck. For the surface S_0, the local coordinates on the neck region depend on the ordering of the pair $w^\pm$. On the strip-like neck region on the left in Figure 7, choose horizontal coordinate s and vertical coordinate t. The local complex coordinate is $s + it$ if w_- is numbered first, and $-s - it$ if w_+ is numbered second. On the other hand, the coordinates on the cylindrical neck on the right in Figure 7 are (on the intersection of the two necks) $t - is$. The identifications in (43), (44) are related by multiplication by $\pm i$ if $w_\mp$ is ordered first. It follows that the gluing map on determinant lines $\det(D_{E_0,F_0}) \to \det(D_{E_1,F_1})$ acts by the sign $(\mp 1)^{\mathrm{rank}(F)}$, if $w_\mp$ is ordered first.

Cases (d), Gluing of strip-like ends: First, we consider the case of a disconnected surface $S = S_- \cup S_+$ with a single pair of strip-like ends, and $E \to S$ a complex vector bundle over

S equipped with totally real boundary conditions F. Let e_-, e_+ be ends and $(E_{e_+}, F_{e_+}) \to (E_{e_-}, F_{e_-})$ an identification of the corresponding fibers. Let $\tilde{S}$ denote the surface obtained by gluing S together along the ends, and $(\tilde{E}, \tilde{F})$ the elliptic boundary value problem obtained by gluing E, F. See for example Figure 3 of [26]. Adding in the points at infinity gives surfaces without strip-like ends

$$\overline{S} = S \cup \bigcup_e s_e, \quad \overline{\tilde{S}} = \tilde{S} \cup \bigcup_{e \neq e^\pm} s_e.$$

Choose an ordering of the boundary components of $\overline{\tilde{S}}$. The strip-like ends of S inherit an ordering from the ordering of the ends of S. We claim that the isomorphism of determinant lines from S to $\tilde{S}$ has the same sign as the isomorphisms of determinant lines from $\overline{S}$ to $\overline{\tilde{S}}$.

To compute the gluing sign, we compare the deformation used in the definition of the orientations on the surface with strip like ends to the isomorphism of determinant lines induced by gluing strip-like ends. Consider the diagram of indices shown in Figure 8. The top left picture represents $\det(D_{E,F})$. The maps are defined as follows:

(ϕ_1, ϕ_6) The map ϕ_1 represents the isomorphism of determinant lines induced by deforming the boundary conditions $F_\pm$ as in the proof of Proposition 4.2.4. This deformation results in a boundary problem that is obtained from the nodal surface on the upper right of Figure 8 by gluing. The map ϕ_6 represents a similar isomorphism of determinant lines induced by a deformation to split form.

(ϕ_2, ϕ_7) The map ϕ_2 represents the isomorphism of determinant lines induced by gluing at the two boundary nodes in the upper right surface. Similarly the map ϕ_7 represents isomorphism induced by gluing along two nodes $w_1 = (w_{1,+}, w_{1,-})$ and $w_2 = (w_{2,+}, w_{2,-})$ on the boundary.

(ϕ_3, ϕ_4, ϕ_5) The maps ϕ_3, ϕ_4, ϕ_5 are gluing isomorphisms for the gluing of strip-like ends e_+, e_-.

In order to compute the gluing sign we establish commutativity of the diagram. The first square in the diagram commutes because deformation commutes with gluing. The second commutes by associativity of gluing for determinant lines in Proposition 2.4.8:

$$\phi_4\phi_1 = \phi_6\phi_3, \quad \phi_5\phi_2 = \phi_7\phi_4.$$

By definition the composition of ϕ_1, ϕ_2 is orientation preserving. ϕ_5 is orientation preserving by construction, and ϕ_6 is orientation preserving since it is induced by a deformation. Hence ϕ_3 has the same sign as ϕ_7. By definition ϕ_7 is the composition of gluing isomorphisms for resolution of the first node $w_1 = (w_{1,+}, w_{1,-})$, then second boundary node $w_2 = (w_{2,+}, w_{2,-})$. Choose the ordering of the boundary components so that the disk boundary $\partial S_{\text{disk}} \subset \partial S$ is ordered first and boundary nodes so that the node on the disk $w_{k,+} \in \partial S_{\text{disk}}$ is ordered first. Then the first gluing isomorphism is orientation preserving, and the resulting surface is $\overline{S}$. Hence ϕ_3 has the same sign as the isomorphism of determinants induced by the second gluing operation. By part (b), this has the sign claimed in the statement of part (c).

The additional signs in the case of multiple ends arise from permuting the remaining outgoing ends and nodes of the S_-, with the incoming ends and nodes of S_+, and

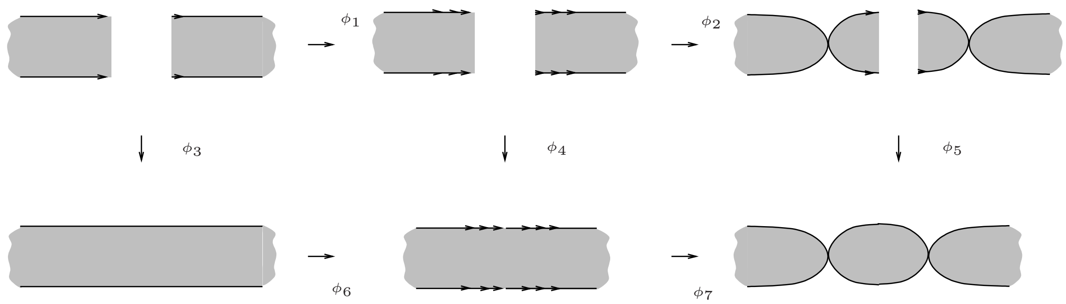

FIGURE 8. Orientations for gluing strip-like ends

also the outgoing ends of S_- with the nodes associated to the outgoing ends of S_+, per the convention (40). The sign resulting from permuting the remaining incoming ends of S_+ past the caps and nodes for $e_\pm$ and determinant line on the closed surface $\overline{S}_-$ is $(-1)^{3\,\mathrm{rank}(F)+\mathrm{Ind}(D_{e_-})+\mathrm{Ind}(D_{e_+})} = (-1)^{4\,\mathrm{rank}(F)} = 1$. This completes the proof. □

Finally we check the signs for various special cases needed later.

Remark 4.3.4. (a) (Annulus) Let $A = [0,1] \times S^1$ with trivial bundle E and two transverse, constant boundary conditions $F = (F_0, F_1)$. By definition the orientation on the determinant line $\det(D_{E,F})$ in induced from the isomorphism with the sphere with two bubbled-off disks,

$$\det(D_{E,F}) \to \Lambda^{\max}((E^2)^\vee) \otimes \Lambda^{\max}(F_0 \oplus E \oplus F_1),$$

see Figure 6. The operator $D_{E,F}$ has trivial kernel and cokernel, and so $\det(D_{E,F}) = \mathbb{R}$. The induced orientation on the determinant line $\det(D_{E,F}) = \mathbb{R}$ is the standard one if and only if the isomorphism $F_0 \oplus F_1 \to E$ is orientation preserving. The quilted case is similar and left to the reader.

(b) (Strip) Let S denote the strip $[0,1] \times \mathbb{R}$. Let e^0 denote the incoming end, and e^1 the outgoing end of S. Let E be the trivial bundle and $F^j, j = 0,1$ denote constant, transverse boundary conditions. Choose a path

$$\Gamma = (\Gamma^t)_{t\in[-\infty,\infty]}, \quad \Gamma^{-\infty} = F^0, \quad \Gamma^\infty = F^1$$

from F^0 to F^1, and compatible orientations on the resulting operators on the once-punctured disks $D_{e^j}, j = 0,1$. Compatibility means that the gluing map for strip-like ends

$$\det(D_{e^0}) \otimes \det(D_{e^1}) \to \det(D_{\mathrm{disk}})$$

to the operator D_{disk} with homotopically trivial boundary conditions is orientation preserving. The orientation on $\det(D)$ is defined so that the gluing isomorphism

$$\det(D_{e^0}) \otimes \Lambda^{\max}((\Gamma(0))^\vee) \otimes \det(D_{\mathrm{disk}}) \otimes \Lambda^{\max}((\Gamma(0))^\vee) \otimes \det(D_{e^1}) \to \det(D_{E,F}) \tag{46}$$

is orientation preserving. Permuting the factor $\det(D_{e^1})$ to the beginning produces a factor of

$$(-1)^{\mathrm{Ind}(D_{e^1})(\mathrm{rank}(F)+\mathrm{Ind}(D_{e^0}))} = (-1)^{\mathrm{Ind}(D_{e^1})}.$$

Using compatibility of orientations and gluing to the annulus gives an orientation preserving isomorphism

$$\det(D_{e^1}) \otimes \det(D_{e^0}) \otimes \Lambda^{\max}((\Gamma(0))^\vee) \otimes \det(D_{\text{disk}}) \otimes \Lambda^{\max}((\Gamma(0))^\vee) \\ \to \det(D_{\text{disk}}) \otimes \Lambda^{\max}((\Gamma(0))^\vee) \otimes \det(D_{\text{disk}}) \otimes \Lambda^{\max}((\Gamma(0))^\vee). \tag{47}$$

By gluing this tensor product is isomorphic to the determinant line for the Cauchy-Riemann operator on the annulus. By the previous item, the gluing isomorphism has orientation $(-1)^{\operatorname{Ind}(D_{e^1})}$. The arguments above give a total sign of $+1$ for the isomorphism of the determinant line of the operator on the strip with the trivial line.

(c) (Cup and Cap) Suppose that $S_\cup, S_\cap$ are the disks with two outgoing resp. incoming ends of Example 4.1.7 of [26]. Suppose these are equipped with constant vector bundles $E_\cup, E_\cap$ and constant real boundary conditions $(F^0_\cup, F^1_\cup) = (F^1_\cap, F^0_\cap)$. Let $D_\cup, D_\cap$ be the corresponding Cauchy-Riemann operators. For the two ends of $S_\cup$ we can choose the paths $\Gamma_\cup, \Gamma_\cap$ on the two ends $e^0_\cup, e^1_\cup$ to be related by time-reversal. Choose the orientations on $D_{e^0_\cup}, D_{e^1_\cup}$ so that the gluing map

$$\det(D_{\text{disk}}) \otimes \Lambda^{\max}(\Gamma_\cup(0)^\vee)^2 \otimes \det(D_{e^1_\cup}) \otimes \det(D_{e^0_\cup}) \to \det(D_\cup) \tag{48}$$

induces the standard orientation on $\det(D_\cup) = \mathbb{R}$. Note that we have isomorphisms

$$\det(D_{e^1_\cap}) \cong \det(D_{e^0_\cup}), \quad \det(D_{e^0_\cap}) \cong \det(D_{e^1_\cup}).$$

The compatibility condition for $D_{e^0_\cup}, D_{e^0_\cap}$ differs from that for $D_{e^1_\cup}, D_{e^1_\cap}$ by a sign

$$(-1)^{\operatorname{Ind}(D_{e^1_\cup})\operatorname{Ind}(D_{e^0_\cup})} = (-1)^{(\operatorname{rank}(F) - \operatorname{Ind}(D_{e^0_\cup}))\operatorname{Ind}(D_{e^0_\cup})}$$

given by the transposition of factors. Changing the order of factors in (48) to that in (46) produces a sign $(-1)^{\operatorname{Ind}(D_{e^0_\cup})^2}$ by (63). The choice of sign orientation for $\det(D_{e^1_\cup})$ that makes the orientations positive is this sign times the induced orientation from $\det(D_{e^0_\cap})$. The orientation on $D_\cap$ is defined by the gluing isomorphism

$$\det(D_{e^0_\cap}) \otimes \Lambda^{\max}(\Gamma_\cup(0)^\vee) \otimes \det(D_{e^1_\cap}) \otimes \Lambda^{\max}(\Gamma_\cup(0)^\vee) \otimes \det(D_{\text{disk}}) \to \det(D_\cap). \tag{49}$$

Changing the order of factors to match that of (47) produces by (63) a sign

$$(-1)^{\operatorname{Ind}(D_{e^0_\cap})^2 + \operatorname{Ind}(D_{e^1_\cap})\operatorname{rank}(F)}.$$

Hence the orientation on $\det(D_\cap) = \mathbb{R}$ is related to the standard one by a sign

$$(-1)^{\operatorname{rank}(F)\operatorname{Ind}(D_{e^0_\cup}) + 2\operatorname{Ind}(D_{e^0_\cup})^2 + (\operatorname{rank}(F) - \operatorname{Ind}(D_{e^0_\cup}))\operatorname{Ind}(D_{e^0_\cup})} = (-1)^{\operatorname{Ind}(D_{e^0_\cup})}$$

as in Example 4.1.7 of [26].

4.4. **Orientations for quilted surfaces.** In this section we define orientations for Cauchy-Riemann operators on quilted surfaces. Let $\underline{S} \to B$ be a family of quilted surfaces possibly with strip-like ends and $D_{\underline{E},\underline{F}}$ be a family of Cauchy-Riemann operators for $(\underline{E}, \underline{F})$, and suppose that the $\underline{F}$ is equipped with a collection of relative spin structures. We define an orientation for $D_{\underline{E},\underline{F}}$ by deforming the seam conditions to split form.

Proposition 4.4.1. (Orientations for quilted Cauchy-Riemann operators via relative spin structures) *Let $\underline{S}, \underline{E}, \underline{F}$ be a family of quilted surfaces with bundles and boundary/seam conditions over a base B. A relative spin structure on $\underline{F}$ and orientations for the ends of each component together induce an orientation on the determinant line bundle* $\det(D_{\underline{E},\underline{F}}) \to B$.

Proof. The proof uses a deformation of the seam conditions to split form, after which we may apply the construction in the unquilted case. For simplicity, we assume that the Hamiltonian perturbations on the strip-like ends vanish. We may assume that the ranks of the bundles are at least two, after stabilizing by adding trivial bundles. Note that the map of Grassmannians of totally real subspaces

$$\frac{U(n_1)}{SO(n_1)} \times \frac{U(n_2)}{SO(n_2)} \to \frac{U(n_1+n_2)}{SO(n_1+n_2)}$$

induces a surjection of first and second homotopy groups, by the long exact sequence of homotopy groups and the isomorphisms

$$\pi_1(SO(n_1)) \to \pi_1(SO(n_1+n_2)), n_1 > 1; \quad \pi_1(U(n_1)) \to \pi_1(U(n_1+n_2)).$$

It follows that there exists a deformation of the seam conditions on the strip-like ends to split form in the product of Grassmannians $(U(\sum n_j)/SO(\sum n_j))^2$ (where $n_1, \ldots, n_k$ are the dimensions of the boundary and seam conditions) such that the path has Maslov index zero:

$$[\underline{F}_{e,\pm,\delta}] \in (U(\Sigma_j n_j)/SO(\Sigma_j n_j)), \quad \delta \in [0,1], \quad [\underline{F}_{e,\pm,1}] \in \prod_j U(n_j)/SO(n_j).$$

Any such path has a deformation with no crossing points, that is, so that every set of conditions in the deformation are transversal:

$$\underline{F}_{e,-,\delta} \cap \underline{F}_{e,+,\delta} = \{0\}, \quad \forall \delta \in [0,1].$$

This deformation produces a family of Fredholm operators, and hence an isomorphism of the determinant line of the original problem with the problem with split form on each strip-like end. The given path can be deformed into a path in partially split form, that is, a path into

$$U(n_1)/SO(n_1) \times U(n_1+n_2)/SO(n_1+n_2) \times \ldots \times U(n_k)/SO(n_k)$$

uniquely up to homotopy of homotopies. Finally we homotope the seam conditions $\underline{F}$ via a homotopy

$$\underline{F}_\delta \subset \underline{E}|_{\partial \underline{S}}, \quad \underline{F}_0 = \underline{F}, \quad \underline{F}_1 = \underline{F}^{\text{split}}$$

to a set of boundary and seam conditions $\underline{F}^{\text{split}}$ of split form over the entire surface.

Having completed the deformation to split form, we now reduce to the unquilted case. The index problem on $(\underline{E}, \underline{F}^{\text{split}})$ splits into a sum of problems on the various components:

$$D_{\underline{E},\underline{F}^{\text{split}}} \cong \bigoplus_{p \in \mathcal{P}} D_{E_p, F_p^{\text{split}}}$$

splits into a sum of problems on the various patches. The constructions in the unquilted case give orientations on the various determinant lines

$$\det(D_{E_p, F_p^{\text{split}}}), \quad p \in \mathcal{P}.$$

These are then pulled back under the deformations to an orientation on the determinant line on the original family of operators $\det(D_{\underline{E},\underline{F}})$. The resulting orientation on $\det(D_{\underline{E},\underline{F}})$ is independent of the choice of deformation $\underline{F}_\delta$ to split form, since any two deformations to split form are homotopic. □

Remark 4.4.2. (a) (Dependence on ordering of components) Recall that for a disconnected unquilted surface S with boundary value problem E, F, the orientation constructed on a Cauchy-Riemann operator $D_{E,F}$ depends on an ordering of the components of S. In particular for a quilted surface, the orientation depends on an ordering of the patches

$$\mathcal{P} = \{p_1, \ldots, p_l\}, \quad l = |\mathcal{P}|.$$

However, if $\underline{S}$ is connected and $D_{\underline{E},\underline{F}}$ has index zero resp. one then the orientation on $D_{\underline{E},\underline{F}}$ is independent of the ordering of the connected components of the patches $\underline{S}$. Indeed, since the orientation constructed is independent of the choice of deformation to split form, we may deform $\underline{F}$ to boundary bundles of split form such that the index is zero on each resp. all but one patch of $\underline{S}$. Then the determinant lines for all connected components commute, see Remark 4.2.5 (c).

(b) (Orientations for the constant bundle on a quilted surface) Suppose that $\underline{S}$ is a quilted strip and $\underline{E}, \underline{F}$ are trivial, and $D_{\underline{E},\underline{F}}$ is the trivial Cauchy-Riemann operator. Then the kernel and cokernel of $D_{\underline{E},\underline{F}}$ are trivial, hence $\det(D_{\underline{E},\underline{F}})$ are trivial. We claim that the orientation on $D_{\underline{E},\underline{F}}$ constructed by deformation to seam conditions of split form is also trivial. Indeed, by the proof of Proposition 4.4.1, the seam conditions can be deformed to split form through a path of seam conditions

$$\underline{F}_\delta \subset \underline{E}, \quad \delta \in [0, 1]$$

that are still transversal at each end $e \in \mathcal{E}(\underline{S})$, for all $\delta \in [0, 1]$. Then the determinant line is isomorphic to the determinant lines on the patches:

$$\det(D_{\underline{E},\underline{F}}) \cong \bigotimes_{p \in \mathcal{P}} \det(D_{E_p, F_p^1}).$$

The orientation on the determinant lines $\det(D_{E_p,F_p^1})$ for each patch is trivial by (46). This proves the claim.

(c) (Effect of gluing on orientations) In the quilted case, there are four types of gluing to consider: gluing at the interior, gluing on the true boundary, gluing at the seams, and gluing along strip-like ends. Suppose that $D_{\underline{E},\underline{F}}$ has index zero or one and suppose that $D_{\underline{\tilde{E}},\underline{\tilde{F}}}$ is obtained by gluing of one of these types. We claim that the gluing sign in the isomorphism

$$\det(D_{\underline{E},\underline{F}}) \to \det(D_{\underline{\tilde{E}},\underline{\tilde{F}}})$$

produced by Corollary 2.4.7 is the product of gluing signs for the unquilted connected components. Indeed, in this case we can find a deformation of $\underline{F}$ to split form $\underline{F}^{\mathrm{split}}$ so that the index of $D_{\underline{E},\underline{F}^{\mathrm{split}}}$ is one on at most one of the unquilted connected components:

$$\#\{p \in \mathcal{P} \mid \operatorname{Ind}(D_{E_p, F_p^{\mathrm{split}}}) = 1\} \leq 1.$$

The gluing operations then reduce to the corresponding gluing operations on disconnected unquilted surfaces, after deformation of the boundary conditions to split form. The determinant lines corresponding to the various unquilted operators commute, by the index assumption. Permuting the connected components to be glued adjacent in the ordering

$$\mathcal{P} = \{\ldots, p_-, p_+, \ldots\}$$

and applying the gluing operation for the unquilted case results in a collection of operators that again have at most one with index 1, and permuting the connected components into the desired ordering does not change the gluing sign. In particular, in the case that $\underline{S}_-, \underline{S}_+$ are obtained by thickening the boundary of an unquilted surface, and $\underline{S}_-$ has a single outgoing end, this convention leads to a positive sign in the gluing rule. This argument gives the associativity relation in the generalized Fukaya category, see Section 5.3 below.

4.5. **Inserting a diagonal for Cauchy-Riemann operators.** In this section we explore the effect of adding an additional seam with seam condition given by a diagonal.

Definition 4.5.1. (a) (Adding a seam to a quilted surface) Let $\underline{S}$ be a quilted surface, S_p a patch, and $I \subset S_p$ an embedded one-manifold. The *surface obtained by adding a seam* is the surface $\underline{S}^\vartriangle$ dividing the patch $S_p = S_p' \cup_I S_p''$ into two patches S_p', S_p'' joined by the seam σ with image I:

$$\underline{S}^\vartriangle = \underline{S}/(S_p \mapsto S_p', S_p'').$$

The ordering of the connected components of the patches $\underline{S}^\vartriangle$ is such that S_p'' follows S_p' immediately (or vice-versa), and the new boundary component of S_p' (resp. S_p'') is last (resp. first) in the ordering of boundary components. See Figure 9 below.

(b) (Adding a diagonal seam condition) Let $(\underline{E}, \underline{F})$ be an collection of bundles with totally real seam and boundary conditions on $\underline{S}$, and suppose that $\underline{S}^\vartriangle$ is obtained by adding a seam. The pair

$$\underline{E}^\vartriangle := (\underline{E}/(E_p \mapsto (E_p, E_p)), \quad F^\vartriangle = (\underline{F}, \Delta)$$

obtained by *adding a diagonal seam condition* is the pair obtained from $\underline{E}, \underline{F}$ by replacing E_p with two copies and assigning to the new seam the diagonal sub-bundle Δ of $E_p \oplus E_p$.

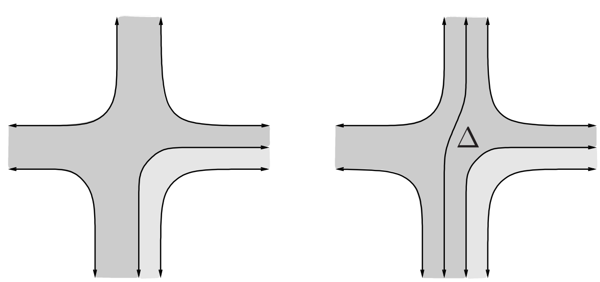

FIGURE 9. Inserting a seam in a quilted surface

Remark 4.5.2. (Identification of determinant lines obtained by adding a seam) Suppose that $\underline{F}$ is equipped with a relative spin structure. Let $\underline{D}$ be a Cauchy-Riemann operator for $(\underline{E}, \underline{F})$, $\underline{S}^\vartriangle, \underline{E}^\vartriangle, \underline{F}^\vartriangle$ are obtained by adding a seam with diagonal seam condition, and $\underline{D}^\vartriangle$ is the Cauchy-Riemann operator for $(\underline{E}^\vartriangle, \underline{F}^\vartriangle)$ obtained from $\underline{D}$. There is a canonical identification of kernels and cokernels

$$\ker(\underline{D}) \to \ker(\underline{D}^\vartriangle), \quad \operatorname{coker}(\underline{D}) \to \operatorname{coker}(\underline{D}^\vartriangle)$$

given by patching together the restrictions to the two components obtained by the division. Hence we have an isomorphism of determinant lines

$$\det(\underline{D}) \to \det(\underline{D}^\vartriangle).$$

Definition 4.5.3. (Relative spin structures for bundles obtained by inserting seams) Let $\underline{S}$ be a quilted surface and $\underline{S}^\vartriangle$ the surface obtained by inserting a seam into a patch S_p. Let $\underline{E}, \underline{F}$ be bundles with boundary/seam conditions on $\underline{S}$.

(a) (The inserted seam is separating). Suppose that the inserted seam σ divides $\underline{S}^\vartriangle$ into quilted surfaces $\underline{S}^{\mathrm{ps}\,\Delta} = \underline{S}_+ \cup \underline{S}_-$. Let $\underline{S}$ be connected. The collection $\underline{F}^\vartriangle$ has a canonical relative spin structure, given a choice of component of the complement of σ. Indeed, the diagonal Δ_p is isomorphic to E_p, via projection on the second factor, hence has a canonical relative spin structure as in 3.1.6. The background classes

$$b(F_\sigma^\vartriangle) = b(F_\sigma) + (w_2(E_{p-(\sigma)}), w_2(E_{p+(\sigma)})), \quad F_\sigma \subset \underline{S}_\pm$$

for the relative spin structure on the components F_σ of $\underline{F}^\vartriangle$ corresponding to seams in $\underline{S}_\pm$ differ from those of $\underline{F}$ by adding $w_2(E_p)$ to all the background classes for components on one side:

$$b_p \mapsto b_p + w_2(E_p), S_p \subset \underline{S}_\pm$$

where $\underline{S}_\pm$ is either $\underline{S}_+$ or $\underline{S}_-$, one side of the new seam σ.

(b) (The inserted seam is not separating) The same construction assigns to $\underline{F}^\vartriangle$ a canonical relative spin structure after adding *two* new seams, separating $\underline{S}$ into two components. The patches in one of the components have shifted background classes.

We wish to show that in each of these cases the isomorphism in 4.5.2 preserves orientations. We begin with the following simple case:

Proposition 4.5.4. (Preservation of orientations for insertion of a separating circle) *Suppose that $\underline{S}^\vartriangle$ is a quilted surface obtained by inserting a seam σ into a quilted surface $\underline{S}$. Suppose that σ is separating and diffeomorphic to a circle, that is, does not meet any strip-like ends. Let $(\underline{E}, \underline{F})$ be a family of bundles with totally real seam and boundary conditions, and $(\underline{E}^\vartriangle, \underline{F}^\vartriangle)$ the family for $\underline{S}^\vartriangle$ obtained by inserting a diagonal. Equip $\underline{F}^\vartriangle$ with with either of the canonical relative spin structures defined in Definition 4.5.3. The isomorphism of 4.5.2 maps the orientation $o_{\underline{E},\underline{F}}$ given by the relative spin structure on $\underline{E}, \underline{F}$ to the orientation $o_{\underline{E}^\vartriangle,\underline{F}^\vartriangle}$ determined by either of the relative spin structures on $\underline{E}^\vartriangle, \underline{F}^\vartriangle$.*

Proof. We deal first with the case of a single unquilted two-sphere. That is, suppose that $\underline{S}$ has a single component S isomorphic to the two-sphere with standard complex structure, $E \to S$ is trivial, and D is the standard Cauchy-Riemann operator. The orientation for $\underline{D}^\vartriangle$ is defined by deforming Δ to a condition of split form as in Proposition 4.4.1. Let

$F_t^\triangle \to \sigma, t \in [0,1]$ denote the family of seam conditions in the deformation. If $F_t^\triangle$ to be constant along the seam (that is, a trivial bundle for each $t \in [0,1]$) then the corresponding family $\underline{D}_t^\triangle$ of Cauchy-Riemann operators is surjective, with kernel isomorphic to any fiber of $F_t^\triangle$ by evaluation at a point $z \in S$ on the seam:

$$\ker(\underline{D}_t^\triangle) \cong (F_t^\triangle)_z, \quad \xi \mapsto \xi(z).$$

Hence the orientation on $\underline{D_0}^\triangle$ is induced by evaluation at a point on the seam, and the orientation on the fibers of Δ. On the other hand, the orientation on $\det(D)$ is induced by the complex structure on E. The proposition follows since the projection of Δ on either factor is orientation preserving.

We reduce the general case to case of a single two-sphere by deforming the surface to a surface with nodes, so that an unquilted two-sphere is created by the deformation. Suppose that S_p is the component of $\underline{S}$ containing σ. Choose a trivialization of E_p in a neighborhood of σ. Let $\sigma_\pm$ be small translates of the seam σ to either side. Contracting the lines $\sigma_\pm$ to nodes one obtains a nodal surface

$$\underline{S}_\delta = (\underline{S}_{-,\delta} \sqcup \underline{S}_{0,\delta} \sqcup \underline{S}_{+,\delta})/\sim$$

consisting of quilted surfaces $\underline{S}_{-,\delta}, \underline{S}_{0,\delta}, \underline{S}_{+,\delta}$, with $\underline{S}_{0,\delta}$ a sphere. Applying the same construction to $\underline{S}^\triangle$ yields a surface

$$\underline{S}_\delta^\triangle = (\underline{S}_{-,\delta} \sqcup \underline{S}_{0,\delta}^\triangle \sqcup \underline{S}_{+,\delta})/\sim$$

with $\underline{S}_{0,\delta}^\triangle$ a quilted sphere. The deformation is illustrated in Figure 10. By gluing for quilted

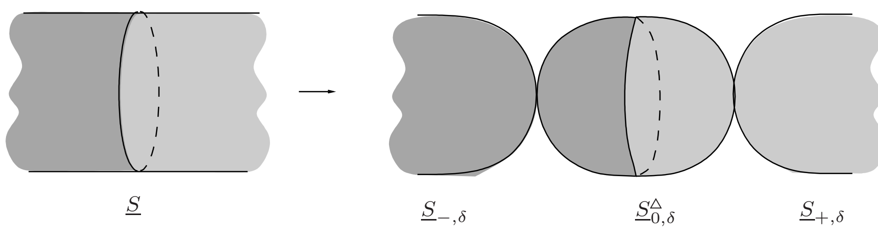

FIGURE 10. Pinching off a seam

surfaces, the isomorphisms of determinant lines induced by gluing

$$\det(\underline{D}) \to \det(\underline{D}_\delta), \quad \det(\underline{D}^\triangle) \to \det(\underline{D}_\delta^\triangle)$$

are orientation preserving. By the previous paragraph, the gluing isomorphism $\det(D_{0,\delta}) \to \det(D_{0,\delta}^\triangle)$ is orientation preserving. Since the isomorphisms of determinant lines induced by gluing commute with the isomorphisms induced by deformation, this proves the Proposition. □

We prove a similar result when the added seam labelled with the diagonal meets the strip-like ends.

Definition 4.5.5. (Ordering of components and boundary components of a surface with an inserted seam) Suppose that $\underline{S}^\triangle$ is obtained from $\underline{S}$ by inserting a new seam connecting two ends in a patch S_j, as in Figure 9.

(a) The ordering of the components of $\underline{S}$ induces an ordering of the components of $\underline{S}^{\vartriangle}$, by replacing the index of the old components with those of the new component and ordering the component $S_{j,-}$ before $S_{j,+}$.
(b) An ordering of the ends of the components of $\underline{S}$ induces an ordering of the ends of $\underline{S}^{\vartriangle}$, since these are in bijection.
(c) The ordering of boundary components of $\underline{S}$ induces an ordering of the boundary components for each component of $\underline{S}^{\vartriangle}$: For each old component, the ordering is the same, while for the new components $S^{\vartriangle}_{j,\pm}$ one puts the new seam last (resp. first) for $S_{j,-}$ resp. $S_{j,+}$, and the other components ordered as before.

Proposition 4.5.6. (Preservation of orientations for insertion of separating diagonals) *Suppose $\underline{S}^{\vartriangle}$ is obtained by adding a seam so that the new seam σ is separating and diffeomorphic to $\mathbb{R}$. Suppose that $\underline{E}^{\vartriangle}, \underline{F}^{\vartriangle}$ are obtained from (E, F) by labelling the new seam by the diagonal. Suppose that the orientations for the ends of $\underline{S}^{\vartriangle}$ as well as the orderings of the components and boundary components are induced from the orientations and orderings from $\underline{S}$ as in Definition 4.5.5. Then the isomorphism of determinant lines* $\det(\underline{D}) \to \det(\underline{D}^{\vartriangle})$ *is orientation preserving.*

Proof. The proof is by a reduction to the case that the new seam is a circle in Proposition 4.5.4. First, suppose that $\underline{S} = S$ is an unquilted surface with strip-like ends. Gluing the ends $e_\pm$, produces a surface $\underline{S}^{\#}$ with two fewer strip-like ends and a Cauchy-Riemann operator $\overline{D}^{\#}$, see Figure 11.

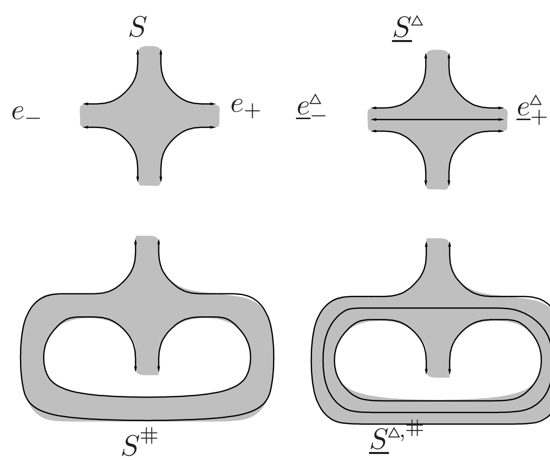

FIGURE 11. Inserting a diagonal and gluing the ends together

We compute the effect of adding a seam by studying the gluing signs for gluing the two quilted ends. Suppose that the ordering of the boundary components of the glued surface is such that the first boundary component of the new surface corresponds to the boundary of S between e_- and e_+. Let $\underline{D}$ denote the Cauchy-Riemann operator on the quilted surface $\underline{S}$ obtained by inserting a seam labelled by a diagonal. Consider the gluing isomorphisms from (17)

$$\mathcal{G} : \det(D) \to \det(D^{\#}), \quad \underline{\mathcal{G}} : \det(\underline{D}) \to \det(\underline{D}^{\#}).$$

The isomorphism $\mathcal{G}$ is orientation preserving by Proposition 5.1.2. By assumption on the ordering of the boundary components, the gluing isomorphism $\underline{\mathcal{G}}$ is orientation preserving. Now by Remark 4.4.2 (b) the natural isomorphism

$$\det(D^{\#}) \to \det(\underline{D}^{\#})$$

is orientation preserving, and similarly for the glued surfaces. Since gluing along the strip-like ends $\underline{e}^{\triangle}_{\pm}$ commutes with these isomorphisms, this proves the Proposition in this case.

In general, the orientations on the quilted Cauchy-Riemann operator are defined by deforming the seam conditions to split form. After deforming all seam conditions except the inserted seam to split form, the Cauchy-Riemann operator splits as a sum of unquilted Cauchy-Riemann operators for the patches. This argument reduces the proof to the previous case. □

4.6. Orientations for compositions of totally real correspondences.

Definition 4.6.1. (Smooth composition of linear seam conditions) Let $\underline{S}$ be a quilted surface with two adjacent patches S_0, S_2, equipped with complex vector bundles $\underline{E}$ and boundary and seam conditions $\underline{F}$. Suppose that E_1 is a complex vector bundle over the seam joining S_0, S_2. Let

$$F_{01} \subset \overline{E}_0 \times E_1, \quad F_{12} \subset \overline{E}_1 \times E_2$$

be totally real seam conditions. We say that

$$F_{02} := \pi_{02}(F_{01} \times_{\Delta_1} F_{12})$$

is a *smooth composition* of totally real subbundles if the intersection

$$(F_{01} \times F_{12}) \pitchfork (E_0 \times \Delta_1 \times E_2)$$

is transverse.

Remark 4.6.2. Let F_{02} be a smooth composition of seam conditions F_{01} and F_{12}.

(a) (Relative spin structure for the composition) Relative spin structures for F_{01}, F_{12}, and the diagonal induce a relative spin structure for F_{02}, because of the isomorphism

$$\pi_{02}^* F_{02} \oplus \Delta_1^{\perp} \to F_{01} \oplus F_{12}, \quad \Delta_1^{\perp} := \{(e,-e) \in E_1 \oplus E_1, e \in E_1\}$$

and the discussion in Proposition 3.1.6.

(b) (Quilted surface obtained by composition) Let $\underline{S}'$ denote the quilted surface with two additional seams separating S_0, S_2. The surface $\underline{S}'$ contains, in comparison with $\underline{S}$, two additional patches S_1^-, S_1^+ each isomorphic to strips. Let $\underline{E}'$ be the collection of complex vector bundles, equal to $\overline{E}$ on all but the new components where given by E_0 (pulled back by projection onto the seam). Let

$$\underline{F}' = \underline{F}/(F_{02} \mapsto (F_{01}, \Delta_1, F_{12}))$$

be the collection of boundary and seam conditions obtained by replacing F_{02} with F_{01}, Δ_1, F_{12}.

(c) (Identification of determinant lines) Let $\underline{D}, \underline{D}'$ denote the corresponding Cauchy-Riemann operators. The natural identifications

$$\ker(\underline{D}) \to \ker(\underline{D}'), \quad \operatorname{coker}(\underline{D}) \to \operatorname{coker}(\underline{D}')$$

induce an identification $\det(\underline{D}) \to \det(\underline{D}')$.

(d) (Orientations for the ends) Orientations for the ends of $\underline{S}$ induce orientations for the ends of $\underline{S}'$: Given a orientation at the end $\underline{e}_\pm$, define an orientation for $\underline{e}'_\pm$ as follows. First choose a path of subspaces connecting $(F_{01} \oplus F_{12})_{\underline{e}_\pm}$ to $(F_{02} \oplus \Delta_1^\perp)_{\underline{e}_\pm}$. Choose a deform of the subspaces $(\Delta_1)_{\underline{e}_\pm}, (\Delta_1^\perp)_{\underline{e}_\pm}$ to split form so that every space in the family has transverse intersection. We obtain from the deformation to split form orientations of $\det(\underline{D}_{\underline{e}_\pm})$ resp. $\det(\underline{D}'_{\underline{e}_\pm})$.

Proposition 4.6.3. (Preservation of orientations for composition of linear seam conditions) *Suppose that the orientations for the ends of $\underline{S}'$ are induced from a choice of orientations for $\underline{S}$, the ordering of components, ends, and boundary components of $\underline{S}$ is induced from those of $\underline{S}'$, the components S_1 and S_1' are adjacent in the ordering. Then the isomorphism of determinant lines* $\det(\underline{D}) \to \det(\underline{D}')$ *is orientation preserving.*

Proof. To compare orientations we deform the seam conditions to the composed seam conditions plus a trivial factor. Namely there is a canonical deformation of $F_{01} \oplus F_{12} \oplus \Delta_1$ to $\sigma_{1423}(F_{02} \oplus \Delta_1^\perp) \oplus \Delta_1$ within the space of totally real sub-bundles, where σ_{1423} is the isomorphism

$$E_0 \oplus E_1 \oplus E_1 \oplus E_2 \to E_0 \oplus E_2 \oplus E_1 \oplus E_1$$

given by permutation of factors. Indeed, any complex vector bundle admits a Hermitian, hence a symplectic structure. The fiber bundle of totally real subspaces of maximal dimension is canonically isomorphic to the Lagrangian Grassmannian. Hence the claim follows from the symplectic case, considered in [26, Lemma 3.1.9].

As a result of this deformation, the determinant lines for the original problem are identified with the determinant lines for the corresponding problem with composed seam conditions. More precisely, the orientations for $D_{\underline{E}',\underline{F}'}$ are those induced by the deformation of the totally real subbundles $F_{01} \oplus \Delta_1 \oplus F_{12}$ resp. F_{02} to split form. Now the orientation is independent of the deformation of $F_{01} \oplus \Delta_1 \oplus F_{12}$ to split form; hence we may take the deformation to be induced by a deformation of $F_{02}, \Delta_1^\perp$ and Δ_1 to split form. In this way we obtain an identification of the determinant line $\det(\underline{D}')$ with the tensor product of $\det(\underline{D})$ with that for the problem on S_1, S_1' with boundary conditions $\Delta_1, \Delta_1^\perp$. The latter has trivial index and orientation by definition (recall S_1, S_1' are strips or annuli) so the orientation on $\det(\underline{D})$ is that induced by $\det(\underline{D}')$. □

4.7. **Orientability of families of quilted Cauchy-Riemann operators.**

Proposition 4.7.1. (Trivializability of the orientation double cover of a family with nodal degeneration) *Let $\underline{S}_b, \underline{E}_b, \underline{F}_b, b \in B$ be a family of complex vector bundles with totally real boundary conditions on quilted surfaces with strip-like ends over a stratified space B. Suppose that $\underline{F}$ are equipped with relative spin structures, and the link of each stratum B_Γ in B is connected. Then the determinant line bundle* $\det(D_{\underline{E},\underline{F}}) \to B$ *is trivializable.*

Proof. A trivialization is given by multiplying the trivialization given by the relative spin structures and Proposition 4.4.1 by the gluing signs of Section 4.3. Since the links are connected, these gluing signs are well-defined, and since the gluing signs are associative, the resulting trivializations are continuous. □

Remark 4.7.2. (a) (Example of a family with non-trivial link) Let S_0 be a nodal disk with three components and two nodes. Consider a family of nodal disks S_δ, $\delta \in$

$\mathbb{R}/\mathbb{Z}$ extending S_0 where the two directions $\delta \in (0,\epsilon)$ resp. $\delta \in (0,\epsilon)$, correspond to deforming the two different nodes, and the family is extended to a family over the circle by identifying $S_{1/2} \sim S_{-1/2}$. The link in this case is a discrete space two points, and the gluing signs for the two components of the link are in general different. Therefore, a family of Cauchy-Riemann operators D_δ over such a space may not be orientable. That is, the family of determinant lines $\det(D_\delta)$ over $\mathbb{R}/\mathbb{Z}$ may be non-trivial.

(b) (Allowing strip-shrinking) One can allow strip-shrinking in the degenerations in $\underline{S}_b, b \in B$ as well as neck-stretching. Families of quilts of this kind are used in [12].

5. Orientations for holomorphic quilts

In this section we apply the orientations for Cauchy-Riemann operators developed in the previous sections to the case of quilted pseudoholomorphic maps.

5.1. **Construction of orientations for linearized operators with Lagrangian boundary conditions.** First we describe the Cauchy-Riemann operators we would like to orient. Let M be a symplectic manifold equipped with a compatible almost complex structure

$$J : TM \to TM, \quad J^2 = -\operatorname{Id}_{TM}.$$

Let S be a Riemann surface with complex structure

$$j : TS \to TS, \quad j^2 = -\operatorname{Id}_{TS}.$$

Definition 5.1.1. (Pseudoholomorphic maps with Lagrangian boundary conditions)

(a) A smooth map $u : S \to M$ is *pseudoholomorphic* if

$$\mathrm{d}u \circ j = J \circ \mathrm{d}u \in \Omega^1(S, u^*TM).$$

(b) A *Lagrangian submanifold* of M is an embedded submanifold $L \subset M$ of half the dimension of M, such that the restriction of the symplectic form to L vanishes:

$$\dim(L) = \dim(M)/2, \quad \omega|_L = 0.$$

(c) Suppose that ∂S has components $I_1, \ldots, I_k$. Given Lagrangian submanifolds

$$L_1, \ldots, L_k \subset M$$

a *pseudoholomorphic map with Lagrangian boundary conditions* in $L_1, \ldots, L_k$ is a pseudoholomorphic map

$$u : S \to M, \quad u(I_j) \subset L_j, j = 1, \ldots, k.$$

Recall the linearized Cauchy-Riemann operators associated to pseudoholomorphic maps and their Fredholm properties:

Remark 5.1.2. (Fredholm nature of linearized operators)

(a) (Maps with boundary) Associated to any pseudoholomorphic map $u : S \to M$ with strip-like ends and Lagrangian boundary conditions in L is a *linearized real Cauchy-Riemann operator*

$$D_u : \Omega^0(S, u^*TM, (u|_{\partial S})^*TL) \to \Omega^{0,1}(S, u^*TM), \tag{50}$$

as in Definition 2.3.6.

(b) (Quilted maps) Similarly, if $\underline{S}$ is a quilted surface with strip-like ends, $\underline{M}$ is a collection of symplectic manifolds associated to the patches of $\underline{S}$, and $\underline{L}$ is a collection of Lagrangian submanifolds and correspondences associated to the boundary and seam components of $\underline{S}$, and $u : \underline{S} \to \underline{M}$ is a pseudoholomorphic map with Lagrangian boundary conditions transverse at infinity on each strip-like end, we denote by D_u the associated linearized Cauchy-Riemann operator D_u.

(c) (Quilted sections) Let S be a surface with boundary and strip-like ends. A *symplectic Lefschetz-Bott fibration* [16], [23] is a space E equipped with a closed two-form ω_E non-degenerate near the fibers and compatible almost complex structure near the singularities of π and a projection $E \to S$ with singularities of Morse-Bott type that is locally holomorphic near the singularities of π. Given a Lagrangian boundary condition $Q \subset \partial E$ (that is, a sub-fiber-bundle of $E|\partial S$ that is Lagrangian in each fiber) and a pseudoholomorphic section $u : S \to E$ let

$$D_u : \Omega^0(S, \partial S; u^*T^{\text{vert}}E, u^*T^{\text{vert}}Q) \to \Omega^{0,1}(S, u^*T^{\text{vert}}E)$$

denote the corresponding linearized Cauchy-Riemann operator.

Definition 5.1.3. (a) (Relative spin structures for collections of Lagrangians) Let M be a symplectic manifold and $\underline{L} = (L_0, \ldots, L_d)$ be a sequence of oriented Lagrangian submanifolds in M. A *relative spin structure* for $(L_0, \ldots, L_d)$ is a stable relative spin structure for the immersion $L_0 \cup \ldots \cup L_d \to M$. In particular, this means that each L_j has a relative spin structure with the same background class.

(b) (Relative spin structures for collections of Lagrangian correspondences) Let $\underline{S}$ be a (possibly quilted) surface (possibly) with boundary and strip-like ends, $\underline{M}$ a collection of symplectic manifolds, and $\underline{L}$ a collection of boundary and seam conditions for $\underline{S}$. A *relative spin structure* for $\underline{L}$ with background classes $w_p, p \in \mathcal{P}$ is a relative spin structure for the immersion

$$\bigcup_{\sigma \in \mathcal{S}} L_\sigma \to \bigcup_{p_1, p_2 \in \mathcal{P}} M_{p_1} \times M_{p_2}$$

with respect to the background classes $\pi_1^* b_{p_1} + \pi_2^* b_{p_2}$.

Remark 5.1.4. (a) (Moduli spaces of quilted trajectories) Let $\underline{L}$ be a periodic sequence of Lagrangian correspondences equipped with admissible brane structures with symplectic manifolds $M_0, \ldots, M_m$, and

$$H_j \in C^\infty([0,1] \times M_j), \quad Y_j \in \text{Map}([0,1], \text{Vect}(M_j)), \quad j = 0, \ldots, m$$

a time-dependent Hamiltonian resp. their Hamiltonian vector fields for each patch. Define

$$\mathcal{I}(\underline{L}) := \left\{ \underline{x} = \big(x_j : [0, \delta_j] \to M_j\big)_{j=0,\ldots,r} \;\middle|\; \begin{array}{r} \dot{x}_j(t) = Y_j(x_j(t)), \\ (x_j(\delta_j), x_{j+1}(0)) \in L_{j(j+1)} \end{array} \right\}.$$

the set of generalized intersection points. As in standard Floer theory, the moduli spaces of "quilted holomorphic strips" $\mathcal{M}(\underline{x}^-, \underline{x}^+)$ arise from quotienting out by simultaneous $\mathbb{R}$-shift in all components u_j. The moduli spaces are regular for generic domain-dependent almost complex structures and Hamiltonian perturbations, see [28].

(b) (Moduli spaces of quilts) Given a quilted surface $\underline{S}$ with patch labels $\underline{M}$ and seam/boundary conditions $\underline{L}$ and a collection of limits $\underline{x}_e \in \mathcal{I}(\underline{L}_e)$ for each strip-like end e, let

$$\mathcal{M}(\underline{M}, \underline{L}, \underline{x}) = \left\{ \underline{u} : \underline{S} \to \underline{M} \;\middle|\; \underline{u}(\partial \underline{S}) \subset \underline{L},\ (1),\ \lim_{s \to \pm\infty} \underline{u}(\epsilon_e(s,t))\underline{u} = \underline{x}_e, \forall e \in \mathcal{E} \right\}$$

denote the moduli space of quilted pseudoholomorphic maps with limits $\underline{x}_e$ along each strip-like end $e \in \mathcal{E}$. The moduli spaces $\mathcal{M}(\underline{M}, \underline{L}, \underline{x})$ are regular for generic almost complex structures and Hamiltonian perturbations, see [27], in the sense that $\mathcal{M}(\underline{M}, \underline{L}, \underline{x})$ is cut out of a Banach space of maps by Fredholm equation with surjective linearized operator D_u. Suppose that $\mathcal{M}(\underline{M}, \underline{L}, \underline{x})$ is regular. By definition its tangent space at a pseudoholomorphic map $u : \underline{S} \to \underline{M}$ is the kernel of the linearized operator D_u:

$$T_u \mathcal{M}(\underline{M}, \underline{L}, \underline{x}) = \ker(D_u).$$

If D_u is oriented, then so is $\mathcal{M}(\underline{M}, \underline{L}, \underline{x})$ at u.

(c) (Orientations for quilted trajectories) Let $\mathcal{M}(\underline{x}_+, \underline{x}_-)$ denote the moduli space of quilted pseudoholomorphic trajectories from $\underline{x}_+$ to $\underline{x}_-$. If $\mathcal{M}(\underline{x}_+, \underline{x}_-)$ is regular at a trajectory u then the tangent space

$$T_{[u]} \mathcal{M}(\underline{x}_+, \underline{x}_-) = \ker(D_u)/\mathbb{R}, \quad 0 = \operatorname{coker}(D_u)$$

where the first is the quotient of the linearized operator D_u by the $\mathbb{R}$-action given by translation. Thus any orientation for D_u induces an orientation on the tangent space $T_{[u]} \mathcal{M}(\underline{x}_+, \underline{x}_-)$.

(d) (Orientations for quilted sections) Let $E \to S$ be a symplectic Lefschetz fibration and Q a Lagrangian boundary condition. Denote by $\mathcal{M}(E, Q; \underline{x})$ the moduli space of pseudoholomorphic sections $u : S \to E$ with boundary values in Q and limits $\underline{x}$. If $\mathcal{M}(E, Q; \underline{x})$ is regular at u then

$$T_u \mathcal{M}(E, Q; \underline{x}) \cong \ker(D_u), \quad 0 = \operatorname{coker}(D_u).$$

So any orientation on D_u induces on an orientation on $\mathcal{M}_u(E, Q; \underline{x})$ at u.

We generalize the discussion in Definition 2.4.11 to holomorphic quilts:

Definition 5.1.5. (a) (Families of holomorphic quilts) Let B be a stratified space as in Definition 2.4.11, and $\underline{S}$ a family of quilts over B. A *family of holomorphic quilts* with domain $\underline{S}$ is a triple (C, f, u) consisting of a space C, a continuous map $f : C \to B$, and a map $u : \underline{S} \times_B C \to \underline{M}$ such that

(i) the restriction of u to each fiber $\underline{S} \times_B \{c\}$ is a holomorphic quilt, and

(ii) u is continuous with respect to the Gromov topology on maps. That is, if $c_\nu \to c$ then $u_{f(c_\nu)}$ Gromov converges to $u_{f(c)}$.

(b) (Determinant line bundle for families of quilts) Given a family of holomorphic quilts, for any $c_\nu \to c$ the linearized operator $D_{u_{c_\nu}}$ is canonically deformable to the operator obtained from D_{u_c} by the gluing construction, by exponential decay on the necks. One obtains a determinant line bundle $\det(D_{u_c}) \to C$ by Proposition 2.4.12.

Theorem 5.1.6. (Orientations via relative spin structures for families of quilts) *Let $\underline{S}$ be a family of quilted surfaces of fixed type over a smooth manifold B, and $\underline{M}, \underline{L}$ a collection of symplectic manifolds for the patches and Lagrangian boundary/seam conditions. Let $u : \underline{S} \times_B C \to \underline{M}$ be a family of pseudoholomorphic map with Lagrangian boundary and seam conditions in $\underline{L}$ over C. Suppose that the link of each stratum B_Γ of B is connected. Then a relative spin structure for $\underline{L}$ and orientations for the ends induce a trivialization the determinant line bundle $\cup_{b \in C} \det(D_{u_c}) \to C$.*

Proof. By Proposition 4.7.1 and the identification of determinant lines in Definition 5.1.5. □

Remark 5.1.7. (Gluing) Theorem 5.1.6 is a family version of Theorem 1.0.1, and includes that Theorem as a special case except for the universal signs. The signs for gluing in the interior, gluing at the boundary, and gluing strip-like ends are given in Section 2.2. The signs for gluing Floer trajectories with surfaces with strip-like ends are determined as follows. The orientation on the moduli space $\mathcal{M}(\underline{x}_+, \underline{x}_-)$ of Floer trajectories, induced from the isomorphism

$$T_u \mathcal{M}(\underline{x}_+, \underline{x}_-) \oplus \mathbb{R} \to T_{\tilde{u}} \tilde{\mathcal{M}}(\underline{x}_+, \underline{x}_-)$$

where second factor is the tangent space to the translational $\mathbb{R}$-action and the codomain is the tangent space to the moduli space of parametrized trajectories. There exists a gluing map

$$\mathcal{M}(x_e^-, y)_0 \times \mathcal{M}_S(\underline{x}^- |_{x_e^- \to y}, \underline{x}^+)_0 \times [0, \epsilon) \to \overline{\mathcal{M}_S(\underline{x}^-, \underline{x}^+)_1}$$

that factors through the product

$$\tilde{\mathcal{M}}(x_e^-, y)_1 \times \mathcal{M}_S(\underline{x}^- |_{x_e^- \to y}, \underline{x}^+)_0$$

preserving the orientation on the $\mathbb{R}$ orbits. Taking the conventions of Remark 4.3.4 (c) shows that the sign of the gluing map is positive. A similar description for the outgoing Floer trajectories shows that the sign is negative.

Remark 5.1.8. (Shift of Background Class) Define an involution Υ on the set of relative spin structures on a Lagrangian submanifold L that shifts the background class as follows. The bundle TM has a canonical splitting (up to homotopy) after restriction to any Lagrangian submanifold L:

$$TM|L \cong T(T^*L)|_L \cong TL \oplus T^\vee L \cong TL \oplus TL$$

where the last is canonical up to homotopy. It follows from (3.1.6) that $TM|L$ has a canonical spin structure, up to isomorphism. We say that two relative spin structures for $\underline{L}$ are *equivalent mod TM* corresponding to bundles $R_1, R_2 \to TM$ and spin structures on $TL \oplus R_1|L, TL \oplus R_2|L$ iff

$$R_2 \cong R_1 \oplus TM$$

up to stabilization and the spin structure on $TL \oplus R_2|L$ is that induced by the isomorphism

$$TL \oplus R_2|L \cong TL \oplus R_1|L \oplus TM|_L.$$

Thus the background class of the second spin structure is

$$w_2(R_2) = w_1(R_1) + w_2(TM).$$

Given a sequence $\underline{L}$ of oriented Lagrangian submanifolds in M, we denote by

$$\Upsilon(\underline{L}) = (\Upsilon(L_0), \ldots, \Upsilon(L_d)) \tag{51}$$

the same sequence with shifted relative spin structures. The relative spin structures shifted by Υ induce a new relative spin structure on $(u|\partial S_j)^* TL$. By Lemma 3.1.8, the orientations on the moduli spaces of pseudoholomorphic disks are reversed by the identity map exactly if the Maslov index is equal to 2 mod 4.

5.2. **Lagrangian Floer invariants over the integers.** Let M be a compact symplectic manifold equipped with an N-fold Maslov cover $\mathrm{Lag}^N(M) \to \mathrm{Lag}(M)$ for some even integer N. The cover $\mathrm{Lag}^N(M)$ is by definition an N-fold cover of the bundle $\mathrm{Lag}(M)$ of Lagrangian subspaces of TM that restricts to the standard N-fold cover on any fiber. We assume that the mod 2 reduction

$$\mathrm{Lag}^2(M) := \mathrm{Lag}^N(M) \times_{\mathbb{Z}_N} \mathbb{Z}_2$$

of $\mathrm{Lag}^N(M)$ is the oriented double cover of $\mathrm{Lag}(M)$.

Definition 5.2.1. (a) (Lagrangian branes) Let $L \subset M$ be a Lagrangian submanifold. A *brane structure* on L is an orientations, relative spin structure and *grading*

$$\sigma_L : L \to \mathrm{Lag}^N(M)|L$$

lifting the canonical section

$$L \to \mathrm{Lag}(M), \quad l \mapsto T_l L.$$

A *Lagrangian brane* is a compact Lagrangian submanifold equipped with brane structure.

(b) (Admissibility) A Lagrangian brane is *admissible* if it has minimal Maslov number at least three and torsion fundamental group; these conditions imply monotonicity for pairs, triples etc. in the sense of [24].

(c) (Periodically-graded Floer complex) Let $L_0, L_1 \subset M$ be admissible Lagrangian branes. The grading induces a degree map

$$\mathcal{I}(L_0, L_1) \to \mathbb{Z}_N, \qquad x \mapsto |x| = d(\sigma_{L_0}(x), \sigma_{L_1}(x)).$$

The *Floer cochain group* is the $\mathbb{Z}_N$-graded group

$$CF(L_0, L_1) = \bigoplus_{d \in \mathbb{Z}_N} CF^d(L_0, L_1), \qquad CF^d(L_0, L_1) = \bigoplus_{x \in \mathcal{I}(L_0, L_1), |x| = d} \mathbb{Z}\langle x \rangle.$$

The *Floer coboundary operator* is the map of degree 1,

$$\partial^d : \; CF^d(L_0, L_1) \to CF^{d+1}(L_0, L_1),$$

defined for $|x_-| = d$ by

$$\partial^d \langle x_- \rangle := \sum_{x_+ \in \mathcal{I}(L_0, L_1)} \Big(\sum_{u \in \mathcal{M}(x_-, x_+)_0} \epsilon(u) \Big) \langle x_+ \rangle.$$

where

$$\epsilon : \mathcal{M}_{\underline{S}}(x_-, x_+)_0 \to \{-1, +1\}$$

is defined by comparing the constructed orientation to the canonical orientation of a point.

(d) (Graded Floer complex using a formal variable) There is also a $\mathbb{Z}$-graded version of Floer homology whose differential is defined over $\Lambda = \mathbb{Z}[q]$ the ring of polynomials in a formal variable q that keeps track of the difference in gradings. Let

$$\tilde{\mathcal{I}}(L_0, L_1) = \mathbb{Z} \times_{\mathbb{Z}_N} \mathcal{I}(L_0, L_1)$$

and $\tilde{d}$ the extended degree map

$$\tilde{d} : \tilde{\mathcal{I}}(L_0, L_1) \to \mathbb{Z}, \quad [n, x] \mapsto n.$$

let $\tilde{CF}(L_0, L_1)$ denote the sum over lifted intersection points

$$\tilde{CF}(L_0, L_1) = \bigoplus_{x \in \tilde{\mathcal{I}}(L_0, L_1)} \Lambda \langle x \rangle.$$

Define

$$\tilde{\partial} \langle x_- \rangle := \sum_{x_+ \in \tilde{\mathcal{I}}(L_0, L_1)} \Big(\sum_{\underline{u} \in \mathcal{M}(x_-, x_+)_0} \epsilon(u) q^{(\tilde{d}(x_+) - \tilde{d}(x_-) - 1)} \Big) \langle \underline{x}_+ \rangle.$$

Theorem 5.2.2. *Suppose that M is monotone and a pair of Lagrangian branes (L_0, L_1) in M is monotone and satisfies the monotonicity conditions L1-3 of [27]: each is compact, oriented, monotone, and has minimal Maslov number at least three. Then the Floer differentials $\partial, \tilde{\partial}$ satisfy $\partial^2 = 0, \tilde{\partial}^2 = 0$.*

Proof. That $\partial^2 = 0$ follows from argument in Oh [15] and that gluing along strip-like ends is orientation preserving for the corresponding Cauchy-Riemann operators. The proof for $\tilde{\partial}$ is similar. □

If the assumptions of Theorem 5.2.2 are satisfied then the *Floer cohomology* is defined by

$$HF(L_0, L_1) := \bigoplus_{d \in \mathbb{Z}_N} HF^d(L_0, L_1), \qquad HF^d(L_0, L_1) := \ker(\partial^d)/\operatorname{im}(\partial^{d-1}).$$

Similarly $\widetilde{HF}(L_0, L_1)$ is the cohomology of $\tilde{\partial}$.

Proposition 5.2.3. *$HF(L_0, L_1)$ resp. $\widetilde{HF}(L_0, L_1)$ is a well-defined $\mathbb{Z}_N$-graded resp. $\mathbb{Z}$-graded group, independent, up to isomorphism, of the choices made in the construction of the orientations.*

Proof. Suppose that $(\Gamma_\pm, D_\pm, \epsilon_\pm, \delta_\pm), (\Gamma'_\pm, D'_\pm, \epsilon'_\pm, \delta'_\pm)$ are two orientations for the ends $e \in \mathcal{E}(S)$. Define maps

$$\sigma : CF(L_0, L_1) \to CF(L_0, L_1), \quad \langle x \rangle \mapsto \sigma(x) \langle x \rangle$$

as follows. Let $(E_\pm, F_\pm)$ denote the corresponding elliptic boundary value problems on the once-punctured disk S_1. Let

$$(\overline{E}_\pm, \overline{F}_\pm, \overline{D}_\pm) = (E_\pm, F_\pm, D_\pm) \# (E'_\pm, F'_\pm, D'_\pm)$$

denote the bundles and Cauchy-Riemann operator obtained by gluing together the problems $E_\pm, F_\pm, D_\pm$ and $E'_\pm, F'_\pm, D'_\pm$ along the strip-like ends. By the gluing formula there exists an isomorphism

$$\det(D_\pm) \otimes \det(D'_\pm) \to \det(\overline{D}_\pm).$$

We define $\sigma(x) = \pm 1$ depending on whether the orientation induced by $\epsilon_\pm, \epsilon'_\pm$ and the gluing isomorphism agrees with the orientation induced by the trivialization of $\overline{F}_\pm$. The gluing law for indices implies that the map σ intertwines the relative invariants Φ_S for S associated to the two different choices of orientation. □

Remark 5.2.4. (Conjugates) Suppose that M^- is the symplectic manifold M with symplectic form reversed. Given Lagrangian branes L_k we denote by L^- the corresponding branes in M^-, with background class shifted by 5.1.8:

$$b(L^-) = b(L) + w_2(M).$$

For any intersection point $x_\pm$ we denote by $x_\pm^-$ the corresponding intersection point of L_j^- and $\mathcal{M}(x_+^-, x_-^-)$ the moduli space of Floer trajectories. Each trajectories $u(s,t)$ for (L_0, L_1) defines a trajectory $u(1-s,t)$ for (L_1^-, L_0^-) giving a bijection

$$\mathcal{M}(x_+, x_-) \to \mathcal{M}(x_-^-, x_+^-)$$

from trajectories for (L_0, L_1) to trajectories for (L_1^-, L_0^-). The bijection acts on orientations at a trajectory u by a sign given by

$$(-1)^{(\mathrm{Ind}(D_{x_+}) - \mathrm{Ind}(D_{x_-}) - 1 - \dim T_{\underline{u}} \mathcal{M}(x_+, x_-)))/2}.$$

By Lemma 3.1.8, this shift cancels out the shift in the background class. Hence as oriented manifolds

$$\mathcal{M}(x_+, x_-) \cong \mathcal{M}(x_+^-, x_-^-).$$

Counting points in the zero-dimensional component implies that the Floer operators are equal so that

$$HF(L_1^-, L_0^-) \cong HF(L_0, L_1). \tag{52}$$

In the $\mathbb{Z}$-graded version one can also define an involution in the power series ring

$$\mathbb{Z}[q] \to \mathbb{Z}[q], \quad q \mapsto (-1)^{N/2} q.$$

This involution extends to an involution of $\widetilde{HF}(L_0, L_1)$. The natural identification

$$\widetilde{CF}(L_0, L_1) \to \widetilde{CF}(L_1^-, L_0^-)$$

composed with the involution intertwines with the Floer differentials $\tilde{\partial}, \tilde{\partial}^-$.

The same considerations apply for the Floer homologies $HF(\underline{L}), \widetilde{HF}(\underline{L})$ of a periodic sequence of Lagrangian correspondences equipped with admissible brane structures. In particular, suppose that $\underline{L}$ is the diagonal. The Floer cohomology satisfies

$$HF(\underline{L}) = \widetilde{HF}(\underline{L})/(q-1) = QH(M)/(q-1)$$

where $QH(M)$ is the quantum cohomology. Then $QH(M)/(q-1)$ is *not* isomorphic via the identity map to $QH(M^-)/(q-1)$.

For example let M be complex projective n-space. Then

$$QH(M)/(q-1) = \mathbb{Z}[x]/(x^{n+1} - 1).$$

On the other hand, M^- is also isomorphic to the projective space, but now the hyperplane class is $-x$, and

$$QH(M^-)/(q-1) = \mathbb{Z}[-x]/((-x)^{n+1} - 1).$$

The latter is isomorphic to $QH(M)/(q-1)$ via the map $x \mapsto -x$, but not via the identity if n is even. On the other hand,

$$QH(M) = \mathbb{Z}[x,q]/(x^n - q) \cong QH(M^-) = \mathbb{Z}[-x,q]/((-x)^n - q)$$

via the map $x \mapsto x, q \mapsto -q$.

Remark 5.2.5. (Disk potentials) If the minimal Maslov number of either Lagrangian is only two, then the Floer operator may not square to zero due to the presence of pseudo-holomorphic disks. We denote by

$$\mathcal{M}_1(L) : \{(u : (D, \partial D) \to (X, L), z) | \partial_J u = 0\} / \operatorname{Aut}(D) \cong SL(2,\mathbb{R}).$$

the moduli space of pseudolomorphic disk with a single marking on the boundary. Consider the evaluation map $\mathrm{ev}_1 : \mathcal{M}_1(L) \to L$. A generic element $\ell \in L$ is a regular value and we define following Oh [15]

$$w(L) = \sum_{u \in \mathrm{ev}_1^{-1}(l) \subset \mathcal{M}(L)_{\dim(L)}} \epsilon(u) \in \mathbb{Z} \tag{53}$$

the *disk potential* of L. One sees by a parametrized version of the moduli space that the number $w(L)$ is independent of the choice of regular value. The boundary of $\mathcal{M}(L_0, L_1)_1$ then two extra components corresponding to bubbling off disks at the left and right boundaries of the strip:

$$\partial \mathcal{M}(L_0, L_1)_1 = (\mathcal{M}(L_0, L_1)_0 \times_{L_0 \cap L_1} \mathcal{M}_1(L_0, L_1)_0) \cup (\mathcal{M}_1(L_0) \times_M L_1) \cup (\mathcal{M}_1(L_1) \times_M L_0). \tag{54}$$

The second and third configurations consist of a disks attached via a node to a constant strip. Recall that automorphisms of the disk fixing a point form a two-dimensional subgroup of $\operatorname{Aut}(D)$ generated by translations and dilations, once the complement of the marking is identified with the half-space. Under the gluing map the one-parameter subgroup consisting of translations on the disk becomes identified with the opposite translation resp. translation on the strip. As a result the square of the Floer operator is the difference in disk potentials:

$$\partial^2 = w(L_1) - w(L_0).$$

We investigate the effect of reversing the symplectic form on disk potentials. If we consider $L^- \subset M^-$ the same Lagrangian in the symplectic manifold with reversed symplectic form, then we have a diffeomorphism of moduli space of pseudoholomorphic disks

$$\mathcal{M}_1(L) \to \mathcal{M}_1(L^-), \quad u \mapsto u^-, \ u^-(z) = u(\overline{z})$$

Lemma 5.2.6. *For the shifted relative spin structure on L^- with background class $b(L) + w_2(M)$, the diffeomorphism $\mathcal{M}_1(L) \to \mathcal{M}_1(L^-)$ is orientation reversing, hence the disk potentials for L, L^- are related by $w(L^-) = -w(L)$.*

Proof. The complex conjugation on the domain reverses the orientation on the group of automorphisms of the disk fixing a point. Indeed after identifying the complement of a point with the upper half plane, this group of automorphisms is generated by dilations and translations, while conjugation becomes identified with a reflection. While dilations are invariant under conjugation by the reflection, translations are reversed producing a sign change. On the other hand, complex conjugation on the index of the linearized operator produces a sign change exactly if the Maslov index is equal to 2 mod 4. The additional signs produce by the shift of relative spin structure produced in Lemma 3.1.8 imply that the orientation on the moduli spaces are reversed. □

These results extend to the quilted case as follows.

Definition 5.2.7. (a) (Quilted Floer cohomology) The quilted Floer coboundary operator

$$\partial^d : \ CF^d(\underline{L}) \to CF^{d+1}(\underline{L})$$

is defined by

$$\partial^d \langle \underline{x}_- \rangle := \sum_{\underline{x}_+ \in \mathcal{I}(\underline{L})} \Big(\sum_{\underline{u} \in \mathcal{M}(\underline{x}_-, \underline{x}_+)_0} \epsilon(\underline{u}) \Big) \langle \underline{x}_+ \rangle,$$

where the signs

$$\epsilon : \mathcal{M}(\underline{x}_-, \underline{x}_+)_0 \to \{\pm 1\}$$

are defined by comparing the given orientation to the canonical orientation of a point. By studying the ends of the one-dimensional moduli spaces as in the unquilted case one obtains $\partial^2 = 0$. The *quilted Floer cohomology* defined in [25] is

$$HF(\underline{L}) := \bigoplus_{d \in \mathbb{Z}_N} HF^d(\underline{L}), \qquad HF^d(\underline{L}) := \ker(\partial^d)/\mathrm{im}(\partial^{d-1})$$

and is a $\mathbb{Z}_N$-graded group. In case that the Lagrangians are N-graded the datum associated to each intersection point $\underline{x}$ is equipped with a canonical mod $\mathbb{Z}_{2N}$ orientation given by the path induced by the grading. Similarly $\widetilde{HF}(\underline{L})$ is defined as the $\mathbb{Z}$-graded group over the formal power series ring in a formal variable q.

(b) (Relative invariants) Suppose that $\underline{S}$ is a quilted surface with strip-like ends. Let $\underline{L}$ be a collection of Lagrangian boundary and seam conditions for a collection $\underline{M}$ of compact monotone symplectic manifolds attached to the patches of $\underline{S}$. A *brane structure* for $\underline{L}$ is a collection of gradings and a relative spin structure. Let $\underline{x}$ be a collection of perturbed intersection points at infinity, and $\mathcal{M}(\underline{x})$ the moduli space of perturbed pseudoholomorphic maps $\underline{S} \to \underline{M}$ with boundary values in $\underline{L}$ and limits $\underline{x}$. A choice of relative spin structure for $\underline{L}$, if it exists, together with orientations on the ends, induces an orientation on $\mathcal{M}(\underline{x})$, by Remark 5.1.2. Assuming suitable monotonicity conditions on the tuple $\underline{L}$ that rule out sphere and disk bubbling in zero and one-dimensional moduli space, there is a cochain level relative invariant constructed in [26] defined by

$$C\Phi_{\underline{S}} : \bigotimes_{e \in \mathcal{E}_-(\underline{S})} CF(\underline{L}_e) \to \bigotimes_{e \in \mathcal{E}_+(\underline{S})} CF(\underline{L}_e)$$

$$\otimes_{e \in \mathcal{E}_-(\underline{S})} \langle \underline{x}_e \rangle \to \sum_{(\underline{x}_e)_{e \in \mathcal{E}_+(\underline{S})}, u \in \mathcal{M}(\underline{x})} \epsilon(u) \otimes_{e \in \mathcal{E}_+(\underline{S})} \langle \underline{x}_e \rangle.$$

For rational coefficients we obtain a cohomology level invariant

$$\Phi_{\underline{S}} : \bigotimes_{e \in \mathcal{E}_-(\underline{S})} HF(\underline{L}_e) \to \bigotimes_{e \in \mathcal{E}_+(\underline{S})} HF(\underline{L}_e).$$

Proposition 5.2.8. *The invariants $C\Phi_{\underline{S}}$ descend to cohomology and the resulting cohomological invariants $\Phi_{\underline{S}}$ are independent up to isomorphism of the choice of perturbation data and orientations.*

Proof. Without signs, this is the main result of [24]. The cochain level invariants descend to cohomology by the determination of the gluing signs in Section 5. It follows again from gluing that the maps for two choices of orientations intertwine with the maps in the proof of Proposition 5.2.3. □

Proposition 5.2.9. (Floer invariant of the torus as the graded dimension) *Let $\underline{L}$ be a periodic sequence of Lagrangian correspondences equipped with admissible brane structures. Let $\underline{T}$ denote the quilted torus, with one seam for each element of $\underline{L}$. Then the relative invariant $\Phi_{\underline{T}}$ is the graded dimension of $HF(\underline{L})$,*

$$\Phi_{\underline{T}} = \operatorname{rank} HF^{\mathrm{even}}(\underline{L}) - \operatorname{rank} HF^{\mathrm{odd}}(\underline{L}).$$

Proof. The equality in the proposition follows from the discussion of the annulus signs in Remark 4.3.4. Indeed that remark shows that the contribution to the invariant $C\Phi_{\underline{T}}$ from $x \in \mathcal{I}(\underline{L})$ is $(-1)^{|x|}$. □

We investigate the behavior of the quilted Lagrangian Floer invariants under the following basic operations:

Remark 5.2.10. (a) (Products) Let $\underline{L}_j, j = 0, 1$ denote two periodic sequences of Lagrangian correspondences of the same length equipped with admissible brane structures so that the Floer homology groups $HF(\underline{L}_j), j = 0, 1$ are well-defined. Let $\underline{L}$ denote the sequence obtained from $\underline{L}_j, j = 0, 1$ by direct sum. Let $\underline{x} = (\underline{x}_0, \underline{x}_1)$ be a generalized intersection point for $\underline{L}$. Consider the natural map

$$\mathcal{M}(\underline{x}_0) \times \mathcal{M}(\underline{x}_1) \to \mathcal{M}(\underline{x}), \quad (u_0, u_1) \to u_0 \times u_1. \tag{55}$$

The linearized operator $D_{u_0 \times u_1}$ is naturally isomorphic to the direct sum $D_{u_0} \oplus D_{u_1}$. It follows that the map (55) is orientation preserving, by the discussion on direct sums in Section 2.3. The Floer complex $CF(\underline{L})$ is the graded tensor product of $CF(\underline{L}_0)$ and $CF(\underline{L}_1)$, and similarly for the relative invariants. Thus if the cohomologies are torsion free then $HF(\underline{L})$ is the graded tensor product of $HF(\underline{L}_j), j = 0, 1$.

(b) (Disjoint Unions of domains) let $S_j, j = 0, 1$ be surfaces with strip-like ends. Suppose that $S = S_1 \cup S_2$, and S_j has $d_j^{\pm}$ incoming resp. outgoing ends for $j = 1, 2$. A pair $u = (u_1, u_2)$ of pseudoholomorphic maps of index zero has determinant line with orientation related to the orientations of the determinant lines

$$\begin{aligned} \epsilon(u) &= \epsilon(u_1)\epsilon(u_2)(-1)^{\operatorname{rank}(F)(\#\pi_0(\partial S_2) + d_2^+)(\sum_{e \in \mathcal{E}_{-,1}} (\dim(M)/2 + \operatorname{Ind}(D_e)))} \\ &= \epsilon(u_1)\epsilon(u_2)(-1)^{|\Phi_{S_2}| \sum_{e \in \mathcal{E}_{-,1}} |x_e|}. \end{aligned}$$

This formula implies that the relative invariant $C\Phi_S$ is the graded tensor product

$$\begin{aligned} C\Phi_S(\otimes_{e \in \mathcal{E}_-(S)} \langle x_e \rangle) &= (-1)^{|\Phi_{S_2}| \sum_{e \in \mathcal{E}_{-,1}} |x_e|} C\Phi_{S_1}(\otimes_{e \in \mathcal{E}_-(S_1)} \langle x_e \rangle) \otimes C\Phi_{S_2}(\otimes_{e \in \mathcal{E}_-(S_2)} \langle x_e \rangle) \\ &= (C\Phi_{S_1} \otimes C\Phi_{S_2})(\otimes_{e \in \mathcal{E}_-(S_1) \cup \mathcal{E}_-(S_2)} \langle x_e \rangle). \end{aligned}$$

With rational coefficients, it follows that Φ_S is the graded tensor product of Φ_{S_1} and Φ_{S_2}.

(c) (Folding of quilts) A "quilt folding" isomorphism was considered in [25]. Let

$$\underline{L} = (L_{01}, L_{12}, \ldots, L_{k0})$$

by a cyclic Lagrangian correspondence with k odd. In [25, Section 5] we identified

$$HF(\underline{L}) \cong HF(L_{01} \times L_{23} \times \ldots . L_{(k-1)k}, \sigma(L_{12}^- \times L_{34}^- \times \ldots L_{k0}^-)) \tag{56}$$

where

$$\sigma : M_1^- \times M_2 \times \ldots \times M_0 \to M_0 \times M_1^- \times \ldots \times M_k^-$$

is the cyclic shift. To justify (56) with integer coefficients we note that the orientations are invariant under deformation of the linearized seam conditions to split form. So we may assume $L_{(j-1)j} = L_{j-1} \times L_j'$ for each j, and the isomorphism (56) follows from the identification of both sides with the tensor product of the Floer cohomology groups $HF(L_j', L_j), j = 0, \ldots, k$ and the identifications $HF(L_j', L_j) \cong HF(L_j^-, L_j^{',-})$ from (52).

Remark 5.2.11. (Quilted sphere bubbling) In the case that some correspondences $L_{(j-1)j}$ have minimal Maslov number equal to two, the quilted Floer operator may not square to zero. Let $\underline{S}$ denoted a quilted sphere, with a single marking on the seam, and $\mathcal{M}_1(L_{(j-1)j}, M_{j-1}, M_j)$ the moduli space of holomorphic quilted spheres. Let $l \in L_{(j-1)j}$ be a regular value of the evaluation map

$$\mathrm{ev}_1 : \mathcal{M}_1(L_{(j-1)j}, M_{j-1}, M_j) \to L_{j-1} \times L_j.$$

Define the *quilted sphere potential*

$$w(L_{(j-1)j}, M_{j-1}, M_j) = \sum_{u \in \mathrm{ev}_1^{-1}(l) \subset \mathcal{M}(L_{(j-1)j}, M_{j-1}, M_j)_{\dim(L)}} \epsilon(u) \in \mathbb{Z}.$$

Then the square of the Floer coboundary operator satisfies

$$\partial^2 = w(L_{01}, M_0, M_1) + w(L_{12}, M_1, M_2) + \ldots + w(L_{n0}, M_n, M_0).$$

In the case of the folding identification, folding produces an identification of moduli spaces

$$\mathcal{M}_1(L_{(j-1)j}, M_{j-1}, M_j) \cong \mathcal{M}_1(L_{(j-1)j}) \cong \mathcal{M}_1(L_{(j-1)j}^-)^{\mathrm{op}}$$

where the superscript op denotes the opposite orientation. Hence

$$w(L_{01}, M_0, M_1) = w(L_{01}, M_0^- \times M_1) = -w(L_{01}, M_0 \times M_1^-).$$

This equality is compatible with the folding identification of spaces with Floer operators

$$CF(\underline{L}) \cong CF(L_{01} \times L_{23} \times \ldots . L_{(k-1)k}, \sigma(L_{12} \times L_{34} \times \ldots L_{k0}))$$

from 5.2.10 (c) since

$$\sum w(L_{(j-1)j}, M_{j-1}, M_j) = w(L_{01} \times L_{23} \times \ldots . L_{(k-1)k}) - w(L_{12}^- \times L_{34}^- \times \ldots L_{k0}^-)).$$

5.3. **Fukaya category.** The Fukaya category is defined by counting pseudoholomorphic disks with boundary lying in Lagrangian submanifolds. In this section we explain how the results on orientations naturally define the Fukaya category and its quilted version over the integers.

First we recall the definition of Stasheff's associahedron via nodal disks. A *nodal disk* D is a contractible space obtained from a union of disks $D_i, i = 1, \ldots, l$ (called the components of D) by identifying pairs of points $w_j^+, w_j^-, j = 1, \ldots, k$ on the boundary (the *nodes* in the resulting space)

$$D = \sqcup_{i=1}^l D_i/(w_j^+ \sim w_j^-, j = 1, \ldots, k)$$

so that each node $w_j \in D$ belongs to exactly two disk components $D_{i_-(j)}, D_{i_+(j)}$. A *set of markings* is a set $\{z_0, \ldots, z_d\}$ of the boundary ∂D in counterclockwise order, distinct from the singularities. A *marked* nodal disk is a nodal disk with markings. A *morphism of marked nodal disks* from $(D, \underline{z})$ to $(D', \underline{z}')$ is a homeomorphism $\varphi : D \to D'$ restricting to a holomorphic isomorphism $\varphi|_{D_i}$ on each component $i = 1, \ldots, l$ and mapping the marking z_j to z_j'. A marked nodal disk $(D, \underline{z})$ is *stable* if it has no automorphisms or equivalently if each disk component $D_i \subset D$ contains at least three nodes or markings. The *combinatorial type* of a nodal disk with markings is the tree

$$\Gamma = (\mathrm{Vert}(\Gamma), \mathrm{Edge}(\Gamma)), \quad \mathrm{Edge}(\Gamma) = \mathrm{Edge}_{<\infty}(\Gamma) \sqcup \mathrm{Edge}_\infty(\Gamma)$$

obtained by replacing each disk with a vertex $v \in \mathrm{Vert}(\Gamma)$, each node with a finite edge $e \in \mathrm{Edge}_{<\infty}(\Gamma)$, and each marking with a semi-infinite edge $e \in \mathrm{Edge}_\infty(\Gamma)$. The semi-infinite edges $\mathrm{Edge}_\infty(\Gamma)$ are labelled by $0, \ldots, d$ corresponding to which marking they represent.

We introduce the following notation for moduli spaces of disks. For each combinatorial type Γ let $\mathcal{R}^d_\Gamma$ denote the set of isomorphism classes of semistable nodal $d+1$-marked disks of combinatorial type Γ, and

$$\overline{\mathcal{R}}^d = \bigcup_\Gamma \mathcal{R}^d_\Gamma$$

the moduli space of stable disks. Each stratum is naturally oriented by identifying each open stratum $\mathcal{R}^d \cong (0,1)^{d-3}$ by identifying the unit disk with upper half space, so that the first and last markings map to $0, 1$ while the zeroth maps to infinity.

Associated to a tuple of Lagrangians is a moduli space of pseudoholomorphic polygons. Namely given Lagrangian branes $L_0, \ldots L_d$ intersecting pairwise transversally and intersection points $x_j \in L_{(j-1)} \cap L_j$ for $j = 0, \ldots, d$ let $\mathcal{M}(x_0, \ldots, x_d)$ denote the moduli space of stable disks S equipped with pseudoholomorphic maps $u : S \to M$ mapping the part of the boundary between z_{j-1} and z_j to L_j. For a comeager set of almost complex structures the moduli space is smooth, the zero-dimensional component is finite, and the one-dimensional component is compact up to bubbling off Floer trajectories at the markings. More precisely, the one-dimensional component $\overline{\mathcal{M}}^d(\underline{x})_1$ has a compactification as a one-manifold with boundary the union

$$\partial \overline{\mathcal{M}}^d_1(\underline{x}) = \bigcup_\Gamma \mathcal{M}^d_{\Gamma,1}(\underline{x}) \tag{57}$$

of strata $\mathcal{M}^d_{\Gamma,1}(\underline{x})$ of $\overline{\mathcal{M}}^d_1(\underline{x})$ corresponding to trees with two vertices (where either (1) Γ is stable with two vertices, or (2) Γ is unstable and corresponds to bubbling of a Floer trajectory).

In the stable range each moduli space of polygons is oriented by identifying its determinant line with the product of determinant lines for the associahedron and the linearized operator. By deforming the parametrized linear operator to the linearized operator plus a trivial operator and bubbling off marked disks on each strip like end we may identify the determinant line with the product of determinant lines for the associahedron and the linearized operator for $d \geq 3$:

$$\det(T_{[C,u]}\mathcal{M}^d(x_0,\ldots,x_d)_0) \cong \det(T_{[C]}\mathcal{R}^d) \otimes \det(D_u).$$

Using (40), we re-write the determinant line of the linearized operator as follows. For any intersection point x_j let

$$\mathcal{D}^{\pm}_{x_j} = \det(D^{\pm}_{e_j}) \otimes \Lambda^{\max}(\Gamma_{e_j}(0)^{\vee})$$

denote the determinant line associated to x_j, given as the tensor product of the determinant line of the Lagrangian with the determinant of a Cauchy-Riemann operator on the disk with a single end depending on a choice of path from $T_{x_j}L_{j-1}$ to $T_{x_j}L_j$ in $T_{x_j}M$, as in (40). The determinant line for the surface with boundary but without ends is the product of determinant lines for the trivial operator, isomorphic to $\det(TL)$, and the determinant line on a complex space which may be ignored for the purposes of the sign computation. Thus (omitting tensor products to save space)

$$\det(T\mathcal{M}^d(x_0,\ldots,x_d)_0) \to \det(T\mathcal{R}^d)\mathcal{D}^-_{x_1}\ldots\mathcal{D}^-_{x_d}\det(TL)\mathcal{D}^+_{x_0}. \tag{58}$$

The chosen orientations for the ends, the associahedron, and the Lagrangian induce an orientation on the moduli space $\mathcal{M}^d(x_0,\ldots,x_d)$. In the unstable range $d \leq 2$ a similar construction, after replacing $T\mathcal{R}^d$ with the Lie algebra of the automorphism group of a disk with 1 or 2 markings on the boundary, gives an orientation on the moduli space of holomorphic polygons.

Definition 5.3.1. The *Fukaya category* $\mathrm{Fuk}(M) := \mathrm{Fuk}(M, \mathrm{Lag}^N(M), \omega, b)$ is defined as follows:

(a) The objects of $\mathrm{Fuk}(M)$ are admissible Lagrangian branes in M with background class b.
(b) The morphism spaces of $\mathrm{Fuk}(M)$ are the $\mathbb{Z}_N$-graded Floer cochain groups

$$\mathrm{Hom}(L, L') := CF(L, L').$$

(c) The composition law in the category $\mathrm{Fuk}(M)$ is defined by counting holomorphic polygons:

$$\mu^d : \mathrm{Hom}(L^0, L^1) \times \ldots \times \mathrm{Hom}(L^{d-1}, L^d) \to \mathrm{Hom}(L^0, L^d)$$

by

$$\mu^d(\langle x_1\rangle, \ldots, \langle x_d\rangle) = (-1)^{\heartsuit} \sum_{u \in \mathcal{M}^d(x_0,\ldots,x_d)_0} \epsilon(u)\langle x_0\rangle \tag{59}$$

where

$$\heartsuit = \sum_{i=1}^{d} i|x_i|. \tag{60}$$

Theorem 5.3.2. *The higher compositions in* Fuk(M) *satisfies the* A_∞ *axiom. The resulting category* Fuk(M) *is independent, up to homotopy equivalence of* A_∞ *categories, of choices of perturbation data and orientations.*

Proof. We consider the difference of orientations in the identification

$$\partial \overline{\mathcal{M}}_1^d(\underline{x}) = \bigcup_\Gamma \mathcal{M}^d_{\Gamma,1}(\underline{x})$$

of (57). Let $x_j \in \mathcal{I}(L^j, L^{j+1})$ for $j = 0, \ldots, d$ indexed mod $d+1$. Consider the gluing map

$$\mathcal{M}^m(y, x_{n+1}, \ldots, x_{n+m})_0 \times \mathcal{M}^{d-m+1}(x_0, x_1, \ldots, y, \ldots, x_d)_0 \to \mathcal{M}^d(x_0, \ldots, x_d)_1. \tag{61}$$

The gluing map (61) takes the form (omitting tensor products from the notation to save space)

$$\begin{aligned} \det(T\mathcal{R}^m)\mathcal{D}^-_{x_{n+1}} \ldots \mathcal{D}^-_{x_{n+m}} \det(TL)\mathcal{D}^+_y \\ \det(T\mathcal{R}^{d-m+1})\mathcal{D}^-_{x_1} \ldots \mathcal{D}^-_y \ldots \mathcal{D}^-_{x_d} \det(TL)\mathcal{D}^+_{x_0} \\ \to \det(T\mathcal{R}^d)\mathcal{D}^-_{x_1} \ldots \mathcal{D}^-_{x_d} \det(TL)\mathcal{D}^+_{x_0}. \end{aligned} \tag{62}$$

To determine the sign of this map, first note that the gluing map $(0,\epsilon)\times\mathcal{R}^m\times\mathcal{R}^{d-m+1} \to \mathcal{R}^d$ on the associahedra is given in coordinates (using the automorphisms to fix the location of the first and last marked point for $\mathcal{R}^m$ and $\mathcal{R}^{d-m+1}$) by

$$\begin{aligned} &(\delta, (z_2, \ldots, z_{m-1}), (w_2, \ldots, w_{d-m})) \\ &\quad \to (w_2, w_3, \ldots, w_{n+1}, w_{n+1} + \delta z_2, \ldots, w_{n+1} + \delta z_{m-1}, w_{n+1} + \delta, w_{n+2}, \ldots, w_{d-m}). \end{aligned} \tag{63}$$

This map acts on orientations by $mn + m + n + 1$ mod 2. These signs combine with the contributions $\heartsuit$ in the definition of μ^d, a contribution $m(d-m)$ from permuting $\det(T\mathcal{R}^m)$ with $\mathcal{D}^-_{x_1} \ldots \mathcal{D}^-_y \ldots \mathcal{D}^-_{x_d} \det(TL)\mathcal{D}^+_{x_0}$, and a contribution $m(|y| + \sum_{i\le n} |x_i|)$ from permuting the ends into their correct order. Comparing the contributions from $(-1)^\heartsuit$ with an overall sign of $(-1)^\square$, where $\square = \sum_{k=1}^d k|x_k|$, one obtains a sign contribution of (-1) to the power $|y| + nm + (m-1)(\sum_{k=1}^{d-m-n} |x_{n+m+k}|)$. On the other hand, the sign in the A_∞ axiom contributes $\sum_{k=1}^n (|x_k| - 1)$. Combining the signs one obtains in total

(64)

$$\begin{aligned} m\left(\sum_{k=m+n+1}^{d} |x_k|\right) + mn + 1 - n - m + nm + (m-1)\left(\sum_{k=m+n+1}^{d} |x_k|\right) + |y| + \sum_{k=1}^{n}(|x_k| - 1) \\ = \sum_{k=1}^{n}(|x_k| - 1) + |y| + 1 + \sum_{k=m+n+1}^{d} |x_k| \cong_2 1 + \sum_{k=1}^{d} |x_k| \end{aligned}$$

which is independent of n. The gluing computation in unstable range $d - m + 1 \le 2$ or $m \le 2$ is similar and the A_∞ -associativity relation follows. □

Remark 5.3.3. (a) (Duals) Let M^- denote M with symplectic form reversed. We have a natural identification of objects of $\mathrm{Fuk}(M)$ and $\mathrm{Fuk}(M^-)$ obtained by considering each brane L as a brane L^- for M^-. If all minimal Maslov numbers are divisible by 4 then then this identification extends to isomorphism of categories

$$\mathrm{Fuk}(M) \to \mathrm{Fuk}(M^-).$$

In general, continuing Remark 5.2.4 the category $\mathrm{Fuk}(M,b)$ is isomorphic to the opposite category of $\mathrm{Fuk}(M^-, b+w_2(M))$ as observed by Fukaya. Indeed by Remark 5.2.4 we have isomorphisms

$$\begin{aligned} \mathrm{Hom}_{\mathrm{Fuk}(M,b)}(L_0,L_1) &\cong CF(L_0,L_1) \\ &\cong CF(L_1^-,L_0^-) \\ &\cong \mathrm{Hom}^{\mathrm{op}}_{\mathrm{Fuk}(M^-,b+w_2(M))}(L_0,L_1). \end{aligned}$$

(b) (Disjoint Unions) Let $\mathcal{C}_j, j = 0,1$ be categories enhanced in groups. Define a *disjoint union* category $\mathcal{C}$ by taking an object to be an object in $\mathcal{C}_0$ or $\mathcal{C}_1$ and morphism groups to be trivial unless the two objects are objects of the same category $\mathcal{C}_j, j = 0,1$. Suppose that $M_j, j = 0,1$ are compact monotone symplectic manifolds, and $M = M_0 \sqcup M_1$. Then

$$\mathrm{Fuk}(M_0 \sqcup M_1) = \mathrm{Fuk}(M_0) \sqcup \mathrm{Fuk}(M_1).$$

(c) (Products) Let $\mathcal{C}_j, j = 0,1$ be categories enhanced in $\mathbb{Z}_N$ graded cochain complexes. The *product category* $\mathcal{C}$ is the category whose objects are pairs of objects of $\mathcal{C}_0$ and $\mathcal{C}_1$, and whose morphism spaces are graded tensor product of morphism spaces of $\mathcal{C}_j, j = 0,1$. Let $M_j, j = 0,1$ be compact monotone symplectic manifolds and $M = M_0 \times M_1$. Then $H(\mathrm{Fuk}(M))$ is the category obtained by taking the cohomology of the cochain-level categories underlying $H(\mathrm{Fuk}(M_0)), H(\mathrm{Fuk}(M_1))$. In particular, if all cohomologies are torsion-free (for example, by working over a field) then

$$H(\mathrm{Fuk}(M_0 \times M_1)) = H(\mathrm{Fuk}(M_0)) \otimes H(\mathrm{Fuk}(M_1)).$$

The A-infinity version of this result is addressed in Amorim [1].

Remark 5.3.4. (a) (Extension to quilts) The quilted versions are similar. In particular, there is a quilted Fukaya category $\mathrm{Fuk}^{\#}(M)$ whose objects are generalized Lagrangian branes $\underline{L}$ (sequences of correspondences from a point to M) equipped with relative brane structures, and whose morphism spaces are quilted Floer cochain groups $\mathrm{Hom}(\underline{L}, \underline{L}') = CF(\underline{L}, \underline{L}')$, defined over the integers.

(b) (Extension to Lefschetz fibrations) In the case of a Lefschetz fibration E over S with Lagrangian boundary condition Q, the gluing signs are the same as for pseudoholomorphic surfaces. In the case Q is oriented and has minimal Maslov number at least two, working with rational coefficients (E,Q) defines a relative invariant

$$\Phi_{E,Q} : \otimes_{e \in \mathcal{E}_-(\underline{S})} HF(\underline{L}_e) \to \otimes_{e \in \mathcal{E}_+(\underline{S})} HF(\underline{L}_e)$$

mapping the tensor product of Floer homologies for the incoming ends to the product for the outgoing ends.

5.4. **Inserting a diagonal for pseudoholomorphic quilts.** In this and the following section we investigate the effect of composition of seam conditions on holomorphic quilt invariants. The first step is to investigate the effect of the insertion of a diagonal seam insertion.

Definition 5.4.1. (Inserting a diagonal Lagrangian seam condition) A triple $(\underline{S}^{\vartriangle}, \underline{M}^{\vartriangle}, \underline{L}^{\vartriangle})$ is obtained from a labelled quilted surface $(\underline{S}, \underline{M}, \underline{L})$ by *inserting a diagonal* iff

(a) the quilted surface $\underline{S}^{\vartriangle}$ is obtained from $\underline{S}$ by inserting a new seam σ into a patch S_p of $\underline{S}$;
(b) the labels $\underline{M}^{\vartriangle}, \underline{L}^{\vartriangle}$ are obtained by inserting a diagonal seam condition in the previous subsection. That is, if M_p is the symplectic manifold labelling S_p then the patches S_p', S_p'' are labelled M_p, and the new seam is labelled Δ_{M_p}.

Proposition 5.4.2. (Isomorphism of Floer homologies and relative invariants for insertion of separating diagonals) *Suppose that the new seam is inserted into the component S_p, and that the new seam is separating. Then there exists a collection of isomorphisms*

$$HF(\underline{L}_{\underline{e}}) \to HF(\underline{L}^{\vartriangle}_{\underline{e}^{\vartriangle}}), \quad e \in \mathcal{E}(\underline{S}).$$

In the case of rational coefficients these intertwine with the relative invariants $\Phi_{\underline{S}}, \Phi_{\underline{S}^{\vartriangle}}$ defined by $\underline{S}, \underline{S}^{\vartriangle}$.

Proof. We take the perturbation data for $\underline{L}^{\vartriangle}$ to be induced by perturbation data for $\underline{L}$. Then $\mathcal{I}(\underline{L}_{\underline{e}})$ and $\mathcal{I}(\underline{L}^{\vartriangle}_{\underline{e}^{\vartriangle}})$ are canonically in bijection. The Proposition follows from the linear case in the previous paragraph, taking the map on cochain complexes to be the identity on cochain complexes. □

Remark 5.4.3. (The spin case) In the case that M_p is spin, the diagonal Δ_p is also spin. So Δ_p has a relative spin structure with background classes $(0, 0)$. Thus the periodic Floer cohomology $HF(\Delta_p)$ is well-defined. (In general without the spin assumption, the quilted Floer cohomology $HF(\Delta_p)$ may be defined as the periodic Floer cohomology $HF(\mathrm{Id}_{M_p})$ of the identity on M_p, that is, treating Δ_p as the generalized Lagrangian correspondence of length 0.) The isomorphism of Floer homology groups can be defined as follows from the isomorphisms $HF(\Delta_p) \to QH(M_p)$. Let

$$\phi_e : HF(\underline{L}_{\underline{e}}) \to HF(\underline{L}^{\vartriangle}_{\underline{e}(\vartriangle)}), \quad \psi_e : HF(\underline{L}^{\vartriangle}_{\underline{e}(\vartriangle)}) \to HF(\underline{L}_{\underline{e}})$$

denote the morphism associated to the quilted surface shown in Figure 12 resp. the reversed surface. In other words, to the infinite strip we add a cylindrical end in the component

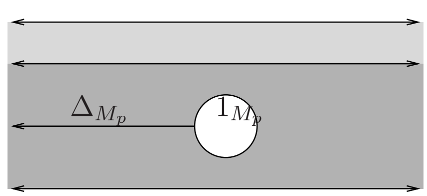

FIGURE 12. Isomorphism of Floer homologies after inserting a seam

separated by the seam σ, and insert at that cylindrical end the identity in $1_{M_p} \in HF(\Delta_{M_p})$. The identities

$$\psi_e \phi_e = 1_{HF(\underline{L}_{\underline{e}})}, \quad \phi_e \psi_e = 1_{HF(\underline{L}_e^{\Delta})}$$

follow from the results of the previous section applied to the surface on the inner circle in Figure 13. Compatibility with the relative invariants is proved in the same way. This ends the remark.

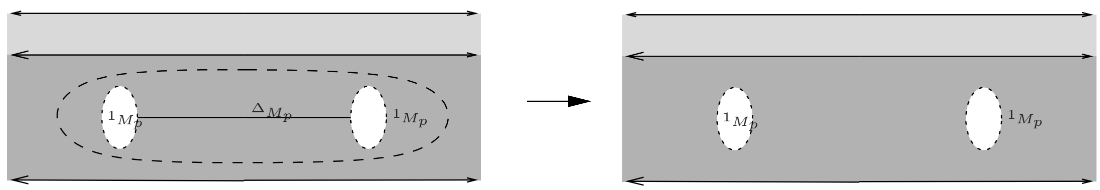

Figure 13. Removing a seam

5.5. **Orientations for compositions of Lagrangian correspondences.** In this final step we investigate the effect of replacing a triple of adjacent seam conditions, the middle of which is a diagonal, with the composed condition.

Definition 5.5.1. (Composed Lagrangian seam conditions) Let $\underline{S}$ denote a quilted surface, $\underline{M}$ a set of symplectic manifolds for the components of $\underline{S}$, and $\underline{L}$ a collection of Lagrangian and seam conditions. Suppose that $\underline{S}$ contains a pair of adjacent components M_1, M_1 diffeomorphic to infinite strips, with boundary conditions L_{01}, Δ_1, L_{12}. Let $\underline{S}^\circ$ denote the surface obtained by removing the M_1 components. The *composed Lagrangian seam conditions*

$$\underline{L}^\circ = \underline{L}/(L_{01}, \Delta_1, L_{12}) \mapsto L_{01} \circ L_{12})$$

assuming that the composition is smooth and embedded by the projection onto $M_0^- \times M_2$. If L_{01}, L_{12} are equipped with relative spin structures, then $L_{02} := L_{01} \circ L_{12}$ inherits a relative spin structure with background class shifted by $w_2(M_2)$.

Proposition 5.5.2. (Geometric composition theorem) *Suppose that $\underline{S}^\circ$ is obtained from $\underline{S}$ and the quilt data for $\underline{S}^\circ$ is obtained from quilt data $\underline{M}, \underline{L}$ for $\underline{S}$ by replacing a triple of seams L_{01}, Δ_1, L_{12} with the geometric composition L_{02}. Suppose that $\underline{M}$ are compact monotone with the same monotonicity constants and $\underline{L}, \underline{L}^\circ$ are admissible correspondences so that the quilted Floer cohomologies and relative invariants are well-defined. For each quilted end $\underline{e}$ changed by the replacement to a quilted end $\underline{e}^\circ$ there exists an isomorphism*

$$HF(\underline{L}_{\underline{e}}) \to HF(\underline{L}^\circ_{\underline{e}^\circ})$$

such that the tensor products over the negative and positive ends of $\underline{S}, \underline{S}^\circ$ intertwine the relative invariants $\Phi_{\underline{S}}, \Phi_{\underline{S}^\circ}$ for $\underline{S}, \underline{S}^\circ$.

Proof. For $\mathbb{Z}_2$ coefficients this was proved in Theorem 5.4.1 of [26]. The map constructed in Section 4 of [26] linearizes to the projection onto the components except the components labelled M_1, up to a small correction. By Lemma 4.6.3 and the identification of the tangent spaces of the various moduli spaces with kernels of Cauchy-Riemann operators with totally real boundary and seam conditions, the isomorphism constructed in [26] is orientation preserving, hence the proposition. □

Corollary 5.5.3. *Given Lagrangian correspondences $L_{01}, L_{12}, L_{02}, L_{20}$ with admissible brane structures such that $L_{02} := L_{01} \circ L_{12}$ is smooth and embedded, the canonical bijection*

$$\mathcal{I}(L_{01}, \Delta_1, L_{12}, L_{20}) \to \mathcal{I}(L_{02}, L_{20})$$

induces an isomorphism

$$HF(L_{01}, \Delta_1, L_{12}, L_{20}) \to HF(L_{02}, L_{20})$$

of quilted Floer cohomology groups with integer coefficients.

Functors for Lagrangian correspondences equipped with brane structures were constructed in [12] by the authors together with S. Ma'u. For each admissible Lagrangian correspondence $L_{01} \subset M_0^- \times M_1$ equipped with a brane structure counting quilts with seam in L_{01} defines an A_∞ functor

$$\Phi(L_{01}) : \mathrm{Fuk}^\#(M_0) \to \mathrm{Fuk}^\#(M_1)$$

acting in the expected way on Floer cohomology: for Lagrangian branes $L_0 \subset M_0, L_1 \subset M_1$ there is an isomorphism with $\mathbb{Z}_2$-coefficients

$$H\,\mathrm{Hom}(\Phi(L_{01})L_0, L_1) \cong HF(L_0 \times L_1, L_{01})$$

Parametrized versions of the arguments for invariance under geometric composition provide the signs necessary for the following theorem:

Theorem 5.5.4. (Geometric composition theorem) *Suppose that M_0, M_1, M_2 are monotone symplectic manifolds with the same monotonicity constant. Let $L_{01} \subset M_0^- \times M_1, L_{12} \subset M_1^- \times M_2$ be admissible Lagrangian correspondences with relative spin structures and gradings such that $L_{01} \circ L_{12}$ is smooth, embedded in $M_0^- \times M_2$, and admissible. Then there exists a homotopy of A_∞ functors*

$$\Phi(L_{12}) \circ \Phi(L_{01}) \cong \Phi(L_{01} \circ L_{12}) \circ \Phi(\Delta_2) \cong \Phi(\Delta_0) \circ \Phi(L_{01} \circ L_{12}).$$

This "composition commutes with categorification" theorem is, in some sense, the main result of [12].

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