

# NON-ABELIAN CONVEXITY BY SYMPLECTIC CUTS

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ABSTRACT. We give new proofs of the convexity and connectedness properties of the moment map using the technique of symplectic cutting and extend these results to the case of orbifolds.

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## 1. INTRODUCTION

In this paper we prove the following theorem, which extends results of Kirwan and others to the case of orbifolds. Recall that a subset  $\Delta$  of a vector space  $V$  is **polyhedral** if it is the intersection of finitely many closed half-spaces, and is **locally polyhedral** if for any point  $x \in \Delta$ , there is a neighborhood  $U$  of  $x$  in  $V$  and a polyhedral set  $P$  in  $V$  such that  $U \cap \Delta = U \cap P$ .

**Theorem 1.1.** *Let  $(M, \omega)$  be a connected symplectic orbifold with a Hamiltonian action of a compact Lie group  $G$  and a proper moment map  $\Phi : M \rightarrow \mathfrak{g}^*$ .*

- a. *Let  $\mathfrak{t}_+^*$  be a closed Weyl chamber for the Lie group  $G$  considered as a subset of  $\mathfrak{g}^*$ . The moment set  $\Delta := \Phi(M) \cap \mathfrak{t}_+^*$  is a convex locally polyhedral set. In particular, if  $M$  is compact then  $\Delta$  is a convex polytope.*
- b. *Each fiber of the moment map  $\Phi$  is connected.*

The non-abelian convexity and connectedness theorem has numerous applications in symplectic geometry: For example, it is used in the classification of Hamiltonian  $G$ -orbifolds, in geometric quantization, and in the study of the existence of invariant Kähler structures. We also extend to orbifolds a theorem of Sjamaar [Sj]:

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**Theorem 1.2.** *Let  $(M, \omega)$  be a connected symplectic orbifold with a Hamiltonian action of a compact Lie group  $G$  and a proper moment map  $\Phi : M \rightarrow \mathfrak{g}^*$ . Then the quotient map  $M/G \rightarrow \Phi(M) \cap \mathfrak{t}_+^*$  is open. That is, for every  $m \in \Phi^{-1}(\mathfrak{t}_+^*)$ , and every  $G$ -invariant neighborhood  $U$  of  $m$  in  $M$ , the image  $\Phi(U) \cap \mathfrak{t}_+^*$  is a relatively open neighborhood of  $\Phi(m)$  in  $\Phi(M) \cap \mathfrak{t}_+^*$ .*

Together with the symplectic slice theorem, Sjamaar's result shows that the shape of the moment set near a point  $x \in \Delta$  is determined by the isotropy representation at *any* point in the fiber  $\Phi^{-1}(x)$ . For an application, see Karshon-Lerman [KL].

Convexity theorems in symplectic geometry have a long history. For the action of a maximal torus on a coadjoint orbit Theorem 1.1 was proved by Kostant [Ko], extending previous results of Schur and Horn. This was generalized to the actions of subgroups by Heckman [H]. For Hamiltonian torus actions on manifolds the theorem was proved independently by Atiyah [A] and by Guillemin and Sternberg [GS2]. In the non-abelian case Guillemin and Sternberg proved that the moment set is a union of convex polytopes and that it is a single polytope for a Kähler manifold. The convexity in the Kähler case was independently proved by Mumford [NM]. The first complete proof for Hamiltonian actions of non-abelian groups on manifolds was given by Kirwan [Ki2], using Morse theory of the Yang-Mills functional and results of Guillemin and Sternberg.

Since Kirwan's proof appeared, there have been several new approaches leading to extensions of the result. The techniques of Condevaux-Dazord-Molino [CDM] and Hilgert-Neeb-Plank [HNP] give a proof which also applies to actions on non-compact manifolds with proper moment maps. Sjamaar [Sj] offers a proof in the compact case which also gives information on the vertices of the polytope. Flaschka-Ratiu [FR] extend the result to the setting of Poisson geometry. Brion [B], Sjamaar [Sj], and Heinzner-Huckleberry [HH] discuss extensions in algebraic and Kähler geometry.

Our motivation for offering a new proof is twofold. First, Theorem 1.1 generalizes the result from manifolds to orbifolds; this category is important in symplectic geometry because, generically, the symplectic quotient of a symplectic manifold is an orbifold. Secondly, our proof is conceptually very simple. The first step is to generalize the abelian version of the theorem to certain non-compact orbifolds. The key idea is to apply the technique of symplectic cutting [L] to reduce to the compact case. In the second step we reduce the non-abelian case to the abelian case by means of a symplectic cross-section, as in [CDM], [HNP] and [FR].

In Meinrenken-Woodward [MW] the methods of this paper are used to prove convexity and connectedness properties for Hamiltonian loop group actions with proper moment maps. Results of this type were observed by S. Chang several years ago.

## 2. SYMPLECTIC CUTS

We begin by recalling the definition of a Hamiltonian action on a symplectic orbifold. An **orbifold**  $M$  is a topological space  $|M|$ , together with an **atlas** of **uniformizing charts**  $(\tilde{U}, \Gamma, \varphi)$ , where  $\tilde{U}$  is open subset of  $\mathbb{R}^n$ ,  $\varphi(\tilde{U})$  is an open subset of  $|M|$ ,  $\Gamma$  is a finite group which acts linearly on  $\tilde{U}$  and fixes a set of codimension at least two, and  $\varphi : \tilde{U} \rightarrow |M|$  induces a homeomorphism from  $\tilde{U}/\Gamma$  to  $\varphi(\tilde{U}) \subset |M|$ . Just as for manifolds, these charts must cover  $|M|$ ; they are subject to certain compatibility conditions; and there is a notion of when two atlases of charts are equivalent. For more details, see Satake [Sa].

A smooth function on  $M$  is a collection of smooth invariant functions on each uniformizing chart  $(\tilde{U}, \Gamma, \varphi)$  which agree on overlaps of the images  $\varphi(\tilde{U})$ . Differential forms, vectors fields, and other objects can be similarly defined. There is also a notion of morphisms (maps) of orbifolds.

Let  $x$  be a point in an orbifold  $M$ , and let  $(\tilde{U}, \Gamma, \varphi)$  be a uniformizing chart with  $x \in \tilde{U}/\Gamma$ . The **(orbifold) structure group** of  $x$  is the isotropy group  $\Gamma_x$  of  $\tilde{x} \in \tilde{U}$ , where  $\varphi(\tilde{x}) = x$ . The group  $\Gamma_x$  is well defined as an abstract group. The tangent space to  $\tilde{x}$  in  $\tilde{U}$ , considered as a representation of  $\Gamma_x$  is called the **uniformized tangent space** at  $x$ , and denoted by  $\tilde{T}_x M$ . The quotient  $\tilde{T}_x M/\Gamma_x$  is  $T_x M$ , the fiber of the **tangent bundle** of  $M$  at  $x$ .

Let  $G$  be a compact connected Lie group with Lie algebra  $\mathfrak{g}$ . A **smooth action**  $a$  of  $G$  on an orbifold  $M$  is a smooth orbifold map  $a : G \times M \rightarrow M$  satisfying the usual laws for an action. Given an action of  $G$  on  $M$ , every vector  $\xi \in \mathfrak{g}$  induces a vector field  $\xi_M$  on  $M$ .

A **symplectic orbifold** is an orbifold  $M$  with a closed non-degenerate 2-form  $\omega$ . A group  $G$  acts **symplectically** on  $(M, \omega)$  if the action preserves  $\omega$ . A **moment map** for a symplectic action of a group  $G$  is an equivariant map  $\Phi : M \rightarrow \mathfrak{g}^*$  such that

$$(1) \quad i_{\xi_M} \omega = -\langle d\Phi, \xi \rangle \quad \text{for all } \xi \in \mathfrak{g}.$$

If a moment map exists, we say that the action of  $G$  on  $(M, \omega)$  is Hamiltonian and refer to  $(M, \omega)$  as a **Hamiltonian  $G$ -space**. If  $b \in \mathfrak{g}^*$  is a regular value of the moment map  $\Phi$ , then the orbifold version of the Marsden-Weinstein-Meyer theorem says that the quotient  $M_b := \Phi^{-1}(b)/G_b$  is a symplectic orbifold, called the **symplectic reduction** of  $M$  at  $b$ . For a proof, see [LT].

We now recall the definition of the technique of symplectic cutting which was introduced in [L]. Let  $(M, \omega)$  be a symplectic orbifold with a Hamiltonian circle action and a moment map  $\mu : M \rightarrow \mathbb{R}$ . Suppose that  $\epsilon$  is a regular value of the moment map. Consider the disjoint union

$$M_{[\epsilon, \infty)} := \mu^{-1}((\epsilon, \infty)) \cup M_\epsilon,$$

obtained from the orbifold with boundary  $\mu^{-1}([\epsilon, \infty))$  by collapsing the boundary under the  $S^1$ -action. We claim that  $M_{[\epsilon, \infty)}$  admits a natural structure of a symplectic orbifold, in such a way that the embeddings of  $\mu^{-1}((\epsilon, \infty))$  and  $M_\epsilon$  are symplectic. Moreover, the induced circle action on  $M_{[\epsilon, \infty)}$  is Hamiltonian, with a moment map coming from the restriction of the original moment map  $\mu$  to  $\{m \in M : \mu(m) \geq \epsilon\}$ .

**Definition 2.1.** We call the symplectic orbifold  $M_{[\epsilon, \infty)}$  the **symplectic cut** of  $M$  with respect to the ray  $[\epsilon, \infty)$  (with symplectic form and moment map understood).

To see that the claim holds consider the product  $M \times \mathbb{C}$  of the orbifold with a complex plane. It has a natural (product) symplectic structure :  $\omega + (-i)dz \wedge d\bar{z}$ . The function  $\nu : M \times \mathbb{C} \rightarrow \mathbb{R}$  given by  $\nu(m, z) = \mu(m) - |z|^2$  is a moment map for the diagonal action of the circle (it commutes with the original action of the circle on  $M$ ). The point  $\epsilon$  is a regular value of  $\mu$  if and only if it is a regular value of  $\nu$ . The map

$$\{m \in M : \mu \geq \epsilon\} \rightarrow \nu^{-1}(\epsilon), \quad m \mapsto (m, \sqrt{\mu(m) - \epsilon})$$

descends to a homeomorphism from the cut space  $M_{[\epsilon, \infty)}$  to the reduced space  $\nu^{-1}(\epsilon)/S^1$ . Moreover the homeomorphism is  $S^1$ -equivariant and it is a symplectomorphism on the set  $\{\mu > \epsilon\}$ .

*Example 2.2.* Consider the complex plane  $\mathbb{C}$  with symplectic form  $-idz \wedge d\bar{z}$ , and let a circle  $U(1) = \{z \in \mathbb{C} : |z| = 1\}$  act by multiplication. The moment map is  $\mu(z) = -|z|^2$ . The symplectic cut of  $\mathbb{C}$  at  $\epsilon < 0$  is a two-sphere.

This construction generalizes to torus actions as follows.<sup>1</sup> Let  $\mu : M \rightarrow \mathfrak{t}^*$  be a moment map for an effective action of a torus  $T$  on a symplectic orbifold  $(M, \omega)$  and let  $\ell \subset \mathfrak{t}$  denote the integral lattice. Choose  $N$  vectors  $v_j \in \ell$ ,  $1 \leq j \leq N$ . The form  $\omega - i \sum_{j=1}^N dz_j \wedge d\bar{z}_j$  is a symplectic form on the orbifold  $M \times \mathbb{C}^N$ . The map  $\nu : M \times \mathbb{C}^N \rightarrow \mathbb{R}^N$  with  $j$ th component

$$\nu_j(m, z) = \langle \mu(m), v_j \rangle - |z_j|^2$$

is a moment map for a  $(S^1)^N$  action on  $M \times \mathbb{C}^N$ . For any  $b \in \mathbb{R}^N$ , define a convex rational polyhedral set

$$P = \{x \in \mathfrak{t}^* \mid \langle x, v_i \rangle \geq b_i \text{ for all } 1 \leq i \leq N\}.$$

The **symplectic cut of  $M$  with respect to a rational polyhedral set  $P$**  is the reduction of  $M \times \mathbb{C}^N$  at  $b$ . We denote it by  $M_P$ .

<sup>1</sup>See [W], [M] for the generalization to non-abelian actions.

*Remark 2.3.* If  $b$  is a regular value of  $\nu$ , then  $M_P$  is a symplectic orbifold. Note that regular values are generic. Thus, for an intersection  $P$  of finitely many rational half-spaces of  $\mathfrak{t}^*$  in general position the cut space  $M_P$  is a symplectic orbifold. Note that if  $P$  is a compact polytope, then the fact that  $P$  is generic implies that  $P$  is **simple**: the number of codimension one faces meeting at a given vertex is the same as the dimension of  $P$ .

There is a natural decomposition of  $M_P$  into a union of symplectic suborbifolds, indexed by the open faces  $F$  of  $P$ :

$$M_P = \bigcup_F \mu^{-1}(F)/T_F,$$

where  $T_F \subset T$  is the torus **perpendicular** to  $F$ . That is, the Lie algebra of  $T_F$  is the annihilator of the linear space defined by the face  $F$ . Thus, one can think of  $M_P$  as  $\mu^{-1}(P)$ , with its boundary collapsed by means of the  $T_F$ -actions. Under this identification, the  $T$ -action on  $M$  descends to a  $T$ -action on  $M_P$ . One can show that this action is smooth and Hamiltonian, with moment map  $\mu_P$  induced from the restriction  $\mu|_{\mu^{-1}(P)}$ . In particular,

$$\mu_P(M_P) = \mu(M) \cap P.$$

It follows immediately that the cut space  $M_P$  is compact exactly if  $\mu^{-1}(P)$  is. Similarly  $M_P$  is connected if and only if  $\mu^{-1}(P)$  is connected. To summarize, we have the following result.

**Proposition 2.4.** *Let  $\mu : M \rightarrow \mathfrak{t}^*$  be a moment map for an effective action of a torus  $T$  on a symplectic orbifold  $(M, \omega)$ . Let  $P \subset \mathfrak{t}^*$  be a generic rational polyhedral set. Let  $\mathcal{F}$  be the set of all open faces of  $P$ . Then the topological space  $M_P$  defined by*

$$M_P = \bigcup_{F \in \mathcal{F}} \mu^{-1}(F)/T_F,$$

where  $T_F \subset T$  is the subtorus of  $T$  perpendicular to  $F$ , is a symplectic orbifold with a natural Hamiltonian action of the torus  $T$ . Moreover, the map  $\mu_P : M_P \rightarrow \mathfrak{t}^*$ , induced by the restriction  $\mu|_{\mu^{-1}(P)}$ , is a moment map for this action. Consequently,

- the cut space  $M_P$  is connected if and only if  $\mu^{-1}(P)$  is connected;
- the fibers of  $\mu_P$  are connected if and only if the fibers of  $\mu|_{\mu^{-1}(P)}$  are connected;
- $M_P$  is compact if and only if  $\mu^{-1}(P)$  is compact.

### 3. THE PRINCIPAL CROSS-SECTION

The main result of this section is the following theorem, which is a variation of results of [CDM], [HNP], and [FR]:

**Theorem 3.1.** *Let  $G$  be a compact connected Lie group, and  $M$  a connected Hamiltonian  $G$ -orbifold, with moment map  $\Phi : M \rightarrow \mathfrak{g}^*$ .*

- There exists a unique open face  $\sigma$  of the Weyl chamber  $\mathfrak{t}_+^*$  with the property that  $\Phi(M) \cap \sigma$  is dense in  $\Phi(M) \cap \mathfrak{t}_+^*$ .
- The preimage  $Y = \Phi^{-1}(\sigma)$  is a connected symplectic  $T$ -invariant suborbifold of  $M$ , and the restriction  $\Phi_Y$  of  $\Phi$  to  $Y$  is a moment map for action of the maximal torus  $T$ .
- The set  $G \cdot Y = \{g \cdot m \mid g \in G, m \in Y\}$  is dense in  $M$ .

We will refer to  $\sigma$  as the **principal face** and to  $Y = \Phi^{-1}(\sigma)$  as the **principal cross-section**.

**3.1. Principal orbit type.** Consider a connected orbifold  $M$ , together with an action of a compact connected Lie group  $G$ . For each  $m \in M$ , let  $\mathfrak{g}_m \subset \mathfrak{g}$  be the corresponding isotropy Lie algebra. Clearly,  $\mathfrak{g}_{g \cdot m} = \text{Ad}(g)(\mathfrak{g}_m)$  for all  $g \in G$ . The set of subalgebras

$$(\mathfrak{g}_m) = \{\text{Ad}(g)(\mathfrak{g}_m) \mid g \in G\}$$

is called the (infinitesimal) **orbit type** of  $m$ . As in the case of manifolds, each point  $m \in M$  has a neighborhood  $U$  such that the number of orbit types  $(\mathfrak{g}_{m'})$ , for  $m' \in U$  is finite and each  $\mathfrak{g}_{m'}$  is subconjugate to  $\mathfrak{g}_m$ .

**Proposition 3.2.** *There exists a unique orbit type  $(\mathfrak{h})$  (called the **principal orbit type**) such that the set*

$$M_{(\mathfrak{h})} = \{m \in M \mid (\mathfrak{g}_m) = (\mathfrak{h})\}$$

*of points of orbit type  $(\mathfrak{h})$  is open, dense and connected.*

*Proof.* The proof is an adaptation of the proof for manifold to orbifolds. The key point is that slices (Definition 3.6) exist for actions of compact groups on orbifolds [LT].  $\square$

**Definition 3.3.** For an action of a compact connected Lie group  $G$  on a connected orbifold  $M$ , we define the **principal stratum**  $M_{prin}$  to be the intersection of the set  $M_{(\mathfrak{h})}$  of points of principal orbit type with the set  $M_{smooth}$  of smooth points of  $M$ .

*Remark 3.4.* By definition, the set  $M_{sing}$  of singular points of an orbifold  $M$  is a union of submanifolds of codimension at least 2. Therefore  $M_{smooth}$  is open, dense and connected, and so is  $M_{prin}$ .

*Remark 3.5.* Let  $M$  be a connected Hamiltonian  $G$ -orbifold. As in the case of manifolds, the definition of the moment map, equation (1), implies that the image of the differential of the moment map at a point  $m$  is the annihilator of the corresponding isotropy Lie algebra  $\mathfrak{g}_m$ . Consequently the restriction of the moment map to  $M_{prin}$  has constant rank.

**3.2. The cross-section theorem.** Theorem 3.8 below is a version of the cross section theorem of Guillemin and Sternberg [GS3, Theorem 26.7] adapted to orbifolds. See also [GLS].

**Definition 3.6.** Suppose that a group  $G$  acts on an orbifold  $M$ . Given a point  $m$  in  $M$  with isotropy group  $G_m$ , a suborbifold  $U \subset M$  containing  $m$  is a **slice** at  $m$  if  $U$  is  $G_m$ -invariant,  $G \cdot U$  is a neighborhood of  $m$ , and the map

$$G \times_{G_m} U \rightarrow G \cdot U, \quad [a, u] \mapsto a \cdot u$$

is an isomorphism. In other words,  $G \cdot y \cap U = G_m \cdot y$  and  $G_y \subset G_m$  for all  $y \in U$ .

*Remark 3.7.* Consider the coadjoint action of a compact connected Lie group  $G$  on  $\mathfrak{g}^*$ . For all  $x \in \mathfrak{g}^*$ , there is a unique largest open subset  $U_x \subset \mathfrak{g}_x^* \subset \mathfrak{g}^*$  which is a slice at  $x$ . We refer to  $U_x$  as the **natural slice** at  $x$  for the coadjoint action. In order to describe the natural slice, we may assume without loss of generality that  $x \in \mathfrak{t}_+^*$ . Let  $\tau \subset \mathfrak{t}_+^*$  be the open face of  $\mathfrak{t}_+^*$  containing  $x$  and let  $G_\tau$  denote the isotropy Lie group of  $x$  (the group is the same for all points of  $\tau$ ). Then

$$U_\tau = G_\tau \cdot \{y \in \mathfrak{t}_+^* \mid G_y \subset G_\tau\} = G_\tau \cdot \bigcup_{\tau \subset \tau'} \tau'$$

is an open subset of  $\mathfrak{g}_x^*$ , and is equal to the natural slice  $U_x$ .

**Theorem 3.8** (Cross-section). *Let  $(M, \omega)$  be a symplectic orbifold with a moment map  $\Phi : M \rightarrow \mathfrak{g}^*$  arising from an action of a compact Lie group  $G$ . Let  $x$  be a point in  $\mathfrak{g}^*$  and let  $U$  be the natural slice at  $x$  (see above). Then the **cross-section**  $R := \Phi^{-1}(U)$  is a  $G_x$ -invariant symplectic suborbifold of  $M$ , where  $G_x$  is the isotropy group of  $x$ . Furthermore the restriction  $\Phi|_R$  is a moment map for the action of  $G_x$  on  $R$ .*

*Proof.* By definition of the slice, coadjoint orbits intersect  $U$  transversally. Since the moment map is equivariant, it is transversal to  $U$  as well. Hence the cross-section  $R = \Phi^{-1}(U)$  is a suborbifold. Since the slice  $U$  is preserved by the action of  $G_x$  and the moment map is equivariant, the cross-section is preserved by  $G_x$ . It remains to show that for all  $r \in R$ , the uniformized tangent space  $\tilde{T}_r R$  is a symplectic subspace of  $\tilde{T}_r M$ . Let  $y = \Phi(r)$ , and let  $\mathfrak{m}$  be the  $G_x$ -invariant complement of  $\mathfrak{g}_x$  in  $\mathfrak{g}$ . Then  $T_y U$  is the annihilator of  $\mathfrak{m}$  for any  $y \in U$ . We claim that

- (a) the tangent space  $\tilde{T}_r R$  is symplectically perpendicular to  $\mathfrak{m}_M(r) = \{\xi_M(r) : \xi \in \mathfrak{m}\}$ , the subspace of the tangent space to the orbit through  $r$  spanned by  $\mathfrak{m}$ ;
- (b)  $\mathfrak{m}_M(r)$  is a symplectic subspace of  $\tilde{T}_r M$ .

Together the two claims establish the theorem because  $\tilde{T}_r M = \tilde{T}_r R \oplus \mathfrak{m}_M(r)$ . To see that (a) holds observe that for  $v \in \tilde{T}_r R$  and  $\xi \in \mathfrak{m}$ ,

$$\omega_r(\xi_M(r), v) = -\langle \xi, d\Phi_r(v) \rangle = 0$$

since  $d\Phi_r(v) \in T_y U = \mathfrak{m}^\circ$  and  $\xi \in \mathfrak{m}$ .

To see that (b) is true, observe that for  $\xi, \eta \in \mathfrak{g}$ ,

$$\omega_r(\xi_M(r), \eta_M(r)) = -\langle \xi, d\Phi_r(\eta_M(r)) \rangle = \langle \xi, ad^\dagger(\eta) \cdot \Phi(r) \rangle = -\langle [\xi, \eta], y \rangle.$$

Thus  $\mathfrak{m}_M(r)$  is symplectic if and only if  $ad^\dagger(\mathfrak{m})y$  is a symplectic subspace of the tangent space  $T_y(G \cdot y)$ . Since  $G_x \cdot y \subset U$  and since  $\mathfrak{m} = (T_y U)^\circ$ , for any  $\xi \in \mathfrak{m}$  and any  $\zeta \in \mathfrak{g}_x$  we have

$$\langle [\xi, \zeta], y \rangle = \langle \xi, ad^\dagger(\zeta)y \rangle = 0,$$

i.e.,  $T_y(G_x \cdot y)$  and  $ad^\dagger(\mathfrak{m})y$  are symplectically perpendicular in  $T_y(G \cdot y)$ . Since  $T_y(G \cdot y) = T_y(G_x \cdot y) \oplus ad^\dagger(\mathfrak{m})y$ , it remains to show that the orbit  $G_x \cdot y$  is a symplectic submanifold of the coadjoint orbit  $G \cdot y$ . Since the natural projection  $\pi : \mathfrak{g}^* \rightarrow \mathfrak{g}_x^*$  is  $G_x$ -equivariant,  $\pi(G_x \cdot y) = G_x \cdot \pi(y)$ . By the definition of the symplectic forms on a coadjoint orbit the restriction of the symplectic form on  $G \cdot y$  to  $G_x \cdot y$  is the pull-back by  $\pi$  of the symplectic form on the  $G_x$  coadjoint orbit  $G_x \cdot \pi(y)$ .  $\square$

*Remarks 3.9.* a. For the theorem to hold for a non-compact Lie group  $G$  one has to assume that a slice  $U$  exists at  $x$  and that the differential of the restriction to  $U$  of the projection  $\mathfrak{g}^* \rightarrow \mathfrak{g}_x^*$  is surjective.

b. The flow-out  $G \cdot R$  is equivariantly diffeomorphic to the associated bundle  $G \times_{G_x} R$  over the coadjoint orbit  $G \cdot x$ .

c. Let  $G$  be a compact connected group, and  $M$  a Hamiltonian  $G$ -orbifold, with moment map  $\Phi : M \rightarrow \mathfrak{g}^*$ . In various applications in this paper, we will use symplectic cross-sections to reduce statements about  $G$ -orbits in  $M$  to the case that the orbit is contained in the zero level set  $\Phi^{-1}(0)$ . The general argument is as follows. Let  $m \in \Phi^{-1}(\mathfrak{t}_+^*)$ , and let  $\tau \subset \mathfrak{t}_+^*$  be the open face containing  $x = \Phi(m)$ . Let  $U_\tau \subset \mathfrak{g}^*$  be the natural slice and  $R_\tau$  the corresponding natural cross-section, which is a Hamiltonian  $G_\tau$ -space. Since  $G_\tau$  contains the maximal torus, there is a unique  $G_\tau$ -invariant decomposition

$$\mathfrak{g} = \mathfrak{z}(\mathfrak{g}_\tau) \oplus [\mathfrak{g}_\tau, \mathfrak{g}_\tau] \oplus \mathfrak{m}_\tau,$$

where  $\mathfrak{z}(\mathfrak{g}_\tau)$  is the center of  $\mathfrak{g}_\tau$ ,  $[\mathfrak{g}_\tau, \mathfrak{g}_\tau]$  its semi-simple part and  $\mathfrak{m}_\tau$  a complement in  $\mathfrak{g}$ . Notice that  $\mathfrak{z}(\mathfrak{g}_\tau)$  can be characterized as the fixed point set of the  $G_\tau$ -action on  $\mathfrak{g}$ . It follows that the linear span of  $\tau$  is equal to  $\mathfrak{z}(\mathfrak{g}_\tau)^*$ . Since  $x = (\Phi|_{R_\tau})(m) \in \tau \subset \mathfrak{z}(\mathfrak{g}_\tau)^*$ , one can shift the moment map  $\Phi|_{R_\tau}$  by  $x$  to obtain a new moment map  $\Phi'_\tau$  for the  $G_\tau$ -action on  $R_\tau$  for which  $m \in (\Phi'_\tau)^{-1}(0)$ .

**3.3. The principal face and cross-section.** Lemma 3.10 below is used to show the existence of the principal face.

**Lemma 3.10.** *Let  $G$  be a compact, connected Lie group, and  $M$  a connected Hamiltonian  $G$ -orbifold, with moment map  $\Phi : M \rightarrow \mathfrak{g}^*$ .*

- a. For all  $m \in M_{prin}$ , the isotropy Lie algebra  $\mathfrak{g}_m$  is an ideal in  $\mathfrak{g}_{\Phi(m)}$ , i.e.,  $[\mathfrak{g}_m, \mathfrak{g}_{\Phi(m)}] \subset \mathfrak{g}_m$ .
- b. All points in the intersection  $\Phi^{-1}(\mathfrak{t}_+^*) \cap M_{prin}$  have the same isotropy Lie algebra  $\mathfrak{h} \subset \mathfrak{g}$ .
- c. Given  $\alpha \in \Phi(M_{prin}) \cap \mathfrak{t}_+^*$  let  $S$  be the affine subspace  $(\alpha + \mathfrak{h}^\circ) \cap \mathfrak{t}^*$ , where  $\mathfrak{h}^\circ$  is the annihilator of  $\mathfrak{h}$  in  $\mathfrak{g}^*$ . The intersection  $\Phi(M_{prin}) \cap \mathfrak{t}_+^*$  is a connected, relatively open subset of  $S \cap \mathfrak{t}_+^*$ .

*Proof.* (1) Consider first the case when  $\Phi(M_{prin})$  intersects  $\mathfrak{z}(\mathfrak{g})^*$ , the fixed point set for the coadjoint action of  $G$ . Let  $m$  be a point in  $\Phi^{-1}(\mathfrak{z}(\mathfrak{g})^*)$  and let  $\mathfrak{h}$  be its isotropy Lie algebra. Since the restriction  $\Phi|_{M_{prin}}$  has constant rank by Remark 3.5, there exists a  $G$ -invariant open neighborhood  $U \subset M_{prin}$  of  $m$ , such that  $\Phi(U) \subset \mathfrak{g}^*$  is a submanifold, and  $T_{\Phi(m)}\Phi(U) = d\Phi_m(T_m M) = \mathfrak{h}^\circ$ . Since  $\Phi(m)$  is fixed by the coadjoint action, the tangent space  $T_{\Phi(m)}\Phi(U)$  is  $G$ -invariant, which proves that  $\mathfrak{h}$  is  $G$ -invariant. Therefore, the isotropy Lie algebras of all points in the principal stratum  $M_{prin}$  are the same. This proves parts (b) and (c) of the lemma in the special case, and also part 1 for the case  $\Phi(m) = 0$ .

(2) We now reduce the general case to (1), using symplectic cross-sections. Given  $m \in \Phi^{-1}(\mathfrak{t}_+^*) \cap M_{prin}$ , let  $\tau$  be the open face containing  $\Phi(m)$ , and  $R_\tau$  be the corresponding natural cross-section. Denote by  $N$  the connected component of  $R_\tau$  containing  $m$ . Then  $N$  is a Hamiltonian  $G_\tau$ -space, with moment map the restriction  $\Phi|_N$ , and  $m$  lies in  $N_{prin} = N \cap M_{prin}$ . Since  $\Phi(m) \in \tau \subset \mathfrak{z}(\mathfrak{g}_\tau)^*$ , part A shows that  $\mathfrak{g}_m$  is an ideal in  $\mathfrak{g}_{\Phi(m)}$ , and that every point in  $N_{prin}$  has the same isotropy algebra. In particular, the isotropy algebra of points in  $\Phi^{-1}(\mathfrak{t}_+^*) \cap M_{prin}$  is locally constant. We claim that  $\Phi^{-1}(\mathfrak{t}_+^*) \cap M_{prin}$  is connected. Indeed, consider the natural surjective map

$$\pi : \Phi^{-1}(\mathfrak{t}_+^*) \cap M_{prin} \rightarrow M_{prin}/G, m \mapsto G \cdot m.$$

The fiber of  $\pi$  over every point  $G \cdot m \in M_{prin}/G$  is equal to the intersection  $G \cdot m \cap \Phi^{-1}(\mathfrak{t}_+^*) \cong G_{\Phi(m)} \cdot m$ , which is connected since  $G_{\Phi(m)}$  is connected. Since  $M_{prin}$  is connected, it follows that  $M_{prin}/G$  is connected. Thus  $\pi$  has connected target space and connected fibers, and it follows that  $\Phi^{-1}(\mathfrak{t}_+^*) \cap M_{prin}$  is connected. This rest of the Lemma is an easy consequence of the fact that the image of  $d\Phi_m$  is the annihilator of  $\mathfrak{g}_m$ .  $\square$

*Proof.* (Theorem 3.1) Let  $S \subset \mathfrak{t}^*$  be the affine subspace described in Lemma 3.10, and  $\sigma \subset \mathfrak{t}_+^*$  the lowest dimensional face such that  $S \cap \mathfrak{t}_+^* = S \cap \bar{\sigma}$ . Since the moment map is continuous, the moment set  $\Phi(M) \cap \mathfrak{t}_+^*$  is contained in the closure of  $\Phi(M_{prin}) \cap \mathfrak{t}_+^*$ . By Lemma 3.10,  $\Phi(M_{prin}) \cap \mathfrak{t}_+^*$  is an open subset of  $S \cap \mathfrak{t}_+^*$ . It follows that  $\Phi(M_{prin}) \cap \sigma$  is non-empty, and that the closure of  $\sigma$  contains the moment set  $\Phi(M) \cap \mathfrak{t}_+^*$ . It is the smallest face with this property: for any face  $\tau$  with  $\Phi(M) \cap \mathfrak{t}_+^* \subset \bar{\tau}$  we have  $\sigma \subset \tau$ . Consequently  $\Phi(M) \cap \sigma = \Phi(M) \cap U_\sigma$  where  $U_\sigma$  is the natural slice (Remark 3.7). Therefore, by the cross-section theorem,

$$Y := \Phi^{-1}(\sigma) = \Phi^{-1}(U_\sigma)$$

is a symplectic  $G_\sigma$ -invariant suborbifold of  $M$  and  $\Phi_Y := \Phi|_Y : Y \rightarrow \sigma \subset U_\sigma$  is a moment map for the action of  $G_\sigma$ . Let  $\mathfrak{h}$  be the stabilizer algebra of points in  $M_{prin} \cap \Phi^{-1}(\mathfrak{t}_+^*)$ . The linear space spanned by  $\sigma$  can be identified with the fixed point set  $\mathfrak{z}(\mathfrak{g}_\sigma)^*$  for the action of  $G_\sigma$  on  $\mathfrak{g}_\sigma^*$ , that is, with the annihilator of the semi-simple part,  $[\mathfrak{g}_\sigma, \mathfrak{g}_\sigma]^\circ \cap \mathfrak{g}_\sigma^*$ . By construction of  $\sigma$ , we have  $\mathfrak{h}^\circ \cap \mathfrak{g}_\sigma^* \subset [\mathfrak{g}_\sigma, \mathfrak{g}_\sigma]^\circ \cap \mathfrak{g}_\sigma^*$ , i.e.  $[\mathfrak{g}_\sigma, \mathfrak{g}_\sigma] \subset \mathfrak{h}$ . This shows that the action on  $Y$  of the semi-simple part of  $G_\sigma$  is trivial, and that the moment map  $\Phi_Y$  is a moment map for the action of the identity component of the center of  $G_\sigma$ . That is, the principal cross-section  $Y$  is a Hamiltonian *torus* orbifold.

We have proved all the assertions of the theorem, except for the fact that  $G \cdot Y$  is dense in  $M$  and that  $Y$  is connected. The complement to  $G \cdot Y \cap M_{prin} = \Phi^{-1}(G \cdot \sigma) \cap M_{prin}$  in  $M_{prin}$  is equal to the union of  $\Phi^{-1}(G \cdot \tau) \cap M_{prin}$  over all  $\tau$  such that  $\tau \subset \bar{\sigma}$  and  $\tau \neq \sigma$ . By Lemma 3.11 below, these are all submanifolds of codimension at least three. It follows that removing these sets from  $M_{prin}$  leaves it dense and connected. Hence  $G \cdot Y$  is connected and dense in  $M$ . Since the quotient map  $\mathfrak{g}^* \rightarrow \mathfrak{t}_+^*$  defined by  $x \mapsto G \cdot x \cap \mathfrak{t}_+^*$  is continuous,  $\Phi(Y)$  is dense in  $\Phi(M) \cap \mathfrak{t}_+^*$ . Since  $G \cdot Y = G \times_{G_\sigma} Y$  (Remark 3.9.b) and since the coadjoint orbit  $G/G_\sigma$  is simply connected, it follows that  $Y$  is connected.  $\square$

In the proof we used the following Lemma.

**Lemma 3.11.** *Let  $G$  be a compact, connected Lie group, and  $M$  a connected Hamiltonian  $G$ -orbifold, with moment map  $\Phi : M \rightarrow \mathfrak{g}^*$ . Let  $\tau$  be a face of  $\mathfrak{t}_+^*$  which is different from the principal face  $\sigma$ . Then the intersection  $\Phi^{-1}(G \cdot \tau) \cap M_{prin}$  is either empty or is a submanifold of codimension at least 3 in  $M_{prin}$ .*

*Proof.* Let  $R_\tau = \Phi^{-1}(U_\tau)$  be the natural cross-section corresponding to  $\tau$ . Let  $\mathfrak{z}(\mathfrak{g}_\tau)$  be the Lie algebra of the center of  $G_\tau$ . Let  $N$  be a connected component of  $R_\tau \cap M_{prin}$  such that  $\Phi(N) \cap \tau \neq \emptyset$ . Let  $m \in N$ , and  $\mathfrak{h} = \mathfrak{g}_m$ . By Lemma 3.10,  $\Phi|_N$  is a submersion onto an open subset of the affine space  $\mathfrak{g}_\tau^* \cap (\Phi(m) + \mathfrak{h}^\circ)$ , which by assumption is not contained in  $\mathfrak{z}(\mathfrak{g}_\tau)^*$ . The fixed point set for the  $G_\tau$ -action on  $\mathfrak{g}_\tau^* \cap (\Phi(m) + \mathfrak{h}^\circ)$  is the proper affine subspace  $\mathfrak{z}(\mathfrak{g}_\tau)^* \cap (\Phi(m) + \mathfrak{h}^\circ)$ . Since  $G_\tau$  is non-abelian, this subspace has codimension at least three. Notice that  $\mathfrak{z}(\mathfrak{g}_\tau)^*$  can be identified with the linear space spanned by  $\tau$ . Since  $\Phi|_N$  has constant rank, it follows that  $\Phi^{-1}(\tau) \cap M_{prin}$  has codimension greater or equal than 3 in  $R_\tau$ . By Remark 3.9.b this implies that

$$\Phi^{-1}(G \cdot \tau) \cap M_{prin} = G \times_{G_\tau} (\Phi^{-1}(\tau) \cap M_{prin}) \subset G \times_{G_\tau} R_\tau$$

has codimension at least 3 in  $M_{prin}$ . □

*Remark 3.12.* In the proof of Theorem 3.1, we have shown that the semi-simple part  $[\mathfrak{g}_\sigma, \mathfrak{g}_\sigma]$  of  $\mathfrak{g}_\sigma$ , where  $\sigma$  is the principal face, is contained in the principal stabilizer algebra  $\mathfrak{h}$  for points in  $M_{prin} \cap \Phi^{-1}(\mathfrak{t}_+^*)$ . We also have  $\mathfrak{h} \subset \mathfrak{g}_\sigma$ , by equivariance of the moment map. Therefore, the commutators  $[\mathfrak{h}, \mathfrak{h}]$  and  $[\mathfrak{g}_\sigma, \mathfrak{g}_\sigma]$  are equal. It follows that the principal stabilizer algebra uniquely determines the principal face.

#### 4. HAMILTONIAN TORUS ACTIONS ON NON-COMPACT ORBIFOLDS

The following theorem is due in the manifold case to Atiyah [A] and to Guillemin and Sternberg [GS2]. The proofs in the orbifold case are in [LT]. Let  $T$  be a torus,  $\mathfrak{t}$  its Lie algebra,  $\mathfrak{t}^*$  the dual of  $\mathfrak{t}$  and  $\ell = \ker\{\exp : \mathfrak{t} \rightarrow T\}$  the **integral lattice**. A polytope in  $\mathfrak{t}^*$  is **rational** if the faces of the polytope are cut out by hyperplanes whose normal vectors are in the lattice  $\ell$ .

**Theorem 4.1.** *Let  $(M, \omega)$  be a compact, connected symplectic orbifold, and let  $\Phi : M \rightarrow \mathfrak{t}^*$  be a moment map for a Hamiltonian torus action on  $M$ .*

- a. *The image  $\Phi(M)$  is a rational convex polytope, and*
- b. *each fiber of  $\Phi$  is connected.*

As we have seen, for non-abelian actions the principal cross-section need not be compact even if the original manifold is compact. Therefore to reduce non-abelian convexity to the abelian case we need to generalize Theorem 4.1 to include a class of torus actions on non-compact orbifolds with not necessarily proper moment maps. Instead, we require that the moment map  $\Phi : M \rightarrow \mathfrak{t}^*$  is proper as a map into a convex open set  $U$  of  $\mathfrak{t}^*$ , i.e., that  $\Phi(M) \subset U$  and that for every compact  $K \subset U$  the preimage  $\Phi^{-1}(K)$  is compact. This criterion is motivated by the following fact: if a Lie group acts on a symplectic orbifold with a proper moment map, then the induced moment map on the principal cross section is proper as a map into the principal face. We extend Theorem 4.1 to this case by using symplectic cuts to “compactify”  $M$ .

First we make an elementary observation.

**Lemma 4.2.** *Let  $(M, \omega)$  be a compact, connected symplectic orbifold, and let  $\Phi : M \rightarrow \mathfrak{t}^*$  be a moment map for a Hamiltonian torus action on  $M$ . Assume that  $\Phi$  is proper as a map into a convex open set  $U \subset \mathfrak{t}^*$ . Then for any compact set  $K \subset U$  there exists a generic rational polytope  $P \subset U$ , such that  $K$  is contained in the interior of  $P$ .*



*Proof.* For any point  $x \in U$  there exists a polytope  $P_x \subset U$  with rational vertices which contains  $x$  in the interior. The collection  $\{\text{int}(P_x) : x \in K\}$  is a cover of  $K$ . Since  $K$  is compact there exists a finite subcover  $\text{int}(P_{x_1}), \dots, \text{int}(P_{x_s})$ . Take the convex hull  $P$  of the union  $P_{x_1} \cup \dots \cup P_{x_s}$ . If  $P$  is generic, it is the desired polytope. If it is not, perturb it to be generic.  $\square$

**Theorem 4.3.** *Let  $(M, \omega)$  be a connected symplectic orbifold, and let  $\Phi : M \rightarrow \mathfrak{t}^*$  be a moment map for a Hamiltonian torus action on  $M$ . If  $\Phi$  is proper as a map into a convex open set  $U \subset \mathfrak{t}^*$ , then*

- a. *the image  $\Phi(M)$  is convex,*
- b. *each fiber of  $\Phi$  is connected, and*
- c. *if for every compact subset  $K$  of  $\mathfrak{t}^*$ , the list of isotropy algebras for the  $T$ -action on  $\Phi^{-1}(K)$  is finite, then the image  $\Phi(M)$  is the intersection of  $U$  with a rational locally polyhedral set.*

*Proof.* a. Consider  $m_0, m_1 \in M$ . Since  $M$  is a connected orbifold, there exists a path  $\gamma : [0, 1] \rightarrow M$  such that  $\gamma(0) = m_0$  and  $\gamma(1) = m_1$ . Since  $\gamma([0, 1])$  is compact, by Lemma 4.2 there exists a generic rational polytope  $P \subset U$  such that  $\Phi(\gamma(t))$  is in the interior of  $P$  for all  $t \in [0, 1]$ . The points  $m_0$  and  $m_1$  are contained in the same connected component  $N$  of the cut space  $M_P$ . Let  $\Phi_N : N \rightarrow \mathfrak{t}^*$  be the induced moment map. Since  $\Phi_N(N)$  is a convex polytope by Theorem 4.1, it contains the line segment joining  $\Phi(m_0)$  to  $\Phi(m_1)$ . Since  $\Phi_N(N) \subset \Phi(M)$ , this proves that  $\Phi(M)$  is convex.

b. A similar argument shows that if  $\Phi(m_0) = \Phi(m_1) = x$ , then  $m_0$  and  $m_1$  are contained in the same connected component of  $\Phi^{-1}(x)$ , since the fibers of  $\Phi_N$  are connected by Theorem 4.1 and since for points  $x$  in the interior of  $P$  the fibers of  $\Phi$  and of  $\Phi_P : M_P \rightarrow \mathfrak{t}^*$  are the same (cf. Proposition 2.4). Therefore the fibers of  $\Phi$  are connected.

c. For each  $x \in \Phi(M)$ , let

$$C_x = \{x + t(y - x) \mid y \in \Phi(M) \text{ and } t \geq 0\}$$

be the cone over  $\Phi(M)$  with vertex  $x$ . Define  $A$  to be the intersection of all the cones  $C_x$  for  $x \in \Phi(M)$ ,  $A := \bigcap_{x \in \Phi(M)} C_x$ . Since  $\Phi(M)$  is convex and is relatively closed in  $U$ , Lemma 4.4 below (applied to the closure  $X$  of  $\Phi(M)$  and to  $S = U$ ) shows that  $\Phi(M) = A \cap U$ .

We claim that  $A$  is the desired rational locally polyhedral set. First we show that all cones  $C_x$  are rational polyhedral cones. Let  $x \in \Phi(M)$ , and let  $P \subset U$  be a generic polytope containing  $x$  in its interior. Since  $\Phi(M)$  is convex,  $C_x$  is also the cone over  $\Phi_P(M_P)$  with vertex  $x$ . In particular,  $C_x$  is a rational polyhedral cone, and the tangent space of each facet is of the form  $\mathfrak{t}_i^\circ$ , where  $\mathfrak{t}_i$  is an isotropy Lie algebra for the action of the torus  $T$  in a neighborhood of  $\Phi^{-1}(x)$ .

Suppose next that  $K \subset \mathfrak{t}^*$  is a compact convex subset with non-empty interior. Since the number of isotropy algebras for the  $T$ -action on  $\Phi^{-1}(K)$  is finite, it follows that up to translation, the list of cones  $C_x$  for  $x \in K \cap \Phi(M)$  is finite. Moreover, if  $C_x$  is a translation of  $C_y$ , then  $C_x = C_y$  since by convexity of  $\Phi(M)$  the line segment  $\overline{xy}$  is contained in both  $C_x$  and  $C_y$ . It follows that the collection of cones  $C_x$ ,  $x \in K \cap \Phi(M)$  itself is a finite list  $C_1, \dots, C_N$ . This shows that  $A$  is a rational locally polyhedral set.  $\square$

In the proof we used the following lemma.

**Lemma 4.4.** *Let  $V$  be a vector space, and  $X, S \subset V$  convex subsets with  $X$  closed. For every  $x \in X$ , let  $C_x = \{x + t(y - x) \mid y \in X, t \geq 0\}$ . Then*

$$X \cap S = \left( \bigcap_{x \in X \cap S} C_x \right) \cap S.$$

*Proof.* The inclusion “ $\subset$ ” is obvious. To prove the opposite inclusion, assume that  $(X \cap S)$  and  $S - (X \cap S)$  are non-empty (since otherwise there is nothing to prove). Let  $y \in S - (X \cap S)$ . We have to show that

$y \notin \bigcap_{x \in X \cap S} C_x$ . Let  $r$  be a ray with vertex  $y$  which intersects  $X \cap S$  nontrivially. Since  $X$  is closed and convex  $r \cap X$  is either a closed ray or a closed line segment. Let  $x$  be the point in  $r \cap X$  closest to  $y$ . Then  $y$  does not lie in  $C_x$ . Since  $S$  is convex,  $x$  is in  $X \cap S$ .  $\square$

In the last section, we will use the following corollary to Theorem 4.3.

**Corollary 4.5.** *Let  $(M, \omega)$  be a connected symplectic orbifold, and let  $\Phi : M \rightarrow \mathfrak{t}^*$  be the moment map for a Hamiltonian torus action on  $M$ . If  $\Phi$  is proper as a map into a convex open set  $U$ , then for every  $\xi \in \mathfrak{t}$ , every local minimum of the function  $\Phi^\xi$  is a global minimum, where  $\Phi^\xi(m) := \langle \Phi(m), \xi \rangle$ .*

*Proof.* Since the moment set  $\Phi(M)$  is convex, its intersection with the affine hyperplanes  $\{x \in \mathfrak{t}^* \mid \xi(x) = a\}$  is connected for all  $a \in \mathbb{R}$ . Since the fibers of  $\Phi$  are connected this implies that

$$(\Phi^\xi)^{-1}(a) = \Phi^{-1}(\{x \in \mathfrak{t}^* \mid \xi(x) = a\})$$

is connected for all  $a$ . The result follows.  $\square$

## 5. PROOF OF NON-ABELIAN CONVEXITY

We now prove the non-abelian convexity and connectedness theorem 1.1. Let  $\sigma$  be the principal face. By Theorem 3.1, the principal cross section  $Y := \Phi^{-1}(\sigma)$  is a connected symplectic orbifold, and the restriction of  $\Phi$  to  $Y$  is a moment map for the action of the maximal torus of  $Y$ . Since  $\Phi : M \rightarrow \mathfrak{g}^*$  is proper, the restriction  $\Phi|_Y : Y \rightarrow \mathfrak{t}^*$  is proper as a map into the open convex set  $\sigma$ . Therefore, by Theorem 4.3, the image  $\Phi(Y)$  is a convex set and is the intersection of  $\sigma$  with a locally polyhedral set  $P$ , that is,  $\Phi(Y) = \sigma \cap P$ . By Theorem 3.1,  $\Phi(M) \cap \mathfrak{t}_+^* = \overline{\Phi(Y)}$ . Since the closure of a convex set is convex, the moment set  $\Phi(M) \cap \mathfrak{t}_+^*$  is convex. Since the closure of  $\sigma$  is a polyhedral cone and since the intersection of the interior of the locally polyhedral set  $P$  with  $\sigma$  is nonempty, the closure of the intersection  $\sigma \cap P$  is the intersection  $\bar{\sigma} \cap P$ . Therefore

$$\Phi(M) \cap \mathfrak{t}_+^* = \overline{\sigma \cap P} = \bar{\sigma} \cap P$$

is a locally polyhedral set. Since both  $P$  and  $\bar{\sigma}$  are rational, the moment set is rational.

It remains to prove that the fiber  $\Phi^{-1}(x)$  is connected for all  $x \in \mathfrak{g}^*$ . Since the fibers of  $\Phi_Y = \Phi|_Y$  are connected, we know that the fibers of the restriction  $\Phi|_{G \cdot Y}$  are connected. Since  $G \cdot Y$  is dense in  $M$ , we would like to conclude that *all* fibers of  $\Phi$  are connected. However, this does not immediately follow. (Consider for example the map  $f$  from  $S^2 \subset \mathbb{R}^3$  to  $S^1 \subset \mathbb{C}$  given by  $f(x_1, x_2, x_3) = e^{i\pi x_3}$ . The map  $f$  is proper and all fibers are connected, except for the fiber over  $z = -1$ .) We need the following topological fact:

**Lemma 5.1.** *Let  $X$  be a normal topological space and  $\{U_i\}_{i=1}^\infty$  a sequence of compact, connected subsets with  $U_{i+1} \subset U_i$ . Then their intersection  $C = \bigcap U_i$  is connected.*

*Proof.* Suppose that  $C = C_1 \cup C_2$  where  $C_1$  and  $C_2$  are closed and disjoint. We have to show that  $C_1$  or  $C_2$  is empty. Since  $X$  is normal, there exist disjoint open neighborhoods  $V_1 \supset C_1$  and  $V_2 \supset C_2$ . The intersection of the sets  $U'_i := U_i - (V_1 \cup V_2)$  is empty. Since the  $U'_i$  are compact and  $U'_{i+1} \subset U'_i$ , this implies that for  $k$  sufficiently large, the set  $U'_k$  is empty. Since  $U_k$  is connected, this means that  $U_k \cap V_1 = \emptyset$  or  $U_k \cap V_2 = \emptyset$ . Consequently, one of the sets  $C \cap V_1 = C_1$ ,  $C \cap V_2 = C_2$  is empty.  $\square$

The proof of the connectivity of the fibers  $\Phi^{-1}(x)$  for  $x \in \mathfrak{g}^*$  is now as follows. We may assume  $x \in \mathfrak{t}_+^*$ . Observe that  $\Phi^{-1}(x)$  is connected if and only if  $\Phi^{-1}(G \cdot x)$  is connected. Indeed, the reverse implication is true since  $\Phi^{-1}(G \cdot x) = G \cdot \Phi^{-1}(x)$  and the group  $G$  is connected. To see the forward implication it suffices to note that  $\Phi^{-1}(G \cdot x)/G = \Phi^{-1}(x)/G_x$  and  $G_x$  is connected.

To show that  $\Phi^{-1}(G \cdot x)$  is connected, it suffices by Lemma 5.1 to show that for any convex open neighborhood  $B$  of  $x$  in  $\mathfrak{t}^*$ , the closure of the open set  $\Phi^{-1}(G \cdot (B \cap \mathfrak{t}_+^*))$  is connected. By 3.1, the intersection

$$\Phi^{-1}(G \cdot (B \cap \mathfrak{t}_+^*)) \cap G \cdot Y = G \cdot \Phi^{-1}(B \cap \sigma)$$

is dense in  $\Phi^{-1}(G \cdot (B \cap \mathfrak{t}_+^*))$  and therefore also dense in its closure. Since  $B \cap \sigma \cap \Phi(M)$  is convex, hence connected and  $\Phi^{-1}(y)$  is connected for any  $y \in \sigma$ , the set  $G \cdot \Phi^{-1}(B \cap \sigma)$  is connected and therefore its closure is connected.

*Remark 5.2.* The assumptions in Theorem 1.1 may be relaxed as in the abelian case: instead of properness it suffices that there exists an invariant open set  $V$  of  $\mathfrak{g}^*$  with  $\Phi(M) \subset V$  such that  $\Phi : M \rightarrow V$  is proper and  $V \cap \mathfrak{t}_+^*$  is convex.

## 6. LOCAL MOMENT CONES

In [Sj] Sjamaar gives a prescription for reading off the local form of the moment set. We reprove one of his results, Theorem 1.2 (see also Karshon [Ka]) in the orbifold setting using symplectic cutting. As in [Sj], an important step is the following result:

**Theorem 6.1.** *Let  $\Phi : M \rightarrow \mathfrak{g}^*$  be a moment map for an action of a compact connected Lie group  $G$  on a symplectic orbifold  $(M, \omega)$ . For every point  $m \in \mathfrak{t}_+^*$ , there exists a rational polyhedral cone  $C_m \subset \mathfrak{t}^*$  with vertex at  $\Phi(m)$ , such that for every sufficiently small  $G$ -invariant neighborhood  $U$  of  $m$ , the image  $\Phi(U) \cap \mathfrak{t}_+^*$  is a neighborhood of the vertex of  $C_m$ . The cone  $C_m$  is called the **local moment cone** for  $m$ .*

*Proof.* We will use the orbifold version of the local normal form theorem due to Marle [Ma] and, independently, to Guillemin and Sternberg [GS1]. The theorem asserts that an invariant neighborhood of a point  $m$  in a Hamiltonian  $G$ -manifold  $M$  is completely determined (up to equivariant symplectomorphism) by two pieces of data: (1) the value of the moment map  $\Phi$  at  $m$  and (2) the symplectic slice representation of the isotropy group  $G_m$  of the point. Recall that the **symplectic slice** at a point  $m$  is the largest symplectic subspace in the fiber at  $m$  of the normal bundle to the orbit  $G \cdot m$ . The analogous result holds for Hamiltonian  $G$ -orbifolds (cf. [LT]) — the only difference is that the symplectic slice is no longer a vector space. Instead it is the quotient of a vector space by a linear action of a finite group.

**Lemma 6.2.** *Let  $G$  be a compact Lie group, let  $(M, \omega)$  be a Hamiltonian  $G$ -orbifold, with moment map  $\Phi$ , and  $m \in \Phi^{-1}(0)$  a point in the zero level set. Let  $\Gamma$  be the orbifold structure group of  $m$  and  $G_m \times V/\Gamma \rightarrow V/\Gamma$  be the symplectic slice representation at  $m$ . For every  $G_m$ -equivariant splitting  $\mathfrak{g}^* \cong \mathfrak{g}_m^\circ \oplus \mathfrak{g}_m^*$ , there is a  $G$ -invariant symplectic form on the orbifold  $F = G \times_{G_m} (\mathfrak{g}_m^\circ \times V/\Gamma)$  such that the moment map  $\Phi_F : F \rightarrow \mathfrak{g}^*$  for the  $G$ -action is given by*

$$\Phi_F([g, \eta, v]) = g \cdot (\eta + \phi_{V/\Gamma}(v)),$$

where  $\phi_{V/\Gamma} : V/\Gamma \rightarrow \mathfrak{g}_m^*$  is the moment map for the slice representation of  $G_m$ . Moreover, the embedding of the orbit  $G \cdot m$  into  $F$  as the zero section is isotropic and the symplectic slice at  $[1, 0, 0]$  is  $V/\Gamma$ . Consequently there exists a  $G$ -equivariant symplectomorphism  $\lambda$  from a neighborhood  $U$  of  $G \cdot m$  in  $M$  to a neighborhood  $U'$  of the zero section in  $F$ , and  $\lambda^* \Phi_F = \Phi$  over  $U$ .

We are now ready to prove Theorem 6.1.

1. We begin by considering the special case  $\Phi(m) = 0$ . Because we are considering arbitrarily small neighborhoods, by Lemma 6.2 it suffices to consider neighborhoods  $U$  of the zero section of the model space  $F$ . Let  $V/\Gamma$  be the symplectic slice at  $m$ . Choose a  $G_m$ -invariant complex structure on  $V/\Gamma$  which is compatible with the symplectic form; let  $\rho$  denote the norm squared of the induced metric. The model  $F$  is a complex orbi-bundle over  $G \times_{G_m} \mathfrak{g}_m^\circ$ . The multiplication action of  $U(1)$  on the fibers of  $F \rightarrow G \times_{G_m} \mathfrak{g}_m^\circ$  is a Hamiltonian action on the orbifold  $F$ , with moment map being the function

$\rho$  defined above. The action of  $U(1)$  commutes with the action of  $G$ . Consequently the moment map  $\hat{\Phi}_F : F \rightarrow \mathbb{R} \times \mathfrak{g}^*$  for the action of  $U(1) \times G$  on  $F$  is given by

$$\hat{\Phi}_F([g, \eta, v]) = \left( \rho([g, \eta, v]), g \cdot (\eta + \phi_{V/\Gamma}(v)) \right).$$

Since  $\hat{\Phi}_F$  is proper, we can apply Theorem 1.1. Therefore,  $\hat{\Phi}_F(F) \cap (\mathbb{R} \times \mathfrak{t}_+^*)$  is a convex rational locally polyhedral set. In fact, since  $\hat{\Phi}_F$  is homogeneous (i.e., equivariant with respect to the action of  $\mathbb{R}_+$  which on  $F$  is given by  $t \cdot [g, \eta, v] = [g, t\eta, \sqrt{t}v]$  and on  $\mathfrak{g}^*$  by multiplication) and since the number of orbit types in  $F$  is finite,  $\hat{\Phi}_F(F) \cap (\mathbb{R} \times \mathfrak{t}_+^*)$  is a convex rational polyhedral cone. Since  $\Phi_F(F) \cap \mathfrak{t}_+^*$  is the image of  $\hat{\Phi}_F(F) \cap (\mathbb{R} \times \mathfrak{t}_+^*)$  under  $\pi : \mathbb{R} \times \mathfrak{t}_+^* \rightarrow \mathfrak{t}_+^*$ , it is also a convex rational polyhedral cone.

2. Choose a  $G$ -invariant metric on  $\mathfrak{g}^*$ , and let  $\hat{\rho} : F \rightarrow \mathbb{R}$  be defined by  $\hat{\rho}([g, \eta, v]) = \rho(v) + \|\eta\|$ . Then  $\hat{\rho}$  is homogeneous and  $\hat{\rho}^{-1}(0)$  is the zero section of  $F$ . Choose  $\epsilon > 0$  sufficiently small so that the set  $\{x \in F \mid \hat{\rho}(x) < \epsilon\}$  is contained in  $U$ . Let  $\lambda$  be the lower bound on  $\hat{\rho}^{-1}(\epsilon)$  of the function

$$f : F \rightarrow \mathbb{R}, f(x) = \rho(x) + \|\Phi_F(x)\|$$

on  $F$ . Since  $f \geq \hat{\rho}$ , we have  $\lambda \geq \epsilon$ . Let  $\hat{\mathcal{V}} := \{(t, \alpha) \in \mathbb{R} \times \mathfrak{g}^* \mid |t| + \|\alpha\| < \lambda\}$ . Since  $\|\Phi_F(x)\| \geq \lambda$  if  $\hat{\rho}(x) \geq \epsilon$ , one has  $\hat{\Phi}_F(U) \supset \hat{\Phi}_F(F) \cap \hat{\mathcal{V}}$ , which is a  $G$ -invariant neighborhood of  $0 \in \hat{\Phi}_F(F)$ . Since  $\hat{\Phi}_F(F) \cap (\mathbb{R} \times \mathfrak{t}_+^*)$  is a convex rational polyhedral cone, the image of every neighborhood of the vertex of this cone under the projection  $\pi : \mathbb{R} \times \mathfrak{t}_+^* \rightarrow \mathfrak{t}_+^*$  is a neighborhood of the vertex of  $\Phi_F(F) \cap \mathfrak{t}_+^* = \pi(\hat{\Phi}_F(F) \cap \mathbb{R} \times \mathfrak{t}_+^*)$ . It follows that  $G \cdot \pi(\hat{\mathcal{V}} \cap (\mathbb{R} \times \mathfrak{t}_+^*))$  is a  $G$ -invariant neighborhood of  $0$  in  $\mathfrak{g}^*$  with the required property.

3. Finally, we use cross sections to reduce the general case to the case that  $\Phi(m) = 0$ . Let  $\tau$  be the (open) face containing  $\Phi(m)$ . Let  $U_\tau$  be the corresponding natural slice and  $R_\tau = \Phi^{-1}(U_\tau)$  the natural cross-section. By equivariance of the moment map, we have

$$\Phi(G \cdot R_\tau) \cap \mathfrak{t}_+^* = \Phi(R_\tau) \cap \mathfrak{t}_+^*.$$

The Cartan subalgebras of  $G$  and  $G_\tau$  are equal, but a Weyl chamber  $(\mathfrak{t}_+^*)_+$  of  $G_\tau$  is the union of certain Weyl chambers of  $G$ . Nonetheless,  $\Phi(R_\tau) \cap (\mathfrak{t}_+^*)_+ = \Phi(R_\tau) \cap \mathfrak{t}_+^*$ , which means that we can apply Remark 3.9.c to reduce to the case  $\Phi(m) = 0$ .  $\square$

We now prove Theorem 1.2.

*Proof.* It suffices to consider an arbitrarily small  $G$ -invariant neighborhood  $U$  of  $m \in M$ . By Theorem 6.1, there exists an open neighborhood  $\mathcal{V}$  of  $\Phi(m)$  and a rational polyhedral cone  $C_m \subset \mathfrak{t}^*$  such that  $\Phi(U) \cap \mathfrak{t}_+^* \cap \mathcal{V} = C_m \cap \mathcal{V}$ . Since  $\Phi(M) \cap \mathfrak{t}_+^*$  is a convex rational locally polyhedral set, there exists a cone  $C'_m$  with vertex  $\Phi(m)$  such that  $\Phi(M) \cap \mathfrak{t}_+^* \cap \mathcal{V} = C'_m \cap \mathcal{V}$ , after shrinking  $\mathcal{V}$  sufficiently. Clearly,  $C_m \subset C'_m$ , so if  $C'_m \neq C_m$ , then there exists a point  $x \in \mathfrak{t}^*$  which lies on a face of  $C_m$ , but in the interior of  $C'_m$ . Since  $C_m$  is closed and convex there exists  $\xi \in \mathfrak{t}$  such that a point in  $U$  mapping to  $x$  is a minimum for  $\Phi^\xi$  in  $U \cap Y$ , but is not a global minimum for  $\Phi^\xi$  on the principal cross section  $Y$ . This contradicts Corollary 4.3. Therefore  $C_m = C'_m$  and hence

$$(\Phi(M) \cap \mathfrak{t}_+^*) \cap \mathcal{V} = C'_m \cap \mathcal{V} = C_m \cap \mathcal{V} = (\Phi(U) \cap \mathfrak{t}_+^*) \cap \mathcal{V}.$$

$\square$

*Remark 6.3.* If  $G = T$  is abelian, Theorem 1.2 implies that the map  $\Phi : M \rightarrow \Phi(M)$  itself is open, since for any set  $U$ ,  $\Phi(U) = T \cdot \Phi(U) = \Phi(T \cdot U)$ . This is however not true for  $G$  non-abelian.

Let  $M = S^2 \times S^2$  be the product of two spheres, with symplectic forms given by the standard area form. Let  $G = \text{SO}(3)$ , and identify  $\mathfrak{so}(3)^* \cong \mathbb{R}^3$  and  $\mathfrak{t}_+^* \cong \{x \mid x_1 > 0, x_2 = x_3 = 0\}$ . The diagonal action of  $\text{SO}(3)$  on  $M$  is Hamiltonian with moment map  $\Phi(x, y) = x + y$ . The zero level set  $\Phi^{-1}(0)$

is the two-sphere  $\{(x, -x) \mid x \in S^2\}$ , and its intersection with the closure of the principal cross-section  $Y = \Phi^{-1}(\mathfrak{t}_+^* - \{0\})$  is the circle  $\{(x, -x) \mid x \in S^2, x_1 = 0\}$ . Let  $U$  be the open set  $U = M - \bar{Y}$ . Then  $0 \in \Phi(U)$ , but  $\Phi(U)$  does not meet  $\mathfrak{t}_+^* - \{0\}$  and is therefore not an open neighborhood of 0.

*Remark 6.4.* A corollary to the above Theorems (or rather their proofs) is that if  $M$  is a Hamiltonian  $G$ -orbifold, with moment map  $\Phi$ , the local moment cone  $C_m$  for a point  $m \in \Phi^{-1}(\mathfrak{t}_+^*)$  does not vary as  $m$  varies in a connected component of a fixed fiber of  $\Phi$ . Indeed, if  $M$  is connected and  $\Phi$  is proper this follows from Theorem 1.2 and Theorem 6.1, because the local moment cone is equal to the cone with vertex at  $\Phi(m)$  over the moment set  $\Delta$ . For the general case, we note that as above, we can use cross-sections to reduce to the case  $\Phi(m) = 0$ , and since the statement is local, it is enough to check it for the local normal form,  $F$ . Now the moment map  $\Phi_F$  is not proper. However, we may replace  $F$  by its cut  $F_{(-\infty, \epsilon]}$  with respect to the  $S^1$ -action generated by  $\rho$ . Since the moment map for the cut space is proper, the claim follows.

## REFERENCES

- [A] M. F. Atiyah, Convexity and commuting hamiltonians, *Bull. London Math. Soc.* **14** (1982), 1-15.
- [B] M. Brion, Sur l'image de l'application moment, in *Séminaire d'algèbre Paul Dubreuil et Marie-Paule Malliavin* (Paris, 1986) (M.-P. Malliavin, ed.), Lecture Notes in Mathematics, vol 1296, Springer-Verlag, Berlin-Heidelberg-New York, 1987, 177-192.
- [CDM] M. Condevaux, P. Dazord, P. Molino, Géométrie du Moment, in *Travaux du Séminar Sud-Rhodanien I*, Publ. Dép. Math., Univ. Claude Bernard - Lyon I (1988), 131-160.
- [FR] H. Flaschka and T. Ratiu, A convexity theorem for Poisson actions of compact Lie groups. *Annales Sci. Ecole Normale Supérieure* **29** (1996), 757-786.
- [GLS] V. Guillemin, E. Lerman and S. Sternberg, *Symplectic fibrations and multiplicity diagrams*, Cambridge University Press, Cambridge, 1996.
- [GS1] V. Guillemin and S. Sternberg, A normal form for the moment map, in: *Differential Geometric Methods in Mathematical Physics* (S. Sternberg, ed.), Reidel Publishing Company, Dordrecht, 1984.
- [GS2] V. Guillemin and S. Sternberg, Convexity properties of the moment mapping I and II, *Invent. math.* **67** (1982), 491-513, *Invent. math.* **77** (1984), 533-546.
- [GS3] V. Guillemin and S. Sternberg, *Symplectic techniques in physics*, Cambridge University Press, Cambridge, 1990.
- [H] G. Heckman, Projection of orbits and asymptotic behavior of multiplicities for compact Lie groups, Ph.D. thesis, Leiden, 1980.
- [HH] P. Heinzner and A. Huckleberry, Kählerian potentials and convexity properties of the moment map. *Invent. Math.* **126** (1996), 65-84.
- [HNP] J. Hilgert, K.-H. Neeb and W. Plank, Symplectic convexity theorems and coadjoint orbits, *Comp. Math.* **94** (1994), 129-180.
- [Ka] Y. Karshon, unpublished notes.
- [KL] Y. Karshon, E. Lerman, The centralizer of invariant functions and division properties of the moment map, to appear in *Illinois J. of Math.*, [dg-ga/9511008](#).
- [Ki1] F. Kirwan, *Cohomology of Quotients in Symplectic and Algebraic Geometry*, Princeton University Press, Princeton, 1984.
- [Ki2] F. Kirwan, Convexity properties of the moment mapping III, *Invent. math.* **77** (1984), 547-552.
- [Ko] B. Kostant, On convexity, the Weyl group, and the Iwasawa decomposition, *Ann. Sci. Ec. Norm. Sup.* **6** (1973), 413-455.
- [L] E. Lerman, Symplectic cuts, *Math. Research Lett.* **2** (1995), 247-258.
- [LT] E. Lerman and S. Tolman, Hamiltonian torus actions on symplectic orbifolds and toric varieties, to appear in *Trans. Am. Math. Soc.*, [dg-ga/9511008](#).
- [Ma] C. M. Marle, Modèle d'action hamiltonienne d'un groupe de Lie sur une variété symplectique, *Rendiconti del Seminario Matematico* **43** (1985), 227-251, Università e Politecnico, Torino.
- [M] E. Meinrenken, Symplectic surgery and the Spin<sub>c</sub>-Dirac operator, to appear in *Adv. Math.*, [dg-ga/9504002](#).
- [MW] E. Meinrenken and C. Woodward, A symplectic proof of Verlinde factorization, preprint, [dg-ga/9612018](#).
- [NM] L. Ness, A stratification of the null cone via the moment map, *Am. J. of Math.* **106**, (1984), 1281-1329, with an appendix by D. Mumford.
- [Sa] I. Satake, The Gauss-Bonnet theorem for V-manifolds *J. Math. Soc. Japan* **9** (1957), 464-492.

- [Sj] R. Sjamaar, Convexity properties of the moment map re-examined, preprint, to appear in *Adv. Math.*, [dg-ga/9408001](#).
- [W] C. Woodward, The classification of transversal multiplicity-free group actions, *Ann. Global Anal. Geom.* **14** (1996), 3–42.

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