Archimedes' Epitaph and Toric Varieties

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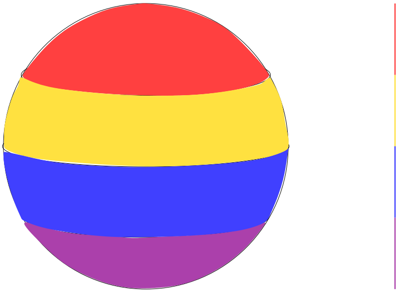
Fall 2013 MAA-NJ Meeting, Rutgers University New Brunswick

1797 Benjamin West: Cicero at Archimedes' tomb

Areas of spheres

Suppose a sphere is cut into four pieces of equal height. (The view of the sphere is slightly from above.)



Which piece has the most surface area?

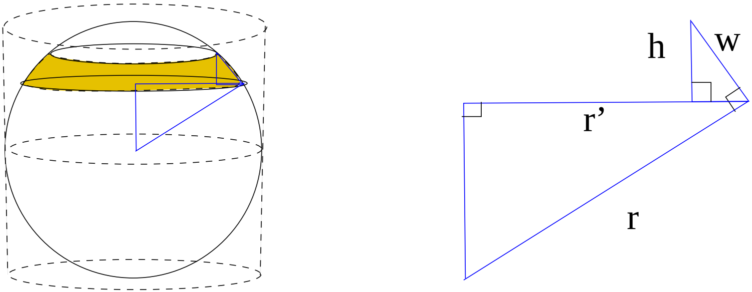
Arguments both ways

On the one hand, the middle pieces “have bigger circumference”.

On the other hand, the top and bottom pieces are “wider” (where width is the difference between the inner and outer radii.)

You can do an experiment using an orange peel.

Each piece can be approximated by a rectangle

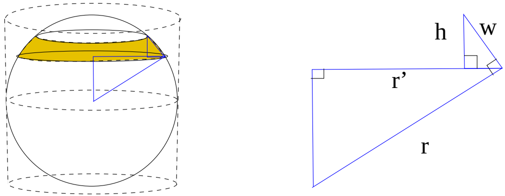


Let r' the radius of the section and w the width of one layer of peel. Each piece, after flattening, is approximately a rectangle of length $2\pi r'$ and width w and so has area

$$A_{\text{piece}} \approx \text{length} \times \text{width} = 2\pi r' w.$$

All pieces have the same area

In the diagram, the little triangle is similar to the big triangle:



Let r be the radius of the sphere, r' the radius of the section, h the height of the section, and w the width of one layer of peel. Then

$$\text{similarity} \implies r'/r = h/w \implies \quad r'w = rh.$$

Each piece, after flattening, is approximately a rectangle of length $2\pi r'$ and width w and so has area

$$A_{\text{piece}} \approx \text{length} \times \text{width} = 2\pi r' w = 2\pi r h.$$

Archimedes' area formula

The formula $A_{\text{piece}} = 2\pi rh$ is *exact* because each piece can be broken up into smaller pieces where the approximation is better. Each piece has height

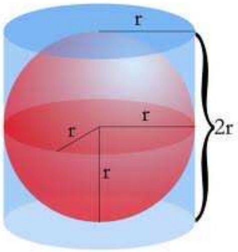
$$h = \text{height of sphere}/\text{number of pieces} = 2r/\text{number of pieces}.$$

So the total area of the sphere is

$$\begin{aligned} A_{\text{sphere}} &= \text{number of pieces} \times \text{area of each piece} \\ &= \text{number of pieces} \times 2\pi rh \\ &= \text{number of pieces} \times 2\pi r(2r/\text{number of pieces}) \\ &= 4\pi r^2. \end{aligned}$$

Areas of spheres versus cylinders

In other words, the area of the sphere is equal to the *lateral surface area* of the cylinder in which it is inscribed, that is:



$$A = \text{perimeter of base} \times \text{height} = 2\pi r \times 2r = 4\pi r^2.$$

Each piece of the sphere has the same area as a piece near the equator. After replacing each piece with a copy of this piece one gets a cylinder instead of a sphere, but with the same height and radius.

Areas of cylinders containing balls

By stacking, we see that the same is true for cylinders circumscribing any number of balls. For example, for a can of tennis balls, the area of the label is the same as the surface area of the balls.

Archimedes epitaph

The formula for the surface area of the sphere is due to Archimedes (c.287 BC c.212 BC) who probably learned mathematics in Alexandria.

Archimedes was so proud of this result that he ordered it to be depicted on his tombstone. When Cicero was appointed governor of Syracuse, he made a special trip to Archimedes' grave to see the tombstone, found it covered in weeds and forgotten, and had it cleaned up.

C H Edwards: "The Romans had so little interest in pure mathematics that this action by Cicero was probably the greatest single contribution of any Roman to the history of mathematics."

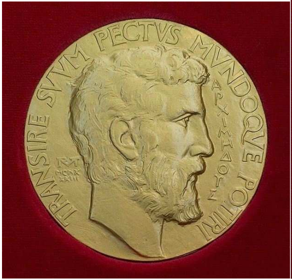
1797 Benjamin West: Cicero at Archimedes' tomb

It's not known how Archimedes' theorem was depicted: as an etching? sculpture? West was just guessing (artistic license).

Mount Aetna is in the background. West is a painter from Philadelphia who became president of the Royal Society in London.

Depiction on the Fields Medal

Archimedes' theorem also appears (behind the laurels) on the Fields Medal on the right below.



The other side has Archimedes' picture (above left.) Besides depicting a great theorem, the Fields Medal also shows a Corolla (Garland) the root of the mathematical term "corollary".

Digression: Google-searching Archimedes' tomb

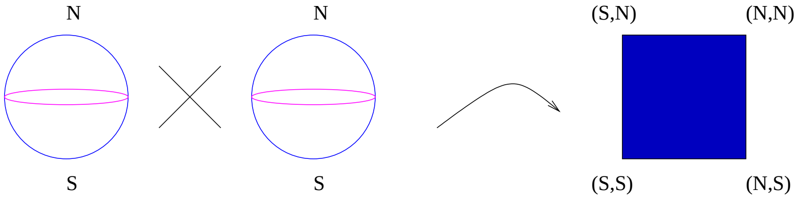
Over the centuries many mathematicians have speculated about the location of Archimedes tomb. The archaelogist Paolo Orsi made an excavation of the Hellenistic graves in the late 1800s and noted the Greek tombs to the west of the city. Using Google streetview one can see what is there now: A shopping center "I Papiri" (The Papyri).

Papyrus is an African plant. Syracuse is the one of the few places outside of Africa where Papyrus grows wild, possibly because it was taken from Egypt by the Greek settlers.

Back to math: Archimedes' formula in higher dimensions

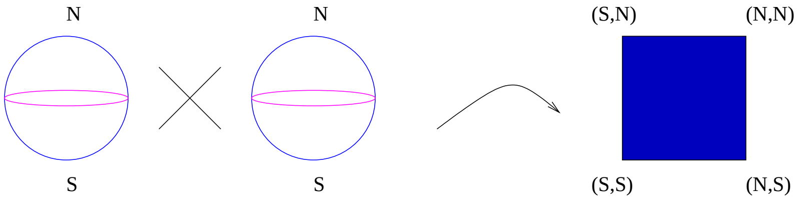
There are higher-dimensional generalizations of Archimedes' formula.

For example, let S^2 be the unit two-sphere, X the product $S^2 \times S^2$ and P the square $[-1, 1] \times [-1, 1]$.



$$\begin{aligned}\mathrm{Vol}(S^2 \times S^2) &= \mathrm{Vol}(S^2) \times \mathrm{Vol}(S^2) \\ &= (4\pi)^2 = (2\pi)^2 4 \\ &= (2\pi)^2\,\mathrm{Vol}(P).\end{aligned}$$

Visualizing $S^2 \times S^2$ as a square



Note that X maps to P by the product of height maps.

$$X \ni (x_1, y_1, z_1), (x_2, y_2, z_2) \mapsto (z_1, z_2) \in P.$$

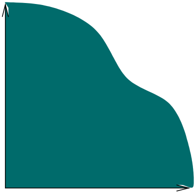
We can "visualize" the four-dimensional space $X = S^2 \times S^2$ as a "thickening" of the square P:

Replace each point in the interior with a two-torus, each point in a face with a circle, and keep each vertex the same.

Toric varieties

The same procedure works for any two (or higher-dimensional) convex polytope P: Replace each point in the interior with a two-torus, each point in a face with a circle, and keep each vertex the same to obtain a space X_P, the *toric variety* corresponding to P.

Question: What is the toric variety corresponding to the first quadrant $P = \mathbb{R}^2_{\geq 0} = \{(\lambda_1, \lambda_2) \in \mathbb{R}^2, \lambda_1 \geq 0, \lambda_2 \geq 0\}$?



Can you think of a space that maps to P whose generic fiber is a two-torus?

Toric varieties

Question: What is the toric variety corresponding to the first quadrant

$$P = \mathbb{R}^2_{\geq 0} = \{(\lambda_1, \lambda_2) \in \mathbb{R}^2, \lambda_1 \geq 0, \lambda_2 \geq 0\}?$$

Answer: The toric variety is $X_P = \mathbb{C}^2$. The map to $P = \mathbb{R}^2_{\geq 0}$ is given by the norm-squares,

$$X_P \to P, \quad (z_1, z_2) \mapsto (|z_1|^2, |z_2|^2).$$

The fiber over $\lambda_1 = |z_1|^2, \lambda_2 = |z_2|^2$ is all pairs of complex numbers with a fixed modulus.

Archimedes' theorem in higher dimension

Theorem: For any convex polytope P, the volume of the toric variety X_P is the same as the volume of the polytope up to a constant:

$$\mathrm{Vol}(X_P) = (2\pi)^{\dim(P)}\,\mathrm{Vol}(P).$$

The argument is basically the same as Archimedes'. Why would anyone care about this? Applications include:

(i) Formulas for the number of lattice points in a polytope; and

(ii) A generalization of the fundamental theorem of algebra to higher dimensions.

A generalization of the fundamental theorem of algebra

Fundamental theorem of algebra: Any complex polynomial of degree d has d roots counted with multiplicity.

Question: What is the generalization of this theorem to more variables? There are many generalizations. But here is a nice one:

If $W(z_1, \ldots, z_n)$ is a Laurent polynomial of degree d, then the zero set $W^{-1}(0)$ is not finite. But the critical set

$$\mathrm{crit}(W) = \left\{ \frac{\partial}{\partial z_i} W(z_1, \ldots, z_n) = 0, i = 1, \ldots, n \right\}$$

is finite for generic W.

A generalization of the fundamental theorem of algebra

Question: How many critical points does a Laurent polynomial W of n variables have?

If $W(z)$ is a Laurent polynomial in a single variable z, then this question is more or less equivalent to the fundamental theorem of algebra.

Example: $W(z) = z + 1/z$, so $\partial W/\partial z = 1 - 1/z^2$ with two solutions $z = \pm 1$.

In general if $W(z) = c_{d_-} z^{d_-} + \ldots + c_{d_+} z^{d_+}$ with $c_{d_\pm}$ non-zero and $d_- < 0 < d_+$ then there are $d_+ - d_-$ non-zero critical points.

An example which comes up in high energy physics

Suppose

$$W(z_1, z_2) = z_1 + z_2 + 1/z_1 z_2.$$

Partial derivatives

$$\partial_{z_1} W(z_1, z_2) = 1 - 1/z_1^2 z_2, \partial_{z_2} W(z_1, z_2) = 1 - 1/z_1 z_2^2.$$

The critical points are solutions to

$$z_1^2 z_2 = 1 = z_1 z_2^2.$$

What are the solutions?

An example which comes up in high energy physics ctd.

Suppose

$$W(z_1, z_2) = z_1 + z_2 + 1/z_1 z_2.$$

Partial derivatives

$$\partial_{z_1} W(z_1, z_2) = 1 - 1/z_1^2 z_2, \partial_{z_1} W(z_1, z_2) = 1 - 1/z_1 z_2^2.$$

The critical points are solutions to

$$z_1^2 z_2 = 1 = z_1 z_2^2$$

so

$$z_1 = z_2, \quad z_1^3 = z_2^3 = 1.$$

There are *three solutions*, the *cubic roots of unity*.

Newton polygons

Suppose that the function W is a sum of monomials in a set $C(W) \subset \mathbb{Z}^n$, so that

$$W(z_1, \dots, z_n) = \sum_{(d_1,\dots,d_n) \in C(W)} c_{d_1,\dots,d_n} z_1^{d_1} \dots z_n^{d_n}, \, c_{d_1,\dots,d_n} \neq 0.$$

The *Newton polygon* $P(W)$ is the convex hull of the elements of $C(W)$.

Example: If $W(z_1, z_2) = z_1 + z_2 + 1/z_1 z_2$ then

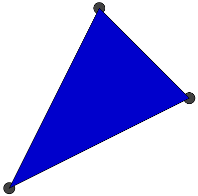
$$C(W) = \{(1, 0), (0, 1), \text{ and (can you fill in the blank)}\}$$

Newton polygons

Example: If $W(z_1, z_2) = z_1 + z_2 + 1/z_1 z_2$ then

$$C(W) = \{(1,0),(0,1),(-1,-1)\}.$$

So $P(W)$ is the convex hull:



The number of critical points of a Laurent polynomial

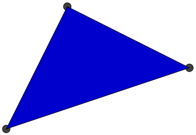
Theorem (Kouchnirenko): For generic W, the number of critical points of W is $\dim(P(W))!\,\mathrm{Vol}(P(W))$.

Example: If $W(z_1, z_2) = z_1 + z_2 + 1/z_1 z_2$ then $\mathrm{Vol}(P(W)) = 3/2$ so the number of critical points is $(3/2)2 = 3$.

Activity: Suppose that $W(z_1, z_2) = z_1^2 + z_2 + 1/z_1 z_2$. Find the Newton polygon $P(W)$, its volume, and the number of critical points of W. Find the critical points explicitly.

The number of critical points of a Laurent polynomial

Answer: For $W(z_1, z_2) = z_1^2 + z_2 + 1/z_1 z_2$ the Newton polygon $P(W)$ is the convex hull of $(2, 0), (0, 1), (-1, -1)$:



Its volume is $5/2$, so there should be $2(5/2) = 5$ critical points. The equations for the critical points are

$$2z_1 - 1/z_1^2 z_2 = 0, \quad 1 - 1/z_1 z_2^2 = 0$$

so

$$2z_1^3 z_2 = 1 = z_1 z_2^2, z_2 = 2z_1^2, 4z_1^5 = 1.$$

So there are five critical points.

Proof of Kouchnirenko's theorem

It boils down to the generalized Archimedes' theorem. Each partial derivative of W define a section of a line bundle over $X_{P(W)}$ usually called $\mathcal{O}(1)$.

Together these line bundles form a vector bundle over the toric variety $X(W)$, so that the critical points correspond to zeroes of the section.

The number of zeroes is called the *Euler number* of the vector bundle and is given by integrating an *Euler class*.

The integral of the Euler class turns out to be the volume of the toric variety divided by $(2\pi)^{\dim(P(W))}$.

The computation in Kouchnirenko's theorem

To do the computation, you have to be a bit familiar with Euler classes.

$$
\begin{aligned}
\#\,\mathrm{Crit}(W) &= \int_{X(W)} \mathrm{Eul}(\mathcal{O}(1)^{\oplus\,\dim(P(W))}) \\
&= \int_{X(W)} \mathrm{Eul}(\mathcal{O}(1))^{\dim(P(W))} \\
&= \dim(P(W))! \int_{X(W)} \exp(\mathrm{Eul}(\mathcal{O}(1))) \\
&= \dim(P(W))!\,\mathrm{Vol}(X(W))/(2\pi)^{\dim(P(W))} \\
&= \dim(P(W))!\,\mathrm{Vol}(P(W)).
\end{aligned}
$$

Ways of making change

Another application of generalized Archimedes' theorem for toric varieties is to computing the number of integer points in polytopes.

Example: How many ways of making change for a dollar are there? (Say, without using a dollar coin?) My daughter was sent home with this question for homework. But the answer given by the textbook was wrong.

The number of ways is the number of *non-negative, integer* solutions to

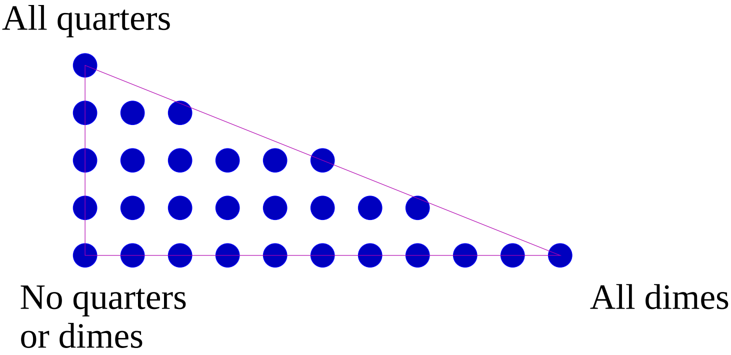
$$p + 5n + 10d + 25q + 50h = 100.$$

A generalization and an answer

More generally let $N(x)$ be the number of ways of making change for x half-dollars, that is, non-negative integer solutions to

$$p + 5n + 10d + 25q + 50h = 50x.$$

For example, if we use only nickels, dimes, and quarters and make change for a dollar, then the possibilities are:



Ways of making change via Archimedes

Let $P(x)$ denote the space of solutions to the equation. It is a convex polytope of dimension four. The number $N(x)$ is the number of integer points in $P(x)$.

$N(x)$ can be expressed in terms of the toric variety $X_{P(x)}$ as the *Euler characteristic* of $\mathcal{O}(x)$. The *Riemann-Roch formula* combined with *Archimedes theorem* gives a formula for $N(x)$, which can be massaged into

$$N(x) = (6 + 55x + 119x^2 + 95x^3 + 25x^4)/6$$

For example, putting $x = 1$ gives 50 ways of making change for fifty cents, or $x = 2$ gives 292 ways of making change for a dollar (without using the dollar coin). This is a special case of *Khovanskii-type* formulas for number of integral points in polytopes.

Further references

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