MORPHISMS OF COHFT ALGEBRAS AND QUANTIZATION OF THE KIRWAN MAP

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Abstract. We introduce a notion of morphism of CohFT algebras, based on the analogy with A_∞ morphisms. We outline the construction of a "quantization" of the classical Kirwan morphism to a morphism of CohFT algebras from the equivariant quantum cohomology of a G-variety to the quantum cohomology of its geometric invariant theory or symplectic quotient, and an example relating to the orbifold quantum cohomology of a compact toric orbifold. Finally we identify the space of Cartier divisors in the moduli space of scaled marked curves; these appear in the splitting axiom.

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1. Introduction

In order to formalize the algebraic structure of Gromov-Witten theory Kontsevich and Manin introduced a notion of *cohomological field theory* (CohFT), see [26, Section IV]. The correlators of such a theory depend on the choice of cohomological classes on the moduli space of stable marked curves and satisfy a splitting axiom for each boundary divisor. In genus zero the moduli space of stable marked curves may be viewed as the complexification of Stasheff's *associahedron* from [30], and the notion of CohFT may be related to the notion of A_∞ algebra: dualizing one of the factors gives rise to a collection of multilinear maps that we call a *CohFT algebra*. The full CohFT is related to the CohFT algebra in the same way that a Frobenius algebra is related to the underlying algebra. Recall that Dubrovin [13] constructed from any CohFT a *Frobenius manifold*, which is a manifold with a family of multiplications on its tangent spaces together with some additional data.

Here we introduce a notion of *morphism of CohFT algebras* which is a "closed string" analog of a morphism of A_∞ algebras. The additional data in the structure maps is the choice of cohomological classes on *moduli space of stable scaled marked lines* introduced in Ziltener [36].

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This space was studied in Ma'u-Woodward [27] and identified with the complexification of Stasheff's multiplihedron appearing in the definition of A_∞ map [30]. The splitting axiom for a morphism of CohFT algebras gives a relation on the structure maps for each divisor relation. Any morphism of CohFT algebras has the property that the linearization at any point is an algebra morphism in the usual sense. This fits in well with the language of Hertling-Manin [20] of *F-manifolds*.

The definition of morphism of CohFT algebra is motivated by an attempt to extend the mirror theorems of Givental [17], Lian-Liu-Yau [25] and others beyond the case of semipositive toric quotients, as has also been discussed by many authors, for example, Iritani [21]. In the second part of the paper we describe a *quantum Kirwan* morphism of CohFT algebras from the equivariant quantum cohomology $QH_G(X)$ of a smooth polarized projective G-variety X to the (possibly orbifold) quantum cohomology $QH(X/\!\!/G)$ of the symplectic/git quotient $X/\!\!/G$. The existence of this morphism depends on results of the last two authors and Venugopalan; see [33]. Morphisms of CohFT algebras provide an "algebraic home" for the counts of "vortex bubbles" that first appeared in the study by Gaio-Salamon [15] of the relationship between gauged Gromov-Witten invariants of a G-variety and the Gromov-Witten invariants of the quotient $X/\!\!/G$ [33]. Applying the quantum Kirwan morphism to the special case of quotients of vector spaces by tori, one obtains a Batyrev-style presentation of the (possibly orbifold) quantum cohomology of a toric Deligne-Mumford stack at a special point; this reproduces partial results by Coates-Corti-Lee-Tseng [8]. We discuss several conjectures (*quantum Kirwan surjectivity* and *quantum reduction in stages*) which arise naturally in this context. In the last part of the paper, we describe which combinations of boundary divisors in the moduli space of stable scaled lines are Cartier, that is, have dual cohomology classes.

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2. Morphisms of CohFT algebras

In this section we describe the definition of morphisms of CohFT algebras. Let $\overline{M}_n$ denote the Grothendieck-Knudsen moduli space of isomorphism classes of genus zero n-marked stable curves [24], which is a smooth projective variety of dimension $\dim(\overline{M}_n) = n - 3$.

Remark 2.1. (Boundary divisors for the Grothendieck-Knudsen space) The boundary of $\overline{M}_n$ consists of the following divisors: for each splitting $\{1, \ldots, n\} = I_1 \cup I_2$ with $|I_1|, |I_2| \geq 2$ a divisor

$$\iota_{I_1 \cup I_2} : D_{I_1 \cup I_2} \to \overline{M}_n$$

corresponding to the formation of a separating node, splitting the curve into irreducible components with markings I_1, I_2. The divisor $D_{I_1 \cup I_2}$ is isomorphic to $\overline{M}_{|I_1|+1} \times \overline{M}_{|I_2|+1}$. Let

$$\delta_{I_1 \cup I_2} \in H^2(\overline{M}_n)$$

denote its dual cohomology class. For any $\beta \in H(\overline{M}_n)$, let

$$\iota^*_{I_1 \cup I_2} \beta = \sum_j \beta_{1,j} \otimes \beta_{2,j} \tag{1}$$

denote the Künneth decomposition of its restriction to $D_{I_1 \cup I_2}$.

Definition 2.2. (CohFT algebras) An *(even, genus zero) cohomological field theory algebra* over a $\mathbb{Q}$-ring Λ is a datum $(V, (\mu^n)_{n\geq 2})$ where V is a Λ-module and $(\mu^n)_{n\geq 2}$ is a collection of multilinear *composition maps*

$$\mu^n : V^n \times H(\overline{M}_{n+1}, \Lambda) \to V$$

such that each μ^n is invariant under the natural action of the symmetric group S_n and the maps $(\mu^n)_{n\geq 2}$ satisfy a *splitting axiom*: for each partition $I_1 \cup I_2 = \{1, \ldots, n\}$,

$$\mu^n(v_1, \ldots, v_n; \beta \wedge \delta_{I_1 \cup I_2}) = \sum_j \mu^{|I_2|+1}(\mu^{|I_1|}(v_i, i \in I_1; \beta_{1,j}), v_i, i \in I_2; \beta_{2,j})$$

where $\beta_{1,j}, \beta_{2,j}$ are as in (1).

Remark 2.3. It would be more natural to use tensor products in the above formula but the use of symbols $\otimes$ instead of commas , makes the formulas substantially longer.

Remark 2.4. (Filtered CohFT algebras) In our applications, Λ will be a *filtered* $\mathbb{Q}$*-ring* by which we mean a union of decreasing rings $\Lambda_a, a \in \mathbb{R}$:

$$\Lambda = \cup_{a\in\mathbb{R}}\Lambda_a, \quad \Lambda_a \supset \Lambda_b \text{ for all } a < b, \quad \cap_{a\in\mathbb{R}}\Lambda_a = \{0\}.$$

A filtered CohFT algebra is a CohFT algebra V with a filtration $(V_a)_{a\in\mathbb{R}}$ compatible with the Λ-module structure, that is, such that $\Lambda_a V_b \subset V_{a+b}, \forall a, b \in \mathbb{R}$, and such that each structure map μ^n maps $V_a^n \times H(\overline{M}_{0,n})$ to V_a, for all $a \in \mathbb{R}$.

Remark 2.5.
(a) (Comparison with A_∞ algebras) The collection of composition maps $(\mu^n)_{n\geq 2}$ (which are termed in Manin [26] Comm_∞-structures) may be viewed as "complex analogs" of the A_∞ -structure maps of Stasheff, in the sense that the relevant moduli spaces have been "complexified".
(b) (Relations via divisor equivalences) The various relations on the divisors in $\overline{M}_n$ give rise to relations on the maps μ^n. In particular the relation $[D_{\{0,3\}\cup\{1,2\}}] = [D_{\{0,1\}\cup\{2,3\}}]$ in $H^2(\overline{M}_4)$ implies that $\mu^2 : V \times V \to V$ is associative.

The notion of *morphism* of CohFT algebras is based on the geometry of the *complexified multiplihedron* $\overline{M}_{n,1}(\mathbb{A})$ introduced in Ziltener [36] and studied further in Ma'u-Woodward [27].

Definition 2.6. (Scalings on smooth curves)
(a) A *non-degenerate scaling* on a smooth genus zero complex projective curve C with *root marking* $z_0 \in C$ is a meromorphic one-form $\lambda : C \to T^\vee C$ with the property that λ has a single pole of order two at z_0, so that λ equips $C - \{z_0\}$ with the structure of an affine line. Denote by $\Sigma(C, z_0)$ the space of scalings on C with pole at z_0, and by $\overline{\Sigma}(C, z_0)$ the compactification $\overline{\Sigma}(C, z_0) = \Sigma(C, z_0) \cup \{0, \infty\}$.
(b) An *n-marked scaled line* is a smooth projective curve of genus zero equipped with a non-degenerate scaling $\lambda \in \Sigma(C, z_0)$ and a collection $z_1, \ldots, z_n \in C$ of points distinct from each other and from the root marking z_0.

Let $M_{n,1}(\mathbb{A})$ denote the moduli space of isomorphism classes of n-marked scaled lines. We may view $M_{n,1}(\mathbb{A})$ as the moduli space of isomorphism classes of n-markings on an affine line $\mathbb{A}$, where two sets of markings are isomorphic if they are related by translation. $M_{n,1}(\mathbb{A})$ admits a compactification by allowing nodal curves with possible degenerate scalings as follows.

Definition 2.7. (a) (Dualizing sheaf and its projectivization) Recall from e.g. [3, p.91] that if C is a genus zero nodal curve then the dualizing sheaf ω_C on C is locally free of rank one, that is, a line bundle. Explicitly, if $\tilde{C}$ denotes the normalization of C (the disjoint union of the irreducible components of C) with nodal points $\{\{w_1^+, w_1^-\}, \ldots, \{w_k^+, w_k^-\}\}$ then ω_C is the sheaf of sections of $\omega_{\tilde{C}} := T^\vee \tilde{C}$ whose residues at the points w_j^+, w_j^- sum to zero, for $j = 1, \ldots, k$. Denote by $\mathbb{P}(\omega_C \oplus \mathbb{C})$ the fiber bundle obtained by adding in a section at infinity.

(b) (Scalings on nodal curves) Let C be a connected projective nodal curve of arithmetic genus zero. A *scaling* on C is a section $\lambda : C \to \mathbb{P}(\omega_C \oplus \mathbb{C})$ such that the restriction of λ to any irreducible component is a (possibly degenerate) scaling as in Definition 2.6.

(c) (Scaled affine lines) A *nodal n-marked scaled line* consists of

(i) a connected projective nodal curve C of arithmetic genus zero,
(ii) a scaling $\lambda : C \to \mathbb{P}(\omega_C \oplus \mathbb{C})$, and
(iii) a collection of markings $z_0, \ldots, z_n \in C$ distinct from the nodes

such that the following *monotonicity conditions* are satisfied:

(i) for each $i = 1, \ldots, n$, there is exactly one irreducible component of $C_{+,i}$ of C on the path of irreducible components between z_0 and z_i on which λ is finite and non-zero, with double pole at the node which disconnects the component from the root marking z_0, and
(ii) the irreducible components other than $C_{+,i}$ on the path of irreducible components between z_i and z_0 have either $\lambda = 0$ (if they can be connected to z_i without passing through $C_{+,i}$) or $\lambda = \infty$ (if they are connected to z_0 without passing through $C_{+,i}$).

A nodal marked scaled affine line is *stable* if each irreducible component with non-degenerate scaling has at least two special points, and each irreducible component with degenerate scaling has at least three special points.

(d) (Combinatorial types of scaled affine lines) The *combinatorial type* of a nodal scaled affine line is the *rooted colored tree* $\Gamma = (\mathrm{V}(\Gamma), \mathrm{E}(\Gamma))$ whose vertices are the irreducible components of C and edges are the nodes and markings, equipped with a bijection from the set of semi-infinite edges $\mathrm{E}_\infty(\Gamma)$ to $\{0, \ldots, n\}$ given by the markings, and a subset of *colored vertices* $\mathrm{V}^+(\Gamma) \subset \mathrm{V}(\Gamma)$ corresponding to irreducible components with non-degenerate scalings. This ends the definition.

Example 2.8. See Figure 1 for an example of a nodal scaled affine line, where irreducible components with $\lambda = 0$ resp. λ finite and non-zero resp. λ is infinite are shown with light resp. medium resp. dark shading. The example shown is not stable, because several of the lightly shaded components and darkly shaded components have less than three special points.

Remark 2.9. (Affine structures on the components with non-degenerate scalings) The monotonicity condition implies that the restriction of λ to any irreducible component $C_{i,+}$ has a unique pole, hence a unique double pole at the nodal point $\hat{z}_i$ connecting $C_{i,+}$ with the component containing z_0, and so defines an affine structure on the complement $C_{i,+} - \hat{z}_i$. The other components have no canonical affine structures.

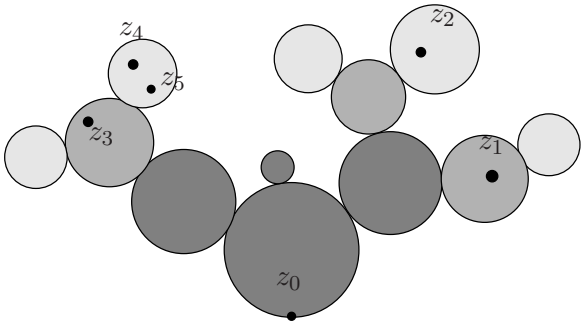

FIGURE 1. An nodal scaled line

Let $\mathcal{M}_{n,1,\Gamma}(\mathbb{A})$ resp. $\overline{M}_{n,1}(\mathbb{A})$ denote the moduli space of isomorphism classes of stable scaled n-marked affine lines of type Γ resp. the union over combinatorial types. We call $\overline{M}_{n,1}(\mathbb{A})$ the *complexified multiplihedron*.

Proposition 2.10. [27] *The spaces $\overline{M}_{n,1}(\mathbb{A})$ admit the structure of quasiprojective resp. projective varieties of dimension*

$$\dim(M_{n,1,\Gamma}(\mathbb{A})) = n - 2 - |\,\mathrm{E}_{<\infty}(\Gamma)| + |\,\mathrm{V}^{+}(\Gamma)|, \quad \dim(\overline{M}_{n,1}(\mathbb{A})) = n - 1. \tag{2}$$

The space $\overline{M}_{n,1}(\mathbb{A})$ was first studied in Ziltener's thesis [36] in the context of gauged Gromov-Witten theory on the affine line.

Example 2.11. (The second complexified multiplihedron) The moduli space $\overline{M}_{n,1}(\mathbb{A})$ in the first non-trivial case $n = 2$ admits an isomorphism

$$\overline{M}_{2,1}(\mathbb{A}) \to \mathbb{P}, \quad [z_1, z_2] \mapsto z_1 - z_2 \tag{3}$$

(here $\mathbb{P}$ is the projective line) with two distinguished points given by nodal scaled affine lines, appearing in the limit where the two markings become infinitely close or far apart, see Figures 2, 3. Here $[z_1, z_2] \in M_{2,1}(\mathbb{A})$ is a point in the open stratum, given by markings at z_1, z_2 modulo translation only on $\mathbb{A}$.

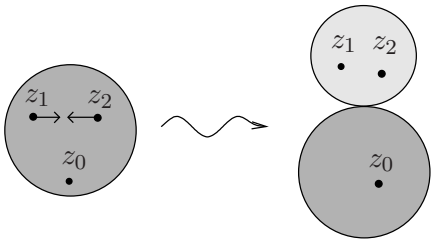

FIGURE 2. Two markings converging

Remark 2.12. (Embedding via forgetful morphisms) More generally, for arbitrary n there exists for any choice $\{i, j\} \subset \{1, \ldots, n\}$ of subset of order 2 a *forgetful morphism*

$$f_{i,j} : \overline{M}_{n,1}(\mathbb{A}) \to \overline{M}_{2,1}(\mathbb{A})$$

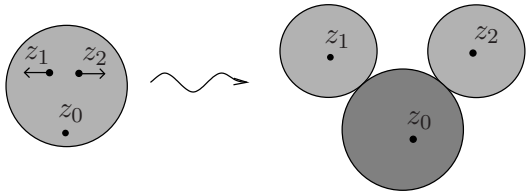

FIGURE 3. Two markings diverging

forgetting the markings other than i, j and collapsing all unstable irreducible components, and for any choice $\{i, j, k, l\} \subset \{0, \ldots, n\}$ of subset of order 4 a forgetful morphism

$$f_{i,j,k,l} : \overline{M}_{n,1}(\mathbb{A}) \to \overline{M}_4$$

given by forgetting the scaling and all markings except i, j, k, l, and collapsing all unstable irreducible components. The product of forgetful morphisms defines an embedding into a product of projective lines.

The variety $\overline{M}_{n,1}(\mathbb{A})$ is not smooth, but rather has toric singularities, see Section 4. The *boundary divisors* are the closures of strata $M_{n,1,\Gamma}$ of codimension one.

Remark 2.13. (Description of the boundary divisors of the complexified multiplihedron) From the dimension formula (2) one sees that there are two types of boundary divisors. First, for any $I \subset \{1, \ldots, n\}$ with $|I| \geq 2$ we have a divisor

$$\iota_I : D_I \to \overline{M}_{n,1}(\mathbb{A})$$

corresponding to the formation of a single bubble containing the markings I. This divisor admits a gluing isomorphism

$$D_I \to \overline{M}_{|I|+1} \times \overline{M}_{n-|I|+1,1}(\mathbb{A}). \tag{4}$$

Call these divisors of *type* I. Second, for any unordered partition $I_1 \cup \ldots \cup I_r$ of $\{1, \ldots, n\}$ with $r \geq 2$ we have a divisor $D_{I_1,\ldots,I_r}$ corresponding to the formation of r bubbles with markings $I_1, \ldots, I_r$, attached to a remaining irreducible component with infinite scaling. This divisor admits a gluing isomorphism

$$D_{I_1,\ldots,I_r} \cong \left(\prod_{i=1}^{r} \overline{M}_{|I_i|,1}(\mathbb{A}) \right) \times \overline{M}_{r+1}. \tag{5}$$

Call these *divisors of type* II. Note that the divisors of type I and type II roughly correspond to the terms in the definition of A_∞ functor.

Recall that a *Weil divisor* on a normal scheme X is a formal, locally finite sum of codimension one subvarieties, while a *Cartier divisor* is a Weil divisor given as the zero set of a meromorphic section of a line bundle with multiplicities given by the order of vanishing of the section [19, Remark 6.11.2]. For smooth varieties, any Weil divisor is Cartier. Since $\overline{M}_{n,1}(\mathbb{A})$ is not smooth, Weil divisors are not necessarily Cartier, in particular, a Weil divisor may not admit a dual cohomology class of degree 2. That is, for a Weil divisor

$$D = \sum_I n_I [D_I] + \sum_{I_1 \sqcup \ldots \sqcup I_r = \{1,\ldots,n\}} n_{I_1,\ldots,I_r} [D_{I_1,\ldots,I_r}] \tag{6}$$

there may or may not exist a class $\delta \in H^2(\overline{M}_{n,1}(\mathbb{A}))$ that satisfies

$$\langle \beta, [D] \rangle = \langle \beta \wedge \delta, [\overline{M}_{n,1}(\mathbb{A})] \rangle.$$

Let $(V, (\mu_V^n)_{n\geq 2})$ and $(W, (\mu_W^n)_{n\geq 2})$ be CohFT algebras over a $\mathbb{Q}$-ring Λ.

Definition 2.14. (Morphisms of CohFT algebras) A *morphism of CohFT algebras* from V to W is a collection of S_n-invariant, multilinear maps

$$\phi^n : V^n \times H(\overline{M}_{n,1}(\mathbb{A})) \to W, \quad n \geq 0$$

such that for any Cartier divisor D of the form (6) with dual class $\delta \in H^2(\overline{M}_{n,1}(\mathbb{A}))$, any $v \in V^n$ and any $\beta \in H(\overline{M}_{n,1}(\mathbb{A}))$

$$\begin{aligned} (7) \quad \phi^n(v, \beta \wedge \delta) = \sum_I n_I \phi^{n-|I|+1}(\mu_V^{|I|}(v_i, i \in I; \cdot), v_j, j \notin I; \cdot)(\iota_I^* \beta) \\ + \sum_{r \leq s, I_1, \ldots, I_r} n_{I_1,\ldots,I_r} \mu_W^s(\phi^{|I_1|}(v_i, i \in I_1; \cdot), \ldots, \\ \phi^{|I_r|}(v_i, i \in I_r; \cdot), \phi^0(1), \ldots, \phi^0(1); \cdot)(\iota_{I_1,\ldots,I_r}^* \beta)/(s-r)! \end{aligned}$$

where $\cdot$ indicates insertion of the Künneth components of $\iota_I^*\beta$, $\iota_{I_1,\ldots,I_r}^*\beta$, using the homeomorphisms (4), (5) and the sum on the right-hand side is, by assumption, finite. The element $\phi^0(1) \in W$ is the *curvature* of the morphism $(\phi^n)_{n\geq 0}$, and $(\phi^n)_{n\geq 0}$ is *flat* if the curvature vanishes. A *morphism of filtered CohFT algebras* V, W is a collection of maps ϕ^n as above such that each ϕ^n preserves the filtrations in the sense that ϕ^n maps $V_a^n \times H(\overline{M}_{n,1}(\mathbb{A}))$ to W_a and (7) is finite modulo W_a for any $a \in \mathbb{R}$.

Example 2.15. $\overline{M}_{2,1}(\mathbb{A}) \cong \mathbb{P}$ and so every Weil divisor is Cartier and any two prime Weil divisors are linearly equivalent. In particular, the equivalence $[D_{\{1,2\}}] = [D_{\{1\},\{2\}}]$ holds in $H^2(\overline{M}_{2,1}(\mathbb{A}), \mathbb{Q}) = \mathbb{Q}$. Hence if $(\phi^n)_{n\geq 0} : (V, (\mu_V^n)_{n\geq 2}) \to (W, (\mu_W^n)_{n\geq 2})$ is a flat morphism of CohFT algebras then $\phi^1 : V \to W$ is a homomorphism, $\phi^1 \circ \mu_V^2 = \mu_W^2 \circ (\phi^1 \times \phi^1)$.

Recall that the notion of CohFT may be reformulated as a Frobenius manifold structure of Dubrovin [13]. Such a structure consists of a datum $(V, g, F, 1, e)$ of an affine manifold V, a metric g on the tangent spaces, a potential F whose third derivatives provide the tangent spaces T_vV with associative multiplications $\star_v : T_v^2 V \to T_v V$, a unity vector field 1 and an Euler vector field e providing a grading. Any CohFT $(V, (\mu^n)_{n\geq 2})$ defines a formal Frobenius manifold [26] with formally associative products

$$(8) \qquad \star_v : T_v^2 V \to T_v V, \quad (w_1, w_2) \mapsto \sum_{n\geq 0} \mu^{n+2}(w_1, w_2, v, \ldots, v)/n!.$$

Formal associativity means that the Taylor coefficients in the expansion of

$$(w_1 \star_v w_2) \star_v w_3 - w_1 \star_v (w_2 \star_v w_3)$$

around $v = 0$ vanish to all orders for any $w_1, w_2, w_3 \in T_vV$; in good cases one has convergence of the corresponding infinite sums. Later, a weaker notion of *F-manifold* was introduced by Manin and Hertling [20], which consists of a pair $(V, \circ)$ where $\circ$ is a family of multiplications on the tangent spaces T_vV satisfying a certain axiom. In other words, one forgets the data $g, 1, e$. This weaker notion is compatible with the notion of morphism of CohFT algebras:

Proposition 2.16. (Algebra homomorphisms on tangent spaces) *Any morphism of CohFT algebras $(\phi^n)_{n\geq 0}$ from V to W defines a formal map*

$$\phi : V \to W, \quad v \mapsto \sum_{n\geq 0} \frac{1}{n!}\phi^n(v,\dots,v;1)$$

with the property that for any $v \in V$ the linearization $D_v\phi : T_vV \to T_{\phi(v)}W$ is a $\star$-homomorphism in the sense that

$$D_v\phi(w_1) \star_{\phi(v)} D_v\phi(w_2) = D_v\phi(w_1 \star_v w_2), \ \forall w_1, w_2 \in T_vV. \tag{9}$$

By a formal map we mean a map from a formal neighborhood of 0 in V to a formal neighborhood of $\phi(0)$ in W. The equation (9) holds in the sense of Taylor expansion around $v = 0$ to all order.

Proof. Consider the divisor relation $D_{\{1,2\}} \sim D_{\{1\},\{2\}}$ on $\overline{M}_{2,1}(\mathbb{A})$. Its pull-back to $\overline{M}_{n,1}(\mathbb{A})$ is the relation

$$\sum_{I_1\ni 1, I_2\ni 2, I_3,\dots,I_r} D_{I_1,I_2,\dots,I_r} \sim \sum_{I\supset\{1,2\}} D_I \tag{10}$$

where the first sum is over unordered partitions $I_1,\dots,I_r$ with $1 \in I_1, 2 \in I_2$ and each $I_j, j = 1,\dots,r$ non-empty, and the second is over subsets $I \subset \{1,\dots,n\}$ with $\{1,2\} \subset I$. Indeed, the map (3) composes with the forgetful map to give a rational function

$$f_{2,1} : \overline{M}_{n,1}(\mathbb{A}) \to \overline{M}_{2,1}(\mathbb{A}) \cong \mathbb{P}.$$

For $n = 2$, this map identifies $D_{\{1,2\}} \to \{0\}$, $D_{\{1\},\{2\}} \to \{\infty\}$. For arbitrary n, one checks using the charts in Ma'u-Woodward [27] that the order of vanishing of $f_{2,1}$ on D_I is 1 if $\{1,2\} \subset I$, -1 if I_1, I_2 separate $1, 2$, and 0 otherwise. Since the partitions are unordered, if I_1, I_2 separate $1, 2$ we may assume that $1 \in I_1$ resp. $2 \in I_2$. Note that the number of ways of choosing the partition on the left-hand-side of (10) with sizes $i_1,\dots,i_r$ is $\begin{pmatrix} n-2 \\ i_1-1\ i_2-1\ i_3\dots i_r \end{pmatrix}$. We compute

$$\begin{aligned}
D_v\phi(w_1 \star_v w_2) &= \sum_{n,i} \frac{1}{(i-2)!(n-i)!}\phi^{n-i+1}\big(\mu_V^i(w_1,w_2,v,\dots,v;1),v,\dots,v;1\big) \\
&= \sum_{n,I}((n-2)!)^{-1}\phi^{n-|I|+1}\big(\mu_V^{|I|}(w_1,w_2,v,\dots,v;1),v,\dots,v;1\big) \\
&= \sum_{I_1\ni 1, I_2\ni 2, I_3,\dots,I_r} ((n-2)!\#\{j \mid I_j = \emptyset\}!)^{-1}\mu_W^r\big(\phi^{|I_1|}(w_1,v,\dots,v;1), \\
&\quad \phi^{|I_2|}(w_2,v,\dots,v;1),\phi^{|I_3|}(v,\dots,v;1),\dots,\phi^{|I_r|}(v,\dots,v;1);1\big) \\
&= \sum_{i_1,i_2\geq 1, i_3\dots,i_r\geq 0} \frac{1}{(i_1-1)!(i_2-1)!i_3!\cdots i_r!(r-2)!}\mu_W^r\big(\phi^{i_1}(w_1,v,\dots,v;1), \\
&\quad \phi^{i_2}(w_2,v,\dots,v;1),\phi^{i_3}(v,\dots,v;1),\dots,\phi^{i_r}(v,\dots,v;1);1\big) \\
&= \sum_r \frac{1}{(r-2)!}\mu_W^r\big(D_v\phi(w_1),D_v\phi(w_2),\phi(v),\dots,\phi(v)\big) \\
&= D_v\phi(w_1) \star_{\phi(v)} D_v\phi(w_2)
\end{aligned}$$

where the right-hand-side is assumed to be a finite sum (modulo any W_a, for a morphism of filtered CohFT algebras). Here the first equality is by definition of $\phi, \star_v$ and the second replaces the sum over i with the sum over subsets I containing $1, 2$. The third follows from the splitting axiom (7), where the elements of the partition $I_1, \ldots, I_r$ may be empty. The fourth equality replaces the sum over unordered partitions $I_1, \ldots, I_r$ with $I_1 \ni 1, I_2 \ni 2$ with the sum over their sizes $i_1, \ldots, i_r$, with the additional factorial $(r-2)!$ arising from the possible orderings of the subsets $I_3, \ldots, I_r$. The fifth equality follows by definition of ϕ, μ_W, and the last equality follows by definition of $\star_{\phi(v)}$. □

Remark 2.17. It would be interesting to characterize which $\star$-morphisms arise from morphisms of CohFT algebras. This would require a study of the cohomology ring of $\overline{M}_{n,1}(\mathbb{A})$ along the lines of Keel [22] for the moduli space of stable marked genus zero curves; this paper is essentially a partial study of the second cohomology group only. The most naive possibility would be an analog of Keel's result [22], namely that $H(\overline{M}_{n,1}(\mathbb{A}))$ is generated by the classes of the Cartier boundary divisors modulo the relations given by the preimages of $D_{\{1\}\{2\}} - D_{\{1,2\}}$ under the forgetful morphism $f_{ij} : \overline{M}_{n,1}(\mathbb{A}) \to \overline{M}_{2,1}(\mathbb{A})$, as i, j range over distinct elements of $\{1, \ldots, n\}$, and the products $D'D''$, if D' and D'' are disjoint Cartier divisors.

3. Quantum Kirwan morphism

In this section we describe the motivating example for the theory of morphisms of CohFT algebras in the previous section, the quantum Kirwan morphism. Let G be a compact Lie group, $G_{\mathbb{C}}$ its complexification, and X be a smooth projective $G_{\mathbb{C}}$-variety equipped with a polarization (ample G-line bundle) such that G acts locally freely on the semistable locus. The classical Kirwan morphism $H_G(X) \to H(X/\!\!/G)$ is surjective by Kirwan's thesis [23]. Computing the kernel of the Kirwan morphism therefore gives a presentation of the cohomology ring of the quotient $X/\!\!/G$. Let $QH_G(X)$ resp. $QH(X/\!\!/G)$ denote the corresponding quantum cohomologies defined over the universal Novikov field. Each has the structure of a CohFT algebra, with products given by suitable counts of genus zero stable maps. The quantum version of the Kirwan morphism is a morphism of CohFT algebras

$$Q\kappa : QH_G(X) \to QH(X/\!\!/G).$$

The virtual fundamental cycles are constructed algebraically in [33]. We describe first the symplectic approach.

From the symplectic point of view the quantum Kirwan morphism is defined by a count of *affine vortices* introduced in Ziltener [36], [35]. There is also an algebro-geometric interpretation, as a count of certain morpisms to the quotient stack $X/G_{\mathbb{C}}$, that we present later. Let $\mathfrak{g}$ denote the Lie algebra of G, and let $\Phi : X \to \mathfrak{g}^\vee$ be a moment map for the action of G on X arising from a unitary connection on the polarization. For any connection $A \in \Omega^1(\mathbb{A}, \mathfrak{g})$, we denote by $F_A \in \Omega^2(\mathbb{A}, \mathfrak{g})$ its curvature. We assume that $\mathfrak{g}$ is equipped with an invariant metric inducing an identification $\mathfrak{g} \to \mathfrak{g}^\vee$.

Definition 3.1. (Affine symplectic vortices) An *n-marked affine symplectic vortex* to X is a datum $(A, u, \underline{z})$, where $A \in \Omega^1(\mathbb{A}, \mathfrak{g})$ is a connection on the trivial bundle, $u : \mathbb{A} \to X$ is a holomorphic with respect to the complex structure determined by A, $\underline{z} = (z_1, \ldots, z_n) \in \mathbb{A}^n$ is a collection of distinct points, and

$$F_A + u^* \Phi \, \mathrm{Vol}_{\mathbb{A}} = 0.$$

Here $\mathrm{Vol}_{\mathbb{A}} = \frac{i}{2}\mathrm{d}z \wedge \mathrm{d}\overline{z}$ is the standard real area form on $\mathbb{A}$.

An *isomorphism* of marked symplectic vortices $(A_j, u_j, \underline{z}_j), j = 0, 1$ is an automorphism of the trivial bundle $\phi : \mathbb{A} \times G \to \mathbb{A} \times G$ such that $\phi^* A_1 = A_0$ and $\phi^* u_1 = u_0$ (thinking of u_0, u_1 as sections of the associated X-bundle) such that ϕ covers a translation on the base, that is, there exists a $\tau \in \mathbb{C}$ such that $\pi \circ \phi(z, g) = z + \tau$ for all $z, g \in \mathbb{A} \times G$, and $z_{i,1} = z_{i,0} + \tau$ for $i = 1, \ldots, n$.

The *energy* of a vortex $(A, u, \underline{z})$ is given by

$$E(A, u) = \frac{1}{2} \int_{\mathbb{A}} \left(\|\mathrm{d}_A u\|^2 + \|F_A\|^2 + \|u^* \Phi\|^2 \right) \mathrm{Vol}_{\mathbb{A}} . \tag{11}$$

This ends the definition.

Let $M^G_{n,1}(\mathbb{A}, X)$ denote the moduli space of isomorphism classes of finite energy n-marked vortices on $\mathbb{A}$ with values in X. The following Hitchin-Kobayashi correspondence gives an algebro-geometric description of the moduli space of affine vortices by work of Venugopalan [31], Xu [34], and Venugopalan-Woodward [32]. By definition [12] a morphism u from the projective line $\mathbb{P}$ to the quotient stack $X/G_{\mathbb{C}}$ consists of a $G_{\mathbb{C}}$-bundle $P \to \mathbb{P}$ together with a section $\mathbb{P} \to P \times_{G_{\mathbb{C}}} X$. By the git quotient $X/\!\!/G_{\mathbb{C}}$ we mean the stack-theoretic quotient of the semi-stable locus by the group action; if stable=semistable then $X/\!\!/G_{\mathbb{C}}$ has coarse moduli space the projective variety considered in Mumford et al [28].

Theorem 3.2. (Classification of affine vortices) *Suppose that X is a smooth polarized projective $G_{\mathbb{C}}$-variety such that $G_{\mathbb{C}}$ acts freely on the semistable locus of X. There is a bijection between isomorphism classes of finite energy affine vortices and isomorphism classes of morphisms u from the projective line $\mathbb{P}$ to the quotient stack $X/G_{\mathbb{C}}$ such that $u(\infty)$ lies in the semistable locus $X/\!\!/G_{\mathbb{C}} \subset X/G_{\mathbb{C}}$.*

The moduli space $M^G_{n,1}(\mathbb{A}, X)$ admits a compactification $\overline{M}^G_{n,1}(\mathbb{A}, X)$ allowing nodal scaled lines as the domain:

Definition 3.3. (Affine scaled gauged maps) An *affine marked nodal scaled gauged map* to X is a marked nodal scaled line $(C, \lambda, \underline{z})$ together with a morphism $u : C \to X/G_{\mathbb{C}}$ such that

(a) (Semistable bundle where the scaling is zero) for each irreducible component C_i with zero scaling $\lambda | C_i = 0$, the G-bundle on C_i is semistable, hence trivializable;
(b) (Semistable point where the scaling is infinite) for each $z \in C$ with $\lambda(z) = \infty$, the image $u(z)$ lies in the semistable locus $X/\!\!/G_{\mathbb{C}}$.

A nodal scaled morphism is *stable* if it has no infinitesimal automorphisms, or equivalently, if each irreducible component on which u is trivial has at least three special points, or two special points and non-degenerate scaling. This ends the definition.

Remark 3.4. (a) (Evaluation and forgetful morphisms) Let $\overline{M}^G_{n,1}(\mathbb{A}, X)$ denote the moduli space of isomorphism classes of stable nodal scaled maps to X. $\overline{M}^G_{n,1}(\mathbb{A}, X)$ admits an evaluation map at the markings, and if the action of G on the semistable locus is free, an additional evaluation map at infinity to $X/\!\!/G$ [36], [35]:

$$\mathrm{ev} \times \mathrm{ev}_\infty : \overline{M}^G_{n,1}(\mathbb{A}, X) \to (X/G_{\mathbb{C}})^n \times X/\!\!/G_{\mathbb{C}}.$$

For $n > 0$, there is a forgetful morphism to the moduli space of scaled lines,

$$f : \overline{M}^G_{n,1}(\mathbb{A}, X) \to \overline{M}^G_{n,1}(\mathbb{A}).$$

(b) (The locally free case) In the case that G acts only *locally* freely on the semistable locus in X, the quotient $X/\!\!/G$ is an orbifold or smooth *Deligne-Mumford stack*. The Hitchin-Kobayashi correspondence in this case relates affine vortices to representable morphisms of a *weighted projective line* $\mathbb{P}(1, r) \to X/G_{\mathbb{C}}$ for some $r > 0$ such that ∞ maps to the semistable locus, so that the evaluation map at infinity

$$\mathrm{ev}_\infty : \overline{M}^G_{n,1}(\mathbb{A}, X) \to \overline{I}_{X/\!\!/G_{\mathbb{C}}}$$

takes values in the *rigidified inertia stack*

$$\overline{I}_{X/\!\!/G_{\mathbb{C}}} := \cup_{r\geq 1} \mathrm{Hom}^{\mathrm{rep}}(\mathbb{P}(r), X/\!\!/G_{\mathbb{C}})/\mathbb{P}(r)$$

of representable morphisms from $\mathbb{P}(r) = B\mathbb{Z}_r$ to $X/\!\!/G_{\mathbb{C}}$ modulo $\mathbb{P}(r)$, for some integer $r \geq 1$. See Abramovich-Olsson-Vistoli [2] and Abramovich-Graber-Vistoli [1] for more on the definition of $\overline{I}_{X/\!\!/G}$.

The quantum Kirwan map is defined by virtual integration over the moduli space of affine vortices introduced in the previous subsection. Existence and axiomatic properties of virtual fundamental classes for the case of smooth projective varieties as target using the Behrend-Fantechi [5] machinery are proved in [33]. Some results in the direction of establishing the existence of fundamental classes for target compact Hamiltonian actions were taken in [35]. Here we review the case of algebraic target.

Definition 3.5. (a) (Novikov field) Given a smooth projective $G_{\mathbb{C}}$-variety X and an equivariant symplectic class $[\omega_G] \in H^G_2(X)$ define the *Novikov field* Λ^G_X for X as the set of all maps $\lambda : H^G_2(X) := H^G_2(X, \mathbb{Q}) \to \mathbb{Q}$ such that for every constant c, the set of classes

$$\left\{d \in H^G_2(X, \mathbb{Q}),\ \langle [\omega], d\rangle \leq c\right\}$$

on which λ is non-vanishing is finite. The delta function at d is denoted q^d. Addition is defined in the usual way and multiplication is convolution, so that $q^{d_1} q^{d_2} = q^{d_1+d_2}$.

(b) (Equivariant quantum cohomology) Define as vector spaces

$$QH^G(X, \mathbb{Q}) := H^G(X, \mathbb{Q}) \otimes \Lambda^G_X.$$

Let $QH(X/\!\!/G)$ denote the quantum cohomology defined over the Novikov field Λ^G_X, that is,

$$QH(X/\!\!/G) := H(\overline{I}_{X/\!\!/G}, \mathbb{Q}) \otimes \Lambda^G_X.$$

(c) (Quantum Kirwan morphism) For each $n \geq 0$ define a map

$$Q\kappa^n : QH_G(X)^n \times H(\overline{M}_{n,1}(\mathbb{A})) \to QH(X/\!\!/G)$$

as follows. For $\alpha \in H_G(X)^n, \beta \in H^*(\overline{M}_{n,1}(\mathbb{A}))$ let

$$(Q\kappa^n(\alpha, \beta), \alpha_\infty) = \sum_{d\in H^G_2(X,\mathbb{Q})} q^d \int_{\overline{M}^G_{n,1}(\mathbb{A},X,d)} \mathrm{ev}^*\, \alpha \cup f^*\beta \cup \mathrm{ev}^*_\infty\, \alpha_\infty$$

using Poincaré duality; the pairing on the left is given by cup product and contraction with the fundamental class of $X/\!\!/G$.

Theorem 3.6. [33] (Quantum Kirwan morphism) *Suppose that X is a smooth polarized projective G-variety such that $G_{\mathbb{C}}$ acts locally freely on the semistable locus of X. The collection $(Q\kappa^n)_{n\geq 0}$ is a morphism of CohFT algebras from $QH_G(X)$ to $QH(X/\!\!/G)$. If X is G-Fano in the sense that c_1^G is positive on all rational curves to the quotient stack $X/G_{\mathbb{C}}$, then the curvature $Q\kappa^0$ vanishes, so $(Q\kappa^n)_{n>0}$ is a flat morphism of CohFT algebras.*

In order to compute presentations of the quantum cohomology of $X/\!\!/G$ one would like to know that the quantum analog of Kirwan's surjectivity theorem, namely that the linearization of map $QH_G(X) \to QH(X/\!\!/G)$ at a generic point is surjective. In the case of free quotients $X/\!\!/G$, the conjecture follows from Kirwan's theorem and linearity over the Novikov ring, using a filtration argument.

Next we describe the quantum Kirwan map in the case that G is a torus acting on a vector space X, so that $X/\!\!/G$ is a toric orbifold. We sketch a proof that the kernel of the linearization of the quantum Kirwan map is Batyrev's quantization of the Stanley-Reisner ideal associated to the toric fan. This reproduces for example the presentation of the quantum cohomology of weighted projective planes described in Coates-Corti-Lee-Tseng [8] (see also [7] [9]).

Example 3.7. (Weighted Projective Line $\mathbb{P}(2,3)$ [18]) Let $\mathbb{C}_2$ resp. $\mathbb{C}_3$ denote the weight space for $G_{\mathbb{C}} = \mathbb{C}^\times$ with weight 2 resp. 3 so that $X = \mathbb{C}_2 \oplus \mathbb{C}_3$ and $X/\!\!/G = \mathbb{P}(2,3)$. Let θ_1 resp. θ_2 resp. θ_3 resp. θ_3^2 denote the generator of the component of $QH(X/\!\!/G) \cong H(\overline{I}_{X/\!\!/G}) \otimes \Lambda_X^G$ with trivial isotropy resp. $\mathbb{Z}_2$ isotropy resp. corresponding to $\exp(\pm 2\pi i/3) \in \mathbb{Z}_3$. Let $\xi \in H_G^2(X)$ denote the integral generator corresponding to the representation with weight 1. Then we have the following table for $Q\kappa^1(\xi^k)$:

(12)

k	0	1	2	3	4	5
$Q\kappa^1(\xi^k)$	1	θ_1	$q^{1/3}\theta_3/6$	$q^{1/2}\theta_2/18$	$q^{2/3}\theta_3^2/36$	$q/108$

Indeed, identifying $H_2^G(X,\mathbb{Q}) \cong \mathbb{Q}$ so that $H_2^G(X,\mathbb{Z}) \cong \mathbb{Z}$ we see from Theorem 3.2 that

$$\begin{aligned}
M_{1,1}^G(\mathbb{A}, X, 0) &= \{(a_0, b_0) \neq 0\}/G_{\mathbb{C}} \cong \mathbb{P}(2,3) \\
M_{1,1}^G(\mathbb{A}, X, 1/3) &= \{(a_0, b_1 z + b_0), b_1 \neq 0\}/G_{\mathbb{C}} \cong \mathbb{C}^2/\mathbb{Z}_3 \\
M_{1,1}^G(\mathbb{A}, X, 1/2) &= \{(a_1 z + a_0, b_1 z + b_0), a_1 \neq 0\}/G_{\mathbb{C}} \cong \mathbb{C}^3/\mathbb{Z}_2 \\
M_{1,1}^G(\mathbb{A}, X, 2/3) &= \{(a_1 z + a_0, b_2 z^2 + b_1 z + b_0), b_2 \neq 0\}/G_{\mathbb{C}} \cong \mathbb{C}^4/\mathbb{Z}_3 \\
M_{1,1}^G(\mathbb{A}, X, 1) &= \{(a_2 z^2 + a_1 z + a_0, b_3 z^3 + b_2 z^2 + b_1 z + b_0), (a_2, b_3) \neq 0\}/G_{\mathbb{C}}
\end{aligned}$$

The map

$$\sigma : \overline{M}_{1,1}^{G,\mathrm{fr}}(\mathbb{A}, X, 1/3) \to \mathbb{C}_2 \oplus \mathbb{C}_3, \quad u \mapsto u(0)$$

defines a section with a single transverse zero, leading to the integral

$$\int_{\overline{M}_{1,1}^G(\mathbb{A},X,1/3)} \mathrm{ev}_1^* 6\xi^2 = \int_{\overline{M}_{1,1}^G(\mathbb{A},X,1/3)} \mathrm{ev}_1^* \mathrm{Eul}(\mathbb{C}_2 \oplus \mathbb{C}_3) = 1/3.$$

The pairing on the sector $H(\mathrm{pt}/\mathbb{Z}_3) \otimes \Lambda_X^G$ in $QH(\mathbb{P}(2,3))$ is defined by contraction with the orbifold fundamental class, that is, $[\mathrm{pt}]/3$, which cancels the factor of $1/3$ in the integral above yielding the $k = 3$ column. (Put another way, $Q\kappa_1(6\xi^2)$ is the push-forward under

$\overline{M}^G_{1,1}(\mathbb{A}, X, 1/3) \to \mathrm{pt}/\mathbb{Z}_3$, whose fiber is $\mathbb{C}_2 \oplus \mathbb{C}_3$.) The other integrals are similar. From (12) one sees that $Q\kappa^1$ is surjective with kernel $\xi^5 - q/108$. Hence

$$QH(\mathbb{P}(2,3)) = \Lambda_X^G[\xi]/(\xi^5 - q/108)$$

where Λ_X^G is the Novikov field of fractional powers of a single formal variable q. Note that the quantum Kirwan map is not surjective in this case without inverting q, that is, over the Novikov ring instead of the Novikov field, and that although the Novikov field involves fractional powers of q, the relations have only integer powers.

More generally let X be a vector space and G a torus acting freely so that $X/\!\!/G$ is a proper Deligne-Mumford toric stack (orbifold). We identify $G = U(1)^k$ and let $\rho_1, \dots, \rho_k \in \mathfrak{g}^\vee$ denote the weights of the action on X with $\dim(X) = k$. We also identify $H_2^G(X, \mathbb{Z})$ with the coweight lattice $\mathfrak{g}_\mathbb{Z} = \exp^{-1}(1)$ in the Lie algebra $\mathfrak{g}$.

Definition 3.8 (Quantum Stanley-Reisner Ideal)**.** Let $QSR^G(X) \subset QH_G(X)$ be the *quantum Stanley-Reisner ideal*, generated by the elements for $d \in H_2^G(X, \mathbb{Z})$

$$\prod_{\rho_j(d) \geq 0} \rho_j^{\rho_j(d)} - q^d \prod_{\rho_j(d) < 0} \rho_j^{-\rho_j(d)}.$$

Batyrev [4] in the case of smooth toric varieties conjectured that the quantum cohomology $QH(X/\!\!/G)$ has a presentation

$$QH^G(X)/QSR^G(X) \cong QH(X/\!\!/G), \tag{13}$$

This conjecture was proved for semipositive toric varieties by Givental in [16], Cox-Katz [10], and is false in general as pointed out by Spielberg [29], at least for the obvious generators. Iritani [21] proved that any smooth projective toric variety has quantum cohomology canonically isomorphic to the Batyrev ring $QH_G(X)/QSR^G(X)$, using corrected generators. Coates-Corti-Lee-Tseng [8] generalized the presentation to the case of weighted projective spaces.

Example 3.9. If $G_\mathbb{C} = \mathbb{C}^\times$ acts on $X = \mathbb{C}^2$ with weights $a, b \in \mathbb{Z}$ so that $X/\!\!/G$ is the weighted projective line $\mathbb{P}(a, b)$ then the quantum Stanley-Reisner ideal is generated by $(a\xi)^a(b\xi)^b - q$. Then with our conventions the quantum cohomology of $\mathbb{P}(a, b)$ has generators ξ and fractional powers of q, the single relation is $(a\xi)^a(b\xi)^b = q$, c.f. [8].

Theorem 3.10. (Orbifold Batyrev conjecture) [18] *After suitable completion, the linearization $D_0Q\kappa$ is surjective and the kernel of $D_0Q\kappa$ is the quantum Stanley-Reisner ideal $QSR^G(X)$, so that $T_{Q\kappa(0)}QH(X/\!\!/G) \cong T_0QH^G(X)/QSR^G(X)$.*

We give a partial proof by showing that for any $d \in H_2^G(X, \mathbb{Z})$,

$$\int_{[\overline{M}_1^G(\mathbb{A}, X, d)]} \mathrm{Eul}(\oplus_j \mathbb{C}_{\rho_j}^{\rho_j(d)}) \cup \mathrm{ev}_\infty^*[\mathrm{pt}] = 1. \tag{14}$$

Let

$$\sigma \;:\; \overline{M}_1^G(\mathbb{A}, X, d) \;\to\; \prod_{\rho_j(d) \geq 0} \mathbb{C}_{\rho_j}^{\max(0, \rho_j(d))}, \quad (u, z) \;\mapsto\; (u_j^{(k)}(z))_{k=1,\dots,\rho_j(d), j=1,\dots,N} \tag{15}$$

denote the map constructed from the derivatives of the evaluation map at the marking z_1. The map σ gives a transverse section with a single zero on the locus $\mathrm{ev}_\infty^{-1}(\mathrm{pt}) \subset M_{1,1}^G(\mathbb{A}, X, d)$ and the remaining factor $\mathrm{Eul}(\oplus_j \mathbb{C}_{\rho_j}^{\min(0,\rho_j(d))})$ is the obstruction bundle.

We claim that σ is non-vanishing on the boundary strata. Let $(C, \lambda, \underline{z}, u)$ be a stable scaled map with reducible domain, and let $d' \neq d$ denote the homology class of the irreducible component containing z_1. Since at least two irreducible components have positive energy, $([\omega_G], d') < ([\omega_G], d)$. By assumption $X/\!\!/G$ is non-empty, which implies that the symplectic class $[\omega_G]$ can be written as a positive combination of the weights ρ_j. Hence $\rho_j(d') < \rho_j(d)$ for at least one j. Furthermore, among j such that $\rho_j(d') < \rho_j(d)$ there must exist at least one such that u_j is non-zero. Indeed the sum of $\mathbb{C}_{\rho_j}$ with $\rho_j(d' - d) \geq 0$ is part of the unstable locus in X, and so no morphism u with only those irreducible components non-zero can be generically semistable. The $\rho_j(d') + 1$-st derivative of u_j is then a non-zero constant, so $\sigma(u) \neq 0$. The equation (14) follows.

Finally we describe a notion of *composition* of morphisms of CohFT algebras. This will make CohFT algebras into an infinity-category, whose higher morphisms are *commutative simplices* of CohFT algebras. This composition plays a natural role in the *quantum reduction with stages* conjecture relating the quantum Kirwan maps for G/K and K with that for G, when $K \subset G$ is a normal subgroup. The definition of composition of morphisms of CohFT algebras involves a moduli space of *s-scaled n-marked* affine lines defined as follows.

Definition 3.11. (Multiply-scaled curves) An *s-scaled, n-marked curve* is a datum $(C, \underline{z}, \underline{\lambda})$ where $(C, \underline{z})$ nodal marked curve and $\underline{\lambda} = (\lambda_1, \ldots, \lambda_s)$ is an s-tuple of scalings as in Definition 2.7 and in addition satisfying the following *balanced condition*:

> For each irreducible component C_i of C and any two scalings λ_j, λ_k not both 0 or both ∞, the ratio $(\lambda_j|C_i)/(\lambda_k|C_i) \in \mathbb{P}$ is independent of the choice of C_i.

An s-scaled, n-marked line C is *stable* if each irreducible component with at least one non-degenerate scaling has at least two marked or nodal points, and each irreducible component with all degenerate scalings has at least three marked or nodal points. The *combinatorial type* of an s-scaled, n-marked affine line C is the tree whose vertices $\mathrm{V}(\Gamma)$ correspond to irreducible components, finite edges $\mathrm{E}_{<\infty}(\Gamma)$ to nodes, and equipped with a labelling of the semi-infinite edges $\mathrm{E}_\infty(\Gamma)$ by $\{0, \ldots, n\}$, and distinguished subsets $\mathrm{V}^i(C) \subset \mathrm{V}(C)$ corresponding to irreducible components on which the i-th scaling λ_i is finite, satisfying combinatorial versions of the monotone and balanced conditions which we leave to the reader to write out. This ends the definition.

Remark 3.12. (More explanation of the balanced condition)

(a) On any irreducible component C_i of C on which λ_j, λ_k are both non-zero and finite, λ_j, λ_k both have a double pole at the same point and so have constant ratio $(\lambda_j|C_i)/(\lambda_k|C_i)$.

(b) The balanced condition is equivalent to the condition that for each marking z_i, if $C_{i,j}^+$ denotes the unique component between z_0 and z_i on which λ_j is finite, then one of the three possibilities holds: $C_{i,j}^+ = C_{i,k}^+$ for all i and the ratio $(\lambda_j|C_{i,j}^+)/(\lambda_k|C_{i,k}^+)$ is independent of i; or $C_{i,j}^+$ is closer (in the sense of trees) to z_0 than $C_{i,k}^+$ for all i; or $C_{i,k}^+$ is closer to z_0 than $C_{i,j}^+$ for all i.

Let $\overline{M}_{n,s}(\mathbb{A})$ denote the moduli space of isomorphism classes of stable s-scaled, n-marked curves.

Remark 3.13. (Boundary divisors of the moduli of multiply-scaled lines) The boundary of $\overline{M}_{n,s}(\mathbb{A})$ can be described as follows.

(a) For any subset $I \subset \{1, \dots n\}$ of order at least two there is a divisor
$$\iota_I : D_I \to \overline{M}_{n,s}(\mathbb{A})$$
and an isomorphism
$$\varphi_I : D_I \to \overline{M}_{|I|+1} \times \overline{M}_{n-|I|+1,s}(\mathbb{A})$$
corresponding to the formation of a bubble containing the markings $z_i, i \in I$ with zero scaling on that bubble, and all scalings zero on that bubble.

(b) For any unordered partition $I_1 \sqcup \ldots \sqcup I_r$ of $\{1, \dots, n\}$ of order at least two with each I_j non-empty and non-empty subset $J \subset \{1, \dots, s\}$ there is a divisor
$$\iota_{I_1,\dots,I_r,J} : D_{I_1,\dots,I_r,J} \to \overline{M}_{n,s}(\mathbb{A})$$
with an isomorphism
$$\varphi_{I_1,\dots,I_r,J} : D_{I_1,\dots,I_r,J} \to \overline{M}_{r+1,s-|J|}(\mathbb{A}) \times \prod_{i=1}^{r} \overline{M}_{|I_i|+1,|J|}(\mathbb{A})$$
corresponding to the formation of r bubbles containing markings $I_j, j = 1, \dots, r$ with the scalings $j \in J$ becoming finite on those bubbles and infinite on the component containing z_0, or, if $J = \{1, \dots, s\}$,
$$\varphi_{I_1,\dots,I_r,J} : D_{I_1,\dots,I_r,J} \to \overline{M}_{r+1} \times \prod_{i=1}^{r} \overline{M}_{|I_i|+1,s}(\mathbb{A}).$$

The union of these divisors is the boundary of $M_{n,s}$:
$$\partial\overline{M}_{n,s}(\mathbb{A}) = \bigcup_{I \subset \{1,\dots n\}} D_I \cup \bigcup_{I_1,\dots,I_r,J} D_{I_1,\dots,I_r,J}.$$

Definition 3.14. (Composition of morphisms of CohFT algebras) Let U_0, U_1, U_2 be CohFT algebras. Given morphisms
$$\phi_{01} : U_0 \to U_1, \quad \phi_{12} : U_1 \to U_2, \phi_{02} : U_0 \to U_2$$
we say that ϕ_{02} is the *composition* of ϕ_{01}, ϕ_{12} if the map
$$\phi_{02} \circ \iota_1 : U_0^n \times H(\overline{M}_{n,2}(\mathbb{A})) \to U_2$$
given by composing ϕ_{02} with the natural restriction map $H(\overline{M}_{n,2}(\mathbb{A})) \to H(\overline{M}_{n,1}(\mathbb{A}))$ agrees with the map

$$(\phi_{12} \circ \phi_{01})^n : U_0^n \times H(\overline{M}_{n,2}(\mathbb{A})) \to U_2 \tag{16}$$
$$(\alpha_1, \dots, \alpha_n, \beta) \mapsto \sum_{r \le s, I_1 \sqcup \ldots \sqcup I_r = \{1,\dots,n\}} \phi_{12}^s(\phi_{01}^{|I_1|}(\alpha_i, i \in I_1; \cdot),$$
$$\dots, \phi_{01}^{|I_r|}(\alpha_i, i \in I_r; \cdot), \phi_{01}^0(1), \dots, \phi_{01}^0(1); \cdot)(\iota_{I_1,\dots,I_r}^* \beta)/(s-r)!,$$

where the dots indicate insertion of the Kunneth components of $\iota^*_{I_1,\ldots,I_r}(\beta)$ with respect to the Kunneth decompositions, and is well-defined if it involves only finite sums on the right-hand-side (modulo $U_{0,a}$ for any $a \in \mathbb{R}$ if all CohFT algebras are filtered.) We call the resulting diagram a *commutative triangle of CohFT*. Similarly one can define *commutative simplices* of CohFT algebras of higher dimension.

We now define a moduli space of *multiply scaled gauged maps* that "lives above" $\overline{M}_{n,s}(\mathbb{A})$. Consider a chain of normal subgroups $G = G_0 \supset G_1 \supset \ldots \supset G_s$. Since G_j is normal and compact, $\mathfrak{g}$ splits as a sum $\mathfrak{g} = \mathfrak{g}_j \oplus \mathfrak{g}'_j$, so there exists a subgroup $G'_j \subset G$ so that $G_j \times G'_j \to G$ is a finite cover. Let X be a smooth projective $G_{\mathbb{C}}$-variety.

Definition 3.15. (Multiply-scaled affine gauged maps) An *s-scaled, n-marked stable affine gauged map* on the affine line $\mathbb{A}$ with values in X is a s-scaled, n-marked nodal curve C equipped with a morphism u from C to the quotient stack $X/G_{\mathbb{C}}$ such that for each $j = 1, \ldots, s$,

(a) ($G'_{j,\mathbb{C}}$-bundle where λ_j is zero) on the irreducible components where λ_j vanishes, the $G_{\mathbb{C}}$-bundle defined by the composition of u with $X/G_{\mathbb{C}} \to B(G_{\mathbb{C}})$ is induced from a $G'_{j,\mathbb{C}}$-bundle;
(b) ($G_{j,\mathbb{C}}$-stable point where λ_j is infinite) if $\lambda_j(z) = \infty$, then $u(z)$ lies in the semistable locus for the action of $G_{j,\mathbb{C}}$.

An s-scaled nodal affine gauged map is *semistable* if each irreducible component with some non-degenerate scalings has at least two special points, and each bubble with all degenerate scalings has at least three special points. A multiply scaled affine gauged map is *stable* if it has finite automorphism group.

Let $\overline{M}^G_{n,s}(\mathbb{A}, X)$ denote the moduli space of isomorphism classes of stable s-scaled, n-marked affine gauged maps on $\mathbb{C}$ with values in X.

Remark 3.16. The divisor relations on $\overline{M}_{n,s}(\mathbb{A})$ naturally induce divisor relations on $\overline{M}^G_{n,s}(\mathbb{A}, X)$. In particular, $\overline{M}_{1,2}(\mathbb{A})$ is a projective line, and the linear equivalence between $D_{\{1\},\{1\}}$, the divisor where the first scaling has become infinite, and the subspace $\overline{M}_{1,1}(\mathbb{A})$ where the two scalings have become equal induces an equivalence in homology in $\overline{M}^G_{n,2}(\mathbb{A}, X)$ between $\overline{M}^G_{n,1}(\mathbb{A}, X)$ (embedded as the subspace where the scalings are equal) and the union of the pre-images of the divisors $D_{[I_1,\ldots,I_r],\{1\}}$.

Remark 3.17. (Equivariant quantum Kirwan morphism) The quantum Kirwan morphism has the following equivariant generalization. If the action of G extends to an action of a group K containing G as a normal subgroup, then the quotient group K/G acts on the moduli space $\overline{M}^G_{n,1}(\mathbb{A}, X)$ and one obtains a morphism

$$\mathrm{ev} \times \mathrm{ev}_\infty : \overline{M}^G_{n,1}(\mathbb{A}, X)/(K/G)_{\mathbb{C}} \to (X/K_{\mathbb{C}})^n \times (X/\!\!/G)/(K/G)_{\mathbb{C}}.$$

Pairing with the virtual fundamental class defines a map

$$QH_K(X, \mathbb{Q})^n \times H(\overline{M}_{n,1}(\mathbb{A}), \mathbb{Q}) \to QH_{G/K}(X/\!\!/G, \mathbb{Q}).$$

After extending the coefficient ring of $QH_{K/G}(X/\!\!/G)$ from Λ^G_X to $\Lambda^{K/G}_{X/\!\!/G}$ one expects this to define a morphism of CohFT algebras

$$(Q\kappa^n_{K,G})_{n\geq 0} : QH_K(X) \to QH_{K/G}(X/\!\!/G). \tag{17}$$

Consider the equivariant quantum Kirwan morphisms

$$(Q\kappa^n_{K,G})_{n\geq 0} : QH_K(X) \to QH_{K/G}(X/\!\!/G)$$

$$(Q\kappa^n_{K/G})_{n\geq 0} : QH_{K/G}(X/\!\!/G) \to QH(X/\!\!/K)$$

defined in (17). The linear equivalence in Remark 3.16 leads naturally to the

Conjecture 3.18. (Quantum reduction in stages) *Suppose that X, K, G are as above, and the symplectic quotients by K and G are locally free. Then there is a commutative triangle of CohFT algebras*

$$\begin{array}{ccc} QH_K(X) & \longrightarrow & QH(X/\!\!/K) \\ & \searrow \quad \nearrow & \\ & QH_{K/G}(X/\!\!/G) & \end{array}$$

In particular, there is an equality of formal, non-linear maps

$$Q\kappa_{K/G} \circ Q\kappa_{G,K} = Q\kappa_K.$$

More generally, given a chain $G = G_0 \supset G_1 \supset \dots G_s$ as above one should obtain a *commutative simplex* of CohFT algebras. We leave it to the reader to formulate the precise conjecture.

4. Local description of boundary divisors

In this section and the next we give a precise description of the group of invariant Cartier divisors on the moduli space of scaled lines $\overline{M}_{n,1}(\mathbb{A})$. We begin with a review of the local description of $\overline{M}_{n,1}(\mathbb{A})$ given in Ma'u-Woodward [27].

Definition 4.1. (Colored trees) A *colored tree* Γ is a finite tree consisting of a set of vertices

$$\mathrm{V}(\Gamma) = \{v_1, \dots, v_m\}$$

a set of (finite and semi-infinite) edges

$$\mathrm{E}(\Gamma) = \mathrm{E}_{<\infty}(\Gamma) \cup \mathrm{E}_{\infty}(\Gamma), \quad \mathrm{E}_{\infty}(\Gamma) = \{e_0, \dots, e_n\}$$

and a subset of *colored vertices*

$$\mathrm{V}^+(\Gamma) \subset \mathrm{V}(\Gamma)$$

such that the following condition is satisfied:

> (Monotonicity condition) Any non-self-crossing path in Γ from the root edge e_0 to any other semi-infinite edge $e_i, i > 0$ crosses exactly one colored vertex $v \in \mathrm{V}^+(\Gamma)$.

The tree Γ is *stable* if the valency of any colored resp. uncolored vertex is at least two resp. three.

We say that a vertex is *above* the colored vertices if it can be connected to the root edge without crossing a colored vertex. Let $\mathrm{V}^{\infty}(\Gamma)$ be the set of vertices above the colored vertices. For any $v \in \mathrm{V}^{\infty}(\Gamma)$, let $\mathrm{V}^+(v)$ be the set of colored vertices $v' \in \mathrm{V}^+(\Gamma)$ that are below v, that is, connected by paths in Γ that move away from the root.

Definition 4.2. (Balanced Labellings) A map $\varphi : \mathrm{E}_{<\infty}(\Gamma) \to \mathbb{C}$ is *balanced* if for all $v \in \mathrm{V}^{\infty}(\Gamma)$ and $v' \in \mathrm{V}^{+}(v)$, the product

$$\prod_{e \in \gamma(v,v')} \varphi(e)$$

over edges e in the non-self-crossing path $\gamma(v, v')$ from v to v' is independent of the choice of a colored vertex v'. Let $V(\Gamma)$ denote the set of balanced labellings:

$$V(\Gamma) := \{\varphi : \mathrm{E}_{<\infty}(\Gamma) \to \mathbb{C} \,|\, \varphi \text{ is balanced}\}. \tag{18}$$

The subset

$$T(\Gamma) := V(\Gamma) \cap \mathrm{Map}(\mathrm{E}_{<\infty}(\Gamma), \mathbb{C}^{\times})$$

of points with non-zero labels is the kernel of the homomorphism

$$\mathrm{Map}(\mathrm{E}_{<\infty}(\Gamma), \mathbb{C}^{\times}) \to \mathrm{Map}(\mathrm{V}^{\infty}(\Gamma), \mathbb{C}^{\times})$$

given by taking the product of labels from the given vertex to the colored vertex above it, and is therefore an algebraic torus.

Example 4.3. The tree Γ in Figure 4 is a balanced colored tree with $n = 4$ and $g = 3$. The space of balanced labellings is

$$V(\Gamma) = \{(x_1, ..., x_6) \in \mathbb{C}^6 \mid x_1x_3 = x_1x_4 = x_2x_5 = x_2x_6, x_3 = x_4, x_5 = x_6\}$$

and admits an action of the torus

$$T(\Gamma) = \{(x_1, ..., x_6) \in V(\Gamma) \mid x_i \neq 0\} \simeq (\mathbb{C}^*)^3.$$

Proposition 4.4. [27] (Local structure of the moduli space of scaled lines) *There exists an isomorphism of a Zariski open neighborhood of $M_{n,1,\Gamma}$ in $M_{n,1,\Gamma} \times V(\Gamma)$ with a Zariski open neighborhood of $M_{n,1,\Gamma}$ in $\overline{M}_{n,1}(\mathbb{A})$.*

We comment briefly on the proof. Given a stable scaled line, one can remove small disks around the nodes and glue together annuli using a map $z \mapsto \varphi(e)/z$ to produce a curve with fewer nodes, where $\varphi(e)$ is the *gluing parameter* associated to the node. In the case of the genus zero curves, the local coordinates used to produce the disks are essentially canonical, and the balanced condition guarantees that the scalings on the resulting curve are well-defined.

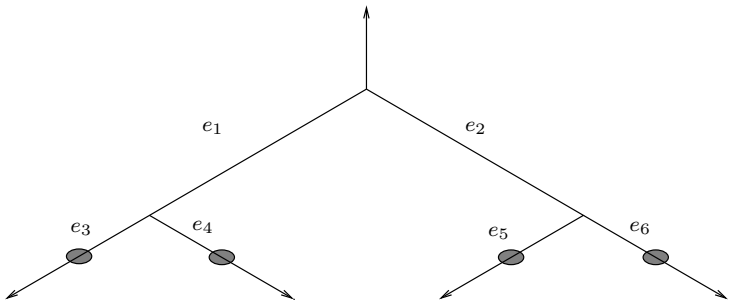

FIGURE 4. A colored tree

Recall that normal affine toric varieties are classified by finitely generated cones [11], [14].

Definition 4.5. (Affine toric variety associated to a cone) Let $V_{\mathbb{Z}} \subset V$ be a lattice, and $C \subset V$ a strictly convex rational cone. The *affine toric variety corresponding to the cone* C is the spectrum $V(C)$ of the ring $R(C^\vee)$ corresponding to the semigroup $C^\vee \cap V_{\mathbb{Z}}^\vee$, that is, the ring generated by symbols f_μ for $\mu \in C^\vee \cap V_{\mathbb{Z}}^\vee$ modulo the ideal generated by relations

$$\sum_i n_i \mu_i = \sum_j m_j \mu_j \implies \prod_i f_{\mu_i}^{n_i} = \prod_j f_{\mu_j}^{m_j}. \tag{19}$$

Any normal affine toric variety is of the form $V(C)$ for some cone C, obtained from X by letting $C^\vee$ be the cone generated by the weights of the action of T on the coordinate ring and C the dual cone of $C^\vee$.

We wish to show that the space $V(\Gamma)$ of balanced labellings (18) is the toric variety associated to some cone $C(\Gamma)$. Note that the part of Γ separated by the colored vertices from the root of Γ trivially affects $V(\Gamma)$ by adding additional independent variables. Hence, for the rest of this section, it suffices to assume that the colored tree Γ does not contain any vertex below any colored vertex. Let $\epsilon_1, \dots, \epsilon_n$ be a basis of $\mathfrak{t}$, and $\epsilon_1^\vee, \dots, \epsilon_n^\vee$ the dual basis of $\mathfrak{t}^\vee$. Define a labelling

$$w : \mathrm{E}_{<\infty}(\Gamma) \to \mathfrak{t}^\vee$$

recursively as follows.

Definition 4.6. (Principal subtree and branch) We say that a subtree $\Gamma' \subset \Gamma$ is a *principal subtree* if it is a component of the tree obtained by removing the vertex adjacent to the root edge. The edge adjacent to the root edge of Γ' is called a *principal branch* of Γ.

Let $\Gamma_1, ..., \Gamma_p$ be the principal subtrees of Γ and $d_1, \dots, d_p$ the principal branches.

Example 4.7. For the example in Figure 4, there are two principal subtrees, with principal branches e_1, e_2.

Definition 4.8. (Sum of Labels) Given a labelling w denote by $s(\Gamma, w)$ the sum of the labels of the edges of a non-self-crossing path from the principal vertex to a colored vertex; a priori this depends on the choice of path but each labelling we construct will have the property that $s(\Gamma, w)$ is independent of the choice of path.

Definition 4.9. (Labelling of edges of a colored tree by weights) Let Γ' be a subtree of Γ.

(Case 1) Γ' is a tree with one non-colored vertex v_i. Label the edges below the vertex v_i by $\epsilon_i^\vee$, that is, define $w(e) = \epsilon_i^\vee$ for every edge e of Γ'.

(Case 2) Γ' has $g > 1$ non-colored vertices. By induction, assume that we have equipped the edges of the principal subtrees $\Gamma'_1, ..., \Gamma'_p$ of Γ' with labellings w_i. We have thus labelled all the edges of Γ except for the principal branches; we denote $s_i := s(\Gamma'_i, w'_i)$. Define

$$s = s(\Gamma', w') = \epsilon_g^\vee + s_1 + ... + s_p. \tag{20}$$

Label the principal branch $d_i \in \mathrm{E}_{<\infty}(\Gamma')$ with

$$w(d_i) = s - s_i = \epsilon_g^\vee + \sum_{j \neq i} s_j. \tag{21}$$

By induction all the edges e of Γ become labelled by weights $w(e)$.

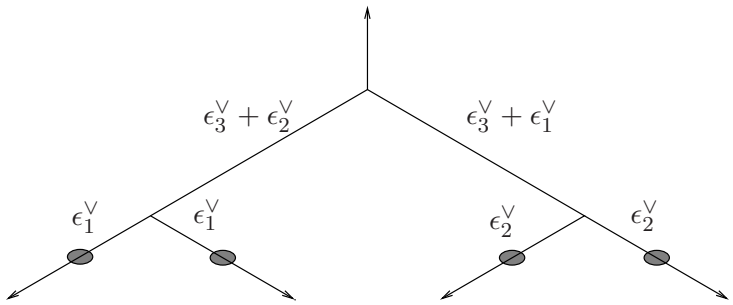

FIGURE 5. An example of a labelling

Example 4.10. Figure 5 illustrates the labels of the edges of Γ from Example 4.3. If we denote the left and right principal branches by d_1 and d_2 respectively, then $s_1 = \epsilon_1^\vee, s_2 = \epsilon_2^\vee, s = \epsilon_3^\vee + \epsilon_2^\vee + \epsilon_1^\vee$.

Lemma 4.11. *$s = s(\Gamma, w)$ is the sum of the labels of the edges of a non-self-crossing path from the principal vertex v_g to a colored vertex and s is independent of the path chosen.*

Proof. By (21) the sum over a path through Γ_i is $s_i + w(d_i) = s$, independent of i. □

Let $C(\Gamma)^\vee$ be the convex cone generated by the labels above,

$$\begin{aligned} C(\Gamma)^\vee &= \mathrm{hull}_{\mathbb{Q}_{\geq 0}} \{w(e) \mid e \in \mathrm{E}_{<\infty}(\Gamma)\} \\ &= \mathrm{hull}_{\mathbb{Q}_{\geq 0}} \bigcup_{j=1}^{p} C(\Gamma_j)^\vee \cup \{w(e_i) \mid 1 \leq i \leq p\} \end{aligned}$$

and let $C(\Gamma)$ denote the dual cone of $C(\Gamma)^\vee$.

Theorem 4.12. (Explicit description of the cone associated to balanced labellings) *The variety $V(\Gamma)$ is the toric variety associated to $C(\Gamma)$ in the sense of Definition 4.5, in particular, $V(\Gamma)$ is normal.*

The proof will be given after the following lemma.

Definition 4.13. (Equivalence of sets of edges) Suppose E', E'' are two disjoint subsets of $\mathrm{E}(\Gamma)$. We write $E' \sim E''$ if there exists a vertex and two non-self-crossing paths γ_1 and γ_2 from that vertex to some two colored vertices so that E' and E'' respectively contain exactly the edges of the paths γ_1 and γ_2.

Example 4.14. The set $E' = \{e_1, e_3\}$ is equivalent to $E'' = \{e_2, e_6\}$ in Example 4.3.

Lemma 4.15. *Suppose E' and E'' are two disjoint multisets of elements of $\mathrm{E}(\Gamma)$. Then*

$$\sum_{e' \in E'} w(e') = \sum_{e'' \in E''} w(e'') \tag{22}$$

if and only if E' and E'' can be partitioned into disjoint unions of $\{E'_1, ..., E'_r\}$ and $\{E''_1, ..., E''_r\}$ where $E'_l \sim E''_l$ for $1 \leq l \leq r$.

Example 4.16. In Example 4.3, let $E' = \{3e_1, 2e_3, e_4, e_5\}$ and $E'' = \{3e_2, 4e_6\}$ which satisfy (22). We can write $E'_1 = \{e_1, e_3\} \sim E''_1 = \{e_2, e_6\}, E'_2 = \{e_1, e_3\} \sim E''_2 = \{e_2, e_6\}, E'_3 = \{e_1, e_4\} \sim E''_3 = \{e_2, e_6\}, E'_4 = \{e_5\} \sim E''_4 = \{e_6\}$.

Proof of Lemma 4.15. One direction of the implication, that the equality (22) holds if E' and E'' can be partitioned, is immediate from the definitions. We only need to show the other direction. As before, it suffices to consider the case that there are no non-colored vertices below the colored vertices. When the number of non-colored vertices is 1, the statement of the lemma is trivial. Assume the proposition holds for any tree with number of vertices less than g. Consider a tree Γ with g non-colored vertices. Denote by $n_1, ..., n_p$ and $m_1, ..., m_p$ the multiplicities of the principal branches $e_1, ..., e_p$ in E' and E''. Since $E' \cap E'' = \emptyset$, we have $n_i m_i = 0$ for all i. Equation (22) and the fact that $\epsilon_g^\vee$ appears only on the edges adjacent to the root edge imply

$$\sum_{i=1}^{p} n_i = \sum_{i=1}^{p} m_i. \tag{23}$$

Similarly, the fact that the labels from each principal branch are independent implies that

$$-n_i s_i + \sum_{e' \in E' \cap \mathrm{E}_{<\infty}(\Gamma_i)} w(e') = -m_i s_i + \sum_{e'' \in E'' \cap \mathrm{E}_{<\infty}(\Gamma_i)} w(e'')$$

for every $1 \leq i \leq p$. For a fixed i, without loss of generality, we can assume $m_i = 0$. Then

$$\sum_{e' \in E' \cap \Gamma_i} w(e') = n_i s_i + \sum_{e'' \in E'' \cap \Gamma_i} w(e'').$$

Noting that s_i is the sum of labels over a non-self-crossing path from v_i to a colored vertex, we may replace $E'^i = E' \cap \mathrm{E}_{<\infty}(\Gamma_i)$ with an equivalent set which contains n_i copies $E'^i_j, j = 1, \ldots, n_i$ of the edges in such a path. The complement of $E'^i_j, j = 1, \ldots, n_i$ in E'^i has the same sum of labels as $E''^i = E'' \cap \mathrm{E}_{<\infty}(\Gamma_i)$, so by the inductive hypothesis there exists a partition of $E' \cap \mathrm{E}_{<\infty}(\Gamma_i)$ and $E'' \cap \mathrm{E}_{<\infty}(\Gamma_i)$ into $\{E'^i_1, ..., E'^i_{r'_i}\}$ and $\{E''^i_1, ..., E''^i_{r''_i}\}$ such that for $1 \leq j \leq n_i$, E'^i_j are equal and for $n_i + 1 \leq j \leq r_i$,

$$E'^i_j \sim E''^i_j. \tag{24}$$

Since E' contains n_i principal branches d_i, we can add one edge d_i in each E'^i_j for every $1 \leq j \leq n_i$. Hence, after the modification, each set E'^i_j contains exactly all the edges of a path from the root of Γ to a colored vertex in Γ_i . Applying the same process for each $1 \leq i \leq p$, by the first equality in (23) and by (24), we can partition E' and E'' into $\{E'_1, ..., E'_r\}$ and $\{E''_1, ..., E''_r\}$ such that $E'_i \sim E''_i$ for every $1 \leq i \leq r$. □

Proof of Theorem 4.12. We must show that the balanced relations for $V(\Gamma)$ in Definition 4.2 are exactly those in the definition of the affine toric variety associated to $C(\Gamma)$ in (19). So suppose that $E' = \{n_1 e_1, ..., n_N e_N\}$ and $E'' = \{m_1 e_1, ..., m_N e_N\}$ are such that $\sum n_i w(e_i) = \sum m_j w(e_j)$, and so define a relation as in (19). Lemma 4.15 yields that E' and E'' can be partitioned into $E'_1, ..., E'_r, E''_1, ..., E''_r$ so that $E'_i \sim E''_i$ for $1 \leq i \leq r$. But these are exactly the balanced relations in 4.2. □

It follows from the theorem that the cone $C(\Gamma)$ corresponding to the toric variety $V(\Gamma)$ is the cone dual to the $\mathbb{Q}_{\geq 0}$-span of $C(\Gamma)^\vee$. Next we find a minimal set $G(\Gamma)$ of generators of $C(\Gamma)$ by an inductive argument on the number of non-colored vertices g of Γ.

Definition 4.17. (Generators of the cone associated to balanced labellings) Define $G(\Gamma)$ inductively as follows for subtrees $\Gamma' \subset \Gamma$:

(a) If $g(\Gamma') = 1$ with vertex v_i, then $G(\Gamma') = \epsilon_i$.
(b) If $g > 1$, then $G(\Gamma') = \{\epsilon_g + n_1(v_1 - \epsilon_g) + ... + n_p(v_p - \epsilon_g) |\ v_i \in G(\Gamma_i'), n_i \in \{0, 1\}\}$.

Note that the elements in $G(\Gamma)$ are in the lattice $\mathbb{Z}^g$ spanned by the vectors $\epsilon_1, ..., \epsilon_g$.

Theorem 4.18. *$G(\Gamma)$ is a minimal set of generators of $C(\Gamma)$.*

Example 4.19. The tree Γ in Figure 5 can be split into two principal subtrees Γ_1 and Γ_2. Since $G(\Gamma_1) = \{\epsilon_1\}$ and $G(\Gamma_2) = \{\epsilon_2\}$, we obtain $G(\Gamma) = \{\epsilon_1, \epsilon_2, \epsilon_3, \epsilon_1 + \epsilon_2 - \epsilon_3\}$. The cone generated by $\{\epsilon_1, \epsilon_2, \epsilon_3, \epsilon_1 + \epsilon_2 - \epsilon_3\}$ is the cone $C(\Gamma)$ corresponding to the toric variety $V(\Gamma)$.

Denote by $\tilde{C}(\Gamma)$ the g-dimensional cone spanned by the vectors in $G(\Gamma)$. To prove Theorem 4.18 we must show that $C(\Gamma) = \tilde{C}(\Gamma)$.

Lemma 4.20. *For s as in (20), for every $v \in G(\Gamma)$, $\langle s, v\rangle = 1$.*

Proof. This follows by induction on the number of vertices from the observation that $\langle s, v\rangle = 1 + \sum_{i=1}^{p} n_i(\langle s_i, v_i\rangle - 1)$. □

Proof of Theorem 4.18. We show $\tilde{C}(\Gamma) \subset C(\Gamma)$ by induction on the number g of non-colored vertices. The case $g = 1$ is obvious. Suppose the claim is true for all colored trees with less than g vertices. Let $v \in G(\Gamma)$ with coefficients $n_i, i = 1, \ldots, p$ and $w \in C(\Gamma)^\vee$. If $v \in G(\Gamma_i)$ then $\langle w, v\rangle = n_i\langle w, v_i\rangle$ and the claim follows by the inductive hypothesis. Otherwise, since $C(\Gamma)^\vee$ is spanned by $w(e), e \in E_{<\infty}(\Gamma)$, we may assume that $w = w(e_i)$ for some $1 \leq i \leq p$. By Lemma 4.20,

$$\begin{aligned} \langle w, v\rangle &= \langle s - s_i, v\rangle \\ &= \langle s, v_i\rangle - n_i\langle s_i, v_i\rangle \\ &= 1 - n_i \end{aligned}$$

Hence $\langle w, v\rangle \geq 0$. Since this holds for all v, w, we have $\tilde{C}(\Gamma) \subset C(\Gamma)$.

Conversely, given $v \in C(\Gamma)$, we claim that v is a non-negative linear combination of elements in $G(\Gamma)$. For $g = 1$, the claim is trivial. Assume the claim is true for all trees with less than g non-colored vertices. Let Γ be a tree with g vertices and $v \in C(\Gamma)$. In particular, v pairs non-negatively with the weights $w(e), e \in \mathrm{E}_{<\infty}(\Gamma_i)$ so by the inductive hypothesis we can write v as a sum

$$v = -c_g\epsilon_g + \sum_{i=1}^{p} \sum_{v \in G(\Gamma_i)} \lambda_v^{(i)} v$$

where $\lambda_v^{(i)} \geq 0$. If $c_g \leq 0$ then the claim follows since $\epsilon_g \in G(\Gamma)$. If $c_g > 0$, let

$$\lambda_i := \sum_{v \in G(\Gamma_i)} \lambda_v^{(i)}.$$

We remark by Lemma 4.20,

$$0 \leq \langle w(e_i), v\rangle = -c_g + \sum_{j\neq i} \lambda_j. \tag{25}$$

To write v as a non-negative linear combination of elements in $G(\Gamma)$, we proceed as follows. Without loss of generality, suppose that λ_p is the minimum of $\{\lambda_j \neq 0\}$, that is, the smallest positive $\lambda_j, j = 1, \ldots, p$. Split each sum

$$\sum_{v\in G(\Gamma_i)} \lambda_v^{(i)} v = \sum_{v\in G(\Gamma_i)} \gamma_v^{(i)} v + \sum_{v\in G(\Gamma_i)} \delta_v^{(i)} v$$

where $\gamma_v^{(i)}, \delta_v^{(i)} \geq 0, \gamma_v^{(i)} + \delta_v^{(i)} = \lambda_v^{(i)}$ and $\sum_{v\in G(\Gamma_i)} \delta_v^{(i)} = \lambda_p$. We can write v as the sum of

$$-(p-1)\lambda_p \epsilon_g + \sum_{i=1}^{p-1} \Big(\sum_{v\in G(\Gamma_i)} \delta_v^{(i)} v\Big) + \sum_{v\in G(\Gamma_p)} \lambda_v^{(p)} v \tag{26}$$

and

$$-c_g' \epsilon_g + \sum_{i=1}^{p-1} \sum_{v\in G(\Gamma_i)} \gamma_v^{(i)} v \tag{27}$$

where

$$-c_g' = -c_g + (p-1)\lambda_p. \tag{28}$$

Since

$$\sum_{v\in G(\Gamma_i)} \delta_v^{(i)} = \sum_{v\in G(\Gamma_p)} \lambda_v^{(p)} = \lambda_p,$$

the expression (26) is a non-negative linear combination of elements of $G(\Gamma)$. If $-c_g' \geq 0$, the expression (27) is already a nonnegative linear combination of elements of $G(\Gamma)$ and we are done. Otherwise, consider the smaller tree Γ' obtained from Γ by removing Γ_p and observe that (27) lies in $C(\Gamma')$. Indeed, by definition, we know $\gamma_v^{(i)} \geq 0$ and therefore it is sufficient to check that

$$-c_g' + \sum_{j\neq i, j=1}^{p-1} \gamma_j \geq 0.$$

However, by definition and the equation (25) we have

$$\begin{aligned} -c_g' + \sum_{j\neq i, j=1}^{p-1} \gamma_j &= -c_g + (p-1)\lambda_p + \sum_{j\neq i, j=1}^{p-1} (\lambda_j - \lambda_p) \\ &= -c_g + \sum_{j=1, j\neq i}^{p} \lambda_i \geq 0. \end{aligned}$$

Thus, the expression (27) is in $C(\Gamma')$. By the inductive hypothesis, (27) is a nonnegative linear combination of elements of $G(\Gamma')$. Hence v is a nonnegative linear combination of elements in $G(\Gamma)$. Thus $C(\Gamma) \subset \tilde{C}(\Gamma)$ and therefore, $C(\Gamma) = \tilde{C}(\Gamma)$.

We argue by induction that $G(\Gamma)$ is a minimal set of generators of $\tilde{C}(\Gamma)$ and $\tilde{C}(\Gamma)$ is non-degenerate, i.e no positive linear combinations of vectors in the cone are 0. It is easy to check the claim when $g(\Gamma) = 1$. Given a tree Γ with $g(\Gamma)$ non-colored vertices and $G(\Gamma_i)$ the constructed minimal set of generators for each nondegenerate cone $\tilde{C}(\Gamma_i)$, suppose $v \in G(\Gamma)$ is a non-negative linear combination of other elements in $G(\Gamma)$. By the induction hypothesis, the projection of v onto the space spanned by $G(\Gamma_i)$ is 0 for each i and thus by the nondegeneracy induction hypothesis, it follows that $v = 0$. Now, suppose that a positive linear combination of some elements in $G(\Gamma)$ is 0. In particular, its projections onto the space spanned by $G(\Gamma_i)$ is 0 for each i. Hence by the nondegeneracy induction hypothesis, all the elements in the combination are 0. Therefore, $G(\Gamma)$ is a minimal set of generators of $\tilde{C}(\Gamma)$ and $\tilde{C}(\Gamma)$ is nondegenerate, concluding the theorem. □

By the description of the cone, the dimension of $V(\Gamma)$ equals the number of non-colored vertices g above the colored vertices plus the number of edges below the colored vertices. On the other hand, by the balanced condition in 4.2,

$$\dim(V(\Gamma)) = \dim(T(\Gamma)) = |\,\mathrm{E}_{<\infty}(\Gamma)| - |\,\mathrm{V}^{+}(\Gamma)| + 1.$$

The two formulas are easily seen to be equivalent, by considering the map from vertices to edges given by taking the adjacent edge in the direction of the root edge. We also have a formula for the number of rays in $C(\Gamma)$, which follows immediately from Theorem 4.18:

Corollary 4.21. *If the number of 1 dimensional faces of $C(\Gamma_i)$ is r_i for $1 \leq i \leq p$, then the number of 1 dimensional faces of $C(\Gamma)$ is $r = (r_1 + 1)...(r_p + 1)$.*

Next we describe the Weil and Cartier divisors in the local toric charts. Recall the description of invariant Weil divisors of an affine toric variety $V(C)$ with cone C (see [14] or in the more general setting of spherical varieties, [6]):

Proposition 4.22. (a) (Classification of invariant Weil divisors) *There is a bijection between invariant prime Weil divisors of $V(C)$ and the one dimensional faces of C.*

(b) (Classification of invariant Cartier divisors) *There is a bijection between invariant Cartier divisors on $V(C)$ and linear functions on C taking integer values on the intersection $C \cap V_{\mathbb{Z}}$.*

We sketch the construction of the bijections. For the classification of invariant Weil divisors, any one-dimensional face C_1 of C corresponds to a codimension-one face $C_1^\vee$ of $C^\vee$. The projection of semigroup rings $R(C^\vee) \to R(C_1^\vee)$ defines an inclusion of the corresponding affine toric varieties $V(C_1) \to V(C)$. For the classification of Cartier divisors, recall that a Weil divisor is Cartier if it is the zero set of a section of a line bundle. On a normal affine toric variety, any line bundle is trivial and any invariant Cartier divisor is defined by a function that is semi-invariant under the torus action. Such functions correspond to lattice points $\lambda \in V_{\mathbb{Z}}^\vee$, where the corresponding function is regular if $\lambda \in C^\vee \subset V_{\mathbb{Z}}^\vee$. If $v \in C$ is any vector generating an extremal ray, then the order of vanishing of λ on the divisor $D(v) \subset V(C)$ corresponding to v is $\lambda(v)$. Thus one sees that a combination $\sum n_v D(v)$ of invariant Weil divisors is Cartier iff there is an element $\lambda \in C^\vee$ such that $\lambda(v) = n_v$ for such $v \in C$. More generally, for a not-necessarily affine toric variety, an invariant Weil divisor is Cartier if there exists a piecewise linear function on the fan whose values on the rays are the multiplicities of the invariant prime Weil divisors.

We now specialize to the case of the toric variety $V(\Gamma)$ associated to the cone $C(\Gamma)$ with generators $G(\Gamma)$ identified in the previous section. We identify the invariant prime Weil divisors of $V(\Gamma)$ as follows.

Definition 4.23 (Minimally complete edge sets)**.** A subset $E \subset \mathrm{E}_{<\infty}(\Gamma)$ is *minimally complete* if and only if each non-self crossing path from v_g to a colored vertex contains exactly one edge in E.

Denote by $\mathcal{E}_{mc}(\Gamma)$ the set of minimally complete subsets $E \subset \mathrm{E}_{<\infty}(\Gamma)$.

Example 4.24. The minimally complete subsets of $\mathrm{E}(\Gamma)$, where Γ is the tree in Figure 4, are $\{e_1, e_2\}$, $\{e_1, e_5, e_6\}$, $\{e_2, e_3, e_4\}$, $\{e_3, e_4, e_5, e_6\}$.

Proposition 4.25. *If the number of minimally complete subsets of* $\mathrm{E}_{<\infty}(\Gamma_i)$ *is* r_i*, then the number of minimally complete sets of* $\mathrm{E}_{<\infty}(\Gamma)$ *is* $r = (r_1 + 1)...(r_p + 1)$.

Proof. Let $d_1, \ldots, d_p$ denote the edges adjacent to the root edge. Each minimally complete set either contains d_i, or induces a minimally complete set in the principal branch Γ_i, for each $i = 1, \ldots, p$. The claim follows. □

From Corollary 4.21 and Proposition 4.25, we obtain

Corollary 4.26. *The number of one dimensional faces of* $C(\Gamma)$ *equals the number of minimally complete subsets of* $\mathrm{E}_{<\infty}(\Gamma)$.

We can now describe the set of invariant Weil divisors of $V(\Gamma)$ as follows.

Proposition 4.27. *There is a bijection between the set of invariant prime Weil divisors and the set* $\mathcal{E}_{mc}(\Gamma)$*. More explicitly, each prime invariant Weil divisor has the form*

$$D_E := \{(x_1, ..., x_N) \in V(\Gamma) \mid x_i = 0 \ \forall e_i \in E\}$$

for some minimally complete subset E.

Proof. Given a minimally complete edge set $E \subset E_{<\infty}(\Gamma)$, for each principal subtree Γ_i of Γ, either $x_{d_i} = 0$ or D_E induces a minimally complete subset $E_i \in \mathcal{D}(\Gamma_i)$. By induction on the number of non-colored vertices, the dimension of D_E is $g(\Gamma_1) + ... + g(\Gamma_p) = g - 1$. Thus D_E is a subvariety of $V(\Gamma)$ of codimension 1. Since $V(\Gamma)$ is the closure of $T(\Gamma)$, the subvariety D_E is the closure of the orbit $D_E \cap T(\Gamma)$ and so a prime Weil divisor. From Proposition 4.22 and Proposition 4.26, the number of prime Weil divisors equals the number of one dimensional faces of $C(\Gamma)$ which equals the number of minimally complete subset of $\mathrm{E}_{<\infty}(\Gamma)$. Therefore the invariant prime Weil divisors of $V(\Gamma)$ are exactly all D_E, where $E \subset \mathrm{E}(\Gamma)$ is minimally complete. □

We now describe inductively the correspondence between the rays of $C(\Gamma)$ and elements in $\mathcal{E}_{mc}(\Gamma)$. Let $D = D_E$ be a Weil divisor as above. Unless $e_{d_i} \in E$, the principal subtree Γ_i has an induced Weil divisor $D_{E_i} \subset V(\Gamma_i)$. Suppose that the one dimensional face of $C(\Gamma_i)$ corresponding to the Weil divisor D_{E_i} is generated by $v_i \in G(\Gamma_i) \subset G(\Gamma)$. Let

$$I(E) = \{i \mid e_{d_i} \notin E, 1 \leq i \leq p\}.$$

Proposition 4.28. *Let $E \in \mathcal{E}_{mc}(\Gamma)$. The one-dimensional face of $C(\Gamma)$ corresponding to the Weil divisor $D_E \subset V(\Gamma)$ is generated by*

$$v_E = \epsilon_g + \sum_{i \in I(E)} (v_i - \epsilon_g). \tag{29}$$

Proof. We must show that v_E is non-zero exactly on the weights $w(e)$ for $e \in E$. This is automatically true by the inductive hypothesis for the edges except for the principal branches, that is, if $e \in \mathrm{E}_{<\infty}(\Gamma_i)$ then $\langle v_E, w(e)\rangle = \langle v_i, w(e)\rangle \neq 0$ iff $e \in E_i$. For the principal branches, the claim follows from 4.20. □

Next we identify the invariant Cartier divisors in $V(\Gamma)$. For each vertex v_k of Γ, denote by Γ_k the subtree below v_k in Γ and which contains the edge right above v_k as its distinguished root. That is, Γ_k is the connected component of $\Gamma - \{v_k\}$ not containing the root edge. We define

$$\mathcal{D}_k = \{D_E \mid E \in \mathcal{E}_{mc}(\Gamma),\ E \cap \mathrm{E}_{<\infty}(\Gamma_k) \neq \emptyset\}, \quad D_k = \sum_{D \in \mathcal{D}_k} D.$$

Hence if $D_E \in \mathcal{D}_k$, E does not contain edges of Γ which are above v_k.

Proposition 4.29. *The group of invariant Cartier divisors is generated by $D_1, ..., D_g$.*

Example 4.30. The group of invariant Cartier divisors of $V(\Gamma)$, where Γ is the tree in Figure 1, is generated by

$$\{D_{\{1,2\}} + D_{\{1,5,6\}} + D_{\{2,3,4\}} + D_{\{3,4,5,6\}}, D_{\{2,3,4\}} + D_{\{3,4,5,6\}}, D_{\{1,5,6\}} + D_{\{3,4,5,6\}}\}.$$

Thus $n_{\{1,2\}}D_{\{1,2\}} + n_{\{3,4,5,6\}}D_{\{3,4,5,6\}} + n_{\{1,5,6\}}D_{\{1,5,6\}} + n_{\{2,3,4\}}D_{\{2,3,4\}}$ is a Cartier divisor if and only if $n_{\{1,2\}} + n_{\{3,4,5,6\}} = n_{\{1,5,6\}} + n_{\{2,3,4\}}$.

Proof of Proposition 4.29. We first check that D_k is a Cartier divisor. Recall the notation in (20), $s_k = s(\Gamma_k) \in \mathfrak{t}^\vee_{\mathbb{Z}}$ for each vertex v_k. Note that s_k satisfies $\langle s_k, v_E\rangle = 1$ if $E \in \mathcal{D}_k$ and $\langle s_k, v_E\rangle = 0$ otherwise. Indeed, for each $E \in \mathcal{D}_k$, D_E induces a Cartier divisor D_{E_k} for the toric variety corresponding to Γ_k and by 4.20, we have $\langle s_k, v_{E_k}\rangle = 1$. This implies $\langle s_k, v_E\rangle = 1$. On the other hand, if $E \notin \mathcal{D}_k$, then E does not contain any edges in Γ_k. Thus, $\langle s_k, v_E\rangle = 0$. Hence D_k is a Cartier divisor.

Next we check that $D_k, k = 1, \ldots, g$ generate the group of invariant Cartier divisors. Note that $s_k = {\epsilon_k}^\vee \mod \epsilon_1^\vee, \ldots, \epsilon_{k-1}^\vee$. It follows by an inductive argument that $s_1, \ldots, s_g$ generate $\mathfrak{t}^\vee_{\mathbb{Z}}$ so that $D_1, ..., D_g$ generate the group of Cartier divisors of $V(\Gamma)$. □

We have the following description of the group of invariant Cartier divisors of $V(\Gamma)$. Let w be the number of prime Weil invariant divisors of $V(\Gamma)$.

Theorem 4.31. *$\sum_D n_D D$ is a Cartier divisor of $V(\Gamma)$ if and only if $\sum_D m_D n_D = 0$ for every $(m_D)_D \in \mathbb{Z}^w$ that satisfies $\sum_{E: e \in E} m_{D_E} = 0$ for every edge $e \in \mathrm{E}_{<\infty}(\Gamma)$.*

Proof. The group of Cartier boundary divisors of $V(\Gamma)$ forms a sublattice of the group of Weil boundary divisors $\mathbb{Z}^w$, isomorphic to the weight lattice $\mathfrak{t}^\vee_{\mathbb{Z}} \cong \mathbb{Z}^g$. Suppose $(m_D)_D \in \mathbb{Z}^w$ satisfies the condition in the Theorem,

$$\sum_{E: e \in E} m_{D_E} = 0, \quad \forall e \in \mathrm{E}_{<\infty}(\Gamma).$$

Consider a non-self-crossing path γ from a vertex v_k to a colored vertex. By summing over edges of γ, we obtain

$$\sum_{D\in\mathcal{D}_k} m_D = \sum_{e\in\gamma}\sum_{E:e\in E} m_{D_E} = 0.$$

Thus, if $\sum_D n_D D$ is a Cartier divisor, then D is a combination of $D_1,\dots,D_g$ and so $\sum_D m_D n_D = 0$. On the other hand, the set of $(m_D)_D \in \mathbb{Z}^w$ that satisfies the condition in the Theorem forms a sublattice with dimension at least $w-g$, since the conditions are linearly independent. Therefore, the space of n_D satisfying the condition in the Theorem form a lattice of dimension g, which is the same as that of the space of Cartier boundary divisors. This shows that the two spaces are the same, up to torsion.

To show that the lattices are in fact the same, suppose that $(n_D)_D \in \mathbb{Z}^w$ satisfies the condition in the Theorem. By the previous paragraph,

$$\sum_D n_D D = \frac{1}{s}\sum_{i=1}^{g} r_i D_i$$

for some integers r_i and s such that $s>0$. It is suffices now to show that $s|r_i$ for every $1\le i\le g$. To see this, note that

$$\sum_D n_D D = \frac{1}{s}\left(\sum_{i=1}^{g}(r_i \sum_{D\in\mathcal{D}_i} D)\right) = \frac{1}{s}\left(\sum_D(\sum_{i:D\in\mathcal{D}_i} r_i)D\right).$$

Thus

$$n_D = \frac{1}{s}\sum_{i:D\in\mathcal{D}_i} r_i.$$

For the principal vertex v_g, define $D^0 = D_E$ where $E=\{d_1,...,d_p\}$. More generally, for any vertex v_k, let γ_k be the down-path from v_g to v_k, and let D^k to be the divisor $D^k = D_E$ where E is the set of edges immediately below the vertices in γ_k. Then

$$n_{D^k} = \frac{1}{s}\sum_{v_i\in\gamma_k} r_i \in \mathbb{Z}. \tag{30}$$

Since $n_{D^1}\in\mathbb{Z}$, we obtain $s|r_1$, and similarly for any vertices adjacent to the semi-infinite edges besides the root edge. Induction on the length of the path γ_k gives $s|r_k$. □

One can reformulate the result of Theorem 4.31 as follows. Given an element $E\in\mathcal{E}_{mc}(\Gamma)$, the set of colored vertices $V^+(\Gamma)$ is partitioned by the subsets of colored vertices below $e\in E$. Denote by $\mathrm{Par}(\Gamma)$ the set of such partitions of $V^+(\Gamma)$. Also, define $\mathcal{P}(\Gamma)$ the power set of $V^+(\Gamma)$.

Example 4.32. The invariant prime Weil divisors of $V(\Gamma)$ from Example 4.3 are $D_{\{1,2\}}, D_{\{1,5,6\}}$, $D_{\{2,3,4\}}, D_{\{3,4,5,6\}}$, corresponding to the partitions

$$\{\{1,2\},\{3,4\}\},\{\{1,2\},\{3\},\{4\}\},\{\{1\},\{2\},\{3,4\}\},\{\{1\},\{2\},\{3\},\{4\}\}$$

of the labels of the markings $\{1,2,3,4\}$.

Theorem 4.31 can be reformulated as

Corollary 4.33. *A sum* $\sum_{\{I_1,\dots,I_r\}\in \mathrm{Par}(\Gamma)} n_{I_1,\dots,I_r} D_{I_1,\dots,I_r}$ *is a Cartier divisor of* $V(\Gamma)$ *if and only if* $(n_{I_1,\dots,I_r})_{\{I_1,\dots,I_r\}\in\mathrm{Par}(\Gamma)}$ *is in the orthogonal complement of the kernel of* t_Γ*, where*

$$t_\Gamma : \mathbb{Z}^{\mathrm{Par}(\Gamma)} \to \mathbb{Z}^{\mathcal{P}(\Gamma)}, \quad (t_\Gamma(m))(S) = \sum_{S\in\{I_1,\dots,I_r\}} m_{I_1,\dots,I_r}.$$

5. Global description of boundary divisors

In this section we give a criterion for a boundary divisor in the moduli space of scaled lines $\overline{M}_{n,1}(\mathbb{A})$ to be Cartier. By the local description of the moduli space in Section 4, any divisor of type I is Cartier, so it suffices to consider divisors of type II. To describe the answer, let $I = \{1,\dots,n\}$, let $\mathrm{Par}(I)$ be the set of *non-trivial* partitions of I, and $\mathcal{P}(I)$ the power set of *non-empty* subsets of I. We identify the set of prime Weil boundary divisors of type I with the subset of elements of $\mathcal{P}(I)$ of size at least two, and the prime Weil boundary divisors of type II with $\mathrm{Par}(I)$. Thus in particular the space of Weil boundary divisors of type II becomes identified with $\mathbb{Z}^{\mathrm{Par}(I)}$, by the map

$$\left\{\sum_P l(P) D_P\right\} \to \mathbb{Z}^{\mathrm{Par}(I)}, \quad \sum_P l(P) D_P \mapsto l.$$

Let $Z(I)$ denote the natural incidence relation,

$$Z(I) = \{(S,P) \in \mathcal{P}(I) \times \mathrm{Par}(I) \,|\, S \in P\}.$$

We have a natural map from the space of functions on $\mathrm{Par}(I)$ to functions on $\mathcal{P}(I)$ given by pullback and push-forward:

$$t : \mathbb{Z}^{\mathrm{Par}(I)} \to \mathbb{Z}^{\mathcal{P}(I)}, \quad (t(h))(S) = \sum_{S\in P} h(P). \tag{31}$$

A *relation* on the group of Cartier boundary divisors is a collection of coefficients $\{m_{I_1,\dots,I_r}\} \in \mathbb{Z}^{\mathrm{Par}(\{1,\dots,n\})}$ such that

$$\sum_{I_1,\dots,I_r} m_{I_1,\dots,I_r} l_{I_1,\dots,I_r} = 0$$

for every Cartier divisor $D = \sum_{I_1,\dots,I_r} l_{I_1,\dots,I_r} D_{I_1,\dots,I_r}$. The space of relations forms a subgroup of $\mathbb{Z}^{\mathrm{Par}(I)}$.

Theorem 5.1. (Relations on Cartier boundary divisors) *The group of relations on the group of Cartier boundary divisors of type II is the kernel of* t*.*

Example 5.2. For $n = 2$ there are two boundary divisors, and there is only the zero relation. For $n = 3$ there are eight boundary divisors, and there is only the zero relation. For $n = 4$ there are $|\mathcal{P}(\{1,2,3,4\})| - 4 = 11$ boundary divisors of type I, and $|\mathrm{Par}(\{1,2,3,4\})| = 1+6+3+4 = 14$ boundary divisors of type II. A divisor

$$D = \sum l_{I_1,\dots,I_r} D_{I_1,\dots,I_r}$$

is Cartier only if the three relations (as i,j,k,l vary)

$$l_{\{i,j\},\{k\},\{l\}} + l_{\{i\},\{j\},\{k,l\}} - l_{\{i,j\},\{k,l\}} - l_{\{i\},\{j\},\{k\},\{l\}} \tag{32}$$

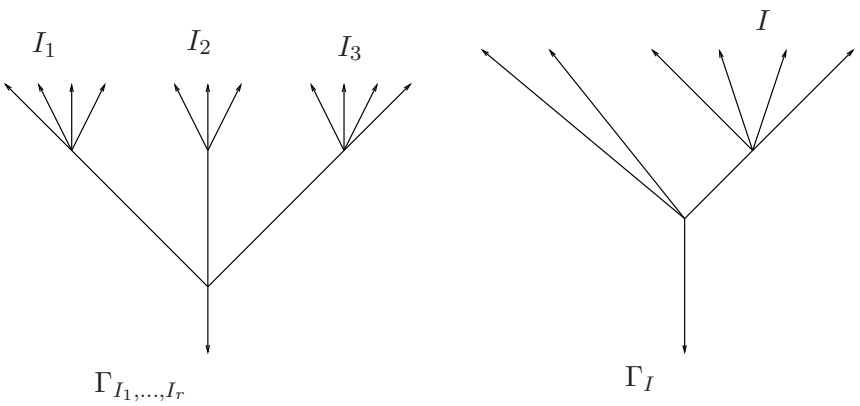

FIGURE 6. The trees Γ_I and $\Gamma_{I_1,\ldots,I_r}$

hold. Thus the space of Cartier boundary divisors of type II is an 11-dimensional subspace of the space of the 14-dimension space of Weil boundary divisors of type II.

Definition 5.3. (Compatible subsets and partitions with a tree)

(a) A tree Γ is *simple* if it has a single vertex.
(b) For each partition $\{I_1, ..., I_r\}$ of $I = \{1, ..., n\}$, define the tree $\Gamma_{I_1,\ldots,I_r}$ as follows: $\Gamma_{I_1,\ldots,I_r}$ has r principal subtrees which are respectively the simple colored trees $\Gamma_j, j = 1, \ldots, r$ whose semi-infinite edges labeled by $i \in I_j$.
(c) For each subset $I \subset \{1, \ldots, n\}$, let Γ_I denote the colored tree with a single colored vertex and a single non-colored vertex with semi-infinite edges labelled by $i \in I$.
(d) Given $v, \tilde{v} \in \mathrm{Vert}(\Gamma)$, we write $vE\tilde{v}$ if there is an edge connecting v and $\tilde{v}$. A *tree homomorphism* $f : \mathrm{Vert}(\Gamma) \to \mathrm{Vert}(\Gamma')$ is a map that maps the vertices and edges of Γ to the vertices and edges of Γ' respectively and satisfies:
  (i) f maps the principal vertex v_g of Γ to the principal vertex v'_0 of Γ'.
  (ii) If $v, \tilde{v} \in \mathrm{Vert}(\Gamma)$ satisfies $vE\tilde{v}$, then either $f(v)Ef(\tilde{v})$ or $f(v) = f(\tilde{v})$.
  (iii) f maps the colored vertices of Γ to the colored vertices of Γ'.
(e) A subset $I \subset \{1, \ldots n\}$ is *compatible* with Γ if there exists a tree homomorphism $\Gamma \to \Gamma_I$.
(f) A partition $\{I_1, ..., I_r\}$ of $\{1, ..., n\}$ is *compatible* with Γ if there exists a tree homomorphism $f : \Gamma \to \Gamma_{I_1,\ldots,I_r}$.

See Figure 6 for the trees $\Gamma_{I_1,\ldots,I_r}$ and Γ_I. Denote by $\mathrm{Par}(\Gamma)$ the set of compatible partitions of $\{1, \ldots, n\}$, and by $\mathcal{P}(\Gamma)$ the set of compatible subsets of $\{1, \ldots, n\}$.

Proposition 5.4. *There is a canonical bijection between the set of minimally complete subsets of* $\mathrm{E}_{<\infty}(\Gamma)$ *and the set of compatible partitions* $\mathrm{Par}(\Gamma)$.

Example 5.5. For the tree Γ in Figure 4, the correspondence between the minimally complete subsets of $\mathrm{E}_{<\infty}(\Gamma)$ and the compatible partitions of $\{1, ..., n\}$ is

$$\{e_1, e_2\} \longleftrightarrow \{1, 2\}, \{3, 4\}, \quad \{e_1, e_5, e_6\} \longleftrightarrow \{1, 2\}, \{3\}, \{4\}$$

$$\{e_2, e_3, e_4\} \longleftrightarrow \{1\}, \{2\}, \{3, 4\}, \quad \{e_3, e_4, e_5, e_6\} \longleftrightarrow \{1\}, \{2\}, \{3\}, \{4\}.$$

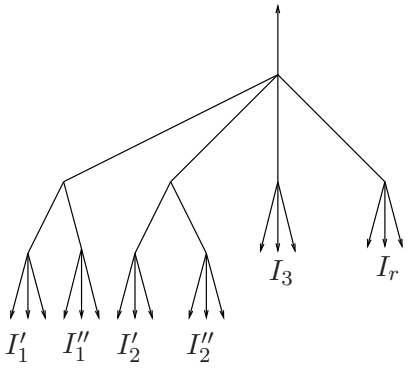

FIGURE 7. The tree Γ compatible with the four-term relation

Proof. Given a minimally complete subset E, we obtain a partition by removing the edges in E and considering the partition of the semi-infinite edges induced by the decomposition into connected components; that is, two semi-infinite edges are in the same set in the partition if they can be connected by a path in the complement of E. There is a morphism of trees $\Gamma \to \Gamma_{I_1,\dots,I_r}$ given by collapsing each connected component of $\Gamma - E$ to a point, which shows that the partition is compatible. Conversely, given a compatible partition, consider the corresponding morphism of trees $\Gamma \to \Gamma_{I_1,\dots,I_r}$ and let E denote the subset of edges of Γ that are not collapsed under the morphism. Since the finite edges of $\Gamma_{I_1,\dots,I_r}$ form a minimally complete subset of $\mathrm{E}_{<\infty}(\Gamma_{I_1,\dots,I_r})$, the set E is also minimally complete. The reader may check that these two maps of sets are inverses. □

From Proposition 5.4 we obtain a bijective correspondence between compatible partitions and the prime invariant Weil divisors of $V(\Gamma)$. For each compatible partition $\{I_1, ..., I_r\} \in \mathrm{Par}(\Gamma)$, denote by $D_{I_1,\dots,I_r}$ the corresponding invariant prime Weil divisor of $V(\Gamma)$.

Lemma 5.6. (Four-term relation) *Suppose that $\{I_1, \dots, I_r\}$ is a partition with at least two elements of size at least two. Then there exists a colored tree Γ so that $\{I_1, \dots, I_r\} \in \mathrm{Par}(\Gamma)$, and a relation $m \in \operatorname{Ker} t_\Gamma$ so that*

(a) $m(\{I_1, \dots, I_r\}) = 1$.
(b) *For any partition $\{J_1, \dots, J_{r'}) \in \mathrm{Par}(\Gamma)$ distinct from $\{I_1, \dots, I_r\}$, we have $m(\{J_1, \dots, J_{r'}\}) = 0$ unless $r' > r$.*

Proof. Without loss of generality suppose that $|I_1|, |I_2|$ are both at least two, and so admit partitions $I_1 = I_1' \cup I_1''$, $I_2 = I_2' \cup I_2''$. Let Γ be the tree with $r+2$ colored vertices, as in Figure 7. Then the sum of delta functions

$$\delta_{I_1',I_1'',I_2',I_2'',I_3,\dots,I_r} - \delta_{I_1,I_2',I_2'',I_3,\dots,I_r} - \delta_{I_1',I_1'',I_2,I_3,\dots,I_r} + \delta_{I_1,I_2,I_3,\dots,I_r} \in \operatorname{Ker} t_\Gamma$$

is a relation since each subset in each partition occurs an equal number of times with opposite signs. □

Lemma 5.7. *Let $m \in \mathrm{Ker}(t)$ be a relation and $r \in \{1, \dots, n-1\}$. Assume that m vanishes on every partition of length less than r. Then m vanishes on every partition that consists of $r-1$ singletons and a set of size $n-r+1$. If $r = n-1$ then m is constantly equal to zero.*

Proof. To prove the first assertion, let S be a set of size $n-r+1$. We denote by P the unique partition consisting of S and $r-1$ singletons. Every partition containing S, other than P, has length less than r. Hence by assumption, m vanishes on such a partition. Using the hypothesis $tm = 0$ and the definition (31) of t, it follows that $m(P) = 0$. The first assertion follows.

To prove the second assertion, consider the case $r = n-1$. Every partition of length $n-1$ consists of $n-2$ singletons and one set of size two. By the first assertion, m vanishes on every such partition. Since by hypothesis m vanishes on every partition of length less than $n-1$, it follows that m vanishes on all partitions, except possibly on $\big\{\{1\},\dots,\{n\}\big\}$. However, since $tm = 0$, equality (31) with $S = \{1\}$ implies that m vanishes on this partition, as well. This proves the second assertion. □

Proof of Theorem 5.1. A Weil divisor is Cartier if and only if its restriction to every affine Zariski open subset is Cartier. Hence it suffices to check if a divisor

$$D = \sum l_{I_1,\dots,I_r} D_{I_1,\dots,I_r}$$

is Cartier in every chart in Proposition 4.4. Note that each $D_{I_1,\dots,I_r}$ corresponds to a non-empty Weil boundary divisor in $V(\Gamma)$ iff $\{I_1,\dots,I_r\} \in \mathrm{Par}(\Gamma)$. A criterion for a Weil boundary divisor in $V(\Gamma)$ to be Cartier is given above in Lemma 4.33. There is a natural embedding π_Γ of $\operatorname{Ker} t_\Gamma$, as in 4.33, in $\operatorname{Ker} t$ which preserves $m_{I_1,\dots,I_r}$ if $\{I_1,\dots,I_r\} \in \mathrm{Par}(\Gamma)$ and maps the other $m_{I_1,\dots,I_r}$, where $\{I_1,\dots,I_r\} \notin \mathrm{Par}(\Gamma)$, to 0. The image of π_Γ is a subspace of $\operatorname{Ker} t$ and D is a Cartier divisor of $V(\Gamma)$ if and only if $(l_{I_1,\dots,I_r}) \in \operatorname{coker} \pi_\Gamma$. Hence, D is a Cartier divisor of all $V(\Gamma)$ if and only if $(l_{I_1,\dots,I_r})$ is in the orthogonal complement of the image π_Γ in $\mathbb{Z}^{\mathrm{Par}(I)}$ for all Γ. Thus, it suffices to show that

$$\operatorname{Ker} t \subset \operatorname{hull}_{\mathbb{Z}} \operatorname{image} \pi_\Gamma. \tag{33}$$

For this let $m \in \operatorname{Ker} t$ be a relation. Assume that m is nonzero on some partition of length $\leq n-2$, and let $\{I_1,\dots,I_r\}$ a partition of minimal length, on which m is nonzero. It follows from Lemma 5.7 that $\{I_1,\dots,I_r\}$ contains at least two sets of size at least two. Hence by Lemma 5.6 there exists a colored tree Γ such that $\{I_1,\dots,I_r\} \in \mathrm{Par}(\Gamma)$, and a relation $m' \in \operatorname{Ker} t_\Gamma$ that attains the value 1 on $\{I_1,\dots,I_r\}$ and vanishes on all other partitions of length at most r. The relation $m - m_{I_1,\dots,I_r} m'$ is non-zero on fewer partitions of length r than m. Continuing in this way we obtain a relation which vanishes on all partitions of length less than $n-1$. By the second statement in Lemma 5.7, any such relation must be zero. It follows that m is a linear combination of elements of $\operatorname{Ker} t_\Gamma$, where Γ ranges over all colored trees. This proves the inclusion (33) and completes the proof of Theorem 5.1. □

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