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AN OPEN QUANTUM KIRWAN MAP

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Abstract

We construct, under semipositivity assumptions, an A-infinity morphism from the equivariant Fukaya algebra of a Lagrangian brane in the zero level set of a moment map to the Fukaya algebra of the quotient brane. The map induces a map between Maurer–Cartan solution spaces, and intertwines the disk potentials. As an application, we show the weak unobstructedness and Floer nontriviality of various Lagrangians in symplectic quotients. In the semi-Fano toric case we give another proof of the results of Chan–Lau–Leung–Tseng [CLLT17], by showing that the potential of a Lagrangian toric orbit in a toric manifold is related to the Givental–Hori–Vafa potential by a change of variable. We also reprove the results of Fukaya–Oh–Ohta–Ono [FOOO10] on weak unobstructedness of these toric orbits. In the case of polygon spaces we show the existence of weakly unobstructed and Floer nontrivial products of spheres.

Contents

1. Introduction

The *gauged sigma model* in physics, see for example, Witten [Wit93], is a theory unifying Gromov–Witten theory of hypersurfaces or complete intersections and A-model Landau–Ginzburg theory. The model is related to the non-linear sigma model of the symplectic quotient by a *renormalization* procedure. In mathematics, there have been a number of versions of this relationship. Givental's work produced a formula relating the closed-string correlation functions, using fixed point techniques. A geometric approach to defining the and relating invariants was introduced by Cieliebak–Gaio–Salamon [CGS00] and Mundet i Riera [Mun99, Mun03],

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and followed by [MT09], in the case analogous to symplectic Gromov–Witten theory. In this approach, the correlators of the gauged sigma model are defined by counting *vortices*, which are equivariant analogues of pseudoholomorphic curves. In the low energy/large area limit, this theory "converges" in a rough sense to the Gromov–Witten theory of the symplectic quotients, as studied in [GS05]. An important advantage of the gauged theory over the usual Gromov–Witten theory of the quotients lies in the fact that the equivariant target space (in many interesting cases just a Euclidean space) often has simpler geometry and topology than the symplectic quotient. It simplifies many technicalities, and especially bypasses the formidable virtual technique treatment of transversality ([LT98] [FO99] [HWZ07] [Par16], etc.). For example, this idea has been applied in Floer theory (see [Fra04a] [Xu16]).

In this paper we use vortices with Lagrangian boundary conditions to study the relationship between the Lagrangian Floer theories in the equivariant target space and in the quotient, extending the work in [Woo11], [WX17], [VXne], [Xu]. The precise set-up is as follows. Let X be a Kähler manifold having symplectic form ω_X and complex structure J_X. Let K be a compact Lie group with Lie algebra $\mathfrak{k}$ acting on X preserving the complex structure and the symplectic form. Let $\mu : X \to \mathfrak{k}^\vee$ be a proper moment map of the action so that K acts freely on $\mu^{-1}(0)$. Let

$$\bar{X} = \mu^{-1}(0)/K$$

be the symplectic quotient, which is a smooth compact Kähler manifold. The action of K on X extends to a holomorphic action on X of a complex reductive Lie group G with maximal compact K. The Kempf–Ness theorem [KN79] identifies the symplectic quotient $\bar{X}$ with the geometric invariant theory quotient $X /\!\!/ G$, which is the variety $\mathrm{Proj}(\mathbb{C}[X]^G)$ whose coordinate ring is the invariant part $\mathbb{C}[X]^G$ of the coordinate ring $\mathbb{C}[X]$ of X. The coordinate ring $\mathbb{C}[X]$ depends on a choice of a linearization i.e. a G-equivariant ample line bundle over X equipped with a K-invariant connection whose curvature resp. moment map is the symplectic form ω_X resp. the moment map μ. In particular X and $\bar{X}$ are rational symplectic manifolds.

There is a natural correspondence between Lagrangians in the quotient and invariant Lagrangians contained in the zero level set $\mu^{-1}(0)$, and we wish to compare the Fukaya algebras associated to these data. Let $L \subset X$ be a compact K-invariant Lagrangian submanifold that is contained in $\mu^{-1}(0)$. The quotient $\bar{L} = L/K$ is then a Lagrangian submanifold in $\bar{X}$. We assume that $\bar{L}$ is equipped with a brane structure: an orientation, a spin structure and a local system; each of these structures induces a corresponding equivariant structure on L. Let $\mathrm{Fuk}(\bar{L})$ denote the Fukaya A_∞ algebra in the quotient $\bar{X}$, with composition maps

$$\boldsymbol{n}_k : \mathrm{Fuk}(\bar{L})^k \to \mathrm{Fuk}(\bar{L}), \quad k \geqslant 0$$

defined by counting treed holomorphic disks in $\bar{X}$. The Fukaya algebra has a family of deformations parametrized by quantum cohomology classes $\mathfrak{a} \in QH^*(\bar{X})$, denoted by $\mathrm{Fuk}^{\mathfrak{a}}(\bar{L})$, called the *bulk deformation* as in [FOOO11], defined by counting holomorphic disks with interior markings mapping to a cycle representative of $\mathfrak{a}$. The A_∞ structure contains many symplectic geometric properties of the Lagrangian $\bar{L}$. For example, when the Fukaya algebra is weakly unobstructed one can define a Floer cohomology of $\bar{L}$, which generalizes Floer's original definition [Flo88a]. The nontriviality of the Floer cohomology is often related to the critical points of the so-called potential function of the A_∞ algebra. The family of bulk deformations is also related to the B-side Landau–Ginzburg theory of the superpotential via mirror symmetry (see [FOOO16]). On the other hand, the Lagrangian brane upstairs $L \subset X$ has an equivariant Fukaya algebra $\mathrm{Fuk}^K(L)$ with composition maps

$$\boldsymbol{m}_k : \mathrm{Fuk}^K(L)^{\otimes k} \to \mathrm{Fuk}^K(L), \quad k \geqslant 0$$

in Woodward [Woo11], by counting holomorphic disks in X modulo the K-action. The equivariant Fukaya algebra and the Fukaya algebra of the quotient have the same underlying space of chains, but are defined using different counts. In the toric case, the work of [FOOO11] [CLLT17] shows that the potential function of $\mathrm{Fuk}(\bar{L})$ is related to the Givental–Hori–Vafa potential via a change of variable. This relation was shown in [CLLT17] in the semi-Fano case, via the Seidel representation; it was shown for general compact toric manifolds in [FOOO11]. The Givental–Hori–Vafa potential is the potential function of $\mathrm{Fuk}^K(\bar{L})$ as shown in [Woo11].

Our aim is to construct an A_∞ morphism from $\mathrm{Fuk}^K(L)$ to certain *bulk deformation* of $\mathrm{Fuk}(\bar{L})$. Using the construction of [CW15] both of the A_∞ algebras are constructed with strict units. We also make the following assumptions to simplify the transversality arguments.

Definition 1.1. The pair (X, L) is *admissible* if

(a) the manifold X is convex at infinity (see Definition 2.1) and either aspherical or monotone with minimal Chern number 2;
(b) the unstable locus $X^{\mathrm{us}} \subset X$ has real codimension at least four;
(c) the Lagrangian L is equivariantly rational (see Definition 3.1); and
(d) the target X and Lagrangian L satisfy the semi-Fano conditions in (2.4).

The main result of this paper relates the quasimap Fukaya algebra upstairs with a vortex bulk-deformation of the Fukaya algebra of the quotient. Since we are in the rational case, we can use the Novikov field with integral energy filtrations, namely the field of Laurent series in a formal variable $\boldsymbol{q}$:

$$\Lambda = \mathbb{Q}((\boldsymbol{q})) = \left\{ \sum_{i=m}^{+\infty} a_i \boldsymbol{q}^i \;\middle|\; a_i \in \mathbb{Q},\ m \in \mathbb{Z} \right\}.$$

Let $\Lambda_{\geqslant 0}$ resp. $\Lambda_{>0}$ be the subring of series starting from nonnegative resp. positive powers of $\boldsymbol{q}$. On the other hand, by the *vortex invariant* we mean a cohomology class $\mathfrak{c} \in QH^*(\bar{X}; \Lambda_{>0})$ that arises, roughly, as the image of the moduli space of affine vortices under the evaluation map at infinity. Denote by $\mathrm{Fuk}^{\mathfrak{c}}(\bar{L})$ the bulk-deformed Fukaya algebra in the quotient with bulk-deformation $\mathfrak{c}$.

Theorem 1.2. *Let (X, L) be an admissible pair as above. There exists a unital A_∞ morphism (called the* open quantum Kirwan morphism*)*

$$\boldsymbol{\varphi} = (\boldsymbol{\varphi}_k : \mathrm{Fuk}^K(L)^{\otimes k} \to \mathrm{Fuk}^{\mathfrak{c}}(\bar{L}), \quad k \geqslant 0)$$

defined by counting affine vortices over the upper half plane $\boldsymbol{H}$. The open quantum Kirwan morphism is a higher order deformation of the identity in the sense that

$$\boldsymbol{\varphi}_1 = \mathrm{Id} + O(\boldsymbol{q}); \quad \boldsymbol{\varphi}_k = O(\boldsymbol{q}), k \neq 1.$$

The unitality of the open quantum Kirwan map has important consequences for disk potentials of Lagrangians in quotient. Recall that an A_∞ algebra A with compositions $\boldsymbol{m}_k$ and strict unit $\boldsymbol{e}$ is called *weakly unobstructed* if the following *weak*

Maurer–Cartan equation has a solution

$$\sum_k \boldsymbol{m}_k(\underbrace{\boldsymbol{b},\ldots,\boldsymbol{b}}_{k}) \in \Lambda \boldsymbol{e}.$$

The solution set MC is equipped with a *potential function* $W : MC \to \Lambda$ defined by

$$W(\boldsymbol{b}) = \sum_{k\geqslant 0} \boldsymbol{m}_k(\underbrace{\boldsymbol{b},\ldots,\boldsymbol{b}}_{k})/\boldsymbol{e},\ \forall \boldsymbol{b} \in MC.$$

In the situation of Theorem 1.2, the A_∞ morphism $\boldsymbol{\varphi}$ induces a total map (formally)

$$\underline{\varphi} : \mathrm{Fuk}^K(L) \to \mathrm{Fuk}^{\mathfrak{c}}(\bar{L}), \qquad \underline{\varphi}(\boldsymbol{a}) = \sum_{k\geqslant 0} \varphi_k(\underbrace{\boldsymbol{a},\ldots,\boldsymbol{a}}_{k}).$$

By the A_∞ axiom for $\boldsymbol{\varphi}$ and unitality the total map $\underline{\varphi}$ maps the Maurer–Cartan solution spaces

$$\mathrm{Fuk}^K(L) \supset MC^K(L) \to MC^{\mathfrak{c}}(\bar{L}) \subset \mathrm{Fuk}^{\mathfrak{c}}(\bar{L})$$

and intertwines with the potential functions

$$W_L^K : MC^K(L) \to \Lambda, \qquad W_{\bar{L}}^{\mathfrak{c}} : MC^{\mathfrak{c}}(\bar{L}) \to \Lambda.$$

Corollary 1.3. *$\underline{\varphi}$ maps $MC^K(L)$ injectively into $MC^{\mathfrak{c}}(\bar{L})$. Therefore,* $\mathrm{Fuk}^{\mathfrak{c}}(\bar{L})$ *is weakly unobstructed provided that* $\mathrm{Fuk}^K(L)$ *is weakly unobstructed. Moreover,*

$$W_{\bar{L}}^{\mathfrak{c}}(\underline{\varphi}(\boldsymbol{b})) = W_L^K(\boldsymbol{b}),\ \forall \boldsymbol{b} \in MC^K(L).$$

Another consequence of Theorem 1.2 is the relation between the quasimap Floer cohomology of L and the bulk-deformed Floer cohomology of $\bar{L}$. Given a weakly bounding cochain $\boldsymbol{b} \in MC^K(L)$ and its image $\bar{\boldsymbol{b}} = \underline{\varphi}(\boldsymbol{b}) \in MC^{\mathfrak{c}}(\bar{L})$, the deformed A_∞ morphism $\varphi_k^{\boldsymbol{b}}$ gives a cochain map

$$\varphi_1^{\boldsymbol{b}} : (\mathrm{Fuk}^K(L), \boldsymbol{m}_1^{\boldsymbol{b}}) \to (\mathrm{Fuk}^{\mathfrak{c}}(\bar{L}), \boldsymbol{n}_1^{\bar{\boldsymbol{b}}}).$$

Since $\boldsymbol{\varphi}$ is a higher order deformation of the identity, $\varphi_1^{\boldsymbol{b}}$ is an invertible linear map. It implies the following result.

Corollary 1.4. *$\varphi_1^{\boldsymbol{b}}$ is an invertible cochain map and induces an isomorphism*

$$H^*(\mathrm{Fuk}^K(L), \boldsymbol{m}_1^{\boldsymbol{b}}) \simeq H^*(\mathrm{Fuk}^{\mathfrak{c}}(\bar{L}), \boldsymbol{n}_1^{\bar{\boldsymbol{b}}}).$$

These results have the following consequences for the existence of weakly bounding cochains. We set $\boldsymbol{b} = 0$ and for $b \in H^1(\bar{L}; \Lambda_{\geqslant 0})$ labelling a local system over $\bar{L}$ denote by $\boldsymbol{m}_{0,b}(1)$ resp. $\boldsymbol{n}_{0,b}(1)$ the curvatures of the corresponding Fukaya algebras $\mathrm{Fuk}^K(L)$resp. $\mathrm{Fuk}^{\mathfrak{c}}(\bar{L})$. Suppose that only Maslov two quasidisks contribute to $\boldsymbol{m}_{0,b}(1)$ or $\boldsymbol{n}_{0,b}(1)$. Denote the deformed curvature

$$\boldsymbol{m}_{0,b}(1) = W_L^K(b)\boldsymbol{x}_M \quad \big(\text{resp. } \boldsymbol{n}_{0,b}(1) = W_{\bar{L}}^{\mathfrak{c}}(b)\boldsymbol{x}_M\big).$$

Here $W_L^K(b)$ or $W_{\bar{L}}^{\mathfrak{c}}(b)$ is an element of $\Lambda_{>0}$ and $\boldsymbol{x}_M \in CF(\bar{L})$ is the Morse cochain represented by the unique maximum of the chosen Morse function on $\bar{L}$. In remainder of the introduction, abusing terminology, we also call the function

$$W_L^K : H^1(\bar{L}; \Lambda_{\geqslant 0}) \to \Lambda, \qquad W_{\bar{L}}^{\mathfrak{c}} : H^1(\bar{L}; \Lambda_{\geqslant 0}) \to \Lambda$$

defined above the *potential function*. (By the trick of homotopy units we can pretend that $\boldsymbol{x}_M = \boldsymbol{e}$ is the strict unit.) Our first result on unobstructedness uses the following assumption:

Hypothesis 1.5. All nonconstant J_X-holomorphic disks in (X, L) have positive Maslov indices.

Theorem 1.6. *Suppose X, L satisfy the conditions in Definition 1.1 and Hypothesis 1.5. Then for all $b \in H^1(\bar{L}; \Lambda_{\geqslant 0})$ labelling a local system, only Maslov index two quasidisks contribute to $\boldsymbol{m}_{0,b}(1)$. In particular, both $\mathrm{Fuk}_b^K(L)$ and $\mathrm{Fuk}_b^{\mathfrak{c}}(\bar{L})$ are weakly unobstructed and their potential functions are related as in Corollary 1.3.*

Our second main result on unobstructedness concerns a particular weakly bounding cochain and requires stronger assumptions:

Hypothesis 1.7.

(a) All nonconstant J_X-affine vortices over $\boldsymbol{H}$ have positive Maslov indices.
(b) All nonconstant $\bar{J}_X$-holomorphic disks in $(\bar{X}, \bar{L})$ have positive Maslov indices.
(c) The anticanonical divisor S in Definition 2.4 does not intersect L.

Together with Definition 2.4, Hypothesis 1.7 is roughly a generalization of the semi-Fano condition in the toric case.

Theorem 1.8. *Assume that X, L satisfy conditions of Definition 1.1, Hypothesis 1.5 and Hypothesis 1.7. Then for any $b \in H^1(\bar{L}; \Lambda_{\geqslant 0})$ labelling a local system, the equality $W_L^K(b) = W_{\bar{L}}^{\mathfrak{c}}(b)$ holds.*

We now turn to concrete examples in which we prove the weak unobstructedness of certain Lagrangians in symplectic quotients. The first case, which is well-studied, is that of compact toric manifolds. Let k be a positive integer. A $2k$-dimensional compact toric manifold $\bar{X}$ whose moment polytope has N faces can be realized as the geometric invariant theory quotient of $X = \mathbb{C}^N$ by an action of $G = (\mathbb{C}^*)^{N-k}$. For each Lagrangian torus fibre $\bar{L} \subset \bar{X}$, we can calculate explicitly the potential function of the equivariant Fukaya algebra $\mathrm{Fuk}_b^K(L)$ as in [Woo11]. Indeed for the standard complex structure on $X = \mathbb{C}^N$, the moduli spaces of stable holomorphic disks in X (of any tree configurations) is regular. The potential function is essentially determined by the counting of Maslov two disks, which gives rise to the Givental–Hori–Vafa potential. The combining with Corollary 1.3, we show that the potential function of $\mathrm{Fuk}_b^{\mathfrak{c}}(\bar{L})$ is related to the Givental–Hori–Vafa potential. The "coordinate change" of the Kähler parameter in this relation is essentially the class $\mathfrak{c}$ counting affine vortices above. This result extends the work of Chan–Lau–Leung–Tseng [CLLT17] using the Seidel representations, and also reformulated a result in [FOOO16]. We formulate our result in the following theorem.

Theorem 1.9. *Let $\bar{X}$ be a rational compact toric manifold viewed as a geometric invariant theory quotient of $X = \mathbb{C}^N$ by $G = (\mathbb{C}^*)^{N-n}$. Suppose X is* equivariantly semi-Fano. *Let $\bar{L} \subset \bar{X}$ be a rational toric orbit and $L \subset X$ be its invariant lift.*

(a) X and L satisfy Assumption 1.1, the semi-Fano condition of Definition 2.4 and Hypothesis 1.5, and for each $b \in H^1(\bar{L}; \Lambda_{\geqslant 0})$, the Fukaya algebras $\mathrm{Fuk}_b^K(L)$ and $\mathrm{Fuk}_b^{\mathfrak{c}}(\bar{L})$ are both weakly unobstructed.
(b) X and L satisfy Hypothesis 1.7, hence

$$W_L^K(b) = W_{\bar{L}}^{\mathfrak{c}}(b), \quad \forall\ b \in H^1(\bar{L}; \Lambda_{\geqslant 0}).$$

(c) $W_L^K(b)$ coincides with the Givental–Hori–Vafa potential (see [Giv95] [HV00]).

A second example is the case of polygon spaces, that is, quotients of products of two-spheres whose classical cohomology rings were studied by Kirwan [Kir84]. View

S^2 as the set of unit vectors $\boldsymbol{v} \in \mathbb{R}^3$ which is acted upon by the group of rotations $SO(3)$. The inclusion $S^2 \hookrightarrow \mathbb{R}^3 \simeq \mathfrak{so}(3)^\vee$ is a moment map for the action. Let X be the product of $(2k+3)$ copies of S^2 acted by the diagonal $SO(3)$-action, with a moment map

$$\mu(v_1, \ldots, v_{2k+3}) = v_1 + \cdots + v_{2k+3}.$$

The corresponding symplectic quotient can be viewed as the moduli space of equilateral $(2k+3)$-gons in $\mathbb{R}^3$ up to rigid body motion. Let $L \subset X$ be the Lagrangian

$$L = \underbrace{\bar{\Delta}_{S^2} \times \cdots \times \bar{\Delta}_{S^2}}_{k} \times \Delta_3$$

where $\bar{\Delta}_{S^2} \subset S^2 \times S^2$ is the anti-diagonal and $\Delta_3 \subset S^2 \times S^2 \times S^2$ is the set of triples of unit vectors (v_1, v_2, v_3) in $\mathbb{R}^3$ satisfying $v_1 + v_2 + v_3 = 0$. The submanifold L is an $SO(3)$-invariant Lagrangian of X which projects to a Lagrangian $\bar{L} \subset \bar{X}$ that is diffeomorphic to the product of k spheres.

Theorem 1.10. *The Fukaya algebras* $\mathrm{Fuk}^K(L)$ *and* $\mathrm{Fuk}^{\mathfrak{c}}(\bar{L})$ *are weakly unobstructed and the corresponding Floer cohomologies for suitable choice of weakly bounding cochain are isomorphic to* $H^*(\bar{L}; \Lambda)$.

The transversality of various relevant moduli spaces is achieved by generalizing the stabilizing divisor technique introduced by Cieliebak–Mohnke [CM07]. Their original framework allows one to construct genus zero Gromov–Witten invariants for compact rational symplectic manifolds. It is extended by by [CW17], [CW15] to Lagrangian Floer theory for rational Lagrangian submanifolds. In all the previous approaches, a single stabilizing divisor is sufficient. In our case we use multiple stabilizing divisors, in order to rule out objects having positive energy whose images are contained in the divisor, or those having less than three intersection points with the divisor. In the case of Cieliebak–Mohnke [CM07] or Charest–Woodward [CW15], one can rule out holomorphic spheres contained in the smooth divisor by perturbing the almost complex structure, via the deformation theory of holomorphic maps into the divisor. In our case, it becomes extremely technical to rule out affine vortices that are contained in the (singular) stabilizing divisor. We include the semi-Fano condition (Definition 2.4) to simplify the argument. However this condition is still not enough to rule out affine vortices that only intersect with the divisor at the infinity. Using multiple divisors guarantees the existence of at least one finite intersection point with the divisor. Results on hyperbolicity in the sense of algebraic geometry are also helpful in simplifying the arguments.

The organization of the paper is as follows. In Section 2 we recall many basic facts, including the analytic and geometric knowledge about vortices. In Section 3 we give some technical discussion about the Lagrangian, the stabilizing divisor and almost complex structures. In Section 4 we give a detailed account of the combinatorial structures we need, especially the definition of treed disks and coherent system of perturbations. In Section 5 we discuss the concept of coherent perturbation data. In Section 6 and Section 7 we study the properties of the moduli space of stable treed scaled vortices and proved the transversality result. In Section 8 we define the Fukaya algebras and prove our main theorem. In Section 9 and Section 10 we consider the examples of semi-Fano toric manifolds and polygon spaces. In Section 11 we explain how to achieve strict unitality for the A_∞ algebras by using weighted treed disks (originally introduced in [CW15]).

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2. Vortices

Symplectic vortices were introduced in Cieliebak–Gaio–Salamon [CGS00] and Mundet i Riera [Mun99, Mun03], as equivariant analogues of pseudoholomorphic curves. In this section we review the preliminary knowledge about vortices and pseudoholomorphic curves, as well as setting up the notations. Several basic assumptions about the target manifold geometry will also be provided.

2.1. **The target manifold.** We work under the assumptions listed in (1.1). In particular the K-action on the symplectic manifold X admits a moment map

$$\mu : X \to \mathfrak{k}^\vee.$$

Denote by $\Gamma(TX)$ the space of sections of TX, that is, vector fields. For $a \in \mathfrak{k}$, let $\mathcal{X}_a \in \Gamma(TX)$ be the vector field generating the infinitesimal action, defined in such a way that

$$\mathfrak{k} \to \Gamma(TX), \quad a \mapsto -\mathcal{X}_a$$

is a Lie algebra homomorphism. By our convention the moment map satisfies

$$\mathrm{d}(\mu \cdot a) \cdot Y = \omega_X(\mathcal{X}_a, Y)$$

for all $a \in \mathfrak{k}$ and all tangent vectors $Y \in TX$.

We identify $\mathfrak{k}$ with $\mathfrak{k}^\vee$ by an Ad-invariant metric, and consider the map μ to be $\mathfrak{k}$-valued. We denote by

$$\Omega_K(X) = \mathrm{Hom}(\mathfrak{k}, \Omega(X))^K, \quad \omega_X^K = \omega_X + \mu \in \Omega_K^2(X)$$

the space of equivariant de Rham forms resp. the equivariant symplectic form on X. By Cartan's equivariant de Rham theory (see [GS99] or [BGV03, Chapter 7]), the equivariant de Rham operator

$$\mathrm{d}_K : \Omega_K(X) \to \Omega_K(X), \quad (\mathrm{d}_K \alpha)(\xi) = \mathrm{d}(\alpha(\xi)) + \iota_{\mathcal{X}_\xi} \alpha$$

squares to zero and has cohomology identified with the equivariant cohomology

$$H_K(X; \mathbb{R}) \cong H(X_K; \mathbb{R}) \cong H_K(EK \times_K X; \mathbb{R}).$$

In particular, the equivariant symplectic form defines a cohomology class in the second equivariant cohomology $H_K^2(X; \mathbb{R})$. We assume as in Definition 1.1 that $0 \in \mathfrak{k}$ is a regular value of μ and the symplectic quotient

$$\bar{X} := X /\!\!/ K := \mu^{-1}(0)/K$$

is a free quotient, equipped with the induced Kähler form $\omega_{\bar{X}}$ and integrable almost complex structure $J_{\bar{X}}$.

[1] A complete argument would require a proof that the symplectic vortex invariants defined here are the same as the algebraic vortex invariants.

According to the Kempf–Ness theorem [KN79], the symplectic quotient $\bar{X}$ may be identified with Mumford's geometric invariant theory quotient

$$X /\!\!/ G = \operatorname{Proj}\left(\bigoplus_k H^0(R^{\otimes k})^G \right).$$

Here we assume that $R \to X$ is a *linearization* of the G-action, namely a holomorphic G-equivariant line bundle. We assume that $R \to X$ is equipped with a K-invariant Hermitian metric whose Chern connection is denoted by $\mathbb{A}$. Assume further that the equivariant curvature given by k times the equivariant symplectic form for some positive integer k:

$$\operatorname{curv}_K(\mathbb{A}) = (2\pi k/i)\omega_X^K \in \Omega_K^2(X).$$

In particular implies the rationality on the cohomology class of $[\omega_X^K] \in H_K^2(X;\mathbb{R})$.

Mumford's geometric invariant theory quotient [MFK94] is defined as follows. For the given linearization R, the semistable locus $X^{\rm ss}$ of X is the set of points

$$X^{\rm ss} = \left\{ x \ \middle|\ \exists k > 0,\ s \in H^0(R^{\otimes k})^G \text{ such that } s(x) \neq 0 \right\}$$

where some equivariant section of a tensor power of the linearization is non-vanishing. The complement of the semistable locus $X^{\rm ss}$ is the unstable locus, denoted by $X^{\rm us}$. Inside the semistable locus, the polystable locus $X^{\rm ps}$ consists of points x for which the orbits Gx is closed in $X^{\rm ss}$, and the stable locus $X^{\rm s} \subset X^{\rm ps}$ consists of polystable points x whose stabilizers G_x are finite. The GIT quotient is then the obtained from the semistable locus $X^{\rm ss}$ by quotienting by the orbit equivalence relation

$$x_1 \sim x_2 \iff \overline{Gx_1} \cap \overline{Gx_2} \cap X^{\rm ss} \neq \emptyset.$$

The Kempf–Ness theorem [KN79] is that the symplectic quotient $\bar{X}$ is homeomorphic to the GIT quotient $X/\!\!/G$ defined by Mumford. The assumption that 0 is a regular value of the moment map is equivalent to the equalities

$$X^{\rm ss} = X^{\rm st} = X^{\rm ps} = G\mu^{-1}(0).$$

We will then only use the symbol $X^{\rm st}$. There is a holomorphic submersion

$$\pi_X : X^{\rm st} \to \bar{X}$$

obtained by mapping any semistable point $x \in X^{\rm st}$ to the intersection $Gx \cap \mu^{-1}(0)$. The kernel of the linearization of this projection and its complement provides a G-equivariant orthogonal splitting

$$TX|_{X^{\rm st}} \simeq H_X \oplus G_X$$

where $G_X = K_X \oplus J_X K_X$ is the (integrable) holomorphic distribution spanned by infinitesimal G-actions, and H_X, which is the orthogonal complement of G_X, is isomorphic to $\pi_X^* T\bar{X}$.

In order to construct perturbations we also use almost complex structures that are compatible with the above projection in the following sense. Let $\mathcal{J}(X)$ be the space of smooth K-invariant, ω_X-compatible almost complex structures J on X whose restriction to $\mu^{-1}(0)$ respects the splitting $H_X \oplus G_X$ and that coincide with J_X outside some compact subset of the stable locus $X^{\rm st}$. Then each $J \in \mathcal{J}(X)$ induces an $\omega_{\bar{X}}$-compatible almost complex structure $\bar{J}$ on the symplectic quotient. Moreover, any $\bar{J}$ on the quotient can also be lifted to certain J in $\mathcal{J}(X)$.

We will assume the following condition that controls the compactness of the moduli spaces of vortices.

Definition 2.1. (cf. [CGMS02, Section 2.5]) Let ∇ be the Levi–Civita connection of the Kähler metric on X. X is called *convex at infinity* if there exists a K-invariant smooth proper function $f_X : X \to [0, +\infty)$ and a number $c_X > 0$ such that

$$\forall\, x \in X,\ f_X(x) \geqslant c_X \Longrightarrow \begin{cases} \langle \nabla_\xi \nabla f_X(x), \xi \rangle + \langle \nabla_{J_X\xi} \nabla f_X(x), J_X \xi \rangle \geqslant 0,\ \forall \xi \in T_x X, \\ \mathrm{d} f_X(x) \cdot J_X \mathcal{X}_{\mu(x)} \geqslant 0. \end{cases}$$

2.2. **Lagrangian branes.** In this section we describe the assumptions and additional structures on the Lagrangian submanifolds we consider. A Lagrangian submanifold $L \subset X$ is called a *K-Lagrangian* if L is contained in $\mu^{-1}(0)$ and is K-invariant. This class of submanifolds was studied in Woodward [Woo11] and Xu [Xu13], but is not the same as that studied in Frauenfelder [Fra04a, Fra04b]. For each K-Lagrangian L, the projection of L in $\bar{X}$ is a Lagrangian submanifold $\bar{L}$, and the correspondence $L \mapsto \bar{L}$ established a bijection between K-Lagrangians in X and Lagrangians in $\bar{X}$.

The structure maps in the Fukaya algebra depend on certain holonomies of local systems on the boundaries of the pseudoholomorphic disks. A Λ-local system on $\bar{L}$ is a flat Λ-bundle over $\bar{L}$ with structure group being the group of units $\Lambda^\times$ of Λ. This is equivalent to a holonomy homomorphism

$$\mathrm{Hol} : H_1(\bar{L}; \mathbb{Z}) \to \Lambda^\times.$$

We only consider the holonomy induced by an element $b \in H^1(\bar{L}; \Lambda_{\geqslant 0})$, i.e.,

$$\mathrm{Hol}(\gamma) = \exp\langle b, \gamma \rangle,\ \forall \gamma \in H_1(\bar{L}; \mathbb{Z}).$$

From now on we fix a compact embedded Lagrangian submanifold $\bar{L} \subset X$ together with an orientation and a spin structure on $\bar{L}$.

2.3. **Pseudoholomorphic maps.** We recall the definition of pseudoholomorphic maps and quasimaps. Let Σ be an oriented surface with possibly nonempty boundary $\partial\Sigma$ and complex structure $j : T\Sigma \to T\Sigma$. A *domain-dependent almost complex structure* on X is a map

$$J : \Sigma \to \mathcal{J}(X), \quad z \mapsto J_z.$$

A J-holomorphic map $u : \Sigma \to X$ with boundary in L is a smooth map satisfying the conditions

$$\overline{\partial}_J u(z) = \frac{1}{2}\big(\mathrm{d}u(z) + J_z \mathrm{d}u(z) j(z)\big),\ \forall z \in \Sigma, \quad u(\partial\Sigma) \subset L.$$

In the given situation that X has an action of a group K fixing the Lagrangian and almost complex structures J_z, K acts on the set of holomorphic maps by pointwise multiplication $(ku)(z) = ku(z)$. A K-orbit of pseudoholomorphic maps $[u] = \{ku, k \in K\}$ from Σ to X is called a *quasimap*.

A local model for the moduli space of quasimaps is obtained by a modification of the local model for pseudoholomorphic maps. It suffices for this paper to consider the case that Σ is $\boldsymbol{D}^2$, the unit disk. Choose a class $B \in H_2(X, L; \mathbb{Z})$. For a fixed integer $p > 2$, let $\tilde{\mathcal{B}}_{\boldsymbol{D}^2}(X, L; B)$ be the space of $W^{1,p}$ maps $u : \boldsymbol{D}^2 \to X$ with boundary condition $u(\partial \boldsymbol{D}^2) \subset L$ that represent the relative class B. Since the K-action is free on L, one has a free K-action on $\tilde{\mathcal{B}}_{\boldsymbol{D}^2}(X, L; B)$. The quotient, which is still a smooth Banach manifold, is denoted by $\mathcal{B}_{\boldsymbol{D}^2}(X, L; B)$. Consider the Banach vector bundle

$$\tilde{\mathcal{E}}_{\boldsymbol{D}^2}(X, L; B) \to \tilde{\mathcal{B}}_{\boldsymbol{D}^2}(X, L; B)$$

whose fibre over u is

$$\tilde{\mathcal{E}}_{\boldsymbol{D}^2}(X,L;B)_u = L^p(\boldsymbol{D}^2, \Lambda^{0,1} \otimes u^*TX).$$

If $g \in K$, then g induces a natural isomorphism between the fibres over u and $g \cdot u$. Therefore $\tilde{\mathcal{E}}_{\boldsymbol{D}^2}(X,L;B)$ descends to a Banach vector bundle

$$\mathcal{E}_{\boldsymbol{D}^2}(X,L;B) \to \mathcal{B}_{\boldsymbol{D}^2}(X,L;B).$$

Given a smooth domain dependent almost complex structure J on X, the Cauchy–Riemann operator defines a section

$$\overline{\partial}_J : \mathcal{B}_{\boldsymbol{D}^2}(X,L;B) \to \mathcal{E}_{\boldsymbol{D}^2}(X,L;B).$$

Its linearization at $[u] \in \mathcal{B}_{\boldsymbol{D}^2}(X,L;B)$ is a Fredholm operator denoted $D_{[u]}$ whose Fredholm index is $\mathbf{ind}(D_{[u]}) = \mathbf{dim}\bar{L} + \mathrm{Mas}(B)$ where

$$\mathrm{Mas} : H_2(X,L;\mathbb{Z}) \to \mathbb{Z}$$

is the Maslov index.

Pseudoholomorphic maps to the symplectic quotient are locally cut out by a Fredholm Cauchy-Riemann operator. We only consider the cases of the two-sphere $\Sigma = \boldsymbol{S}^2$ or disk $\Sigma = \boldsymbol{D}^2$. We view

$$\boldsymbol{S}^2 = \boldsymbol{C} \cup \{\infty\}. \quad \text{resp.} \quad \boldsymbol{D}^2 = \boldsymbol{H} \cup \{\infty\}$$

as the compactification of the complex plane $\boldsymbol{C}$ resp. upper half plane $\boldsymbol{H}$. Choose a class $B \in H_2(\bar{X},\bar{L};\mathbb{Z})$. Let $\mathcal{B}_{\boldsymbol{S}^2}(\bar{X};B)$ be the space of $W^{1,p}$ maps from $\boldsymbol{S}^2$ to $\bar{X}$ which represent the class B. Over $\mathcal{B}_{\boldsymbol{S}^2}(\bar{X};B)$ there is a bundle $\mathcal{E}_{\boldsymbol{S}^2}(\bar{X};B)$ whose fibre over $\bar{u} \in \mathcal{B}_{\boldsymbol{S}^2}(\bar{X};B)$ is

$$\mathcal{E}_{\boldsymbol{S}^2}(\bar{X};B)_u = L^p(\boldsymbol{S}^2, \Lambda^{0,1} \otimes \bar{u}^*T\bar{X}).$$

Similarly, let $\mathcal{B}_{\boldsymbol{D}^2}(\bar{X},\bar{L};B)$ be the space of $W^{1,p}$ maps from $\boldsymbol{D}^2$ to $\bar{X}$ with boundary condition $\bar{u}(\partial\boldsymbol{D}^2) \subset \bar{L}$, which represent the class B. Over this Banach manifold there is a bundle $\mathcal{E}_{\boldsymbol{D}^2}(\bar{X},\bar{L};B)$ whose fibre over $\bar{u} \in \mathcal{B}_{\boldsymbol{D}^2}(\bar{X},\bar{L};B)$ is $L^p(\boldsymbol{D}^2, \Lambda^{0,1} \otimes \bar{u}^*T\bar{X})$. Each domain-dependent almost complex structure J on X induces a domain-dependent almost complex structure $\bar{J}$ on $\bar{X}$. Then the Cauchy–Riemann equation defines a Fredholm section

$$\overline{\partial}_{\bar{J}} : \mathcal{B}_{\boldsymbol{S}^2}(\bar{X};B) \to \mathcal{E}_{\boldsymbol{S}^2}(\bar{X};B), \text{ or } \quad \overline{\partial}_{\bar{J}} : \mathcal{B}_{\boldsymbol{D}^2}(\bar{X},\bar{L};B) \to \mathcal{E}_{\boldsymbol{D}^2}(\bar{X},\bar{L};B).$$

The Fredholm index of its linearization at $\bar{u}$ is

$$\mathbf{ind}(D_{\bar{u}}) = \mathbf{dim}\bar{X} + \mathrm{Mas}(B) \quad \text{resp.} \quad \mathbf{dim}\bar{L} + \mathrm{Mas}(B)$$

in the closed case resp. open case. The moduli spaces of pseudoholomorphic spheres and pseudoholomorphic disks have a well-known topology and compactifications by stable maps [MS04].

2.4. **Affine vortices.** Vortices are equivariant generalizations of pseudoholomorphic maps, obtained from what we call *gauged maps* by minimizing a certain energy functional. Let Σ be a Riemann surface, not necessarily compact and possibly having a nonempty boundary $\partial\Sigma$. By a *gauged map* from $(\Sigma, \partial\Sigma)$ to (X, L) we mean a triple $\boldsymbol{v} = (P, A, u)$ where

$P \to \Sigma$ is a smooth K-bundle,
$A \in \mathcal{A}(P)$ is a smooth connection on P and
$u \in \Gamma(P(X))$ is a section of the associated X-bundle $P(X) = P \times_K X$

satisfying the boundary condition that $u(\partial\Sigma) \subset P \times_K L$. The group $\mathrm{Aut}(P)$ of automorphisms (i.e. gauge transformations) of P acts (from the right) on the space of gauged maps (P, A, u) by

$$g^*(P, A, u) = (P, g^* A, g(X)^{-1} u),\ g \in \mathrm{Aut}(P)$$

where $g(X) : P(X) \to P(X)$ is the associated fiber bundle automorphism.

Given an invariant almost complex structure, there is a natural analogue of the Cauchy–Riemann operator on the space of gauged maps. Given a gauged map (P, A, u), A induces a covariant derivative

$$d_A u \in \Omega^1(\Sigma, u^* P(TX)),$$

where $P(TX) \to P(X)$ is the vertical tangent bundle. Let J be a domain-dependent K-invariant almost complex structure on X. Then it induces a complex structure on $P(TX)$. A gauged map (P, A, u) is called *pseudoholomorphic* if the $(0, 1)$-part of $d_A u$, denoted by

$$\overline{\partial}_A u \in \Omega^{0,1}(\Sigma, u^* P(TX))$$

vanishes. The set of pseudoholomorphic gauged maps with fixed P is preserved by the group $\mathrm{Aut}(P)$ of gauge transformations; choosing an area form $\nu \in \Omega^2(\Sigma)$ on the surface, this set becomes formally a symplectic manifold and a moment map for the action of $\mathrm{Aut}(P)$ is

$$(P, A, u) \mapsto F_A + u^* P(\mu)\nu \in \Omega^2(\Sigma, P(\mathfrak{k})) \tag{2.1}$$

where $\nu \in \Omega^2(\Sigma)$ is an area form on Σ and $P(\mu) : P(X) \to P(\mathfrak{k})$ is the map on associated bundles induced by the moment map μ. A pseudoholomorphic gauged map (P, A, u) is called a *vortex* if (2.1) vanishes.

In this paper we only consider gauged maps over the complex plane, upper half plane or open subsets of such. We always view P as trivialized and fix the $z = s + \mathbf{i}t$. Then a gauged map can be written as a triple $\boldsymbol{v} = (u, \phi, \psi)$ where $u : \Sigma \to X$ is an ordinary map, and ϕ, ψ are the components of the gauge field A, i.e., $A = \mathrm{d} + \phi \mathrm{d}s + \psi \mathrm{d}t$. Let the area form be $\nu = \sigma ds \wedge dt$. Then a gauged map $\boldsymbol{v} = (u, \phi, \psi)$ is a vortex if it satisfies the equations

$$\boldsymbol{v}_s + J_z \boldsymbol{v}_t = 0, \qquad\qquad F_{\phi,\psi} + \sigma\mu(u) = 0 \tag{2.2}$$

where

$$\boldsymbol{v}_s = \partial_s u + \mathcal{X}_\phi(u),\ \boldsymbol{v}_t = \partial_t u + \mathcal{X}_\psi(u),\ F_{\phi,\psi} = \partial_s \psi - \partial_t \phi + [\phi, \psi].$$

Just as pseudoholomorphic maps are minimizers of their energy in their homotopy class, vortices are minimizers of the *Yang–Mills–Higgs* functional, defined by

$$\mathcal{YMH}(P, A, u) := \frac{1}{2}\left[\|\boldsymbol{v}_s\|_{L^2}^2 + \|\boldsymbol{v}_t\|_{L^2}^2 + \|F_{\phi,\psi}\|_{L^2}^2 + \|\mu(u)\|_{L^2}^2\right].$$

Here the norms are defined using the metric on Σ determined by the complex structure on Σ and the volume form $\nu \in \Omega^2(\Sigma)$, the metric on $\mathfrak{k}$ and the family of almost Kähler metrics on X determined by $\omega_X(\cdot, J_z \cdot)$. Vortices over a surface Σ are always assumed to have finite Yang–Mills–Higgs action, and if Σ is noncompact, we assume $u(\Sigma)$ has compact closure in X.

The linearization of (2.2), modulo gauge transformations, is an elliptic first-order differential operator. The space of (formal) infinitesimal deformations of a gauged map $\boldsymbol{v}$ is a triple $\boldsymbol{\xi} = (\xi, \eta, \zeta)$ where $\xi \in \Gamma(u^*TX)$ and $\eta, \zeta : \Sigma \to \mathfrak{k}$. The linearization of (2.2) at $\boldsymbol{v}$ is then

$$D_{\boldsymbol{v}}(\boldsymbol{\xi}) = \left[\begin{array}{c} \nabla_s \xi + \nabla_\xi \mathcal{X}_\phi + J\big(\nabla_t \xi + \nabla_\xi \mathcal{X}_\psi\big) + (\nabla_\xi J)\boldsymbol{v}_t + \mathcal{X}_\eta(u) + J\mathcal{X}_\zeta(u) \\ \partial_s \zeta + [\phi, \zeta] - \partial_t \eta - [\psi, \eta] + \mathrm{d}\mu(u) \cdot \xi \end{array} \right].$$

We introduce an additional operator to take care of the gauge fixing. Define the *augmented linearization*

$$\hat{\mathcal{D}}_{\boldsymbol{v}}(\boldsymbol{\xi}) = \left[\begin{array}{c} \nabla_s \xi + \nabla_\xi \mathcal{X}_\phi + J\big(\nabla_t \xi + \nabla_\xi \mathcal{X}_\psi\big) + (\nabla_\xi J)\boldsymbol{v}_t + \mathcal{X}_\eta(u) + J\mathcal{X}_\zeta(u) \\ \partial_s \eta + [\phi, \eta] + \partial_t \zeta + [\psi, \zeta] + \mathrm{d}\mu(u) \cdot J\xi \\ \partial_s \zeta + [\phi, \zeta] - \partial_t \eta - [\psi, \eta] + \mathrm{d}\mu(u) \cdot \xi \end{array} \right]. \tag{2.3}$$

Our main interest is in the compactification and regularization of *affine vortices*. Let $\boldsymbol{A}$ be either the complex plane $\boldsymbol{C}$ or the upper half plane $\boldsymbol{H}$. Let J be a domain-dependent almost complex structure over $\boldsymbol{A}$. A J-affine vortex over $\boldsymbol{A}$ is a vortex over $\boldsymbol{A}$ with respect to J and the standard area form $\nu = \mathrm{d}s \wedge \mathrm{d}t$, that satisfies the boundary condition $u(\partial \boldsymbol{A}) \subset L$ when $\boldsymbol{A} = \boldsymbol{H}$.

We recall the results on removable singularities of affine vortices at infinity. For affine vortices over $\boldsymbol{C}$ it is due to Ziltener (see [Zil09, Zil05, Zil14], and a more general result in the recent [CWW17]); for affine vortices over $\boldsymbol{H}$ it is due to Dongning Wang and the second named author [WX17, Theorem 2.11].

Proposition 2.2. *Let $\boldsymbol{v} = (u, \phi, \psi)$ be an affine vortex over $\boldsymbol{C}$ (resp. over $\boldsymbol{H}$), then there exists $\bar{x} \in \bar{X}$ (resp. $\bar{x} \in \bar{L}$) such that as points in the orbit space X/K,*

$$\lim_{z \to \infty} K \cdot u(z) = \bar{x}.$$

Furthermore, when $\boldsymbol{A} = \boldsymbol{C}$, then up to gauge transformation, there exists a smooth loop $\gamma : S^1 \to K$ and $x \in \mu^{-1}(0)$ such that

$$\lim_{r \to +\infty} u(re^{\mathbf{i}\theta}) = \gamma(\theta)x.$$

The above proposition allows one to associate an equivariant homology class to any affine vortex. For the case $\boldsymbol{A} = \boldsymbol{C}$, given the loop γ above, define a K-bundle $P \to \boldsymbol{S}^2$ using γ as a clutching function. Let $\tilde{u} \in \Gamma(P(X))$ be the section that with respect trivialization of $P|_{\boldsymbol{C}}$, $\tilde{u}$ coincides with the map u. Then section $\tilde{u}$ represents an equivariant homology class $B \in H_2^K(X; \mathbb{Z})$. By [Zil14, page 5] one has the energy identity

$$E(\boldsymbol{v}) = \langle [\omega_X^K], B \rangle. \tag{2.4}$$

Similarly, an affine vortex over $\boldsymbol{H}$ represents a relative class $B \in H_2(X, L; \mathbb{Z})$ and a similar energy identity holds.

We are particularly interested in vortices that map to an orbit of the complex group, since these are the most difficult to deal with in transversality:

Lemma 2.3. *Let $\boldsymbol{v} = (u, \phi, \psi)$ be an affine vortex over $\boldsymbol{H}$. Suppose there exists $x \in \mu^{-1}(0)$ such that $u(\boldsymbol{C}) \subset Gx$ (not the closure of Gx), then $E(\boldsymbol{v}) = 0$.*

Proof. Let $\gamma : S^1 \to K$ be the loop given by Proposition 2.2. By the condition on $\boldsymbol{v}$, there is a smooth map $g : \boldsymbol{C} \to G$ such that $u(z) = g(z)x$. Since the G-action on $G\mu^{-1}(0)$ is free, it follows that $\gamma(\theta)$ is contractible in K. Then the induced bundle $P \to \boldsymbol{S}^2$ (see the paragraph above) is trivial and the section $\tilde{u}$ is homotopic to an ordinary map from $\boldsymbol{S}^2$ to the orbit Gx, written as $\tilde{g}(z)x$ for some map $\tilde{g} : \boldsymbol{S}^2 \to G$.

Recall that the second homotopy groups of connected Lie groups are trivial. Hence $\tilde{u}$ is homotopic to a constant. Since the energy of affine vortices is topological, it follows that the energy of the original affine vortex is zero. □

In order to regularize moduli spaces of affine vortices we impose the following semi-positivity condition for affine vortices and holomorphic spheres, which simplifies the transversality arguments. We remark that assuming suitable regularization, the arguments in this paper do not require any semi-positivity condition and apply to any symplectic quotient.

Definition 2.4. A Kähler Hamiltonian G-manifold (X, ω_X, J_X, μ) is *semi-Fano* if the following hold:

(a) All J_X-affine vortices $\boldsymbol{v} = (u, \phi, \psi)$ over $\boldsymbol{C}$ have nonnegative equivariant Chern numbers $\langle c_1^K(TX), [\boldsymbol{v}]\rangle$.
(b) All $\bar{J}_X$-holomorphic spheres $\bar{u} : \boldsymbol{S}^2 \to \bar{X}$ in $\bar{X}$ have nonnegative Chern numbers $\langle c_1(T\bar{X}), \bar{u}(\boldsymbol{S}^2)\rangle$.
(c) There exists a nonempty G-invariant normal crossing anticanonical divisor

$$S = S_1 \cup \cdots \cup S_N \subset X - L$$

that is, a divisor whose class $[S]$ is dual to the equivariant Chern class $c_1^K(TX)$. We require that $X - S$ contains no nontrivial J_X-affine vortices or nonconstant holomorphic spheres in $\bar{X}$.

Given the semi-Fano condition, the equivariant Poincaré dual of the anti-canonical divisor S represents the equivariant first Chern class. The quotient of the divisor

$$\bar{S} = \bar{S}_1 \cup \cdots \cup \bar{S}_N = (S \cap \mu^{-1}(0))/K \subset \bar{X}$$

is an anti-canonical divisor in the quotient $\bar{X}$. That is, its homology class $[\bar{S}]$ represents the first Chern class $c_1(T\bar{X})$ of $\bar{X}$. The semi-Fano condition also implies that, if a nontrivial J_X-affine vortices over $\boldsymbol{C}$ resp. nonconstant holomorphic sphere in $\bar{X}$ has zero equivariant Chern number resp. ordinary Chern number, then its image is contained in S resp. $\bar{S}$.

The following lemma is a useful result related to choosing the support of perturbations for affine vortices.

Lemma 2.5. *Let $J : \boldsymbol{C} \to \mathcal{J}(X)$ be a family of almost complex structures on X depending continuously on $z \in \boldsymbol{C}$. Let $\boldsymbol{v} = (u, \phi, \psi)$ be a J-affine vortex with energy $E > 0$. For any K-invariant open neighborhood $U \subset X$ of $\mu^{-1}(0)$, let $c_U > 0$ be such that $|\mu(x)| \leqslant c_U \Longrightarrow x \in U$. Define*

$$A_E = \frac{2E}{c_U^2}. \tag{2.5}$$

Then for any measurable subset $B \subset \boldsymbol{C}$ with area greater than or equal to A_E, $u(B) \cap U \neq \varnothing$.

Proof. Suppose $u(B) \cap U = \varnothing$. Then $|\mu(u)|_B > c_U$. By the definition of energy, we have

$$E \geqslant \frac{1}{2}\int_{\boldsymbol{C}} |\mu(u)|^2 \mathrm{d}s\mathrm{d}t \geqslant \frac{1}{2}\int_B |\mu(u)|^2 \mathrm{d}s\mathrm{d}t > \frac{1}{2} A_E c_U^2 \geqslant E,$$

which is a contradiction. □

2.5. **Local model of affine vortices.** A topology on the moduli space of affine vortices is defined as follows. Given a class $B \in H_2^K(X;\mathbb{Z})$ resp. $B \in H_2(X, L;\mathbb{Z})$ and a domain-dependent almost complex structure $J : \boldsymbol{A} \to \mathcal{J}(X)$, let $\mathcal{M}_J(\boldsymbol{A}; B)$ be the set of gauge equivalence classes of J-affine vortices over $\boldsymbol{A}$. There is a natural topology, called the *compact convergence topology*, abbreviated as c.c.t., on $\mathcal{M}_J(\boldsymbol{A}; B)$ defined as follows. We say that a sequence of smooth gauged maps $\boldsymbol{v}_i = (u_i, \phi_i, \psi_i)$ *converges* in c.c.t. to a gauged map $\boldsymbol{v}_\infty = (u_\infty, \phi_\infty, \psi_\infty)$ if for any precompact open subset $Z \subset \boldsymbol{A}$, $\boldsymbol{v}_i|_Z$ converges to $\boldsymbol{v}_\infty|_Z$ uniformly with all derivatives. We say that $\boldsymbol{v}_i$ converges in c.c.t. to $\boldsymbol{v}_\infty$ modulo gauge transformation if there exist a sequence of smooth gauge transformations $g_i : \boldsymbol{A} \to K$ such that $g_i \cdot \boldsymbol{v}_i$ converges to $\boldsymbol{v}_\infty$ in c.c.t.. The notion of sequential convergence in c.c.t. modulo gauge descends to a notion of sequential convergence in the moduli space $\mathcal{M}_J(\boldsymbol{A}; B)$, and induces a topology on $\mathcal{M}_J(\boldsymbol{A}; B)$ (see [MS04, Section 5.6]). This topology is Hausdorff, first countable, however not necessarily compact. A natural compactification for $\boldsymbol{A} = \boldsymbol{C}$ is constructed in by Ziltener [Zil05, Zil14], while the compactification in the case of $\boldsymbol{A} = \boldsymbol{H}$ is also essentially constructed by Wang and the second named author [WX17].

The analytic local models of the moduli spaces of affine vortices are not as straightforward as the case of pseudoholomorphic curves, largely due to the fact that the domain $\boldsymbol{A}$ is noncompact and asymptotically Euclidean. In [VXne], a local model is constructed using the following weighted Sobolev spaces. Let $\boldsymbol{A}$ be either affine space $\boldsymbol{C}$ or the half-space $\boldsymbol{H}$. Choose a smooth function $\rho : \boldsymbol{A} \to [1, +\infty)$ that coincides with the radial coordinate r outside a compact subset of $\boldsymbol{A}$. For $\delta \in \mathbb{R}$ and an open subset $U \subset \boldsymbol{A}$, introduce the following weighted Sobolev norms:

$$\|f\|_{L^{p,\delta}(U)} = \left[\int_U |f(z)|^p \rho(z)^{p\delta} \mathrm{d}s\mathrm{d}t \right]^{\frac{1}{p}}.$$

$$\|f\|_{L_g^{1,p,\delta}(U)} = \|f\|_{L^{p,\delta}(U)} + \|\nabla f\|_{L^{p,\delta}(U)}, \qquad \|f\|_{L_h^{1,p,\delta}(U)} = \|f\|_{L^\infty(U)} + \|\nabla f\|_{L^{p,\delta}(U)}.$$

When $U = \boldsymbol{A}$ in the notation we drop the dependence on the domain U. These norms have the property that if $p > 2$, $\delta \in (1 - \frac{2}{p}, 2)$, then for $f \in L_g^{1,p,\delta}$, the limit of f at infinity is zero; while any $f \in L_h^{1,p,\delta}$ has a well-defined limit $f(\infty)$, not necessarily zero. We use the specialization of [Xu] where we set

$$p \in (2, 4), \qquad \delta = \delta_p = 2 - \frac{4}{p}.$$

In this case the above three norms are abbreviated as $\tilde{L}^p$, $\tilde{L}_g^{1,p}$, $\tilde{L}_h^{1,p}$.

We use the above weighted norms to define Sobolev norms on the tangent space to the space of gauged maps. Given a gauged map $\boldsymbol{v} = (u, a) = (u, \phi, \psi)$ from $\boldsymbol{A}$ to X (where $\boldsymbol{A}$ is either $\boldsymbol{H}$ or $\boldsymbol{C}$), consider the space of formal infinitesimal deformations

$$\Gamma(\boldsymbol{A}, u^*TX \oplus \mathfrak{k} \oplus \mathfrak{k})$$

whose elements are denoted by $\boldsymbol{\xi} = (\xi, \eta, \zeta)$. Define the covariant derivative of $\boldsymbol{\xi}$ by

$$\hat{\nabla}\boldsymbol{\xi} = \mathrm{d}s \otimes \hat{\nabla}_s \boldsymbol{\xi} + \mathrm{d}t \otimes \hat{\nabla}_t \boldsymbol{\xi}$$

(this differs from the notations used in [Xu, VXne]) where

$$\hat\nabla_s \boldsymbol{\xi} = \begin{bmatrix} \nabla_s \xi + \nabla_\xi \mathcal{X}_\phi \\ \nabla_s \eta + [\phi, \eta] \\ \nabla_s \zeta + [\phi, \zeta] \end{bmatrix}, \qquad \hat\nabla_t \boldsymbol{\xi} = \begin{bmatrix} \nabla_t \xi + \nabla_\xi \mathcal{X}_\psi \\ \nabla_t \eta + [\psi, \eta] \\ \nabla_t \zeta + [\psi, \zeta] \end{bmatrix}.$$

Given $p > 2$ and $\delta \in (1 - \frac{2}{p}, 1)$, define the following norm on $\boldsymbol{\xi} = (\xi, \eta, \zeta)$:

$$\|\boldsymbol{\xi}\|_{\tilde L^{1,p}_m} := \|\xi\|_{L^\infty} + \|\hat\nabla \boldsymbol{\xi}\|_{\tilde L^p} + \|\mathrm{d}\mu(u)\cdot \xi\|_{\tilde L^p} + \|\mathrm{d}\mu(u) \cdot J_X \xi\|_{\tilde L^p}. \tag{2.6}$$

The subscript "m" represents "mixed." This norm can be viewed as the combination of the norms on $\tilde L^{1,p}_h$ and $\tilde L^{1,p}_g$. Namely, near infinity in $\boldsymbol{A}$ in the direction of $G_X \oplus \mathfrak{k} \oplus \mathfrak{k}$, finiteness of the above norm requires $\boldsymbol{\xi}$ has finite $\tilde L^{1,p}_g$-norm; in the direction of H_X, finiteness requires that $\boldsymbol{\xi}$ has finite $\tilde L^{1,p}_h$-norm. On the other hand, the norm (2.6) is gauge invariant in the following sense. If we have two gauged maps $\boldsymbol{v} = (u, a)$, $\boldsymbol{v}' = (u', a')$ and a gauge transformation $g : \boldsymbol{A} \to K$ such that $\boldsymbol{v}' = g \cdot \boldsymbol{v}$. Then for any infinitesimal deformation $\boldsymbol{\xi} = (\xi, \eta, \zeta)$ along $\boldsymbol{v}$, $\boldsymbol{\xi}' := g \cdot \boldsymbol{\xi} = (g\xi, \mathrm{Ad}_g \eta, \mathrm{Ad}_g \zeta)$ is an infinitesimal deformation of $\boldsymbol{v}'$ and $\|\boldsymbol{\xi}\|_{\tilde L^{1,p}_m} = \|\boldsymbol{\xi}'\|_{\tilde L^{1,p}_m}$.

A Banach manifold of gauged maps near a given affine vortex is constructed using the above norms. Suppose $\boldsymbol{v}$ is an affine vortex over $\boldsymbol{A}$. Let $\hat{\mathcal{B}}_{\boldsymbol{v}}$ be the Banach space of $W^{1,p}_{\mathrm{loc}}$-sections $\boldsymbol{\xi} = (\xi, \eta, \zeta)$ that have finite $\tilde L^{1,p}_m$-norm and satisfies the boundary condition

$$\xi|_{\partial \boldsymbol{A}} \subset u^* TL, \qquad \eta|_{\partial \boldsymbol{A}} = 0$$

For any $\epsilon > 0$ let $\hat{\mathcal{B}}^\epsilon_{\boldsymbol{v}} \subset \hat{\mathcal{B}}_{\boldsymbol{v}}$ be the ϵ-ball centered at the origin. Then we can identify $\boldsymbol{\xi} \in \hat{\mathcal{B}}^\epsilon_{\boldsymbol{v}}$ with a nearby gauged map defined as

$$\exp_{\boldsymbol{v}} \boldsymbol{\xi} = (\exp_u \xi, \phi + \eta, \psi + \zeta).$$

Here the exponential map exp is defined using a K-invariant Riemannian metric h_X on X such that both L and $\mu^{-1}(0)$ are totally geodesic. Therefore, $\exp_{\boldsymbol{v}} \boldsymbol{\xi}$ also satisfies the boundary condition $\exp_{\boldsymbol{v}} \boldsymbol{\xi}(\partial \boldsymbol{A}) \subset L$.

Local slices for the action of the group of gauge transformations are given by enforcing a Coulomb gauge condition. With respect to the central $\boldsymbol{v}$, an element $\boldsymbol{v}' = \exp_{\boldsymbol{v}} \boldsymbol{\xi}$ is said to be *in Coulomb gauge* if

$$\hat\nabla_s \eta + \hat\nabla_t \zeta + \mathrm{d}\mu(u) \cdot J\xi = 0.$$

Let $\mathcal{B}^\epsilon_{\boldsymbol{v}} \subset \hat{\mathcal{B}}^\epsilon_{\boldsymbol{v}}$ be the set of elements in Coulomb gauge. The set $\mathcal{B}^\epsilon_{\boldsymbol{v}}$ is an open subset of an affine subspace and so also a Banach manifold. The Coulomb gauge condition is gauge invariant, so that

$$g \cdot \mathcal{B}^\epsilon_{\boldsymbol{v}} = \mathcal{B}^\epsilon_{g\cdot \boldsymbol{v}} \quad \forall g : \boldsymbol{A} \to K.$$

Let $\mathcal{E} \to \mathcal{B}^\epsilon_{\boldsymbol{v}}$ be a Banach space bundle whose fibre over a gauged map $\boldsymbol{v}' = (u', \phi', \psi')$ is

$$\mathcal{E}|_{\boldsymbol{v}'} = \tilde L^p((u')^* TX \oplus \mathfrak{k}).$$

The affine vortex equation (2.2) with respect to certain J defines a Fredholm section

$$\mathcal{F}_{\boldsymbol{v}} : \mathcal{B}^\epsilon_{\boldsymbol{v}} \to \mathcal{E}.$$

By Ziltener [Zil14, (1.27)] if $\boldsymbol{v}$ represents an equivariant homology class $B \in H^K_2(X; \mathbb{Z})$, then one has the index formula

$$\mathbf{ind} \mathcal{F}_{\boldsymbol{v}} = \mathbf{dim} \bar X + 2\langle c_1^K(TX), B\rangle.$$

Moreover, in [VXne, Appendix] we proved the index formula for affine vortices over $\boldsymbol{H}$. Namely, for every $B \in H_2(X, L; \mathbb{Z})$, we have

$$\mathbf{ind}\mathcal{F}_{\boldsymbol{v}} = \mathbf{dim}\bar{L} + \mathrm{Mas}(B).$$

A local model of affine vortices was obtained by the second named author and Venugopalan [VXne]. We state a slight generalization in which the almost complex structure is allowed to vary with $z \in \boldsymbol{A}$ in a compact set, which has the same proof. Let $W \subset \mathbb{R}^k$ be an open neighborhood of the origin. Let $\tilde{J} : W \times \boldsymbol{A} \to J_X$ be a smooth family of almost complex structures on X such that for some compact subset $Z \subset \boldsymbol{A}$, $\tilde{J}(w, z) = J_X$ for $z \notin Z$. Consider the moduli space $\mathcal{M}_W$ of pairs $(w, [\boldsymbol{v}])$ where $w \in W$ and $[\boldsymbol{v}]$ is an gauge equivalence class of J_w-affine vortices in class B. Equip $\mathcal{M}_W$ with the compact convergence topology. Choose a point $(0, [\boldsymbol{v}])$ and a representative $\boldsymbol{v}$.

Theorem 2.6. [VXne] *Suppose $(w_n, [\boldsymbol{v}_n])$ converges to $(0, [\boldsymbol{v}])$. Then for n sufficiently large, there exists a gauge transformation g_n of regularity $W^{2,p}_{\mathrm{loc}}$, such that $g_n \cdot \boldsymbol{v}_n \in \mathcal{B}^\epsilon_{\boldsymbol{v}}$. Moreover, for ϵ sufficiently small, the natural inclusion*

$$\mathcal{F}_W^{-1}(0) \cap (W^\epsilon \times \mathcal{B}^\epsilon_{\boldsymbol{v}}) \hookrightarrow \mathcal{M}_W$$

is a homeomorphism onto an open neighborhood of $(0, [\boldsymbol{v}])$ in $\mathcal{M}_W$.

Each Coulomb slice gives a Banach manifold chart for the moduli space. Indeed, for different smooth affine vortices $\boldsymbol{v}$ and $\boldsymbol{v}'$ and possibly different ϵ, ϵ', there can be a nonempty overlap between $\mathcal{B}^\epsilon_{\boldsymbol{v}}$ and $\mathcal{B}^{\epsilon'}_{\boldsymbol{v}'}$ as a set of gauge equivalence classes of gauged maps and the overlaps can be identified with open subset of $\mathcal{B}^\epsilon_{\boldsymbol{v}}$ and $\mathcal{B}^{\epsilon'}_{\boldsymbol{v}'}$. Hence the overlap defines a transition function between these two open subsets. It is proved as [VXne, Proposition 3.8] that the transition function is a diffeomorphism of Banach manifolds. The collection of all such charts and transition functions clearly satisfy the cocycle condition and hence define a Banach manifold of gauge equivalence classes, which is denoted as

$$\mathcal{B}_{\boldsymbol{A}}(B) = \mathcal{B}_{\boldsymbol{A}}(X, L, \mu; B), \ B \in H_2^K(X, L; \mathbb{Z}). \tag{2.7}$$

A point of $\mathcal{B}_{\boldsymbol{A}}(B)$ is defined as $[u, \phi, \psi]$ where $\boldsymbol{v} = (u, \phi, \psi)$ is a concrete representative. Moreover, we can also glue the Banach vector bundles to obtain a bundle

$$\mathcal{E}_{\boldsymbol{A}}(B) = \mathcal{E}_{\boldsymbol{A}}(X, L, \mu; B) \to \mathcal{B}_{\boldsymbol{A}}(X, L, \mu; B), \ B \in H_2^K(X, L; \mathbb{Z}). \tag{2.8}$$

Then for the chosen domain-dependent almost complex structure J, there is a smooth Fredholm map

$$\mathcal{F}_{J,B} : \mathcal{B}_{\boldsymbol{A}}(B) \to \mathcal{E}_{\boldsymbol{A}}(B).$$

Then Theorem 2.6 implies that $\mathcal{F}_{J,B}^{-1}(0)$ is homeomorphic to the moduli space of J-affine vortices over $\boldsymbol{A}$ that represent the class B. In particular, when a vortex $\boldsymbol{v}$ is regular, a neighborhood of its gauge equivalence class $[\boldsymbol{v}]$ in $\mathcal{M}_J(\boldsymbol{A}; B)$ is homeomorphic to an open subset in the kernel of the linearization of $\mathcal{F}_{J,B}$ at $[\boldsymbol{v}]$.

2.6. **Orientations.** Orientations on the moduli space of affine vortices may be constructed as in the case of pseudoholomorphic curves assuming a relative spin structure on the Lagrangian. It is well-known that the Cauchy–Riemann operator $\overline{\partial}_{\bar{J}} : \mathcal{B}_{\boldsymbol{S}^2}(\bar{X}, \bar{L}; B) \to \mathcal{E}_{\boldsymbol{S}^2}(\bar{X}, \bar{L}; B)$ over $\boldsymbol{S}^2$ has a canonical orientation (on the determinant line of its linearizations) by deformation of its linearization to a complex linear operator. On the other hand, recall from Fukaya–Oh–Ohta–Ono [FOOO09] the orientation and the spin structure on $\bar{L}$ induces an orientation on

$$\overline{\partial}_J : \mathcal{B}_{\boldsymbol{D}^2}(\bar{X}, \bar{L}; B) \to \mathcal{E}_{\boldsymbol{D}^2}(\bar{X}, \bar{L}; B).$$

The orientation and the spin structure on $\bar{L}$ also induces a canonical orientation on the operator

$$\overline{\partial}_J : \mathcal{B}_{\boldsymbol{D}^2}(X, L; B) \to \mathcal{E}_{\boldsymbol{S}^2}(X, L; B)$$

by using the canonical spin structure and orientation on the Lie algebra.

The moduli space of affine vortices with Lagrangian boundary condition is oriented by a similar discussion.

Proposition 2.7. *Given a spin structure and an orientation on $\bar{L}$, there is a naturally induced orientation on the linearization at every affine vortex over $\boldsymbol{H}$.*

Before proving the proposition we introduce the following notation. Let $E \to \boldsymbol{D}^2$ be a complex vector bundle and $\overline{\partial}^E : \Omega^0(E) \to \Omega^{0,1}(E)$ be a real linear Cauchy–Riemann operator. Let $\lambda \to \partial \boldsymbol{D}^2$ be a totally real subbundle of $E|_{\partial \boldsymbol{D}^2}$, meaning that $J_E \lambda \cap \lambda = \{0\}$ and λ has maximal dimension, where J_E is the complex structure on E. The pair (E, λ) is called a *bundle pai*r over $(\boldsymbol{D}^2, \partial \boldsymbol{D}^2)$. Let $p > 2$ and $W^{1,p}_\lambda(E)$ be the Sobolev space of $W^{1,p}$-sections of E whose boundary values lie in λ. Then we have a Fredholm operator

$$\overline{\partial}{}^E_\lambda : W^{1,p}_\lambda(E) \to L^p(E).$$

Lemma 2.8. [FOOO09, Proposition 8.1.4] *Suppose λ is trivial. Then each trivialization of λ canonically induces an orientation on $\overline{\partial}{}^E_\lambda$.*

A similar discussion holds in the case of boundary value problems on the half-space, fixed outside a compact set. Let (E, λ) be a bundle pair over $\boldsymbol{H}$. Fix a trivialization of E outside a compact set $Z \subset \boldsymbol{H}$, i.e.,

$$\rho : E|_{\boldsymbol{H} \smallsetminus Z} \simeq (\boldsymbol{H} \smallsetminus Z) \times \mathbb{C}^m$$

whose restriction to $\partial \boldsymbol{H} \smallsetminus Z$ maps λ to $\mathbb{R}^m$. If we regard $\boldsymbol{D}^2 \simeq \boldsymbol{H} \cup \{\infty\}$, then ρ induces a bundle pair on $\boldsymbol{D}^2$, which is still denoted by (E, λ).

Choose a Hermitian metric h on E such that

$$h - h_0 \circ \rho \in \tilde{L}^{1,p}(\boldsymbol{H} \smallsetminus Z, \mathbb{C}^{n \times n})$$

where h_0 is the standard metric on $\mathbb{C}^n$. Here $\tilde{L}^{1,p}$ is just the weighted Sobolev space used in the previous section for $p \in (2,4)$ and weighted by $|z|^{2 - \frac{4}{p}}$. Choose a metric connection ∇^E on E such that near infinity it is almost the ordinary differentiation for $\mathbb{C}^n$-valued functions. Then, using h_E and ∇^E, one can define weighted Sobolev space of sections of E

$$\tilde{L}^{1,p}_h(\boldsymbol{H}, E)_\lambda.$$

Here the subscript λ indicates the boundary values should be in $\lambda \subset E|_{\partial \boldsymbol{H}}$. Consider a real linear Cauchy–Riemann operator $\overline{\partial}^E$ on E such that with respect to ρ

$$\overline{\partial}^E = \overline{\partial} + \alpha, \text{ where } \alpha \in W^{1,p,\delta}(\boldsymbol{H} \smallsetminus Z, \Lambda^{0,1} \otimes \mathbb{C}^{m \times m}),\ \delta > \delta_p = 2 - \frac{4}{p}.$$

Consider the bounded linear operator

$$\overline{\partial}{}^E_\lambda : \tilde{L}^{1,p}_m(\boldsymbol{H}, E)_\lambda \to \tilde{L}^p(\boldsymbol{H}, \Lambda^{0,1} \otimes E). \tag{2.9}$$

The operator $\overline{\partial}{}^E_\lambda$ is Fredholm, by an argument similar to the proof of the Fredholm property of linearizations of affine vortices over $\boldsymbol{H}$ given in [VXne, Appendix]. Moreover, by straightforward verification, the Banach norm $\tilde{L}^{1,p}_h$ over $\boldsymbol{H}$ and the Banach norm $W^{1,p}$ over $\boldsymbol{D}^2$ are equivalent. Hence Lemma 2.8 implies the following lemma.

Lemma 2.9. *Suppose the real bundle λ over $\partial \boldsymbol{D}^2$ is trivial. Then each trivialization of λ canonically induces an orientation of the operator $\overline{\partial}{}^E_\lambda$ of (2.9).*

Now we prove Proposition 2.7. Given an affine vortex $\boldsymbol{v} = (u, \phi, \psi)$ over $\boldsymbol{H}$. By Proposition 2.2, u has a well-defined limit at ∞ modulo K-action. Since a neighborhood of $\infty \in \boldsymbol{H}$ is contractible, by gauge transformation, we can assume that u has a well-defined limit $x_\infty \in L$ at ∞. Moreover, the convergence can be made stronger, i.e., up to gauge transformation, we may assume that in local coordinates around x_∞,

$$\nabla u \in W^{1,p,\delta}(\boldsymbol{H} \smallsetminus Z),\ \forall \delta \in \Big(\frac{\delta_p}{2}, 1\Big).$$

Therefore, $E := u^* TX$ has a continuous extension to $\boldsymbol{D}^2$, which is still denoted by E. Let the boundary restriction of u be $l : \partial \boldsymbol{H} \to L$, which projects to $\bar{l} : \partial \boldsymbol{H} \to \bar{L}$. Both l and $\bar{l}$ extends to continuous maps on $\partial \boldsymbol{D}^2$. Denote $\lambda := l^* TL$ and $\bar\lambda := \bar{l}^* T\bar{L}$. There is a projection $\lambda \to \bar\lambda$ whose kernel is spanned by infinitesimal K-actions. We write

$$\lambda \simeq \lambda_K \oplus \bar\lambda.$$

The direct sum $\lambda_K \oplus J_E \lambda_K$ is a complex subbundle of $E|_{\partial \boldsymbol{D}^2}$, which is canonically trivialized. The sum $\lambda_K \oplus J_E \lambda_K$ extends unique to a trivial complex subbundle $E_K \subset E$ over $\boldsymbol{D}^2$, uniquely up to homotopy. Extend $\bar\lambda \oplus J_E \bar\lambda$ to a complement of E_K, denoted by $E_K^\perp$. With respect to the splitting $E \simeq E_K^\perp \oplus E_K$, the linear operator (2.3) may be written, up to compact operators, as

$$\begin{bmatrix} \overline{\partial}{}^{E_K^\perp} & 0 & 0 \\ 0 & \overline{\partial}{}^{E_K} & I_K \\ 0 & I_K & \partial^{E_K} \end{bmatrix}.$$

Note that adding compact operators does not alter the orientation problem. Moreover, the lower right two by two matrix, denoted by D_K, is invertible. Therefore the orientation of the augmented linearized operator $\hat{\mathcal{D}}_{\boldsymbol{v}}$ is reduced to the orientation of $\overline{\partial}{}^{E_K^\perp}$, which is a kind of operator treated in Lemma 2.9. Hence a trivialization of $\bar\lambda \simeq \bar{l}^* T\bar{L}$ induces a canonical orientation. Finally, the spin structure and the orientation of $\bar{L}$ canonically induce an oriented trivialization of $\bar\lambda$ (see discussions in [FOOO09, Chapter 8] for details).

Lastly, the orientation problem for affine vortices over $\boldsymbol{C}$ is relatively simpler, as the linearized operator has a complex linear symbol. Hence we leave to the reader to check that for affine vortices over $\boldsymbol{C}$ there is always a canonical orientation.

3. Stabilizing divisors

To regularize the moduli spaces we choose invariant divisors so that the additional intersection points stabilize the domains. The choice of stabilizing divisor also determines the set of almost complex structures we can use to perturb the equation. This technique was firstly introduced by Cieliebak–Mohnke [CM07] and many results presented here have parallel discussions in [CW17, CW15]. However, since $\bar{X}$

is an algebraic variety, the proofs of some of these results can be simplified using results from algebraic geometry in this case.

The theory of stabilizing divisors requires a rationality assumption. Recall that $R \to X$ is a G-equivariant holomorphic line bundle that carries a K-invariant Hermitian metric, and the Chern connection A on R, which is K-invariant, extends to an equivariant (super)connection A_{eq} whose equivariant curvature form is a multiple of the equivariant symplectic form

$$\omega_X^K = \omega_X + \mu \in \Omega_K^2(X).$$

The bundle R descends to a holomorphic line bundle $\bar{R} \to \bar{X}$ whose Chern class is a multiple of the class $[\omega_{\bar{X}}] \in H_K^2(X)$. Hence $\bar{X}$ is a rational symplectic manifold.

The notion of rational Lagrangians used in [CW17] generalizes to the equivariant case as follows.

Definition 3.1. (cf. [CW17, Definition 3.5]) The K-Lagrangian submanifold $L \subset X$ is called *equivariantly rational* if the following conditions are satisfied.

(a) For some positive integer k, the restriction of $(R^{\otimes k}, A_{\text{eq}}^{\otimes k})$ to L is isomorphic to a trivial K-equivariant line bundle with the trivial connection. Notice that then $\bar{R}^{\otimes k}|_{\bar{L}}$ is trivialized. Without loss of generality assume that $k = 1$.

(b) Notice that every G-equivariant section $f \in \Gamma(R)^G$ induces a smooth section $\bar{f} \in \Gamma(\bar{R})$. We require that there is a holomorphic section $f_0 \in H_G^0(R)$ such that $\bar{f}_0|_{\bar{L}}$ is nonvanishing and defines a trivialization of $\bar{R}|_{\bar{L}}$ which is homotopic to the trivialization in the first condition.

Equivariant rationality in particular implies that the quotient $\bar{L} = L/K$ is a rational Lagrangian (i.e., "Bohr–Sommerfeld") of $\bar{X}$. Indeed, the bundle-with-connection R descends to a bundle

$$\bar{R} = R|_{\mu^{-1}(0)}/K$$

on $\bar{X}$ with curvature a multiple of the symplectic form. From now on we assume that L is equivariantly rational.

3.1. **Stabilizing divisors.** We construct invariant divisors with mild singularities in the unstable locus, used later to construct regularizations of the moduli space of vortices. Take a sufficiently large integer k and denote

$$\Gamma_k = H^0(X, R^{\otimes k})^G, \quad \text{resp. } \bar{\Gamma}_k = H^0(\bar{X}, \bar{R}^{\otimes k})$$

the space of G-equivariant holomorphic sections on X resp. holomorphic sections on $\bar{X}$. Restriction to the semistable locus defines a natural map

$$\Gamma_k \to \bar{\Gamma}_k.$$

Let $\Gamma_k(L) \subset \Gamma_k$ be the subset of sections $f \in \Gamma_k$ such that $f|_L$ is nonvanishing and such that $f|_L$ is in the same homotopy class of $f_0^{\otimes k}|_L$ where f_0 is the section in Definition 3.1. Therefore $\Gamma_k(L)$ is a nonempty open subset of Γ_k. Similarly, define $\bar{\Gamma}_k(\bar{L}) \subset \bar{\Gamma}_k$. The following can be proved in the same way as in [CW17]. Given $f \in \Gamma_k(L)$ denote

$$D_f = f^{-1}(0), \quad \bar{D}_{\bar{f}} = \bar{f}^{-1}(0).$$

Then L is exact in $X \setminus D_f$ and $\bar{L}$ is exact in $\bar{X} \setminus \bar{D}_f$.

The existence of sections transverse to the zero section follows from Bertini's theorem [Har77, II.8.18]). By Hartog's theorem and the assumption that the unstable locus is complex codimension at least two, restriction induces an identification of

$\mathit{\Gamma}_k$ with $\bar{\mathit{\Gamma}}_k$ and induces an identification $\mathit{\Gamma}_k(L) \simeq \bar{\mathit{\Gamma}}_k(\bar{L})$. For a generic $\bar{f} \in \bar{\mathit{\Gamma}}_k(\bar{L})$, $\bar{D}_{\bar{f}} \subset \bar{X}$ is a smooth divisor representing the Poincaré dual of an integer multiple of $[\omega_{\bar{X}}]$. For the corresponding $f \in \mathit{\Gamma}_k(L)$, its zero locus D_f is a G-invariant subvariety, smooth inside $X^{\rm st}$. Any such D_f is called a *stabilizing divisor*. Let PD be the Poincaré dual operator acting on (pseudo)cycles in X. The integer $\mathrm{PD}(\bar{D}_{\bar{f}})/[\omega_{\bar{X}}]$ obtained by pairing the symplectic class $[\omega_{\bar{X}}]$ with the Poincaré dual $\mathrm{PD}(\bar{D}_{\bar{f}})$ is called the *degree* of $D_{\bar{f}}$. We note, for later use, the fact that any stabilizing divisor defined using an invariant section contains the unstable locus. Indeed, by Mumford's definition [MFK94], $X^{\rm us}$ is the locus where all equivariant sections of $R^{\otimes n}$ vanish.

3.2. **Intersection numbers.** Our transversality scheme uses a generalization of the approach of Cieliebak–Mohnke [CM07] to the case of multiple stabilizing divisors, as follows. Suppose $\bar{X}$ has complex dimension $\bar{n}$. Choose a sufficiently large k and $\bar{n}+1$ generic sections with the corresponding stabilizing divisors

$$f_0, \cdots, f_{\bar{n}} \in \mathit{\Gamma}_k(L) \quad D_0, \ldots, D_{\bar{n}} \subset \bar{X}.$$

For generic choices, the union

$$D = D_0 \cup \cdots \cup D_{\bar{n}}$$

is a normal crossing divisor, and the intersection of all components of D is $X^{\rm us}$. Denote

$$\bar{D} = \bar{D}_0 \cup \cdots \cup \bar{D}_{\bar{n}}.$$

Any disk with boundary in L has a well-defined intersection number with the stabilizing divisor. Let $u : \boldsymbol{D}^2 \to X$ be a smooth map such that $u(\partial \boldsymbol{D}^2) \cap D = \varnothing$ and $z \in u^{-1}(D)$. Let $V \subset \boldsymbol{D}^2$ be an open neighborhood of z containing no other element of $u^{-1}(D)$. The *local intersection number* $(u \cap D)(z)$ of u with D at z is defined to be the sum

$$(u \cap D)(z) = \sum_{z' \in V} (u' \cap D)(z')$$

of topological intersection numbers $u' \cap D(z')$ for $z' \in V$ between a generic perturbation u' supported in the interior of $\boldsymbol{D}^2$ of u with all $D_a^{\rm st}$, for any sufficiently small perturbation u'. Because $X^{\rm us}$ and all intersections $D_a^{\rm st} \cap D_b^{\rm st}$ have codimension at least four, a generic perturbation u' avoids the singularities of D. Moreover, given two transverse perturbations u', u'' of u that are homotopic, a generic homotopy between them can also avoid singularities of D. Hence the intersection number is well-defined and is homotopy invariant. Since D and all its strata are K-invariant, if $g : \boldsymbol{D}^2 \to K$ is a continuous map, then $(g \circ u)^{-1}(D) = u^{-1}(D)$. Therefore, if $P \to \boldsymbol{D}^2$ be a continuous K-bundle and u is a continuous section of $P(X)$ that has an isolated intersection at $0 \in \boldsymbol{D}^2$, then the local intersection number can be defined by using locally trivializations.

In particular the local intersection number between an affine vortex mapping infinity to the divisor is well-defined. Let $\boldsymbol{v} = (u, \phi, \psi)$ be a gauged map from $\boldsymbol{C}$ to X which extends to a continuous gauged map $\tilde{\boldsymbol{v}} = (P, A, \tilde{u})$ over $\boldsymbol{S}^2 = \boldsymbol{C} \cup \{\infty\}$. We denote by $\boldsymbol{v}^{-1}(D) \subset \boldsymbol{S}^2$ the subset of points that are mapped by $\tilde{u}$ into $P(D) \subset P(X)$. We say that ∞ is an *isolated intersection* with D if $\infty \in \tilde{\boldsymbol{v}}^{-1}(D)$ while u does not intersect D outside some compact subset of $\boldsymbol{C}$. Suppose $\boldsymbol{v}$ has isolated intersection points with D both in the interior and at ∞. Then the intersection number $\boldsymbol{v} \cap D$ is defined as the sum of local intersection numbers of $\boldsymbol{v}$ plus the local intersection number of $\tilde{\boldsymbol{v}}$ at ∞.

To study the intersection numbers of holomorphic spheres or vortices with the divisor in the algebraic case, recall the following results from algebraic geometry.

> A hypersurface $V \subset \mathbb{P}^n$ of degree d is *general* if its defining section belongs to a nonempty Zariski open subset of the projectivization of the space of sections $PH^0(\mathcal{O}(d))$.
>
> A hypersurface $V \subset \mathbb{P}^n$ is *very general* if its defining section belongs to a countable intersection of nonempty Zariski open subsets of $PH^0(\mathcal{O}(d))$.

Theorem 3.2.

(a) [Cle86][Voi96][Voi98] *Let $V \subset \mathbb{P}^n$ be a general hypersurface of degree $d \geqslant 2n-1$. Then V contains no rational curves.*

(b) [PR07, Corollary 4] *Let $V \subset \mathbb{P}^n$ be a very general hypersurface of degree $d \geqslant 2n+1$, and $C \subset \mathbb{P}^n$ a curve not contained in V, then*

$$2g(\tilde{C}) - 2 + i(C,V) \geqslant (d-2n)\mathrm{deg}C.$$

where $\nu : \tilde{C} \to C$ is the normalization of C, $g(\tilde{C})$ its genus, and $i(C,V)$ is the number of distinct points in $\nu^{-1}(V)$.

These results imply the existence of stabilizing divisors in the following sense:

Corollary 3.3. *There exists $N > 0$ such that for a generic G-invariant divisor $D_f \subset X$ of degree $n_D \geqslant N$, the following conditions are satisfied.*

(a) *$\bar{D}_f$ contains no nonconstant $\bar{J}_X$-holomorphic sphere.*

(b) *Suppose $\bar{u} : \boldsymbol{S}^2 \to \bar{X}$ is a nonconstant $\bar{J}_X$-holomorphic sphere that is not contained in $\bar{D}_f$. Then $(\bar{u})^{-1}(\bar{D}_f)$ contains at least three points.*

(c) *Suppose $\boldsymbol{v} = (u, \phi, \psi)$ is a nonconstant J_X-affine vortex whose image is not contained in D_f. Then $u^{-1}(D_f)$ contains at least one finite point in $\boldsymbol{C}$.*

Proof. The claims follow by equivariant embedding in projective space. The sections of the positive line bundle $\bar{R} \to \bar{X}$ (after taking a tensor power) embed $\bar{X}$ into projective space $\mathbb{P}^n$. Let $\bar{D}$ be the intersection of $\bar{X}$ with a divisor $V \subset \mathbb{P}^n$. Given a nonconstant $\bar{J}_X$-holomorphic sphere in $\bar{X}$, the underlying simple holomorphic sphere is in particular a rational curve in $\mathbb{P}^n$. Theorem 3.2 implies that there is no nonconstant holomorphic sphere in $\bar{D}$, namely Item (a) holds; Theorem 3.2 implies that any nonconstant holomorphic sphere in $\bar{X}$ must intersect at least three points with $\bar{D}$, namely Item (b) holds.

For Item (c), let $\boldsymbol{v}$ be a nonconstant J_X-affine vortex over $\boldsymbol{C}$ that is not contained in D. The image of u is contained in the stable locus except at isolated points. Hence using the projection $X^{\mathrm{st}} \to \bar{X}$ and the removal of singularity, u projects to a holomorphic sphere $\bar{u} : \boldsymbol{S}^2 \to \bar{X}$. If $\bar{u}$ is nonconstant, then by Theorem 3.2, $\bar{u}$ has to intersect with $\bar{D}$ at at least three points, two of which must be finite points. Since D is G-invariant, $\boldsymbol{v}$ also intersects with D at these points. If $\bar{u}$ is constant, then u is contained in the closure of a single G-orbit $\overline{Gx}$. By Lemma 2.3, u has to intersect the unstable locus X^{us} at a finite point. Since $X^{\mathrm{us}} \subset D$, $u^{-1}(D)$ then contains a finite point. So Item (c) holds. □

Later we will see that given an energy bound, any almost complex structure sufficiently close to J_X also satisfies the conditions of Corollary 3.3 .

3.3. **Almost complex structures.** The almost complex structures used for the construction of perturbations are given as follows. Recall the set of almost complex structures $\mathcal{J}(X)$ in Subsection 2.1 can be viewed as the set of certain "compactly supported" perturbation of J_X. However a common support of the perturbations must be specified in order to have a complete space. Furthermore, any almost complex structure J must make D almost complex in order to identify local intersection numbers with contact orders. Finally to control the singularities of the divisor we require $J = J_X$ near the unstable locus. So now we fix a space of more restricted almost complex structures. We fix a K-invariant open neighborhood U of $\mu^{-1}(0)$ where the K-action is free. Define

$$\begin{aligned}\mathcal{J}(U, D) =& \big\{ J \in \mathcal{J}(X) \mid J|_D = J_X,\ J|_{X \smallsetminus U} = J_X \big\},\\ \mathcal{J}(\bar X, \bar D) =& \big\{ \bar J \in \mathcal{J}(\bar X) \mid \bar J|_{\bar D} = \bar J_X \big\}.\end{aligned}$$

Clearly there is a smooth map $\mathcal{J}(U, D) \to \mathcal{J}(\bar X, \bar D)$. Further, the following proposition implies nonempty intersections between D and holomorphic objects on open domains.

Proposition 3.4. (cf. [CW17, Equation (15)]) *Given $J \in \mathcal{J}(X, D)$. Suppose $\boldsymbol{v}$ is either a $\bar J$-holomorphic disk, or a J-holomorphic quasidisk, or a J-affine vortex over $\boldsymbol{H}$. If $\boldsymbol{v}$ has positive energy, then $\boldsymbol{v}^{-1}(D) \neq \varnothing$.*

Proof. The equivariant rationality of L implies that L is exact in $X \smallsetminus D$. Reducing to $\bar X$ it implies that $\bar L$ is exact in $\bar X \smallsetminus \bar D$. Then the case of $\boldsymbol{v}$ being a holomorphic disk is proved in [CW17]. For the case of $\boldsymbol{v}$ being a quasidisk, if $\boldsymbol{v}^{-1}(D) = \varnothing$, then the exactness implies that $E(\boldsymbol{v}) = 0$. For the case of $\boldsymbol{v}$ being an affine vortex over $\boldsymbol{H}$, suppose $\boldsymbol{v}^{-1}(D) = \varnothing$. Assume that in some gauge (see [WX17]), the connection part $a \in \Omega^1(\boldsymbol{H})$ of $\boldsymbol{v}$ is well-behaved at infinity. Then by the energy identity for affine vortices (2.4), one has

$$E(\boldsymbol{v}) = \int_{\boldsymbol{H}} \big[u^* \omega_X + {\rm d}(a \cdot \mu) \big] = \int_{\boldsymbol{H}} u^* \omega_X = 0..$$

Hence $\boldsymbol{v}$ is trivial. □

The above proposition implies that the intersection points with the divisor stabilize the domain in the case of disks. However, the corresponding discussion for spheres is more involved since in this case we need at least three points. For $E > 0$, define

$$\mathcal{J}_E(U, D) = \big\{ J \in \mathcal{J}(U, D) \bigm| J \text{ satisfies condition } {\rm Stab}_E \big\}$$

where ${\rm Stab}_E$ is the condition

(${\rm Stab}_E$) Every nonconstant $\bar J$-holomorphic sphere in $\bar X$ having energy at most E intersects $\bar D$ at at least three distinct points.

The following is similar to Cieliebak–Mohnke [CM07, Proposition 8.11].

Lemma 3.5. *For every $E > 0$, $\mathcal{J}_E(U, D)$ contains an open neighborhood of J_X in $\mathcal{J}(U, D)$ in the C^2-topology.*

Proof. The proof of the statement of the lemma is a Gromov compactness argument. First, by Corollary 3.3, we see that $J_X \in \mathcal{J}_E(U, D)$ for all $E > 0$. Suppose $\mathcal{J}_E(U, D)$ does not contain an open neighborhood of J_X for some $E > 0$. Then there exists a sequence $J_i \in \mathcal{J}(U, D) \smallsetminus \mathcal{J}_E(U, D)$ that converges in C^2-topology to J_X, such that there is a sequence of nonconstant $\bar J_i$-holomorphic spheres $\bar u_i : \boldsymbol{S}^2 \to \bar X$ that is not contained in $\bar{D}$ but each $\bar{u}_i$ only intersects with $\bar{D}$ at at most two points. Then by Gromov compactness [MS04, Chapter 5] a subsequence of $\bar{u}_i$ converges to a stable $\bar{J}_X$-holomorphic sphere in $\bar{X}$, and at least one component of the limit has positive energy. By Corollary 3.3, this component, denoted by $\bar{u} : \boldsymbol{S}^2 \to \bar{X}$, cannot lie in $\bar{D}$ and must intersects with $\bar{D}$ at at least three points. Then in the subsequence, for i sufficiently large, $\bar{u}_i$ is not contained in $\bar{D}$. We claim that for i sufficiently large, $\bar{u}_i$ intersects with $\bar{D}$ at at least three points. Indeed, by reparametrizing the maps $\bar{u}_i$, we may assume that $\bar{u}_i$ converges to $\bar{u}$ as maps away from nodal points. Then for each $z \in \bar{u}^{-1}(\bar{D})$, if z is not a node, then the local intersection number n_z of $\bar{u}$ restricted to a small neighborhood B_z of z is positive; if z is a node, then the local intersection number of $\bar{u}$ restricted to a small neighborhood B_z of z, plus the intersection number of the bubble tree attached at z is also positive. Denote the sum as n_z. In both cases, by the homotopy invariance of local intersection numbers, for i sufficiently large, n_z coincides with the local intersection number of $\bar{u}_i$ restricted to B_z. Since all these small neighborhoods B_z are disjoint in the domains of $\bar{u}_i$, we see that for i sufficiently large, $\bar{u}_i^{-1}(\bar{D})$ contains at least three points, which is a contradiction. □

To ensure that negative index vortices do not occur, we also need the following result to make use of Hypothesis 1.5 and Hypothesis 1.7.

Lemma 3.6. *For any $E > 0$, there exists $\epsilon_E > 0$ satisfying the following conditions. Let Σ be either $\boldsymbol{D}^2$, $\boldsymbol{S}^2$, $\boldsymbol{C}$ or $\boldsymbol{H}$, and let $J : \Sigma \to \mathcal{J}(U, D)$ be a smooth map such that $\|J - J_X\|_{C^0(\Sigma, C^2(X))} \leqslant \epsilon_E$.*

- *(a) Suppose Hypothesis 1.5 is satisfied. Then every nonconstant J-holomorphic disk $u : \boldsymbol{D}^2 \to X$ with boundary in L whose energy is at most E has positive Maslov index.*
- *(b) Suppose both Hypothesis 1.5 and Hypothesis 1.7 are satisfied. Then in addition*
  - *(i) For a nonconstant $\bar{J}$-holomorphic disk $u : \boldsymbol{D}^2 \to \bar{X}$ with boundary in $\bar{L}$ (resp. a nonconstant J-affine vortex over $\boldsymbol{H}$ with boundary in $\bar{L}$) whose energy is at most E, its Maslov index $\mathrm{Mas}(u)$ is positive.*
  - *(ii) For any $\bar{J}$-holomorphic sphere $\bar{u} : \boldsymbol{S}^2 \to \bar{X}$ (resp. any J-affine vortex $\boldsymbol{v}$ over $\boldsymbol{C}$) whose energy is at most E, its Chern number $\langle c_1(T\bar{X}), [\bar{u}]\rangle$ (resp. equivariant Chern number $\langle c_1^K(TX), [\boldsymbol{v}]\rangle$) is nonnegative.*
  - *(iii) For any nonconstant $\bar{J}$-holomorphic sphere in $\bar{X}$ (resp. any nontrivial J-affine vortex over $\boldsymbol{C}$) whose energy is at most E, if its Chern number (resp. equivariant Chern number) is zero, then its image is disjoint from $\bar{L}$ (resp. L).*

Proof. We only prove (bii) and the case of affine vortex over $\boldsymbol{H}$. The other cases are similar and simpler. Suppose the conclusion is not true. Then there exists a sequence of smooth maps $J_i : \boldsymbol{H} \to \mathcal{J}(U, D)$ converging to J_X uniformly, and a sequence of nonconstant J_i-affine vortices $\boldsymbol{v}_i$ over $\boldsymbol{H}$ with uniformly bounded energy. Then by Gromov compactness, a subsequence (still indexed by i) converges to a stable J_X-holomorphic disk. Moreover, by energy quantization, the energy of $\boldsymbol{v}_i$ is bounded from below. By the conservation of energy, the limit object has positive energy. Therefore, by Hypothesis 1.5, Hypothesis 1.7, and the monotonicity condition of X, the total Maslov index of the limit object is nonnegative. Moreover, if the Maslov index is zero, then all holomorphic disks or affine vortices over $\boldsymbol{H}$ in the limit object have to be constants, hence their images are contained in L. Moreover, in the limit object there is only one nonconstant J_X-holomorphic sphere in $\bar{X}$ or a nonconstant J_X-affine vortex whose Maslov index is zero. However, by Hypothesis 1.7, both possibilities are impossible because such a sphere or an affine vortex has to be disjoint from $\bar{L}$ or L. The conclusion follows. □

For later use we introduce the following notation for sequences of spaces almost complex structures satisfying the required perturbation conditions. Fix a positive number $E_0 > 0$ such that the energies of nonconstant holomorphic curves or affine vortices are integral multiples of E_0. Choose a sequence of open neighborhoods

$$J_X \in \cdots \mathcal{J}_n(U,D) \subset \mathcal{J}_{n-1}(U,D) \subset \cdots \subset \mathcal{J}_1(U,D) \subset \mathcal{J}(U,D) \tag{3.1}$$

such that $\mathcal{J}_n(U,D) \subset \mathcal{J}_{nE_0}(U,D)$ where $\mathcal{J}_{nE_0}(U,D)$ is the one given in Lemma 3.5, and for each $J \in \mathcal{J}_n(U,D)$,

$$\|J - J_X\|_{C^2(X)} \leqslant \epsilon_{nE_0}$$

where ϵ_{nE_0} is given by Lemma 3.6.

Given one of the admissible almost complex structures above, the intersection multiplicity of any gauged map with the divisor may be identified with the order of vanishing along the divisor. Let $J \in \mathcal{J}(U,D)$ and suppose there is one irreducible component $D_i \subset D$. If $u : \boldsymbol{D}^2 \to X$ is a J-holomorphic map and $0 \in \boldsymbol{D}^2$ is an isolated intersection point with $D_i^{\rm st}$, denote the contact order of u at 0 with D^0 by

$$l(u,0) = \left(\min_{k \geqslant 1} \operatorname{Im} D^k u(0) \subset T_{u(0)} D^0 \right) - 1 \in \mathbb{N} \cup \{0\}.$$

By [CM07, Proposition 7.1] if $0 \in \boldsymbol{D}^2$ is an isolated intersection point with $D_i^{\rm st}$ and $l \geqslant 0$ be the contact order of u at 0. Then the local intersection number of u with D_i at zero is $l+1$. Since D_i are all G-invariant, the notion of contact order can also be extended to pseudoholomorphic gauged maps and it is invariant under gauge transformations.

The intersection number of any gauged map with the stabilizing divisor is nonnegative:

Lemma 3.7. *Given $J \in \mathcal{J}(U,D)$ and let $\boldsymbol{v} = (u,\phi,\psi)$ be a J-holomorphic gauged map from $\boldsymbol{D}^2$ to X such that $u(\partial \boldsymbol{D}^2) \cap D = \varnothing$. Then the intersection number $u \cap D \geqslant 0$.*

Proof. Indeed, we can regard u as a $\tilde{J}_{\phi,\psi}$-holomorphic map $\tilde{u} : \boldsymbol{D}^2 \to \boldsymbol{D}^2 \times X$, where $\tilde{J}_{\phi,\psi}$ is an almost complex structure on the product such that $\tilde{J}_{\phi,\psi}(0,V) = (0,JV)$ for any $V \in TX$. The product $\tilde{D} = \boldsymbol{D}^2 \times D$ is of codimension 2 and $\tilde{J}_{\phi,\psi}$-complex by the G-invariance of D. The inverse image of the divisor is $u^{-1}(D) = \tilde{u}^{-1}(\tilde{D})$, and, by the positivity of intersection numbers between pseudoholomorphic curves and almost complex hypersurfaces, (see discussions in [MS04, Appendix E]), the conclusion follows. □

4. Combinatorics

In this section we give a comprehensive treatment of the combinatorics of trees used in this paper. Similar treatments have appeared in several previous papers [Woo11], [CW17], [CW15]. This version is adapted for vortices.

4.1. **Trees.** We begin with the combinatorics of trees. First we look at rooted trees which are combinatorial models for nodal spheres. A *rooted tree* Γ consists of a tree with its set of vertices $\mathrm{V}(\Gamma)$ and its set of edges $\mathrm{E}(\Gamma)$, a set $\mathrm{T}(\Gamma)$ of *tails* with a *tail map* $\mathrm{T}(\Gamma) \to \mathrm{V}(\Gamma)$, and a distinguished tail $t_{\mathrm{out}} \in \mathrm{T}(\Gamma)$ called the *output.* The image $\mathfrak{v}_\infty \in \mathrm{V}(\Gamma)$ of t_{out} is called the *root.* The remaining tails $\mathrm{T}(\Gamma) - \{t_{\mathrm{out}}\}$ are called the *inputs.* The output determines the orientation on edges so that all edges are pointing towards the output. The orientation can also be viewed as a partial order $\leqslant$ among vertices: we denote $\mathfrak{v}_\beta \leqslant \mathfrak{v}_\alpha$ if $\mathfrak{v}_\beta$ is closer to the root $\mathfrak{v}_\infty$. We write $\mathfrak{v}_\alpha \succ \mathfrak{v}_\beta$ if $\mathfrak{v}_\beta \leqslant \mathfrak{v}_\alpha$ and $\mathfrak{v}_\alpha$ and $\mathfrak{v}_\beta$ are adjacent. Associated to a rooted tree is a topological space, also denoted Γ, obtained by replacing vertices with points, edges and tails with line segments. The *boundary* Γ_∂ of Γ is a finite union of points, one " at infinity" on each tail.

The combinatorial type of a nodal disk with both disk and sphere components is a *based tree*, in which a distinguished subtree corresponds to the disk components. To set up the definition, we say that a rooted subtree of a rooted tree Γ is a subtree Γ' of Γ which contains the root $\mathfrak{v}_\infty$. Γ' becomes a rooted tree by setting its set of tails $\mathrm{T}(\Gamma')$ as the subset of $\mathrm{T}(\Gamma)$ consisting of tails that are attached to vertices in Γ'. In the remainder of this paper, all trees are rooted and the term "rooted" will be skipped. On the other hand, a *ribbon tree*, denoted by $\underline{\Gamma}$ in this paper, is a tree equipped with an isotopy class of topological embeddings of Γ into the unit 2-disk with Γ_∂ mapped to the boundary of the disk. With the above notions understood, a *based tree* consists of a rooted tree Γ with set of tails $\mathrm{T}(\Gamma)$, a nonempty subtree $\underline{\Gamma}$ with a subset $\underline{\mathrm{T}}(\underline{\Gamma}) \subset \mathrm{T}(\underline{\Gamma})$ of tails containing t_{out}, such that the rooted tree $(\underline{\Gamma}, \underline{\mathrm{T}}(\underline{\Gamma}))$, called the *base*, is equipped with the structure of a ribbon tree. To clarify different tails, we call elements of $\underline{\mathrm{T}}(\underline{\Gamma})$ *boundary tails* and elements of

$$\mathrm{L}(\Gamma) := \mathrm{T}(\Gamma) \smallsetminus \underline{\mathrm{T}}(\underline{\Gamma})$$

interior leaves of Γ. The numbers of boundary tails and interior leaves are usually denoted by $\underline{k}$ and k respectively. To model nodal spheres with marked points, we also call a rooted tree Γ without specifying a base a *baseless tree.* In this case the set $\mathrm{L}(\Gamma) = \mathrm{T}(\Gamma) \smallsetminus \{t_{\mathrm{out}}\}$ is also called the set of interior leaves.

Inclusions of strata in the moduli spaces of vortices correspond to certain morphisms of trees with additional data. A *morphism* between two based resp. baseless trees Γ' and Γ, denoted by $\rho : \Gamma' \to \Gamma$, consists of a surjective map $\rho_{\mathrm{V}} : \mathrm{V}(\Gamma') \to \mathrm{V}(\Gamma)$ between the set of vertices and a bijection $\rho_{\mathrm{T}} : \mathrm{T}(\Gamma) \to \mathrm{T}(\Gamma')$ between the sets of leaves, both of which satisfy the following conditions.

(a) $\rho_{\mathrm{V}} : \mathrm{V}(\Gamma') \to \mathrm{V}(\Gamma)$ gives a morphisms of trees such that $\rho_{\mathrm{V}}(\mathrm{V}(\underline{\Gamma}')) = \mathrm{V}(\underline{\Gamma})$.
(b) If a boundary tail or an interior leaf $t' \in \mathrm{T}(\Gamma')$ is attached to a vertex $\mathfrak{v}_{\underline{\alpha}'} \in \mathrm{V}(\underline{\Gamma}')$, then $\rho_{\mathrm{T}}(t')$ is attached to $\rho_{\mathrm{V}}(\mathfrak{v}_{\underline{\alpha}'})$.
(c) $\rho_{\mathrm{T}}(t'_{\mathrm{out}}) = t_{\mathrm{out}}$ and $\rho_{\mathrm{T}}(\underline{\mathrm{T}}(\underline{\Gamma}')) = \underline{\mathrm{T}}(\underline{\Gamma})$.

Hence a morphism ρ induces a bijection between the set of leaves $\rho_{\mathrm{L}} : \mathrm{L}(\Gamma') \to \mathrm{L}(\Gamma)$. Given a morphism $\rho : \Gamma' \to \Gamma$, for any vertex $\mathfrak{v} \in \mathrm{V}(\Gamma)$, the vertices in the preimage $\rho_{\mathrm{V}}^{-1}(\mathfrak{v})$ together with all edges among them and all leaves, boundary tails attached to them form a subtree of Γ'. An edge in this subtree is said to be *contracted* by ρ. Any edge that does not belong to such a subtree is said to be *preserved* by ρ.

We introduce the following terminology for broken trees. A *broken* ribbon tree is a ribbon tree $\underline{\Gamma}$ with a set of vertices of valence 2, called "breakings." A broken based tree is a tree Γ such that $\underline{\Gamma}$ is broken and the breakings are not adjacent to any vertices in $\mathrm{V}(\Gamma) \smallsetminus \mathrm{V}(\underline{\Gamma})$. However, we will view a broken based tree in a different way, namely, we view a broken based tree as the union of finitely many unbroken based trees glued along the breakings. Each unbroken component of a broken tree is called a *basic part*. In notation, we do not treat the breakings as elements of $\mathrm{V}(\Gamma)$, but instead, each breaking gives two elements in $\underline{\mathrm{T}}(\underline{\Gamma})$, an input of some basic part and an output of another basic part. There is a naturally induced order on the set of basic parts, denoted by $\Gamma_i \succ \Gamma_j$. Given a broken tree, one can *glue* it at a subset of breakings, and obtain a tree with fewer breakings.

We put various extra discrete structures to a based or baseless tree Γ, consisting of a *coloring*, a *metric type*, and a *contact datum*. The tree Γ together with these structures will be called a *combinatorial type*.

Definition 4.1. (Coloring) Let Γ be an unbroken based or baseless tree. Let $\triangle$, $\diamond$, $\triangledown$ be ordered as $\triangle \leqslant \diamond \leqslant \triangledown$. A *coloring* on Γ is an order-reversing map

$$\mathfrak{s} : \mathrm{V}(\Gamma) \to \{\triangle, \diamond, \triangledown\}$$

satisfying

(Monotonicity condition) within any non-self-crossing path $\alpha_1 \succ \cdots \succ \alpha_l$ in Γ with $\mathfrak{s}(\alpha_1) \leqslant \diamond$, $\mathfrak{s}(\alpha_l) \geqslant \diamond$, there is a unique α_j with $\mathfrak{s}(\alpha_j) = \diamond$.

For $a = \triangle, \diamond, \triangledown$, we denote $\mathrm{V}_a(\Gamma) = \mathfrak{s}^{-1}(a)$. A coloring on Γ induces a partition

$$\mathrm{T}(\Gamma) = \mathrm{T}_\triangle(\Gamma) \sqcup \mathrm{T}_\triangledown(\Gamma)$$

as follows. $t \in \mathrm{T}_\triangle(\Gamma)$ if the vertex to which t is attached is in $\mathrm{V}_\triangle(\Gamma) \sqcup \mathrm{V}_\diamond(\Gamma)$; otherwise $t \in \mathrm{T}_\triangledown(\Gamma)$. A *colored tree* $(\Gamma, \mathfrak{s})$ consists of a based tree Γ and a coloring $\mathfrak{s}$. Given a colored tree $(\Gamma, \mathfrak{s})$, denote by

$\mathrm{E}_\triangle(\Gamma)$ be the set of edges between vertices in $\mathrm{V}_\triangle(\Gamma) \sqcup \mathrm{V}_\diamond(\Gamma)$
$\mathrm{E}_\triangledown(\Gamma)$ the set of edges between vertices in $\mathrm{V}_\triangledown(\Gamma)$, and
$\mathrm{E}_\diamond(\Gamma)$ the set of edges between a vertex in $\mathrm{V}_\diamond(\Gamma)$ and a vertex in $\mathrm{V}_\triangledown(\Gamma)$.

Elements of the first two sets of edges are called *non-special edges* and the elements of the last set of edges are called *special edges*. A colored tree $(\Gamma, \mathfrak{s})$ is of *type* $\triangle$ if $\mathfrak{s} \equiv \triangle$; $(\Gamma, \mathfrak{s})$ is of *type* $\triangledown$ if it is based and $\mathfrak{s}|_{\mathrm{V}(\underline{\Gamma})} \equiv \triangledown$, or if it is baseless and $\mathfrak{s} \equiv \triangledown$; otherwise we say that $(\Gamma, \mathfrak{s})$ is of *mixed type* or *type* $\diamond$. In the last case there could be incoming boundary tails of both type $\triangle$ and type $\triangledown$.

The set of maximal vertices is denoted as follows. When an unbroken colored tree $(\Gamma, \mathfrak{s})$ is of type $\diamond$, denote by $\mathrm{V}_{\partial\triangledown}(\Gamma) \subset \mathrm{V}_\triangledown(\Gamma)$ the set of vertices in $\mathrm{V}_\triangledown(\Gamma)$ which are maximal (with respect to the partial order $\leqslant$). When $(\Gamma, \mathfrak{s})$ is of type $\triangledown$, define $\mathrm{V}_{\partial\triangledown}(\Gamma) \subset \mathrm{V}_\triangledown(\Gamma) \smallsetminus \mathrm{V}(\underline{\Gamma})$ to be the set of vertices in $\mathrm{V}_\triangledown(\Gamma)$ which are maximal (with respect to the partial order $\leqslant$). When $(\Gamma, \mathfrak{s})$ is of type $\triangle$, define $\mathrm{V}_{\partial\triangledown}(\Gamma) = \varnothing$.

If Γ is a broken tree, then a coloring on Γ consists a collection of colorings $\mathfrak{s}_1, \ldots, \mathfrak{s}_m$ on all its basic parts $\Gamma_1, \ldots, \Gamma_m$, satisfying the following conditions: 1) the induced map $\mathfrak{s} : \mathrm{V}(\Gamma) \to \{\triangle, \diamond, \triangledown\}$ is order-reversing; 2) if $\Gamma_1 \succ \cdots \succ \Gamma_l$ is a chain of basic parts and $(\Gamma_1, \mathfrak{s}_1)$ is of type $\triangle$ or $\diamond$, $(\Gamma_l, \mathfrak{s}_l)$ is of type $\diamond$ or $\triangledown$, then there is a unique basic part Γ_j in this chain such that $(\Gamma_j, \mathfrak{s}_j)$ is of type $\diamond$.

We list a few special colored trees which play certain roles in our construction.

(a) An *infinite edge* is a tree Γ with $\mathrm{V}(\Gamma) = \mathrm{E}(\Gamma) = \varnothing$ (hence the coloring is empty) and $\mathrm{T}(\Gamma) = \{t_{\mathrm{in}}, t_{\mathrm{out}}\}$.
(b) A *Y-shape* is a colored tree $(\Gamma, \mathfrak{s})$ with $\Gamma = \underline{\Gamma}$, $\mathrm{V}(\Gamma) = \{\mathfrak{v}_\infty\}$, $\mathrm{E}(\Gamma) = \varnothing$, $\mathrm{T}(\Gamma) = \{t'_{\mathrm{in}}, t''_{\mathrm{in}}, t_{\mathrm{out}}\}$ and $\mathfrak{s}(\mathfrak{v}_\infty)$ being $\triangle$ or $\triangledown$.
(c) A *Φ-shape* is a colored tree $(\Gamma, \mathfrak{s})$ with $\Gamma = \underline{\Gamma}$, $\mathrm{V}(\Gamma) = \{\mathfrak{v}_\infty\}$, $\mathrm{E}(\Gamma) = \varnothing$, $\mathrm{T}(\Gamma) = \{t_{\mathrm{in}}, t_{\mathrm{out}}\}$ and $\mathfrak{s}(\mathfrak{v}_\infty) = \diamond$.

The moduli space of vortices we consider will be a union of strata corresponding to *stable types*, defined as follows. In this paper the valence of a vertex $\mathfrak{v}$ in a tree Γ is the number of vertices in Γ that are attached to $\mathfrak{v}$ (tails attached to $\mathfrak{v}$ do not count).

Definition 4.2. (Stability) A colored tree $(\Gamma, \mathfrak{s})$ that does not consist of a single infinite edge is called *stable* if the following conditions are satisfied.

(a) For each $\mathfrak{v}_\alpha \in \mathrm{V}_\triangledown(\Gamma) \sqcup \mathrm{V}_\vartriangle(\Gamma) \smallsetminus \mathrm{V}(\underline{\Gamma})$, the valence of $\mathfrak{v}_\alpha$ plus the number of tails that are attached to $\mathfrak{v}_\alpha$ is at least 3.

(b) For each $\mathfrak{v}_{\underline{\alpha}} \in \mathrm{V}_\triangledown(\underline{\Gamma}) \sqcup \mathrm{V}_\vartriangle(\underline{\Gamma})$, the valence of $\mathfrak{v}_{\underline{\alpha}}$ in $\underline{\Gamma}$ plus the number of boundary tails that are attached to $\mathfrak{v}_{\underline{\alpha}}$ plus twice the number of vertices in $\mathrm{V}(\Gamma) \smallsetminus \mathrm{V}(\underline{\Gamma})$ that are attached to $\mathfrak{v}_{\underline{\alpha}}$ plus twice the number of interior leaves that are attached to $\mathfrak{v}_{\underline{\alpha}}$ is at least 3.

(c) For each $\mathfrak{v}_\alpha \in \mathrm{V}_\diamond(\Gamma)$, the valence of $\mathfrak{v}_\alpha$ plus the number of tails that are attached to $\mathfrak{v}_\alpha$ is at least 2.

Definition 4.3. (Metric) Let $(\Gamma, \mathfrak{s})$ be an unbroken colored based tree. A *metric type* on $(\Gamma, \mathfrak{s})$ consists of two functions $\mathfrak{m} = (\mathfrak{m}', \mathfrak{m}'')$, $\mathfrak{m}' : \mathrm{E}(\underline{\Gamma}) \to \{0, +\}$ and $\mathfrak{m}'' : \mathrm{V}_{\partial\triangledown}(\underline{\Gamma}) \to \{0, -\}$. These functions induce two partitions

$$\mathrm{E}(\underline{\Gamma}) = \mathrm{E}^0(\underline{\Gamma}) \sqcup \mathrm{E}^+(\underline{\Gamma}), \qquad \mathrm{V}_{\partial\triangledown}(\underline{\Gamma}) = \mathrm{V}^0_{\partial\triangledown}(\underline{\Gamma}) \sqcup \mathrm{V}^-_{\partial\triangledown}(\underline{\Gamma}).$$

A *metric* on Γ of type $\mathfrak{m} = (\mathfrak{m}', \mathfrak{m}'')$ is a map $\boldsymbol{\lambda} : \mathrm{E}(\underline{\Gamma}) \to [0, \infty)$ satisfying $\mathrm{E}^0(\underline{\Gamma}) = \boldsymbol{\lambda}^{-1}(0)$ and the following condition.

(Balanced condition) For each $\mathfrak{v}_{\underline{\alpha}} \in \mathrm{V}_\diamond(\underline{\Gamma}) \sqcup \mathrm{V}_\triangledown(\underline{\Gamma})$, let $P(\mathfrak{v}_{\underline{\alpha}}, \mathfrak{v}_\infty)$ be the unique path in $\underline{\Gamma}$ connecting $\mathfrak{v}_{\underline{\alpha}}$ with the root, viewed as a subset of $E(\underline{\Gamma})$. Consider the function $\tilde{\boldsymbol{\lambda}} : \mathrm{V}_\diamond(\underline{\Gamma}) \sqcup \mathrm{V}_\triangledown(\underline{\Gamma}) \to \mathbb{R}$ defined by

$$\tilde{\boldsymbol{\lambda}}(\mathfrak{v}_{\underline{\alpha}}) := \sum_{\mathfrak{e} \in P(\mathfrak{v}_{\underline{\alpha}}, \mathfrak{v}_\infty)} \boldsymbol{\lambda}(\mathfrak{e}).$$

We require that $\tilde{\boldsymbol{\lambda}}$ restricted to $\mathrm{V}_\diamond(\underline{\Gamma})$ is a constant (say b), $\tilde{\boldsymbol{\lambda}}(\mathfrak{v}_{\underline{\alpha}}) < b$ for $\mathfrak{v}_{\underline{\alpha}} \in \mathrm{V}^-_{\partial\triangledown}(\underline{\Gamma})$ and $\tilde{\boldsymbol{\lambda}}(\mathfrak{v}_{\underline{\alpha}}) = b$ for $\mathfrak{v}_{\underline{\alpha}} \in \mathrm{V}^0_{\partial\triangledown}(\underline{\Gamma})$. (The balanced condition implies that if $\mathfrak{v}_{\underline{\alpha}} \in \mathrm{V}_\diamond(\underline{\Gamma})$ is adjacent to $\mathfrak{v}_{\underline{\beta}} \in \mathrm{V}^0_{\partial\triangledown}(\underline{\Gamma})$, then $\lambda(\mathfrak{e}_{\underline{\alpha}\underline{\beta}}) = 0$.)

The balanced condition is nonempty only for colored trees of mixed type. A metric type (resp. metric) on a broken colored tree is a collection of metric types (resp. metrics) on all its basic parts. Notice that for a broken tree, the balanced condition only applies to its basic parts of type $\diamond$.

Since we are using Cieliebak–Mohnke's [CM07] stabilizing divisor technique, we need to record the contact orders of holomorphic curves or vortices at interior markings with respect to a (singular) divisor. For a colored tree $(\Gamma, \mathfrak{s})$, a *contact datum* is a function

$$\mathfrak{o} : \mathrm{L}(\Gamma) \to \mathbf{N}.$$

Definition 4.4. (Combinatorial types)

(a) An *combinatorial type* is a quadruple $(\Gamma, \mathfrak{s}, \mathfrak{m}, \mathfrak{o})$ where Γ is a based tree, $\mathfrak{s}$ is a coloring on Γ, $\mathfrak{m}$ is a metric type on $(\Gamma, \mathfrak{s})$, and $\mathfrak{o}$ is a contact data. The type $(\Gamma, \mathfrak{s}, \mathfrak{m}, \mathfrak{o})$ is called *stable* if $(\Gamma, \mathfrak{s})$ is stable. In many cases we use Γ to abbreviate the quadruple.

(b) Two combinatorial types $(\Gamma, \mathfrak{s}, \mathfrak{m}, \mathfrak{o})$, $(\Gamma', \mathfrak{s}', \mathfrak{m}', \mathfrak{o}')$ are *isomorphic* if there is a tree isomorphism $\rho : \Gamma' \to \Gamma$ which respects all the extra structures. Let $\mathbf{T}$ be the set of isomorphism classes of types and $\mathbf{T}^{\mathbf{st}} \subset \mathbf{T}$ be the subset of stable ones. Without making any confusion, we drop the term "isomorphism class" and call an element $\mathbf{T}$ a combinatorial type.

4.2. **Degeneration and broken trees.** The possible degenerations of colored metric trees involving "degenerating" an edge in to obtain a broken tree, or extend an edge of length zero to one with positive length. The balanced condition in Definition 4.3 imposes some restrictions on such operations.

Definition 4.5. (Elementary morphisms of types) Let $\Gamma', \Gamma \in \mathbf{T}$ be types such that Γ is *unbroken* and let $\mathfrak{s}'$, $\mathfrak{s}$ be their colorings respectively. We say that Γ is obtained from Γ' by an *elementary morphism* if one of the following situations holds. (From Figure 1 to Figure 7, ■ represents a vertex not in the base, • represents a vertex in the base, and ▾ represents a breaking.)

(a) (Collapsing a non-special edge not in the base) There is a morphism $\rho : (\Gamma', \mathfrak{s}') \to (\Gamma, \mathfrak{s})$ which collapses exactly one edge in $\mathrm{E}(\Gamma') \smallsetminus (\mathrm{E}(\underline{\Gamma}') \cup \mathrm{E}_\diamond(\Gamma'))$. Geometrically the morphism corresponds to inclusion of a stratum bubbling off a sphere. See Figure 1; in the figures square vertices represent spherical components while the round vertices represent disk components.

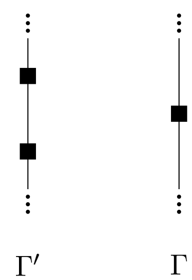

FIGURE 1. Collapsing a non-special edge not in the base.

(b) (Collapsing a non-special edge of length zero in the base) There is a morphism $\rho : (\Gamma', \mathfrak{s}') \to (\Gamma, \mathfrak{s})$ such that ρ collapses exactly one edge in $\mathrm{E}^0(\underline{\Gamma}') \smallsetminus \mathrm{E}_\diamond(\underline{\Gamma}')$. Geometrically the morphism corresponds to inclusion of a stratum corresponds to bubbling off a disk. See Figure 2.

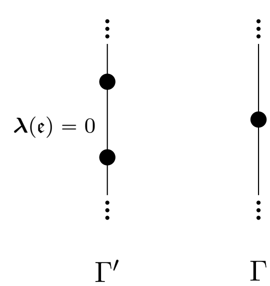

FIGURE 2. Collapsing a non-special edge of length zero in the base.

(c) (Collapsing special edges of length zero) There is a morphism $\rho : \Gamma' \to \Gamma$ and a vertex $\mathfrak{v}'_{\underline{\alpha}} \in \mathrm{V}^0_{\partial\triangledown}(\underline{\Gamma}')$, such that ρ collapses (and only collapses) the subtree consisting of $\mathfrak{v}'_{\underline{\alpha}}$ and all $\mathfrak{v}'_\beta$ with $\mathfrak{v}'_\beta \succ \mathfrak{v}'_{\underline{\alpha}}$ to a vertex $\mathfrak{v}_{\diamond\alpha} \in \mathrm{V}_\diamond(\underline{\Gamma})$. Geometrically the morphism corresponds to the gluing of stable affine vortices over $\boldsymbol{H}$ (i.e. the case considered in [Xu]). See Figure 3.

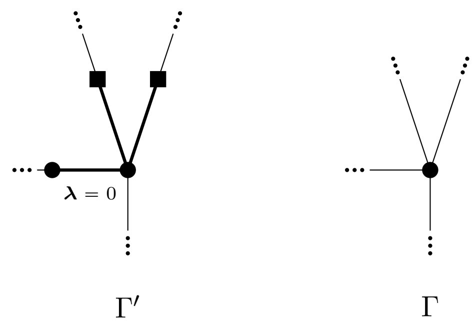

FIGURE 3. Collapsing special edges of length zero in the base.

(d) (Collapsing special edges not in the base) There is a morphism $\rho : \Gamma' \to \Gamma$ and a vertex $\mathfrak{v}'_\alpha \in \mathrm{V}_{\partial\triangledown}(\Gamma') \smallsetminus \mathrm{V}(\underline{\Gamma}')$, such that ρ collapses (and only collapses) the subtree consisting of $\mathfrak{v}'_\alpha$ and all $\mathfrak{v}'_\beta$ with $\mathfrak{v}'_\beta \succ \mathfrak{v}'_\alpha$ to a single vertex $\mathfrak{v}_\alpha \in \mathrm{V}_\diamond(\Gamma) \smallsetminus \mathrm{V}(\underline{\Gamma})$. This type of morphism corresponds to the gluing of stable affine vortices over $\boldsymbol{C}$. See Figure 4.

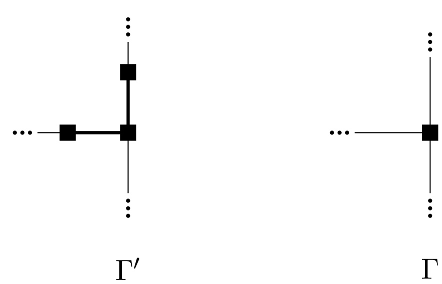

FIGURE 4. Collapsing special edges not in the base.

(e) (Extending a non-special edge of length zero) There is a morphism $\rho : \Gamma' \to \Gamma$ that is an isomorphism on the underlying tree and preserves all extra structures except that there is one edge $\underline{\mathfrak{e}}' \in \mathrm{E}^0(\underline{\Gamma}')$ identified with an edge $\underline{\mathfrak{e}} \in \mathrm{E}^+(\underline{\Gamma})$. See Figure 5.

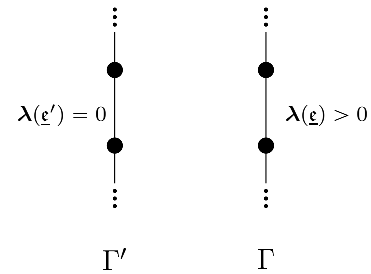

FIGURE 5. Extending a non-special edge of length zero.

(f) (Extending special edges of length zero) There is an isomorphism

$$\rho : (\Gamma', \mathfrak{o}', \mathfrak{s}', \mathfrak{w}') \to (\Gamma, \mathfrak{o}, \mathfrak{s}, \mathfrak{w})$$

that preserves metric types on all edges with the following exceptions. There is a vertex $\mathfrak{v}'_{\underline{\alpha}} \in \mathrm{V}^0_{\partial\triangledown}(\underline{\Gamma}')$ such that for all $\mathfrak{v}'_{\underline{\beta}} \succ \mathfrak{v}'_{\underline{\alpha}}$, $\mathfrak{e}'_{\underline{\beta\alpha}} \in \mathrm{E}^0(\underline{\Gamma}')$ and the corresponding edge $\mathfrak{e}_{\underline{\beta\alpha}} \in \mathrm{E}^+(\underline{\Gamma})$. See Figure 6.

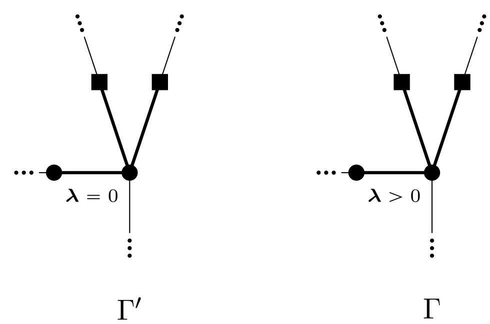

FIGURE 6. Extending special edges of length zero.

(g) (Gluing one breaking) When both Γ' and Γ are of type $\triangle$ or type $\diamond$, Γ' is obtained from Γ by degenerating an edge $\underline{\mathfrak{e}} \in \mathrm{E}^+(\underline{\Gamma}) \cap \mathrm{E}_\triangle(\underline{\Gamma})$. When both Γ' and Γ are of type $\triangledown$, Γ' is obtained from Γ by degenerating an edge $\underline{\mathfrak{e}} \in \mathrm{E}^+(\underline{\Gamma})$. See Figure 7.

FIGURE 7. Gluing a breaking to a non-special edge.

(h) (Gluing several breakings) Both Γ' and Γ are of type $\diamond$. Γ' has basic parts $\Gamma'_1, \dots, \Gamma'_m, \Gamma'_\infty$ where Γ'_i are of type $\diamond$ and Γ'_∞ is of type $\triangledown$. Γ' is obtained from Γ by degenerating edges $\mathfrak{e}_1, \dots, \mathfrak{e}_m \in \mathrm{E}^+(\underline{\Gamma}) \cap \mathrm{E}_\triangledown(\underline{\Gamma})$.

Using elementary morphisms define a partial order on the set of types as follows. For $\Pi, \Gamma \in \mathbf{T}$ with Γ unbroken, denote $\Pi \trianglelefteq \Gamma$, if there are types $\Pi = \Gamma'_0, \Gamma'_1, \dots, \Gamma'_{k-1}, \Gamma'_k = \Gamma \in \mathbf{T}$ such that Γ'_i is obtained from Γ'_{i-1} by an elementary transformation. This notion can be extended in a straight forward way to the case that Γ is possibly broken. The proof of the following lemma is left to the reader:

Lemma 4.6. *The relation $\Pi \trianglelefteq \Gamma$ is a partial order. Moreover, if $\Pi \trianglelefteq \Gamma$, then there exists a unique subset of breakings of Π, and, if we denote by Γ' the combinatorial type obtained from Π by gluing this subset of breakings, then there is a unique morphism $\rho : \Gamma' \to \Gamma$.*

4.3. **Treed disks.** The domains of the configurations of vortices in our compactification are constructed from colored trees by replacing each vertex with a nodal disk or sphere, or an affine space or half-space in the case of a colored vertex.

Definition 4.7. (Treed scaled disks) Given an unbroken type $\Gamma = (\Gamma, \mathfrak{s}, \mathfrak{m}, \mathfrak{o})$ a *treed disk* modelled on Γ consists of a collection (Σ_α) of surfaces indexed by $\mathfrak{v}_\alpha \in \mathrm{V}(\Gamma)$, a collection of markings and nodes Z, and a metric $\boldsymbol{\lambda}$ of type $\mathfrak{m}$ such that

(a) (Upper half plane for vertices in the base) If $\mathfrak{v}_\alpha \in \mathrm{V}(\underline{\Gamma})$, then $\Sigma_\alpha = \boldsymbol{H}$;
(b) (Complex plane for verties not in the base) if $\mathfrak{v}_\alpha \notin \mathrm{V}(\underline{\Gamma})$, then $\Sigma_\alpha = \boldsymbol{C}$.

Each Σ_α admits a compactification to a disk $\overline{\Sigma_\alpha} \cong \boldsymbol{D}^2$ or sphere $\overline{\Sigma_\alpha} \cong \boldsymbol{S}^2$ by adding a "point at infinity" $\infty_\alpha \in \overline{\Sigma_\alpha}$. The metric, markings, and nodes form a tuple

$$\mathcal{Z} = \big[\boldsymbol{\lambda},\ \underline{\boldsymbol{z}} = (\underline{z_i})_{1 \leqslant \underline{i} \leqslant \underline{k}},\ \boldsymbol{z} = (z_i)_{1 \leqslant i \leqslant k},\ \boldsymbol{w} = (w_{\alpha\alpha'})_{\mathfrak{e}_{\alpha\alpha'} \in \mathrm{E}(\Gamma)}\big]$$

where $\underline{\boldsymbol{z}}$ (resp. $\boldsymbol{z}$) is the collection of boundary (resp. interior) marked points, and $\boldsymbol{w}$ is the collection of nodes. These data are required to satisfy the following conditions/conventions.

(a) (Boundary and interior nodes) If $\mathfrak{e}_{\underline{\alpha\alpha'}} \in \mathrm{E}(\underline{\Gamma})$, $w_{\underline{\alpha\alpha'}} \in \partial\Sigma_{\underline{\alpha'}}$; if $\mathfrak{e}_{\alpha\alpha'} \notin \mathrm{E}(\underline{\Gamma})$, $w_{\alpha\alpha'} \in \mathbf{Int}\Sigma_{\alpha'}$.
(b) (Boundary and interior markings) $\underline{z_{\underline{i}}} \in \partial\Sigma_{\underline{\alpha_{\underline{i}}}}$, $z_i \in \mathbf{Int}\Sigma_{\alpha_i}$;

For each $\mathfrak{v}_\alpha \in \mathrm{V}(\Gamma)$, the collection of special points

$$W_\alpha := \big\{\underline{z_{\underline{i}}} \mid \underline{\alpha_{\underline{i}}} = \alpha\big\} \cup \big\{z_i \mid \alpha_i = \alpha\big\} \cup \big\{w_{\beta\alpha} \mid \mathfrak{e}_{\beta\alpha} \in \mathrm{E}(\Gamma)\big\} \subset \Sigma_\alpha$$

are required to be distinct, and the order of boundary markings and nodes $w_{\underline{\alpha\alpha'}} \in \partial\Sigma_{\underline{\alpha'}}, \underline{z_{\underline{i}}} \in \partial\Sigma_{\underline{\alpha_{\underline{i}}}}$ corresponding to edges meeting any vertex respect the ribbon structure of $\underline{\Gamma}$. If Γ is broken and has basic parts $(\Gamma_s)_{s=1,\dots,d}$, then a treed disk modelled on Γ is a collection of treed disks $\mathcal{Z}_s$ modelled on every Γ_s. Note that a basic part of $\mathcal{Z}$ could be an infinite edge.

The set of treed disks naturally forms a category. If $\mathcal{Z}$ and $\mathcal{Z}'$ are treed disks modelled on Γ and Γ' respectively, an *isomorphism* from $\mathcal{Z}'$ to $\mathcal{Z}$ consists of an isomorphism $\rho : \Gamma' \to \Gamma$ (which identifies each $\mathfrak{v}'_\alpha \in \mathrm{V}(\Gamma')$ with $\mathfrak{v}_\alpha = \rho_{\mathrm{V}}(\mathfrak{v}'_\alpha) \in \mathrm{V}(\Gamma)$), translations on infinite edges, and biholomorphic maps $\varphi_\alpha : \Sigma'_\alpha \to \Sigma_\alpha$ such that the nodes and markings transform correspondingly. We require that when $\mathfrak{v}_\alpha \in \mathrm{V}_\vartriangle(\Gamma) \sqcup \mathrm{V}_\triangledown(\Gamma)$, φ_α is a Möbius transformation fixing the infinity; when $\mathfrak{v}_\alpha \in \mathrm{V}_\diamond(\Gamma)$, then φ_α is a translation of Σ_α. The automorphism group of a treed disk $\mathcal{Z}$ modelled on $\Gamma \in \mathbf{T}$ is trivial if and only if Γ is stable (see Definition 4.2).

We think of treed disks as topological spaces via the following *realization* construction. Given a treed disk $\mathcal{Z}$ for each $\underline{\mathfrak{e}} \in \mathrm{E}(\underline{\Gamma})$ let $I_{\underline{\mathfrak{e}}} \subset \mathbb{R}$ be a closed interval of length $\boldsymbol{\lambda}(\underline{\mathfrak{e}})$; to each boundary tail $t \in \underline{\mathrm{T}}(\underline{\Gamma})$ not belong to an infinite edge, let I_t be a semi-infinite interval being either $(-\infty, 0]$ or $[0, +\infty)$; to an infinite edge $\{t_{\rm in}, t_{\rm out}\}$ we assign $I_{t_-} = I_{t_+} = (-\infty, +\infty)$. These intervals and the surfaces Σ_α are glue together in a natural way, and form a connected topological space called the *realization* of $\mathcal{Z}$. In the realization, all intervals with positive or infinite lengths are replaced by a finite closed interval and zero-length intervals are replaced by a point. So the realization is well-defined up to homeomorphism.

We denote the moduli spaces of stable treed disks as follows. For $\Gamma \in \mathbf{T}^{\mathbf{st}}$, let $\mathcal{W}_\Gamma$ be the set of isomorphism classes of stable treed disks of type Γ. Denote by the bar notation the union over subordinate types

$$\overline{\mathcal{W}}_\Gamma := \bigsqcup_{\substack{\Pi \trianglelefteq \Gamma \\ \Pi \text{ stable}}} \mathcal{W}_\Pi.$$

Introduce a topology on this union by the following notion of sequential convergence, which generalizes that for stable genus zero curves with $k+1$ markings in McDuff–Salamon [MS04] and for trees in Boardman–Vogt [BV73]. We first consider the simple case where $(\Gamma, \mathfrak{s})$ is the colored baseless tree with a single vertex colored by $\diamond$ and interior leaves $l_1, \ldots, l_k$ $(k \geqslant 1)$. Let $\mathfrak{o}$ be any contact data. For $\Gamma = (\Gamma, \mathfrak{o}, \mathfrak{s})$. Let $\mathcal{Z}_j$ be a sequence of treed disks modelled on $(\Gamma, \mathfrak{s})$, which are equivalent to distinct points $z_{j,1}, \ldots, z_{j,k} \in \boldsymbol{C}$ modulo translations. As j goes to ∞, the points may come together or separate from each other. Consider $\Pi \trianglelefteq \Gamma$ and let $\mathcal{Z}_\infty$ be a tree disk modelled on Π, which is described as in Definition 4.7. We temporarily fix the following notations. Let $\mathfrak{v}_\alpha \in \mathrm{V}_\vartriangle(\Pi)$. Then there exists a unique vertex $\mathfrak{v}_{\alpha^\diamond} \in \mathrm{V}_\diamond(\Pi)$ connecting $\mathfrak{v}_\alpha$ to the root (see Definition 4.1), and a point $w_{\alpha\alpha^\diamond} \in \boldsymbol{C} \simeq \Sigma_{\alpha^\diamond}$ corresponding to the node in the path connecting $\mathfrak{v}_\alpha$ and $\mathfrak{v}_{\alpha^\diamond}$.

Definition 4.8. A sequence $\mathcal{Z}_j$ *converges* to $\mathcal{Z}_\infty$ if the following conditions hold.

- For each $\mathfrak{v}_\alpha \in \mathrm{V}_\vartriangle(\Pi)$, there exist a sequence of Möbius transformations $\phi_{j,\alpha} : (\overline{\Sigma}_\alpha, \infty) \to (\overline{\boldsymbol{C}}, \infty)$ that converge to the constant map with value $w_{\alpha\alpha^\diamond}$ uniformly with all derivatives away from special points of $\overline{\Sigma}_\alpha$, such that for each leaf $l_i \in \mathrm{L}_\alpha(\Pi)$, $\phi_{j,\alpha}^{-1}(z_{j,i})$ converges to $z_{\infty,i} \in \Sigma_\alpha$.
- For each $\mathfrak{v}_\alpha \in \mathrm{V}_\triangledown(\Pi)$, there exist a sequence of Möbius transformations $\phi_{j,\alpha} : (\overline{\Sigma}_\alpha, \infty) \to (\overline{\boldsymbol{C}}, \infty)$ that converges to the constant map with value ∞ away from special points on $\overline{\Sigma}_\alpha$.
- For each $\mathfrak{v}_\alpha \in V_\diamond(\Pi)$, there exist a sequence of translations $\phi_{j,\alpha} : \Sigma_\alpha \to \boldsymbol{C}$ such that for each leaf $l_i \in \mathrm{L}_\alpha(\Pi)$, $\phi_{j,\alpha}^{-1}(z_{j,i})$ converges to $z_{\infty,i} \in \Sigma_\alpha$.

The sequences of Möbius transformations satisfy the following condition. For each edge $\mathfrak{e}_{\alpha\beta} \in \mathrm{E}(\Pi)$ having a corresponding node $w_{\alpha\beta} \in \Sigma_\beta$, the sequence of maps

$$\phi_{j,\beta}^{-1} \circ \phi_{j,\alpha} : \Sigma_\alpha \to \Sigma_\beta$$

converges uniformly with all derivatives to the constant $w_{\alpha\beta}$ away from ∞. Here ends this definition.

A topology on moduli spaces of colored treed disks is defined similarly. When $(\Gamma, \mathfrak{s})$ is the colored based tree with a single vertex colored by $\diamond$, boundary tails $t_1, \ldots, t_{\underline{k}}$ and interior leaves $l_1, \ldots, l_k$ $(\underline{k} + 2k \geqslant 1)$. Let $\mathfrak{m}$ be the trivial metric type.

Let $\mathfrak{o}$ be any contact data. Then for $\Gamma = (\Gamma, \mathfrak{s}, \mathfrak{m}, \mathfrak{o})$, the topology of $\overline{\mathcal{W}}_\Gamma$ is defined in a way similar to Definition 4.8 (cf. [Xu, Section 2] for detailed discussion of a special case). We omit the details.

For a general stable colored based or baseless tree Γ, the notion of sequential convergence in $\overline{\mathcal{W}}_\Gamma$ can be obtained from the above two special cases combined with the notion of convergence of stable marked spheres or stable marked disks, and the notion of convergence of metric trees. Again we omit the details. The sequential convergence actually determines a compact Hausdorff topology, because of the existence of local distance functions as in McDuff–Salamon [MS04]. We leave it to the reader to check the following statement.

Lemma 4.9. *The moduli space $\overline{\mathcal{W}}_\Gamma$ is compact and Hausdorff with respect to the topology defined in Definition 4.8.*

4.4. **Dimension formula.** We give a formula for the dimension of the moduli spaces of stable treed disks. Given $\underline{k}, k \geqslant 0$ and $\mathbf{t} \in \{\triangle, \triangledown, \diamond\}$, there are finitely many unbroken types $\Gamma_i \in \mathbf{T}^{\mathbf{t}}_{\underline{k},k}$, $i = 1, \dots, M$, satisfying the following conditions.

(a) There are no edges of length zero.
(b) All vertices in $\underline{\Gamma}$.
(c) $V^0_{\partial\triangle}(\hat{\underline{\Gamma}}) = \varnothing$.

The collection of $\mathcal{W}_{\Gamma_i}$'s form the top stratum. Define

$$\mathcal{W}^{\mathbf{t}}_{\underline{k},k} := \bigsqcup_{i=1}^{M} \mathcal{W}_{\Gamma_i},\ \ \mathbf{t} \in \{\triangle, \diamond, \triangledown\}.$$

Then one has the dimension formula for the top stratum:

$$\mathbf{dim}\big[\mathcal{W}^{\triangle}_{\underline{k},k}\big] = \mathbf{dim}\big[\mathcal{W}^{\triangledown}_{\underline{k},k}\big] = \mathbf{dim}\big[\mathcal{W}^{\diamond}_{\underline{k},k}\big] + 1 = 2\underline{k} + 2k - 2.$$

On the other hand, for a general $\Gamma \in \mathbf{T}^{\mathbf{t}}_{\underline{k},k}$, the dimension formula is

$$\mathbf{dim}\mathcal{W}_\Gamma = \mathbf{dim}\mathcal{W}^{\mathbf{t}}_{\underline{k},k} - \#\Big(\mathrm{E}^0_{\triangle}(\underline{\Gamma}) \sqcup \mathrm{E}^0_{\triangledown}(\underline{\Gamma})\Big) - 2\#\Big(\mathrm{E}_{\triangle}(\Gamma) \sqcup \mathrm{E}_{\triangledown}(\Gamma) \smallsetminus \mathrm{E}(\underline{\Gamma})\Big)$$
$$- \#\mathrm{V}^0_{\partial\triangledown}(\underline{\Gamma}) - 2\#\Big(\mathrm{V}_{\partial\triangledown}(\Gamma) \smallsetminus \mathrm{V}_{\partial\triangledown}(\underline{\Gamma})\Big) - \mathrm{b}(\underline{\Gamma}). \quad (4.1)$$

Here $\mathrm{b}(\underline{\Gamma})$ is a number characterizing how many breakings $\underline{\Gamma}$ has defined as

$$\mathrm{b}(\underline{\Gamma}) = \begin{cases} \#\text{breakings in } \underline{\Gamma}, & \text{if all basic parts are of type } \triangle \text{ or } \triangledown; \\ \#\text{basic parts of type } \triangle + \#\text{basic parts of type } \triangledown, & \text{in other cases}. \end{cases}$$

In particular, if $\underline{\Gamma}$ is unbroken, then $\mathrm{b}(\underline{\Gamma}) = 0$.

4.5. **Forests and superstructures.** In this section we discuss moduli spaces treed spheres which are modelled on forests, that is, disconnected graphs where each connected component is a baseless tree. Given a forest Γ denote the connected components by $\Gamma_1, \dots, \Gamma_s$. We require that the labelling of interior leaves is a bijection between $\{1, \dots, k\}$ for certain $k \geqslant 0$ and $\mathrm{L}(\Gamma_1) \sqcup \cdots \sqcup \mathrm{L}(\Gamma_s)$. Then the labelling determines an order among the components, by the smallest label of each component.

We always assume forests are baseless. Then for the structures introduced in Subsection 4.1, only the coloring $\mathfrak{s}$ and the contact data $\mathfrak{o}$ are nonempty. A forest Γ is of type $\triangle$ or type $\triangledown$ if all components are of type $\triangle$ or type $\triangledown$, otherwise it is of type $\diamond$. A forest is stable if every component is stable. Almost all notions about colored trees extend to forests componentwise. Let $\mathbf{F}$ be the set of all forests and $\mathbf{F^{st}}$ the set of stable forests.

For each forest Γ with components $\Gamma_1, \ldots, \Gamma_s$, a treed disk (more precisely treed spheres) of combinatorial type Γ is roughly the disjoint union of treed disks of type $\Gamma_1, \ldots, \Gamma_s$, with interior markings indexed by the labelling data in Γ. Let $\mathcal{W}_\Gamma$ be the moduli space . We have natural isomorphisms

$$\mathcal{W}_\Gamma \simeq \mathcal{W}_{\Gamma_1} \times \cdots \times \mathcal{W}_{\Gamma_s}, \qquad \overline{\mathcal{W}}_\Gamma \simeq \overline{\mathcal{W}}_{\Gamma_1} \times \cdots \times \overline{\mathcal{W}}_{\Gamma_s}.$$

Let $\Gamma = (\Gamma, \mathfrak{s}, \mathfrak{m}, \mathfrak{o})$ be a combinatorial type. Every connected component of $\Gamma \smallsetminus \underline{\Gamma}$ is called a *superstructure*, which is an element of $\mathbf{F}$. Any subset of superstructures of Γ naturally form a forest where the order of interior leaves (and hence the order of components) is inherited from the order of interior leaves of Γ.

5. Perturbations

In this section we define the notion of coherent perturbation data and prove certain properties of the space of coherent perturbation data. In Subsection 5.1 we discuss the geometry and topology of the universal curve over the moduli space of stable treed disks. In Subsection 5.2 we define the notion of coherent nodal neighborhoods, which is prerequisite for defining coherent perturbations. In Subsection 5.3 we introduce the notion of coherent maps. In Subsection 5.4 we specify the target spaces of coherent maps and give a system of Banach spaces of perturbation data. In Subsection 5.5 we consider the technical notion of special perturbations.

5.1. The universal curves.

Our perturbation scheme requires us to study the geometry and topology of the universal curve of stable treed disks. The following proposition explains the basic properties of the universal curve. To state the coherence condition we also allow disconnected types (with finitely many components). For each stable type $\Gamma \in \mathbf{T^{st}} \cup \mathbf{F^{st}}$, there is a compact and metrizable topological space $\overline{\mathcal{U}}_\Gamma$, the *universal curve* of disks of type Γ together with two closed subspaces $\overline{\mathcal{U}}_\Gamma^2$, $\overline{\mathcal{U}}_\Gamma^1$, the *two* and *one-dimensional parts* of the universal curve, satisfying the following conditions.

(a) The universal curve $\overline{\mathcal{U}}_\Gamma$ is the union of the parts $\overline{\mathcal{U}}_\Gamma^2 \cup \overline{\mathcal{U}}_\Gamma^1$. Moreover if Γ is baseless, then $\overline{\mathcal{U}}_\Gamma^1 = \varnothing$.

(b) There is a subjective continuous map $\pi_\Gamma : \overline{\mathcal{U}}_\Gamma \to \overline{\mathcal{W}}_\Gamma$ such that $\mathcal{U}_\Gamma := \pi_\Gamma^{-1}(\mathcal{W}_\Gamma)$ is an open and dense subset.

(c) Denote $\mathcal{U}_\Gamma^2 := \mathcal{U}_\Gamma \cap \overline{\mathcal{U}}_\Gamma^2$ and $\mathcal{U}_\Gamma^1 := \mathcal{U}_\Gamma \cap \overline{\mathcal{U}}_\Gamma^1$. Then the restrictions of π_Γ to $\mathcal{U}_\Gamma^2$ and $\mathcal{U}_\Gamma^1$ are fibre bundles over $\mathcal{W}_\Gamma$.

(d) For each stable treed disk $\mathcal{Z}$ representing a point $p \in \overline{\mathcal{W}}_\Gamma$, there is a homeomorphism $\pi_\Gamma^{-1}(p) \simeq \mathcal{Z}$ unique in the following sense. If $\mathcal{Z}$ is isomorphic to $\mathcal{Z}'$, then the canonical homeomorphism $\mathcal{Z} \simeq \mathcal{Z}'$ between their realizations intertwines between the homeomorphisms $\pi_\Gamma^{-1}(p) \simeq \mathcal{Z}$ and $\pi_\Gamma^{-1}(p) \simeq \mathcal{Z}'$.

We denote $\pi_\Gamma^{-1}(p)$ by $\mathcal{Z}_p$ and identify $\mathcal{Z}_p$ with its realization.

To state the coherence condition of perturbations, consider the following types of maps between different universal curves.

(a) For each type $\Pi \trianglelefteq \Gamma$, there are natural injective maps $\rho_{\Gamma,\Pi} : \overline{\mathcal{W}}_{\Pi} \to \overline{\mathcal{W}}_{\Gamma}$ and $\tilde{\rho}_{\Gamma,\Pi} : \overline{\mathcal{U}}_{\Pi} \to \overline{\mathcal{U}}_{\Gamma}$ such that the following diagram commutes.

$$\begin{array}{ccccc} \overline{\mathcal{U}}^i_{\Pi} & \longrightarrow & \overline{\mathcal{U}}_{\Pi} & \xrightarrow{\pi_\Pi} & \overline{\mathcal{W}}_{\Pi} \\ {\scriptstyle \tilde{\rho}_{\Gamma,\Pi}}\downarrow & & {\scriptstyle \tilde{\rho}_{\Gamma,\Pi}}\downarrow & & \downarrow{\scriptstyle \rho_{\Gamma,\Pi}} \\ \overline{\mathcal{U}}^i_{\Gamma} & \longrightarrow & \overline{\mathcal{U}}_{\Gamma} & \xrightarrow[\pi_\Gamma]{} & \overline{\mathcal{W}}_{\Gamma} \end{array}, \qquad i = 1, 2. \tag{5.1}$$

(b) Let a baseless forest $\Gamma \in \mathbf{F^{st}}$ have connected components $\Gamma_1, \ldots, \Gamma_s$. Let $\overline{\mathcal{U}}_{\Gamma_1} \boxplus \cdots \boxplus \overline{\mathcal{U}}_{\Gamma_s}$ be the fibration over $\overline{\mathcal{W}}_{\Gamma_1} \times \cdots \times \overline{\mathcal{W}}_{\Gamma_s}$ whose fibre over $(p_1, \ldots, p_s)$ is the disjoint union of $\mathcal{Z}_{p_1}, \ldots, \mathcal{Z}_{p_s}$. Then we have the following commutative diagram

$$\begin{array}{ccc} \overline{\mathcal{U}}_{\Gamma_1} \boxplus \cdots \boxplus \overline{\mathcal{U}}_{\Gamma_s} & \longrightarrow & \overline{\mathcal{W}}_{\Gamma_1} \times \cdots \times \overline{\mathcal{W}}_{\Gamma_s} \\ \downarrow & & \downarrow \\ \overline{\mathcal{U}}_{\Gamma} & \longrightarrow & \overline{\mathcal{W}}_{\Gamma} \end{array}. \tag{5.2}$$

(c) Let $\Gamma \in \mathbf{T^{st}}$ be a broken type and with basic parts $\Gamma_1, \ldots, \Gamma_s$. Let $\overline{\mathcal{U}}_{\Gamma_1} \dot{\boxplus} \cdots \dot{\boxplus} \overline{\mathcal{U}}_{\Gamma_s}$ be the fibration over $\overline{\mathcal{W}}_{\Gamma_1} \times \cdots \times \overline{\mathcal{W}}_{\Gamma_s}$ whose fibre over $(p_1, \ldots, p_s)$ is the union of $\mathcal{Z}_{p_1}, \ldots, \mathcal{Z}_{p_s}$ where we glue at the breakings. Then we have the following commutative diagram whose vertical lines are both homeomorphisms.

$$\begin{array}{ccc} \overline{\mathcal{U}}_{\Gamma_1} \dot{\boxplus} \cdots \dot{\boxplus} \overline{\mathcal{U}}_{\Gamma_s} & \longrightarrow & \overline{\mathcal{W}}_{\Gamma_1} \times \cdots \times \overline{\mathcal{W}}_{\Gamma_s} \\ {\scriptstyle \tilde{\rho}_\Gamma}\downarrow & & \downarrow{\scriptstyle \rho_\Gamma} \\ \overline{\mathcal{U}}_{\Gamma} & \xrightarrow[\pi_\Gamma]{} & \overline{\mathcal{W}}_{\Gamma} \end{array}. \tag{5.3}$$

(d) For any subset $\mathrm{S} \subset \mathrm{V}(\Gamma) \setminus \mathrm{V}(\underline{\Gamma})$ which is either disjoint from $\mathrm{V}(\underline{\Gamma})$ or contains $\mathrm{V}(\underline{\Gamma})$, let Γ_{S} be the subtree or forest that contains all vertices in S, with all nodes between vertices in S and vertices in $\mathrm{V}(\Gamma) \setminus \mathrm{S}$ as leaves. Then there is a closed subset $\overline{\mathcal{U}}_{\Gamma,\Gamma_{\mathrm{S}}} \subset \overline{\mathcal{U}}_{\Gamma}$, and a commutative diagram

$$\begin{array}{ccc} \overline{\mathcal{U}}_{\Gamma,\Gamma_{\mathrm{S}}} & \xrightarrow{\tilde{\pi}_{\mathrm{S}}} & \overline{\mathcal{U}}_{\Gamma_{\mathrm{S}}} \\ \downarrow & & \downarrow \\ \overline{\mathcal{W}}_{\Gamma} & \xrightarrow{\pi_{\mathrm{S}}} & \overline{\mathcal{W}}_{\Gamma_{\mathrm{S}}} \end{array}. \tag{5.4}$$

Here the lower horizontal arrow is the map forgetting all vertices not in S.

5.2. **Coherent system of nodal neighborhoods.** In the following subsection we define Banach manifolds of coherent perturbations on the universal curves $\overline{\mathcal{U}}_{\Gamma}$. Our perturbations are fibrewise smooth maps from $\overline{\mathcal{U}}_{\Gamma}$ to certain Banach spaces that satisfy the coherence condition we just defined. However to obtain a complete norm, we need to specify the neighborhoods of nodal points and breakings over which the perturbations are zero or constants. In the total space of the universal curve $\overline{\mathcal{U}}_{\Gamma}$, there is a closed subset $\overline{\mathcal{U}}^{\mathbf{nd}}_{\Gamma}$ corresponding to all interior nodes, all edges in $\mathrm{E}^0(\underline{\Gamma})$, and all interior leaves in $\mathrm{L}(\Gamma)$. There is another closed subset $\overline{\mathcal{U}}^{\mathbf{bk}}_{\Gamma}$ corresponding to all breakings and all infinities of infinite edges. We use $\overline{\mathcal{U}}_\Gamma^{\rm thin}$ to denote a neighborhood of $\overline{\mathcal{U}}_\Gamma^{\bf nd}$ (also called the neck region), and use $\overline{\mathcal{U}}_\Gamma^{\rm long}$ to denote a neighborhood of $\overline{\mathcal{U}}_\Gamma^{\bf bk}$.

Definition 5.1. A collection of nodal neighborhoods $\{\overline{\mathcal{U}}_\Gamma^{\rm thin} \mid \Gamma \in {\bf T^{st}} \cup {\bf F^{st}}\}$ are called coherent if the following conditions are satisfied.

(a) For any $\Pi \trianglelefteq \Gamma$, we have $\overline{\mathcal{U}}_\Pi^{\rm thin} = \overline{\mathcal{U}}_\Gamma^{\rm thin} \cap \overline{\mathcal{U}}_\Pi$.

(b) The nodal area on each irreducible component does not depend on the moduli of other irreducible components. More precisely, for any subset $\mathrm{S} \subset \mathrm{V}(\Gamma)$ which either contains $\mathrm{V}(\underline{\Gamma})$ or is disjoint from $\mathrm{V}(\underline{\Gamma})$, we have $\overline{\mathcal{U}}_\Gamma^{\rm thin} \cap \overline{\mathcal{U}}_{\Gamma,\Gamma_{\rm S}} = \tilde{\pi}_{\rm S}^{-1}(\overline{\mathcal{U}}_{\Gamma_{\rm S}}^{\rm thin})$ (see 5.4).

(c) For every vertex $\mathfrak{v}_\alpha \in V_\diamond(\Gamma) \smallsetminus \mathrm{V}(\underline{\Gamma})$, define

$$E_\alpha = \frac{1}{N_D} \sum_{l_i \in \mathrm{L}(\mathfrak{v}_\alpha)} \mathfrak{o}(l_i) > 0, \qquad A_\alpha = \frac{2E_\alpha}{c_U^2}. \tag{5.5}$$

Then for every $p \in \mathcal{W}_\Gamma$, $\overline{\mathcal{U}}_\Gamma^{\rm thin} \cap \mathcal{Z}_p \cap \overline{\mathcal{U}}_{\Gamma,\mathfrak{v}_\alpha}$ is disjoint from some open set of area at least A_α. (This condition is to respect Lemma 2.5).

It is important to prove the existence of such collections.

Lemma 5.2. *There exists a coherent collection of nodal neighborhoods.*

Proof. We first construct a coherent system of nodal neighborhoods for all $\Gamma \in {\bf F^{st}}$. By Item (b) of Definition 5.1, the choice of a nodal neighborhood for a stable forest $\Gamma \in {\bf F^{st}}$ is completely determined by the case of its connected components. Hence it suffices to construct the neighborhoods for connected elements of ${\bf F^{st}}$. The construction is inductive. First consider the case that Γ is of type $\triangledown$. The case for spheres with two leaves is obvious. Suppose we have constructed a coherent system of nodal neighborhoods for all configurations with at most k interior leaves. Let Γ be a configuration with exactly $k+1$ interior leaves. If Γ is disconnected, then each component has at most k leaves, and the requirement for coherence uniquely determines a nodal neighborhood $\overline{\mathcal{U}}_\Gamma^{\rm thin} \subset \overline{\mathcal{U}}_\Gamma$. Now we consider a connected Γ. If Γ has at least one node, then by Item (b), the nodal neighborhoods we have already constructed uniquely determines a nodal neighborhood $\overline{\mathcal{U}}_\Gamma^{\rm thin} \subset \overline{\mathcal{U}}_\Gamma$. It remains to consider the unique Γ which has $k+1$ interior leaves and no node. In this case, the nodal neighborhoods $\overline{\mathcal{U}}_\Pi^{\rm thin}$ for all $\Pi \triangleleft \Gamma$ (which have been determined) define an open subset $\overline{\mathcal{U}}_{\partial\Gamma}^{\rm thin} \subset \partial\mathcal{U}_\Gamma$. For each point $x \in \overline{\mathcal{U}}_{\partial\Gamma}^{\rm thin}$, choose an open subset $U_x \subset \overline{\mathcal{U}}_\Gamma$ such that $U_x \cap \partial\mathcal{U}_\Gamma \subset \overline{\mathcal{U}}_{\partial\Gamma}^{\rm thin}$. Then the open set

$$\overline{\mathcal{U}}_\Gamma^{\rm thin,\prime} := \bigcup_{x \in \overline{\mathcal{U}}_{\partial\Gamma}^{\rm thin}} U_x$$

contains $\overline{\mathcal{U}}_\Gamma^{\bf nd} \cap \partial\mathcal{U}_\Gamma$. Then we can find another open set $\overline{\mathcal{U}}_\Gamma^{\rm thin,\prime\prime}$ which is disjoint from $\partial\mathcal{U}_\Gamma$ such that $\overline{\mathcal{U}}_\Gamma^{\rm thin} := \overline{\mathcal{U}}_\Gamma^{\rm thin,\prime} \cup \overline{\mathcal{U}}_\Gamma^{\rm thin,\prime\prime}$ contains all $\overline{\mathcal{U}}_\Gamma^{\bf nd}$. This finishes the induction step which grants a nodal neighborhood for all forests with exactly $k+1$ interior leaves. Inductively this construction gives a coherent system of nodal neighborhoods for all type $\triangledown$ stable forests.

The construction of coherent nodal neighborhoods for stable forests of type $\vartriangle$ is accomplished in a similar way. We consider type $\diamond$ stable forests in ${\bf F^{st}}$. First, for the stable tree Γ of type $\diamond$ with a single vertex and a single interior leaf (with arbitrary

$\mathfrak{o}(l_i)$), it is easy to find a neighborhood $\overline{\mathcal{U}}_\Gamma^{\rm thin}$; this is essentially a neighborhood of infinity of $\boldsymbol{C}$ that becomes smaller and smaller as $\mathfrak{o}(l_i)$ goes larger. Given Γ, suppose we have chosen $\overline{\mathcal{U}}_\Pi^{\rm thin}$ for all $\Pi \lhd \Gamma$ and all Π that has strictly fewer interior leaves. Then if Γ has more than one vertex, then the nodal neighborhoods for every vertices determine uniquely a nodal neighborhood $\overline{\mathcal{U}}_\Gamma^{\rm thin}$. So it suffices to consider the case that Γ has only one vertex. By the same method as the case of type $\triangledown$ trees, there exists a neighborhood $\overline{\mathcal{U}}_\Gamma^{\rm thin,\prime} \subset \overline{\mathcal{U}}_\Gamma$ such that $\overline{\mathcal{U}}_\Gamma^{\rm thin,\prime} \cap \partial \mathcal{U}_\Gamma = \overline{\mathcal{U}}_{\partial\Gamma}^{\rm thin}$. Moreover, one can guarantee that the intersection of $\overline{\mathcal{U}}_\Gamma^{\rm thin,\prime}$ with each fibre of $\overline{\mathcal{U}}_\Gamma \to \overline{\mathcal{W}}_\Gamma$ satisfies Item (c). Lastly, we can choose $\overline{\mathcal{U}}_\Gamma^{\rm thin,\prime\prime}$ sufficiently small so that $\overline{\mathcal{U}}_\Gamma^{\rm thin} := \overline{\mathcal{U}}_\Gamma^{\rm thin,\prime} \cup \overline{\mathcal{U}}_\Gamma^{\rm thin,\prime\prime}$ still satisfies Item (c).

One can carry out similar inductive constructions to construct coherent collection of nodal neighborhoods for all types $\Gamma \in \mathbf{T}^{\mathbf{st}}$, while preserving the choice of nodal neighborhoods for all baseless forests we have constructed above. The details are omitted. □

One also needs to construct coherent neighbourhoods of the breakings.

Lemma 5.3. *There exists a collection of open neighborhoods* $\overline{\mathcal{U}}_\Gamma^{\rm long}$ *of* $\overline{\mathcal{U}}_\Gamma^{\mathbf{br}} \subset \overline{\mathcal{U}}_\Gamma^1$ *for all* $\Gamma \in \mathbf{T}^{\mathbf{st}}$ *satisfying the following conditions.*

(a) If $\Pi \trianglelefteq \Gamma$*, then* $\overline{\mathcal{U}}_\Pi^{\rm long} = \overline{\mathcal{U}}_\Gamma^{\rm long} \cap \overline{\mathcal{U}}_\Pi$.

(b) For any $\mathrm{S} \subset \mathrm{V}(\Gamma)$ *containing the base,* $\overline{\mathcal{U}}_\Gamma^{\rm long} = \tilde\pi_{\mathrm{S}}^{-1}(\overline{\mathcal{U}}_{\Gamma_{\mathrm{S}}}^{\rm long})$.

Proof. Left to the reader. □

From now on we fix coherent collections

$$\overline{\mathcal{U}}_\Gamma^{\rm thin} \subset \overline{\mathcal{U}}_\Gamma,\ \forall \Gamma \in \mathbf{T}^{\mathbf{st}} \cup \mathbf{F}^{\mathbf{st}}, \qquad \overline{\mathcal{U}}_\Gamma^{\rm long} \subset \overline{\mathcal{U}}_\Gamma,\ \forall \Gamma \in \mathbf{T}^{\mathbf{st}}. \tag{5.6}$$

5.3. **Coherent maps.** We first introduce the following concept which will be used frequently in many inductive constructions.

Definition 5.4. A nonempty subset $\mathbf{S} \subset \mathbf{T}^{\mathbf{st}} \cup \mathbf{F}^{\mathbf{st}}$ is called a *basic subset* if the following conditions are satisfied.

(a) If $\Gamma \in \mathbf{S}$ and $\Gamma' \trianglelefteq \Gamma$, then $\Gamma' \in \mathbf{S}$.
(b) If $\Gamma \in \mathbf{S}$ and $\mathrm{S} \subset \mathrm{V}(\Gamma)$ either contains $\mathrm{V}(\underline{\Gamma})$ or is disjoint from $\mathrm{V}(\underline{\Gamma})$, then $\Gamma_{\mathrm{S}} \in \mathbf{S}$.
(c) If $\Gamma \in \mathbf{S} \cap \mathbf{T}^{\mathbf{st}}$ and Γ' is a basic part of Γ, then $\Gamma' \in \mathbf{S}$.

In other words, a subset of the set of stable based trees or stable forests is a basic subset if it is closed under the operations of degeneration, removing a subtree containing the base, removing a subtree disjoint from the base, and taking a basic part for broken types.

Given $\Gamma \in \mathbf{T}^{\mathbf{st}}$, let $\mathbf{S}(\Gamma)$ be the smallest basic subset that contains Γ, and denote $\mathbf{S}^*(\Gamma) = \mathbf{S}(\Gamma) \smallsetminus \{\Gamma\}$. Notice that $\mathbf{S}^*(\Gamma)$ is also a basic subset. It is also easy to see that $\mathbf{T}^{\mathbf{st}}_\triangle$, $\mathbf{T}^{\mathbf{st}}_\triangledown$, and $\mathbf{F}^{\mathbf{st}}$ are basic subsets.

Definition 5.5. Let Z^1, Z^2 be topological vector spaces and $\mathbf{S} \subset \mathbf{T}^{\mathbf{st}} \cup \mathbf{F}^{\mathbf{st}}$ be a basic subset. For $i = 1, 2$, a system of continuous maps $\{g_\Gamma^i : \overline{\mathcal{U}}_\Gamma^i \to Z^i \mid \Gamma \in \mathbf{S}\}$ is called *coherent* if the following conditions are satisfied.

(a) (Neck region) g_Γ^1 vanishes in $\overline{\mathcal{U}}_\Gamma^{\rm long}$ and g_Γ^2 vanishes on $\overline{\mathcal{U}}_\Gamma^{\rm thin}$.
(b) (Degeneration) For all pairs $\Gamma, \Pi \in \mathbf{S}$ with $\Pi \trianglelefteq \Gamma$, with respect to the commutative diagram (5.1), one has $g_\Gamma^i \circ \tilde\rho_{\Gamma,\Pi}^i = g_\Pi^i$.

(c) (Cutting edge) For any broken $\Gamma \in \mathbf{S}$ with all basic parts $\Gamma_1, \ldots, \Gamma_s$, with respect to the commutative diagram (5.3), one has $g^i_\Gamma \circ \tilde{\rho}^i_\Gamma = \sqcup g^i_{\Gamma_\alpha}$.

(d) (Components) For $\Gamma \in \mathbf{S}$ and $\mathrm{S} \subset \mathrm{V}(\Gamma)$ that either contains $\mathrm{V}(\underline{\Gamma})$ or is disjoint from $\mathrm{V}(\underline{\Gamma})$, there exist continuous maps $h^i_{\Gamma_{\mathrm{S}}} : \overline{\mathcal{U}}_{\Gamma^i_{\mathrm{S}}} \to Z^i$ such that

$$g^i_\Gamma|_{\overline{\mathcal{U}}_{\Gamma,\Gamma_{\mathrm{S}}}} = h^i_{\Gamma_{\mathrm{S}}} \circ \tilde{\pi}_{\mathrm{S}}.$$

We remark that Item (d) implies that the restriction of g^2_Γ to any two-dimensional components only depends on the positions of the special points on that component.

5.4. **Banach manifolds of perturbations.** For the purpose of applying the Sard-Smale theorem we define Banach spaces of perturbation data. First we recall Floer's norm on smooth functions. Choose a sequence of positive numbers ϵ_l converging to zero. On any Riemannian manifold M, let $C^\epsilon(M)$ denote the class of smooth functions that has finite C^ϵ-norm, which is defined by

$$\|f\|_\epsilon := \sum_{l \geqslant 0} \epsilon_l \|f\|_{C^l}.$$

This is a complete norm. Floer (see [Flo88b, Lemma 5.1]) observed and proved the fact that if ϵ_l decays fast enough (which is a dimensionless condition), then the space of C^ϵ-functions contain certain bumped smooth functions whose supports can be arbitrarily small.

We extend Floer's norm to almost complex structures as follows. Recall the set $\mathcal{J}(U, D)$ specified in Subsection 3.3. Let $\mathcal{J}^\epsilon(U, D)$ denote the subset of almost complex structures of the form $J_X + h$ for some $h \in \Gamma(\mathrm{End}(TX))$ of finite C^ϵ-norm (with respect to the Kähler metric on X). Then $\mathcal{J}^\epsilon(U, D)$ is a Banach manifold and $J_X \in \mathcal{J}^\epsilon(U, D)$. Take a small neighborhood $\mathcal{J} \subset \mathcal{J}^\epsilon(U, D)$ of J_X which can be identified with a small neighborhood of the origin of the tangent space $T_{J_X}\mathcal{J}^\epsilon(U, D)$. An element $J \in \mathcal{J}$ is also denoted by $J_X + h$. On the other hand, let $\mathcal{F} = C^\epsilon(\bar{L})$ be the set of C^ϵ-functions on $\bar{L}$.

The perturbations are almost complex structures and Morse functions that equal the given ones on certain open subsets. Recall the open sets $\overline{\mathcal{U}}^{\mathrm{thin}}_\Gamma$ and $\overline{\mathcal{U}}^{\mathrm{long}}_\Gamma$ fixed in (5.6). Denote by $\mathbf{Map}(\overline{\mathcal{U}}^1_\Gamma, \mathcal{F})$ and $\mathbf{Map}(\overline{\mathcal{U}}^2_\Gamma, \mathcal{J})$ the set of maps which satisfy the (Neck region) condition of Definition 5.5 whose difference with the base maps is of class C^ϵ. Denote the Banach space of perturbations

$$\mathcal{P}(\overline{\mathcal{U}}_\Gamma) = \mathbf{Map}(\overline{\mathcal{U}}^1_\Gamma, \mathcal{F}) \times \mathbf{Map}(\overline{\mathcal{U}}^2_\Gamma, \mathcal{J}).$$

Definition 5.6. Let $\mathbf{S} \subset \mathbf{T^{st}} \cup \mathbf{F^{st}}$ be a basic subset. A *coherent system of perturbation data over* $\mathbf{S}$ consists of a collection

$$\underline{P}_{\mathbf{S}} := \Big\{ P_\Gamma = (f_\Gamma, h_\Gamma) \in \mathcal{P}(\overline{\mathcal{U}}_\Gamma) \mid \Gamma \in \mathbf{S} \Big\}$$

that satisfies the following conditions.

(a) The collection P_Γ is coherent in the sense of Definition 5.5.

(b) For each $\mathfrak{v}_{\underline{\alpha}} \in \mathrm{V}(\underline{\Gamma})$, let $\Gamma_{\underline{\alpha}}$ be the subtree of Γ consisting of $\mathfrak{v}_{\underline{\alpha}}$ and all superstructures attached to $\mathfrak{v}_{\underline{\alpha}}$. Denote

$$n_{\underline{\alpha}} = \frac{1}{N_D E_0} \sum_{l_i \in \mathrm{L}(\Gamma_{\underline{\alpha}})} \mathfrak{o}(l_i).$$

Then the restriction of J_Γ to components in the subtree $\Gamma_{\underline{\alpha}}$ takes value in the open subset $\mathcal{J}_{n_{\underline{\alpha}}} \subset \mathcal{J}$, where for an integer n, $\mathcal{J}_n$ is chosen in (3.1).

If $\mathbf{S} = \mathbf{T}^{\mathbf{st}} \cup \mathbf{F}^{\mathbf{st}}$, then we drop the subscript $\mathbf{S}$ in the notation. We will also identify f_Γ with $F_\Gamma := F + f_\Gamma$ and h_Γ with $J_\Gamma := J_X + h_\Gamma$. Occasionally for convenience we will also write the perturbation data for $\Gamma \in \mathbf{S}$ as $P_\Gamma = (F_\Gamma, J_\Gamma)$.

The Banach norms on the spaces of perturbations are adapted from the Floer norm. Consider $\Gamma \in \mathbf{T}^{\mathbf{st}} \cup \mathbf{F}^{\mathbf{st}}$. The norm for $f_\Gamma \in \mathbf{Map}(\overline{\mathcal{U}}^1_\Gamma, \mathcal{F})$ is defined in terms of a metric on $\overline{\mathcal{U}}^1_\Gamma$ by

$$\|f_\Gamma\|_\epsilon := \sup_{\mathfrak{e}} \|f_\Gamma|_{I_\mathfrak{e}}\|_{C^\epsilon(I_\mathfrak{e} \times \bar{L})}. \tag{5.7}$$

However, the definition of a C^ϵ norm on $\mathbf{Map}(\overline{\mathcal{U}}^2_\Gamma, \mathcal{J})$ is a bit subtle, since the way of measuring derivatives depends on how one parametrizes the fibres of $\overline{\mathcal{U}}_\Gamma$. For any $p \in \overline{\mathcal{W}}_\Gamma$, we know that there is a local universal family $\mathcal{U}_p \to \mathcal{W}_p$ where $\mathcal{W}_p$ is an open neighborhood of p in $\overline{\mathcal{W}}_\Gamma$. For any open subset $O_p \subset \mathcal{Z}^2_p$ whose closure is disjoint from the special points, one can find $\mathcal{W}_p$ sufficiently small such that there is a trivialization

$$\mathcal{U}^2_p \setminus \overline{\mathcal{U}}^{\rm thin}_p \simeq \mathcal{W}_p \times O_p.$$

For any $q \in \mathcal{W}_p$, let $\phi_{pq} : O_p \to \mathcal{Z}^2_q \subset \mathcal{U}^2_p$ the corresponding open subset. We may assume that O_p is chosen in such a way that $\mathcal{Z}^2_p = O_p \cup (\overline{\mathcal{U}}^{\rm thin}_\Gamma \cap \mathcal{Z}^2_p)$ and such that $\overline{\mathcal{U}}^{\rm thin}_p$ is contained in $\overline{\mathcal{U}}^{\rm thin}_\Gamma$. For $h_\Gamma \in \mathbf{Map}(\overline{\mathcal{U}}^2_\Gamma, \mathcal{J})$, define in terms of a metric on the complement of the nodes in $\overline{\mathcal{U}}^2_\Gamma$ the Floer norm $\|h_\Gamma\|_\epsilon$. The subspace of maps vanish on $\overline{\mathcal{U}}^{\rm thin}$ is a Banach space which we take to be our space of perturbations.

5.5. **Forgetting leaves.** In order to ensure that the regularized moduli space of vortices are compact, we wish to rule out certain configurations involving ghost bubbles in the compactification. For this, we need to ensure that certain stabilization operations respect the perturbations. Let $\Gamma = (\Gamma, \mathfrak{s}, \mathfrak{m}, \mathfrak{w}, \mathfrak{o}) \in \mathbf{T} \cup \mathbf{F}$ (not necessarily stable). Let $\mathrm{S} \subset \mathrm{V}(\Gamma) \setminus \mathrm{V}(\underline{\Gamma})$ be a nonempty subset which has connected components $\Xi_1, \ldots, \Xi_m$, such that each connected component has at least one interior leaf. In the following we define a new object $\Gamma' = (\Gamma', \mathfrak{s}', \mathfrak{m}', \mathfrak{w}', \mathfrak{o}') \in \mathbf{T}^{\mathbf{st}} \cup \mathbf{F}^{\mathbf{st}}$ called the *S-stabilization* of Γ.

First, there are two types of connected components Ξ_s of S, i.e., the non-separating and separating ones, depending on whether the complement of Ξ_s is connected or not. Suppose Ξ_s is connected via an edge $\mathfrak{e}_s$ to a vertex $\mathfrak{v}_{\alpha_s}$. For each non-separating Ξ_s, remove Ξ_s together with the edge $\mathfrak{e}_s$ connecting Ξ_s to the rest of the tree, and add a new leaf l'_s to be attached to $\mathfrak{v}_{\alpha_s}$. Label l'_s by the largest labelling of all interior leaves on Ξ_s. To each separating Ξ_s, remove all but the one leaf on Ξ_s that has the largest labelling, and regard this leaf as a new leaf. This separation may result in an unstable type, so after the above operations we stabilize and obtain a stable colored tree $(\Gamma', \mathfrak{s}')$. The set of leaves of Γ' can be decomposed as

$$\mathrm{L}(\Gamma') = \mathrm{L}_{\rm old}(\Gamma') \sqcup \mathrm{L}_{\rm new}(\Gamma').$$

The leaves are not labelled by consecutive numbers. We relabel them by consecutive numbers $1, 2, \ldots$ while keep the order unchanged. The metric type $\mathfrak{m}$ is naturally inherited by Γ', denoted by $\mathfrak{m}'$. Lastly, for each old leaf l_i, define $\mathfrak{o}'(l_i) = \mathfrak{o}(l_i)$. For each new leaf l'_s obtained from a connected component Ξ_s, define

$$\mathfrak{o}'(l'_s) = \sum_{l_i \in \mathrm{L}(\Xi_s)} \mathfrak{o}(l_i).$$

Lastly, if there is a vertex $\mathfrak{v} \in \mathrm{V}_\diamond(\Gamma') \setminus \mathrm{V}(\underline{\Gamma}')$ which has valence 1 and only one interior leaf, and if $\mathfrak{v}$ is attached to $\mathfrak{w} \in \mathrm{V}_\triangledown(\Gamma')$, then we contract $\mathfrak{v}, \mathfrak{v}$ and replace the leaf by a new leaf attached to $\mathfrak{w}$. The contact order at the new leaf is set to be the same as the contact order at the leaf attached to $\mathfrak{v}$. This completes the definition of the S-stabilization $\Gamma' = (\Gamma', \mathfrak{s}', \mathfrak{m}', \mathfrak{w}', \mathfrak{o}')$.

Definition 5.7. Consider an arbitrary $\Gamma \in \mathbf{T^{st}} \cup \mathbf{F^{st}}$. A subset $\mathrm{S} \subset \mathrm{V}(\Gamma) \smallsetminus \mathrm{V}(\underline{\Gamma})$ is called *special* if for each connected component of S, there is at most one interior leaf attached to this component. For a special subset S, a map $J_\Gamma \in \mathbf{Map}^\epsilon(\overline{\mathcal{U}}_\Gamma, \mathcal{J})$ is called *special* for S if its restriction to $\overline{\mathcal{U}}_{\Gamma,\Gamma_{\mathrm{S}}} \subset \overline{\mathcal{U}}_\Gamma^2$ equals the constant J_X. A perturbation $P_\Gamma = (F_\Gamma, J_\Gamma)$ is called an S-*special* perturbation if J_Γ is special for S. Let $\mathcal{P}_{\mathrm{S}}(\overline{\mathcal{U}}_\Gamma)$ the space of S-special perturbations. Then $\mathcal{P}(\overline{\mathcal{U}}_\Gamma) = \mathcal{P}_\varnothing(\overline{\mathcal{U}}_\Gamma)$.

Notice that as a closed subspace of $\mathcal{P}(\overline{\mathcal{U}}_\Gamma)$, $\mathcal{P}_{\mathrm{S}}(\overline{\mathcal{U}}_\Gamma)$ is a separable Banach space.

Now we define an operation on perturbation data induced by the stabilization procedure defined above. Consider $\Gamma \in \mathbf{T^{st}} \cup \mathbf{F^{st}}$ and any subset $\mathrm{S} \subset \mathrm{V}(\Gamma) \smallsetminus \mathrm{V}(\underline{\Gamma})$. Let Π be the S-stabilization of Γ. Then the union of all non-separating components of S becomes a special subset R of $\mathrm{V}(\Pi) \smallsetminus \mathrm{V}(\underline{\Pi})$ and the S-stabilization yields a map $\overline{\mathcal{W}}_\Gamma \to \overline{\mathcal{W}}_\Pi$. Moreover, for the complement of S or R, there are closed subsets $\overline{\mathcal{U}}_{\Gamma,\mathrm{S}^c} \subset \overline{\mathcal{U}}_\Gamma$ and $\overline{\mathcal{U}}_{\Pi_{\mathrm{S}},\mathrm{R}^c} \subset \overline{\mathcal{U}}_\Pi$ and a commutative diagram

$$\begin{array}{ccc} \overline{\mathcal{U}}_{\Gamma,\mathrm{S}^c} & \longrightarrow & \overline{\mathcal{U}}_{\Pi,\mathrm{R}^c} \\ \downarrow & & \downarrow \\ \overline{\mathcal{W}}_\Gamma & \longrightarrow & \overline{\mathcal{W}}_\Pi \end{array} .$$

An arbitrary perturbation $P_\Gamma = (F_\Gamma, J_\Gamma) \in \mathcal{P}(\overline{\mathcal{U}}_\Gamma)$ induces a R-special perturbation on Π. Hence such a perturbation yields a map

$$\pi_{\mathrm{S}} : \mathcal{P}(\overline{\mathcal{U}}_\Gamma) \to \mathcal{P}_{\mathrm{R}}(\overline{\mathcal{U}}_\Pi) \tag{5.8}$$

between Banach manifolds. If we identify (F, J_X) with the origin of the corresponding Banach space, then this map is linear. Furthermore, a right inverse to this map is given by using the constant almost complex structure on all components of $\overline{\mathcal{U}}_\Gamma$ corresponding to vertices in S.

This discussion has the following consequence for generic perturbations. Recall that in a Baire space a *comeager* subset (a countable intersection of open dense subsets) is dense.

Lemma 5.8. *Suppose* $\Gamma \in \mathbf{T^{st}} \cup \mathbf{F^{st}}$.

(a) *For any* $\mathrm{S} \subset \mathrm{V}(\Gamma) \smallsetminus \mathrm{V}(\underline{\Gamma})$, *let* Π *be the* S*-stabilization of* Γ. *With notation as above, the preimage of any comeager subset of* $\mathcal{P}_{\mathrm{R}}(\overline{\mathcal{U}}_\Pi)$ *under the restriction map is a comeager subset in* $\mathcal{P}(\overline{\mathcal{U}}_\Gamma)$.
(b) *For any* $\Gamma' \trianglelefteq \Gamma$, *the preimage of any comeager subset of* $\mathcal{P}(\overline{\mathcal{U}}_{\Gamma'})$ *under the restriction map is a comeager subset in* $\mathcal{P}(\overline{\mathcal{U}}_\Gamma)$.

Proof. Both claims follow from the existence of a continuous right inverse and Lemma 5.9 below. □

Lemma 5.9. *Let* X, Y *be separable Banach spaces. Let* $f : X \to Y$ *be a continuous linear map that admits a continuous right inverse. Then for any comeager subset* $U \subset Y$, $f^{-1}(U)$ *is also a comeager subset of* X.

Proof. By definition, we can write $U = \bigcap_{i=1}^{\infty} U_i$ where U_i is an open dense subset of Y. Then

$$f^{-1}(U) = \bigcap_{i=1}^{\infty} f^{-1}(U_i).$$

Since preimages of open sets are open sets, the conclusion of this lemma follows if we can prove that the preimage of any dense subset of Y is dense in X. Let $S \subset Y$ be a dense subset. Since X is Hausdorff and first countable, it suffices to prove that every point $x \in X$ can be approximated by a sequence in $f^{-1}(S)$. Denote $y = f(x)$ and $x' = g(y)$. Then $x - x' \in \mathrm{ker} f$. On the other hand, there is a sequence $y_i \in S$ converging to y in Y. Denote $x_i' = g(y_i)$. Then the sequence $x_i' + (x - x') = x + (x_i' - x') = x + g(y_i' - y)$ converges to x. Moreover, $x_i' + (x - x') \in x_i' + \mathrm{ker} f \subset f^{-1}(S)$. Hence $f^{-1}(S)$ is dense. □

6. Moduli Spaces

In this section we construct moduli space of curves and vortices appearing in definitions of the A_∞ algebras and the A_∞ morphism. In Subsection 6.1 we define the notion of vortex types, which are combinatorial types for the moduli spaces of affine vortices. In Subsection 6.2 we define the notion of treed scaled vortices and the notion of adaptedness with respect to the stabilizing divisor. In Subsection 6.3 we define the topology on the moduli space and prove the compactness result.

6.1. **Vortex types.** The combinatorial type of a vortex is defined as follows. Remember that we have chosen a Morse function $F : \bar{L} \to \mathbb{R}$ which has a unique maximum $\boldsymbol{x}_M$ (since L is connected). For $\boldsymbol{x} \in \mathbf{crit}$ we assign the degree

$$\mathbf{i}(\boldsymbol{x}) = \mathbf{dim}(\bar{L}) - \mathbf{ind}(\boldsymbol{x})$$

where **ind** is the Morse index. We use homology classes to label two-dimensional components. There is a commutative diagram

$$\begin{array}{ccc} H_2(\bar{X}) & \longrightarrow & H_2^K(X) \\ \downarrow & & \downarrow \\ H_2(\bar{X}, \bar{L}) & \longrightarrow & H_2^K(X, L) \end{array}.$$

We use B to denote an element of $H_2^K(X, L)$. Denote the Maslov index

$$\mathrm{Mas} : H_2^K(X, L; \mathbb{Z}) \to \mathbb{Z}.$$

We say that a holomorphic sphere in $\bar{X}$ (resp. a holomorphic disk in $\bar{X}$ resp. an affine vortex over $\boldsymbol{C}$) *represents* B if its homology class in $H_2(\bar{X})$ (resp. $H_2(\bar{X}, \bar{L})$ resp. $H_2^K(X)$) is mapped to B.

We use certain submanifolds to label pointwise constraints at interior markings. For each nonempty subset $I \subset \{0, 1, \ldots, \bar{n}\}$, denote

$$D_I = \bigcap_{a \in I} D_a, \qquad D_I^0 = \Big(\bigcap_{a \in I} D_a^{\mathrm{st}} \Big) \smallsetminus \Big(\bigcup_{b \notin I} D_b \Big).$$

The normal crossing divisor $S = S_1 \cup \cdots \cup S_N$ of Definition 2.4 can also be stratified: for each $J \in \{1, \ldots, N\}$, define

$$S_J^0 = \Big(\bigcap_{a \in J} S_a \Big) \smallsetminus \Big(\bigcup_{b \notin J} S_b \Big) \subset X^{\mathrm{st}}.$$

We also have stratified the unstable locus X^{us} as $X^{\mathrm{us}} = X_1^{\mathrm{us}} \sqcup \cdots \sqcup X_m^{\mathrm{us}}$. Then let

$$\mathcal{V} = \{X_c^{\mathrm{us}} \mid c = 1, \ldots, m\} \sqcup \{D_I^0 \cap S_J^0 \mid I \subset \{0, 1, \ldots, \bar{n}\},\ J \subset \{1, \ldots, N\}\}.$$

Notice that when $I = J = \varnothing$, $D_I^0 \cap S_J^0 = X$. Each $V \in \mathcal{V}$ defines a smooth submanifold $\bar{V} = (\mu^{-1}(0) \cap V)/K \subset \bar{X}$ which is possibly empty.

Definition 6.1. An unbroken *vortex type* $\tilde{\Gamma}$ is a tuple $\tilde{\Gamma} = (\Gamma, \tilde{\boldsymbol{x}}, \boldsymbol{B}, \boldsymbol{V}, \tilde{\mathfrak{o}})$ where

- $\Gamma \in \mathbf{T} \cup \mathbf{F}$ is an unbroken type.
- $\tilde{\boldsymbol{x}} = (\boldsymbol{x}_1, \ldots, \boldsymbol{x}_{\underline{k}}; \boldsymbol{x}_\infty)$ is a sequence of elements in the set of critical points of F, where $\underline{k}$ is the number of incoming boundary tails of Γ.
- $\boldsymbol{B}$ is collection of classes $B_\alpha \in H_2^K(X, L; \mathbb{Z})$ labelled by vertices of Γ.
- $\boldsymbol{V} = (V_1, \ldots, V_k)$ is a sequence of K-invariant smooth submanifolds of X belonging to $\mathcal{V}$.
- $\tilde{\mathfrak{o}} : \mathrm{L}(\Gamma) \to \mathbb{Z}$ is a function such that $\tilde{\mathfrak{o}}(l_i)$ is between 1 and $\mathfrak{o}(l_i)$.

The notion of unbroken vortex types can be easily extended to the notion of *broken* vortex types, for which we skip the details. Every vortex type $\tilde{\Gamma}$ has an underlying combinatorial type Γ and the correspondence $\tilde{\Gamma} \to \Gamma$ will be used frequently without explicit explanation. The type $\tilde{\Gamma}$ is then called a *refinement* of Γ. The total energy $E(\tilde{\Gamma})$ and the total Maslov index $M(\tilde{\Gamma})$ of a vortex type $\tilde{\Gamma}$ is defined as

$$E(\tilde{\Gamma}) := \sum_{\mathfrak{v}_\alpha \in \mathrm{V}(\Gamma)} E(B_\alpha), \qquad \mathrm{Mas}(\tilde{\Gamma}) = \sum_{\mathfrak{v}_\alpha \in \mathrm{V}(\Gamma)} \mathrm{Mas}(B_\alpha).$$

For two vortex types $\tilde{\Gamma}$, $\tilde{\Gamma}'$, we say $\tilde{\Gamma}' \tilde{\leqslant} \tilde{\Gamma}$ if the following conditions are satisfied:

- $\Gamma' \leqslant \Gamma$, which implies a tree map $\rho : \Gamma' \to \Gamma$.
- For any boundary tail $t'_{\underline{i}} \in \underline{\mathrm{T}}(\underline{\Gamma}')$ and $t_{\underline{i}} = \rho_{\mathrm{T}}(t_{\underline{i}}) \in \underline{\mathrm{T}}(\underline{\Gamma})$, $\boldsymbol{x}'_{\underline{i}} = \boldsymbol{x}_{\underline{i}}$.
- For any vertex $\mathfrak{v}_\alpha \in \mathrm{V}(\Gamma)$, we require
$$B_\alpha = \sum_{\mathfrak{v}'_\beta \in \rho_{\mathrm{V}}^{-1}(\mathfrak{v}_\alpha)} B'_\beta \in H_2^K(X, L; \mathbb{Z}).$$
- For any interior leaf $l'_i \in \mathrm{L}(\Gamma')$ and $l_i = \rho_{\mathrm{L}}(l'_i) \in \mathrm{L}(\Gamma)$, if $V'_i = D_a^0$ for some $a \in \{0, 1, \ldots, \bar{n}\}$, then we require
$$V_i = D_a^0,\ \tilde{\mathfrak{o}}'(l'_i) \geqslant \tilde{\mathfrak{o}}(l_i).$$
If $V'_i \neq D_a^0$ for any $a \in \{0, 1, \ldots, \bar{n}\}$, then we require that $V'_i \subset \overline{V_i}$.

6.2. **Treed vortices.** A treed vortex is a combination of disks, spheres, and vortices, sometimes with Lagrangian boundary condition, pseudoholomorphic with respect to a given collection of domain-dependent almost complex structures, together with gradient segments of the given domain-dependent Morse function. Given a collection of coherent perturbation data $\underline{P} := \{P_\Gamma = (J_\Gamma, F_\Gamma) \mid \Gamma \in \mathbf{T}^{\mathbf{st}} \cup \mathbf{F}^{\mathbf{st}}\}$, for each (not necessarily stable) treed disk $\mathcal{Z}$ with underlying type $\Gamma \in \mathbf{T} \cup \mathbf{F}$, there are an induced family of almost complex structures $J_{\mathcal{Z}} : \mathcal{Z}^2 \to \mathcal{J}$ and an induced family of domain-dependent smooth functions $F_{\mathcal{Z}} : \mathcal{Z}^1 \to \mathcal{F}$. Indeed, if Γ is stable, then $(J_{\mathcal{Z}}, F_{\mathcal{Z}})$ is induced by pull back P_Γ via the embedding of $\mathcal{Z}$ into a fibre of the universal curve $\overline{\mathcal{U}}_\Gamma$. If $\mathcal{Z}$ is unstable, let $\mathcal{Z}^{\mathrm{st}}$ (with underlying type Γ^{st}) be the stabilization. Then on each stable two-dimensional component of $\mathcal{Z}$, $J_{\mathcal{Z}}$ is induced from $J_{\mathcal{Z}^{\mathrm{st}}}$; on the edges of $\mathcal{Z}$ that persist under the stabilization, $F_{\mathcal{Z}}$ is also induced from $F_{\mathcal{Z}^{\mathrm{st}}}$; on edges that are contracted by the stabilization, $F_{\mathcal{Z}}$ is equal to the unperturbed F.

Definition 6.2. Given a system of coherent perturbation data $\underline{P} = \{P_\Gamma \mid \Gamma \in \mathbf{T}^{\mathbf{st}} \cup \mathbf{F}^{\mathbf{st}}\}$, given a (not necessarily stable) vortex type $\tilde{\Gamma}$,
A *treed scaled vortex* of vortex type $\tilde{\Gamma}$ is a collection

$$\mathcal{C} := \{\mathcal{Z}, (\boldsymbol{v}_\alpha)_{\mathfrak{v}_\alpha \in \mathrm{V}(\Gamma)}, (x_{\mathfrak{e}})_{\mathfrak{e} \in \mathrm{E}(\Gamma) \sqcup \mathrm{T}(\Gamma)}\} \tag{6.1}$$

where

- $\mathcal{Z}$ is a treed disk with type Γ;
- for each $\mathfrak{v}_\alpha \in \mathrm{V}(\Gamma)$, $\boldsymbol{v}_\alpha$ is a gauged map from $(\Sigma_\alpha, \partial\Sigma_\alpha)$ to (X, L);
- for each $\mathfrak{e} \in \mathrm{E}(\Gamma) \sqcup \mathrm{T}(\Gamma)$, $x_{\mathfrak{e}} : I_e \to \bar{L}$ is a smooth map

satisfying the following conditions:

- $\boldsymbol{v}_\alpha$ is a generalized vortex with respect to $J_{\mathcal{Z}}|_{\Sigma_\alpha}$. More precisely, if $\mathfrak{v}_\alpha \in \mathrm{V}_\vartriangle(\Gamma)$, then $\boldsymbol{v}_\alpha$ is a K-orbit of $J_{\mathcal{Z}}|_{\Sigma_\alpha}$-holomorphic disk or sphere in $\bar{X}$; if $\mathfrak{v}_\alpha \in \mathrm{V}_\diamond(\Gamma)$, then $\boldsymbol{v}_\alpha$ is a gauge equivalence class of $J_{\mathcal{Z}}|_{\Sigma_\alpha}$-affine vortex over $\boldsymbol{C}$ or $\boldsymbol{H}$; if $\mathfrak{v}_\alpha \in \mathrm{V}_\triangledown(\Gamma)$, then $\boldsymbol{v}_\alpha$ is a $\bar{J}_{\mathcal{Z}}|_{\Sigma_\alpha}$-holomorphic disk or sphere in $\bar{X}$.
- Each $x_{\mathfrak{e}}$ is a solution to $x_{\mathfrak{e}}'(s) + \nabla(F_{\mathcal{Z}}|_{I_{\mathfrak{e}}})(x_{\mathfrak{e}}(s)) = 0$ on $I_{\mathfrak{e}}$. Moreover, if t_- and t_+ form an infinite edge in Γ, then (recall $I_{t_-} = I_{t_+} = \mathbb{R}$) $x_{t_-} = x_{t_+}$.
- For each $\mathfrak{e} \in \mathrm{T}(\Gamma)$, $x_{\mathfrak{e}}(t)$ converges as $t \to \pm\infty$ to the prescribed limit in **crit**.
- There are obvious matching conditions at nodes and points where one dimensional components are attached to two dimensional components.

A treed scaled vortex $\mathcal{C}$ as described above is said to satisfy the *contact order condition* at a leaf $l_i \in \mathrm{L}(\Gamma)$ if the following conditions are satisfied.

- Suppose $l_i \in \mathrm{L}^\vartriangle(\Gamma)$ and is attached to a vertex $\mathfrak{v}_\alpha \in \mathrm{V}_\vartriangle(\Gamma) \sqcup \mathrm{V}_\diamond(\Gamma)$, then $z_i \in \boldsymbol{v}_\alpha^{-1}(V_i)$. Moreover, suppose $V_i = D_a^0$ for some $a \in \{0, 1, \ldots, \bar{n}\}$ and $\boldsymbol{v}_\alpha$ is not entirely contained in D_a^0, then the contact order of $\boldsymbol{v}_\alpha$ at z_i with D_a^0 is $\tilde{\mathfrak{o}}(l_i)$.
- Suppose $l_i \in \mathrm{L}^\triangledown(\Gamma)$ and is attached to a vertex $\mathfrak{v}_\alpha \in \mathrm{V}_\triangledown(\Gamma)$, then $z_i \in \boldsymbol{v}_\alpha^{-1}(\bar{V}_i)$. Moreover, suppose $V_i = D_a^0$ for some $a \in \{0, 1, \ldots, \bar{n}\}$ and $\boldsymbol{v}_\alpha$ is not entirely contained in $\bar{D}_a$, then the contact order of $\boldsymbol{v}_\alpha$ at z_i with $\bar{D}$ is $\tilde{\mathfrak{o}}(l_i)$.

A treed scaled vortex $\mathcal{C}$ as described above is *stable* if the following condition is satisfied.

- For each unstable vertex $\mathfrak{v}_\alpha$, $\boldsymbol{v}_\alpha$ has positive energy.
- For each infinite edge $\{t_-, t_+\}$ in Γ, $x_{t_-} = x_{t_+}$ is nonconstant.

The notion of isomorphisms between treed scaled vortices can be defined in an ordinary way, namely, by incorporating domain symmetries and the gauge symmetry.

Given a vortex type $\tilde{\Gamma}$, let $\mathcal{M}_{\tilde{\Gamma}}(P_\Gamma)$ be a moduli space of treed scaled vortices of type $\tilde{\Gamma}$ that satisfy the contact order condition at all interior leaves.

6.3. **Convergence and compactness.** We describe a notion of Gromov convergence for sequences of vortices. Let $(\Gamma, \mathfrak{s})$ be the colored baseless tree with a single vertex colored by $\diamond$ and a finite set of leaves $l_1, \ldots, l_k$ with $k \geqslant 1$. Let $\mathcal{Z}_k$ be a sequence of marked curves representing a sequence of points $p_k \in \mathcal{W}_\Gamma$. Let $\Pi \trianglelefteq \Gamma$ be a vortex type and let $\mathcal{Z}_\infty$ be a marked stable curve representing a point $p_\infty \in \mathcal{W}_\Pi \subset \overline{\mathcal{W}}_\Gamma$.

Definition 6.3. (Gromov convergence of vortices) Let $\boldsymbol{v}_k = (u_k, \phi_k, \psi_k)$ be a sequence of vortices from $\boldsymbol{C}$ to X and $\boldsymbol{v}_\infty$ be a stable J_∞-affine vortex on $\boldsymbol{C}$ having components $\boldsymbol{v}_{\infty,\alpha}$. We say that $\boldsymbol{v}_k$ *Gromov converges* to $\boldsymbol{v}_\infty$ if the following conditions are satisfied: there exist sequences of Möbius transformations or translations

$$\phi_{\alpha,k} : \Sigma_\alpha \to \boldsymbol{C}$$

such that

(a) (Holomorphic spheres or disks or vortices in X) For $\mathfrak{v}_\alpha \in \mathrm{V}_\triangle(\Pi) \sqcup \mathrm{V}_\diamond(\Pi)$, there exists a sequence of gauge transformations $\zeta_k \in \mathrm{Aut}(P_\alpha)$ such that $\boldsymbol{v}_k \circ \phi_{\alpha,k}$ converges uniformly with all derivatives to $\boldsymbol{v}_{\infty,\alpha}$ on compact subsets of $\Sigma_\alpha \smallsetminus W_\alpha$.
(b) (Holomorphic spheres or disks in $\bar{X}$) For $\mathfrak{v}_\alpha \in \mathrm{V}_\triangledown(\Pi)$, $u_k \circ \phi_{\alpha,k}$ converges uniformly to $u_{\infty,\alpha}$ on compact subsets of $\Sigma_\alpha \smallsetminus W_\alpha$.
(c) (No energy loss) $E(\boldsymbol{v}_k)$ converges to $E(\boldsymbol{v}_\infty)$.

This notion of convergence is essentially due to Ziltener (see [Zil14]). Slightly generalizing this notion to the bordered case, we can define the convergence of a sequence of affine vortices over $\boldsymbol{H}$ with markings to a stable affine vortex over $\boldsymbol{H}$. We omit the details but assume that the reader understand the role of the Möbius transformations (or translations) in the convergence and different requirements between components in $\mathrm{V}_\triangle(\Gamma) \cup \mathrm{V}_\diamond(\Gamma)$ and components in $\mathrm{V}_\triangledown(\Gamma)$.

Now we define the notion of convergence for unbroken objects stable treed scaled vortices. The definition for the case of a sequence of broken objects can be easily derived. Besides infinite edges, we only need to consider the compactification of $\mathcal{M}_{\tilde{\Gamma}}$ for which the underlying type Γ is stable, although the boundary of $\mathcal{M}_{\tilde{\Gamma}}$ may contain objects with unstable underlying types.

Definition 6.4. Let $(\mathcal{C}_k)_{k=1}^\infty$ be a sequence of stable treed scaled vortices of *unbroken* vortex type $\tilde{\Gamma}_k$ with the same underlying *stable* type Γ. Let $\mathcal{C}_\infty$ be another stable treed scaled vortex. We say that $\mathcal{C}_k$ *converges modulo gauge* to $\mathcal{C}_\infty$ if the following conditions are satisfied.

(a) Let the underlying treed disks of $\mathcal{C}_k$ be $\mathcal{Z}_k$ and the underlying treed disk of $\mathcal{C}_\infty$ be $\mathcal{Z}_\infty^\circ$. $\mathcal{Z}_\infty^\circ$ may not be stable and let its stabilizations be $\mathcal{Z}_\infty$. Then we require that $\mathcal{Z}_\nu$ converges to $\mathcal{Z}_\infty$ in the sense of Definition 4.8 (we use the same notations as Definition 4.8). Let Γ and Γ_∞ be the underlying combinatorial types of $\mathcal{Z}_\nu$ and $\mathcal{Z}_\infty$, respectively. Let Γ' be the combinatorial type obtained from Γ_∞ by gluing new breakings. Then there is a morphism $\rho : \Gamma' \to \Gamma$.
(b) For each $\mathfrak{v}_\alpha \in \mathrm{V}(\Gamma)$, consider the sequence of generalized vortices $\boldsymbol{v}_{\alpha,k}$ with domain Σ_α. The preimage of $\mathfrak{v}_\alpha$ under ρ_{V} corresponds to a subtree of Γ', which also corresponds to a stable generalized vortex $\mathcal{C}_{\alpha,\infty}$. Then the sequence $\boldsymbol{v}_{\alpha,k}$ converges modulo gauge to $\mathcal{C}_{\alpha,\infty}$, in the sense we just discussed before this definition.
(c) For an edge of positive length or boundary tail $\mathfrak{e} \in \mathrm{E}^+(\underline{\Gamma}) \sqcup \underline{\mathrm{T}}(\underline{\Gamma})$, its preimage under ρ_{E} corresponds to a (chain of) edges in Γ_∞°, which in $\mathcal{C}_\infty$ corresponds to a (broken) perturbed gradient line (with possibly finite ends). If the lengths of $\mathfrak{e}$ in $\mathcal{Z}_k$ do not converge to zero, then the sequence of perturbed gradient lines $x_{\mathfrak{e},k}$ converges uniformly in all derivatives to the (broken) perturbed gradient line in $\mathcal{C}_\infty$. If the lengths of $\mathfrak{e}$ converge to zero, then there is no extra requirement for the sequence of perturbed gradient lines $x_{\mathfrak{e},k}$.

One can easily extend the above definition to the case that the underlying vortex type $\tilde{\Gamma}$ of the sequence is broken. We also leave the following statements to the reader to check.

Proposition 6.5. *The sequential limits defined as in Definition 6.4, are unique. Namely, if a sequence $\mathcal{C}_k$ stable treed scaled vortices converges to two stable treed scaled vortices $\mathcal{C}_\infty, \mathcal{C}'_\infty$, then there exists an isomorphism from $\mathcal{C}_\infty$ to $\mathcal{C}'_\infty$.*

Proposition 6.6. *If in the sequence each $\mathcal{C}_k$ satisfies the contact order condition at all interior leaves, then the limit $\mathcal{C}_\infty$ also satisfies the contact order condition at all interior leaves.*

Proposition 6.7. *Suppose $\mathcal{C}_k$ converges to $\mathcal{C}_\infty$. Suppose $\mathcal{C}_\infty$ has underlying vortex type $\tilde{\Gamma}_\infty$ and for all k, $\mathcal{C}_k$ has the same underlying vortex type $\tilde{\Gamma}$, then $\tilde{\Gamma}_\infty \tilde{\leqslant} \tilde{\Gamma}$.*

Lastly we state the compactness theorem. We do not provide the detailed proof since as long as the convergence notion is defined clearly, the proof is a combination of results on compactness for broken Morse trajectories, Gromov compactness for holomorphic spheres and disks, and affine vortices due to Ziltener in [Zil14]; the compactness result of affine vortices over $\boldsymbol{H}$ is essentially proved by Wang–Xu in [WX17] (in Wang–Xu [WX17] the authors actually proved the compactness of vortices over the unit disk under adiabatic limit, and the case of vortices over $\boldsymbol{H}$ can be reproduced from the same argument).

Theorem 6.8. (Compactness of fixed type) *Let $\mathcal{C}_k$ be a sequence of stable treed scaled vortices with isomorphic stable underlying combinatorial types $\Gamma \in \mathbf{T}^{\mathbf{st}} \cup \mathbf{F}^{\mathbf{st}}$, while the equations are defined by a common perturbation $P_\Gamma \in \mathcal{P}^\epsilon(\overline{\mathcal{U}}_\Gamma)$. Suppose the energy $E(\mathcal{C}_k)$ are uniformly bounded. Then there is a subsequence of $\mathcal{C}_k$ (still indexed by k) that have isomorphic underlying vortex types and that converge to a stable treed scaled vortex $\mathcal{C}_\infty$.*

In particular, consider a *stable* type $\Gamma \in \mathbf{T}^{\rm st} \cup \mathbf{F}^{\mathbf{st}}$ and a refinement $\tilde{\Gamma}$. Consider the moduli space $\mathcal{M}_{\tilde{\Gamma}}(P_\Gamma)$. We define

$$\overline{\mathcal{M}_{\tilde{\Gamma}}(P_\Gamma)} := \bigsqcup_{\tilde{\Pi} \tilde{\leqslant} \tilde{\Gamma}} \mathcal{M}_{\tilde{\Pi}}(P_\Gamma|_{\overline{\mathcal{U}}_{\Pi^{\rm st}}}). \tag{6.2}$$

Notice that there might be stable refinements $\tilde{\Pi}$ appearing in the disjoint union on the right hand side, whose underlying Π is unstable. *A priori* Theorem 6.8 does not immediately imply the sequential compactness of $\overline{\mathcal{M}_{\tilde{\Gamma}}(P_\Gamma)}$. To show the sequential compactness, first notice that there are only finitely many $\tilde{\Pi}$ appearing in the disjoint union on the right hand side of (6.2). Second, given a sequence of elements in $\mathcal{M}_{\tilde{\Pi}}(P_\Gamma|_{\Pi^{\rm st}})$ with Π unstable, we can add a fixed number of markings to stabilize all unstable components of Π. Then Theorem 6.8 implies that a subsequence of the stabilized sequence converge to a limit. It is routine to show that after removing the added marked points, the convergence still holds.

Theorem 6.9. *Let $\Gamma \in \mathbf{T}^{\rm st}$ be a stable type and $\tilde{\Gamma}$ be a refinement. Let P_Γ be a perturbation data on $\overline{\mathcal{U}}_\Gamma$. Then the moduli space $\overline{\mathcal{M}_{\tilde{\Gamma}}(P_\Gamma)}$ has a unique compact and Hausdorff topology for which sequential convergence is the same as the sequential convergence defined by Definition 6.4.*

7. Transversality

In this section we discuss the regularity of various moduli spaces. To achieve transversality while maintaining certain properties of the countings, we impose several additional conditions on the perturbation data.

Definition 7.1. Let $\tilde{\Gamma} = (\Gamma, \tilde{\boldsymbol{x}}, \boldsymbol{B}, \tilde{\mathfrak{o}}, \boldsymbol{V})$ be a vortex type (see Definition 6.1).

(a) Γ is called *reduced* if $\mathrm{V}_{\triangle}(\Gamma) = \mathrm{V}_{\triangle}(\underline{\Gamma})$, i.e., there is no vertices corresponding to holomorphic spheres in X.

(b) In $\tilde{\Gamma}$, a *ghost vertex* is a vertex $\mathfrak{v}_\alpha$ with $B_\alpha = 0$. A *ghost tree* is a subtree whose vertices are all ghost vertices.

(c) A vortex type $\tilde{\Gamma}$ is called *uncrowded* if for each ghost tree Ξ, there is at most one interior leaf attached to vertices in Ξ. Otherwise $\tilde{\Gamma}$ is called *crowded*.

(d) Let $\mathrm{S} \subset \mathrm{V}(\Gamma) \smallsetminus \mathrm{V}(\underline{\Gamma})$ be a special subset (see Definition 5.7). A refinement $\tilde{\Gamma}$ of Γ is called S-*special* if $B_\alpha = 0$ for all $\mathfrak{v}_\alpha \in \mathrm{S}$.

(e) Suppose the degree of the stabilizing divisor D is N_D. The vortex type $\tilde{\Gamma}$ is said to be *controlled* if: 1) for each superstructure Π of Γ

$$N_D E(\tilde{\Pi}) = N_D \sum_{\mathfrak{v}_\alpha \in \mathrm{V}(\tilde{\Pi})} \langle \omega_X^K, B_\alpha \rangle \leqslant \sum_{l_i \in \mathrm{L}(\Pi)} \mathfrak{o}(l_i); \tag{7.1}$$

2) For each maximal subtree Π_α of Γ that contains only one vertex in $\mathfrak{v}_\alpha \in \mathrm{V}(\underline{\Gamma})$, one has

$$N_D E(\tilde{\Pi}_\alpha) = N_D \sum_{\mathfrak{v}_\alpha \in \mathrm{V}(\tilde{\Pi}_\alpha)} \langle \omega_X^K, B_\alpha \rangle \leqslant \sum_{l_i \in \mathrm{L}(\Pi_\alpha)} \mathfrak{o}(l_i); \tag{7.2}$$

We remark that in this paper transversality is only achieved for moduli spaces $\mathcal{M}^*_{\tilde{\Gamma}}(P_\Gamma) \subset \mathcal{M}_{\tilde{\Gamma}}(P_\Gamma)$ in which $\tilde{\Gamma}$ is reduced, uncrowded, and controlled in the sense of Definition 7.1. Here $\mathcal{M}^*_{\tilde{\Gamma}}(P_\Gamma)$ is the open subset consisting of configurations for which no nonconstant component is mapped entirely into the stabilizing divisor. In order to obtain nice compactifications of the moduli spaces (the refined compactness theorem, Proposition 7.12), the transversality needs to be stronger than the naive one, in which the discussion involves S-special refinements and S-stabilizations (see Subsection 5.5).

The remaining of this section is organized as follows. In Subsection 7.1 we give the analytic local models for holomorphic curves and affine vortices with prescribed tangency conditions at interior markings. In Subsection 7.2 we set up the analytical framework for the associated universal Fredholm problem and state the definition of regular perturbation data. In Subsection 7.3 we consider the transversality for moduli spaces over closed domains. In Subsection 7.4 we consider the transversality for bordered domains. In Subsection 7.5 we comment on a special condition on the perturbation data, which has implications on the theories on the classical level. In Subsection 7.6 we prove a compactness results for certain moduli spaces of expected dimension zero or one.

7.1. Local model with tangency conditions.

As in [CM07] we need to discuss holomorphic maps or vortices having constraints of derivatives at interior markings. One should be able to describe the tangency condition on the level of Banach manifolds. In [CM07], one can use higher Sobolev space $W^{k,p}$ to model maps and consider the l-th order derivative of any maps of regularity $W^{k,p}$, thanks to the Sobolev embedding $W^{k,p} \hookrightarrow C^l$ if $k \geqslant l+1$. However if we would like to use higher Sobolev spaces to model vortices, we need to re-work the result of the second name author and Venugopalan [VXne], which was already quite technical. Instead, we still use the $W^{1,p}$ norm and use the twisting trick which appeared in [Zin11]. We hereby thank A. Zinger for explaining this method to us.

Define a Banach space of sections that vanish to a certain order at a given point as follows. Recall $p \in (2,4)$. Let Σ be a Riemann surface, $E \to \Sigma$ be a complex line bundle equipped with a metric and a metric connection. Let $Z = o_1 z_1 + \cdots + o_m z_m$ be an effective divisor with multiplicity $o_i \geqslant 1$. Then denote by $W^{k,p}_Z(\Sigma, E) \subset W^{k,p}_{\rm loc}(\Sigma \smallsetminus Z, E)$ the space of sections s of E such that for every $z_i \in Z$ there exists a neighborhood U_i of z_i and a coordinate $w : U_i \to \mathbb{C}$ such that

$$w(z_i) = 0, \qquad\qquad \frac{\nabla^l s}{w^{k-l+o_i-1}} \in L^p(U_i, E),\ \forall l = 0, \dots, k.$$

In other words, s is of class $W^{k,p,o_i - \frac{2}{p}}$ with respect to the cylindrical metric near Z. It is straightforward to check that if s is holomorphic near Z with respect to certain local holomorphic structure of E and $s(w) = O(|w|^{o_i})$ near all z_i, then s lies in the space $W^{k,p}_Z(\Sigma, E)$. Moreover, the following lemma is standard.

Lemma 7.2. *$W^{k,p}_Z(\Sigma, E)$ is a Banach space. Moreover, if Σ is a compact Riemann surface without boundary, $D : \Omega^0(E) \to \Omega^{0,1}(E)$ is a real linear Cauchy–Riemann operator, then D induces a Fredholm operator*

$$D : W^{k,p}_Z(\Sigma, E) \to W^{k-1,p}_Z(\Sigma, \Lambda^{0,1} \otimes E),$$

whose index is

$$\mathbf{ind}_{\mathbb{R}} D = 2 - 2g + 2{\rm deg} E - 2{\rm deg} Z.$$

A similar index formula, left to the reader, holds for the case of Cauchy–Riemann operators on a surface with boundary with totally real boundary conditions.

Consider the following tangency conditions for pseudoholomorphic curves or vortices. First consider the case that the domain is $\boldsymbol{S}^2$. Let Z_a be an effective divisor for $a \in \{0, 1, \dots, \bar{n}\}$. Assume all Z_a pairwise disjoint and denote $Z = (Z_0, Z_1, \dots, Z_{\bar{n}})$. Let $\mathcal{B}_{\boldsymbol{S}^2, Z}(\bar{B})$ be the space of $W^{1,p}$-maps $\bar{u} : \boldsymbol{S}^2 \to \bar{X}$ that represent the class $\bar{B} \in H_2(\bar{X}, \bar{L}; \mathbb{Z})$, such that, if $\bar{D}_a$ is the vanishing locus of $\bar{s}_a \in \Gamma(\bar{R})$, then

$$\bar{u}^* \bar{s}_a \in W^{1,p}_{Z_a}(\boldsymbol{S}^2, \bar{u}^* \bar{R}).$$

Define the Banach space bundle $\mathcal{E}_{\boldsymbol{S}^2, Z}(B) \to \mathcal{B}_{\boldsymbol{S}^2, Z}(B)$ whose fibre over $\bar{u}$ is the space

$$\mathcal{E}_{\boldsymbol{S}^2, Z}(B)|_{\bar{u}} = \Big\{ \eta \in L^p(\boldsymbol{S}^2, \Lambda^{0,1} \otimes \bar{u}^* T\bar{X}) \mid d\bar{s}_a \circ \eta \in L^p_{Z_a}(\boldsymbol{S}^2, \Lambda^{0,1} \otimes \bar{u}^* \bar{R}),\ 0 \leqslant a \leqslant \bar{n} \Big\}.$$

Let $J \in \mathcal{J}(U, D)$ and consider the Cauchy–Riemann operator $\overline{\partial}_{\bar{J}}$. We claim that for each $\bar{u} \in \mathcal{B}_{\boldsymbol{S}^2, Z}(B)$, $\overline{\partial}_{\bar{J}} \bar{u} \in \mathcal{E}_{\boldsymbol{S}^2, Z}(B)$. Indeed, by the condition $\bar{u}^* \bar{s}_a \in W^{1,p}_{Z_a}(\boldsymbol{S}^2, \bar{u}^* \bar{R})$ and the fact that $\bar{s}_D$ is holomorphic with respect to $\bar{J}_X$, we see

$$d\bar{s}_a \circ \overline{\partial}_{\bar{J}} \bar{u} = \overline{\partial}(\bar{s}_a \circ \bar{u}) + \frac{1}{2} d\bar{s}_a \circ (\bar{J} - \bar{J}_X) \circ d\bar{u} \circ j.$$

The first term lies in $\mathcal{E}_{\boldsymbol{S}^2, Z}(B)$ automatically. For the second term, notice that $\bar{J}|_{\bar{D}_a} = \bar{J}_X|_{\bar{D}_a}$ and the distance between $\bar{u}$ and $\bar{D}_a$ can be estimated. It is not hard to show that the second term is also in $\mathcal{E}_{\boldsymbol{S}^2, Z}(B)$. Hence we have a smooth section

$$\overline{\partial}_{\bar{J}} : \mathcal{B}_{\boldsymbol{S}^2, Z}(B) \to \mathcal{E}_{\boldsymbol{S}^2, Z}(B)$$

of Banach space bundles. The local expression for the Cauchy–Riemann operator implies that $\overline{\partial}_{\bar J}$ is a Fredholm section and Lemma 7.2 implies its index is equal to

$$\dim \bar X + \mathrm{Mas}(\bar B) - 2\sum_{a=0}^{\bar n} \deg Z_a.$$

Moreover, every point in the zero locus is a holomorphic sphere representing the class $\bar B$, which has the prescribed tangency conditions at the interior markings.

Similar Fredholm result and index computations hold if the almost complex structure is domain-dependent; if there are zeroth order interior constraints at other markings; or if there Lagrangian boundary condition; these are left to the reader.

We state a similar Fredholm result and index computation for affine vortices with tangency conditions. Choose $B \in H_2^K(X, L; \mathbb{Z})$. Recall the Banach manifolds $\mathcal{B}_{\boldsymbol{C}}(B)$ of gauged equivalence classes of triples (u, ϕ, ψ) we introduced in (2.7), for $\boldsymbol{A} = \boldsymbol{C}$. Let $\mathcal{B}_{\boldsymbol{C},Z}(B) \subset \mathcal{B}_{\boldsymbol{C}}(B)$ be the subset of triples gauge equivalence classes of gauged maps (u, ϕ, ψ) from $\boldsymbol{C}$ to X satisfying the conditions that

(a) for each $z_i \in Z_a$, $u(z_i) \in D_a^0$; and
(b) for certain small neighborhood U_i around z_i, $u^* s_a \in W^{1,p}_{Z_a}(U_i, u^*R)$.

Moreover, let $\mathcal{E}_{\boldsymbol{C}}(B) \to \mathcal{B}_{\boldsymbol{C}}(B)$ be the bundle (2.8) and let $\mathcal{E}_{\boldsymbol{C},Z}(B) \subset \mathcal{E}_{\boldsymbol{C}}(B)$ be the subset of $\boldsymbol{\eta} = [\eta', \eta'', \eta''']$ where near each $z_i \in Z$, $ds_a \circ \eta' \in L^p_Z(U_i, u^*R)$. Then for any $J \in \mathcal{J}(U, D)$, the affine vortex equation defines a Fredholm section

$$\mathcal{F}_{J,B,Z} : \mathcal{B}_{\boldsymbol{C},Z}(B) \to \mathcal{E}_{\boldsymbol{C},Z}(B)$$

with

$$\mathbf{ind}\mathcal{F}_{J,B,Z} = \mathbf{dim}_{\mathbb{R}} \bar X + 2\deg B - 2\sum_{a=0}^{\bar n} \deg Z_a.$$

The corresponding statements for affine vortices over $\boldsymbol{H}$ are left to the reader.

The above index computations lead to the following formula for the expected dimension of $\mathcal{M}_{\tilde\Gamma}$ where $\tilde\Gamma$ is a reduced, unbroken, uncrowded type $\tilde\Gamma$. We *define* the expected dimension of $\mathcal{M}_{\tilde\Gamma}$ as

$$\mathbf{ind}\tilde\Gamma = \mathbf{dim}\mathcal{W}_\Gamma + \mathbf{i}(\tilde{\boldsymbol{x}}) + \mathrm{Mas}(\tilde\Gamma) - \sum_{i=1}^{k} \delta_i(\tilde\Gamma). \tag{7.3}$$

Here the first term is given in (4.1),

$$\mathbf{i}(\tilde{\boldsymbol{x}}) = \mathbf{i}(\boldsymbol{x}_\infty) - \sum_{\underline{i}=1}^{\underline{k}} \mathbf{i}(\boldsymbol{x}_{\underline{i}}),$$

and $\delta_i(\tilde\Gamma)$ is defined as

$$\delta_i(\tilde\Gamma) = \begin{cases} \mathbf{codim}_{\mathbb{R}}(V_i, X), & B_{\alpha_i} = 0 \text{ or } V_i \neq D_a^0 \text{ for all } a \in \{0, 1, \ldots, \bar n\}; \\ 2\tilde{\mathfrak{o}}(l_i), & B_{\alpha_i} \neq 0 \text{ and } V_i = D_a^0 \text{ for some } a \in \{0, 1, \ldots, \bar n\}. \end{cases} \tag{7.4}$$

The formula for broken vortex types that are reduced and uncrowded can be derived easily from (7.3).

7.2. **Definition of regularity.**

7.2.1. *The Fredholm problem setup.* The moduli spaces of stable treed scaled vortices of any given type is locally cut out by a Fredholm map of Banach spaces as follows. Given a *reduced* vortex type $\tilde\Gamma$ whose underlying type Γ is unbroken and stable; let $\mathcal{Z}$ be any stable treed disk representing a point of $\mathcal{W}_\Gamma$. To define various Sobolev spaces, we need to choose a Riemannian metric on every two-dimensional component of $\mathcal{Z}$. If $\mathfrak{v}_\alpha \in V_\triangle(\Gamma) \sqcup V_\triangledown(\Gamma)$, there is no canonical choice of a metric on Σ_α because the group of Möbius transformations do not preserve any metric. Nonetheless we choose an arbitrary one. If $\mathfrak{v}_\alpha \in V_\diamond(\Gamma)$, choose the translation invariant metric on Σ_α, which is diffeomorphic to either $\boldsymbol{C}$ or $\boldsymbol{H}$. For each vertex $\mathfrak{v}_\alpha \in \mathrm{V}(\Gamma)$, let Z_α be the set of interior markings on Σ_α, and let $Z_\alpha^0 = (Z^0_{\alpha,0}, \ldots, Z^0_{\alpha,\bar n})$ be the union of divisors given by all interior markings z_i labelled by some D_a^0 together with the contact orders $\tilde{\mathfrak{o}}(l_i)$. Let $\mathcal{B}_\alpha$ be the Banach manifold consisting of equivalence classes of maps defined as follows.

(a) If $\mathfrak{v}_\alpha \in \mathrm{V}_\triangle(\Gamma)$, set
$$\mathcal{B}_\alpha = \Big\{ [u] \in \mathcal{B}_{\Sigma_\alpha, Z_\alpha^0}(X, L; B_\alpha) \mid u(\Sigma_\alpha) \subsetneq D,\ u(z_i) \in V_i,\ \forall z_i \in Z_\alpha \smallsetminus Z_\alpha^0 \Big\};$$

(b) If $\mathfrak{v}_\alpha \in \mathrm{V}_\triangledown(\Gamma)$, set
$$\mathcal{B}_\alpha = \Big\{ \bar u \in \mathcal{B}_{\Sigma_\alpha, Z_\alpha^0}(\bar X, \bar L; B_\alpha) \mid \bar u(\Sigma_\alpha) \subsetneq \bar D,\ u(z_i) \in \bar V_i,\ \forall z_i \in Z_\alpha \smallsetminus Z_\alpha^0 \Big\};$$

(c) If $\mathfrak{v}_\alpha \in \mathrm{V}_\diamond(\Gamma)$, set
$$\mathcal{B}_\alpha = \Big\{ [\boldsymbol{v}] = [u, \phi, \psi] \in \mathcal{B}_{\Sigma_\alpha, Z_\alpha^0}(X, L, \mu; B_\alpha) \mid u(\Sigma_\alpha) \subsetneq D,\ u(z_i) \in V_i,\ \forall z_i \in Z_\alpha \smallsetminus Z_\alpha^0 \Big\}.$$

To unify the notations, for all $\mathcal{B}_\alpha$, its elements are denoted as gauge equivalence classes $[\boldsymbol{v}] = [u, \phi, \psi]$ of gauged maps from Σ_α to X[2]. On the other hand, for each one-dimensional component $I_\mathfrak{e}$ of $\mathcal{Z}$ of positive length $\lambda(\mathfrak{e}) > 0$, let $\mathcal{B}_\mathfrak{e}$ be the Banach manifold of $W^{1,p}$-maps from $I_\mathfrak{e}$ to $\bar L$. If $I_\mathfrak{e}$ is unbounded, then $\tilde\Gamma$ induces asymptotic constraints and we can define $\mathcal{B}_\mathfrak{e}$ consisting of $W^{1,p}_{\mathrm{loc}}$ maps from $I_\mathfrak{e}$ into $\bar L$ satisfying the asymptotic constraints in a suitable sense. We omit the details.

Before discussing the sections defined by the equations, we consider the matching conditions at nodes. Notice that there are well-defined evaluation maps defined on $\mathcal{B}_\alpha$ or $\mathcal{B}_\mathfrak{e}$. Let
$$\mathcal{B}_{\tilde\Gamma}(\mathcal{Z}) \subset \prod_\alpha \mathcal{B}_\alpha \times \prod_\mathfrak{e} \mathcal{B}_\mathfrak{e}$$
to be the open subset of the product such that for each node of $\mathcal{Z}$, the distance between the evaluations at the two sides of the nodes is less than a fixed very small constant; for example, a number small than the injectivity radii of $\bar X$ and $\bar L$. An element of $\mathcal{B}_{\tilde\Gamma}(\mathcal{Z})$ is denoted by $((\boldsymbol{v}_\alpha), (x_\mathfrak{e}))$. Abbreviate $\bar X_2 = \bar X \times \bar X$, $\bar L_2 = \bar L \times \bar L$. Let $\Delta_{\bar X} \subset \bar X_2$, $\Delta_{\bar L} \subset \bar L_2$ be the diagonals. Denote
$$X_\Gamma = (\bar X_2)^{\mathrm{E}_\triangledown(\Gamma) \smallsetminus \mathrm{E}_\triangledown(\underline{\Gamma})} \times (\bar L_2)^{\mathrm{E}^0(\underline{\Gamma})} \times (\bar L_2)^{\partial \mathrm{E}^+(\underline{\Gamma})};$$
Then there is a smooth evaluation map
$$\mathrm{ev}_\Gamma : \mathcal{B}_{\tilde\Gamma}(\mathcal{Z}) \to X_\Gamma.$$
Its image is contained in a small neighborhood of the "diagonal" Δ_Γ, defined as
$$\Delta_\Gamma = (\Delta_{\bar X})^{\mathrm{E}_\triangledown(\Gamma) \smallsetminus \mathrm{E}_\triangledown(\underline{\Gamma})} \times (\Delta_{\bar L})^{\mathrm{E}^0(\underline{\Gamma})} \times (\Delta_{\bar L})^{\partial \mathrm{E}^+(\underline{\Gamma})}.$$

[2]Because $\tilde\Gamma$ is reduced, we do not need to define Banach manifolds for maps from $\boldsymbol{S}^2$ to X.

Let N_Γ be the normal bundle of Δ_Γ inside X_Γ. Using parallel transport we can regard N_Γ as a bundle over a small neighborhood of the diagonal Δ_Γ inside X_Γ. Let $\mathcal{E}_{\tilde\Gamma}(\mathcal{Z})$ be a Banach space bundle over $\mathcal{B}_{\tilde\Gamma}(\mathcal{Z})$ whose fibre over $((\boldsymbol{v}_\alpha),(x_{\mathfrak{e}}))$ is

$$\Big(\bigoplus_\alpha \mathcal{E}_{\boldsymbol{v}_\alpha}\Big) \oplus \Big(\bigoplus_{\mathfrak{e}} \mathcal{E}_{x_{\mathfrak{e}}}\Big) \oplus \mathrm{ev}_\Gamma^* N_\Gamma.$$

Here for each vertex $\mathfrak{v}_\alpha$, $\mathcal{E}_{\boldsymbol{v}_\alpha}$ is the corresponding (weighted) Sobolev space considered in Subsection 7.1; for each $\mathfrak{e} \in \mathrm{E}^+(\underline{\Gamma})$, $\mathcal{E}^{k,p}_{x_{\mathfrak{e}}} = L^p(I_{\mathfrak{e}}, x_{\mathfrak{e}}^* T\bar{L})$.

We define a Fredholm section of a Banach vector bundle locally as follows. Let $\mathcal{Z}$ be a stable treed disk representing a point of $p \in \mathcal{W}_\Gamma$. Points near p parametrize the deformation of the lengths of finite length edges and the position of interior and boundary markings (modulo Möbius transformations or translations). Let $W_p \subset \mathcal{W}_\Gamma$ be such a small neighborhood, which is homeomorphic to an open ball in $\mathbb{R}^{\dim \mathcal{W}_\Gamma}$. The union of two-dimensional components of fibres of $\mathcal{U}_\Gamma$ for all $q \in W_p$ can be identified with a product, in which the deformation is realized as deforming the positions of the special points of each component. The union of all one-dimensional components can also be identified with the product of W_p with the disjoint union of different intervals corresponding to edges of positive lengths. By choosing such identifications, we can define a Banach manifold

$$\mathcal{B}_{\tilde\Gamma}(W_p) \simeq W_p \times \mathcal{B}_{\tilde\Gamma}(\mathcal{Z}).$$

The union of Banach vector bundles $\mathcal{E}_{\tilde\Gamma}$ defined above also gives a Banach vector bundle

$$\mathcal{E}_{\tilde\Gamma}(W_p) \to \mathcal{B}_{\tilde\Gamma}(W_p).$$

Define the regularized moduli space as a quotient of a subset of the zero set of the Fredholm section above, as follows. Consider a perturbation $P_\Gamma \in \mathcal{P}_\Gamma$. Its restriction to $\mathcal{U}_\Gamma|_{W_p}$ induces a section

$$\mathcal{F}_{P_\Gamma, W_p} : \mathcal{B}_{\tilde\Gamma}(W_p) \to \mathcal{E}_{\tilde\Gamma}(W_p).$$

Choose for each $p \in \mathcal{W}_\Gamma$ a local patch W_p and define the Banach manifold, Banach vector bundles and sections as above. We remark that $\mathcal{W}_\Gamma$ can be covered by *countably* many such patches. Denote

$$\mathcal{M}^*_{\tilde\Gamma}(P_\Gamma, W_p) = \mathcal{F}^{-1}_{P_\Gamma, W_p}(0).$$

Then there is a natural equivalence relation $\sim$ on the disjoint union of all $\mathcal{M}^*_{\tilde\Gamma}(P_\Gamma, W_p)$. Define a topological space

$$\mathcal{M}^*_{\tilde\Gamma}(P_\Gamma) = \bigsqcup_{p \in \mathcal{W}_\Gamma} \mathcal{M}^*_{\tilde\Gamma}(P_\Gamma, W_p)/\sim. \tag{7.5}$$

The usual regularity results for holomorphic curves and the regularity of vortices (see [CGMS02, Section 3]) imply that there is a natural inclusion

$$\mathcal{M}^*_{\tilde\Gamma}(P_\Gamma) \hookrightarrow \mathcal{M}_{\tilde\Gamma}(P_\Gamma), \tag{7.6}$$

where the latter, defined in Section 6, is the moduli space of gauge equivalence classes of stable treed scaled vortices with vortex type $\tilde\Gamma$ that satisfy the contact order conditions at all interior markings, where the equations are defined with respect to the perturbation data P_Γ. Here the superscript * only indicates that we take out those objects that have some nonconstant components mapped entirely into the stabilizing divisor D. Notice that *a priori* the topologies on the two sides of (7.6) are different: the topology on the domain is induced from the topology of Banach manifolds, while the topology on the target is induced from sequential limit. However it is a standard process to show that the natural inclusion (7.6) is a homeomorphism on to an open subset.

7.2.2. *Regular and strongly regular perturbations.* We now define the notion of regular and strongly perturbations and coherent system of perturbations.

Definition 7.3. Let $\Gamma \in \mathbf{T}^{\rm st} \cup \mathbf{F}^{\rm st}$ be a stable type.

(a) Given $P_\Gamma \in \mathcal{P}_\Gamma$, an element $\mathcal{C} \in \mathcal{M}^*_{\tilde\Gamma}(P_\Gamma, W_p) \subset \mathcal{M}^*_{\tilde\Gamma}(P_\Gamma)$ is called regular if the linearization of $\mathcal{F}_{P_\Gamma, W_p}$ at $\mathcal{C}$ is surjective. (Notice that this condition descends under the equivalence relation $\sim$ in (7.5).)
(b) A perturbation datum P_Γ is *regular* if for all reduced, uncrowded and controlled refinement $\tilde\Gamma$ of Γ, all elements in $\mathcal{M}^*_{\tilde\Gamma}(P_\Gamma)$ are regular.
(c) Let $\mathrm{S} \subset \mathrm{V}(\Gamma)$ be a special subset. Then an S-special perturbation P_Γ (see Definition 5.7) is S-regular if for any controlled and S-special refinement $\tilde\Gamma$ of Γ (see Definition 7.1), all elements in $\mathcal{M}^*_{\tilde\Gamma}(P_\Gamma)$ are regular.
(d) A perturbation P_Γ is *strongly regular* if the following condition is satisfied. For any subset $\mathrm{S} \subset \mathrm{V}(\Gamma) \smallsetminus \mathrm{V}(\underline{\Gamma})$, let Π be the S-stabilization of Γ and $\mathrm{R} \subset \mathrm{V}(\Pi)$ be the induced special subset (see Subsection 5.5) and (5.8). Then $\pi_{\mathrm{S}}(P_\Gamma) \in \mathcal{P}_{\mathrm{R}}(\overline{\mathcal{U}}_\Pi)$, which is an R-special perturbation, is R-regular.
(e) Let $\underline{P} = \{P_\Gamma \mid \Gamma \in \mathbf{T}^{\rm st} \cup \mathbf{F}^{\rm st}\}$ be a coherent collection of perturbations. We say that $\underline{P}$ is strongly regular all its elements are strongly regular.

7.3. **Transversality for closed domains.** In this subsection we prove the following proposition about regular and strongly regular perturbations on closed domains. This result is crucial for defining the bulk deformation term $\mathfrak{c} \in H^*(\bar X; \Lambda_{>0})$.

Proposition 7.4. *For every stable forest $\Gamma \in \mathbf{F}^{\rm st}$ and a special subset $S \subset \mathrm{V}(\Gamma)$, there exist comeager subsets*

$$\mathcal{P}^{\mathbf{reg}}(\overline{\mathcal{U}}_\Gamma) \subset \mathcal{P}(\overline{\mathcal{U}}_\Gamma); \qquad \mathcal{P}^{\mathbf{reg}}_{\mathrm{S}}(\overline{\mathcal{U}}_\Gamma) \subset \mathcal{P}_{\mathrm{S}}(\overline{\mathcal{U}}_\Gamma)$$

that satisfy the following conditions.

(a) Each $J_\Gamma \in \mathcal{P}^{\mathbf{reg}}(\overline{\mathcal{U}}_\Gamma)$ is regular.
(b) Each $J_\Gamma \in \mathcal{P}^{\mathbf{reg}}_{\mathrm{S}}(\overline{\mathcal{U}}_\Gamma)$ is S-regular.
(c) For $\Gamma' \trianglelefteq \Gamma$, recall the inclusion $\tilde\rho_{\Gamma,\Gamma'} : \overline{\mathcal{U}}_{\Gamma'} \to \overline{\mathcal{U}}_\Gamma$ (see (5.1)). Then we have

$$\mathcal{P}^{\mathbf{reg}}(\overline{\mathcal{U}}_\Gamma) \subset \tilde\rho^{-1}_{\Gamma,\Gamma'} \mathcal{P}^{\mathbf{reg}}(\overline{\mathcal{U}}_{\Gamma'}). \tag{7.7}$$

(d) For every $\mathrm{S} \subset \mathrm{V}(\Gamma)$, let Π be the S-stabilization and $\mathrm{R} \subset \mathrm{V}(\Pi)$ be the induced special subset (see notations explained in Subsection 5.5). Then we have

$$\mathcal{P}^{\mathbf{reg}}(\overline{\mathcal{U}}_\Gamma) \subset \pi_{\mathrm{S}}^{-1} \mathcal{P}^{\mathbf{reg}}_{\mathrm{R}}(\overline{\mathcal{U}}_\Pi). \tag{7.8}$$

(e) Let Γ have n connected components $\Gamma_1, \dots, \Gamma_n$. Then for each $J_\Gamma \in \mathcal{P}^{\mathbf{reg}}(\overline{\mathcal{U}}_\Gamma)$, for any subset $I = \{i_1, \dots, i_s\} \subset \{1, \dots, n\}$, the evaluation map

$$\mathrm{ev}_I : \mathcal{M}^*_{\tilde\Gamma}(J_\Gamma) \to \bar X^I$$

is transversal to

$$\Delta^I \bar L = \{(x, \dots x) \in \bar X^I \mid x \in \bar L\}.$$

Remark 7.5. (a) The second collection of comeager subsets (those labelled by S) only plays an auxiliary role. The perturbations we use to construct moduli spaces are picked from $\mathcal{P}^{\mathbf{reg}}(\overline{\mathcal{U}}_\Gamma)$.

(b) Let Γ' be a connected component of $\Gamma \in \mathbf{F}^{\mathbf{st}}$. If J_Γ and $J_{\Gamma'}$ belong to a coherent collection of perturbations, then it does *not* follow that the restriction to J_Γ to $\overline{\mathcal{U}}_{\Gamma,\Gamma'}$ coincides with $J_{\Gamma'} \circ \tilde{\pi}_{\Gamma,\Gamma'}$. However, Item (d) above implies that the restriction J_Γ to $\overline{\mathcal{U}}_{\Gamma,\Gamma'}$ comes from a comeager subset of regular perturbations on $\overline{\mathcal{U}}_{\Gamma'}$.

Proof of Proposition 7.4. We first construct the collection $\mathcal{P}_{\mathrm{S}}^{\mathbf{reg}}(\overline{\mathcal{U}}_\Gamma)$ by using the Sard–Smale theorem. Given a reduced, controlled, uncrowded and S-special refinement $\tilde{\Gamma}$ of Γ, we consider the universal moduli problem

$$\mathcal{F}_{\tilde{\Gamma}} : \mathcal{P}_{\mathrm{S}}(\overline{\mathcal{U}}_\Gamma) \times \mathcal{B}_{\tilde{\Gamma}} \to \mathcal{E}_\Gamma.$$

Claim. $\mathcal{F}_{\tilde{\Gamma}}$ is transverse to zero.

Proof of the claim. Consider any point $(J_\Gamma, p, \boldsymbol{v}) \in \mathcal{F}_{\tilde{\Gamma}}^{-1}(0)$, where $p \in \overline{\mathcal{W}}_\Gamma$ is of certain type $\Pi \trianglelefteq \Gamma$, and $\boldsymbol{v}$ is a stable affine vortex with domain isomorphic to $\mathcal{Z}_p$ with respect to the almost complex structure $J_\Gamma|_{\mathcal{Z}_p}$. We prove that without using the deformation of p, the linearization of $\mathcal{F}_{\tilde{\Gamma}}$ at this point is surjective onto the corresponding fibre of $\mathcal{E}_\Gamma$. Let an infinitesimal deformation of J_Γ be $h_\Gamma = (h_\alpha)_{\alpha \in \mathrm{V}(\Gamma)}$ and an infinitesimal deformation of $\boldsymbol{v} = (\boldsymbol{v}_\alpha)_{\mathfrak{v}_\alpha \in \mathrm{V}(\Gamma)}$ be $\boldsymbol{\xi} = (\boldsymbol{\xi}_\alpha)_{\alpha \in \mathrm{V}(\Gamma)}$. Then we see

$$\mathcal{D}\mathcal{F}_{\tilde{\Gamma}}(e, \boldsymbol{\xi}) = \big(\mathcal{D}\mathcal{F}_\alpha(\boldsymbol{\xi}_\alpha) + h_\alpha(\boldsymbol{v}_\alpha), B_\Gamma(\boldsymbol{\xi})\big).$$

Here $B_\Gamma(\boldsymbol{\xi})$ takes value in the normal bundle N_Γ to the diagonal Δ_Γ, defined by the differences of the evaluations of components of $\boldsymbol{\xi}$ at nodal points.

Consider any vertex $\mathfrak{v}_\alpha$ that is not a ghost vertex; namely the object $\boldsymbol{v}_\alpha$ has positive energy. We want to show that the map

$$(\boldsymbol{\xi}_\alpha, h_\alpha) \mapsto \big(\mathcal{D}\mathcal{F}_\alpha(\boldsymbol{\xi}_\alpha) + h_\alpha(\boldsymbol{v}_\alpha), \mathrm{ev}_\alpha(\xi_\alpha)\big) \tag{7.9}$$

is surjective onto $\mathcal{E}_\alpha|_{\boldsymbol{v}_\alpha} \oplus \bigoplus_{z \in Z_\alpha^{\mathrm{node}}} T_{\boldsymbol{v}_\alpha(z)} \bar{X}$. This surjectivity follows if we can show that the map $h_\alpha \mapsto h_\alpha(\boldsymbol{v}_\alpha)$ has dense range. If $\boldsymbol{v}_\alpha$ is a non-constant holomorphic sphere in $\bar{X}$, then since $\boldsymbol{v}_\alpha$ is not a ghost vertex, the points in its domain where $\boldsymbol{v}_\alpha$ is locally an immersion is dense and open, in particular intersecting the complement of $\overline{\mathcal{U}}_\Gamma^{\mathrm{thin}}$ (the nodal neighborhood) at a nonempty open subset. Hence it is standard to show that (7.9) has dense range and is surjective. On the other hand, suppose $\boldsymbol{v}_\alpha$ is an affine vortex over $\boldsymbol{C}$. Let $U_{p,\alpha} \subset \Sigma_\alpha$ be the complement of the nodal neighborhood, namely the domain where we can adjust the almost complex structure. Then by the definition of nodal neighborhoods (see Definition 5.1), the area of $U_{p,\alpha}$ is greater than or equal to the number A_α defined by (5.5). Moreover, since $\tilde{\Gamma}$ is controlled, the energy of $\boldsymbol{v}_\alpha$ is less than or equal to the number E_α given by (5.5). By Lemma 2.5, we have

$$u_\alpha(U_{p,\alpha}) \cap U \neq \varnothing.$$

Moreover, the subset of Σ_α where $\boldsymbol{v}_{\alpha,s} = \boldsymbol{v}_{\alpha,t} \neq 0$ is open and dense. Hence one can use the deformation of $J_\Gamma|_{U_{p,\alpha}}$ to show that $\mathcal{D}\mathcal{F}_\alpha$ is surjective.

It remains to prove transversality in the presence of ghost vertices. Let Π be a maximal ghost subtree of Γ,

$$\bar{X}_\Pi = (\bar{X} \times \bar{X})^{\mathrm{E}(\Pi)}$$

and $\Delta_\Pi \subset \bar{X}_\Pi$ be the product of diagonals. Then the evaluation of $\boldsymbol{v}$ at nodes corresponding to edges of Π is a point $\bar{x}_\Pi \in \Delta_\Pi$ (notice that $\mathrm{V}_\triangle(\Pi) = \varnothing$ because

Γ is reduced). If $\mathrm{V}_\diamond(\Pi) = \varnothing$, then Π corresponds to a maximal ghost tree of holomorphic spheres in $\bar{X}$. Consider the linear map

$$\bigoplus_{\mathfrak{v}_\alpha \in \mathrm{V}(\Pi)} W^{1,p}_{Z_\alpha}(\Sigma_\alpha, \bar{u}_\alpha^* T\bar{X}) \to \bigoplus_{\mathfrak{v}_\alpha \in \mathrm{V}(\Pi)} L^p_{Z_\alpha}(\Sigma_\alpha, \Lambda^{0,1} \otimes \bar{u}_\alpha^* T\bar{X}) \oplus \frac{T_{\bar{x}_\Pi}\bar{X}_\Pi}{T_{\bar{x}_\Pi}\Delta_\Pi}.$$

Here $W^{1,p}_{Z_\alpha}(\Sigma_\alpha, \bar{u}_\alpha^* T\bar{X})$ is the tangent space of $\mathcal{B}_{\boldsymbol{S}^2, Z_\alpha}(B_\alpha)$ at $\bar{u}_\alpha$ and $B_\alpha = 0$, and $L^p_{Z_\alpha}(\Sigma_\alpha, \Lambda^{0,1} \otimes \bar{u}_\alpha^* T\bar{X})$ is the fibre of $\mathcal{E}_{\boldsymbol{S}^2, Z_\alpha}(B_\alpha)$ over $\bar{u}_\alpha$. By the uncrowdedness condition, among all divisors Z_α there is at most one nontrivial one whose degree is one. Therefore this linear map is surjective. On the other hand, if $\mathrm{V}_\diamond(\Pi) \neq \varnothing$, then by the stability condition $\mathrm{V}_\diamond(\Pi)$ only contains one element $\mathfrak{v}_\beta$ which has one interior leaf corresponding to a marked point $z_\beta \in \Sigma_\beta$. Then we have the surjectivity of the linearization

$$\mathcal{D}_{\boldsymbol{v}_\beta} : T_{\boldsymbol{v}_\beta} \mathcal{B}_{\boldsymbol{C}, Z_\beta}(\bar{X}) \to \mathcal{E}_{\boldsymbol{C}, Z_\alpha}(\bar{X})|_{\boldsymbol{v}_\beta}.$$

is also surjective. Moreover, for any other vertex $\mathfrak{v}_\alpha \in \mathrm{V}(\Pi) \smallsetminus \{\mathfrak{v}_\beta\}$, we know $Z_\alpha = \varnothing$ and hence the linear map

$$\bigoplus_{\mathfrak{v}_\alpha \in \mathrm{V}(\Pi) \smallsetminus \{\mathfrak{v}_\beta\}} W^{1,p}(\Sigma_\alpha, \bar{u}_\alpha^* T\bar{X}) \to \bigoplus_{\mathfrak{v}_\alpha \in \mathrm{V}(\Pi) \smallsetminus \{\mathfrak{v}_\beta\}} L^p(\Sigma_\alpha, \Lambda^{0,1} \otimes \bar{u}_\alpha^* T\bar{X}) \oplus \frac{T_{\bar{x}_\Pi}\bar{X}_\Pi}{T_{\bar{x}_\Pi}\Delta_\Pi}$$

is surjective. Combining the surjectivity we have proved for non-ghost components, this implies that $\mathcal{F}_{\tilde{\Gamma}}$ is surjective. ☐

The above claim implies that the zero locus of $\mathcal{F}_{\tilde{\Gamma}}$, denoted by $\mathcal{UM}_{\tilde{\Gamma}}$, is a smooth Banach manifold of $\mathcal{P}_{\mathrm{S}}(\overline{\mathcal{U}}_\Gamma) \times \mathcal{B}_{\tilde{\Gamma}}$. Then the projection $\mathcal{UW}_{\tilde{\Gamma}} \to \mathcal{P}_{\mathrm{S}}(\overline{\mathcal{U}}_\Gamma)$ is a Fredholm map whose index coincides with the index of this problem. By Sard–Smale theorem, the set of regular values, denoted by $\mathcal{P}^{\mathbf{reg}}_{\mathrm{S},\tilde{\Gamma}}(\overline{\mathcal{U}}_\Gamma)$, is a comeager subset of $\mathcal{P}_{\mathrm{S}}(\overline{\mathcal{U}}_\Gamma)$. Then since there are at most countably many reduced, uncrowded, controlled, and S-special vortex types $\tilde{\Gamma}$ whose underlying combinatorial type is Γ, the intersections of all such regular subsets of perturbations is still a comeager subset, denoted by

$$\mathcal{P}^{\mathbf{reg}}_{\mathrm{S}}(\overline{\mathcal{U}}_\Gamma) \subset \mathcal{P}_{\mathrm{S}}(\overline{\mathcal{U}}_\Gamma).$$

Then every element $J_\Gamma \in \mathcal{P}^{\mathbf{reg}}_{\mathrm{S}}(\overline{\mathcal{U}}_\Gamma)$ is S-regular.

We construct the desired set of regular perturbations $\mathcal{P}^{\mathbf{reg}}(\overline{\mathcal{U}}_\Gamma)$ as an intersection of special subsets. Using the same argument as in the previous paragraph, one can find a comeager subset of $\mathcal{P}(\overline{\mathcal{U}}_\Gamma)$ containing regular elements J_Γ. Using induction, we intersect this comeager subset with all sets on the right hand sides of (7.7) and (7.8), and define $\mathcal{P}^{\mathbf{reg}}(\overline{\mathcal{U}}_\Gamma)$ to be the intersection. By Lemma 5.8, the right hand sides of (7.7) and (7.8) are comeager subsets. Hence $\mathcal{P}^{\mathbf{reg}}(\overline{\mathcal{U}}_\Gamma)$ is a countable intersection of comeager subsets, which is still comeager. ☐

7.4. **Transversality for bordered domains.** Now we start to construct strongly regular perturbation data that restricts to the chosen perturbation $\underline{J_{\mathbf{F}}}$ for closed domains.

7.4.1. *An induction lemma.* Let $\Gamma \in \mathbf{T}^{\mathbf{st}}$ be a stable *unbroken* type such that $\mathrm{V}(\Gamma) = \mathrm{V}(\underline{\Gamma})$. Denote $\mathbf{S} = \mathbf{S}^*(\Gamma)$. Let $\underline{P}^*_\Gamma := \{P_\Pi \mid \Pi \in \mathbf{S}\}$ be a coherent system of perturbation data for all $\Pi \lhd \Gamma$. Let

$$\mathcal{P}(\overline{\mathcal{U}}_\Gamma, \underline{P}^*_\Gamma) \subset \mathcal{P}(\overline{\mathcal{U}}_\Gamma)$$

be the subset of perturbations on $\overline{\mathcal{U}}_\Gamma$ which coincides with $\underline{P}_\Gamma^*$ over all boundary strata. Then as a closed subspace, $\mathcal{P}(\overline{\mathcal{U}}_\Gamma, \underline{P}_\Gamma^*)$ is also a Banach manifold.

Lemma 7.6. *Then there exists a comeager subset*

$$\mathcal{P}^{\mathbf{reg}}(\overline{\mathcal{U}}_\Gamma, \underline{P}_\Gamma^*) \subset \mathcal{P}(\overline{\mathcal{U}}_\Gamma, \underline{P}_\Gamma^*)$$

containing only strongly regular perturbations.

Proof. The proof is an application of the Sard–Smale theorem to a universal moduli space. We first prove that tor any reduced, uncrowded, and controlled vortex type $\tilde{\Gamma}$ that refines Γ, the universal problem

$$\mathcal{F}_{\tilde{\Gamma}} : \mathcal{P}(\overline{\mathcal{U}}_\Gamma, \underline{P}_\Gamma^*) \times \mathcal{B}_{\tilde{\Gamma}} \to \mathcal{E}_\Gamma \tag{7.10}$$

is transverse to zero. To prove the claim consider $(J_\Gamma, F_\Gamma, p, \tilde{\mathcal{C}}) \in \mathcal{F}_{\tilde{\Gamma}}^{-1}(0)$ where $(J_\Gamma, F_\Gamma) \in \mathcal{P}(\overline{\mathcal{U}}_\Gamma, \underline{P}_\Gamma^*)$, $p \in \mathcal{W}_\Gamma$ and $\tilde{\mathcal{C}}$ is a stable perturbed treed vortex with domain $\mathcal{Z}_p$ (the fibre over p). It suffices to prove the surjectivity of the linearization of $\mathcal{F}_{\tilde{\Gamma}}$ onto the subspace where we do not deform the underlying stable treed disk $\mathcal{Z}$. A general element of the tangent space of $\mathcal{P}(\overline{\mathcal{U}}_\Gamma, \underline{P}_\Gamma^*)$, denoted by (f_Γ, h_Γ), induce on each component $\Sigma_\alpha \subset \mathcal{Z}_p$ a deformation h_α of the almost complex structure and on each edge of positive length $I_{\mathfrak{e}} \subset \mathcal{Z}_p$ a deformation of the function $f_{\mathfrak{e}}$. An infinitesimal deformation $\boldsymbol{\xi}$ of $\tilde{\mathcal{C}}$ contains on each component Σ_α an infinitesimal deformation ξ_α of the gauged map $\boldsymbol{v}_\alpha$ and on each edge of positive length $I_{\mathfrak{e}}$ an infinitesimal deformation $\xi_{\mathfrak{e}}$ of the curve $x_{\mathfrak{e}} : I_{\mathfrak{e}} \to \bar{L}$. Then the linearization of (7.10) reads

$$\mathcal{D}\mathcal{F}_{\tilde{\Gamma}}((f_\Gamma, h_\Gamma), \boldsymbol{\xi}) = \big(\mathcal{D}\mathcal{F}_\alpha(\xi_\alpha) + h_\alpha(\boldsymbol{v}_\alpha), \mathcal{D}\mathcal{F}_{\mathfrak{e}}(\xi_{\mathfrak{e}}) + \nabla f_{\mathfrak{e}}(x_{\mathfrak{e}}), B_\Gamma(\boldsymbol{\xi})\big).$$

Here $B_\Gamma(\boldsymbol{\xi})$ takes value in the normal bundle N_Γ to the diagonal, defined by the difference of the evaluations of components of $\boldsymbol{\xi}$ at nodal points. Notice that one can have nontrivial $f_{\mathfrak{e}}$ on all edges $\mathfrak{e}$ with positive length. The Morse operator

$$(f_{\mathfrak{e}}, \xi_{\mathfrak{e}}) \mapsto \big(\mathcal{D}\mathcal{F}_{\mathfrak{e}}(\xi_{\mathfrak{e}}) + \nabla f_{\mathfrak{e}}(x_{\mathfrak{e}}), \xi_{\mathfrak{e}}|_{\partial I_{\mathfrak{e}}}\big) \in W^{k-1,p}(I_{\mathfrak{e}}, x_{\mathfrak{e}}^* T\bar{L}) \oplus T_{x_{\mathfrak{e}}(\partial I_{\mathfrak{e}})}(\bar{L})^{\partial I_{\mathfrak{e}}} \tag{7.11}$$

has cokernel consisting of sections satisfying a first-order equation that are perpendicular to all variations of $f_{\mathfrak{e}}$ over a nonempty interval, which therefore vanishes. For any subgraph $\Pi \subset \Gamma$, denote $N_\Pi^0 := (N_{\bar{L}})^{E^0(\Pi)} = (N_{\bar{L}})^{E^0(\underline{\Pi})}$ which is a subspace of N_Γ. Denote by $B_\Pi^0(\boldsymbol{\xi})$ the N_Π^0-component of $B_\Gamma(\boldsymbol{\xi})$. Introduce

$$\mathcal{D}\mathcal{F}_\Pi^0(p, \boldsymbol{\xi}) = \Big(\big(\mathcal{D}\mathcal{F}_\alpha(\xi_\alpha) + h_\alpha(\boldsymbol{v}_\alpha)\big)_{\mathfrak{v}_\alpha \in \mathrm{V}(\Pi)},\ B_0(\boldsymbol{\xi})\Big) \in \bigoplus_{\mathfrak{v}_\alpha \in \mathrm{V}(\Pi)} \mathcal{E}_{\boldsymbol{v}_\alpha} \oplus N_\Pi^0. \tag{7.12}$$

By the surjectivity of (7.11), it remains to prove the surjectivity of the above operator for any maximal subtree $\Pi \subset \Gamma$ that contains no edge of positive length.

We use an induction argument to prove the surjectivity of the linearized operator. Recall that there is a partial order among vertices of Π. Consider a vertex $\mathfrak{v}_\alpha \in \mathrm{V}(\Pi)$ that is farthest to the root of Π among all vertices of Π, then $\mathfrak{v}_\alpha$ is connected to only one vertex in $V(\Pi)$. Since the type is reduced, $\boldsymbol{v}_\alpha$ cannot be a holomorphic sphere in X. We then discuss the following possibilities.

(a) If $\boldsymbol{v}_\alpha$ has zero energy, then $\mathcal{D}\mathcal{F}_\alpha$ plus the evaluation at ∞ is surjective, even without using deformations of the complex structure.

(b) Suppose $\boldsymbol{v}_\alpha = (v_\alpha, \phi_\alpha, \psi_\alpha)$ has nonzero energy. Then the 1-form

$$(\partial_s v_\alpha + \mathcal{X}_{\phi_\alpha})\mathrm{d}s + (\partial_t v_\alpha + \mathcal{X}_{\psi_\alpha})\mathrm{d}t$$

is nonvanishing over an open dense subset $V_\alpha \subset \Sigma_\alpha$. If $\mathfrak{v}_\alpha \in V_\triangle(\Gamma) \sqcup V_\diamond(\Gamma)$, then either $\mathfrak{v}_\alpha$ is a quasidisk, or an affine vortex over $\boldsymbol{H}$, so that V_α has nonempty intersection with the preimage of U. Then using deformations of J_Γ (which lies in $\mathcal{J}(U, D)$) one can kill all possible obstructions. The case when $\boldsymbol{v}_\alpha$ is a quasidisk, the linearized operator is a Cauchy–Riemann operator (modulo constant K-actions), so the situation is similar to that of pseudoholomorphic curves discussed in [MS04, Chapter 3]; when $\boldsymbol{v}_\alpha$ is an affine vortex over $\boldsymbol{H}$, the way to deal with the augmented linearization is similar to that in [Xu16, Section 6]. In summary, the map $(\xi_\alpha, h_\alpha) \mapsto (\mathcal{D}\mathcal{F}_\alpha(\xi_\alpha) + h_\alpha(\boldsymbol{v}_\alpha), \mathrm{ev}_\infty(\xi_\alpha))$ is surjective.

Therefore, we reduce the problem of proving surjectivity for Π to the surjectivity of the smaller subtree which removes $\mathfrak{v}_\alpha$ from Π. Then using induction one knows that the operator (7.12) is surjective. Hence the section (7.10) is transverse.

It follows that the zero set, denoted by $\mathcal{UM}_{\tilde\Gamma}$, is a smooth Banach submanifold of $\mathcal{P}(\overline{\mathcal{U}}_\Gamma, \underline{P}^*_\Gamma) \times \mathcal{B}_{\tilde\Gamma}$. Consider the projection $\mathcal{UM}_{\tilde\Gamma} \to \mathcal{P}(\overline{\mathcal{U}}_\Gamma, \underline{P}^*_\Gamma)$. By Sard–Smale theorem, the set of regular values of this projection is a comeager subset, while being a regular value implies being regular for the vortex type $\tilde\Gamma$. There are at most countably many uncrowded, reduced, and controlled refinements of Γ, which give at most countably many comeager subsets of $\mathcal{P}(\overline{\mathcal{U}}_\Gamma, \underline{P}^*_\Gamma)$. Denote the intersection of the countably many comeager subsets by $\mathcal{P}^{\mathbf{reg}}(\overline{\mathcal{U}}_\Gamma, \underline{P}^*_\Gamma)$, which is still a comeager subset, containing only regular perturbations. Notice that in this case $\mathrm{V}(\Gamma) = \mathrm{V}(\underline{\Gamma})$, hence by definition, regular coincides with strongly regular (see Definition 7.3). $\square$

Now we construct perturbation data for type $\triangle$. The construction is based on an induction process. To start, we consider the base cases for type $\triangle$ which corresponds to types $\Gamma \in \mathbf{T}^{\mathrm{st}}_\triangle$ that has no continuous moduli, namely, a Y-shape, denoted by Y. The following transversality result for perturbed Morse trajectories is similar to that in Abouzaid [Abo11] and Mescher [Mes18].

Lemma 7.7. *There exists a comeager subset $\mathcal{P}^{\mathbf{reg}}(\overline{\mathcal{U}}_{\mathrm{Y}}) \subset \mathcal{P}(\overline{\mathcal{U}}_{\mathrm{Y}})$ whose elements are all regular (at the same time strongly regular).*

Then using the Lemma 7.6 we can inductively construct a (strongly) regular coherent system of perturbation data $\underline{P}_\triangle = \{P_\Gamma \mid \Gamma \in \mathbf{T}^{\mathbf{st}}_\triangle\}$. Notice that we only need to construct for unbroken types $\Gamma \in \mathbf{T}^{\mathbf{st}}_\triangle$ because the perturbations for broken ones are determined by the perturbations for unbroken ones via the (Cutting edge) axiom of Definition 5.5. Given an unbroken $\Gamma \in \mathbf{T}^{\mathbf{st}}_\triangle$, assume that one has chosen a coherent system of perturbation data $\underline{P}^*_\Gamma$ over $\mathbf{S} = \mathbf{S}^*(\Gamma)$. Then Lemma 7.6 implies that one can choose from a comeager subset of certain Banach manifold a regular perturbation data P_Γ for Γ whose boundary values are given by $\underline{P}^*_\Gamma$. Via induction this gives a regular coherent system $\underline{P}_\triangle$.

Now we construct perturbations form type $\triangledown$ and type $\diamond$ domains. Such domains may contain superstructures corresponding to affine vortices over $\boldsymbol{C}$ or holomorphic spheres in $\bar X$. To achieve transversality, we use the perturbations for closed domains chosen in Subsection 7.3 and vary the perturbation data only on the bases.

Consider an arbitrary stable type $\Gamma \in \mathbf{T}^{\mathrm{st}}_\triangledown$. We consider an induction that is indexed by the number of superstructure components on Γ. Given $k \geqslant 0$, let $\mathbf{T}^k_\triangledown \subset \mathbf{T}^{\mathbf{st}}_\triangledown$ be the subset of stable types Γ such that to each $\mathfrak{v}_{\underline{\alpha}} \in \mathrm{V}(\underline{\Gamma})$ there are at most k superstructures attached. Here an interior leaf of type $\triangledown$ attached to a vertex $\mathfrak{v}_{\underline{\alpha}} \in \mathrm{V}(\underline{\Gamma})$ is also regarded as a superstructure attached to $\mathfrak{v}_\alpha$. Then obviously

$\mathbf{T}_{\triangledown}^{k} \subset \mathbf{T}_{\triangledown}^{k+1}$. Moreover, each $\mathbf{T}_{\triangledown}^{k}$ is a basic set in the sense of Definition 5.5. We want to construct inductively a regular coherent family of perturbation data $\underline{P}_{\triangledown}^{k} = \{P_\Gamma \mid \Gamma \in \mathbf{T}_{\triangledown}^{k}\}$.

Firstly, when $k = 0$, the regularity for $\underline{P}_{\triangledown}^{k}$ only requires considering moduli spaces of objects with zero energy. This corresponds to the classical situation and has no difference from the case for type $\triangle$. Indeed one can also copy from perturbations for type $\triangle$ (cf. Subsection 7.5). Therefore the base case of the induction is established.

Proposition 7.8. *Let $\underline{J}_{\mathbf{F}}$ be the strongly regular coherent system of perturbation data we chose for closed domains in Subsection 7.3. Let $k \geqslant 0$. Suppose we have chosen a strongly regular coherent system of perturbation data $\underline{P}_{\triangledown}^{k} = \{P_\Gamma \mid \Gamma \in \mathbf{T}_{\triangledown}^{k}\}$ which restricts to $\underline{J}_{\mathbf{F}}$ on superstructures. Then the system $\underline{P}_{\triangledown}^{k}$ can be extended to a strongly regular coherent system $\underline{P}_{\triangledown}^{k+1}$ which restricts to $\underline{J}_{\mathbf{F}}$ on superstructures.*

Proof. The strategy of our proof is similar to that for the cases considered in the previous three subsections. Take $\Gamma \in \mathbf{T}_{\triangledown}^{k+1} \smallsetminus \mathbf{T}_{\triangledown}^{k}$. Suppose we have chosen a strongly regular coherent system of perturbation data

$$\underline{P}_{\Gamma}^{*} := \{P_\Pi \mid \Pi \in \mathbf{S}^*(\Gamma) \cup \mathbf{T}_{\triangledown}^{k}\}$$

that restricts to $\underline{J}_{\mathbf{F}}$ on superstructures. Then it induces a regular coherent family of perturbation data $\underline{P}_{\underline{\Gamma}} := \{P_\Pi \mid \Pi \in \mathbf{S}^*(\underline{\Gamma})\}$.

Then consider the Banach manifold $\mathcal{P}(\overline{\mathcal{U}}_{\underline{\Gamma}})$ of perturbation data on $\overline{\mathcal{U}}_{\underline{\Gamma}}$ whose values at the boundary strata are given by $\underline{P}_{\underline{\Gamma}}$. Consider a universal problem

$$\mathcal{F}_\Gamma : \mathcal{P}(\overline{\mathcal{U}}_{\underline{\Gamma}}) \times \mathcal{B}_\Gamma \to \mathcal{E}_\Gamma \tag{7.13}$$

that is defined in a similar way as (7.10). Now we prove that the linearization of $\mathcal{F}_\Gamma$ at any point in its zero locus $\mathcal{F}_\Gamma^{-1}(0)$ is surjective. First, by considering the operator (7.11) for all edges in the base that have positive lengths, the problem reduces to proving the surjectivity of the operator $\mathcal{D}\mathcal{F}_\Pi^0$ defined by (7.12) for all maximal subtrees of Π that contains no edge in the base having positive length. For such a subtree Π, divide Π into components such that each component is either a single vertex $\mathfrak{v}_\alpha$ supporting a nonconstant holomorphic disk, or a maximal subtree of $\underline{\Pi}$ supporting objects with zero energy. For the former case, the deformation of almost complex structures on Σ_α can give enough resource to make things transverse. For the latter case, the deformation of almost complex structures on all the vertices does not contribute to the linearized operator. However, since $\underline{J}_{\mathbf{F}}$ satisfies Item (e) of Proposition 7.4, it is not hard to see the transversality already holds. Hence the universal problem (7.13) is transverse to zero. Then by Sard–Smale theorem one can construct a strongly regular perturbation data P_Γ as required. By carrying out the induction the statement of the proposition is proved. □

Now we construct perturbation data P_Γ for all $\Gamma \in \mathbf{T}_{\diamond}$, under the assumption that certain coherent systems of perturbation data $\underline{P}_{\triangle}$ and $\underline{P}_{\triangledown}$ have been chosen for types of type $\triangle$ and $\triangledown$.

Proposition 7.9. *Given a coherent system of perturbation data $\underline{P}_{\triangle} = \{P_\Gamma \mid \Gamma \in \mathbf{T}_{\triangle}^{\mathbf{st}}\}$ and a coherent system of perturbation data $\underline{P}_{\triangledown} = \{P_\Gamma \mid \Gamma \in \mathbf{T}_{\triangledown}^{\mathbf{st}}\}$, both of which are strongly regular and restrict to $\underline{J}_{\mathbf{F}}$ on superstructures, there exists a strongly regular coherent system of perturbation data $\underline{P} = \{P_\Gamma \mid \Gamma \in \mathbf{T}^{\mathbf{st}}\}$ that restricts to $\underline{J}_{\mathbf{F}}$ on superstructures and that extends both $\underline{P}_{\triangle}$ and $\underline{P}_{\triangledown}$,* *Proof.* The proof is based on the same induction process as been used in the previous subsections. Notice that we only need to consider unbroken types because perturbations for broken types are determined by those on unbroken ones via the (Cutting edge) axiom of Definition 5.5. First we need to consider unbroken types $\Gamma \in \mathbf{T}_\diamond^{\mathbf{st}}$ that has no continuous moduli. There is only one such type that is the Φ-shape (see Subsection 4.1), and it is easy to see that the trivial perturbation is (strongly) regular. Then using Lemma 7.6 one can construct perturbations inductively over all unbroken $\Gamma \in \mathbf{T}_\diamond^{\mathbf{st}}$. □

We confirm that we have finished the construction of a regular coherent system of perturbation data for the set consisting of all stable

7.5. **The classical level theories.** In the next section we prove a result stating that $\mathrm{Fuk}^K(L)$ and $\mathrm{Fuk}^{\mathfrak{c}}(\bar{L})$ are isomorphic on the classical level, and the quantum Kirwan map is the identity on the classical level. This condition requires an additional restriction on the perturbation data. In this subsection we explain that this restriction can always be satisfied during the construction.

The condition involves the following forgetful map. Let $\mathbf{T}^0$ be the set of stable types that have no interior leaves. Let $\mathbf{T}^0_\triangle$, $\mathbf{T}^0_\triangledown$, and $\mathbf{T}^0_\diamond$ be the subsets of $\mathbf{T}^0$ of types $\triangle$, $\triangledown$ and $\diamond$ respectively. There is a one-to-one correspondence $\mathbf{T}^0_\triangle \simeq \mathbf{T}^0_\triangledown$, denoted by $\Gamma_\triangle \mapsto \Gamma_\triangledown$, inducing commutative diagrams

$$\begin{array}{ccc} \overline{\mathcal{U}}_{\Gamma_\triangle} & \longrightarrow & \overline{\mathcal{U}}_{\Gamma_\triangledown} \\ \downarrow & & \downarrow \\ \overline{\mathcal{W}}_{\Gamma_\triangle} & \longrightarrow & \overline{\mathcal{W}}_{\Gamma_\triangledown} \end{array} . \tag{7.14}$$

There is a forgetful map $\mathbf{T}^0_\diamond \to \mathbf{T}^0_{\triangle/\triangledown}$, denoted by $\Gamma_\diamond \mapsto \Gamma_{\triangle/\triangledown}$, that changes the coloring to $\mathfrak{s} \equiv \triangle$ or $\mathfrak{s} \equiv \triangledown$, and then possibly contracts unstable vertices.[3]. For $\Gamma_\diamond \in \mathbf{T}^0_\diamond$, let the corresponding type be $\Gamma_{\triangle/\triangledown} \in \mathbf{T}^0_{\triangle/\triangledown}$. Then there is a projection

$$\begin{array}{ccc} \overline{\mathcal{U}}_{\Gamma_\diamond} & \xrightarrow{\mu_{\triangle/\triangledown}} & \overline{\mathcal{U}}_{\Gamma_{\triangle/\triangledown}} \\ \downarrow & & \downarrow \\ \overline{\mathcal{W}}_{\Gamma_\diamond} & \longrightarrow & \overline{\mathcal{W}}_{\Gamma_{\triangle/\triangledown}} \end{array} . \tag{7.15}$$

This map $\mu_{\triangle/\triangledown}$ is defined by identifying each two-dimensional component $\boldsymbol{H} \simeq \Sigma_\alpha$ corresponding to $\mathfrak{v}_\alpha \in V_\diamond(\Gamma_\diamond)$ with boundary markings the corresponding marked disk, and possibly stabilizing. Since the translation group is contained in the group of real Möbius transformations, this is a well-defined map. We remark that $\mu_{\triangle/\triangledown}$ has one-dimensional fibres.

Lemma 7.10. *One can choose the strongly regular coherent family of perturbations $\underline{P}$ in such a way that the following conditions are satisfied.*

(a) *For any $\Gamma_\triangle \in \mathbf{T}^0_\triangle$, $P_{\Gamma_\triangle} = P_{\Gamma_\triangledown}$ with respect to the homeomorphism* (7.14).
(b) *For any $\Gamma_\diamond \in \mathbf{T}^0_\diamond$, $P_{\Gamma_\diamond} = P_{\Gamma_{\triangle/\triangledown}} \circ \mu_{\triangle/\triangledown}$.*

[3]For example, there could be a vertex $\mathfrak{v}_\alpha \in \mathrm{V}_\diamond(\Gamma_\diamond)$ connecting to only one vertex in $\mathrm{V}_\triangle(\Gamma_\triangledown)$. This operation will contract the two-dimensional component corresponding to $\mathfrak{v}_\alpha$. See Figure 12.

Proof. In the previous construction, it is easy to choose the strongly regular coherent family for types in $\mathbf{T}^0_\triangle \sqcup \mathbf{T}^0_\triangledown$ first, such that they satisfy the first condition. Then by pulling back via $\mu_{\triangle/\triangledown}$, one obtains $P_{\Gamma_\diamond}$ for all $\Gamma_\diamond \in \mathbf{T}^0_\diamond$. One easily checks that this family defined for all types in $\mathbf{T}^0$ is coherent. The last thing to check is the strong regularity for types in $\mathbf{T}^0_\diamond$. First, since the types considered here have no leaves and no superstructures, strong regularity is equivalent to regularity. Second, for all controlled refinement $\tilde\Gamma$ of $\Gamma \in \mathbf{T}^0_\diamond$ and any representative $\mathcal{C}$ of a point in $\mathcal{M}^*_{\tilde\Gamma}(P_\Gamma)$, $\mathcal{C}$ is trivial over two-dimensional components. So $\mathcal{C}$ is a perturbed gradient tree of (F, H) in $\bar L$. It then descends to an element $\mathcal{C}_{\triangle/\triangledown}$ representing a point in $\mathcal{M}^*_{\tilde\Gamma_{\triangle/\triangledown}}(P_{\Gamma_{\triangle/\triangledown}})$, where $\tilde\Gamma_{\triangle/\triangledown}$ has the obvious meaning. Moreover, $\mathcal{C}_{\triangle/\triangledown}$ is regular. Since $\mu_{\triangle/\triangledown}$ is a submersion, it implies that $\mathcal{C}$ is also regular. This finishes the proof. □

7.6. **Vortex compactness.** We prove a vortex version of the compactness theorem (Theorem 6.8) for components of the moduli space of expected dimension at most one satisfying the following conditions:

Definition 7.11. A vortex type $\tilde\Gamma \in \tilde{\mathbf{T}} \cup \tilde{\mathbf{F}}$ is called *admissible* if the following conditions are satisfied.

(a) $\tilde\Gamma$ is uncrowded.
(b) For each $l_i \in \mathrm{L}(\Gamma)$, $V_i = D^0_a$ for some $a \in \{0, 1, \dots, \bar n\}$, $\mathfrak{o}(l_i) = \tilde{\mathfrak{o}}(l_i) = 1$.
(c) For any vertex $\mathfrak{v}_\alpha \in \mathrm{V}(\Gamma)$ with $B_\alpha \neq 0$, the number of leaves is given by $\#\mathrm{L}(\mathfrak{v}_\alpha) = N_D E(B_\alpha)$. Here N_D is the degree of the stabilizing divisor.
(d) $\mathrm{V}(\Gamma) \smallsetminus \big[\mathrm{V}(\underline{\Gamma}) \cup \mathrm{V}_\diamond(\Gamma)\big] = \varnothing$, i.e., there is no component corresponding to holomorphic spheres.

We remark that if $\tilde\Gamma$ is admissible, then $\tilde\Gamma$ is reduced and controlled, and the underlying Γ is stable. $\tilde\Gamma$ is called *essential* if it is admissible, unbroken and has no edges of length zero.

The vortex compactness result is the following proposition.

Proposition 7.12. *Let $\tilde\Gamma$ be an essential vortex combinatorial type and suppose $\mathcal{C}$ is a stable treed scaled vortex of vortex type $\tilde\Pi$ that represents an element in the closure $\overline{\mathcal{M}^*_{\tilde\Gamma}(P_\Gamma)}$. If $\mathbf{ind}\tilde\Gamma = 0$, then $\tilde\Pi = \tilde\Gamma$. If $\mathbf{ind}\tilde\Pi = 1$, then $\tilde\Pi$ is still admissible is obtained from $\tilde\Gamma$ by applying exactly one of the following operations.*

- `(bu)` *Breaking a finite edge in $E_\triangle(\underline{\Gamma})$.*
- `(bd)` *Breaking a collection of finite edges downstairs in $E_\triangledown(\underline{\Gamma})$.*
- `(f1)` *Shrinking a finite edge in $E_\triangle(\underline{\Gamma})$ to zero.*
- `(f2)` *Bubbling off a holomorphic disk in X.*
- `(f3)` *Shrinking a finite edges in $E_\triangledown(\underline{\Gamma})$ to zero.*
- `(f4)` *Bubbling off a holomorphic disk in $\bar X$.*
- `(f5)` *Shrinking a collection of finite edges in $E_\diamond(\underline{\Gamma})$ to zero or a vertex $\mathfrak{v}_{\underline{\alpha}} \in \mathrm{V}^-_{\partial\triangledown}(\underline{\Gamma})$ is changed to $\mathrm{V}^0_{\partial\triangledown}(\underline{\Gamma})$.*
- `(f6)` *Degeneration of an affine vortex over $\boldsymbol{H}$.*

7.7. **Proof of Proposition 7.12.** We first fix a few notations. Suppose $\mathcal{C}$ represents a point in the closure $\overline{\mathcal{M}^*_{\tilde\Gamma}(P_\Gamma)}$ that is the limit of a sequence $\mathcal{C}_k$ representing a sequence of points in the uncompactified part $\mathcal{M}^*_{\tilde\Gamma}(P_\Gamma)$. Then by the definition of convergence, there is a tree morphism

$$\rho : \Pi \to \Gamma$$

that is surjective on the level of vertices. Then for each $\mathfrak{v}_\alpha \in \mathrm{V}(\Pi)$, let the image of $\mathfrak{v}_\alpha$ be $\mathfrak{v}_{\breve\alpha} \in \mathrm{V}(\Gamma)$.

We first give a simple but useful result.

Lemma 7.13. *Suppose $\mathfrak{v}_\alpha \in \mathrm{V}(\Pi)$ and the corresponding vortex or holomorphic map is $\boldsymbol{v}_\alpha$. Suppose $\boldsymbol{v}_\alpha$ has positive energy and $z_\alpha \in \Sigma_\alpha$ is an isolated intersection of $\boldsymbol{v}_\alpha$ with D, then z_α is either a node or an interior marking.*

Proof. Suppose z_α is not a node. Take a small disk $U_\alpha \subset \Sigma_\alpha$ containing z_α that is disjoint from other special points of Σ_α and other isolated intersections with D. Because the map $\boldsymbol{v}_\alpha|_{U_\alpha}$ is pseudoholomorphic, the local intersection number of $\boldsymbol{v}_\alpha|_{U_\alpha}$ with D is positive. For k sufficiently large, there is a small disk $U_{\breve\alpha,k} \subset \Sigma_{\breve\alpha}$ such that $\boldsymbol{v}_k|_{U_{\breve\alpha,k}}$ converges (after reparametrization and gauge transformation) to $\boldsymbol{v}_\alpha|_{U_\alpha}$. By the conservation of local intersections, for k sufficiently large, the local intersection number of $\boldsymbol{v}_k|_{U_{\breve\alpha,k}}$ is also positive. Since $\tilde\Pi$ is admissible, this local intersection number is equal to the number of interior markings of $\mathcal{Z}_k$ contained in $U_{\breve\alpha,k}$, which is positive. Hence in the limit these markings converge to markings on the limit. Since inside U_α there are no other markings or nodes, the only possibility is that the local intersection number is one and z_α is an interior marking of the limit domain. □

An important condition for achieving transversality is the controlledness condition for a vortex type in the sense of Definition 7.1. We first prove that the vortex type $\tilde\Pi$ of the limiting object is controlled.

Lemma 7.14. *In the situation of Proposition 7.12, the vortex type $\tilde\Pi$ is controlled.*

Proof. By Definition 7.1, the controlledness condition holds for every vertex $\mathfrak{v}_{\underline{\alpha}}$ in the base and every superstructure. Since $\mathfrak{o}(l_i) = 1$ for all interior leaves l_i, we see that the controlledness condition reads

$$N_D E(\tilde\Xi) \leqslant \#\mathrm{L}(\Xi) \tag{7.16}$$

for Ξ being a superstructure or a subtree of the type $\Pi_{\underline{\alpha}}$.

We first check the condition for $\Xi = \Pi_{\underline{\alpha}}$. In fact the vertex $\mathfrak{v}_{\underline{\breve\alpha}}$ "splits" into a subtree $\Xi_{\underline{\breve\alpha}}$ of Π in the limit. Since $\tilde\Gamma$ is admissible, we know that

$$N_D E(\Gamma_{\underline{\breve\alpha}}) = \#\mathrm{L}(\Gamma_{\underline{\breve\alpha}}).$$

Therefore we obtain a sequence of generalized vortices $\boldsymbol{v}_{k,\underline{\breve\alpha}}$ defined on the domain $\Sigma_{\underline{\breve\alpha}}$. By the property of the convergence, choosing a subsequence if necessary, we can find a collection of sequences of arcs $C_{1,k}, \ldots, C_{m,k}$ contained in $\Sigma_{\underline{\breve\alpha}}$ satisfying the following conditions.

(a) $C_{s,k}$ is disjoint from interior markings or nodes.
(b) For all k, $C_{1,k}, \ldots, C_{m,k}$ are mutually disjoint.
(c) For all k, the complement of the arcs in $\Sigma_{\underline{\breve\alpha}}$ has connected components $U_{\underline{\alpha}_1,k}, \ldots, U_{\underline{\alpha}_{m+1},k}$ corresponding to vertices $\mathfrak{v}_{\alpha_1}, \ldots, \mathfrak{v}_{\alpha_{m+1}}$ of $\Xi_{\underline{\breve\alpha}}$. Moreover,
$$\lim_{k\to\infty} E(\boldsymbol{v}_k|_{U_{\underline{\alpha}_s,k}}) = E(\tilde\Pi_{\underline{\alpha}_s}).$$
(d) The lengths of the images of the arcs $C_{s,k}$ converge to zero and disjoint from D.

Then for k sufficiently large, the restriction of $\boldsymbol{v}_k$ to each connected component $U_{\underline{\alpha}_s,k}$ defines a relative class $B_{s,k}$ in $H_2(X,L)$ or $H_2(\bar{X},\bar{L})$, and $B_{s,k}$ is the same class as that of $\tilde{\Pi}_{\underline{\alpha}_s}$. Since for all k, the intersections with the stabilizing divisor are in the smooth part of D (or $\bar{D}$) and are transverse, by the relation between topological pairing and intersection number, we see that the number of interior markings that are attached to a connected component $U_{\underline{\alpha}_s,k}$ (including all superstructures attached to this connected component) is the topological pairing between $\mathrm{PD}(D)$ (or $\mathrm{PD}(\bar{D})$) and the relative class, hence is the same as $N_D E(\tilde{\Pi}_{\underline{\alpha}_s})$. This proves (7.16) (indeed an equality) for subtrees of type $\Pi_{\underline{\alpha}}$.

Now we check (7.16) for a superstructure Ξ of Π. There can only be at most one vertex $\mathfrak{v}_{\underline{\breve{\alpha}}} \in \mathrm{V}(\underline{\Gamma})$ contained in $\rho_{\mathrm{V}}(\mathrm{V}(\Xi))$. If there is no such vertex $\mathfrak{v}_{\underline{\breve{\alpha}}}$, then conservation of homology class implies that

$$N_D E(\tilde{\Xi}) = \#\mathrm{L}(\Xi).$$

If there is such a vertex $\mathfrak{v}_{\underline{\breve{\alpha}}}$, then assume $\rho_{\mathrm{V}}^{-1}(\mathfrak{v}_{\underline{\breve{\alpha}}}) = \{\mathfrak{v}_1, \ldots, \mathfrak{v}_m\}$, which is a subtree Ξ' of Ξ that contains the root of Ξ. This subtree appears because of interior bubbling. Suppose the bubble tree in $\mathcal{C}$ corresponding to Ξ is attached to the node $z \in \Sigma_{\underline{\alpha}}$, then $\rho_{\mathrm{V}}(\mathfrak{v}_{\underline{\alpha}}) = \mathfrak{v}_{\underline{\breve{\alpha}}}$. Choose a small disk $U_{\underline{\alpha}} \subset \Sigma_{\underline{\alpha}}$ around z that does not contain any other special point. Then for k sufficiently large, $U_{\underline{\alpha}}$ corresponds to a small disk $U_{k,\underline{\breve{\alpha}}}$ in $\Sigma_{\underline{\breve{\alpha}}}$, and the local intersection of $\boldsymbol{v}_k|_{U_{k,\underline{\breve{\alpha}}}}$ with D is equal to the sum of the local intersection number $n_{\underline{\alpha}}$ of $\boldsymbol{v}_{\underline{\alpha}}|_{U_{\underline{\alpha}}}$ plus the intersection number of the bubble tree corresponding to Ξ. This sum is also equal to the number of interior markings contained in $U_{k,\underline{\breve{\alpha}}}$,

$$\#\mathrm{L}(\Xi) = n_{\underline{\alpha}} + N_D E(\tilde{\Xi}) \geqslant N_D E(\tilde{\Xi}).$$

So (7.16) holds for the superstructure Ξ. □

We aim to show that there cannot be codimension two or higher bubblings in the limit, by using transversality. However the limiting triple $(\Pi, \tilde{\Pi}, \mathcal{C})$ may not belong to the class of objects for which transversality holds. We first would like to a related triple for which we can use the transversality. Consider subtrees Ξ of Π that satisfies the following conditions.

- All vertices of Ξ are contained in $\mathrm{V}(\Pi) \smallsetminus \mathrm{V}(\underline{\Pi})$.
- The complement of Ξ in Π is connected.
- The total energy of $\tilde{\Xi}$ is positive.
- Ξ satisfies one of the five conditions below.
  - (a) The total Chern number of $\tilde{\Xi}$ is greater than or equal to two and $\mathrm{L}(\Xi) \neq \varnothing$;
  - (b) The total Chern number of $\tilde{\Xi}$ is one, the evaluation of $\mathcal{C}$ at the infinity of Ξ is in D or $\bar{D}$, and $\mathrm{L}(\Xi) \neq \varnothing$;
  - (c) The total Chern number of $\tilde{\Xi}$ is zero, the evaluation of $\mathcal{C}$ at the infinity of Ξ is in $D \cap S$ or $\bar{D} \cap \bar{S}$, and $\mathrm{L}(\Xi) \neq \varnothing$.
  - (d) The total energy of $\tilde{\Xi}$ is positive and $\mathrm{L}(\Xi) = \varnothing$.
- Ξ is maximal with respect to the above conditions.

Let a maximal subtree Ξ satisfying condition (a), (b), (c), or (d) above be called a type (a), (b), (c), or (d) subtree. All such subtrees, denoted by $\Xi_1, \ldots, \Xi_m$, are mutually disjoint. Moreover, all components corresponding to either nonconstant holomorphic spheres in X or nonconstant affine vortices over $\boldsymbol{C}$ whose images are contained in D are contained in the union of these subtrees. Denote the edge that connects Ξ_s to the rest of the tree by $\mathfrak{e}_s$, where the other end is a vertex $\mathfrak{v}_{\alpha_s}$, and the node corresponding to this edge is $z_s \in \Sigma_{\alpha_s}$.

Lemma 7.15. *There is no type (d) maximal subtrees.*

Proof. Let Ξ be a type (d) maximal subtree. By definition Ξ has no interior leaves. For any vertex $\mathfrak{v}_\alpha$ in $\tilde{\Xi}$ that has positive energy, its image must be contained in D, otherwise $\mathfrak{v}_\alpha$ will have an isolated intersection with D. Therefore the image of Ξ is contained in D. Moreover, if $\tilde{\Xi}$ has total Chern number zero, then the image of Ξ is contained in $D \cap S$.

Now consider the vertex $\mathfrak{v}_\alpha$ below Ξ. We claim that $\mathfrak{v}_\alpha$ has zero energy. Suppose on the contrary that $\mathfrak{v}_\alpha$ has positive energy, then it has an isolated intersection with D at the node connecting $\mathfrak{v}_\alpha$ and Ξ. Choose a small loop around this node. Then for large k, the local intersection number of the disk surrounded by the loop is positive. Hence inside the loop there must be at least one interior marking, which is a contradiction. Hence $\mathfrak{v}_\alpha$ must have zero energy.

Then consider all subtrees of Π that are attached to $\mathfrak{v}_\alpha$. The evaluations of any such subtree Π' is in D and $\mathfrak{v}_\alpha$ has positive energy. Then Π' is a larger subtree of type (a), (b), (c) or (d). This contradicts the maximality of Ξ. Hence there cannot be any type (d) maximal subtrees. $\square$

We construct a new colored tree $(\Pi^{[1]}, \mathfrak{s}^{[1]})$ by removing all subtrees Ξ_s of type (a), (b) and (c) together with the edge $\mathfrak{e}_s$, and then attaching an interior leaves $l_s^{[1]}$ to $\mathfrak{v}_{\alpha_s}$. Moreover, the leaf $l_s^{[1]}$ is labelled by the largest labelling of all leaves in $\mathrm{L}(\Xi_s)$. The set of leaves $\mathrm{L}(\Pi')$ decomposes as

$$\mathrm{L}(\Pi^{[1]}) = \mathrm{L}_{\mathrm{old}}(\Pi^{[1]}) \sqcup \mathrm{L}_{\mathrm{new}}(\Pi^{[1]}) = \mathrm{L}_{\mathrm{old}}(\Pi^{[1]}) \sqcup \mathrm{L}^a_{\mathrm{new}}(\Pi^{[1]}) \sqcup \mathrm{L}^b_{\mathrm{new}}(\Pi^{[1]}) \sqcup \mathrm{L}^c_{\mathrm{new}}(\Pi^{[1]}).$$

Here $\mathrm{L}_{\mathrm{old}}(\Pi^{[1]})$ is the set of original leaves of Π that survive this operation; $\mathrm{L}_{\mathrm{new}}(\Pi^{[1]})$ is the union $\mathrm{L}^a_{\mathrm{new}}(\Pi^{[1]}) \sqcup \mathrm{L}^b_{\mathrm{new}}(\Pi^{[1]}) \sqcup \mathrm{L}^c_{\mathrm{new}}(\Pi^{[1]})$, while each of the three components is the set of new leaves obtained by removing type (a), (b) or (c) subtrees. We relabel all leaves of $\Pi^{[1]}$ by consecutive integers $1, 2, \ldots$ without changing the original order of the labelling.

Moreover, the original metric type $\mathfrak{m}$ can all be inherited by $(\Pi^{[1]}, \mathfrak{s}^{[1]})$. We can also define $\mathfrak{o}^{[1]}$ in the following way. If $l_i^{[1]}$ is an old leaf, then $\mathfrak{o}^{[1]}(l_i^{[1]}) = \mathfrak{o}(l_i)$. If $l_s^{[1]}$ is a new leaf, then

$$\mathfrak{o}^{[1]}(l_s^{[1]}) = \sum_{l_i \in \mathrm{L}(\Xi_s)} \mathfrak{o}(l_i).$$

All type (d) and type (e) subtrees of Π are unchanged and are still called type (d) or type (e) subtrees of $\Pi^{[1]}$.

Lemma 7.16. *Every vertex of* $(\Pi^{[1]}, \mathfrak{s}^{[1]})$ *is stable.*

Proof. We first consider a vertex $\mathfrak{v}_{\underline{\alpha}}$ in the base. By Proposition 3.4, $\boldsymbol{v}_{\underline{\alpha}}$ intersect D at at least one point in the interior and the intersection is isolated. Then by Lemma 7.13, this isolated intersection must be a special point on $\Sigma_{\underline{\alpha}}$. Hence $\mathfrak{v}_{\underline{\alpha}}$ is stable.

Now consider a vertex $\mathfrak{v}_\alpha$ that is not in the base of $\Pi^{[1]}$. The vertex $\boldsymbol{v}_\alpha$ cannot be a holomorphic sphere in X since the vertex $\mathfrak{v}_\alpha$ is then contained in a type (a) maximal subtree of Π and has been removed by the operation $(\Pi, \mathfrak{s}) \mapsto (\Pi^{[1]}, \mathfrak{s}^{[1]})$. If $\boldsymbol{v}_\alpha$ is an affine vortex over $\boldsymbol{C}$, then its evaluation at ∞ cannot lie in every irreducible component $\bar{D}_a$ of $\bar{D}$, hence there is at least one irreducible component $D_a \subset D$ such that $\boldsymbol{v}_\alpha$ is not entirely contained in D and the evaluation at ∞ is not in $\bar{D}_a$. However $\boldsymbol{v}_\alpha$ has to have positive topological intersection number with D_a. Hence $\boldsymbol{v}_\alpha$ has at least one finite isolated intersection with D_a. Therefore by Lemma 7.13 there is at least one finite special point on Σ_α. So $\mathfrak{v}_\alpha$ is stable.

Lastly, suppose $\boldsymbol{v}_\alpha$ is a holomorphic sphere in $\bar{X}$. Suppose $\mathfrak{v}_\alpha$ is unstable. Then $\boldsymbol{v}_\alpha$ is holomorphic with respect to a constant almost complex structure $\bar{J}$. By Definition 5.6 and Lemma 3.5, $\boldsymbol{v}_\alpha$ must intersect D at at least three isolated intersection points. By Lemma 7.13 these points must be special points, which contradicts the assumption that $\mathfrak{v}_\alpha$ is unstable. □

Then we define a refinement $\tilde{\Pi}^{[1]} = (\Pi^{[1]}, \tilde{\boldsymbol{x}}^{[1]}, \boldsymbol{B}^{[1]}, \boldsymbol{V}^{[1]}, \tilde{\mathfrak{o}}^{[1]})$. $\tilde{\boldsymbol{x}}^{[1]}$ and $\boldsymbol{B}^{[1]}$ are naturally inherited from $\tilde{\Pi}$; define $\tilde{\mathfrak{o}}^{[1]}(l_i^{[1]}) = 1$ for all leave $l_i^{[1]} \in \mathrm{L}(\Pi^{[1]})$; the labelling submanifolds on the surviving leaves $\mathrm{L}_{\mathrm{old}}(\Pi^{[1]})$ are inherited from $\tilde{\Pi}$. Lastly, if $l_s^{[1]} \in \mathrm{L}^a_{\mathrm{new}}(\Pi^{[1]})$, then define $V_s^{[1]} = X$; if $l_s^{[1]} \in \mathrm{L}^b_{\mathrm{new}}(\Pi^{[1]})$, then define $V_s^{[1]} = D_0$; if $l_s^{[1]} \in \mathrm{L}^c_{\mathrm{new}}(\Pi^{[1]})$, then define $V_s^{[1]} = D_0 \cap S$ (indeed we should take an irreducible component of S, however we may pretend that S is smooth).

Lemma 7.17. $\tilde{\Pi}^{[1]}$ *is controlled.*

Proof. Follows from the construction of $\tilde{\Pi}^{[1]}$ and Lemma 7.14. □

Let $\mathcal{C}^{[1]} \in \mathcal{M}^*_{\tilde{\Pi}^{[1]}}(P_{\Pi^{\mathrm{st}}, \Pi^{[1]}})$ be the point in the moduli space corresponding to $\tilde{\Pi}^{[1]}$. This configuration might be crowded, which obstructs the transversality. We need to do another operation to obtain a transverse configuration. Let

$$\Xi_s^{[1]},\ s = 1, \ldots, n$$

be the collection of maximal ghost subtrees of $\tilde{\Pi}^{[1]}$. Denote

$$\mathrm{S}^{[1]} = \bigcup_{s=1}^{n} \mathrm{V}(\Xi_s^{[1]}).$$

Let $\Pi^{[2]}$ be the $\mathrm{S}^{[1]}$-stabilization of $\Pi^{[1]}$. Its set of leaves decomposes as

$$\mathrm{L}(\Pi^{[2]}) = \mathrm{L}_{\mathrm{new}}(\Pi^{[2]}) \sqcup \mathrm{L}_{\mathrm{int}}(\Pi^{[2]}) \sqcup \mathrm{L}_{\mathrm{old}}(\Pi^{[2]}).$$

Here $\mathrm{L}_{\mathrm{new}}(\Pi^{[2]})$ is the set of new leaves obtained from a maximal ghost subtree of $\tilde{\Pi}^{[1]}$, $\mathrm{L}_{\mathrm{int}}(\Pi^{[1]})$, called the set of intermediate leaves, consists of new leaves on $\Pi^{[1]}$ that survive the $S^{[1]}$-stabilization, and $\mathrm{L}_{\mathrm{old}}(\Pi^{[2]})$ consists of leaves on the original Π that remains unchanged. We can also further decompose

$$\mathrm{L}_{\mathrm{new}}(\Pi^{[2]}) = \mathrm{L}^s_{\mathrm{new}}(\Pi^{[2]}) \sqcup \mathrm{L}^n_{\mathrm{new}}(\Pi^{[2]}),$$

the set of new leaves obtained from a separating (resp. nonseparating) maximal ghost subtree.

We can also define a refinement $\tilde{\Pi}^{[2]} = (\Pi^{[2]}, \tilde{\boldsymbol{x}}^{[2]}, \boldsymbol{B}^{[2]}, \boldsymbol{V}^{[2]}, \tilde{\mathfrak{o}}^{[2]})$ as follows. First $\tilde{\boldsymbol{x}}^{[2]}$ and $\boldsymbol{B}^{[2]}$ are naturally inherited from $\tilde{\boldsymbol{x}}^{[1]}$ and $\boldsymbol{B}^{[1]}$. The labelling submanifolds and the values of $\tilde{\mathfrak{o}}^{[2]}$ on the old and intermediate leaves are also inherited. On the other hand, for each maximal ghost subtree $\Xi_s^{[1]}$, take $V_s^{[2]}$ to be the intersection of all labelling submanifolds on that maximal ghost subtree. Also, define $\tilde{\mathfrak{o}}^{[2]}$ to be 1 on all new leaves except for one case: if $l_s^{[2]} \in \mathrm{L}^n_{\mathrm{new}}(\Pi^{[2]})$ and $V_s^{[2]} = D_a^0$ for some $a \in \{0, 1, \ldots, \bar{n}\}$, then set $\tilde{\mathfrak{o}}^{[2]}(l_s^{[2]})$ to be the number of interior leaves attached to $\Xi_s^{[1]}$, which is at least two.

Then we obtain an element $\mathcal{C}^{[2]} \in \mathcal{M}^*_{\tilde{\Pi}^{[2]}}(P_{\Pi^{\rm st},\Pi^{[2]}})$ corresponding to $\Pi^{[2]}$. Since $\Pi^{[2]}$ is controlled, uncrowded, and the perturbation is strongly regular, $\mathcal{M}^*_{\tilde{\Pi}^{[2]}}(P_{\Pi^{\rm st},\Pi^{[2]}})$ is transversely cut out. Hence $\mathbf{ind}\tilde{\Pi}^{[2]} \geqslant 0$. We compare this with the explicit index formula (7.3) and (4.1). Abbreviate $(\Pi^{[2]}, \tilde{\Pi}^{[2]}, \mathcal{C}^{[2]}) = (\Pi', \tilde{\Pi}', \mathcal{C}')$. We have

$$\begin{split} 0 \leqslant \mathbf{ind}\tilde{\Pi}' = \mathbf{dim}\mathcal{W}_{\Pi'} + \mathbf{i}(\tilde{\boldsymbol{x}}) + {\rm Mas}(\tilde{\Pi}') - \sum_{l_i \in {\rm L}(\Pi')} \delta_i(\tilde{\Pi}') \\ = \mathbf{dim}\mathcal{W}_\Gamma - \#\big({\rm E}^0_\triangle(\underline{\Pi}') \sqcup {\rm E}^0_\triangledown(\underline{\Pi}')\big) - 2\#\big({\rm E}_\triangle(\Pi') \sqcup {\rm E}_\triangledown(\Pi') \smallsetminus {\rm E}(\underline{\Pi}')\big) \\ - \#{\rm V}^0_{\partial\triangledown}(\underline{\Pi}') - 2\#\big({\rm V}_{\partial\triangledown}(\Pi') \smallsetminus {\rm V}_{\partial\triangledown}(\underline{\Pi}')\big) - \#{\rm b}(\underline{\Pi}') + \mathbf{i}(\tilde{\boldsymbol{x}}) \\ + {\rm Mas}(\tilde{\Pi}) - \big({\rm Mas}(\tilde{\Pi}) - {\rm Mas}(\tilde{\Pi}')\big) - 2\big(\#{\rm L}(\Gamma) - \#{\rm L}(\Pi')\big) - \sum_{l_i \in {\rm L}(\Pi')} \delta_i(\tilde{\Pi}') \\ = \mathbf{ind}\tilde{\Gamma} - \#\big({\rm E}^0_\triangle(\underline{\Pi}') \sqcup {\rm E}^0_\triangledown(\underline{\Pi}')\big) - 2\#\big({\rm E}_\triangle(\Pi') \sqcup {\rm E}_\triangledown(\Pi') \smallsetminus {\rm E}(\underline{\Pi}')\big) \\ - \#{\rm V}^0_{\partial\triangledown}(\underline{\Pi}') - 2\#\big({\rm V}_{\partial\triangledown}(\Pi') \smallsetminus {\rm V}_{\partial\triangledown}(\underline{\Pi}')\big) - \#{\rm b}(\underline{\Pi}') - \big({\rm Mas}(\tilde{\Pi}) - {\rm Mas}(\tilde{\Pi}')\big) \\ - \sum_{l_i' \in {\rm L}_{\rm old}(\Pi')} \big(\delta_i(\tilde{\Pi}') - 2\big) - \sum_{l_i' \in {\rm L}_{\rm int}(\Pi') \sqcup {\rm L}_{\rm new}(\Pi')} \big(\delta_i(\tilde{\Pi}') - 2\big). \end{split} \tag{7.17}$$

The missing Maslov indices $({\rm Mas}(\tilde{\Pi}) - {\rm Mas}(\tilde{\Pi}'))$ decompose according to the formula

$${\rm Mas}(\tilde{\Pi}) - {\rm Mas}(\tilde{\Pi}') = \sum_{l_i' \in {\rm L}^a_{\rm int}(\Pi')} a_i + \sum_{l_i' \in {\rm L}^b_{\rm int}(\Pi')} b_i + \sum_{l_i' \in {\rm L}^c_{\rm new}(\Pi')} c_i + \sum_{l_i' \in {\rm L}^n_{\rm new}(\Pi')} f_i + \sum_{l_i'' \in {\rm L}^s_{\rm new}(\Pi'')} g_i,$$

where we know that $a_i \geqslant 4$, $b_i = 2$, $c_i = 0$, and $f_i, g_i \geqslant 0$. Moreover, from the construction we know the following facts.

(a) For $l_i' \in {\rm L}^a_{\rm int}(\Pi')$, $\delta_i(\tilde{\Pi}') = 0$;
(b) For $l_i' \in {\rm L}^b_{\rm int}(\Pi')$, $\delta_i(\tilde{\Pi}') = 2$;
(c) For $l_i \in {\rm L}^c_{\rm new}(\Pi')$, $\delta_i(\tilde{\Pi}') = 4$;
(d) For $l_i' \in {\rm L}^n_{\rm new}(\Pi')$, $\delta_i(\tilde{\Pi}') \geqslant 4$;
(e) For $l_i' \in {\rm L}^s_{\rm new}(\Pi')$, $\delta_i(\tilde{\Pi}') \geqslant 2$.

Inserting these conditions into (7.17), one obtains

$$\begin{split} 0 \leqslant \mathbf{ind}\tilde{\Pi}' \leqslant \mathbf{ind}\tilde{\Gamma} - 2\#\Big({\rm E}_\triangle(\Pi') \sqcup {\rm E}_\triangledown(\Pi') \smallsetminus {\rm E}(\underline{\Pi}')\Big) - 2\#\Big({\rm V}_{\partial\triangledown}(\Pi') \smallsetminus {\rm V}_{\partial\triangledown}(\underline{\Pi}')\Big) \\ - \sum_{l_i' \in {\rm L}_{\rm old}(\Pi')} \Big(\delta_i(\tilde{\Pi}') - 2\Big) - 2\#{\rm L}_{\rm int}(\Pi') - 2{\rm L}^n_{\rm new}(\Pi'). \end{split}$$

We remark that the argument so far has not used the condition that $\mathbf{ind}\tilde{\Gamma} \leqslant 1$. Using this condition, then the above inequality implies the following conclusions.

(a) For each $l_i' \in {\rm L}_{\rm old}(\Pi')$, $\delta_i(\Pi') = 2$. This equality implies that $V_i' = D_0$ and $\tilde{\mathfrak{o}}'(l_i') = 1$.
(b) ${\rm E}_\triangle(\Pi') \sqcup {\rm E}_\triangledown(\Pi') \smallsetminus {\rm E}(\underline{\Pi}') = {\rm V}_{\partial\triangledown}(\Pi') \smallsetminus {\rm V}_{\partial\triangledown}(\underline{\Pi}') = \varnothing$, namely there is no bubbling of holomorphic spheres and no bubbling of affine vortices over $\boldsymbol{C}$. In other words, in Π' there are no components corresponding to holomorphic spheres.
(c) ${\rm L}_{\rm int}(\Pi') = {\rm L}^n_{\rm new}(\Pi') = \varnothing$. Moreover, ${\rm L}^s_{\rm new}(\Pi')$ is also empty. Indeed, if this is not the case, then there is a holomorphic sphere components, which contradicts item (b) above.

The last two items imply in particular

$$\Pi = \Pi^{[1]} = \Pi^{[2]} = \Pi^{[3]}.$$

Hence $\tilde{\Pi}$ is admissible, and $[\mathcal{C}] \in \mathcal{M}^*_{\tilde{\Pi}}(P_\Pi)$. So using (7.17) again, we see

$$0 \leqslant \mathbf{ind}\tilde{\Pi} = \mathbf{ind}\tilde{\Gamma} - \#\Big(\mathrm{E}^0_\vartriangle(\underline{\Pi}) \sqcup \mathrm{E}^0_\triangledown(\underline{\Pi})\Big) - \#\mathrm{V}^0_{\partial\triangledown}(\underline{\Pi}) - \#\mathrm{b}(\underline{\Pi}).$$

All the last four terms in the equation are nonnegative. Notice that we have $\Pi \trianglelefteq \Gamma$. Then if $\mathbf{ind}\tilde{\Gamma} = 0$, all the last four terms vanish, which implies $\tilde{\Pi} = \tilde{\Gamma}$. If $\mathbf{ind}\tilde{\Gamma} = 1$, then all but one of the last four numbers vanish, and the nonzero number must be one. Then we have the following possibilities.

- $\#\mathrm{E}^0_\vartriangle(\underline{\Pi}) = 1$, corresponding to bubbling of one holomorphic disk in X.
- $\#\mathrm{E}^0_\triangledown(\underline{\Pi}) = 1$, corresponding to bubbling of one holomorphic disk in $\bar{X}$.
- $\#V^0_{\partial\triangledown}(\underline{\Pi}) = 1$, corresponding to two different subcases: 1) the degeneration of an affine vortex over $\boldsymbol{H}$; 2) the lengths of all special edges connecting to the same $\mathfrak{v}_{\underline{\alpha}} \in \mathrm{V}^-_{\partial\triangledown}(\underline{\Gamma})$ shrinks to zero.
- $\#\mathrm{b}(\underline{\Pi}) = 1$. This corresponds to the case of breaking edges.

This finishes the proof of Proposition 7.12.

8. Fukaya algebras

In this section we wrap up all the previous analytical constructions and discuss their algebraic implications. In particular we complete the definitions of the A_∞ algebras $\mathrm{Fuk}^K(L)$, $\mathrm{Fuk}^{\mathfrak{c}}(\bar{L})$ and the A_∞ morphism, and prove their unitality and other properties. A more detailed outline is as follows. In Subsection 8.1 we recall basic knowledge about A_∞ algebras. In Subsection 8.2 we define the quantum cohomology class $\mathfrak{c}$. In Subsection 8.3 we use various gluing results and the vortex compactness theorem (Proposition 7.12) to give a description of one-dimensional moduli spaces. In Subsection 8.4 we define the A_∞ algebra $\mathrm{Fuk}^K(L)$. In Subsection 8.5 we define the A_∞ algebra $\mathrm{Fuk}^{\mathfrak{c}}(\bar{L})$. In Subsection 8.6 we define the A_∞ morphism $\boldsymbol{\varphi}$. In Subsection 8.7 we briefly describe how to equip the A_∞ algebras $\mathrm{Fuk}^K(L)$ and $\mathrm{Fuk}^{\mathfrak{c}}(\bar{L})$ with strict units such that the A_∞ morphism $\boldsymbol{\varphi}$ is strictly unital (the detailed construction is given in Appendix 11). At the end we also prove Corollary 1.3, Corollary 1.4, Theorem 1.6 and Theorem 1.8.

The counts that define the algebraic structures rely on orientations of the moduli spaces. Recall from Subsection 2.6 that we have obtained canonical orientations on moduli spaces of holomorphic disks in X and in $\bar{X}$, as well as the moduli space of affine vortices over $\boldsymbol{H}$. These moduli space orientations are induced from the orientation and the spin structure on $\bar{L}$. On the other hand, the moduli spaces of affine vortices over $\boldsymbol{C}$ is canonically oriented. The appearance of interior markings and the contact order condition does not alter the orientation since they are defined in a complex linear way. Hence in the same way as orienting moduli spaces of stable treed disks in the ordinary Lagrangian Floer theory (such as in [BC07] [CW17]). More precisely, for an admissible vortex type $\tilde{\Gamma} \in \tilde{\mathbf{T}}$ of dimension zero, whose moduli space $\mathcal{M}_{\tilde{\Gamma}}$ consists of isolated points, we can assign a sign to each point $\mathcal{C} \in \mathcal{M}_{\tilde{\Gamma}}$, denoted as

$$\mathbf{Sign}(\mathcal{C}) \in \{1, -1\}.$$

8.1. A_∞ **algebras.** In the rational case the energy filtration is discrete. Let Λ be the Novikov field of formal Laurent series, i.e.,

$$\Lambda = \left\{ \sum_{i \geqslant i_0} a_i \boldsymbol{q}^i \;\middle|\; a_i \in \mathbb{C} \right\}.$$

Let $\Lambda_{\geqslant 0}$ (resp. $\Lambda_{>0}$) be the subring of series with only nonnegative (resp. positive) powers. Let $\Lambda^\times \subset \Lambda_{\geqslant 0}$ be the group of invertible elements.

We recall the basic notions about A_∞ algebras over the Novikov field Λ. Let $\underline{A}$ be a finite dimensional $\mathbb{Z}$-graded $\mathbb{Q}$-vector space, and denote $A = \underline{A} \otimes \Lambda$. The grading of a homogeneous element of $a \in A$ is denoted by $|a| \in \mathbb{Z}$. For $k \in \mathbb{Z}$, $A[k]$ is the shifted graded module whose degree l factor is $A[k]^l = A^{k+l}$.

Definition 8.1. (a) A (curved) A_∞ algebra structure over A consists of a sequence of graded, Λ-linear maps

$$\boldsymbol{m}_{\underline{k}} : \overbrace{A \underset{\Lambda}{\otimes} \cdots \underset{\Lambda}{\otimes} A}^{\underline{k}} \to A[2-\underline{k}],\ \underline{k} \geqslant 0$$

satisfying the A_∞ relation: for any $\underline{k} \geqslant 1$ and a $\underline{k}$-tuple $(\boldsymbol{a}_1, \ldots, \boldsymbol{a}_{\underline{k}})$ of homogeneous elements of A, denoting $\mathcal{N}_j = |\boldsymbol{a}_1| + \cdots + |\boldsymbol{a}_j| + j$, then

$$0 = \sum_{\underline{r}=0}^{\underline{k}} \sum_{j=0}^{\underline{k}-\underline{r}} (-1)^{\mathcal{N}_j} \boldsymbol{m}_{\underline{k}-\underline{r}+1}\big(\boldsymbol{a}_1, \ldots, \boldsymbol{a}_j, \boldsymbol{m}_{\underline{r}}(\boldsymbol{a}_{j+1}, \ldots, \boldsymbol{a}_{j+\underline{r}}), \boldsymbol{a}_{j+\underline{r}+1}, \ldots, \boldsymbol{a}_{\underline{k}}\big).$$

(b) A *strict unit* of an A_∞ algebra is an element $\boldsymbol{e} \in A$ of degree zero satisfying

$$\boldsymbol{m}_2(\boldsymbol{e}, \boldsymbol{a}) = (-1)^{|\boldsymbol{a}|} \boldsymbol{m}_2(\boldsymbol{a}, \boldsymbol{e}) = \boldsymbol{a},\ \forall \boldsymbol{a} \in A;$$

$$\boldsymbol{m}_{\underline{k}}(\boldsymbol{a}_1, \ldots, \boldsymbol{a}_{j-1}, \boldsymbol{e}, \boldsymbol{a}_{j+1}, \ldots, \boldsymbol{a}_{\underline{k}}) = 0,\ \forall \underline{k} \neq 2,\ \boldsymbol{a}_1, \ldots, \boldsymbol{a}_{j-1}, \boldsymbol{a}_{j+1}, \ldots, \boldsymbol{a}_{\underline{k}} \in A.$$

An A_∞ algebra with a strict unit is called a *strict unital* A_∞ algebra.

8.1.1. *Maurer–Cartan equation and potential function.* Denote $A_{>0} = \underline{A} \otimes_{\mathbb{Q}} \Lambda_{>0}$. The *weak Maurer–Cartan* equation for an A_∞ algebra $\mathcal{A}$ reads

$$\boldsymbol{m}_0^{\boldsymbol{b}}(1) := \sum_{\underline{k} \geqslant 0} \boldsymbol{m}_{\underline{k}}(\underbrace{\boldsymbol{b}, \ldots, \boldsymbol{b}}_{\underline{k}}) \equiv 0 \bmod \Lambda \boldsymbol{e}, \quad \text{where } \boldsymbol{b} \in A_{>0} \cap A^{\text{odd}}.$$

A solution $\boldsymbol{b}$ is called a *weakly bounding cochain.* Let $MC(\mathcal{A})$ be the set of all weakly bounding cochains. Define the *potential function* $W : MC(\mathcal{A}) \to \Lambda$ by

$$W(\boldsymbol{b})\boldsymbol{e} = \boldsymbol{m}_0^{\boldsymbol{b}}(1).$$

8.1.2. A_∞ *morphisms.*

Definition 8.2. Let $\mathcal{A}_i = (A, \boldsymbol{m}_{i,0}, \boldsymbol{m}_{i,1}, \ldots)$, $i = 1, 2$ be two A_∞ algebras over A.

(a) A *morphism* from $\mathcal{A}_1$ to $\mathcal{A}_2$ is a collection of Λ-linear maps

$$\boldsymbol{\varphi} = \Big(\varphi_{\underline{k}} : \overbrace{A \underset{\Lambda}{\otimes} \cdots \underset{\Lambda}{\otimes} A}^{\underline{k}} \to A[1-\underline{k}] \mid \underline{k} = 0, 1, \ldots \Big)$$

satisfying the A_∞ relation

$$0 = \sum_{i+j\leqslant \underline{k}} (-1)^{\mathcal{N}_j} \varphi_{\underline{k}-i+1}\big(\boldsymbol{a}_1, \ldots, \boldsymbol{a}_j, \boldsymbol{m}_{1,i}\big(\boldsymbol{a}_{j+1}, \ldots, \boldsymbol{a}_{j+i}\big), \boldsymbol{a}_{j+i+1}, \ldots, \boldsymbol{a}_{\underline{k}}\big)$$
$$+ \sum_{\underline{r}\geqslant 1} \sum_{i_1,\ldots,i_{\underline{r}}\geqslant 0}^{i_1+\cdots+i_{\underline{r}}=\underline{k}} \boldsymbol{m}_{2,\underline{r}}\big(\varphi_{i_1}\big(\boldsymbol{a}_1, \ldots, \boldsymbol{a}_{i_1}\big), \ldots, \varphi_{i_{\underline{r}}}\big(\boldsymbol{a}_{\underline{k}-i_{\underline{r}}+1}, \ldots, \boldsymbol{a}_{\underline{k}}\big)\big). \quad (8.1)$$

(b) If $\mathcal{A}_1$ and $\mathcal{A}_2$ are strictly unital with strict units $\boldsymbol{e}_1, \boldsymbol{e}_2$, then a morphism $\boldsymbol{\varphi} : \mathcal{A}_1 \to \mathcal{A}_2$ is called *unital* if

$$\varphi_1(\boldsymbol{e}_1) = \boldsymbol{e}_2,\ \varphi_{\underline{k}}\big(\boldsymbol{a}_1, \ldots, \boldsymbol{a}_i, \boldsymbol{e}_1, \boldsymbol{a}_{i+2}, \ldots, \boldsymbol{a}_{\underline{k}}\big) = 0,\ \forall \underline{k} \neq 1,\ 0 \leqslant i \leqslant \underline{k} - 1.$$

From now on we make the following assumption in the abstract discussion of A_∞ algebras. We assume that $\mathcal{A}_1, \mathcal{A}_2$ be two A_∞ algebras over the same finite dimensional Λ-module $A = \underline{A} \otimes_{\mathbb{Q}} \Lambda$ with a common strict unit $\boldsymbol{e} \in \underline{A} \subset A$ and $\boldsymbol{\varphi} : \mathcal{A}_1 \to \mathcal{A}_2$ be a unital A_∞ morphism. An A_∞ morphism $\boldsymbol{\varphi} : \mathcal{A}_1 \to \mathcal{A}_2$ is called a *higher order deformation of the identity*, if

$$\varphi_1(\boldsymbol{a}) - \boldsymbol{a} \in A_{>0},\ \forall \boldsymbol{a} \in \underline{A}; \qquad \varphi_{\underline{k}}\big(\overbrace{\underline{A} \underset{\mathbb{Q}}{\otimes} \cdots \underset{\mathbb{Q}}{\otimes} \underline{A}}^{\underline{k}}\big) \subset A_{>0},\ \forall \underline{k} \neq 1.$$

Lemma 8.3. *There is a well-defined map*

$$\underline{\varphi} : A_{>0} \to A_{>0},\ \underline{\varphi}(\boldsymbol{a}) = \sum_{\underline{k}\geqslant 0} \varphi_{\underline{k}}(\underbrace{\boldsymbol{a}, \ldots, \boldsymbol{a}}_{\underline{k}}). \quad (8.2)$$

Moreover, $\varphi_1 : A \to A$ is a linear isomorphism and $\underline{\varphi} : A_{>0} \to A_{>0}$ is a bijection.

Proof. The convergence of $\underline{\varphi}$ follows from the definition. φ_1 is invertible because it differs from the identity by higher order terms. $\underline{\varphi}$ is bijective because we can always solve $\underline{\varphi}(\boldsymbol{a}) = \boldsymbol{b}$ order by order. □

Now we consider how A_∞ morphisms interact with potential functions.

Proposition 8.4. *Let $\mathcal{A}_1, \mathcal{A}_2$ be A_∞ algebras over the same Λ-module A with a common strict unit $\boldsymbol{e}$. Let $\boldsymbol{\varphi} : \mathcal{A}_1 \to \mathcal{A}_2$ be a unital A_∞ morphism which is a higher order deformation of the identity. Then the map $\underline{\varphi} : A_{>0} \to A_{>0}$ defined by (8.2) maps $MC(\mathcal{A}_1)$ into $MC(\mathcal{A}_2)$. Moreover one has*

$$W_1(\boldsymbol{b}_1) = W_2(\underline{\varphi}(\boldsymbol{b}_1)),\ \forall \boldsymbol{b}_1 \in MC(\mathcal{A}_1)$$

where $W_1 : MC(\mathcal{A}_1) \to \Lambda$ and $W_2 : MC(\mathcal{A}_2) \to \Lambda$ are potential functions.

Proof. It is easy to see that $\underline{\varphi}$ maps odd degree elements to odd degree elements. Moreover, by the A_∞ axiom of the morphism $\boldsymbol{\varphi}$ and its unitality, one has

$$\begin{aligned}
&\sum_{\underline{k}\geqslant 0} \boldsymbol{m}_{2,\underline{k}}\big(\underbrace{\underline{\varphi}(\boldsymbol{b}_1),\ldots,\underline{\varphi}(\boldsymbol{b}_1)}_{\underline{k}}\big)\\
&= \sum_{\underline{k}\geqslant 0}\sum_{i_1,\ldots,i_{\underline{k}}\geqslant 0} \boldsymbol{m}_{2,\underline{k}}\big(\varphi_{i_1}(\underbrace{\boldsymbol{b}_1,\ldots,\boldsymbol{b}_1}_{i_1}),\ldots,\varphi_{i_{\underline{k}}}(\underbrace{\boldsymbol{b}_1,\ldots,\boldsymbol{b}_1}_{i_{\underline{k}}})\big)\\
&= \sum_{\underline{r}\geqslant 0}\sum_{\underline{k}\geqslant 0}\sum_{i_1,\ldots,i_{\underline{k}}\geqslant 0}^{i_1+\cdots+i_{\underline{k}}=\underline{r}} \boldsymbol{m}_{2,\underline{k}}\big(\varphi_{i_1}(\underbrace{\boldsymbol{b}_1,\ldots,\boldsymbol{b}_1}_{i_1}),\ldots,\varphi_{i_{\underline{k}}}(\underbrace{\boldsymbol{b}_1,\ldots,\boldsymbol{b}_1}_{i_{\underline{k}}})\big)\\
&= \sum_{\underline{r}\geqslant 0}\sum_{\underline{l}=0}^{\underline{r}}\sum_{j=0}^{\underline{r}-\underline{l}} (-1)^{\mathcal{N}_j}\varphi_{\underline{r}-\underline{l}+1}\big(\underbrace{\boldsymbol{b}_1,\ldots,\boldsymbol{b}_1}_{j},\boldsymbol{m}_{1,\underline{l}}(\boldsymbol{b}_1,\ldots,\boldsymbol{b}_1),\underbrace{\boldsymbol{b}_1,\ldots,\boldsymbol{b}_1}_{\underline{r}-\underline{l}-j}\big)\\
&= \sum_{j\geqslant 0}(-1)^{\mathcal{N}_j}\sum_{\underline{k}\geqslant j}\varphi_{\underline{k}+1}\big(\underbrace{\boldsymbol{b}_1,\ldots,\boldsymbol{b}_1}_{j},\sum_{\underline{l}\geqslant 0}\boldsymbol{m}_{1,\underline{l}}(\boldsymbol{b}_1,\ldots,\boldsymbol{b}_1),\underbrace{\boldsymbol{b}_1,\ldots,\boldsymbol{b}_1}_{\underline{k}-j}\big)\\
&= \sum_{j\geqslant 0}(-1)^{\mathcal{N}_j}\varphi_{\underline{k}+1}\big(\underbrace{\boldsymbol{b}_1,\ldots,\boldsymbol{b}_1}_{j},W_1(\boldsymbol{b}_1)\boldsymbol{e},\underbrace{\boldsymbol{b}_1,\ldots,\boldsymbol{b}_1}_{\underline{k}-j}\big)\\
&= \varphi_1\big(W_1(\boldsymbol{b}_1)\boldsymbol{e}\big)\\
&= W_1(\boldsymbol{b}_1)\boldsymbol{e}.
\end{aligned}$$

This finishes the proof. ∎

8.1.3. *Deformations and cohomology.* Let $\mathcal{A}$ be a strictly unital A_∞ algebra and $\boldsymbol{b}\in MC(\mathcal{A})$. Define

$$\boldsymbol{m}^{\boldsymbol{b}}_{\underline{k}}(\boldsymbol{a}_1,\ldots,\boldsymbol{a}_{\underline{k}}) = \sum_{i_0,\ldots,i_{\underline{k}}} \boldsymbol{m}_{\underline{k}+i_0+\cdots+i_{\underline{k}}}(\underbrace{\boldsymbol{b},\ldots,\boldsymbol{b}}_{i_0},\boldsymbol{a}_1,\underbrace{\boldsymbol{b},\ldots,\boldsymbol{b}}_{i_1},\cdots,\boldsymbol{a}_{\underline{k}},\underbrace{\boldsymbol{b},\ldots,\boldsymbol{b}}_{i_{\underline{k}}}).$$

Similarly, suppose $\boldsymbol{\varphi}:\mathcal{A}_1\to\mathcal{A}_2$ is a unital A_∞ morphism which is a higher order deformation of the identity, and $\boldsymbol{b}_1\in MC(\mathcal{A}_1)$. Define

$$\varphi^{\boldsymbol{b}}_{\underline{k}}(\boldsymbol{a}_1,\ldots,\boldsymbol{a}_{\underline{k}}) = \sum_{i_0,\ldots,i_{\underline{k}}} \varphi_{\underline{k}+i_0+\cdots+i_{\underline{k}}}(\underbrace{\boldsymbol{b},\ldots,\boldsymbol{b}}_{i_0},\boldsymbol{a}_1,\underbrace{\boldsymbol{b},\ldots,\boldsymbol{b}}_{i_1},\cdots,\boldsymbol{a}_{\underline{k}},\underbrace{\boldsymbol{b},\ldots,\boldsymbol{b}}_{i_{\underline{k}}}).$$

Lemma 8.5. *Let $\mathcal{A}_1$ and $\mathcal{A}_2$ be A_∞ algebras over A. Let $\boldsymbol{\varphi}:\mathcal{A}_1\to\mathcal{A}_2$ be an A_∞ morphism that is a higher order deformation of the identity. Let $\boldsymbol{b}_1\in MC(\mathcal{A}_1)$ be a weakly bounding cochain and denote $\boldsymbol{b}_2=\underline{\varphi}(\boldsymbol{b}_1)\in MC(\mathcal{A}_2)$.*

(a) For $i=1,2$, the maps $\boldsymbol{m}^{\boldsymbol{b}_i}_{i,0},\boldsymbol{m}^{\boldsymbol{b}_i}_{i,1},\ldots$ define a unital A_∞ algebra $\mathcal{A}^{\boldsymbol{b}_i}_i$ over A.

(b) The maps $\varphi^{\boldsymbol{b}_1}_0,\varphi^{\boldsymbol{b}_1}_1,\ldots$ define a unital A_∞ morphism $\boldsymbol{\varphi}^{\boldsymbol{b}_1}:\mathcal{A}^{\boldsymbol{b}_1}_1\to\mathcal{A}^{\boldsymbol{b}_2}_2$.

(c) For $i=1,2$, one has $\boldsymbol{m}^{\boldsymbol{b}_i}_{i,1}\circ\boldsymbol{m}^{\boldsymbol{b}_i}_{i,1}=0$; moreover, $\varphi^{\boldsymbol{b}_1}_1$ is an isomorphism of cochain complexes $\varphi^{\boldsymbol{b}_1}_1:(A,\boldsymbol{m}^{\boldsymbol{b}_1}_{1,1})\simeq(A,\boldsymbol{m}^{\boldsymbol{b}_2}_{2,1})$ and hence induces an isomorphism in cohomology

$$H(\boldsymbol{\varphi},\boldsymbol{b}_1): H(\mathcal{A}_1,\boldsymbol{b}_1)\simeq H(\mathcal{A}_2,\boldsymbol{b}_2).$$

Proof. It is routine to check the first two items. For the third item, by the A_∞ axiom and the fact that $\boldsymbol{b}_i \in MC(\mathcal{A}_i)$, one sees that

$$\boldsymbol{m}_{i,1}^{\boldsymbol{b}_i}\big(\boldsymbol{m}_{i,1}^{\boldsymbol{b}_i}(\boldsymbol{a})\big) = \boldsymbol{m}_{i,2}\big(\boldsymbol{m}_{i,0}^{\boldsymbol{b}_i}(1), \boldsymbol{a}\big) - (-1)^{|\boldsymbol{a}|}\boldsymbol{m}_{i,2}\big(\boldsymbol{a}, \boldsymbol{m}_{i,0}^{\boldsymbol{b}_i}(1)\big) = 0.$$

Hence $(A, \boldsymbol{m}_{i,1}^{\boldsymbol{b}_i})$ is a cochain complex. Further, by the A_∞ relations, we have (since the degree of $\boldsymbol{b}_1$ is odd, the signs in the (8.1) are all positive)

$$\begin{aligned}
&\varphi_1^{\boldsymbol{b}_1}\big(\boldsymbol{m}_{1,1}^{\boldsymbol{b}_1}(\boldsymbol{a})\big)\\
&= \sum_{i_0, i_1 \geqslant 0} \varphi_{i_0+i_1+1}\Big(\underbrace{\boldsymbol{b}_1, \ldots, \boldsymbol{b}_1}_{i_0}, \sum_{j_0, j_1 \geqslant 0} \boldsymbol{m}_{1, j_0+j_1+1}\big(\underbrace{\boldsymbol{b}_1, \ldots, \boldsymbol{b}_1}_{j_0}, \boldsymbol{a}, \underbrace{\boldsymbol{b}_1, \ldots, \boldsymbol{b}_1}_{j_1}\big), \underbrace{\boldsymbol{b}_1, \ldots, \boldsymbol{b}_1}_{i_1}\Big)\\
&= -\sum_{\underline{r}_0, \underline{r}_1, \underline{r}_2 \geqslant 0} \sum_{j_0 \geqslant 0} \varphi_{\underline{r}_0+\underline{r}_1+\underline{r}_2+1}\Big(\underbrace{\boldsymbol{b}_1, \ldots, \boldsymbol{b}_1}_{\underline{r}_0}, \boldsymbol{m}_{1,j_0}\big(\underbrace{\boldsymbol{b}_1, \ldots, \boldsymbol{b}_1}_{j_0}\big), \underbrace{\boldsymbol{b}_1, \ldots, \boldsymbol{b}_1}_{\underline{r}_1}, \boldsymbol{a}, \underbrace{\boldsymbol{b}_1, \ldots, \boldsymbol{b}_1}_{\underline{r}_2}\Big)\Big)\\
&\quad -\sum_{\underline{s}_0, \underline{s}_1, \underline{s}_2 \geqslant 0} \sum_{j_1 \geqslant 0} \varphi_{\underline{s}_0+\underline{s}_1+\underline{s}_2+1}\Big(\underbrace{\boldsymbol{b}_1, \ldots, \boldsymbol{b}_1}_{\underline{s}_0}, \boldsymbol{a}, \underbrace{\boldsymbol{b}_1, \ldots, \boldsymbol{b}_1}_{\underline{s}_1}, \boldsymbol{m}_{1,j_1}\big(\underbrace{\boldsymbol{b}_1, \ldots, \boldsymbol{b}_1}_{j_1}\big), \underbrace{\boldsymbol{b}_1, \ldots, \boldsymbol{b}_1}_{\underline{s}_2}\Big)\\
&\quad + \sum_{\underline{r}, \underline{s} \geqslant 0} \sum_{i_1, \ldots, i_{\underline{r}} \geqslant 0} \sum_{j_1, \ldots, j_{\underline{s}} \geqslant 0} \boldsymbol{m}_{2, \underline{r}+\underline{s}+1}\Big(\varphi_{i_1}(\underbrace{\boldsymbol{b}_1, \ldots, \boldsymbol{b}_1}_{i_1}), \ldots, \varphi_{i_{\underline{r}}}(\underbrace{\boldsymbol{b}_1, \ldots, \boldsymbol{b}_1}_{i_{\underline{r}}})\\
&\quad , \sum_{\underline{k}_0, \underline{k}_1 \geqslant 0} \varphi_{\underline{k}_0+\underline{k}_1+1}\big(\underbrace{\boldsymbol{b}_1, \ldots, \boldsymbol{b}_1}_{\underline{k}_0}, \boldsymbol{a}, \underbrace{\boldsymbol{b}_1, \ldots, \boldsymbol{b}_1}_{\underline{k}_1}\big), \varphi_{j_0}(\underbrace{\boldsymbol{b}_1, \ldots, \boldsymbol{b}_1}_{j_0}), \ldots, \varphi_{j_{\underline{s}}}(\underbrace{\boldsymbol{b}_1, \ldots, \boldsymbol{b}_1}_{j_{\underline{s}}})\Big)\\
&= -\sum_{\underline{r}_0, \underline{r}_1, \underline{r}_2 \geqslant 0} \varphi_{\underline{r}_0+\underline{r}_1+\underline{r}_2+1}\Big(\underbrace{\boldsymbol{b}_1, \ldots, \boldsymbol{b}_1}_{\underline{r}_0}, W_1(\boldsymbol{b}_1)\boldsymbol{e}_1, \underbrace{\boldsymbol{b}_1, \ldots, \boldsymbol{b}_1}_{\underline{r}_1}, \boldsymbol{a}, \underbrace{\boldsymbol{b}_1, \ldots, \boldsymbol{b}_1}_{\underline{r}_2}\Big)\Big)\\
&\quad -\sum_{\underline{s}_0, \underline{s}_1, \underline{s}_2 \geqslant 0} \varphi_{\underline{s}_0+\underline{s}_1+\underline{s}_2+1}\Big(\underbrace{\boldsymbol{b}_1, \ldots, \boldsymbol{b}_1}_{\underline{s}_0}, \boldsymbol{a}, \underbrace{\boldsymbol{b}_1, \ldots, \boldsymbol{b}_1}_{\underline{s}_1}, W_1(\boldsymbol{b}_1)\boldsymbol{e}_1, \underbrace{\boldsymbol{b}_1, \ldots, \boldsymbol{b}_1}_{\underline{s}_2}\Big)\\
&\quad + \sum_{\underline{r}, \underline{s} \geqslant 0} \boldsymbol{m}_{2, \underline{r}+\underline{s}+1}\Big(\underbrace{\boldsymbol{b}_2, \ldots, \boldsymbol{b}_2}_{\underline{r}}, \varphi_1^{\boldsymbol{b}_1}(\boldsymbol{a}), \underbrace{\boldsymbol{b}_2, \ldots, \boldsymbol{b}_2}_{\underline{s}}\Big)\\
&= \boldsymbol{m}_{2,1}^{\boldsymbol{b}_2}\big(\varphi_1^{\boldsymbol{b}_1}(\boldsymbol{a})\big).
\end{aligned}$$

Here to obtain the last equality we used the unitality of $\boldsymbol{\varphi}$. Hence $\varphi_1^{\boldsymbol{b}_1}$ is a chain map. It is an isomorphism because it is an invertible linear map (see Lemma 8.3). □

8.2. **Closed quantum Kirwan map.** In this subsection we define a cohomology class $\mathfrak{c} \in H^*(\bar{X}; \Lambda_{>0})$ which will be used to deform the Fukaya algebra downstairs. We remark that $\mathfrak{c}$ can be viewed as the image of $0 \in H_K^*(X; \Lambda_{>0})$ under the closed quantum Kirwan map, although we do not need a complete definition of it. The definition of the closed quantum Kirwan map in the algebraic case is due to the first named author [Woo15].

Recall the construction of coherent perturbations. From now on we fix a coherent system of perturbation data $\underline{P}$ for all stable types $\Gamma \in \mathbf{T}^{\mathbf{st}} \cup \mathbf{F}^{\mathbf{st}}$. Let $\underline{J_{\mathbf{F}}}$ be the restriction of $\underline{P}$ to the superstructures. For every positive integer k let $\mathbf{F}_k$ be the set of stable colored forests of type $\diamond$ that has k interior leaves of type $\triangle$ and no leaves of type $\triangledown$. Let $\mathbf{F}_k' \subset \mathbf{F}_k$ be the subset of maximal elements with respect to the partial order relation $\trianglelefteq$. It is easy to see that $\mathbf{F}_k'$ consists of all forests Γ for which each connected component only has one vertex and k interior leaves. For all k and each $\Gamma \in \mathbf{F}_k$, the chosen perturbation $\underline{J_{\mathbf{F}}}$ gives a strongly regular family of almost complex structures $J_\Gamma \in \mathbf{Map}(\overline{\mathcal{U}}_\Gamma, \mathcal{J})$. Let $\Gamma_k \in \mathbf{F}_k'$ be the stable tree having a single vertex and k interior leaves with all contact orders being 1. Then an admissible refinement of Γ_k is just a homology class $B \in H_2^K(X;\mathbb{Z})$ such that the energy satisfies

$$(1+\bar{n})N_D E(B) = k.$$

Recall here $1+\bar{n}$ is the number of stabilizing divisors we have.

Proposition 8.6. *For every admissible refinement $\tilde{\Gamma}$ of Γ_k, the evaluation map $\mathcal{M}_{\tilde{\Gamma}}(J_{\Gamma_k}) \to \bar{X}$ is an oriented pseudocycle of dimension* $2\dim\bar{X} + 2\deg(B) - 2$.

Proof. Consider a sequence of elements $\mathcal{C}_k \in \mathcal{M}_{\tilde{\Gamma}}(J_{\Gamma_k})$ with representatives $\boldsymbol{v}_k = (u_k, \phi_k, \psi_k)$. Then by Ziltener's compactness theorem, there is a subsequence (still indexed by i) that converges to a perturbed stable affine vortices representing a point $\mathcal{C} \in \overline{\mathcal{M}}_{\tilde{\Gamma}}(J_{\Gamma_k})$. Let Π be the combinatorial type of the limit configuration and $\tilde{\Pi}$ the refinement. As in the proof of Proposition 7.12, we construct another triple $(\Pi', \tilde{\Pi}', \mathcal{C}')$ out of $(\Pi, \tilde{\Pi}, \mathcal{C})$ such that $\mathcal{C}'$ lies in a transverse moduli space. This construction proceeds as follows.

First we may define type (a), (b), (c), (d) maximal subtrees of Π as in the proof of Proposition 7.12 (see Subsection 7.7). For the same reason as Lemma 7.15, there is actually no type (d) maximal subtrees. Then by removing type (a), (b), (c) maximal subtrees from $(\Pi, \tilde{\Pi}, \mathcal{C})$, and contracting crowded ghost subtrees, we obtain a new triple $(\Pi', \tilde{\Pi}', \mathcal{C}')$ such that $\mathcal{C}'$ lies in a transversal moduli space $\mathcal{M}_{\tilde{\Pi}'}(J_{\Pi,\Pi'})$. Here $J_{\Pi,\Pi'}$ is the perturbation induced from J_Π via the S-stabilization operation (see Subsection 5.5). From the index formula, we see that either $\Pi' = \Pi = \Gamma_k$, or we have

$$\dim\mathcal{M}_{\tilde{\Pi}'}(P_{\Pi,\Pi'}) = \mathbf{ind}\tilde{\Pi}' \leqslant \mathbf{ind}\Gamma_k - 2.$$

Since the evaluation map at infinity is continuous with respect to our notion of convergence, by definition, this means the evaluation map $\mathrm{ev}_\infty : \mathcal{M}_{\tilde{\Gamma}}(J_{\Gamma_k}) \to \bar{X}$ is a pseudocycle of dimension equal to $\mathbf{ind}\tilde{\Gamma}_k$. □

Recall that in a closed manifold, a pseudocycle has a well-defined Poincaré dual as a cohomology class in rational coefficients, and the Poincaré dual only depends on the bordism class of the pseudocycle (see [MS04, Section 6.5]). Define the vortex invariant as the homology class of the pseudocycle given by evaluation at infinity on the moduli space of vortices:

$$\mathfrak{c} := \sum_{k\geqslant 1}\sum_{\tilde{\Gamma}\in\tilde{\mathbf{F}}'_k} \frac{\boldsymbol{q}^k}{k!}\mathrm{PD}\Big(\mathrm{ev}_\infty(\mathcal{M}_{\tilde{\Gamma}}(J_{\Gamma_k}))\Big) \in H^*(\bar{X};\Lambda_{>0}).$$

By the same argument as in the proof of Proposition 8.6, one sees that the cohomology class of $\mathfrak{c}$ is independent of J_Γ.

There is a similar invariant for disconnected types with different perturbations for the connected components, as follows. For any $s \geqslant 1$ and integers $k_1, \ldots, k_s \geqslant 1$, let $\Gamma_{k_1,\ldots,k_s}$ be the colored forest which has s components whose number of leaves are $k_1, \ldots, k_s$ respectively, and each component of which has only one vertex of type $\diamond$. Let $\tilde{\mathbf{F}}_{k_1,\ldots,k_s}$ be the set of admissible refinements $\tilde{\Gamma}$ of $\Gamma_{k_1,\ldots,k_s}$, i.e., essentially the set of sequences of homology classes $B_1, \ldots, B_s \in H_2^K(X;\mathbb{Z})$ such that the energy satisfies

$$(1+\bar{n})N_D E(B_i) = k_i.$$

Then for each $\tilde{\Gamma} \in \tilde{\mathbf{F}}_{k_1,\dots,k_s}$ we obtain a pseudocycle in $\mathrm{ev}_\infty : \mathcal{M}_{\tilde{\Gamma}}(J_\Gamma) \to \bar{X} \times \cdots \times \bar{X}$. Then define

$$\mathfrak{c}^{(s)} := \sum_{k_1,\dots,k_s \geqslant 1} \sum_{\tilde{\Gamma} \in \tilde{\mathbf{F}}_{k_1,\dots,k_s}} \frac{\boldsymbol{q}^{k_1+\cdots+k_s}}{k_1! \cdots k_s!} \mathrm{PD}\Big(\mathrm{ev}_\infty(\mathcal{M}_{\tilde{\Gamma}}(J_\Gamma))\Big) \in H^*(\bar{X}; \Lambda_{>0})^{\otimes s}.$$

Using the same argument as in the proof of Proposition 8.6, one can prove the following result.

Lemma 8.7. *As cohomology classes, one has*

$$\mathfrak{c}^{(s)} = \overbrace{\mathfrak{c} \otimes \cdots \otimes \mathfrak{c}}^{s}, \ \forall s \geqslant 1.$$

8.3. **Fake and real boundaries of one-dimensional moduli spaces.** In this section we study the boundary of the moduli spaces. Consider moduli spaces $\overline{\mathcal{M}_{\tilde{\Gamma}}}(P_\Gamma)$ for $\Gamma \in \mathbf{T^{st}}$ and $\tilde{\Gamma}$ an admissible refinement of Γ. We remember our choice of perturbation data $\underline{P}$ and omit it in all notations. Let $\mathbf{t}$ be an element of $\{\triangle, \diamond, \triangledown\}$. Given a sequence of elements $\tilde{\boldsymbol{x}} = (\boldsymbol{x}_1, \dots, \boldsymbol{x}_{\underline{k}}; \boldsymbol{x}_\infty)$ of **crit** (here k could be zero), let

$$\tilde{\mathbf{T}}^i_{\mathbf{t}}(\tilde{\boldsymbol{x}}), \ i = 0, 1$$

be the set of essential (see Definition 7.11) vortex combinatorial types $\tilde{\Gamma}$ of type $\mathbf{t}$ with $\mathbf{ind}\tilde{\Gamma} = 0, 1$, $\underline{k}$ boundary inputs and one output, where the labelling of its boundary inputs and output are given by $\tilde{\boldsymbol{x}}$. For each $\tilde{\Gamma} \in \tilde{\mathbf{T}}^1_{\mathbf{t}}(\tilde{\boldsymbol{x}})$, Proposition 7.12 gives a description of the compactification of the moduli space $\mathcal{M}_{\tilde{\Gamma}}$.

Lemma 8.8. *For each $\tilde{\Gamma} \in \tilde{\mathbf{T}}^1_{\mathbf{t}}(\tilde{\boldsymbol{x}})$, $\overline{\mathcal{M}_{\tilde{\Gamma}}}$ is a compact one-dimensional manifold with boundary. Moreover, the following are true.*

(a) *If $\mathbf{t} = \triangle$, then*

$$\partial \overline{\mathcal{M}_{\tilde{\Gamma}}} = \partial_{\mathtt{bu}} \overline{\mathcal{M}_{\tilde{\Gamma}}} \sqcup \partial_{\mathtt{f1}} \overline{\mathcal{M}_{\tilde{\Gamma}}} \sqcup \partial_{\mathtt{f2}} \overline{\mathcal{M}_{\tilde{\Gamma}}}.$$

(b) *If $\mathbf{t} = \triangledown$, then*

$$\partial \overline{\mathcal{M}_{\tilde{\Gamma}}} = \partial_{\mathtt{bd}} \overline{\mathcal{M}_{\tilde{\Gamma}}} \sqcup \partial_{\mathtt{f3}} \overline{\mathcal{M}_{\tilde{\Gamma}}} \sqcup \partial_{\mathtt{f4}} \overline{\mathcal{M}_{\tilde{\Gamma}}}.$$

(c) *If $\mathbf{t} = \diamond$, then*

$$\partial \overline{\mathcal{M}_{\tilde{\Gamma}}} = \partial_{\mathtt{bu}} \overline{\mathcal{M}_{\tilde{\Gamma}}} \sqcup \partial_{\mathtt{bd}} \overline{\mathcal{M}_{\tilde{\Gamma}}} \sqcup \partial_{\mathtt{f1}} \overline{\mathcal{M}_{\tilde{\Gamma}}} \sqcup \partial_{\mathtt{f2}} \overline{\mathcal{M}_{\tilde{\Gamma}}} \sqcup \partial_{\mathtt{f3}} \overline{\mathcal{M}_{\tilde{\Gamma}}} \sqcup \partial_{\mathtt{f4}} \overline{\mathcal{M}_{\tilde{\Gamma}}} \sqcup \partial_{\mathtt{f5}} \overline{\mathcal{M}_{\tilde{\Gamma}}} \sqcup \partial_{\mathtt{f6}} \overline{\mathcal{M}_{\tilde{\Gamma}}}.$$

Here $\partial_{\mathtt{bu}} \overline{\mathcal{M}_{\tilde{\Gamma}}}$ etc. are the boundary components corresponding to the vortex types listed in Proposition 7.12.

Proof. Transversality implies that the interior of $\overline{\mathcal{M}_{\tilde{\Gamma}}}$ is a one-dimensional manifold. Proposition 7.12 implies that the boundary of $\overline{\mathcal{M}_{\tilde{\Gamma}}}$ has the above description. It remains to put boundary charts on $\overline{\mathcal{M}_{\tilde{\Gamma}}}$. The existence of these charts is the corollary of various gluing results. Here gluing of holomorphic disks and gluing of gradient flow lines are standard, which give boundary charts near type (`f2`), type (`f4`), type (`bu`) and type (`bd`) boundary components. For type (`f1`), type (`f3`) and type (`f5`) boundary components, using the transversality, one can also put boundary charts parametrized by the lengths of the edges that are shrunk to zero. Lastly, for type (`f6`) boundary components, one can put boundary charts using the gluing result of [Xu]. □

For two different $\tilde{\Gamma}_1, \tilde{\Gamma}_2 \in \tilde{\mathbf{T}}^1_{\mathbf{t}}(\tilde{\boldsymbol{x}})$, the compactified moduli spaces $\overline{\mathcal{M}_{\tilde{\Gamma}_1}}$ and $\overline{\mathcal{M}_{\tilde{\Gamma}_2}}$ may share common boundary components. Indeed, the boundary components $\partial_{\mathtt{bu}}\overline{\mathcal{M}_{\tilde{\Gamma}}}$ and $\partial_{\mathtt{bd}}\overline{\mathcal{M}_{\tilde{\Gamma}}}$ are components that cannot be glued in two different ways and are called *real boundaries*; other boundary components in the list of Proposition 7.12 can be glued in two different ways and are called *fake boundaries* (as indicated by their labellings). More precisely, by using two different types of gluings, the fake boundary points are the ends of one-dimensional components of the moduli space in two different ways. The various one-dimensional strata glue together to a topological one-manifold whose boundary is the union of true boundary points:

Lemma 8.9. *For each $\tilde{\Gamma} \in \tilde{\mathbf{T}}^1_{\mathbf{t}}(\tilde{\boldsymbol{x}})$ and $l = 1, \ldots, 6$, if $\partial_{fl}\overline{\mathcal{M}_{\tilde{\Gamma}}} \neq \emptyset$, then there exists exactly one essential vortex type $\tilde{\Gamma}_l \in \tilde{\mathbf{T}}^1_{\mathbf{t}}(\tilde{\boldsymbol{x}})$ (which has the same number of interior leaves) such that*

$$\overline{\partial}_{fl}\overline{\mathcal{M}_{\tilde{\Gamma}}} = \begin{cases} \overline{\partial}_{fl+1}\overline{\mathcal{M}_{\tilde{\Gamma}_l}}, & \text{if } l = 1,3,5; \\ \overline{\partial}_{fl-1}\overline{\mathcal{M}_{\tilde{\Gamma}_l}}, & \text{if } l = 2,4,6. \end{cases}$$

Moreover, if we take the union $\overline{\mathcal{M}_{\tilde{\Gamma}}} \cup \overline{\mathcal{M}_{\tilde{\Gamma}_l}}$ by identifying their common boundaries, then the union is an oriented one-dimensional manifold (with or without boundary) whose orientation agrees with the orientations on $\overline{\mathcal{M}_{\tilde{\Gamma}}}$ and $\overline{\mathcal{M}_{\tilde{\Gamma}_l}}$.

Now for each $k \geqslant 0$, let $\tilde{\mathbf{T}}^i_{\mathbf{t}}(\tilde{\boldsymbol{x}})_k \subset \tilde{\mathbf{T}}^i_{\mathbf{t}}(\tilde{\boldsymbol{x}})$ be the subset of vortex types that has exactly k interior leaves. Then we define

$$\overline{\mathcal{M}^i_{\mathbf{t}}(\tilde{\boldsymbol{x}})_k} = \Big[\bigsqcup_{\tilde{\Gamma} \in \tilde{\mathbf{T}}^i_{\mathbf{t}}(\tilde{\boldsymbol{x}})_k} \overline{\mathcal{M}_{\tilde{\Gamma}}} \Big] / \sim .$$

Here the equivalence relation is defined as follows: when $i = 1$, we identify different fake boundaries as in Lemma 8.9. Since k dictates the total energy, it implies that $\overline{\mathcal{M}_{\mathbf{t}}(\tilde{\boldsymbol{x}})_k}$ is a compact i-dimensional manifold with boundary. We can write

$$\partial \overline{\mathcal{M}^i_{\mathbf{t}}(\tilde{\boldsymbol{x}})_k} = \partial_{\mathtt{bu}} \overline{\mathcal{M}^i_{\mathbf{t}}(\tilde{\boldsymbol{x}})_k} \sqcup \partial_{\mathtt{bd}} \overline{\mathcal{M}^i_{\mathbf{t}}(\tilde{\boldsymbol{x}})_k}. \tag{8.3}$$

8.4. **Fukaya algebra upstairs.** Consider $\mathbf{t} = \triangle$. For each $k \geqslant 0$, consider the compact zero-dimensional moduli space $\mathcal{M}^0_{\triangle}(\tilde{\boldsymbol{x}})_k$. Using the orientation induced from the spin structure, we have a well-defined count of vortices

$$\langle \boldsymbol{x}_1, \ldots, \boldsymbol{x}_{\underline{k}}; \boldsymbol{x}_\infty \rangle_k = \tilde{\#} \mathcal{M}^0_{\triangle}(\tilde{\boldsymbol{x}})_k := \sum_{\mathcal{C} \in \mathcal{M}^0_{\triangle}(\tilde{\boldsymbol{x}})_k} \mathbf{Sign}(\mathcal{C}) \cdot \mathrm{Hol}_{\bar{L}}(\mathcal{C}) \in \mathbb{C}.$$

Here if the local system is given by $b \in H^1(\bar{L}; \Lambda_{\geqslant 0})$, then the holonomy $\mathrm{Hol}_{\bar{L}}(\mathcal{C})$ is defined as

$$\mathrm{Hol}_{\bar{L}}(\mathcal{C}) = \prod_{\mathfrak{v}_\alpha \in \mathrm{V}(\Gamma)} \exp\langle b, \partial B_\alpha \rangle \in \Lambda_\times \subset \Lambda.$$

Then define

$$\langle \boldsymbol{x}_1, \ldots, \boldsymbol{x}_{\underline{k}}; \boldsymbol{x}_\infty \rangle = (-1)^{\heartsuit(\tilde{\boldsymbol{x}})} \sum_{k \geqslant 0} \frac{\boldsymbol{q}^k}{k!} \langle \boldsymbol{x}_1, \ldots, \boldsymbol{x}_{\underline{k}}; \boldsymbol{x}_\infty \rangle_k \in \Lambda, \tag{8.4}$$

where $\tilde{\boldsymbol{x}}$ denotes the sequence $(\boldsymbol{x}_1, \ldots, \boldsymbol{x}_{\underline{k}}, \boldsymbol{x}_\infty)$ and the symbol $\heartsuit(\tilde{\boldsymbol{x}})$ is defined by

$$\heartsuit(\tilde{\boldsymbol{x}}) = \sum_{\underline{i}=1}^{\underline{k}} \underline{i} \cdot \mathbf{i}(\boldsymbol{x}_{\underline{i}}) \in \mathbb{Z};$$

Now we define the composition maps $\boldsymbol{m}_{\underline{k}}$. For $\underline{k} = 1$, define

$$\boldsymbol{m}_1(\boldsymbol{x}) = \delta_{\mathrm{Morse}}(\boldsymbol{x}) + \sum_{\boldsymbol{x}_\infty \in \mathbf{crit}} \langle \boldsymbol{x}; \boldsymbol{x}_\infty \rangle \cdot \boldsymbol{x}_\infty.$$

For $\underline{k} \neq 1$, define

$$\boldsymbol{m}_{\underline{k}}(\boldsymbol{x}_1, \ldots, \boldsymbol{x}_{\underline{k}}) = \sum_{\boldsymbol{x}_\infty \in \mathbf{crit}} \langle \boldsymbol{x}_1, \ldots, \boldsymbol{x}_{\underline{k}}; \boldsymbol{x}_\infty \rangle \cdot \boldsymbol{x}_\infty. \tag{8.5}$$

The $\underline{k} = 1$ case has additional terms coming from Morse trajectories because by definition an infinite edge is not a stable combinatorial type.

Theorem 8.10. (cf. [Woo11, Theorem 3.6]) $\mathrm{Fuk}^K(L) := (CF^*(\bar{L}) \otimes \Lambda, \boldsymbol{m}_0, \boldsymbol{m}_1, \ldots)$ *is an* A_∞ *algebra.*

Proof. Recall that the A_∞ relation for $\boldsymbol{m}_{\underline{k}}$ reads

$$\sum_{\underline{r}=0}^{\underline{k}} \sum_{j=0}^{\underline{k}-\underline{r}} (-1)^{\mathcal{N}_j} \boldsymbol{m}_{\underline{k}-\underline{r}+1}\big(\boldsymbol{a}_1, \ldots, \boldsymbol{a}_{j-1}, \boldsymbol{m}_{\underline{r}}\big(\boldsymbol{a}_{j+1}, \ldots, \boldsymbol{a}_{j+\underline{r}}\big), \boldsymbol{a}_{j+\underline{r}+1}, \cdots, \boldsymbol{a}_{\underline{k}}\big) = 0.$$

It suffices to verify for all $\boldsymbol{a}_i$ being generators $\boldsymbol{x}_i \in \mathbf{crit}$. Then it is equivalent to verify that all $k \geqslant 0$ and $\boldsymbol{x}_\infty \in \mathbf{crit}$, one has

$$\sum_{\underline{r}=0}^{\underline{k}} \sum_{j=0}^{\underline{k}-\underline{r}} (-1)^{\mathcal{N}_j} \langle \boldsymbol{x}_1, \ldots, \boldsymbol{x}_{j-1}, \boldsymbol{m}_{\underline{r}}(\boldsymbol{x}_{j+1}, \ldots, \boldsymbol{x}_{j+\underline{r}}), \boldsymbol{x}_{j+\underline{r}+1}, \cdots, \boldsymbol{x}_{\underline{k}}; \boldsymbol{x}_\infty \rangle_k = 0.$$

We will only verify these relations modulo signs. The verification with signs is the same as the case of [Woo11] and [CW17] as the almost complex constraints at interior markings do not affect the signs.

Denote $\tilde{\boldsymbol{x}} = (\boldsymbol{x}_1, \ldots, \boldsymbol{x}_{\underline{k}}; \boldsymbol{x}_\infty)$. Specializing (8.3) to the case $i = 1$ and $\mathbf{t} = \triangle$, we obtain

$$\partial \overline{\mathcal{M}^1_\triangle(\tilde{\boldsymbol{x}})}_k = \partial_{\mathtt{bu}} \overline{\mathcal{M}^1_\triangle(\tilde{\boldsymbol{x}})}_k.$$

Let $\mathtt{BU}$ be the set of pairs $(\tilde{\boldsymbol{y}}, \tilde{\boldsymbol{z}})$ where for some $j, \underline{r}$ and $\boldsymbol{z} \in \mathbf{crit}$,

$$\tilde{\boldsymbol{y}} = (\boldsymbol{x}_{j+1}, \ldots, \boldsymbol{x}_{j+\underline{r}}; \boldsymbol{z}), \qquad \tilde{\boldsymbol{z}} = (\boldsymbol{x}_1, \ldots, \boldsymbol{x}_j, \boldsymbol{z}, \boldsymbol{x}_{j+\underline{r}+1}, \ldots, \boldsymbol{x}_{\underline{k}}; \boldsymbol{x}_\infty).$$

(in the notation $\mathtt{BU}$ we omit the dependence on $\tilde{\boldsymbol{x}}$). Then we have

$$\partial_{\mathtt{bu}} \overline{\mathcal{M}^1_\triangle(\tilde{\boldsymbol{x}})}_k \simeq \bigsqcup_{(\tilde{\boldsymbol{y}}, \tilde{\boldsymbol{z}}) \in \mathtt{BU}} \bigsqcup_{l+m=k} \frac{k!}{l!m!} \Big(\mathcal{M}^0_\triangle(\tilde{\boldsymbol{y}})_l \times \mathcal{M}^0_\triangle(\tilde{\boldsymbol{z}})_m \Big).$$

Then we have (modulo signs)

$$\begin{aligned}
&\sum_{\underline{r}=0}^{\underline{k}} \sum_{j=1}^{\underline{k}-\underline{r}} \langle \boldsymbol{x}_1, \ldots, \boldsymbol{x}_j, \boldsymbol{m}_{\underline{r}}(\boldsymbol{x}_{j+1}, \ldots, \boldsymbol{x}_{j+\underline{r}}), \boldsymbol{x}_{j+\underline{r}+1}, \ldots, \boldsymbol{x}_{\underline{k}}); \boldsymbol{x}_\infty \rangle_k \\
&\quad = \sum_{(\tilde{\boldsymbol{y}}, \tilde{\boldsymbol{z}}) \in \mathtt{BU}} \sum_{l+m=k} \langle \boldsymbol{x}_{j+1}, \ldots, \boldsymbol{x}_{j+\underline{r}}; \boldsymbol{z} \rangle_l \cdot \langle \boldsymbol{x}_1, \ldots, \boldsymbol{x}_j, \boldsymbol{z}, \boldsymbol{x}_{j+\underline{r}+1}, \ldots, \boldsymbol{x}_{\underline{k}}; \boldsymbol{x}_\infty \rangle_m \\
&\qquad = \sum_{(\tilde{\boldsymbol{y}}, \tilde{\boldsymbol{z}}) \in \mathtt{BU}} \sum_{l+m=k} \left(\frac{\tilde{\#} \mathcal{M}^0_\triangle(\tilde{\boldsymbol{y}})_l}{l!} \cdot \frac{\tilde{\#} \mathcal{M}^0_\triangle(\boldsymbol{z})_m}{m!} \right) = \frac{1}{k!} \Big(\tilde{\#} \big(\partial \overline{\mathcal{M}^1_\triangle(\tilde{\boldsymbol{x}})}_k \big) \Big) = 0.
\end{aligned}$$

This finishes the proof of the A_∞ relation for $\mathrm{Fuk}^K(L)$. ∎

An important situation is when quasidisks have minimal Maslov index two.

Proposition 8.11. *When Hypothesis 1.5 is satisfied, there is a function* $W_L^K : H^1(\bar{L}; \Lambda_{\geqslant 0}) \to \Lambda_{>0}$ *such that for any local system* $b \in H^1(\bar{L}; \Lambda_{\geqslant 0})$,

$$\boldsymbol{m}_{0,b}(1) = W_L^K(b)\boldsymbol{x}_M.$$

8.5. **Fukaya algebra corrected by $\mathfrak{c}$.** Similar to the above case, let $\tilde{\mathbf{T}}^0_{\triangledown}(\tilde{\boldsymbol{x}})$ be the set of essential vortex types $\tilde{\Gamma}$ with boundary tails labelled by $\tilde{\boldsymbol{x}}$ and $\mathbf{ind}\tilde{\Gamma} = 0$. Notice that for any such $\tilde{\Gamma}$, $\mathrm{V}_{\triangledown}(\underline{\Gamma}) = \mathrm{V}(\underline{\Gamma})$ but $\mathrm{V}_{\triangledown}(\Gamma) \neq \mathrm{V}(\Gamma)$, as there must be components corresponding to affine vortices over $\boldsymbol{C}$. Define

$$\mathcal{M}^0_{\triangledown}(\tilde{\boldsymbol{x}}) = \bigsqcup_{\tilde{\Gamma} \in \tilde{\mathbf{T}}^0_{\triangledown}(\tilde{\boldsymbol{x}})} \mathcal{M}^*_{\tilde{\Gamma}}(P_\Gamma)$$

and $\mathcal{M}^0_{\triangledown}(\tilde{\boldsymbol{x}})_k$ be the subset corresponding to configurations having exactly k interior markings (all markings are of type $\triangle$). Then Proposition 7.12 and the regularity implies that $\mathcal{M}^0_{\triangledown}(\tilde{\boldsymbol{x}})_k$ is a compact zero-dimensional manifold, hence consists of finitely many points. Using the orientation induced from the spin structure of $\bar{L}$, we have a well-defined counting

$$\overline{\langle \boldsymbol{x}_1, \ldots, \boldsymbol{x}_{\underline{k}}; \boldsymbol{x}_\infty \rangle_k} := \sum_{\mathcal{C} \in \mathcal{M}^0_{\triangledown}(\tilde{\boldsymbol{x}})_k} \mathbf{Sign}(\mathcal{C}) \cdot \mathrm{Hol}_{\bar{L}}(\mathcal{C}) \in \boldsymbol{C}.$$

Then define

$$\overline{\langle \boldsymbol{x}_1, \ldots, \boldsymbol{x}_{\underline{k}}; \boldsymbol{x}_\infty \rangle} := (-1)^{\heartsuit(\tilde{\boldsymbol{x}})} \sum_k \frac{\boldsymbol{q}^k}{k!} \overline{\langle \boldsymbol{x}_1, \ldots, \boldsymbol{x}_{\underline{k}}; \boldsymbol{x}_\infty \rangle_k} \in \Lambda. \tag{8.6}$$

Now we define the composition maps $\boldsymbol{n}^{\mathfrak{c}}_{\underline{k}}$. For $\underline{k} = 1$, define

$$\boldsymbol{n}^{\mathfrak{c}}_1(\boldsymbol{x}) = \delta_{\mathrm{Morse}}(\boldsymbol{x}) + \sum_{\boldsymbol{x}_\infty \in \mathbf{crit}} \overline{\langle \boldsymbol{x}; \boldsymbol{x}_\infty \rangle} \cdot \boldsymbol{x}_\infty.$$

For $\underline{k} \neq 1$, define

$$\boldsymbol{n}^{\mathfrak{c}}_{\underline{k}}(\boldsymbol{x}_1, \ldots, \boldsymbol{x}_{\underline{k}}) = \sum_{\boldsymbol{x}_\infty \in \mathbf{crit}} \overline{\langle \boldsymbol{x}_1, \ldots, \boldsymbol{x}_{\underline{k}}; \boldsymbol{x}_\infty \rangle}. \tag{8.7}$$

Theorem 8.12. $\mathrm{Fuk}^{\mathfrak{c}}(\bar{L}) := \left(CF(\bar{L}) \otimes \Lambda, \boldsymbol{n}^{\mathfrak{c}}_*\right)$ *is an* A_∞ *algebra over* Λ.

Proof. The proof of the A_∞ relation for $\mathrm{Fuk}^{\mathfrak{c}}(\bar{L})$ is similar to the case of $\mathrm{Fuk}^K(L)$, once one identifies of boundary components of all relevant one-dimensional moduli spaces. □

Remark 8.13. We have not defined the bulk-deformed Fukaya algebra of $\bar{L}$ in general. The general theory of bulk-deformations of Lagrangian Floer theory is given in the book [FOOO09] using Kuranishi structures, while a T^n-equivariant version for the toric case is given in [FOOO11]. In this remark we explain in what sense the A_∞ algebra $\mathrm{Fuk}^{\mathfrak{c}}(\bar{L})$ is a bulk deformation, and how to construct a general bulk-deformed Fukaya algebra using similar perturbation schemes.

First we introduce a suitable class for the bulk deformation. Let $\mathfrak{a} \in H^*(\bar{X}; \Lambda_{>0})$ be a cohomology class having positive energy level and for simplicity assume that $\boldsymbol{q}^{-m}\mathfrak{a} \in H^*(\bar{X}; \mathbb{Q})$ and has a pure degree. One can choose a pseudocycle representative[4] $e_0 : V_0 \to \bar{X}$ of the Poincaré dual of $\boldsymbol{q}^{-m}\mathfrak{a}$, and choose "boundaries" $e_i : V_i \to \bar{X}$, $i = 1, \ldots, s$ such that

[4]Indeed by the theorem of Thom [Tho54] one can choose a smooth representative of an integer multiple of the same class.

(a) The boundary $\overline{e_0(V_0)} \smallsetminus e_0(V_0)$ is contained in the union of $e_i(V_i)$ for all $i = 1, \ldots, s$ and the dimensions of V_i for all $i \geqslant 1$ are at most $\dim V_0 - 2$.

(b) For any $l \geqslant 1$ and any sequence $\boldsymbol{i} := (i_1, \ldots, i_l)$ where $i_1, \ldots, i_l \in \{0, 1, \ldots, s\}$, the map

$$e_{\boldsymbol{i}} := (e_{i_1}, \ldots, e_{i_l}) : V_{i_1} \times \cdots \times V_{i_l} \to \bar{X}^l$$

is transverse to the product $\Delta^l \bar{L} \subset \bar{X}^l$.

We allow perturbation data over for treed disks of types Γ for which 1) $\mathrm{V}(\Gamma) = \mathrm{V}_{\triangledown}(\Gamma)$; 2) the treed disks contains two types of interior markings, called auxiliary markings and real markings. Similar arguments to the ones for the regular Fukaya algebra show the existence of coherent perturbation data such that the moduli spaces of perturbed stable treed disks for which the auxiliary markings are mapped into the prescribed loci of the stabilizing divisor, and the real markings are mapped into the prescribed images of $e_{\boldsymbol{i}}$ are all regular. Then the counts of zero dimensional moduli spaces define the composition maps for an A_∞ algebra structure over the module $CF(\bar{L}; \Lambda)$. One can further show that the A_∞ algebra has a well-defined homotopy type that is independent of the choices of perturbation data, and only depends on the bordism class of the maps $e_0, e_1, \ldots, e_s$. On the other hand, as shown in [Zin08], bordism classes of pseudocycle representatives are the same as homology classes. Hence the homotopy type of the A_∞ algebra only depends on the homology class $\mathfrak{a}$. Then this A_∞ algebra (or vaguely its homotopy type) is called the $\mathfrak{a}$-deformed Fukaya algebra.

Note that for every $k \geqslant 1$, we choose a collection of not-necessarily identical-pseudocycle representatives

$$(e_0^{\gamma,k}, e_1^{\gamma,k}, \ldots, e_s^{\gamma,k}),\ \gamma = 1, \ldots, k$$

all of which represent the same class $\boldsymbol{q}^{-m}\mathfrak{a}$ (see Lemma 8.7). For different real markings we use different constraints given by the above pseudocycles. One can still prove that the homotopy types of the resulting A_∞ algebras are independent of such choices. By arguments similar to those of Proposition 7.4, Proposition 8.6, Lemma 8.7 and the above construction, the A_∞ algebra we constructed is a concrete representative of the $\mathfrak{c}$-deformed Fukaya algebra. This ends the Remark.

Next we show that Hypothesis 1.7 implies the unobstructedness of $\mathrm{Fuk}^{\mathfrak{c}}(\bar{L})$.

Proposition 8.14. *When Hypothesis 1.7 is satisfied, there is a function*

$$W_{\bar{L}}^{\mathfrak{c}} : H^1(\bar{L}; \Lambda_{\geqslant 0}) \to \Lambda_{>0}$$

such that for any local system labelled by $b \in H^1(\bar{L}; \Lambda_{\geqslant 0})$,

$$\boldsymbol{n}_{0,b}(1) = W_{\bar{L}}^{\mathfrak{c}}(b)\boldsymbol{x}_M.$$

Proof. The proof is a dimension count. By the positivity of disks in $\bar{X}$ given by Hypothesis 1.7 and the positivity of affine vortices over $\boldsymbol{C}$ contained in the semi-Fano condition (see Definition 2.4), only Maslov zero or Maslov two stable treed disks contribute to $\boldsymbol{n}_{0,b}(1)$. We claim that there is no zero-dimensional moduli space of Maslov zero stable treed disks that contribute to $\boldsymbol{n}_{0,b}(1)$. Indeed, such configurations can only be trivial holomorphic disks combined with affine vortices with zero equivariant Chern number. However the trivial disks are contained in $\bar{L}$ and the evaluations of such affine vortices are away from $\bar{L}$, by Item (c) of Definition 2.4 and our choice of perturbation data (cf. Lemma 3.6). Hence only the counting of Maslov two stable treed holomorphic disks contribute to $\boldsymbol{n}_{0,b}(1)$, which gives a multiple of $\boldsymbol{x}_M$. □

8.6. **The open quantum Kirwan map.** Given $\tilde{\boldsymbol{x}} = (\boldsymbol{x}_1, \dots, \boldsymbol{x}_{\underline{k}}; \boldsymbol{x}_\infty)$. For each $k \geqslant 0$, consider the compact zero-dimensional moduli space $\mathcal{M}^0_\diamond(\tilde{\boldsymbol{x}})_k$. Define

$$\langle\!\langle \boldsymbol{x}_1, \dots, \boldsymbol{x}_{\underline{k}}; \boldsymbol{x}_\infty \rangle\!\rangle_k := \sum_{\mathcal{C} \in \mathcal{M}^0_\diamond(\tilde{\boldsymbol{x}})_k} \mathbf{Sign}(\mathcal{C}) \cdot \mathrm{Hol}_{\bar{L}}(\mathcal{C}) \in \mathbb{C},$$

and

$$\varphi_{\underline{k}}(\boldsymbol{x}_1, \dots, \boldsymbol{x}_{\underline{k}}) = \sum_{\boldsymbol{x}_\infty \in \mathbf{crit}} (-1)^{\heartsuit(\tilde{\boldsymbol{x}})} \sum_{k \geqslant 0} \frac{\boldsymbol{q}^k}{k!} \langle\!\langle \boldsymbol{x}_1, \dots, \boldsymbol{x}_{\underline{k}}; \boldsymbol{x}_\infty \rangle\!\rangle_k \cdot \boldsymbol{x}_\infty. \tag{8.8}$$

Theorem 8.15. *The collection of multilinear maps $\varphi_{\underline{k}}$ is an A_∞ morphism from $\mathrm{Fuk}^K(L)$ to $\mathrm{Fuk}^{\mathfrak{c}}(\bar{L})$. Moreover, it is a higher order deformation of the identity.*

Proof. Recall that the A_∞ axiom for $\boldsymbol{\varphi}$ holds if for all positive integers $\underline{k}$ and homogeneous elements $\boldsymbol{a}_1, \dots, \boldsymbol{a}_{\underline{k}}$ of $CF(\bar{L})$,

$$\begin{aligned} &\sum (-1)^{\mathcal{N}_j} \varphi_{\underline{k}-\underline{r}+1}(\boldsymbol{a}_1, \dots, \boldsymbol{a}_j, \boldsymbol{m}_{\underline{r}}(\boldsymbol{a}_{j+1}, \dots, \boldsymbol{a}_{j+\underline{r}}), \boldsymbol{a}_{j+\underline{r}+1}, \dots, \boldsymbol{a}_{\underline{k}}) \\ &\qquad = \sum \boldsymbol{n}^{\mathfrak{c}}_{\underline{l}}(\varphi_{\underline{r}_1}(\boldsymbol{a}_1, \dots, \boldsymbol{a}_{\underline{r}_1}), \dots, \varphi_{\underline{r}_{\underline{l}}}(\boldsymbol{a}_{\underline{k}-\underline{r}_{\underline{l}}-1}, \dots, \boldsymbol{a}_{\underline{k}})). \end{aligned} \tag{8.9}$$

Here the first summation is over all $\underline{r} \in \{0, 1, \dots, \underline{k}\}$ and all $j \in \{0, 1, \dots, \underline{k} - \underline{r}\}$; the second summation is over all $\underline{l} \in \mathbb{N}$ and all $\underline{r}_1, \dots, \underline{r}_{\underline{l}} \in \mathbb{N} \cup \{0\}$ such that $\underline{k} = \underline{r}_1 + \dots + \underline{r}_{\underline{l}}$.

To prove (8.9), it suffices consider the case that all $\boldsymbol{a}_i = \boldsymbol{x}_i$ are arbitrary elements of **crit**. Choose $\boldsymbol{x}_\infty \in \mathbf{crit}$ and denote $\tilde{\boldsymbol{x}} = (\boldsymbol{x}_1, \dots, \boldsymbol{x}_{\underline{k}}; \boldsymbol{x}_\infty)$. For $k \geqslant 0$, specialize (8.3) to the case $\mathbf{t} = \diamond$, $i = 1$. We have

$$\partial \overline{\mathcal{M}^1_\diamond(\tilde{\boldsymbol{x}})}_k = \partial_{\mathtt{bu}} \overline{\mathcal{M}^1_\diamond(\tilde{\boldsymbol{x}})}_k \sqcup \partial_{\mathtt{bd}} \overline{\mathcal{M}^1_\diamond(\tilde{\boldsymbol{x}})}_k.$$

To better describe these boundary components, introduce the following notations.

(a) Let $\mathtt{BU}$ be the set of pairs $(\tilde{\boldsymbol{y}}, \tilde{\boldsymbol{z}})$, where $\tilde{\boldsymbol{y}}$ (resp. $\tilde{\boldsymbol{z}}$) is a sequence of elements of **crit** of the form

$$\tilde{\boldsymbol{y}} = (\boldsymbol{x}_{j+1}, \dots, \boldsymbol{x}_{j+\underline{r}}; \boldsymbol{z}) \ \big(\text{resp. } \tilde{\boldsymbol{z}} = (\boldsymbol{x}_1, \dots, \boldsymbol{x}_j, \boldsymbol{z}, \boldsymbol{x}_{j+\underline{r}+1}, \dots, \boldsymbol{x}_{\underline{k}}; \boldsymbol{x}_\infty)\big)$$

for all $\boldsymbol{z} \in \mathbf{crit}$, $\underline{r} \in \{0, 1, \dots, \underline{k}\}$, and $j \in \{0, 1, \dots, \underline{k} - \underline{r}\}$. Then modulo orientations, we have

$$\partial_{\mathtt{bu}} \overline{\mathcal{M}^1_\diamond(\tilde{\boldsymbol{x}})}_k \simeq \bigsqcup_{(\tilde{\boldsymbol{y}}, \tilde{\boldsymbol{z}}) \in \boldsymbol{b}_\triangle(\tilde{\boldsymbol{x}})} \bigsqcup_{l+m=k} \frac{k!}{l!m!} \Big(\mathcal{M}^0_\triangle(\tilde{\boldsymbol{y}})_l \times \mathcal{M}^0_\diamond(\tilde{\boldsymbol{z}})_l \Big).$$

Here we have the factor $k!/l!m!$ because there are such a number of ways to distribute k leaves into the two basic parts, and, the perturbation data is independent of the distribution.

(b) Let $\mathtt{BD}$ be the set of $(\underline{l}+1)$-tuples $(\tilde{\boldsymbol{y}}_1, \dots, \tilde{\boldsymbol{y}}_{\underline{l}}; \tilde{\boldsymbol{z}})$ where

$$\tilde{\boldsymbol{y}}_i = (\boldsymbol{x}_{j_i}, \dots, \boldsymbol{x}_{j_i+\underline{r}_i}; \boldsymbol{z}_i), \ i = 1, \dots, \underline{l}, \ j_{i+1} = j_i + \underline{r}_i + 1; \ \tilde{\boldsymbol{z}} = (\boldsymbol{z}, \dots, \boldsymbol{z}_{\underline{l}}; \boldsymbol{x}_\infty)$$

for all $\underline{l} \in \mathbb{N}$, all partitions $\underline{k} = \underline{r}_1 + \dots + \underline{r}_{\underline{l}}$ into $\underline{l}$ nonnegative integers, and all choices of $(\boldsymbol{z}_1, \dots, \boldsymbol{z}_{\underline{l}}) \in (\mathbf{crit})^{\underline{l}}$. Then similar to the above case, modulo orientation one has

$$\partial_{\mathtt{bd}} \overline{\mathcal{M}^1_\diamond(\tilde{\boldsymbol{x}})}_k \simeq \bigsqcup_{(\tilde{\boldsymbol{y}}_1, \dots, \tilde{\boldsymbol{y}}_{\underline{l}}; \tilde{\boldsymbol{z}}) \in \mathtt{BD}} \bigsqcup_{r+s_1+\dots+s_{\underline{l}}=k} \frac{k!}{r! s_1! \cdots s_{\underline{l}}!} \Big(\mathcal{M}^0_\diamond(\tilde{\boldsymbol{x}}_1)_{s_1} \times \dots \times \mathcal{M}^0_\diamond(\tilde{\boldsymbol{y}}_{\underline{l}})_{s_{\underline{l}}} \times \mathcal{M}^0_\triangledown(\tilde{\boldsymbol{x}}_\triangledown)_r \Big).$$

Now we prove the A_∞ relation. We leave to the reader the check of orientations. For all $k \geqslant 0$, one has

$$\sum_{\underline{r}=0}^{\underline{k}} \sum_{j=1}^{\underline{k}-\underline{r}} \langle\!\langle \boldsymbol{x}_1, \ldots, \boldsymbol{x}_j, \boldsymbol{m}_{\underline{r}}(\boldsymbol{x}_{j+1}, \ldots, \boldsymbol{x}_{j+\underline{r}}), \boldsymbol{x}_{j+\underline{r}+1}, \ldots, \boldsymbol{x}_{\underline{k}}); \boldsymbol{x}_\infty \rangle\!\rangle_k$$

$$= \sum_{(\tilde{\boldsymbol{y}}, \tilde{\boldsymbol{z}}) \in \mathtt{BU}} \sum_{l+m=k} \langle \boldsymbol{x}_{j+1}, \ldots, \boldsymbol{x}_{j+\underline{r}}; \boldsymbol{z} \rangle_l \cdot \langle\!\langle \boldsymbol{x}_1, \ldots, \boldsymbol{x}_j, \boldsymbol{z}, \boldsymbol{x}_{j+\underline{r}+1}, \ldots, \boldsymbol{x}_{\underline{k}}; \boldsymbol{x}_\infty \rangle\!\rangle_m$$

$$= \sum_{(\tilde{\boldsymbol{y}}, \tilde{\boldsymbol{z}}) \in \mathtt{BU}} \sum_{l+m=k} \left(\frac{\tilde{\#} \mathcal{M}^0_\triangle(\tilde{\boldsymbol{y}})_l}{l!} \cdot \frac{\tilde{\#} \mathcal{M}^0_\diamond(\boldsymbol{z})_m}{m!} \right)$$

$$= \frac{1}{k!} \Big(\tilde{\#} \big(\partial_{\mathtt{bu}} \overline{\mathcal{M}^1_\diamond(\tilde{\boldsymbol{x}})}_k \big) \Big)$$

$$= \frac{1}{k!} \Big(\tilde{\#} \big(\partial_{\mathtt{bd}} \overline{\mathcal{M}^1_\diamond(\tilde{\boldsymbol{x}})}_k \big) \Big)$$

$$= \sum_{(\tilde{\boldsymbol{y}}_1, \ldots, \tilde{\boldsymbol{y}}_{\underline{l}}; \boldsymbol{z}) \in \mathtt{BD}} \sum_{r+s_1+\cdots+s_{\underline{l}}=k} \left(\frac{\tilde{\#} \mathcal{M}^0_\triangledown(\tilde{\boldsymbol{z}})_r}{r!} \times \frac{\tilde{\#} \mathcal{M}^0_\diamond(\tilde{\boldsymbol{y}}_1)_{s_1}}{s_1!} \times \cdots \times \frac{\tilde{\#} \mathcal{M}^0_\diamond(\tilde{\boldsymbol{y}}_{\underline{l}})_{s_{\underline{l}}}}{s_{\underline{l}}!} \right)$$

$$= \sum_{(\tilde{\boldsymbol{y}}_1, \ldots, \tilde{\boldsymbol{y}}_{\underline{l}}; \boldsymbol{z}) \in \mathtt{BD}} \sum_{r+s_1+\cdots+s_{\underline{l}}=k} \overline{\langle \boldsymbol{z}_1, \ldots, \boldsymbol{z}_{\underline{l}}; \boldsymbol{x}_\infty \rangle}_r \cdot \prod_{i=1}^{\underline{l}} \langle\!\langle \boldsymbol{x}_{j_i}, \ldots, \boldsymbol{x}_{j_i+\underline{r}_i}; \boldsymbol{z}_i \rangle\!\rangle_{s_i}$$

$$= \sum_{\underline{r}_1+\cdots+\underline{r}_{\underline{l}}=\underline{k}} \overline{\langle \varphi_{\underline{r}_1}(\boldsymbol{x}_1, \ldots, \boldsymbol{x}_{\underline{r}_1}), \cdots, \varphi_{\underline{r}_{\underline{l}}}(\boldsymbol{x}_{\underline{r}_1+\cdots+\underline{r}_{\underline{l}-1}+1}, \ldots, \boldsymbol{x}_{\underline{k}}) \rangle}_k$$

This finishes the proof of the A_∞ axioms for $\boldsymbol{\varphi}$. □

Remark 8.16. We comment on orientations. The cases for the two A_∞ algebras have been discussed in previous literature ([Woo11] for the quasimap Fukaya algebra and [CW15] for the Fukaya algebra of $\bar{L}$). The bulk-deformation does not affect the orientation problem since the equations on closed domains are canonically oriented.

Consider the signs in the axioms of the A_∞ morphism (see also [CW15, Theorem 3.6]). Consider a one-dimensional moduli space and its boundary components. Let us call the boundary components corresponding to degeneration of gradient lines *real boundaries* and call other boundary components *fake boundaries*. Indeed, fake boundaries can be continued to another moduli space of dimension one. The reason that gluing common fake boundaries still gives an oriented one-dimensional manifold is because of the coherence of orientations of the corresponding linearizations. Therefore, the glued one-dimensional manifold (which is compact below any given energy level) is oriented.

The consideration of the signs related to the real boundaries is the same as the case of [CW15]. Indeed, for a given type of real boundaries of a one-dimensional moduli space, which is a finite set, it is identified with a product of moduli spaces of zero dimension. As argued in the proof of [CW15, Theorem 3.6], the comparison between the orientation of the interior of the one-dimensional moduli space and the product orientation of the different factors of a real boundary only involves elementary sign checking, but not involve the form of the differential equations (holomorphic curves or vortices). Therefore, the verification of the signs can be reproduced by exactly the same way.

It is also important to consider the situation when the positivity condition of Hypothesis 1.5 and Hypothesis 1.7 are satisfied.

Proposition 8.17. *When Hypothesis 1.5 and Hypothesis 1.7 are satisfied, we have* $\varphi_0(1) = 0$.

Proof. We know that $\varphi_0(1) = O(\boldsymbol{q})$. Furthermore, by Hypothesis 1.5, Hypothesis 1.7 and the property of our perturbation data (see Lemma 3.6 and (3.1)), any treed scaled vortices that may contribute to $\varphi_0(1)$ has a nonnegative Maslov index. Indeed only objects of Maslov index zero can contribute since the lowest degree of generators of $CF(\bar{L})$ is 0. For any such configuration, the energy of components with boundary must be zero, so their images are contained in L; the energy of components without boundary are either from spheres in $\bar{X}$ with zero Chern number or affine vortices over $\boldsymbol{C}$ with zero equivariant Chern number. However, these objects are contained in S and does not intersect L. Hence there is no such configurations. So $\varphi_0(1) = 0$. □

8.7. **Strict unitality.** Strict unitality is an important property for A_∞ algebras. It plays a similar role as the *Fundamental Class* axiom of Gromov–Witten invariants. However, due to the complexity of the chain level theory in Lagrangian Floer theory the naive construction usually does not grant strict units. A usual method is to construct a weaker alternative, called a *homotopy unit* (see [FOOO09, Section 3]).

We state our theorem of such extension as follows. The detailed construction and proof are given in the appendix.

Theorem 8.18. *Let* $\mathrm{Fuk}^K(L)$ *and* $\mathrm{Fuk}^{\mathfrak{c}}(\bar{L})$ *be the* A_∞ *algebras and* $\boldsymbol{\varphi} : \mathrm{Fuk}^K(L) \to \mathrm{Fuk}^{\mathfrak{c}}(\bar{L})$ *be the* A_∞ *morphism, all of which have been constructed by choosing a coherent system of perturbation data* $\underline{P}$*. Let*

$$\widetilde{CF}^*(\bar{L}; \Lambda) = CF^*(\bar{L}) \oplus \Lambda \boldsymbol{e} \oplus \Lambda \boldsymbol{p}$$

be the graded Λ*-vector space where the degree of* $\boldsymbol{e}$ *is* 0 *and the degree of* $\boldsymbol{p}$ *is* -1*. Then there exist* A_∞ *algebra structures* $\widetilde{\mathrm{Fuk}}^K(L)$ *and* $\widetilde{\mathrm{Fuk}}^{\mathfrak{c}}(\bar{L})$ *over* $\widetilde{CF}(\bar{L}; \Lambda)$ *satisfying the following properties.*

(a) *The composition maps* $\tilde{\boldsymbol{m}}_{\underline{k}}$ *of* $\widetilde{\mathrm{Fuk}}^K(L)$ *extend* $\boldsymbol{m}_{\underline{k}}$*, namely*

$$\tilde{\boldsymbol{m}}_{\underline{k}}(\boldsymbol{a}_1, \ldots, \boldsymbol{a}_{\underline{k}}) = \boldsymbol{m}_{\underline{k}}(\boldsymbol{a}_1, \ldots, \boldsymbol{a}_{\underline{k}}),\ \forall \underline{k} \geqslant 0,\ \boldsymbol{a}_1, \ldots, \boldsymbol{a}_{\underline{k}} \in CF(\bar{L}) \subset \widetilde{CF}(\bar{L}).$$

(b) *The composition maps* $\tilde{\boldsymbol{n}}_{\underline{k}}$ *of* $\widetilde{\mathrm{Fuk}}^{\mathfrak{c}}(\bar{L})$ *extend* $\boldsymbol{n}_{\underline{k}}$*, namely*

$$\tilde{\boldsymbol{n}}_{\underline{k}}(\boldsymbol{a}_1, \ldots, \boldsymbol{a}_{\underline{k}}) = \boldsymbol{n}_{\underline{k}}(\boldsymbol{a}_1, \ldots, \boldsymbol{a}_{\underline{k}}),\ \forall \underline{k} \geqslant 0,\ \boldsymbol{a}_1, \ldots, \boldsymbol{a}_{\underline{k}} \in CF(\bar{L}) \subset \widetilde{CF}(\bar{L}).$$

(c) $\boldsymbol{e}$ *is a strict unit for both* $\widetilde{\mathrm{Fuk}}^K(L)$ *and* $\widetilde{\mathrm{Fuk}}^{\mathfrak{c}}(\bar{L})$*.*

(d) *There is a unital* A_∞ *morphism* $\tilde{\boldsymbol{\varphi}} : \widetilde{\mathrm{Fuk}}^K(L) \to \widetilde{\mathrm{Fuk}}^{\mathfrak{c}}(\bar{L})$ *which extends* $\boldsymbol{\varphi}$*, namely,*

$$\tilde{\varphi}_{\underline{k}}(\boldsymbol{a}_1, \ldots, \boldsymbol{a}_{\underline{k}}) = \varphi_{\underline{k}}(\boldsymbol{a}_1, \ldots, \boldsymbol{a}_{\underline{k}}),\ \forall \underline{k} \geqslant 0,\ \boldsymbol{a}_1, \ldots, \boldsymbol{a}_{\underline{k}} \in CF^*(\bar{L}) \subset \widetilde{CF}^*(\bar{L}).$$

Moreover, $\tilde{\boldsymbol{\varphi}}$ *is a higher order deformation of the identity.*

(e) *When Hypothesis 1.5 is satisfied, i.e., all nonconstant* J_X*-holomorphic disks upstairs have positive Maslov indices, we have*

$$\tilde{\boldsymbol{m}}_1(\boldsymbol{p}) = \boldsymbol{e} - \boldsymbol{x}_M, \qquad \tilde{\boldsymbol{m}}_{\underline{k}}(\underbrace{\boldsymbol{p}, \ldots, \boldsymbol{p}}_{\underline{k}}) = 0,\ \forall \underline{k} > 1. \tag{8.10}$$

(f) *When Hypothesis 1.5 and Hypothesis 1.7 are satisfied, we have*

$$\tilde{\boldsymbol{n}}_1(\boldsymbol{p}) = \boldsymbol{e} - \boldsymbol{x}_M, \qquad \tilde{\boldsymbol{n}}_{\underline{k}}(\underbrace{\boldsymbol{p}, \ldots, \boldsymbol{p}}_{\underline{k}}) = 0, \ \forall \underline{k} > 1, \tag{8.11}$$

and

$$\tilde{\varphi}_1(\boldsymbol{p}) = \boldsymbol{p}, \qquad \tilde{\varphi}_{\underline{k}}(\underbrace{\boldsymbol{p}, \ldots, \boldsymbol{p}}_{\underline{k}}) = 0, \ \forall \underline{k} > 1. \tag{8.12}$$

The detailed proof is based on the construction of a coherent system of perturbation data $\underline{\hat{P}}$ on the universal curve of moduli spaces of *weighted* treed disks, which is provided in Appendix 11.

8.7.1. *Potential functions.* By Proposition 8.4 a corollary is that $\underline{\varphi}$ intertwines with the potential functions of $\widetilde{\mathrm{Fuk}}^K(L)$ and $\widetilde{\mathrm{Fuk}}^{\mathfrak{c}}(\bar{L})$.

Corollary 8.19. *Let $MC^K(L)$ and $MC^{\mathfrak{c}}(\bar{L})$ be the Maurer–Cartan solution spaces of the A_∞ algebras $\widetilde{\mathrm{Fuk}}^K(L)$ and $\widetilde{\mathrm{Fuk}}^{\mathfrak{c}}(\bar{L})$ respectively. Then $\underline{\varphi}(MC^K(L)) \subset MC^{\mathfrak{c}}(\bar{L})$ and for each weakly bounding cochain $\boldsymbol{b} \in MC^K(L)$,*

$$W_L^K(\boldsymbol{b}) = W_{\bar{L}}^{\mathfrak{c}}(\underline{\varphi}(\boldsymbol{b})) \in \Lambda.$$

On the other hand, given a weakly bounding cochain $\boldsymbol{b} \in MC^K(L)$, denote $\bar{\boldsymbol{b}} := \underline{\varphi}(\boldsymbol{b}) \in MC^{\mathfrak{c}}(\bar{L})$. Then by Lemma 8.5, one has Floer cochain complexes $(\widetilde{CF}(\bar{L}) \otimes \Lambda, \boldsymbol{m}_1^{\boldsymbol{b}})$ and $(\widetilde{CF}(\bar{L}) \otimes \Lambda, \boldsymbol{n}_1^{\bar{\boldsymbol{b}}})$, and an isomorphism

$$\underline{\varphi} : \big(\widetilde{CF}^*(\bar{L}; \Lambda), \boldsymbol{m}_1^{\boldsymbol{b}}\big) \simeq \big(\widetilde{CF}^*(\bar{L}; \Lambda), \boldsymbol{n}_1^{\bar{\boldsymbol{b}}}\big).$$

Their cohomologies

$$\widetilde{QHF}^*(L, \boldsymbol{b}; \Lambda) = H^*(\widetilde{CF}^*(\bar{L}; \Lambda), \tilde{\boldsymbol{m}}_1^{\boldsymbol{b}}), \qquad \widetilde{HF}^*(\bar{L}, \bar{\boldsymbol{b}}; \Lambda) = H^*(\widetilde{CF}^*(\bar{L}; \Lambda), \tilde{\boldsymbol{n}}_1^{\bar{\boldsymbol{b}}}).$$

are called the quasimap Floer cohomology and Lagrangian Floer cohomology, respectively. The identification of Floer cohomology follows from Theorem 8.18 and Lemma 8.5:

Theorem 8.20. *For any weakly bounding cochain $\boldsymbol{b} \in MC^K(L)$, the open quantum Kirwan map gives a weakly bounding cochain $\bar{\boldsymbol{b}} \in MC^{\mathfrak{c}}(\bar{L})$ and an isomorphism of cohomology groups*

$$\widetilde{QHF}^*(L, \boldsymbol{b}; \Lambda) \simeq \widetilde{HF}^*(\bar{L}, \bar{\boldsymbol{b}}; \Lambda).$$

8.7.2. *Proof of Theorem 1.6.* Firstly, by Hypothesis 1.5 and Lemma 3.6, only quasidisks with Maslov index two contribute to $\boldsymbol{m}_{0,b}(1)$. Hence by dimension counting $\boldsymbol{m}_{0,b}(1)$ is a multiple of $\boldsymbol{x}_M$ and the coefficient is in $\Lambda_{>0}$. Then by Item (a) of Theorem 8.18, we can write

$$\tilde{\boldsymbol{m}}_{0,b}(1) = \boldsymbol{m}_{0,b}(1) = W_L^K(b)\boldsymbol{x}_M.$$

Then by Item (e) of Theorem 8.18, one has (see also [CW15, Lemma 2.40])

$$\begin{aligned}\sum_{\underline{k} \geqslant 0} \tilde{\boldsymbol{m}}_{\underline{k},b}\big(\overbrace{W_L^K(b)\boldsymbol{p}, \ldots, W_L^K(b)\boldsymbol{p}}^{\underline{k}}\big) &= \tilde{\boldsymbol{m}}_{0,b}(1) + \tilde{\boldsymbol{m}}_{1,b}(W_L^K(b)\boldsymbol{p}) \\ &= W_L^K(b)\boldsymbol{x}_M + W_L^K(b)\boldsymbol{e} - W_L^K(b)\boldsymbol{x}_M = W_L^K(b)\boldsymbol{e}. \end{aligned}\tag{8.13}$$

By definition, $W_L^K(b)\boldsymbol{p}$ is a weakly bounding cochain for $\widetilde{\mathrm{Fuk}}_b^K(L)$.

Definition 8.21. We call $W_L^K(b)\boldsymbol{p} \in \widetilde{CF}^{-1}(\bar{L}; \Lambda_{>0})$ the *canonical weakly bounding cochain* associated to b.

Theorem 1.6 then follows from the above lemma and Corollary 8.19.

8.7.3. *Proof of Theorem 1.8.* By Item (f) of Theorem 8.18,

$$\underline{\tilde{\varphi}}_b(W_L^K(b)\boldsymbol{p}) = \sum_{\underline{k}\geqslant 0} \tilde{\varphi}_{\underline{k},b}\Big(\overbrace{W_L^K(b)\boldsymbol{p}, \ldots, W_L^K(\boldsymbol{b})}^{\underline{k}}\Big) = \tilde{\varphi}_{1,b}\Big(W_L^K(b)\boldsymbol{p}\Big) = W_L^K(b)\boldsymbol{p}.$$

Proposition 8.4 tells us that $W_L^K(b)\boldsymbol{p}$ is a weakly bounding cochain of $\widetilde{\mathrm{Fuk}}_b^{\mathfrak{c}}(\bar{L})$ and $\underline{\tilde{\varphi}}_b$ intertwines with the potential functions. Hence

$$W_L^K(b)\boldsymbol{e} = \sum_{\underline{k}\geqslant 0} \tilde{\boldsymbol{n}}_{\underline{k},b}\Big(\overbrace{W_L^K(b)\boldsymbol{p}, \ldots, W_L^K(b)\boldsymbol{p}}^{\underline{k}}\Big)$$

$$= \tilde{\boldsymbol{n}}_{0,b}(1)+\tilde{\boldsymbol{n}}_{1,b}\Big(W(b)\boldsymbol{p}\Big) = \boldsymbol{n}_{0,b}(1)+\tilde{\boldsymbol{n}}_{1,b}\Big(W(b)\boldsymbol{p}\Big) = W_{\bar{L}}^{\mathfrak{c}}(b)\boldsymbol{x}_M - W(b)\boldsymbol{x}_M + W(b)\boldsymbol{e}.$$

The last three equalities follow from Theorem 8.18 and Proposition 8.14. Hence

$$W_{\bar{L}}^{\mathfrak{c}}(b) = W_L^K(b).$$

This finishes the proof of Theorem 1.8.

9. Toric Manifolds

In the last two sections of this paper we apply the open quantum Kirwan map to concrete situations. In this section we consider toric manifolds satisfying certain a nonnegativity condition. More precisely, let $\bar{X}$ be a compact toric manifold with moment polytope $P \subset \mathbb{R}^n$. The polytope gives a canonical presentation of $\bar{X}$ as GIT quotient. In particular, if P has N faces, then there is a subtorus $G = (\mathbb{C}^*)^{N-n}$ such that

$$\bar{X} = \mathbb{C}^N /\!\!/ G.$$

This nonnegativity condition stated in Definition 2.4 implies that the quotient $\bar{X}$ is semi-Fano, namely, its anti-canonical line bundle $K_{\bar{X}}$ is nef. On the other hand, the Lagrangian $\bar{L}$ is the preimage of an interior point of P under the residual moment map $\bar{X} \to P$, whose lift L is the product of N circles in $X = \mathbb{C}^N$.

There have been many mathematical works about Lagrangian Floer theory in the toric case preceding our result, see for example [CO06], [CP14], [FOOO10, FOOO11, FOOO16], [Woo11], [CL14], [GI17], [CLLT17]. Our result gives a conceptual explanation of the work of Chan–Lau–Leung–Tseng [CLLT17].

This section is organized as follows. In Subsection 9.1 we recall the basics of toric manifolds as symplectic reductions of the Euclidean space, and the Lagrangian submanifolds we are interested in. In Subsection 9.2 we prove the first two items of Theorem 1.9. In Subsection 9.3–Subsection 9.6 we calculate the potential function of the quasimap A_∞ algebra.

9.1. **Compact toric manifolds as GIT quotients.** Let us first recall the basics about compact toric manifolds. We introduce toric manifolds using Batyrev's approach [Bat93] but change a few notations. Let M be the lattice $\mathbb{Z}^n$ and $M^\vee = \mathrm{Hom}_{\mathbb{Z}}(M, \mathbb{Z})$ its dual lattice. Denote $M_{\mathbb{R}} = M \otimes_{\mathbb{Z}} \mathbb{R}$, $M^\vee_{\mathbb{R}} = M^\vee \otimes_{\mathbb{Z}} \mathbb{R}$. Let $k \geqslant 1$. A convex subset $\sigma \subset M_{\mathbb{R}}$ is a *regular k-dimensional cone* if there are k linearly independent vectors $v_1, \ldots, v_k \in M$ such that

$$\sigma = \{a_1 v_1 + \cdots + a_k v_k \mid a_1, \ldots, a_k \geqslant 0\}$$

and $v_1, \ldots, v_k$ can be extended to a $\mathbb{Z}$-basis of M. A regular cone σ' is a *face* of another regular cone σ, denoted by $\sigma' \prec \sigma$, if the set of generators of σ' is a subset of that of σ. A *complete n-dimensional fan* is a finite collection $\Sigma = \{\sigma_1, \ldots, \sigma_s\}$ of regular cones in $M_{\mathbb{R}}$ satisfying the following conditions.

(a) If $\sigma \in \Sigma$ and $\sigma' \prec \sigma$, then $\sigma' \in \Sigma$.
(b) If $\sigma, \sigma' \in \Sigma$, then $\sigma \cap \sigma' \prec \sigma$, $\sigma \cap \sigma' \prec \sigma'$.
(c) $M_{\mathbb{R}} = \sigma_1 \cup \cdots \cup \sigma_s$.

Given a complete n-dimensional fan Σ, let $G(\Sigma) = \{v_1, \ldots, v_N\} \subset M$ be the set of generators of 1-dimensional fans in Σ. $\gamma = \{v_{i_1}, \ldots, v_{i_p}\} \subset G(\Sigma)$ is called a *primitive collection* if $\{v_{i_1}, \ldots, v_{i_p}\}$ does not generate a p-dimensional cone in Σ but any proper subset of γ generate a cone in Σ. Then consider the exact sequence

$$0 \longrightarrow \mathbb{K}_\Sigma \longrightarrow \mathbb{Z}^N \xrightarrow{e_i \mapsto v_i} \mathbb{Z}^n \longrightarrow 0\ ,$$

Let $K^+ = (S^1)^N$ be the N-dimensional torus and $G^+ = (\mathbb{C}^*)^N$ be its complexification. Then $\mathbb{K}_\Sigma \simeq \mathbb{Z}^{N-n}$ generates a subgroup $G := G_\Sigma \subset G^+$, which acts on $\mathbb{C}^N$ via weights $\boldsymbol{w}_1, \ldots, \boldsymbol{w}_N \in (\mathbb{Z}^{N-n})^\vee$. The unstable locus of this action is

$$X^{\mathrm{us}} = \bigcup_\gamma X^\gamma = \bigcup_\gamma \{(x_1, \ldots, x_N) \in \mathbb{C}^N \mid x_i = 0\ \forall\ v_i \in \gamma\}. \tag{9.1}$$

Let $X = \mathbb{C}^N$. Then the G-action on $X^{\mathrm{st}} = X \smallsetminus X^{\mathrm{us}}$ is free and the quotient $\bar{X} := X^{\mathrm{st}}/G$ is a compact toric manifold, with a residual $(\mathbb{C}^*)^n$-action. On the other hand, a moment map for the K^+-action is

$$\mu_0^+(x_1, \ldots, x_N) = \left(\mu_{0,1}^+, \ldots, \mu_{0,N}^+\right) = \left(-\frac{\mathbf{i}^*|x_1|^2}{2}, \ldots, -\frac{\mathbf{i}^*|x_N|^2}{2}\right).$$

Here we view $\mathbf{i}^*$ as the generator of the dual of the Lie algebra $\mathbf{i}\mathbb{R} \simeq \mathrm{Lie}S^1$ with $\langle \mathbf{i}^*, \mathbf{i}\rangle = 1$. Then via the map $\mathfrak{k} \hookrightarrow \mathfrak{k}^+$, μ_0^+ restricts to a moment map of the G-action, which is

$$\mu_0(x_1, \ldots, x_N) = \sum_{i=1}^N |x_i|^2 \boldsymbol{w}_i \in \mathfrak{k}^\vee.$$

The Lagrangians we are interested in are Lagrangian tori $L \subset X$ of the form

$$L = \left\{(x_1, \ldots, x_N) \in X \;\middle|\; |x_i|^2 = \frac{c_i}{\pi}\right\} \simeq (S^1)^N,\ c_1, \ldots, c_N > 0. \tag{9.2}$$

Indeed the Lagrangian is the level set of the moment map μ_0^+ at

$$\mathbf{c}^+ := \left(-\frac{\mathbf{i}^* c_1}{2\pi}, \ldots, -\frac{\mathbf{i}^* c_N}{2\pi}\right).$$

Via the inclusion $\mathfrak{k} \subset \mathfrak{k}^+$, $\mathbf{c}^+$ reduces to $\mathbf{c} \in \mathfrak{k}^\vee$. Denote

$$\mu^+ = \mu_0^+ - \mathbf{c}^+, \qquad\qquad \mu = \mu_0 - \mathbf{c}.$$

Any smooth projective toric variety has a presentation as a GIT quotient. We first construct a linearization of the G-action, such that the induced moment map coincides with μ and such that L is equivariantly rational in the sense of Definition 3.1. Assume $c_i/2\pi \in \mathbb{Q}$ for all i. We define a linearization with respect to the G^+-action as follows. Let $X_i \simeq \mathbb{C}$ be the i-th factor of $X \simeq \mathbb{C}^N$ with coordinate x_i. Let $R_i \to X_i$ be the trivial line bundle with a constant Hermitian metric, equipped with the $\mathbb{C}^*$-action of weight 1. Consider the connection A_i on R_i given by

$$A_i = \mathrm{d} - \pi \mathbf{i} r_i^2 \mathrm{d}\theta_i.$$

This is an S^1-invariant connection whose curvature form is $-2\pi \boldsymbol{i} \omega_i$ where ω_i is the standard symplectic form on $\mathbb{C}$. The equivariant extension of A_i has equivariant first Chern form

$$\omega_i + \mu_{0,i}^+.$$

In our convention the constant equivariantly closed form $\mathbf{i}^*/2\pi$ is an integer class. Suppose $k > 0$ is an integer such that $kc_i/2\pi \in \mathbb{Z}$ for all i. Take the k-th tensor power of R_i and twist the $\mathbb{C}^*$-action by additional weight $kc_i/2\pi$. Notice that $A_i^{\otimes k}$ is still an invariant connection, and the corresponding equivariant connection has Chern form

$$k\omega_i + k\mu_{0,i}^+ - \frac{\mathbf{i}^* k c_i}{2\pi}.$$

Then define the equivariant line bundle

$$R = R_1 \boxtimes \cdots \boxtimes R_N \to X.$$

It is a linearization of the G^+-action whose equivariant first Chern form reads

$$\omega_X + \left(\mu_0^+ - \mathbf{c}^+\right) = \omega_X + \mu^+.$$

Restricting from G^+ to G, we obtain a linearization of the G-action on X. It is easy to verify that the unstable locus of the G-action with respect to this linearization is still X^{us} given by (9.1).

Lemma 9.1. *L is K-equivariantly rational with respect to the above linearization in the sense of Definition 3.1.*

Proof. Because $kc_i/2\pi \in \mathbb{Z}$, the connection $A_i^{\otimes k}$ restricted to $L_i \subset X_i$ has trivial holonomy. Hence $(R_i^{\otimes k}, A_i^{\otimes k})|_{L_i}$ is isomorphic to the trivial bundle with the trivial connection. The isomorphisms for all i induces an isomorphism of $(R^{\otimes k}, A^{\otimes k})|_L$ with the trivial line bundle (as both G^+-equivariant bundles and G-equivariant bundles) with the trivial connection over L. Hence Item (a) is satisfied. Moreover, an equivariant holomorphic section s_i of $R_i^{\otimes k}$ which can be written as a monomial in x_i vanishes at the origin of X_i and hence is nowhere vanishing over L_i, and certainly the trivialization of $R_i|_{L_i}$ induced by $s_i|_{L_i}$ agrees with the above trivialization of $R_i|_{L_i}$ up to homotopy. Then $s_1 \boxtimes \cdots \boxtimes s_N$ is a G^+-equivariant, hence G-equivariant holomorphic section of R which is nowhere vanishing over L, and the trivialization induced from $s|_L$ agrees with the trivialization of $R|_L$ up to homotopy. Hence Item (b) of Definition 3.1 is satisfied. □

9.2. **The nonnegativity condition.** We assume that all nontrivial J_X-affine vortices in $X = \mathbb{C}^N$ have nonnegative equivariant Chern numbers. Indeed this condition implies that all nontrivial $\bar{J}_X$-holomorphic sphere in the toric manifold $\bar{X}$ has nonnegative Chern numbers, i.e. $\bar{X}$ is semi-Fano. There is an obvious normal crossing divisor $S = S_1 \cup \cdots \cup S_N$ which is the union of the coordinate hyperplanes, and it descends to an anti-canonical divisor of $\bar{X}$. Since any nontrivial J_X-affine vortex that is not contained in S has a positive intersection number with S and any nonconstant $\bar{J}_X$-holomorphic spheres that is not contained in $\bar{S}$ has a positive intersection number with $\bar{S}$, we see that the divisor S satisfies the requirement of Definition 2.4.

Remark 9.2. We do not know if the equivariant semi-Fano condition of X is equivalent to the semi-Fano condition of $\bar{X}$. Namely, if $\bar{X}$ is a compact toric manifold with nef anti-canonical divisor, then for the GIT presentation $\bar{X} = \mathbb{C}^N /\!\!/ (\mathbb{C}^*)^{N-n}$, $X = \mathbb{C}^N$ is also equivariantly semi-Fano. This is a purely combinatorial problem and one can verify this equivalence for semi-Fano toric surfaces directly.

Since the condition in Definition 2.4 is verified, we can define the homology class $\mathfrak{c} \in H^*(\bar{X}; \Lambda_{>0})$ as well as the A_∞ algebras $\mathrm{Fuk}_b^K(L)$ and $\mathrm{Fuk}_b^{\mathfrak{c}}(\bar{L})$ for any local system $b \in H^1(\bar{L}; \Lambda_{\geqslant 0})$. Further, there is a unital A_∞ morphism

$$\boldsymbol{\varphi} = (\varphi_0, \varphi_1, \ldots) : \mathrm{Fuk}_b^K(L) \to \mathrm{Fuk}_b^{\mathfrak{c}}(\bar{L}).$$

On the other hand, by Proposition 9.6 below, (X, L) satisfies Hypothesis 1.5. Moreover, Theorem 1.6 implies:

Theorem 9.3. *For any local system $b \in H^1(\bar{L}; \Lambda_{\geqslant 0})$, both $\mathrm{Fuk}_b^K(L)$ and $\mathrm{Fuk}_b^{\mathfrak{c}}(\bar{L})$ are weakly unobstructed.*

This proves the first assertion of Theorem 1.9. To prove Item (b) of Theorem 1.9, we need to verify Hypothesis 1.7. Indeed, except for Item (a) and Item (e), other items follows either from the semi-Fano condition or Proposition 9.6 below.

Lemma 9.4. *Every nonconstant J_X-affine vortex over $\boldsymbol{H}$ has positive Maslov index.*

Proof. Let $\boldsymbol{v} = (u, \phi, \psi)$ be such an affine vortex, written in coordinates as

$$u = (u_1, \ldots, u_N),\ \phi = (\phi_1, \ldots, \phi_N),\ \psi = (\psi_1, \ldots, \psi_N).$$

Moreover, the Lagrangian L is the product $L_1 \times \cdots \times L_N$. Then since the action is linear, we see that $\boldsymbol{v}_i = (u_i, \psi_i, \psi_i)$ is actually an affine vortex over $\boldsymbol{H}$ with target (X_i, L_i). Since (X_i, L_i) is monotone, the assertion follows immediately. □

Therefore Hypothesis 1.7 is verified. By Theorem 1.8, we derive Item (b) of Theorem 1.9. We restate the conclusion.

Theorem 9.5. *For any local system $b \in H^1(\bar{L}; \Lambda_{\geqslant 0})$, $W_L^K(b) = W_{\bar{L}}^{\mathfrak{c}}(b)$.*

The remaining of this section is to explicitly calculate the potential function $W_L^K(b)$. Indeed by [CO06] and [Woo11], we see that the potential function should be equal to the Givental–Hori–Vafa potential. Their calculations are based on the fact that the standard complex structure J_X is enough to make relevant moduli spaces regular. However, if we want to keep using the standard J_X and repeat their calculations, we still need to show that for a generic choice of stabilizing divisor D, all relevant moduli spaces are still regular (including those with tangency conditions). In Subsection 9.3–9.6 we will show that such a stabilizing divisor exists.

9.3. **Quasidisks in the toric case.** Now we start to calculate the potential function in the toric case. The key point is that one can achieve transversality upstairs by using only the standard complex structure J_X on X. This requires us to choose a generic stabilizing divisor.

Firstly, one has the Blaschke formula, which classified all quasidisks in the toric case with respect to the standard complex structure.

Proposition 9.6. *Every J_X-holomorphic quasidisk $u : (\boldsymbol{D}^2, \partial \boldsymbol{D}^2) \to (X, L)$ is of the form*

$$\boldsymbol{D}^2 \ni z \mapsto \big[u_j(z)\big]_{j=1}^N := \Big[\frac{c_j e^{\mathbf{i}\theta_j}}{\pi} \prod_{k=1}^{d_j} \Big(\frac{z - \alpha_{j,k}}{1 - \overline{\alpha}_{j,k} z}\Big)\Big]_{j=1}^N. \tag{9.3}$$

Here $\alpha_{j,k}$ is in the interior of the unit disk. Its Maslov index is $2(d_1 + \ldots + d_N)$.

One sees that a disk class $B \in H_2(X, L; \mathbb{Z})$ is equivalent to a N-tuple of integers $(d_1, \ldots, d_N)$. Let $\widetilde{\mathcal{M}}_{\boldsymbol{D}^2}(B)$ be the set of quasidisks $u : (\boldsymbol{D}^2, \partial \boldsymbol{D}^2) \to (X, L)$ representing B. Then it is easy to see that $\widetilde{\mathcal{M}}_{\boldsymbol{D}^2}(B)$ is cut out transversely and has dimension $N + \mathrm{Mas}(B)$. Let $\widetilde{\mathcal{M}}_1(B)$ be the set of pairs (u, z_0) where u is a quasidisks of class B and $z_0 \in \mathbf{Int}(\boldsymbol{D}^2)$. Then $\widetilde{\mathcal{M}}_1(B)$ is regular and has dimension $N + \mathrm{Mas}(B) + 2$. Consider the following singular loci of $\widetilde{\mathcal{M}}_1(B)$.

(a) Let $\widetilde{\mathcal{M}}_1^m(B) \subset \widetilde{\mathcal{M}}_1(B)$ be the subset of pairs (u, z_0) such that for some j, u_j vanishes at z_0 with multiplicity two or more.
(b) Let $\widetilde{\mathcal{M}}_1^{\mathrm{us}}(B) \subset \widetilde{\mathcal{M}}_1(B)$ be the subset of pairs (u, z_0) such that $u(z_0)$ is in the unstable locus X^{us}, i.e., $u(z_0)$ is contained in X_γ for some primitive collection $\gamma \subset G(\Sigma)$.

The following lemma is an easy consequence of Proposition 9.6.

Lemma 9.7. *$\widetilde{\mathcal{M}}_1^m(B)$ and $\widetilde{\mathcal{M}}_1^{\mathrm{us}}(B)$ are both unions of smooth manifolds of dimensions at most $N + \mathrm{Mas}(B) - 2$.*

On the other hand, for $i = 1, \ldots, N$ and $d_i \geqslant 0$, let $\widetilde{\mathcal{M}}_{\boldsymbol{D}^2}(L_i, d_i)$ the space of maps from $(\boldsymbol{D}^2, \partial \boldsymbol{D}^2)$ to (X_i, L_i) of degree d_i. Elements of $\widetilde{\mathcal{M}}_{\boldsymbol{D}^2}(L_i, d_i)$ can be parametrized by θ_i and $\alpha_{i,1}, \ldots, \alpha_{i,d_i}$.

9.4. **The stabilizing divisor.** We choose a generic stabilizing divisor of sufficiently high degree such that the higher codimension strata of our moduli spaces are of expected dimensions so do not affect the definition of our invariants. Our stabilizing divisor will be defined as the zero set of an invariant section of the linearizing bundle. Let $R \to X$ be the line bundle linearizing the G-action, which descends to a holomorphic line bundle $\bar{R} \to \bar{X}$. For any $k \geqslant 1$, denote $\varGamma_k = H^0_G(X, R^{\otimes k})$ the space of G-equivariant holomorphic sections of $R^{\otimes k}$, which are certain subspace of polynomials on $\mathbb{C}^N$. For sufficiently large k, $\varGamma_k$ is isomorphic to $\bar{\varGamma}_k := H^0(\bar{X}, \bar{R}^{\otimes k})$. By standard knowledge of complex geometry, there exists $\underline{k} \geqslant 1$ such that for all $k \geqslant \underline{k}$, for all $\bar{x} \in \bar{X}$, one has the surjectivity of two linear maps

$$\bar{\varGamma}_k \mapsto \bar{R}_{\bar{x}}^{\otimes k}, \ \bar{s} \mapsto \bar{s}(\bar{x}); \tag{9.4}$$

$$\bar{\varGamma}_{k,\bar{x}} \mapsto T^*_{\bar{x}} \bar{X} \otimes \bar{R}_{\bar{x}}^{\otimes k}, \ \bar{s} \mapsto d\bar{s}(\bar{x}). \tag{9.5}$$

Here $\bar{\varGamma}_{k,\bar{x}} \subset \bar{\varGamma}_k$ is the subset of sections that vanish at $\bar{x}$. We assume $k \geqslant \underline{k}$. Given $f \in \varGamma_k$, denote by $D_f = f^{-1}(0)$ the equivariant stabilizing divisor and $D_f^{\mathrm{st}} := D_f \cap X^{\mathrm{st}}$ the semi-stable part. Let $\varGamma_k(L) \subset \varGamma_k$ be the open subset of f such that $D_f \cap L = \varnothing$.

The following lemma is used to show that the strata of maps meeting both the stabilizing divisor and the boundary divisor at the same time are expected dimensions. Let $B = (d_1, \ldots, d_N) \in H_2(X, L; \mathbb{Z})$ be a nonzero tuple. Let $I_B \subset \{1, \ldots, N\}$ be the subset of indices i such $d_i \neq 0$. Define

$$S_B = \Big\{ (x_1, \ldots, x_N) \ \Big| \ \|x_i\|^2 = \frac{c_i}{\pi} \ \forall i \in I_B \Big\}, \qquad S_B^* = \big\{ (x_1, \ldots, x_N) \in S_B \mid x_j \neq 0 \ \forall j \notin I_B \big\}$$

where c_i are the constants from (9.2). Since $L \subset S_B^*$, for any $f \in \Gamma_k(L)$, S_B^* is not contained in D_f.

Lemma 9.8. *There exists $i \in I_B$ such that $\frac{\partial}{\partial x_i}|_{S_B^*} \notin G_X|_{S_B^*}$.*

Proof. At any $p \in S_B^*$, the infinitesimal G-action gives a linear map

$$L_p : \mathfrak{g} \to T_pX \simeq \mathbb{C}^{I_B} \oplus \mathbb{C}^{I_B^c}, \text{ where } I_B^c = \{1, \ldots, N\} \smallsetminus I_B.$$

If the statement is not true, then the image of L_p contains $\mathbb{C}^{I_B} \oplus \{0\}$. Hence there is a subgroup of G of complex dimension at least I_B which fixes coordinates x_j for $j \in I_B^c$, whose Lie algebra is mapped onto $\mathbb{C}^{I_B}$ under L_p. Then for any $i \in I_B$, there is a one-dimensional torus in G which fixes all other coordinates but not x_i. Then the G-action on $S_i \subset X$, which is defined by $x_i = 0$, has at least a one-dimensional stabilizer. Hence S_i is contained in the unstable locus. This contradicts the fact that the unstable locus of X has complex codimension at least two. □

For each $B \neq 0$, let $i_B \in I_B$ be the smallest index such that the condition of the above lemma holds. Now for all $f \in \Gamma_k(L)$, consider the space

$$\widetilde{\mathcal{M}}_1^a(f; B) = \left\{ (u, z_0) \in \widetilde{\mathcal{M}}_1(B) \;\middle|\; u(z_0) \in D_f^{\rm st},\ \frac{\partial f}{\partial x_{i_B}}(u(z_0)) = 0 \right\}. \tag{9.6}$$

$$\widetilde{\mathcal{M}}_1^b(f; B) = \left\{ (u, z_0) \in \widetilde{\mathcal{M}}_1(B) \smallsetminus \widetilde{\mathcal{M}}_1^m(B) \;\middle|\; u(z_0) \in D_f^{\rm st},\ u_{i_B}(z_0) = 0 \right\}. \tag{9.7}$$

Lemma 9.9. *There is a comeager subset $\Gamma_k^{\rm reg} \subset \Gamma_k(L)$ such that for all $f \in \Gamma_k^{\rm reg}$ and all $B \in \pi_2(X, L)$, $\widetilde{\mathcal{M}}_1^a(f; B)$ and $\widetilde{\mathcal{M}}_1^b(f; B)$ are smooth manifolds of dimension $N + \mathrm{Mas}(B) - 2$.*

Proof. Consider the smooth map

$$\mathcal{F}_a : \Gamma_k(L) \times \widetilde{\mathcal{M}}_1(B) \to \mathbb{C} \times \mathbb{C},$$
$$(f, u, z_0) \mapsto \left(f(u(z_0)),\ \frac{\partial f}{\partial x_{i_B}}(u(z_0)) \right).$$

Let the infinitesimal deformations of (f, u, z_0) be denoted by (s, ξ, w). Suppose $(f, u, z_0) \in \mathcal{F}_a^{-1}(0)$ and $p_0 := u(z_0) \in D_f^{\rm st}$. The partial derivative of the above map at (f, u, z_0) in the f direction is

$$s \mapsto \left(s(p_0),\ \frac{\partial s}{\partial x_{i_B}}(p_0) \right) \tag{9.8}$$

Since $\frac{\partial}{\partial x_{i_B}}(p_0) \notin G_X|_{p_0}$, by the surjectivity of (9.4) and (9.5), one sees that the linear map (9.8) is surjective if $p_0 \in D_f^{\rm st}$. So $\mathcal{F}_a^{-1}(0)$ is a smooth manifold in the locus where $p_0 \in D_f^{\rm st}$. Then consider the projection $\mathcal{F}_a^{-1}(0) \to \Gamma_k(L)$. By Sard's theorem, the set of regular values in $\Gamma_k(L)$ is a comeager subset $\Gamma_k^{\rm reg}(B)$. Hence all $f \in \Gamma_k^{\rm reg}(B)$, the condition for $\widetilde{\mathcal{M}}_1^a(f; B)$ holds.

Now consider

$$\mathcal{F}_b : \Gamma_k(L) \times \left[\widetilde{\mathcal{M}}_1(B) \smallsetminus \widetilde{\mathcal{M}}_1^m(B) \right] \to \mathbb{C} \times \mathbb{C}$$
$$(f, u, z_0) \mapsto \big(f(u(z_0)), u_{i_B}(z_0) \big).$$

Suppose $(f, u, z_0) \in \mathcal{F}_b^{-1}(0)$ and $p_0 := u(z_0) \in D_f^{\rm st}$. The derivative in the (f, u) direction reads

$$(s, \xi) \mapsto \big(s(p_0) + \mathrm{d}f(p_0) \cdot \xi(z_0),\ \xi_{i_B}(z_0) \big). \tag{9.9}$$

We want to show that this linear map is surjective. By the surjectivity of (9.4), it suffices to show that the linear map $\xi_{i_B} \mapsto \xi_{i_B}(z_0)$, from $T_{u_{i_B}}\widetilde{\mathcal{M}}_{\boldsymbol{D}^2}(L_{i_B}, d_{i_B})$ to $\mathbb{C}$ is surjective. Indeed, suppose

$$u_{i_B}(z) = \sqrt{\tau_{i_B}} e^{\mathbf{i}\theta_{i_B}} \prod_{k=1}^{d_{i_B}} \left(\frac{z - \alpha_{i_B,k}}{1 - \overline{\alpha}_{i_B,k} z} \right).$$

By reparametrizing the domain, we can assume $z_0 = 0$. Then since $(u, z_0) \notin \widetilde{\mathcal{M}}_1^m(B)$, there is at most one $\alpha_{i_B,k}$ equal to zero. Then by (9.14) below, one sees that for the infinitesimal deformation ξ_{i_B} corresponding to the deformation of $\alpha_{i_B,k}$, its evaluation at $z_0 = 0$ is surjective onto $\mathbb{C}$. Therefore (9.9) is surjective and hence $\mathcal{F}_b^{-1}(0)$ is a smooth manifold in the locus where $u(z_0) \in D_f^{\mathrm{st}}$. By considering the projection onto $\Gamma_k(L)$ and using Sard's theorem, one obtain another comeager subset of $\Gamma_k(L)$ consisting of elements f for which $\widetilde{\mathcal{M}}_1^b(f; B)$ satisfies the condition in the statement of this lemma. Since there are only countably many different classes B and the intersection of countably many comeager subsets is still a comeager subset, the lemma holds true. ☐

Now consider another moduli space for each $f \in \Gamma_k(L)$. Define

$$\widetilde{\mathcal{M}}_1^c(f; B) = \left\{ (u, z_0) \in \widetilde{\mathcal{M}}_1(B) \;\middle|\; \begin{array}{c} u(z_0) \in D_f^{\mathrm{st}},\ u_{i_B}(z_0) \neq 0, \\ \dfrac{\partial f}{\partial x_{i_B}}(u(z_0)) \neq 0,\ \mathrm{d}f(u(z_0)) \cdot du(z_0) = 0 \end{array} \right\}. \tag{9.10}$$

Lemma 9.10. *For all $k \geqslant \underline{k}$, there exists a comeager subset $\Gamma_k^{\mathrm{reg}} \subset \Gamma_k(L)$ such that for all $f \in \Gamma_k^{\mathrm{reg}}$, $\widetilde{\mathcal{M}}_1^c(f; B)$ is a smooth manifold of dimension $N + \mathrm{Mas}(B) - 2$.*

Proof. Consider the differentiable map

$$\begin{aligned} \mathcal{F}_c : \Gamma_k(L) \times \widetilde{\mathcal{M}}_1(B) &\to \mathbb{C} \times \mathbb{C} \\ (f, u, z_0) &\mapsto \left(f(u(z_0)),\ \mathrm{d}f(u(z_0)) \cdot \frac{\partial u}{\partial z}(z_0) \right). \end{aligned} \tag{9.11}$$

Suppose $(f, u, z_0) \in \mathcal{F}_c^{-1}(0)$ and $p_0 := u(z_0) \in D_f^{\mathrm{st}}$. We discuss in two separate cases.

(a) Suppose $\frac{\partial u}{\partial z}(z_0) \notin G_X$. The derivative of $\mathcal{F}_b$ at (f, u, z_0) in the f direction is

$$s \mapsto \left(s(p_0),\ \mathrm{d}s(p_0) \cdot \frac{\partial u}{\partial z}(z_0) \right)$$

Since $p_0 \in X^{\mathrm{st}}$, one can decompose $\frac{\partial u}{\partial z}(0) = \mathcal{X}_{\mathfrak{a}}(p_0) + \eta$ where $\mathfrak{a} \in \mathfrak{g}$ and $\eta \in G_X^{\perp}|_{p_0}$. Then $\eta \neq 0$ and

$$\left(s(p_0),\ \mathrm{d}s(p_0) \cdot \frac{\partial u}{\partial z}(z_0) \right) = \left(s(p_0),\ \rho_R^k(\mathfrak{a}) s(p_0) + \mathrm{d}s(p_0) \cdot \eta \right). \tag{9.12}$$

The surjectivity of this map follows from the surjectivity of (9.4) and (9.5).

(b) Suppose $\frac{\partial u}{\partial z}(z_0) = \mathcal{X}_{\mathfrak{a}}(p_0)$. Then by (9.12) the partial derivative in the f direction has image equal to

$$H = \left\{ (w_1, w_2) \in \mathbb{C} \times \mathbb{C} \mid w_2 = \rho_R^k(\mathfrak{a}) w_1 \right\}$$

We will use the partial derivative of $\mathcal{F}_c$ in the u direction to cover a complement of H.

Now we calculate the derivative of $\mathcal{F}_c$ in the u direction. Firstly, we write the action of $e^{t\mathfrak{a}}$ on $\mathbb{C}^N$ as

$$e^{t\mathfrak{a}}(x_1, \dots, x_N) = (e^{ta_1}x_1, \dots, e^{ta_N}x_N),\ a_1, \dots, a_N \in \mathbb{C}.$$

Since f is homogeneous, one has $(e^{t\mathfrak{a}})^*\mathrm{d}f = e^{t\rho_R^k(\mathfrak{a})}\mathrm{d}f$. Equivalently,

$$\frac{\partial f}{\partial x_i}(e^{ta_1}x_1, \dots, e^{ta_N}x_N)\mathrm{d}(e^{ta_i}x_i) = e^{t\rho_R^k(\mathfrak{a})}\frac{\partial f}{\partial x_i}\mathrm{d}x_i.$$

Differentiating in t gives

$$\frac{\partial^2 f}{\partial x_i \partial x_j}a_j + \frac{\partial f}{\partial x_i}a_i = \rho_R^k(\mathfrak{a})\frac{\partial f}{\partial x_i}.$$

Then the linearization in ξ direction is

$$\begin{aligned}
\xi \mapsto & \left(\frac{\partial f}{\partial x_i}(p_0)\cdot\xi_i(z_0),\ \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0}\left(\frac{\partial f}{\partial x_i}(p_0 + t\xi(z_0))\cdot\frac{\partial(u_i + t\xi_i)}{\partial z}(z_0)\right)\right)\\
&= \left(\frac{\partial f}{\partial x_i}(p_0)\cdot\xi_i(z_0),\ \frac{\partial^2 f}{\partial x_i\partial x_j}(p_0)a_i\xi_j(z_0) + \frac{\partial f}{\partial x_i}(p_0)\cdot\frac{\partial \xi_i}{\partial z}(z_0)\right)\\
&= \left(\frac{\partial f}{\partial x_i}(p_0)\cdot\xi_i(z_0),\ \frac{\partial f}{\partial x_i}(p_0)\cdot\left(\rho_R^k(\mathfrak{a})\xi_i(z_0) - a_i\xi_i(z_0) + \frac{\partial \xi_i}{\partial z}(z_0)\right)\right).
\end{aligned}$$

Since $\frac{\partial f}{\partial x_{i_B}}(p_0) \neq 0$, compare to the description of $H \subset \mathbb{C}\times\mathbb{C}$, it suffices to show that the linear map from $\widetilde{\mathcal{M}}_{\boldsymbol{D}^2}(L_{i_B}, d_{i_B})$ to $\mathbb{C}$, defined by

$$\xi_{i_B} \mapsto \frac{\partial \xi_{i_B}}{\partial z}(z_0) - a_{i_B}\xi_{i_B}(z_0) \tag{9.13}$$

is surjective. By reparametrizing the domain, we may assume that $z_0 = 0$. By Lemma 9.11 below, one sees that restricting to the subspace of deformations that parametrizes the deformation of $\alpha_{i_B,k}$, the linear map (9.13) can be expressed as

$$\left(\frac{\partial}{\partial \alpha_{i_B,k}},\ \frac{\partial}{\partial \overline{\alpha}_{i_B,k}}\right) \mapsto (-1)^{d_{i_B}}\sqrt{\tau_{i_B}}e^{\mathbf{i}\theta_{i_B}}\prod_{l\neq k}\alpha_{i_B,l}\left(\frac{1}{\alpha_{i_B,k}},\ \alpha_{i_B,k}\right).$$

Since all $\alpha_{i_B,l}$ are nonzero and $\|\alpha_{i_B,k}\| < 1$, it corresponds to a nondegenerate real 2×2 matrix. Therefore (9.13) is surjective. Hence the partial derivative of (9.11) in the f and u directions is surjective.

We have proved that $\mathcal{F}_c^{-1}(0)$ is a smooth manifold in the locus where $u(z_0) \in D_f^{\mathrm{st}}$, $u_{i_B}(z_0) \neq 0$ and $\frac{\partial f}{\partial x_{i_B}}(u(z_0)) \neq 0$. Using Sard's theorem and the implicit function theorem, one finds the comeager subset $\Gamma_k^{\mathrm{reg}} \subset \Gamma_k(L)$ that satisfies our requirement. □

The following technical result was used in the above proof.

Lemma 9.11. *Suppose $u : \boldsymbol{D}^2 \to \mathbb{C}$ is given by*

$$u(z) = \sqrt{\tau}e^{\mathbf{i}\theta}\prod_{k=1}^{d}\left(\frac{z - \alpha_k}{1 - \overline{\alpha}_k z}\right).$$

Then one has

$$\frac{\partial u}{\partial \alpha_k}(0) = (-1)^d\sqrt{\tau}e^{\mathbf{i}\theta}\prod_{l\neq k}\alpha_l, \qquad \frac{\partial u}{\partial \overline{\alpha}_k}(0) = 0. \tag{9.14}$$

Moreover, if $\alpha_1, \dots, \alpha_d$ are all nonzero, then

$$\frac{\partial}{\partial \alpha_k}\big(u'(0)\big) = (-1)^d \sqrt{\tau} e^{\mathbf{i}\theta} \Big[\prod_{l \neq k} \alpha_l\Big] \Big[\frac{u'(0)}{u(0)} + \frac{1}{\alpha_k}\Big], \tag{9.15}$$

$$\frac{\partial}{\partial \overline{\alpha}_k}\big(u'(0)\big) = (-1)^d \sqrt{\tau} e^{\mathbf{i}\theta} \prod_{l=1}^d \alpha_l. \tag{9.16}$$

Proof. The two identities in (9.14) and (9.16) are obvious. For (9.15), denote $a = u'(0)/u(0)$. Then

$$u'(0) = \sqrt{\tau} e^{\mathbf{i}\theta} \sum_{k=1}^d \Big[(-1)^d (|\alpha_k| - 1^2) \prod_{l \neq k} \alpha_l\Big] = a u(0) = (-1)^d a \sqrt{\tau} e^{\mathbf{i}\theta} \prod_{k=1}^d \alpha_k.$$

Therefore,

$$a = \sum_{k=1}^d \frac{|\alpha_k|^2 - 1}{\alpha_k} = \sum_{k=1}^d \overline{\alpha}_k - \frac{1}{\alpha_k}.$$

Therefore

$$\begin{aligned}\frac{\partial}{\partial \alpha_k}\big(u'(0)\big) &= (-1)^d \sqrt{\tau} e^{\mathbf{i}\theta} \overline{\alpha}_k \prod_{l \neq k} \alpha_l + \sum_{l \neq k} (|\alpha_k|^2 - 1) \prod_{l' \neq k, l} \alpha_{l'} \\ &= (-1)^d \sqrt{\tau} e^{\mathbf{i}\theta} \Big[\prod_{l \neq k} \alpha_l\Big] \Big[\overline{\alpha}_k + \sum_{l \neq k}\Big(\overline{\alpha}_l - \frac{1}{\alpha_l}\Big)\Big] \\ &= (-1)^d \sqrt{\tau} e^{\mathbf{i}\theta} \Big[\prod_{l \neq k} \alpha_l\Big] \Big[a + \frac{1}{\alpha_k}\Big].\end{aligned}$$

So (9.15) holds. □

Let us summarize the transversality results for a generic choice of $f \in \Gamma_k(L)$.

Proposition 9.12. *There exists $\underline{k} \geqslant 1$ such that for all $k \geqslant \underline{k}$, there exists a comeager subset $\Gamma_k^{\mathrm{reg}} \subset \Gamma_k(L)$ such that for all $f \in \Gamma_k^{\mathrm{reg}}$ and for all $B \in H_2(X, L; \mathbb{Z})$, the following conditions hold.*

(a) $D_f^{\mathrm{st}} := D_f \cap X^{\mathrm{st}}$ is a smooth submanifold disjoint from L.

(b) $\widetilde{\mathcal{M}}_1^a(f; B)$ defined by (9.6), $\widetilde{\mathcal{M}}_1^b(f; B)$ defined by (9.7), and $\widetilde{\mathcal{M}}_1^c(f; B)$ defined by (9.10) are smooth manifolds of dimension $N + \mathrm{Mas}(B) - 2$.

Now define

$$\widetilde{\mathcal{M}}_1^*(f; B) = \widetilde{\mathcal{M}}_1(f; B) \smallsetminus \Big(\widetilde{\mathcal{M}}_1^m(B) \cup \widetilde{\mathcal{M}}_1^{\mathrm{us}}(B) \cup \widetilde{\mathcal{M}}_1^a(f; B) \cup \widetilde{\mathcal{M}}_1^b(f; B) \cup \widetilde{\mathcal{M}}_1^c(f; B)\Big).$$

This is the moduli space of disks in class B with one marked point such that the disk intersects transversely with D_f^{st} at the marked point.

Now consider a pair $f_1, f_2 \in \Gamma_k^{\mathrm{reg}}$, defining a pair of divisors D_{f_1}, D_{f_2}. For any $B \in H_2(X, L; \mathbb{Z})$, consider

$$\widetilde{\mathcal{M}}_1(f_1, f_2; B) = \Big\{(u, z_0) \in \widetilde{\mathcal{M}}_1^*(f_1; B) \cap \widetilde{\mathcal{M}}_1^*(f_2; B) \mid u(z_0) \in D_{f_1} \cap D_{f_2}\Big\}.$$

Lemma 9.13. *There exists a comeager subset $\Gamma_{k,n+1}^{\mathrm{reg}} \subset \Gamma_k(L)^{n+1}$ such that any $(f_0, f_1, \dots, f_n) \subset \Gamma_{k,n+1}^{\mathrm{reg}}$ satisfies the following conditions.*

(a) Each f_j is in Γ_k^{reg}.

(b) $D_{f_0}, \dots, D_{f_n}$ *intersect transversely in* $X^{\rm st}$.

(c) *For any* f_i, f_j, $\widetilde{\mathcal{M}}_1(f_i, f_j; B)$ *is a smooth manifold of dimension* $N + {\rm Mas}(B) - 2$.

Proof. Notice that the finite product of comeager subsets of $\Gamma_k(L)$ is still a comeager subset, we can take $(f_0, f_1, \dots, f_n)$ from $(\Gamma_k^{\rm reg})^{n+1}$. The second condition is also satisfied by a generic collection. Lastly, the third condition can be achieved by a generic collection because the intersections of the disks with D_{f_1} and D_{f_2} are transversal and the map (9.4) is surjective. □

9.5. **Transversality for stable quasidisks.** Now consider the regularity of stable quasidisks which may have a singular domain. From now on we fix $k \geqslant \underline{k}$ and a choice of $(f_0, f_1, \dots, f_n) \in \Gamma_{k,n+1}^{\rm reg}$ given by Lemma 9.13. Abbreviate D_{f_j} by D_j.

First recall the following regularity result.

Theorem 9.14. [Woo11, Corollary 6.2] (cf. [CO06, Section 6]) *In the toric case every stable quasidisk is regular.*

Next, consider the moduli space of quasidisks with transversal intersections with all $D_j^{\rm st}$. Consider a stable tree $\Gamma \in \mathbf{T}^{\mathbf{st}}$ satisfying: 1) ${\rm T}(\Gamma) = \varnothing$; 2) $\underline{\Gamma} = \Gamma$; 3) ${\rm E}(\Gamma) = {\rm E}^0(\Gamma)$. Let $\tilde{\Gamma}$ be an admissible refinement, i.e., to each vertex $\mathfrak{v}_\alpha \in {\rm V}(\Gamma)$ we assign a class $B_\alpha \in H_2(X, L)$, and to each leaf l_i we assign the submanifold

$$V_i = D^* := D_0^0 \cup D_1^0 \cdots \cup D_n^0$$

and $\tilde{\mathfrak{o}}(l_i) = 1$ such that

$$\#{\rm L}(\mathfrak{v}_\alpha) = n_D \langle \omega_X, B_\alpha \rangle.$$

Let $\mathcal{M}_{\tilde{\Gamma}}$ be the space of marked stable quasidisks with underlying type $\tilde{\Gamma}$. By the positivity of local intersection numbers, for any $\mathcal{C} \in \mathcal{M}_{\tilde{\Gamma}}$, the intersections at all interior markings with D^* are transverse and has no other intersections with D. Combining with Theorem 9.14, we have the following result.

Corollary 9.15. *Given a vortex type* $\tilde{\Gamma}$ *as above, the moduli space* $\mathcal{M}_{\tilde{\Gamma}}$ *is cut out transversely and is a smooth oriented manifold of dimension*

$$\mathbf{dim}_{\mathbb{R}} \mathcal{M}_{\tilde{\Gamma}} = n + \sum_{\mathfrak{v}_\alpha} {\rm Mas}(B_\alpha) - \#{\rm E}(\Gamma) - 3.$$

Now we consider the dimensions of various singular loci. Consider a tree $\Gamma \in \mathbf{T}$ of type $\triangle$ (not necessarily stable) satisfying: 1) ${\rm T}(\Gamma) = \varnothing$; 2) $\underline{\Gamma} = \Gamma$; $E(\Gamma) = E^0(\Gamma)$; 4) $\#{\rm L}(\Gamma) = 1$. Consider a collection $B_\alpha \in H_2(X, L; \mathbb{Z})$ for all $\mathfrak{v}_\alpha \in {\rm V}(\Gamma)$ such that $\langle \omega_X, B_\alpha \rangle > 0$ if $\mathfrak{v}_\alpha$ is unstable. Let $\mathfrak{v}_0$ be the vertex that contain the only leaf l_0.

Let $\mathcal{M}_\Gamma^{\rm sing}(B_\alpha)$ be the moduli space of stable quasidisks with one marked point (u_α, z_∞) such that $u_\alpha \in \widetilde{\mathcal{M}}_{\boldsymbol{D}^2}(B_\alpha)$ and

$$(u_0, z_0) \in \widetilde{\mathcal{M}}_1^m(B_0) \cup \widetilde{\mathcal{M}}_1^{\rm us}(B_0) \cup \Big[\bigcup_{0 \leqslant i < j \leqslant n} \widetilde{\mathcal{M}}_1(f_i, f_j; B_0) \Big]$$
$$\cup \Big[\bigcup_{j=0}^n \big(\widetilde{\mathcal{M}}_1^a(f_j; B_0) \cup \widetilde{\mathcal{M}}_1^b(f_j; B_0) \cup \widetilde{\mathcal{M}}_1^c(f_j; B_0) \big) \Big].$$

By combining the previous result (Proposition 9.12 and Lemma 9.13), one obtains

Corollary 9.16. $\mathcal{M}_\Gamma^{\rm sing}(B_\alpha)$ *is the union of finitely many smooth manifolds whose dimensions are at most* $n + \sum_\alpha m(B_\alpha) - \#{\rm E}(\Gamma) - 5$.

9.6. **Transversality for treed quasidisks.** Now we show that to define $\mathrm{Fuk}^K(L)$, we only need to perturb the gradient flow equation. This is similar to the situation of [Woo11]. We first define a new class of vortex combinatorial types.

Definition 9.17. Let $\tilde{\Gamma}$ be a stable vortex combinatorial type of type $\triangle$. We say that it is *weakly admissible* (compare to Definition 7.11) if the following conditions are satisfied.

(a) $\tilde{\Gamma}$ is reduced (which means $\underline{\Gamma} = \Gamma$). In particular, $\tilde{\Gamma}$ is uncrowded.
(b) For each vertex $\mathfrak{v}_\alpha \in \mathrm{V}(\Gamma)$ one of the following holds.
    (i) All leaves on $\mathfrak{v}_\alpha$ are labelled by the regular labelling triple $\mathcal{V}_0$ and the number of leaves satisfies $\#\mathrm{L}(\mathfrak{v}_\alpha) = N_D E(B_\alpha)$, including the situation that $\mathfrak{v}_\alpha$ has no interior leaves.
    (ii) There is only one leaf attached to $\mathfrak{v}_\alpha$ labelled by one of the singular labelling triples.

Let $\Gamma \in \mathbf{T}^{\mathbf{st}}_{\triangle}$ be a stable type. Recall that one can write

$$\mathcal{W}_\Gamma = \mathcal{W}^2_\Gamma \times \mathcal{W}^1_\Gamma.$$

Here $\mathcal{W}^2_\Gamma$ is the product of certain moduli spaces of marked disks parametrizing conformal structures of the two-dimensional components (with markings), and $\mathcal{W}^1_\Gamma$ is the product of certain copies of $\mathbb{R}_+$ parametrizing lengths of edges. Then it is easy to see that $\mathcal{U}^1_\Gamma \to \mathcal{W}_\Gamma$ is pulled back from a fibration $\mathcal{O}^1_\Gamma \to \mathcal{W}^1_\Gamma$. We say that a perturbation data $P_\Gamma = (F_\Gamma, J_\Gamma)$ is *of restricted type* if the following conditions hold.

(a) $J_\Gamma \equiv J_X$;
(b) $F_\Gamma : \overline{\mathcal{U}}^1_\Gamma \to C^\infty(\bar{L})$ is pulled back from certain $F_\Gamma : \mathcal{O}^1_\Gamma \to C^\infty(\bar{L})$.

In this subsection we only consider perturbation data of restricted type. To save notations we just call them perturbation data and denote one by F_Γ. A system of perturbation data $\underline{F}_\triangle = \{F_\Gamma \mid \Gamma \in \mathbf{T}^{\mathbf{st}}_\triangle\}$ is called coherent if the corresponding system $\{(J_\Gamma, F_\Gamma) \mid \Gamma \in \mathbf{T}^{\mathbf{st}}_\triangle\}$ is coherent in the sense of Definition 5.5.

Recall that in the general case, we only have transversality for reduced, uncrowded and controlled combinatorial types. In the definition of the composition maps and the A_∞ morphisms, we only need more special vortex types called admissible ones (see Definition 7.11). Indeed when considering only vortex combinatorial types of type $\triangle$, we do not need the controlledness.

Definition 9.18. A coherent system of perturbation $\underline{F}_\triangle = \left\{F_\Gamma \mid \Gamma \in \mathbf{T}^{\mathbf{st}}_\triangle\right\}$ of restricted type is called *weakly regular* if the following condition holds.

(a) For all weakly admissible vortex types $\tilde{\Gamma}$ with underlying type Γ, the moduli space $\mathcal{M}_{\tilde{\Gamma}}(F_\Gamma)$ is cut out transversely.
(b) For all admissible vortex types $\tilde{\Gamma}$, there exist finitely many vortex combinatorial types $\tilde{\Gamma}_a \tilde{\triangleleft} \tilde{\Gamma}$ and finitely many other weakly admissible vortex types $\tilde{\Pi}_b$ such that
    - $\mathbf{ind}\tilde{\Gamma}_a = \mathbf{ind}\tilde{\Gamma} - 1$ and $\mathbf{ind}\tilde{\Pi}_b \leqslant \mathbf{ind}\tilde{\Gamma} - 2$.
    - There is a map

    $$f_{\tilde{\Gamma}} : \overline{\mathcal{M}_{\tilde{\Gamma}}(F_\Gamma)} \smallsetminus \left[\mathcal{M}_{\tilde{\Gamma}}(F_\Gamma) \cup \bigsqcup_a \mathcal{M}_{\tilde{\Gamma}_a}(F_{\Gamma_a})\right] \to \bigsqcup_b \mathcal{M}_{\tilde{\Pi}_b}(F_{\Pi_b}).$$

    In particular, if $\mathbf{ind}\tilde{\Gamma} \leqslant 1$, which implies the target set of the above map is empty, then the domain set is also empty.

Using the same induction process as in Section 7, and the transversality for single disks with various singular labelling triples at one interior marking (Corollary 9.15 and Corollary 9.16). One can prove the following transversality result. The details are left to the reader.

Lemma 9.19. *There exists a weakly regular coherent system of perturbation data* $\underline{F}_{\triangle}$ *of restricted type.*

On the other hand one can construct a strongly regular coherent system of perturbation data $\underline{P}_{\triangledown} = \{P_\Gamma \mid \Gamma \in \mathbf{T}^{\mathbf{st}}_{\triangledown}\}$, which is independent of the choice of the one $\underline{F}_{\triangle}$ in Lemma 9.19, in the same way as in Subsection 7.4. They can be extended to a strongly regular coherent system of perturbation data $\underline{P}$ over $\mathbf{T}^{\mathbf{st}}$. They allow us to define the A_∞ algebras $\mathrm{Fuk}^K(L)$, $\mathrm{Fuk}^{\mathfrak{c}}(\bar{L})$ and the A_∞ morphism $\boldsymbol{\varphi}$.

It then implies the following explicit calculation of the potential function. Notice that every local system $b \in H^1(\bar{L}; \Lambda_{\geqslant 0})$ can be expressed in coordinates as $b = (b_1, \ldots, b_N)$. However $b_1, \ldots, b_N$ are not independent in order to descend to $\bar{L}$.

Theorem 9.20. *For any local system* $b \in H^1(\bar{L}; \Lambda_{\geqslant 0})$*, we have*

$$W_L^K(b) = \sum_{i=1}^N \exp(b_i) \boldsymbol{q}^{\tau_i}. \tag{9.17}$$

Here the right hand side coincides with the Givental–Hori–Vafa potential.

Proof. The argument is almost the same as in [CO06] and [Woo11]. By the unobstructedness result of Theorem 9.3 we have $\boldsymbol{m}_{0,b}(1) = W_L^K(b)\boldsymbol{x}_M$. Further, by Proposition 9.6 there are exactly N quasidisks of Maslov index two (modulo reparametrization and K-action) that pass through the K-orbit of $\boldsymbol{x}_M$, whose boundaries corresponding to the generating loops of $H_1(L)$. Then the holonomies of b along these loops are $\exp b_1, \ldots, \exp b_N \in \Lambda^\times$, and their areas are $\tau_1, \ldots, \tau_N$. One can also check the orientations as did in [Woo11] and each disk contributes positively to the counting. Hence by the definition of the composition maps, (9.17) follows. □

Theorem 9.3, Theorem 9.5 and Theorem 9.20 constitute Theorem 1.9 in the introduction.

10. Polygon spaces

In this section we consider the case of a non-abelian symplectic quotient, that of the diagonal action of rotations on a product of two-spheres. The quotient $\bar{X}$ is a moduli space of $(2l+3)$-gons in $\mathbb{R}^3$, and where $\bar{L}$ is diffeomorphic to the product of l spheres.

10.1. **Basic facts about the polygon spaces.** Let (S^2, ω_{S^2}) be the unit sphere in $\mathbb{R}^3$ equipped with the Fubini–Study form ω_{S^2}. Here we distinguish from the notation $\boldsymbol{S}^2$ which is used for domains. Viewing each point of S^2 as a unit vector in $\mathbb{R}^3$, then $K = SO(3)$ acts diagonally on the product

$$X := X_{2l+3} := (S^2)^{2l+3}$$

Thus X is monotone but not aspherical. Since X is Kähler and the $SO(3)$ action preserves the complex structure, the action extends to an action by $PSL(2) = K^{\mathbb{C}}$.

A moment map of the $SO(3)$ action is

$$\mu(x_1, \ldots, x_{2l}, x_{2l+1}, x_{2l+2}, x_{2l+3}) = \sum_{i=1}^{2l+3} x_i \in \mathbb{R}^3 \simeq \mathfrak{so}(3).$$

Then the GIT quotient $\bar{X} = \mu^{-1}(0)/K$ can be viewed as the moduli space of equilateral $(2l+3)$-gons in $\mathbb{R}^3$ modulo rigid body motion.

Let $L_l \subset X_{2l+3}$ be the Lagrangian

$$L_l := (\bar{\Delta}_{S^2})^l \times \Delta_3$$

where each $\bar{\Delta}_{S^2}$ is the anti-diagonal of the product of the $2j-1$ and $2j$-th factors of X, and $\Delta_3 \subset S^2 \times S^2 \times S^2$ is the set of vectors $(x_{2l+1}, x_{2l+2}, x_{2l+3})$ such that $x_{2l+1} + x_{2l+2} + x_{2l+3} = 0$. It is easy to see that L is a compact embedded $SO(3)$-Lagrangian, i.e., it is contained in $\mu^{-1}(0)$ and is $SO(3)$-invariant. Then L descends to $\bar{L} = L/SO(3) \subset \bar{X}$. It is easy to see that the $SO(3)$ action on the Δ_3 factor is free and transitive. Hence $\bar{L}$ is diffeomorphic to the product of l two-spheres in the polygon space $\bar{X}$.

It is a simple exercise to verify the equivariant rationality of L_l.

Lemma 10.1. *The $PSL(2)$-action on X_{2l+3} has a linearization with respect to which L_l is equivariantly rational.*

10.2. **Monotonicity.** It is standard knowledge that $(S^2 \times S^2, \bar{\Delta}_{S^2})$ is monotone, having minimal Maslov index four. If we take the standard Fubini–Study symplectic form with volume 1 on each S^2-factor, then the monotonicity constant of $(S^2 \times S^2, \bar{\Delta}_{S^2})$ is the same as that of S^2.

Now consider $H_2(S^2 \times S^2 \times S^2, \Delta_3)$. Since $\Delta_3 \simeq SO(3) \simeq \mathbb{RP}^3$, the homology exact sequence

$$H_2(\Delta_3) \to H_2(S^2 \times S^2 \times S^2) \to H_2(S^2 \times S^2 \times S^2, \Delta_3) \to H_1(\Delta_3) \to H_1(S^2 \times S^2 \times S^2)$$

in integer coefficients is isomorphic to

$$0 \to \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z} \to H_2(S^2 \times S^2 \times S^2, \Delta_3) \to \mathbb{Z}_2 \to 0.$$

So for any $B \in H_2(S^2 \times S^2 \times S^2, \Delta_3)$, $2B$ can be lifted to a spherical class in $S^2 \times S^2 \times S^2$. Therefore, $(S^2 \times S^2 \times S^2, \Delta_3)$ is also monotone, having minimal Maslov index two, and its monotonicity constant is the same as that of S^2, and hence the same as that of $(S^2 \times S^2, \bar{\Delta}_{S^2})$. Then we have

Lemma 10.2. *The pair (X_{2l+3}, L_l) is monotone with minimal Chern number two.*

Moreover, we know that the unstable locus X^{us} consists of points $(x_1, \ldots, x_{2l+3})$ in which at least $l+2$ coordinates are equal. Since $l \geqslant 1$, the real codimension of X^{us} is at least four. So all conditions of Definition 1.1 are satisfied. Further, one has

Lemma 10.3. *If $B \in H_2^{SO(3)}(X_{2l+3}; \mathbb{Z})$ is a spherical class with $\langle \omega_{X_{2l+3}}^{SO(3)}, B \rangle > 0$, then the equivariant Chern number of B is at least 1.*

Proof. Since $\pi_1(SO(3)) \simeq \mathbb{Z}_2$, there are two isomorphism classes of $SO(3)$-bundles over $\boldsymbol{S}^2$, the trivial and the nontrivial ones. Suppose B is represented by a pair (P, u) where $P \to \boldsymbol{S}^2$ is an $SO(3)$-bundle and u is a section of $P \times_{SO(3)} X_{2l+3}$, then $2B$ is contained in the image of

$$H_2(X; \mathbb{Z}) \to H_2^{SO(3)}(X; \mathbb{Z}).$$

Then $2\langle\omega^{SO(3)}_{X_{2l+3}}, B\rangle = \langle\omega_{X_{2l+3}}, A\rangle$ for some $A \in H_2(X_{2l+3}; \mathbb{Z})$. □

This lemma implies Definition 2.4 for $S = \varnothing$. Then one can define $\mathrm{Fuk}^{SO(3)}(L_k)$ and $\mathrm{Fuk}^{\mathfrak{c}}(\bar{L}_l)$. Notice that because $h^1(\bar{L}_l) = 0$, there is only one local system hence we do not have the dependence on b. By Theorem 1.2, there is an A_∞ morphism between these two A_∞ algebras. Moreover, by Theorem 1.6, one obtains the following unobstructedness result.

Theorem 10.4. *Both* $\mathrm{Fuk}^{SO(3)}(L_l)$ *and* $\mathrm{Fuk}^{\mathfrak{c}}(\bar{L}_l)$ *are weakly unobstructed.*

It is possible to use the standard complex structure J_X on X_{2l+3} to define the quasimap Fukaya algebra $\mathrm{Fuk}^{SO(3)}(L_l)$. Nevertheless, the current situation is enough to determine the Floer cohomology, if we perturb J_X.

Theorem 10.5. *For any weakly bounding cochain* $\boldsymbol{b} \in MC(L_l)$*, the quasimap Floer cohomology* $\widetilde{HQF}(L_l, \boldsymbol{b})$ *is isomorphic to the homology of the product of* l *spheres.*

Proof. Indeed, $\bar{L}_l$ is diffeomorphic to the product of l spheres. One can choose a perfect Morse function on $\bar{L}_l$, i.e., a Morse function $\bar{F} : \bar{L}_l \to \mathbb{R}$ whose number of critical points are equal to the sum of Betti numbers of $\bar{L}_l$. Moreover, since the Maslov indices are all even, the Floer differential must be odd. Hence by Item (e) of Theorem 8.18, the only nonzero result of $\tilde{\boldsymbol{m}}_1^{\boldsymbol{b}}$ acting on generators of $\widetilde{CF}(\bar{L}_l; \Lambda)$ is

$$\tilde{\boldsymbol{m}}_1^{\boldsymbol{b}}(\boldsymbol{p}) \in \boldsymbol{e} + CF(\bar{L}_l; \Lambda).$$

Hence $\boldsymbol{e}$ is cohomologous to another cocycle. Therefore, as Λ-modules, one has

$$\widetilde{HQF}(L_l, \boldsymbol{b}) \simeq H^*(\bar{L}_l; \delta_{\mathrm{Morse}}) \simeq CF(\bar{L}_l) \otimes \Lambda \simeq H^*(\overbrace{S^2 \times \cdots \times S^2}^{l}) \otimes \Lambda.$$

□

Remark 10.6. There are many similar ways to obtain new Lagrangians that have nontrivial Floer cohomology. Firstly, we can generalize $SO(3) = PSU(2)$ to $PSU(k)$. Notice that $PSU(k)$ acts on $\mathbb{P}^{k-1}$. By considering dimensions, the diagonal $PSU(k)$ action on the product of $k+1$ copies of $\mathbb{P}^{k-1}$ has zero dimensional GIT quotient. Therefore, one can cook up a Lagrangian in the GIT quotient of the diagonal $PSU(k)$ actions on the product of $2l + k + 1$ copies of $\mathbb{P}^{k-1}$, where the Lagrangian is diffeomorphic to the product of l copies of $\mathbb{P}^{k-1}$. (cf. [Smi17, Section 5].)

11. Strict units

In this section we describe the combinatorics necessary to equip our Fukaya algebras with strict units. The construction follows from the idea of Charest–Woodward [CW17] using weighted trees. Using this construction we prove Theorem 8.18.

11.1. **Weighted treed disks.** The construction depends on the existence of perturbation data that respect forgetful maps. If a colored tree $(\Gamma, \mathfrak{s})$ is possibly unstable, then there is a *stabilization* of $(\Gamma, \mathfrak{s})$, denoted by $(\Gamma^{\mathrm{st}}, \mathfrak{s}^{\mathrm{st}})$, and a morphism $\rho : \Gamma \to \Gamma^{\mathrm{st}}$ which respects the colorings. It is possible that the stabilization is empty, for example, when $\mathrm{V}(\Gamma) = \{\mathfrak{v}_\infty\}$ and $\mathrm{T}^{\mathrm{in}}(\Gamma) = \mathrm{L}(\Gamma) = \varnothing$. Using the notion of stabilization we define the notion of *forgetful operation*. Let $(\Gamma, \mathfrak{s})$ be a stable colored tree and let $t \in \mathrm{T}(\Gamma)$. It suffices to discuss the case that Γ is unbroken, and extend to the case that Γ is broken. Forgetting t gives another colored tree Γ_t, which is obtained from Γ by removing the boundary tail t and stabilizing the resulting tree. This process may contract some boundary tail, some edge and some vertex. It may happen that after removing t and stabilizing, we obtain an empty tree, for example, when Γ is a Y-shape or Φ-shape (see more discussions in Remark 11.7). In this case we denote $\Gamma_t = \varnothing$. When Γ is broken and after forgetting t one of its basic part is contracted, we do not write $\Gamma_t = \varnothing$.

Consider a commutative monoid with three elements $\circ, \bullet, {\color{gray}\bullet}$. The multiplication is defined in such a way that $\circ$ is the unit, $\bullet$ is zero, and ${\color{gray}\bullet}$ is idempotent.

Definition 11.1. Let $(\Gamma, \mathfrak{s})$ be an unbroken colored based tree. A *weighting type* is a map $\mathfrak{w} : \mathrm{T}(\Gamma) \to \{\circ, \bullet, {\color{gray}\bullet}\}$ such that $\mathfrak{m} \equiv \bullet$ on $\mathrm{L}(\Gamma)$ and

$$\mathfrak{w}(t_{\rm out}) = \prod_{t \in \mathrm{T}(\Gamma) \smallsetminus \{t_{\rm out}\}} \mathfrak{w}(t) \in \{\circ, \bullet, {\color{gray}\bullet}\}. \tag{11.1}$$

The weighting type induces a partition $\mathrm{T}(\Gamma) = \mathrm{T}^{\circ}(\Gamma) \sqcup \mathrm{T}^{\bullet}(\Gamma) \sqcup \mathrm{T}^{{\color{gray}\bullet}}(\Gamma)$. This requirement implies that there are nine weighted Y-shapes and three weighted Φ-shapes, as shown in Figure 8 and Figure 9.

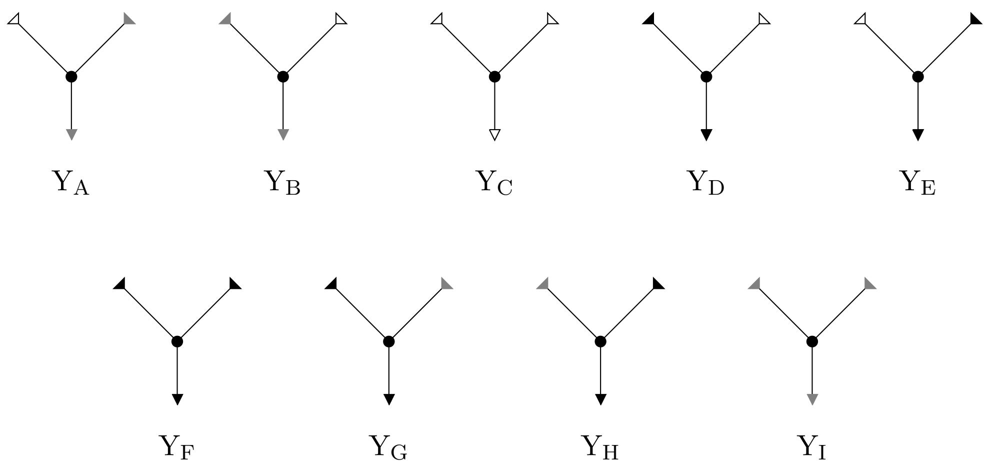

FIGURE 8. The allowed weighted Y-shapes. The grayscales indicate the weighting types on the boundary tails.

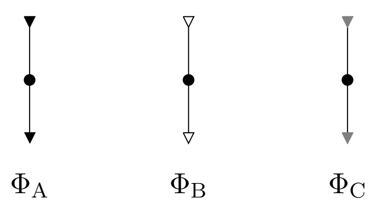

FIGURE 9. The allowed weighting types on a Φ-shape.

Equation 11.1 implies that the values of $\mathfrak{w}$ on boundary inputs determines its value on the output. Hence the notion of weighting types naturally generalizes to the case of possibly broken trees.

Definition 11.2.

(a) A *weighted combinatorial type* (weighted type for short) is a tuple

$$\hat{\Gamma} = (\Gamma, \mathfrak{s}, \mathfrak{m}, \mathfrak{w}, \mathfrak{o}),$$

where $\Gamma = (\Gamma, \mathfrak{s}, \mathfrak{m}, \mathfrak{o})$ is an unweighted type and $\mathfrak{w}$ is a weighting type on $(\Gamma, \mathfrak{s})$. A basic part of a broken weighted type is a basic part of its underlying tree with the inherited set of contact data, coloring, metric type and weighting type. $\hat{\Gamma}$ is *stable* if $(\Gamma, \mathfrak{s})$ is stable.

(b) Two weighted types $\hat{\Gamma}$ and $\hat{\Gamma}'$ are *isomorphic* if there is a tree isomorphism $\rho : \Gamma \simeq \Gamma'$ that preserves all the extra structures. Let $\hat{\mathbf{T}}$ be the set of isomorphism classes of weighted types (with nonempty bases) and let $\hat{\mathbf{T}}^{\mathbf{st}} \subset \hat{\mathbf{T}}$ be the subset of stable ones.

Definition 11.3.

(a) A *weighting* on an *unbroken* weighted type $\hat{\Gamma}$ is a function $\omega : \mathrm{T}(\Gamma) \sqcup \mathrm{L}(\Gamma) \to [0,1]$ that satisfies

$$\omega|_{\mathrm{T}^\circ(\Gamma)} = 1,\ \omega|_{\mathrm{T}^\bullet(\Gamma)\sqcup \mathrm{L}(\Gamma)} = 0,\ \omega|_{\mathrm{T}^*(\Gamma)} \in (0,1),\ \omega(t_{\mathrm{out}}) = \prod_{t \in \mathrm{T}^{\mathrm{in}}(\Gamma)\sqcup \mathrm{L}(\Gamma)} \omega(t).$$

A weighting on a broken weighted type $\hat{\Gamma}$ is a collection of weightings on all its basic parts which coincide at breakings.

(b) Let $\hat{\Gamma}$ and $\hat{\Gamma}'$ be two weighted types and ω, ω' be weightings on them. $(\hat{\Gamma}, \omega)$ is said to be *equivalent* to $(\hat{\Gamma}', \omega')$ if there is an isomorphism $\rho : \hat{\Gamma}' \simeq \hat{\Gamma}$, and one of the following is true.
   (i) Both $\hat{\Gamma}$ and $\hat{\Gamma}'$ have weighted output (which implies that all their inputs are weighted or forgettable), and there is a positive number a such that for all $t \in \mathrm{T}(\Gamma)$, $t' = \rho_{\mathrm{T}}(t)$,
   $$\omega(t)^a = \omega'(t').$$
   (ii) The outputs of $\hat{\Gamma}$ and $\hat{\Gamma}'$ are not weighted and $\omega = \omega' \circ \rho_{\mathrm{T}}$.

(c) Given $\hat{\Gamma} \in \hat{\mathbf{T}}$. A weighted treed disk of type $\hat{\Gamma}$ is a treed disk of type Γ together with a weighting ω on $\hat{\Gamma}$. Two weighted treed disks of type $\hat{\Gamma}$ are isomorphic if the treed disks are isomorphic and the weightings are equivalent.

Now we define a partial order among weighted types.

Definition 11.4. Let $\hat{\Gamma}'$, $\hat{\Gamma}$ be weighted types. We denote $\hat{\Gamma}' \trianglelefteq \hat{\Gamma}$ if $\Gamma' \trianglelefteq \Gamma$, and for each boundary tail $t' \in \mathrm{T}(\hat{\Gamma}')$ with corresponding boundary tail $t \in \mathrm{T}(\hat{\Gamma})$, one has

$$t \in \mathrm{T}^\circ(\hat{\Gamma}) \Longrightarrow t' \in \mathrm{T}^\circ(\hat{\Gamma}'); \qquad t \in \mathrm{T}^\bullet(\hat{\Gamma}) \Longrightarrow t' \in \mathrm{T}^\bullet(\hat{\Gamma}').$$

Using Lemma 4.6 it is not hard to see that $\trianglelefteq$ is still a partial order.

11.1.1. *Moduli spaces.* For a *stable* weighted type $\hat{\Gamma}$, let $\mathcal{W}_{\hat{\Gamma}}$ be the set of all isomorphism classes of stable weighted treed disks modelled on $\hat{\Gamma}$. Define

$$\overline{\mathcal{W}}_{\hat{\Gamma}} := \bigsqcup_{\substack{\hat{\Pi} \trianglelefteq \hat{\Gamma} \\ \hat{\Pi}\ \mathrm{stable}}} \mathcal{W}_{\hat{\Pi}}.$$

The topology on $\overline{\mathcal{W}}_{\hat{\Gamma}}$ is defined via the following notion of sequential convergence.

Definition 11.5. Let $\mathcal{Z}_\nu$ be a sequence of stable weighted treed disks of a stable weighted type $\hat{\Gamma}$ and $\mathcal{Z}_\infty$ is another stable weighted treed disks of a stable weighted type $\hat{\Gamma}_\infty \trianglelefteq \hat{\Gamma}$. We say that $\mathcal{Z}_\nu$ converges to $\mathcal{Z}_\infty$ if the convergence hold for the underlying unweighted treed disks, and, the following conditions are satisfied.

(a) If the outputs t_{out}, t'_{out} of $\hat{\Gamma}$, $\hat{\Gamma}_\infty$ are not weighted, then the weights on boundary inputs converge.

(b) If the outputs t_{out}, t'_{out} of $\hat{\Gamma}$, $\hat{\Gamma}_\infty$ are weighted, then there exist real numbers a_ν, a_∞ such that

$$\big[\omega_\nu(t_{\text{out}})\big]^{a_\nu} = \big[\omega_\infty(t'_{\text{out}})\big]^{a_\infty} = \frac{1}{2}.$$

We require that for all boundary inputs $t_{\underline{i}}$,

$$\lim_{\nu\to\infty} \big[\omega_\nu(t_{\underline{i}})\big]^{a_\nu} = \big[\omega_\infty(t'_{\underline{i}})\big]^{a_\infty} \in (0,1).$$

To better understand the topology of $\overline{\mathcal{W}}_{\hat{\Gamma}}$, we look at a few special cases.

(a) When $\hat{\Gamma}$ is a Y-shape (see Figure 8), by Definition 11.3, for the first six configurations of Figure 8, $\mathcal{W}_{\hat{\Gamma}}$ is an isolated point. $\mathcal{W}_{Y_G}, \mathcal{W}_{Y_H}$ and $\mathcal{W}_{Y_I}$ are all homeomorphic to an open interval parametrized by the weightings on the weighted inputs. One can compactify them by adding boundary configurations as described by Figure 10 and Figure 11.

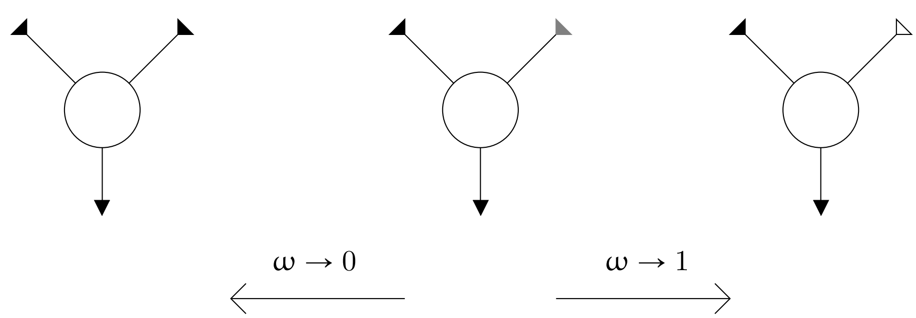

FIGURE 10. Moduli space $\mathcal{W}_{Y_G}$ and its compactification. The case of $\mathcal{W}_{Y_H}$ is similar.

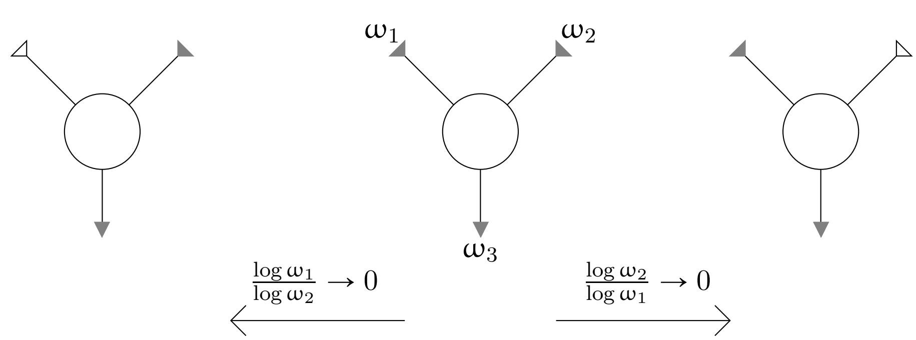

FIGURE 11. Moduli space $\mathcal{W}_{Y_I}$ and its compactification.

(b) When $\hat{\Gamma}$ is a Φ-shape, by Definition 11.1 and Definition 11.3, $\mathcal{W}_{\hat{\Gamma}}$ is a single point.

Now we prove that moduli spaces of stable weighted treed disks are compact and Hausdorff.

Lemma 11.6. *For each $\hat{\Gamma} \in \hat{\mathbf{T}}^{\mathbf{st}}$, the moduli space $\overline{\mathcal{W}}_{\hat{\Gamma}}$ is compact and Hausdorff with respect to the topology defined in Definition 11.5.*

Proof. We first prove the sequential compactness of $\overline{\mathcal{W}}_{\hat{\Gamma}}$. Let $\mathcal{Z}_\nu$ be a sequence of stable weighted tree disks representing a sequence of points in $\mathcal{W}_{\hat{\Gamma}}$. By Lemma 4.9, the underlying sequence $\mathcal{Z}_\nu$ of stable unweighted treed disks has a convergent subsequence (still indexed by ν), converging to certain $\mathcal{Z}_\infty$. Let the underlying unweighted type of $\mathcal{Z}_\infty$ be $\Gamma_\infty \trianglelefteq \Gamma$.

It remains to assign weighting types and weights onto boundary tails and breakings of $\mathcal{Z}_\infty$, which should give a sequential limit $\mathcal{Z}_\infty$ of $\mathcal{Z}_\nu$. There are three possibilities regarding the weighting types of the output $t_{\rm out}$ of $\hat{\Gamma}$.

(a) $t_{\rm out}$ is forgettable. It implies that all inputs of $\hat{\Gamma}$ are forgettable. Then define $\mathcal{Z}_\infty$ to have all boundary tails and breakings forgettable. It follows easily that $\mathcal{Z}_\nu$ converges to $\mathcal{Z}_\infty$.

(b) $t_{\rm out}$ is unforgettable. Let the boundary inputs of $\hat{\Gamma}$ be $t_1, \ldots, t_{\underline{k}}$. Then one can find a subsequence (still indexed by ν) such that for all $\underline{i} = 1, \ldots, \underline{k}$, $\omega_\nu(t_{\underline{i}})$ converges to some number $\omega_\infty(t'_{\underline{i}}) \in [0,1]$. These numbers determine the weighting types of all inputs of Γ_∞. It also determines the weighting types and weightings of all breakings of Γ_∞ successively, and hence a stable combinatorial type $\hat{\Gamma}_\infty$ and a stable weighted treed disk $\mathcal{Z}_\infty$ representing a point in $\mathcal{W}_{\hat{\Gamma}_\infty}$. One can check easily that $\mathcal{Z}_i$ converges to $\mathcal{Z}_\infty$.

(c) $t_{\rm out}$ is weighted. Then the weights in all $\mathcal{Z}_\nu$ are positive. Choosing a sequence of positive numbers a_ν such that $\left[\omega_\nu(t_{\rm out})\right]^{a_\nu} = \frac{1}{2}$. Then one can choose a subsequence (still indexed by ν) such that for all $\underline{i} = 1, \ldots, \underline{k}$, $\left[\omega_\nu(t_{\underline{i}})\right]^{a_\nu}$ converges some number $\omega_\infty(t'_{\underline{i}}) \in [\frac{1}{2}, 1]$. These numbers determine the weighting types of all inputs of Γ_∞. It also determines the weighting types and weightings of all breakings of Γ_∞ successively. On the other hand, set $t'_{\rm out}$ to be weighted and $\omega_\infty(t'_{\rm out}) = \frac{1}{2}$. Then we have found a stable combinatorial type $\hat{\Gamma}_\infty$ and a stable weighted treed disk $\mathcal{Z}_\infty$ representing a point in $\mathcal{W}_{\hat{\Gamma}_\infty}$. One can check easily that $\mathcal{Z}_i$ converges to $\mathcal{Z}_\infty$.

On the other hand, the Hausdorffness of $\overline{\mathcal{W}}_{\hat{\Gamma}}$ follows from the Hausdorffness of the moduli space of stable unweighted treed disks, and the uniqueness of the way of assigning weighting types and weights in a sequential limit. □

Lastly we give the dimension formula for a stable weighted type $\hat{\Gamma}$. Let Γ be the underlying unweighted type. Then (4.1) gives the dimension of $\mathcal{W}_\Gamma$. Therefore,

$$\dim \mathcal{W}_{\hat{\Gamma}} = \dim \mathcal{W}_\Gamma + \#{\rm W}(\hat{\Gamma})$$

where $\#{\rm W}(\hat{\Gamma})$ is the number of weighted inputs minus the number of weighted outputs (see Item (b) of Definition 11.3).

11.1.2. *Forgetting boundary tails.* Let t be a forgettable incoming boundary tail of $\hat{\Gamma}$ and $\hat{\Gamma}_t$ be the combinatorial type obtained by forgetting t and stabilization. For any $\mathcal{Z}$ representing a point in $\overline{\mathcal{W}}_{\hat{\Gamma}}$, this operation gives another stable weighted treed disk $\mathcal{Z}_t$, and induces a continuous map $\mathfrak{f}_t : \overline{\mathcal{W}}_{\hat{\Gamma}} \to \overline{\mathcal{W}}_{\hat{\Gamma}_t}$. It also induces a contraction map $\mathcal{Z} \to \mathcal{Z}_t$, which gives a commutative diagram

$$\begin{array}{ccc} \overline{\mathcal{U}}_{\hat{\Gamma}} & \xrightarrow{\pi_{\hat{\Gamma}}} & \overline{\mathcal{W}}_{\hat{\Gamma}} \\ {\scriptstyle \tilde{\mathfrak{f}}_t}\downarrow & & \downarrow{\scriptstyle \mathfrak{f}_t} \\ \overline{\mathcal{U}}_{\hat{\Gamma}_t} & \xrightarrow[\pi_{\hat{\Gamma}_t}]{} & \overline{\mathcal{W}}_{\hat{\Gamma}_t} \end{array} . \tag{11.2}$$

Remark 11.7. Given a treed disk $\mathcal{Z}$ with combinatorial type $\hat{\Gamma}$ and an incoming forgettable boundary tail t of $\hat{\Gamma}$. Suppose the basic part containing t is not an infinite edge, a Y-shape, or a Φ-shape. There are several possibilities of the change of shapes regarding the forgetful operation $\mathfrak{f}_t$.

(a) When $\mathfrak{v}_t$, the vertex to which t is attached, is still stable after forgetting t, only the interval I_t is contracted by the stabilization.
(b) If $\mathfrak{v}_t$ becomes unstable after forgetting t, then $\mathfrak{v}_t$ has valence three if $\mathfrak{v}_t \in V_\triangle(\hat{\Gamma}) \sqcup V_\triangledown(\hat{\Gamma})$, and has valence two if $\mathfrak{v}_t \in V_\diamond(\hat{\Gamma})$. The two-dimensional component Σ_t corresponding to $\mathfrak{v}_t$ is also contracted by the forgetful map. See Figure 12.

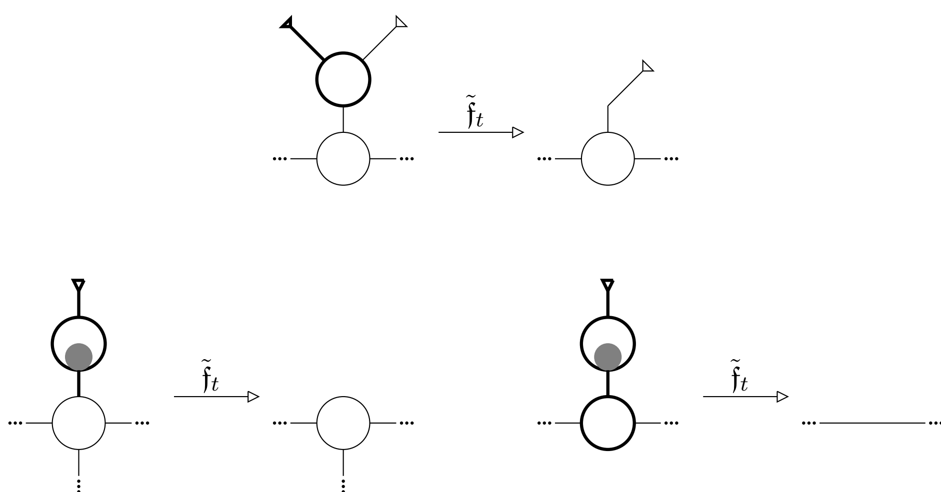

FIGURE 12. When forgetting a boundary tail, we regard the thick part of the treed disk is contracted.

(c) An extremal situation of the above case is described by the Figure 13, where we contract a whole basic part of a broken treed disk.

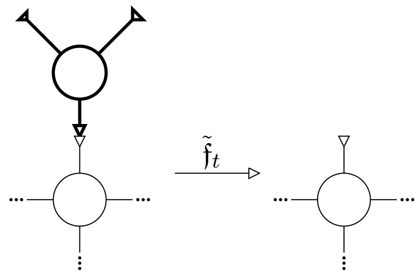

FIGURE 13. If we forgetting the upper-left boundary tail, then we regard the thick part of the treed disk is contracted.

11.1.3. *Coherent perturbations.* For a weighted combinatorial type $\hat{\Gamma}$, let Γ be its underlying unweighted combinatorial type. Then there is a natural forgetful map $\overline{\mathcal{U}}_{\hat{\Gamma}} \to \overline{\mathcal{U}}_{\Gamma}$ which covers another forgetful map $\overline{\mathcal{W}}_{\hat{\Gamma}} \to \overline{\mathcal{W}}_{\Gamma}$. Indeed, $\overline{\mathcal{W}}_{\hat{\Gamma}}$ is homeomorphic to the product of $\overline{\mathcal{W}}_{\Gamma}$ and a cube, while $\overline{\mathcal{U}}_{\hat{\Gamma}}$ is the pull-back of $\overline{\mathcal{U}}_{\Gamma} \to \overline{\mathcal{W}}_{\Gamma}$. Then the coherent collection of nodal neighborhoods $\overline{\mathcal{U}}_{\Gamma}^{\rm thin} \subset \overline{\mathcal{U}}_{\Gamma}^{2} \subset \overline{\mathcal{U}}_{\Gamma}$ we have used (see Definition 5.1 and Lemma 5.2) are pulled back to

$$\overline{\mathcal{U}}_{\hat{\Gamma}}^{\rm thin} \subset \overline{\mathcal{U}}_{\hat{\Gamma}}^{2} \subset \overline{\mathcal{U}}_{\hat{\Gamma}}.$$

The notion of coherent maps

Definition 11.8. (cf. Definition 5.5) Let Z^1, Z^2 be topological vector spaces. For $i = 1, 2$, a system of continuous maps $\{g^i_{\hat{\Gamma}} : \overline{\mathcal{U}}^i_{\hat{\Gamma}} \to Z^i \mid \hat{\Gamma} \in \hat{\mathbf{T}}^{\mathbf{st}} \cup \mathbf{F}^{\mathbf{st}}\}$ is called *coherent*, if the following conditions are satisfied.

(a) (Neck region) $g^1_{\hat{\Gamma}}$ vanishes in $\overline{\mathcal{U}}^{\rm long}_{\hat{\Gamma}} \cap \overline{\mathcal{U}}^1_{\hat{\Gamma}}$ and $g^2_{\hat{\Gamma}}$ vanishes on $\overline{\mathcal{U}}^{\rm thin}_{\hat{\Gamma}} \cap \overline{\mathcal{U}}^2_{\hat{\Gamma}}$.

(b) (Degeneration) For all pairs $\hat{\Gamma}, \hat{\Pi} \in \hat{\mathbf{T}}^{\mathbf{st}} \cup \mathbf{F}^{\mathbf{st}}$ with $\hat{\Pi} \trianglelefteq \hat{\Gamma}$, with respect to the commutative diagram (5.1), one has $g^i_{\hat{\Gamma}} \circ \tilde{\rho}^i_{\hat{\Gamma},\hat{\Pi}} = g^i_{\hat{\Pi}}$.

(c) (Cutting edge) For any broken $\hat{\Gamma} \in \hat{\mathbf{T}}^{\mathbf{st}} \cup \mathbf{F}^{\mathbf{st}}$ with all basic parts $\hat{\Gamma}_1, \dots, \hat{\Gamma}_s$, with respect to the commutative diagram (5.3), one has $g^i_{\hat{\Gamma}} \circ \tilde{\rho}^i_{\hat{\Gamma}} = \sqcup g^i_{\hat{\Gamma}_\alpha}$.

(d) (Components) For $\hat{\Gamma} \in \hat{\mathbf{T}}^{\rm st} \cup \mathbf{F}^{\mathbf{st}}$ and a subset $\mathrm{S} \subset \mathrm{V}(\hat{\Gamma})$ which either contains $\mathrm{V}(\underline{\hat{\Gamma}})$ or is disjoint from $\mathrm{V}(\underline{\hat{\Gamma}})$, there exist continuous maps $h^i_{\hat{\Gamma}_{\rm S}} : \overline{\mathcal{U}}_{\hat{\Gamma}^i_S} \to Z^i$ such that

$$g^i_{\hat{\Gamma}}|_{\overline{\mathcal{U}}_{\hat{\Gamma},\hat{\Gamma}_S}} = h^i_{\hat{\Gamma}_S} \circ \tilde{\pi}_{\hat{\Gamma},\hat{\Gamma}_S}.$$

(e) (Forgetful map) Let $\hat{\Gamma} \in \hat{\mathbf{T}}^{\mathbf{st}}$ be unbroken and t be the first forgettable boundary input and assume that $\hat{\Gamma}_t$ is nonempty. Then with respect to the forgetful map $\tilde{\mathfrak{f}}_t : \overline{\mathcal{U}}_{\hat{\Gamma}} \to \overline{\mathcal{U}}_{\hat{\Gamma}_t}$ in (11.2), on the components that are not contracted by $\tilde{\mathfrak{f}}_t$, one has $g^i_{\hat{\Gamma}} = g^i_{\hat{\Gamma}_t} \circ \tilde{\mathfrak{f}}^i_t$.

Lemma 11.9. *Let $\hat{\Gamma}$ one of the first five* Y*-shapes in Figure 8 and $P_{\hat{\Gamma}} = (f_{\hat{\Gamma}}, h_{\hat{\Gamma}})$ belongs to a coherent system of perturbation data (for all stable weighted types).*

(a) If $\hat{\Gamma} \in \{\mathrm{Y_A}, \mathrm{Y_B}, \mathrm{Y_D}, \mathrm{Y_E}\}$, then $f_{\hat{\Gamma}}$ vanishes on the edges that are not forgettable.

(b) If $\hat{\Gamma} = \mathrm{Y_C}$, then $f_{\hat{\Gamma}}$ vanishes on the outgoing and the second incoming edges.

Proof. Consider an unbroken stable weighted type $\hat{\Xi}$ with more than three boundary inputs. Assume that there is a sequence of stable weighted treed disks $\mathcal{Z}_\nu$ degenerate to a once broken weighted treed disk $\mathcal{Z}$, where one of the basic part is a stable weighted treed disk modelled on one of the Y-shape we are considering. Also assume that the first forgettable boundary input of $\hat{\Xi}$ becomes the (first) forgettable boundary input of the Y-shape $\hat{\Gamma}$. Then the sequence obtained by forgetting the first forgettable boundary input t, denoted by $\mathcal{Z}_{\nu,t}$, also converges to $\mathcal{Z}_t$. Let $\mathfrak{v} \in \mathrm{V}(\hat{\Xi})$ be the vertex to which t is attached and let $\mathfrak{e} \in \mathrm{E}(\hat{\Xi})$ be the edge the connects $\mathfrak{v}$ with the other part of the tree. Then for ν sufficiently large, the lengths $\boldsymbol{\lambda}_\nu(\mathfrak{e})$ are very large. Then after forgetting t, the contraction image of the disk corresponding to $\mathfrak{v}$ is a point $t_\nu \in [0, +\infty)$ on the edge close to infinity (see Figure 12). By Item (a) of Definition 11.8, $f_{\mathcal{Z}_{\nu,t}}$ has to vanish from $[t_\nu - M, +\infty)$ for any finite M and large enough ν. Then the coherence and continuity imply that $f_{\hat{\Gamma}}$ has to vanish on the edges except the (first) forgettable one. □

11.2. **Moduli spaces and transversality.** In this subsection we construct a strongly regular coherent system of perturbation data $\underline{\hat{P}}$ for all stable weighted types that extends the system $\underline{P}$ we have used for the unweighted case.

11.2.1. *Weighted vortex types.* Define the *extended critical point set*

$$\mathbf{crit} := \mathbf{crit} \sqcup \{\boldsymbol{x}_M^\circ, \boldsymbol{x}_M^\bullet\}.$$

Here $\boldsymbol{x}_M^\circ$ is called the *forgettable* $\boldsymbol{x}_M$ (which will be the strict units), and $\boldsymbol{x}_M^\bullet$ is called the *weighted* $\boldsymbol{x}_M$. To distinguish from these two elements, we re-denote the original $\boldsymbol{x}_M \in \mathbf{crit}$ by $\boldsymbol{x}_M^\bullet$, called the *unforgettable* $\boldsymbol{x}_M$. Other elements of **crit** are also regarded as unforgettable. For the two new elements, define their degrees to be $\mathbf{i}(\boldsymbol{x}_M^\circ) = 0$ and $\mathbf{i}(\boldsymbol{x}_M^\bullet) = -1$.

Now we extend the notion of vortex types, given by Definition 6.1, to *weighted* vortex types. A weighted vortex type is a tuple $\tilde{\Gamma} = (\hat{\Gamma}, \tilde{\boldsymbol{x}}, \boldsymbol{B}, \boldsymbol{V}, \tilde{\mathfrak{o}})$ where $\hat{\Gamma} = \hat{\mathbf{T}} \cup \mathbf{F}$ is a weighted combinatorial type (of $\underline{k}$ boundary inputs and one output), $\tilde{\boldsymbol{x}} = (\boldsymbol{x}_1, \ldots, \boldsymbol{x}_{\underline{k}}; \boldsymbol{x}_\infty)$ is a sequence of elements in **crit** such that $\boldsymbol{x}_{\underline{i}}$ has the same weighting type as $t_{\underline{i}} \in \underline{\mathrm{T}}(\hat{\Gamma})$ for $\underline{i} = 1, \ldots, \underline{k}, \infty$. The other elements $\boldsymbol{B}, \boldsymbol{V}, \tilde{\mathfrak{o}}$ are the same as those in Definition 6.1, namely $\boldsymbol{B}$ labels the homology classes of vertices, $\boldsymbol{V}$ labels the asymptotic constrains at interior leaves, and $\tilde{\mathfrak{o}}$ labels the contact orders.

Again the correspondence from each weighted vortex type $\tilde{\Gamma}$ to its underlying weighted combinatorial type $\hat{\Gamma}$ will be indicated from notations. Using the collection of perturbation data $\underline{\hat{P}}$, we can define the moduli spaces $\mathcal{M}_{\tilde{\Gamma}}$ in the same way as Section 6. In order to achieve transversality for $\mathcal{M}_{\tilde{\Gamma}}$, we also look at special weighted vortex types such as reduced, uncrowded, S-special, and controlled weighted vortex types, which are defined in the same way as Definition 7.1.

11.2.2. *Perturbed gradient flows.* In order to achieve transversality for weighted vortex types while maintaining the coherence condition, especially the one about forgetting forgettable boundary tails (see Definition 11.8), we need to carefully choose certain perturbations to the gradient flow equation.

Let $f^\circ \in C_c^\infty((-\infty, 0] \times \bar{L})$ be a time-dependent perturbation of the function $F : \bar{L} \to \mathbb{R}$ and denote $F^\circ = F + f^\circ$. Consider the perturbed gradient flow equation

$$\dot{x}(s) + \nabla F^\circ(x(s)) = 0, \ -\infty < s \leqslant 0, \ \lim_{s \to -\infty} x(s) = \boldsymbol{x}_M. \tag{11.3}$$

Let the moduli space of solutions be $\mathcal{M}_\circ(\boldsymbol{x}_M)$ whose dimension is $\mathbf{dim}\bar{L}$.

Lemma 11.10. *There exists a function* $f^\circ \in C_c^\infty((-\infty, 0] \times \bar{L})$ *satisfying the following condition.*

(a) f° *vanishes in a neighborhood of* $\boldsymbol{x}_M$*;*
(b) For any $\boldsymbol{x} \in \mathbf{crit}$*, there is a unique solution* (11.3) *with* $x(0) = \boldsymbol{x}$*.*
(c) The evaluation at time zero $\mathcal{M}_\circ(\boldsymbol{x}_M) \to \bar{L}$ *is a submersion.*

The proof is left to the reader. Choosing a function $F^\circ = F + f^\circ$ where f° satisfies the conditions of Lemma 11.10, the perturbation data for certain simple weighted types are also determined by the coherence condition. The following two lemmata gives these perturbation data explicitly and prove that they are regular. Their proofs are also very straightforward and are left to the reader.

Lemma 11.11. *Choose* f° *satisfying the conditions of Lemma 11.10. Let* $\hat{\Gamma}$ *be one of the first five* Y*-shapes listed in Figure 8. Then the perturbation* $P_{\hat{\Gamma}}$ *that is equal to* f° *on the (first, if any) forgettable incoming edge and is trivial on other edges and trivial on the two-dimensional component is regular.*

Moreover, if $\tilde{\Gamma}$ *is an uncrowded and controlled refinement of* $\hat{\Gamma}$*, then* $\tilde{\Gamma}$ *is classified by the labellings of the boundary tails. Then we have*

(a) For $\hat{\Gamma} = \mathrm{Y_A}, \mathrm{Y_B}, \mathrm{Y_C}$*, there is only one such refinement* $\tilde{\Gamma}$*, where the boundary tails are labelled by* $\boldsymbol{x}_M^\circ$ *or* $\boldsymbol{x}_M^\bullet$ *according to the weighting types of the boundary tails. In all cases* $\mathbf{ind}\tilde{\Gamma} = 0$ *and* $\mathcal{M}_{\tilde{\Gamma}}$ *contains a single element represented by the trivial solutions.*
(b) For $\hat{\Gamma} = \mathrm{Y_D}, \mathrm{Y_E}$*, a refinement* $\tilde{\Gamma}$ *is essentially a pair of labelling* $(\boldsymbol{x}_{\mathrm{in}}^\bullet, \boldsymbol{x}_{\mathrm{out}}^\bullet)$ *on the two unforgettable boundary tails. If* $\mathbf{ind}\tilde{\Gamma} = 0$*, then* $\mathbf{i}(\boldsymbol{x}_{\mathrm{in}}^\bullet) = \mathbf{i}(\boldsymbol{x}_{\mathrm{out}}^\bullet)$*. In this case, the moduli space* $\mathcal{M}_{\tilde{\Gamma}}$ *is empty if* $\boldsymbol{x}_{\mathrm{in}}^\bullet \neq \boldsymbol{x}_{\mathrm{out}}^\bullet$ *and contains a single element if* $\boldsymbol{x}_{\mathrm{in}}^\bullet = \boldsymbol{x}_{\mathrm{out}}^\bullet$*. In the latter case, the only element is represented by the solution which is constant on the unforgettable edges and constant on the disk component, and is equal to the unique solution to* (11.3) *with* $x(0) = \boldsymbol{x}_{\mathrm{in}}^\bullet = \boldsymbol{x}_{\mathrm{out}}^\bullet$ *(uniqueness follows from Lemma 11.10).*

11.2.3. *Canonical extension of perturbation data to weighted types.* Suppose we are given a coherent family of perturbation data for all unweighted combinatorial types. We would like to extend this family to include weighted types.

Given an unbroken $\hat{\Gamma} \in \hat{\mathbf{T}}^{\mathbf{st}}$ and let $t \in \mathrm{T}^{\mathrm{in}}(\hat{\Gamma})$ be its first forgettable input. Consider the case that $\Gamma_t \neq \varnothing$. Suppose we have chosen a perturbation $P_{\hat{\Gamma}_t}$. Then upon choosing F°, there is a uniquely determined $P_{\hat{\Gamma}} = (F_{\hat{\Gamma}}, J_{\hat{\Gamma}})$, called the F°-extension of $P_{\hat{\Gamma}_t}$, which, together with $P_{\hat{\Gamma}_t}$, respects the forgetful operation forgetting the boundary tail t. We explain the construction as follows.

Let $\mathfrak{v}_t \in \mathrm{V}(\underline{\hat{\Gamma}})$ be the vertex to which t is attached. For each treed disk $\mathcal{Z}$ with underlying type $\hat{\Gamma}$, let $\mathcal{Z}_t$ be the treed disk obtained by forgetting the boundary tail t and stabilizing. Then certain one-dimensional or two-dimensional components of $\mathcal{Z}$ are contracted. The preserved components corresponds to components of $\mathcal{Z}_t$. Then $P_{\hat{\Gamma}_t}$ determines the value of $P_{\hat{\Gamma}}|_{\mathcal{Z}}$ on all the preserved components. To determine the value of $P_{\hat{\Gamma}}|_{\mathcal{Z}}$ on the contracted components, we consider the following three cases.

(a) If the forgetful operation only contracts a boundary tail $I_t \simeq (-\infty, 0]$, then we define $F_{\hat{\Gamma}}|_{I_t}$ to be F°.
(b) If the forgetful operation contracts a boundary tail I_t and a vertex $\mathfrak{v} \in \mathrm{V}_\vartriangle(\hat{\Gamma}) \sqcup \mathrm{V}_\triangledown(\hat{\Gamma})$ (see Figure 12), then we define $F_{\hat{\Gamma}}|_{I_t}$ in the same way as above and define $J_{\hat{\Gamma}}|_{\Sigma_{\mathfrak{v}}}$ to be J_X.

(c) Suppose the forgetful operation contracts a vertex $\mathfrak{v} \in V_\diamond(\underline{\hat{\Gamma}})$. Then $\mathfrak{v}$ has valence two and has no superstructure or leaf. Let $\mathfrak{e}$ be the edge starting from $\mathfrak{v}$ towards the root. Then $\mathfrak{e}$ has finite length. Then $I_t \cup I_\mathfrak{e}$ can be identified with $(-\infty, 0] \cup [0, -\boldsymbol{\lambda}(\mathfrak{e})] \simeq (-\infty, 0]$. Then define $F_{\hat{\Gamma}}|_{I_\mathfrak{e} \cup I_\mathfrak{e}}$ to be F° via this identification. Moreover, the forgetful map may or may not contract another vertex $\mathfrak{v}' \in V_\triangledown(\hat{\Gamma})$ i.e., the other end of $\mathfrak{e}$. In either case, define the restriction of $J_{\hat{\Gamma}}$ on the contracted two-dimensional component(s) to be J_X.

The following lemma shows that $P_{\hat{\Gamma}}$ is (strongly) regular as long as F° satisfies conditions in Lemma 11.10 and $P_{\hat{\Gamma}_t}$ is (strongly) regular.

Lemma 11.12. *Suppose F° satisfies conditions of Lemma 11.10 and $P_{\hat{\Gamma}_t}$ is a strongly regular perturbation on $\overline{\mathcal{U}}_{\hat{\Gamma}}$, then the F°-extension $P_{\hat{\Gamma}}$ of $P_{\hat{\Gamma}_t}$ is also strongly regular.*

Proof. The strong regularity of $P_{\hat{\Gamma}}$ follows from Item (c) of Lemma 11.10. □

Now let $\hat{\mathbf{T}}^\bullet \cup \mathbf{F}^{\mathbf{st}} \simeq \mathbf{T}^{\mathbf{st}} \cup \mathbf{F}^{\mathbf{st}}$ be the set of all stable unweighted combinatorial types, or equivalently, viewed as the set of all stable weighted combinatorial types whose inputs are all unforgettable. We have used a previously chosen strongly regular coherent system of perturbation data $\underline{P}$ for all types in $\hat{\mathbf{T}}^{\mathbf{st}} \cup \mathbf{F}^{\mathbf{st}}$, whose restriction to superstructures coincides with a system $\underline{J_\mathbf{F}}$ of almost complex structures. Now let $\hat{\mathbf{T}}^{\bullet\circ} \subset \hat{\mathbf{T}}^{\mathbf{st}}$ consisting of all stable weighted types whose boundary inputs are unforgettable or forgettable, but not weighted. By forgetting forgettable inputs successively, any unbroken $\hat{\Gamma} \in \hat{\mathbf{T}}^{\bullet\circ}$ can be reduced to some $\hat{\Gamma}' \in \hat{\mathbf{T}}^\bullet$ or one of $\mathrm{Y_A}, \mathrm{Y_B}, \mathrm{Y_C}, \mathrm{Y_D}, \mathrm{Y_E}$. Then upon choosing F°, we can extend $\underline{P}$ to a system of strongly regular perturbation data $\underline{P}^{\bullet\circ}$ to all $\hat{\Gamma} \in \hat{\mathbf{T}}^{\bullet\circ} \cup \mathbf{F}^{\mathbf{st}}$. It is routine to check that this new system is coherent in the sense of Definition 11.8.

Lastly, we can use (a variant of) Lemma 7.6 to inductively construct a strongly regular coherent system of perturbation data $\underline{\hat{P}}$ for all stable weighted types $\hat{\Gamma} \in \hat{\mathbf{T}}^{\mathbf{st}} \cup \mathbf{F}^{\mathbf{st}}$ that extends the strongly regular $\underline{P}^{\bullet\circ}$. The base cases of the induction include choosing regular perturbations (independently) for the types $\mathrm{Y_G}, \mathrm{Y_H}, \mathrm{Y_I}$ (see Figure 8) and the type Φ_C (see Figure 9). Notice that the cases for $\mathrm{Y_A}$ and $\mathrm{Y_B}$ have been taken care of by Lemma 11.9 and Lemma 11.11. The induction can be performed because the partial order among unweighted types can be extended canonically to a partial order for all weighted types that are compatible with the stratification of various moduli spaces. For example, for a stable weighted type $\hat{\Gamma}$, a weighted input turning to forgettable or unforgettable corresponds to two codimension one boundary strata of $\overline{\mathcal{W}}_{\hat{\Gamma}}$. The induction procedure provides us a strongly regular coherent system of perturbation data $\underline{\hat{P}}$, which grants each stable weighted combinatorial type $\hat{\Gamma} \in \hat{\mathbf{T}}^{\mathbf{st}} \cup \mathbf{F}^{\mathbf{st}}$ a strongly regular perturbation $P_{\hat{\Gamma}} = (J_{\hat{\Gamma}}, F_{\hat{\Gamma}})$. Then for each weighted vortex type $\tilde{\Gamma}$ whose underlying weighted combinatorial type has stabilization equal to $\hat{\Gamma}$, we can define the moduli space $\mathcal{M}_{\tilde{\Gamma}}(P_{\hat{\Gamma}})$. The strongly regularity condition of $P_{\hat{\Gamma}}$ implies that when $\tilde{\Gamma}$ is reduced, uncrowded, and controlled (see Definition 7.1, which can be extended to the weighted case without changing a word), the subset $\mathcal{M}^*_{\tilde{\Gamma}}(P_{\hat{\Gamma}}) \subset \mathcal{M}_{\tilde{\Gamma}}(P_{\hat{\Gamma}})$ of stable weighted treed scaled vortices in which no nontrivial component is mapped entirely into D is a smooth manifold of dimension (compare to (7.3))

$$\mathbf{ind}\tilde{\Gamma} = \dim \mathcal{W}_{\hat{\Gamma}} + \mathbf{i}(\tilde{\boldsymbol{x}}) + \mathrm{Mas}(\tilde{\Gamma}) - \sum_{i=1}^{k} \delta_i(\tilde{\Gamma}).$$

11.2.4. *Refined compactness.* Many results about moduli spaces of unweighted configurations can be extended to include weighted configurations. Here we only mention one of the most important one, namely the refined compactness result for moduli spaces of expected dimensions at most one (Proposition 7.12).

Firstly, the notions of admissible and essential refined vortex types (see Definition 7.11) can be extended to include weighted vortex types without changing a word. Then we have the following extension about refined compactness (Proposition 7.12). Its proof is completely the same as before.

Proposition 11.13. *Let $\tilde{\Gamma}$ be an essential weighted vortex type and suppose there is a stable weighted treed scaled vortex of a weighted vortex type $\tilde{\Pi}$ which represents an element of the closure $\overline{\mathcal{M}^*_{\tilde{\Gamma}}(P_{\hat{\Gamma}})}$. If $\mathbf{ind}\tilde{\Gamma} = 0$, then $\tilde{\Pi} = \tilde{\Gamma}$. If $\mathbf{ind}\tilde{\Gamma} = 1$, then either $\tilde{\Pi} = \tilde{\Gamma}$, or $\tilde{\Pi}$ belongs to the types listed in Proposition 7.12, plus the following one more possibility:*

- *(*`cw`*) One weighted input of $\tilde{\Gamma}$ changes to a forgettable or unforgettable input (the weighting type of the output of $\tilde{\Pi}$ is determined by weighting rule (11.1)).*

11.3. **Strict unitality.** Now we start to equip the A_∞ algebras with strict units and prove Theorem 8.18. Let $\widetilde{CF}^*(\bar{L};\Lambda)$ be the free Λ-module

$$\widetilde{CF}^*(\bar{L};\Lambda) = CF^*(\bar{L};\Lambda) \oplus \Lambda \boldsymbol{e} \oplus \Lambda \boldsymbol{p} = CF^*(\bar{L};\Lambda) \oplus \Lambda \boldsymbol{x}_M^{\circ} \oplus \Lambda \boldsymbol{x}_M^{\bullet}.$$

Then we extend the A_∞ compositions $\boldsymbol{m}_{\underline{k}}$, $\boldsymbol{n}_{\underline{k}}$ and the A_∞ morphism defined in Section 8 as multilinear operators on $CF^*(\bar{L};\Lambda)$ to $\widetilde{CF}^*(\bar{L};\Lambda)$. Indeed, by counting elements in zero-dimensional moduli spaces of types $\triangle, \triangledown, \diamond$ one can define in the same way these extensions, except for $\tilde{\boldsymbol{m}}_1(\boldsymbol{x}_M^{\bullet})$ and $\tilde{\boldsymbol{n}}_1(\boldsymbol{x}_M^{\bullet})$. Indeed they are defined as follows.

$$\tilde{\boldsymbol{m}}_1(\boldsymbol{x}_M^{\bullet}) = \boldsymbol{x}_M^{\circ} - \boldsymbol{x}_M^{\bullet} + \sum_{\boldsymbol{x}_\infty \in \mathbf{crit}} \langle \boldsymbol{x}_M^{\bullet}; \boldsymbol{x}_\infty \rangle \cdot \boldsymbol{x}_\infty$$

where $\langle \boldsymbol{x}_M^{\bullet}; \boldsymbol{x}_\infty \rangle$ is defined in the same way as (8.4). Similarly,

$$\tilde{\boldsymbol{n}}_1(\boldsymbol{x}_M^{\bullet}) = \boldsymbol{x}_M^{\circ} - \boldsymbol{x}_M^{\bullet} + \sum_{\boldsymbol{x}_\infty \in \mathbf{crit}} \overline{\langle \boldsymbol{x}_M^{\bullet}; \boldsymbol{x}_\infty \rangle} \cdot \boldsymbol{x}_\infty$$

where $\overline{\langle \boldsymbol{x}_M^{\bullet}; \boldsymbol{x}_\infty \rangle}$ is defined in the same way as (8.6).

The following proposition proves Theorem 8.18 Item (a)–Item (d).

Proposition 11.14.

- *(a) $\widetilde{\mathrm{Fuk}}^K(L) = (\widetilde{CF}^*(\bar{L};\Lambda), \tilde{\boldsymbol{m}}_0, \tilde{\boldsymbol{m}}_1, \ldots)$ and $\widetilde{\mathrm{Fuk}}^{\mathfrak{c}}(\bar{L}) = (\widetilde{CF}^*(\bar{L};\Lambda), \tilde{\boldsymbol{n}}_0, \tilde{\boldsymbol{n}}_1, \ldots)$ are strictly unital A_∞ algebras with $\boldsymbol{x}_M^{\circ}$ being their strict units.*
- *(b) $\tilde{\boldsymbol{\varphi}} = (\tilde{\varphi}_0, \tilde{\varphi}_1, \ldots)$ is a strictly unital A_∞ morphism.*

Proof. The fact that $\tilde{\boldsymbol{m}}_0, \tilde{\boldsymbol{m}}_1, \ldots$ (resp. $\tilde{\boldsymbol{n}}_0, \tilde{\boldsymbol{n}}_1, \ldots$) give an A_∞ algebra structure over $\widetilde{CF}^*(\bar{L}) \otimes \Lambda$, and the fact that $\tilde{\varphi}_0, \tilde{\varphi}_1, \ldots$ define an A_∞ morphism follow from the same argument as before, in which the refined compactness theorem (Proposition 11.13) plays a crucial role. In the following we explain the unitality about $\boldsymbol{e} = \boldsymbol{x}_M^{\circ}$.

We first prove the unitality of $\boldsymbol{x}_M^\circ$ for $\widetilde{\mathrm{Fuk}}^K(L)$. To show that $\tilde{\boldsymbol{m}}_1(\boldsymbol{x}_M^\circ) = 0$, consider essential combinatorial types $\tilde{\Gamma}$ of index zero with only one boundary input t labelled by $\boldsymbol{x}_M^\circ$. For the underlying weighted type $\hat{\Gamma}$, recall that $\hat{\Gamma}_t$ is the weighted type obtained by forgetting the boundary tail t and stabilizing. If $\hat{\Gamma}_t$ is nonempty, we want to prove that the moduli space $\mathcal{M}_{\tilde{\Gamma}}$ is empty. Indeed, if $\mathcal{C} \in \mathcal{M}_{\hat{\Gamma}}$, then since the perturbation data respects forgetting a forgettable tail (see Item (e) of Definition 11.8), by forgetting the tail t, we obtain an element $\mathcal{C}_t \in \mathcal{M}_{\tilde{\Gamma}_t}$. However, the expected dimension of the latter moduli space is one less than that of $\mathcal{M}_{\tilde{\Gamma}}$, which, by Item (e) of Definition 11.8 and the transversality, is impossible. Hence $\mathcal{M}_{\tilde{\Gamma}} = \varnothing$. On the other hand, if $\hat{\Gamma}_t = \varnothing$, then the only possibility is that $\hat{\Gamma}$ is an infinite edge. Therefore, by the requirement of the perturbation data, over an infinite edge, $\mathcal{M}_{\tilde{\Gamma}}$ is just the moduli space of gradient lines of (F, H) that start at $\boldsymbol{x}_M^\circ$. Then the output of $\tilde{\Gamma}$ is labelled by an (unforgettable) element of index $\bar{n} - 1$. Since in the Morse complex of F, $\boldsymbol{x}_M$ is closed, $\tilde{\boldsymbol{m}}_1(\boldsymbol{x}_M^\circ) = \delta_{\mathrm{Morse}}(\boldsymbol{x}_M^\circ) = 0$. Now we show that $\tilde{\boldsymbol{m}}_2(\boldsymbol{x}_M^\circ, \boldsymbol{a}) = \boldsymbol{a} = (-1)^{|\boldsymbol{a}|}\tilde{\boldsymbol{m}}_2(\boldsymbol{a}, \boldsymbol{x}_M^\circ)$ for any generator $\boldsymbol{a}$ of $\widetilde{CF}^*(\bar{L})$. If $\boldsymbol{a} = \boldsymbol{x}_M^\circ$, then the second identity is the same as the first one. If $\boldsymbol{a} \neq \boldsymbol{x}_M^\circ$, then the proof of the second identity is also similar to the proof of the first one. Hence we only show $\tilde{\boldsymbol{m}}_2(\boldsymbol{x}_M^\circ, \boldsymbol{a}) = \boldsymbol{a}$. Indeed, suppose a configuration with vortex type $\tilde{\Gamma}$ and weighted type $\hat{\Gamma}$ contributes to $\tilde{\boldsymbol{m}}_2(\boldsymbol{x}_M^\circ, \boldsymbol{a})$. Let t be the first input of $\hat{\Gamma}$. Then for the same reason as above, $\hat{\Gamma}_t = \varnothing$, which implies that $\hat{\Gamma}$ is a Y-shape. The condition that the index of $\tilde{\Gamma}$ is zero implies that the other two boundary tails are labelled by critical points of the same Morse index, say $\boldsymbol{x}'$ and $\boldsymbol{x}''$. By our construction of the perturbation data on Y-shapes (see Lemma 11.10), $\mathcal{M}_{\tilde{\Gamma}} = \varnothing$ if $\boldsymbol{x}' \neq \boldsymbol{x}''$, and contains a single element if $\boldsymbol{x}' = \boldsymbol{x}''$. The sign of this single element is positive. Hence it shows that $\tilde{\boldsymbol{n}}_2(\boldsymbol{x}_M^\circ, \boldsymbol{a}) = \boldsymbol{a}$ for a generator $\boldsymbol{a}$. Similarly one has $\tilde{\boldsymbol{m}}_{\underline{k}}(\cdots, \boldsymbol{x}_M^\circ, \cdots) = 0$ for $\underline{k} \geqslant 3$ because for any unbroken weighted type with more than three inputs, forgetting one input results in a nonempty weighted type.

The unitality of $\boldsymbol{x}_M^\circ$ for $\widetilde{\mathrm{Fuk}}^{\mathfrak{c}}(\bar{L})$ is exactly the same. We now prove the unitality of the morphism, which means

$$\tilde{\varphi}_1\big(\boldsymbol{x}_M^\circ\big) = \boldsymbol{x}_M^\circ; \quad \tilde{\varphi}_{\underline{k}}\big(\ \cdots, \boldsymbol{x}_M^\circ, \cdots\big) = 0,\ \forall \underline{k} \geqslant 2.$$

To prove the first identity, consider any $\tilde{\Gamma} \in \tilde{\mathbf{T}}^0_\diamond(\tilde{\boldsymbol{x}})$ such that the moduli space $\mathcal{M}_{\tilde{\Gamma}} \neq \varnothing$ may contribute to $\tilde{\varphi}_1(\boldsymbol{x}_M^\circ)$. Suppose t is the boundary input labelled by $\boldsymbol{x}_M^\circ$. If $\omega(\tilde{\Gamma}) > 0$, then $\mathrm{L}(\hat{\Gamma}) \neq \varnothing$. Therefore, $\hat{\Gamma}_t$ is nonempty, which results in a nonempty moduli space $\mathcal{M}_{\tilde{\Gamma}_t}$ with expected dimension -1. Hence $\omega(\tilde{\Gamma}) = 0$. Then since $\mathbf{ind}\tilde{\Gamma} = 0$, the output of $\tilde{\Gamma}$ must be labelled by $\boldsymbol{x}_M^\circ$ or $\boldsymbol{x}_M^\bullet$. However the latter was excluded by our requirement on weighting types (see Figure 9). Therefore, it follows $\tilde{\varphi}_1(\boldsymbol{x}_M^\circ) = \boldsymbol{x}_M^\circ$.

On the other hand, for all admissible vortex type $\tilde{\Gamma}$ with $\underline{k} \geqslant 2$ boundary inputs, one of whose boundary tails is labelled by $\boldsymbol{x}_M^\circ$, forgetting the first one results in a nonempty type $\tilde{\Gamma}_t$ with $\mathbf{ind}\tilde{\Gamma}_t = -1$. Hence $\mathcal{M}_{\tilde{\Gamma}} = \varnothing$ by transversality and the coherence of the perturbation data.

Lastly, by our choice of perturbation data discussed in Subsection 7.5, the A_∞ algebras $\widetilde{\mathrm{Fuk}}^K(L)$ and $\widetilde{\mathrm{Fuk}}^{\mathfrak{c}}(\bar{L})$ are identical on the classical level and $\tilde{\boldsymbol{\varphi}}$ is the identify on the classical level. Hence $\tilde{\boldsymbol{\varphi}}$ is a higher order deformation of the identity (recall its definition in Subsection 8.1). This finishes the proof. $\square$

It remains to prove Item (e) and Item (f) of Theorem 8.18. Indeed, when J_X satisfies the conditions assumed in Hypothesis 1.5 and/or Hypothesis 1.7, in our perturbation data, the perturbed almost complex structure can be made sufficiently close to J_X and satisfy the same conditions. The argument uses various compactness theorems and depends on an energy bound. Such energy bound is dictated by the underlying combinatorial types of the moduli spaces. Then Item (e) and Item (f) of Theorem 8.18 follow from dimension counting, whose proofs are very similar to those of Proposition 8.11, 8.14 and 8.17. The details are left to the reader. This finishes the proof of Theorem 8.18.

References

[Abo11] Mohammed Abouzaid, *A topological model for the Fukaya categories of plumbings*, Journal of Differential Geometry **87** (2011), no. 1, 1–80.

[Bat93] Victor Batyrev, *Quantum cohomology rings of toric manifolds*, Journés de géométrie algébrique d'Orsay (Orsay, 1992), Astérisque, vol. 218, 1993, pp. 9–34.

[BC07] Paul Biran and Octav Cornea, *Quantum structures for Lagrangian submanifolds*, http://arxiv.org/abs/0708.4221, 2007.

[BGV03] Nicole Berline, Ezra Getzler, and Michèle Vergne, *Heat kernels and Dirac operators*, Grundlehren Text Editions, Springer Berlin Heidelberg, 2003.

[BV73] J. M. Boardman and Rainer Vogt, *Homotopy invariant algebraic structures on topological spaces*, Lecture Notes in Mathematics, vol. 347, Springer-Verlag, Berlin-New York, 1973.

[CGMS02] Kai Cieliebak, Ana Gaio, Ignasi Mundet i Riera, and Dietmar Salamon, *The symplectic vortex equations and invariants of Hamiltonian group actions*, Journal of Symplectic Geometry **1** (2002), no. 3, 543–645.

[CGS00] Kai Cieliebak, Ana Gaio, and Dietmar Salamon, *J-holomorphic curves, moment maps, and invariants of Hamiltonian group actions*, International Mathematics Research Notices **16** (2000), 831–882.

[CL14] Kwokwai Chan and Siu-Cheong Lau, *Open Gromov–Witten invariants and superpotentials for semi-Fano toric surfaces*, International Mathematics Research Notices (2014).

[Cle86] Herbert Clemens, *Curves on generic hypersurfaces*, Annales scientifique de l'É.N.S. **19** (1986), no. 4, 629–636.

[CLLT17] Kwokwai Chan, Siu-Cheong Lau, Naichung Leung, and Hsian-Hua Tseng, *Open Gromov–Witten invariants, mirror maps, and Seidel representations for toric manifolds*, Duke Mathematical Journal **166** (2017), 1405–1462.

[CM07] Kai Cieliebak and Klaus Mohnke, *Symplectic hypersurfaces and transversality in Gromov–Witten theory*, Journal of Symplectic Geometry **5** (2007), 281–356.

[CO06] Cheol-Hyun Cho and Yong-Geun Oh, *Floer cohomology and disk instantons of Lagrangian torus fibres in Fano toric manifolds*, Asian Journal of Mathematics **10** (2006), 773–814.

[CP14] Cheol-Hyun Cho and Mainak Poddar, *Holomorphic orbidiscs and Lagrangian Floer cohomology of symplectic toric orbifolds*, Journal of Differential Geometry **98** (2014), 21–116.

[CW15] François Charest and Chris Woodward, *Floer cohomology and flips*, http://arxiv.org/abs/1508.01573, 2015.

[CW17] ———, *Floer trajectories and stabilizing divisors*, Journal of Fixed Point Theory **19** (2017), 1165–1236.

[CWW17] Bohui Chen, Bai-Ling Wang, and Rui Wang, *The asymptotic behavior of finite energy symplectic vortices with admissible metrics*, https://arxiv.org/abs/1706.07027, 2017.

[Flo88a] Andreas Floer, *Morse theory for Lagrangian intersections*, Journal of Differential Geometry **28** (1988), 513–547.

[Flo88b] ———, *The unregularized gradient flow of the symplectic action*, Communications on Pure and Applied Mathematics **41** (1988), no. 6, 775–813.

[FO99] Kenji Fukaya and Kaoru Ono, *Arnold conjecture and Gromov–Witten invariants for general symplectic manifolds*, Topology **38** (1999), 933–1048.

[FOOO09] Kenji Fukaya, Yong-Geun Oh, Hiroshi Ohta, and Kaoru Ono, *Lagrangian intersection Floer theory: anomaly and obstruction, part i & ii*, AMS/IP studies in advanced mathematics, vol. 46, American Mathematical Society, 2009.

[FOOO10] ______, *Lagrangian Floer theory on compact toric manifolds, I*, Duke Mathematical Journal **151** (2010), 23–175.

[FOOO11] ______, *Lagrangian Floer theory on compact toric manifolds II: bulk deformations*, Selecta Mathematica New Series **17** (2011), 609–711.

[FOOO16] ______, *Lagrangian Floer theory and mirror symmetry on compact toric manifolds*, Astérisque **376** (2016), 1–340.

[Fra04a] Urs Frauenfelder, *Floer homology of symplectic quotients and the Arnold–Givental conjecture*, Ph.D. thesis, Swiss Federal Institute of Technology, 2004.

[Fra04b] ______, *Floer homology of symplectic quotients and the Arnold–Givental conjecture*, International Mathematics Research Notices **2004** (2004), no. 42, 2179–2269.

[GI17] Eduardo González and Hiroshi Iritani, *Seidel elements and potential functions of holomorphic disk counting*, Tohoku Mathematical Jounal **69** (2017), 327–368.

[Giv95] Alexander Givental, *Homological geometry and mirror symmetry*, Proceedings of the International Congress of Mathematicians, Vol. 1, 2 (Zürich, 1994), Birkhäuer, 1995, pp. 472–480.

[GS99] Victor Guillemin and Shlomo Sternberg, *Supersymmetry and equivariant de Rham theory*, Mathematics Past and Present, Springer-Verlag, Berlin, 1999.

[GS05] Ana Gaio and Dietmar Salamon, *Gromov–Witten invariants of symplectic quotients and adiabatic limits*, Journal of Symplectic Geometry **3** (2005), no. 1, 55–159.

[Har77] Robin Hartshorne, *Algebraic geometry*, Graduate Texts in Mathematics, vol. 52, Springer-Verlag, 1977.

[HV00] Kentaro Hori and Cumrun Vafa, *Mirror symmetry*, http://arxiv.org/abs/hep-th/0002222, 2000.

[HWZ07] Helmut Hofer, Krzysztof Wysocki, and Eduard Zehnder, *A general Fredholm theory. I. A splicing-based differential geometry*, Journal of European Mathematical Society **9** (2007), 841–876.

[Kir84] Frances Kirwan, *Cohomology of quotients in symplectic and algebraic geometry*, Mathematical Notes, vol. 31, Princeton University Press, 1984.

[KN79] George Kempf and Linda Ness, *The length of vectors in representation spaces*, Algebraic Geometry (Proc. Summer Meeting, Univ. Copenhagen, Copenhagen, 1978), Lecture Notes in Mathematics, vol. 732, Springer, Berline, 1979, pp. 233–243.

[LT98] Jun Li and Gang Tian, *Virtual moduli cycles and Gromov–Witten invariants of general symplectic manifolds*, Topics in symplectic 4-manifolds (Irvine, CA, 1996), First International Press Lecture Series., no. I, International Press, Cambridge, MA, 1998, pp. 47–83.

[Mes18] Stephan Mescher, *Perturbed gradient flow trees and A_∞-algebra structures in Morse cohomology*, Atlantis Studies in Dynamical Systems, Springer, 2018.

[MFK94] David Mumford, John Fogarty, and Frances Kirwan, *Geometric invariant theory*, Ergebnisse der Mathematik und ihrer Grenzgebiete (2), vol. 34, Springer-Verlag, Berline, 1994.

[MS04] Dusa McDuff and Dietmar Salamon, *J-holomorphic curves and symplectic topology*, Colloquium Publications, vol. 52, American Mathematical Society, 2004.

[MT09] Ignasi Mundet i Riera and Gang Tian, *A compactification of the moduli space of twisted holomorphic maps*, Advances in Mathematics **222** (2009), 1117–1196.

[Mun99] Ignasi Mundet i Riera, *Yang–Mills–Higgs theory for symplectic fibrations*, Ph.D. thesis, Universidad Autónoma de Madrid, 1999.

[Mun03] ______, *Hamiltonian Gromov–Witten invariants*, Topology **43** (2003), no. 3, 525–553.

[Par16] John Pardon, *An algebraic approach to virtual fundamental cycles on moduli spaces of pseudo-holomorphic curves*, Geometry and Topology **20** (2016), 779–1034.

[PR07] Gianluca Pacienza and Erwan Rousseau, *On the logarithmic Kobayashi conjecture*, Journal für die Reine und Angewandte Mathematik **611** (2007), 221–235.

[Smi17] Jack Smith, *Symmetry in monotone Lagrangian Floer theory*, Ph.D. thesis, Trinity College, University of Cambridge, 2017.

[Tho54] René Thom, *Quelques propriétés globales des variétés différentiables*, Commentarii Mathematici Helvetici **28** (1954), 17–86.

[Voi96] Claire Voisin, *On a conjecture of Clemens on rational curves on hypersurfaces*, Journal of Differential Geometry **44** (1996), 200–214.

[Voi98] ______, *A correction on "A conjecture of Clemens on rational curves on hypersurfaces"*, Journal of Differential Geometry **49** (1998), 601–611.

[VXne] Sushmita Venugopalan and Guangbo Xu, *Local model for the moduli space of affine vortices*, International Journal of Mathematics (Published Online).

[Wit93] Edward Witten, *Phases of $N = 2$ theories in two dimensions*, Nuclear Physics **B403** (1993), 159–222.

[Woo11] Chris Woodward, *Gauged Floer theory for toric moment fibers*, Geometric and Functional Analysis **21** (2011), 680–749.

[Woo15] ———, *Quantum Kirwan morphism and Gromov–Witten invariants of quotients I, II, III*, Transformation Groups **20** (2015), 507–556, 881–920, 1155–1193.

[WX17] Dongning Wang and Guangbo Xu, *A compactification of the moduli space of disk vortices in adiabatic limit*, Mathematische Zeitschrift **287** (2017), 405–459.

[Xu] Guangbo Xu, *Gluing affine vortices*, https://arxiv.org/abs/1610.09764, submitted.

[Xu13] ———, *Moduli space of twisted holomorphic maps with Lagrangian boundary condition: compactness*, Advances in Mathematics **242** (2013), 1–49.

[Xu16] ———, *Gauged Hamiltonian Floer homology I: definition of the Floer homology groups*, Transactions of the American Mathematical Society **368** (2016), 2967–3015.

[Zil05] Fabian Ziltener, *Symplectic vortices on the complex plane and quantum cohomology*, Ph.D. thesis, Swiss Federal Institute of Technology Zurich, 2005.

[Zil09] ———, *The invariant symplectic action and decay for vortices*, Journal of Symplectic Geometry **7** (2009), no. 3, 357–376.

[Zil14] ———, *A quantum Kirwan map: bubbling and Fredholm theory*, Memiors of the American Mathematical Society **230** (2014), no. 1082, 1–129.

[Zin08] Aleksey Zinger, *Pseudocycles and integral homology*, Transactions of the American Mathematical Society **360** (2008), 2741–2765.

[Zin11] ———, *A comparison theorem for Gromov–Witten invariants in the symplectic category*, Advances in Mathematics **228** (2011), no. 1, 535–574.

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