Notes on Linear Algebra

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Contents

1. Vectors

A *vector* is a quantity with a magnitude and a direction. *Linear algebra* uses vectors to study systems of linear equations. Let's start with a simple example that shows what linear algebra is about.

1.1. **Pancakes and waffles.**
Consider the following recipes for pancakes and waffles.

Recipe	Flour (cups)	Sugar (tablespoons)
Pancakes	2	1
Waffles	3/2	2

Suppose we have 10 tbsp sugar and 10 cups flour. How much of each can be made without waste?

To solve the problem, let p denote the number of batches of pancakes, and w the number of batches of waffles. To make p batches of pancakes, we need $2p$ cups flour and p tbsps sugar. To make w batches of waffles, we need $(3/2)w$ cups flour and $2w$ tbsps sugar. To use up the flour we need

$$2p + (3/2)w = 10.$$

To use up the sugar we need

$$p + 2w = 10.$$

Subtracting one equation from the other gives $(-5/2)w = -10$, so $w = 4$. Substituting $w = 4$ into the second equation gives $p + 8 = 10$, so $p = 2$. What we have done is a simple case of *elimination*, a procedure for solving systems of linear equations that will be covered in detail later.

The system of equations above can be represented geometrically by representing each recipe as a vector. By a 2-*vector* we mean a pair of numbers, called the *components* of the vector. Typically we write the vector vertically, so that the first component appears above the first component. For example,

$$\begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

is the vector whose first component is 2 and second component is 1. The pancake and waffle recipes are represented by the 2-vectors

$$\mathbf{v}_p = \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \quad \mathbf{v}_w = \begin{bmatrix} 3/2 \\ 2 \end{bmatrix}$$

Geometrically a 2-vector $\mathbf{v} = \begin{bmatrix} x \\ y \end{bmatrix}$ is drawn as arrow in the plane. Draw horizontal and vertical axes on the plane, so that the *origin* is the intersection of the axes. The *tail* of the vector can be at any point. The *head* of the vector is drawn x units to the right, and y units above the tail of the vector. The vectors $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 3/2 \\ 2 \end{bmatrix}$, with tails at the origin, are shown in Figure 1.

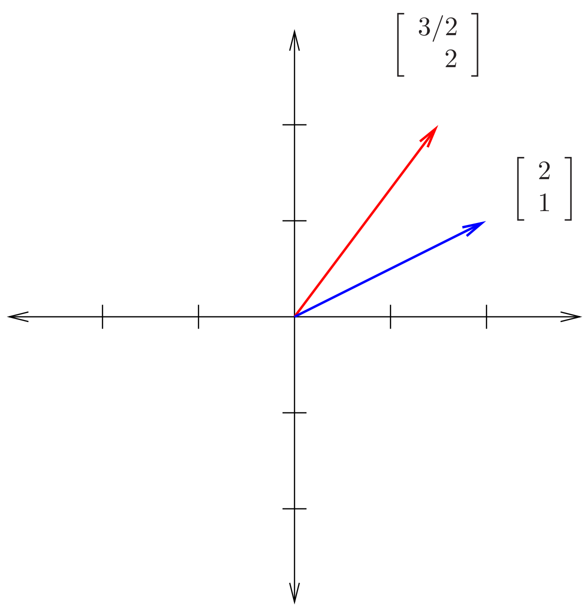


FIGURE 1. Examples of vectors

1.2. **Vector operations.**

1.2.1. *Vector addition.* We add 2-vectors by adding their components:

$$\begin{bmatrix} 2 \\ 1 \end{bmatrix} + \begin{bmatrix} 3/2 \\ 2 \end{bmatrix} = \begin{bmatrix} 2+3/2 \\ 1+2 \end{bmatrix} = \begin{bmatrix} 7/2 \\ 3 \end{bmatrix}.$$

The sum of two vectors can be drawn by putting the vectors *head to tail.* Let $\mathbf{v}$ and $\mathbf{w}$ be 2-vectors. Draw $\mathbf{v}$ with tail at 0, and $\mathbf{w}$ with tail at the head of $\mathbf{v}$. Now draw the vector from tail $(0,0)$ to the head of $\mathbf{w}$. This is the vector $\mathbf{v}+\mathbf{w}$. In Figure 2 the vector $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$ is drawn head to tail with $\begin{bmatrix} 3/2 \\ 2 \end{bmatrix}$.

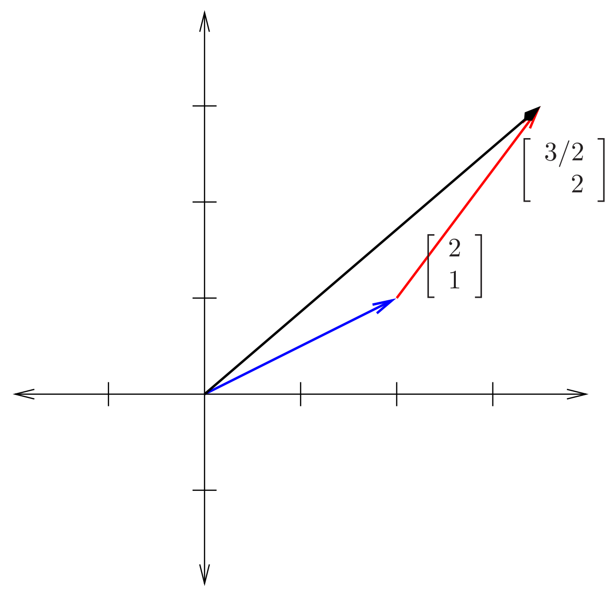


FIGURE 2. Vector addition

1.2.2. *Length.* You can think of a vector as an object that has a *direction* and a *length* (or *magnitude*). The vector $\mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$ forms the hypotenuse of a triangle with adjacent side v_1 and opposite side v_2. By the Pythagorean theorem, the length or *norm* of $\mathbf{v}$ is the square root of the squares of the adjacent and opposite sides of the right triangle formed by $\mathbf{v}$:

$$\|\mathbf{v}\| = \sqrt{v_1^2 + v_2^2}.$$

If the vector has magnitude 1, it is called a *unit vector*.

1.2.3. *Scalar multiplication.* A *scalar* is another name for a number. We multiply a scalar times a vector by multiplying each component. For example, $2\begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \cdot 2 \\ 2 \cdot 1 \end{bmatrix} = \begin{bmatrix} 4 \\ 2 \end{bmatrix}$. We do not use the notation $\times$ for this sort of multiplication, which has a special meaning we will discuss later. Figure 3 shows the vector $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$, and its scalar multiple $\begin{bmatrix} 4 \\ 2 \end{bmatrix}$.

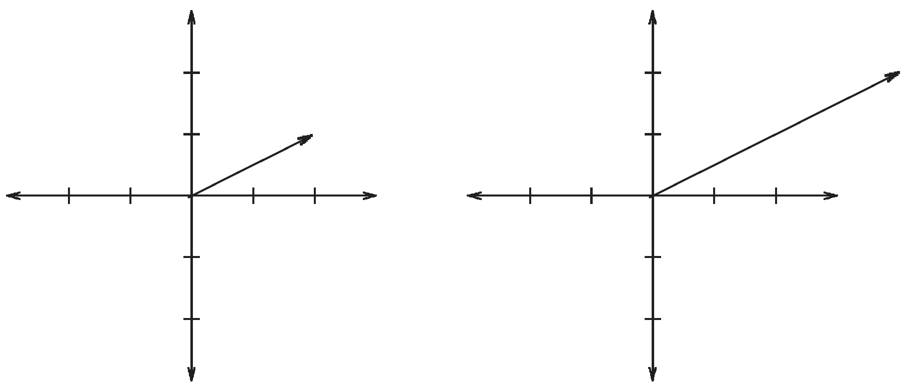


FIGURE 3. Scalar multiplication

Geometrically scalar multiplication by a positive scalar c leaves the direction of the vector unchanged, but multiplies the length by c. If the scalar c is negative, then $c\mathbf{v}$ points in the opposite direction from $\mathbf{v}$.

For any non-zero vector $\mathbf{v}$ there is a *unit vector in the direction* of v, given by the formula

$$\mathbf{u} = \frac{1}{\|\mathbf{v}\|}\mathbf{v}.$$

The unit vector in the direction of $\mathbf{v} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ is

$$\frac{1}{\sqrt{2^2+1^2}}\begin{bmatrix} 2 \\ 1 \end{bmatrix} = \frac{1}{\sqrt{5}}\begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{2}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} \end{bmatrix}.$$

1.2.4. *Vector subtraction.* Subtraction is similar to addition: We subtract each component.

$$\begin{bmatrix} 2 \\ 1 \end{bmatrix} - \begin{bmatrix} 3/2 \\ 2 \end{bmatrix} = \begin{bmatrix} 2-3/2 \\ 1-2 \end{bmatrix} \begin{bmatrix} 1/2 \\ -1. \end{bmatrix}$$

Subtraction is a combination of vector addition and scalar multiplication:

$$\begin{bmatrix} 2 \\ 1 \end{bmatrix} - \begin{bmatrix} 3/2 \\ 2 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix} + (-1)\begin{bmatrix} 3/2 \\ 2 \end{bmatrix}$$

and so is not considered a separate operation.

1.3. **Vector form of a linear system.** Two vectors are *equal* if they have the same components, in order. This means that any system of linear equations can be written as a single vector equation. For instance, the pancake/waffle system

$$\begin{array}{rcrcl} 2p & + & 3/2w & = & 10 \\ p & + & 2w & = & 10 \end{array}$$

can be written as the single vector equation

$$p\begin{bmatrix} 2 \\ 1 \end{bmatrix} + w\begin{bmatrix} 3/2 \\ 2 \end{bmatrix} = \begin{bmatrix} 10 \\ 10 \end{bmatrix}.$$

A sum of scalar multiples of vectors is called a *linear combination.* To solve the system, we have to write the total vector

$$\mathbf{v}_t = \begin{bmatrix} 10 \\ 10 \end{bmatrix}$$

as a linear combination of the pancake and waffle vectors

$$\mathbf{v}_p = \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \quad \mathbf{v}_w = \begin{bmatrix} 3/2 \\ 2 \end{bmatrix}.$$

Figure 4 shows that two of $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$ and four of $\begin{bmatrix} 3/2 \\ 2 \end{bmatrix}$ give $\begin{bmatrix} 10 \\ 10 \end{bmatrix}$. This is what we mean by solving the system geometrically.

1.4. **Vectors in three-dimensions.** A 3-vector $\mathbf{v}$ is a triple of numbers, called the components of $\mathbf{v}$. For example $\mathbf{v} = \begin{bmatrix} 4 \\ 2 \\ 6 \end{bmatrix}$ is a 3-vector whose components are $4, 2$ and 6. Geometrically a 3-vector $\mathbf{v} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ is drawn as an arrow on a plane with three axes. The head of the arrow is drawn x units along the first axis, y units along the second axis, and z units along the third axis from the tail.

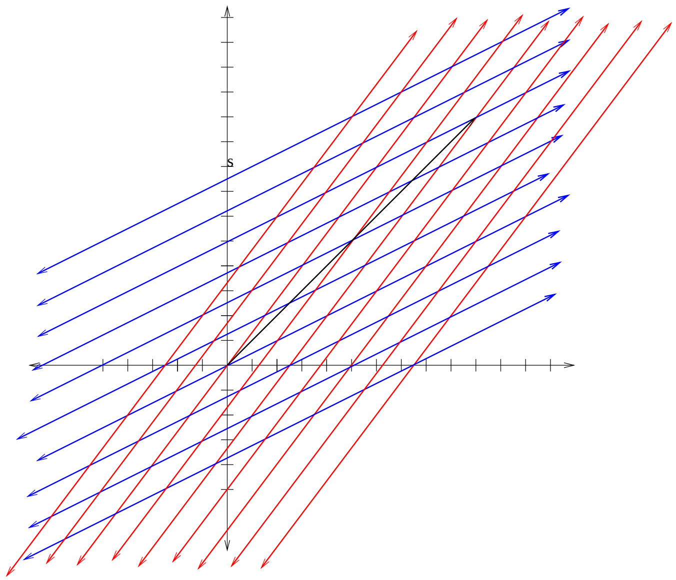


FIGURE 4. Total as a linear combination

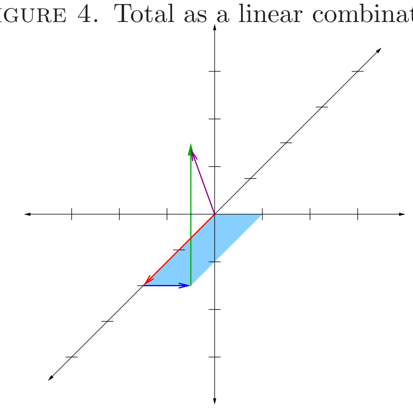


FIGURE 5. A 3-vector

The vector $\begin{bmatrix} 4 \\ 2 \\ 6 \end{bmatrix}$ with tail at $(0,0,0)$ is drawn in Figure 5 in purple. In the other colors it is shown how to arrive at the head: by starting at the origin, moving one unit along the first axis, two units along the second, and three units along the third.

Operations on 3-vectors are defined in the same way as for 2-vectors. Scalar multiplication multiplies a number times each component, for example,

$$2\begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 4 \\ 2 \\ 4 \end{bmatrix}, \quad 4\begin{bmatrix} 3/2 \\ 2 \\ 2 \end{bmatrix} = \begin{bmatrix} 6 \\ 8 \\ 8 \end{bmatrix}.$$

Add vectors by adding their components:

$$\begin{bmatrix} 4 \\ 2 \\ 4 \end{bmatrix} + \begin{bmatrix} 6 \\ 8 \\ 8 \end{bmatrix} = \begin{bmatrix} 4+6 \\ 2+8 \\ 4+8 \end{bmatrix} = \begin{bmatrix} 10 \\ 10 \\ 12 \end{bmatrix}.$$

An unavoidable problem with representing three-vectors on the page is that different vectors look the same when drawn. For instance,

$$\begin{bmatrix} -1/2 \\ 0 \\ \frac{3}{2} \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

are drawn the same on our representation of three-space. In particular, the length of a three-vector is not the length of the vector as it is drawn on the page.

Three-vectors naturally appear in systems of three equations. For instance, in the pancake/waffle problem suppose we also consider how many eggs are needed for each type, so that the table of ingredients becomes

	Flour	Sugar	Eggs
Pancakes	2	1	2
Waffles	3/2	2	2

Suppose we have 10 cups flour, 10 tbsps sugar, and 10 eggs.

$$\begin{array}{rcrcl} 2p & + & 3/2w & = & 10 \\ p & + & 2w & = & 10 \\ 2p & + & 2w & = & 10\,\frac{1}{4}. \end{array} \tag{1}$$

From the first two equations, we found a ove that $p = 2$ and $w = 4$. Plugging these into the third equation we get

$$2p + 2w = 12 \neq 10$$

so there is no solution, that is, no way to use the ingredients without waste. In vector form, the pancake/waffle system is now

$$p\begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix} + w\begin{bmatrix} 3/2 \\ 2 \\ 2 \end{bmatrix} = \begin{bmatrix} 10 \\ 10 \\ 10 \end{bmatrix}.$$

We denote the vectors for each recipe

$$\mathbf{v}_p = \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix}, \quad \mathbf{v}_w = \begin{bmatrix} 3/2 \\ 2 \\ 2 \end{bmatrix}, \quad \mathbf{v}_t = \begin{bmatrix} 10 \\ 10 \\ 10 \end{bmatrix}$$

To solve the system geometrically, we need to find some number of blue arrows $\mathbf{v}_p$ and the number of red arrows $\mathbf{v}_w$ which together give the green arrow $\mathbf{v}_t$. It's not hard to see that the set of combinations of $\mathbf{v}_p$ and $\mathbf{v}_w$ form a plane.

What do we see geometrically that we didn't see before algebraically? The set of all linear combinations of the ingredient vectors forms a plane inside three-dimensional space. The pancake/waffle system is not solvabbe because the total vector of available ingredients $\begin{bmatrix} 10 \\ 10 \\ 10 \end{bmatrix}$ does not lie in the plane given by all combinations of $\mathbf{v}_p$ and $\mathbf{v}_w$. However, if we have 12 eggs instead of 10 then there is a solution.

Definition 1.1. The set of all linear combinations of a set of vectors is called the *span* of the vectors.

A linear system has a solution if, in its vector form, the vector on the right hand side lies in the span of the vectors on the left-hand-side.

Example 1.2. The span of the vectors $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ is all the vectors of the form $a\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + b\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} a \\ b \\ 0 \end{bmatrix}$, that is, the xy-plane.

Example 1.3. The span of the vectors $[1 \ -1 \ 0]$ and $[0 \ -1 \ 1]$ is the set of all combinations

$$a[1 \ -1 \ 0] + b[0 \ -1 \ 1] = [a \ -a-b \ b].$$

Any vector of this form has $x+y+z=0$. Conversely, any vector with $x+y+z=0$ can be written as

$$[x \ -x-y \ z] = a[1 \ -1 \ 0] + b[0 \ 1 \ -1]$$

where $a = x$ and $b = y$. So the span is the plan $x+y+z=0$.

1.5. **Vectors in any dimension.** An n-vector is an n-tuple of numbers

$$\mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}$$

called the *components* of $\mathbf{v}$. We say that $\mathbf{v}$ has *size* (or *length* or *dimension*) n. Add two n-vectors by adding their components:

$$\begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix} + \begin{bmatrix} w_1 \\ w_2 \\ \vdots \\ w_n \end{bmatrix} = \begin{bmatrix} v_1+w_1 \\ v_2+w_2 \\ \vdots \\ v_n+w_n \end{bmatrix}.$$

The product of a scalar c and an n-vector is defined by

$$c\begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix} = \begin{bmatrix} cv_1 \\ cv_2 \\ \vdots \\ cv_n \end{bmatrix}.$$

The *length* of an n-vector is

$$\|\mathbf{v}\| = \sqrt{v_1^2 + v_2^2 + \ldots + v_n^2}.$$

This is the distance from the head to tail. The formula can be justified by repeatedly applying the Pythagorean theorem. For example, for $n = 3$ we have

$$\begin{bmatrix} v_1 \\ v_2 \\ v_2 \end{bmatrix} = \begin{bmatrix} v_1 \\ v_2 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ v_3 \end{bmatrix}$$

which form a right angle, hence

$$\left\|\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}\right\|^2 = \left\|\begin{bmatrix} v_1 \\ v_2 \\ 0 \end{bmatrix}\right\|^2 + \left\|\begin{bmatrix} 0 \\ 0 \\ v_3 \end{bmatrix}\right\|^2 = (v_1^2 + v_2^2) + v_3^2.$$

In particular this is a formula for the distance between two points in n-dimensional space: To find the distance between any two points, take the length of the vector connecting them.

Example 1.4. To find the distance between the points $(3,1,2,4)$ and $(-1,2,1,2)$ in R^4, we take the length of the vector

$$\mathbf{v} = [-1 \ 2 \ 1 \ 2] - [3 \ 1 \ 2 \ 4] = [-4 \ 1 \ -1 \ -2]$$

which is

$$\|\mathbf{v}\| = \sqrt{(-4)^2 + 1^2 + (-1)^2 + (-2)^2} = \sqrt{22}.$$

Here are some of the properties of vector operations. Later we will define new operations which do *not* satisfy some of these axioms.

(1)	$\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$	(Vector Addition is Commutative)
(2)	$\mathbf{u} + (\mathbf{v} + \mathbf{w}) = (\mathbf{u} + \mathbf{v}) + \mathbf{w}$	(Vector Addition is Associative)
(3)	$c(\mathbf{u} + \mathbf{v}) = c\mathbf{u} + c\mathbf{v}$	(Scalar Mult. is Distributive)
(8)	$\|c\mathbf{v}\| = \|c\|\|\mathbf{v}\|$	(Scalar Mult multiplies the Length)

Proofs involving vectors usually involve writing out the general form, using the corresponding properties for numbers.

Example 1.5. Prove that $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$ for any n-vectors $\mathbf{u}, \mathbf{v}$. Answer: Let

$$\mathbf{u} = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix}, \quad \mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}.$$

Then

$$\mathbf{u} + \mathbf{v} = \begin{bmatrix} u_1 + v_1 \\ u_2 + v_2 \\ \vdots \\ u_n + v_n \end{bmatrix} = \begin{bmatrix} v_1 + u_1 \\ v_2 + u_2 \\ \vdots \\ v_n + u_n \end{bmatrix} = \mathbf{v} + \mathbf{u}.$$

To save space, we often write vectors horizontally, so that $v = [1/2]$ is the same vector as $v = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$.

1.6. **Problems.**

(1) Find the sums and differences of the following vectors:
   (a) $[1\ 0]$ and $[0\ 1]$
   (b) $[1\ 0]$ and $[-1\ -1]$
   (c) $[2\ 2\ 1]$ and $[2\ -1\ 2]$ and
   (d) $[1\ 0\ 1]$ and $[1\ 1\ 0]$.

(2) Find the second component of the following vectors
   (a) $[1\ 0]$ and $[0\ 1]$
   (b) $[1\ 0]$ and $[-1\ -1]$
   (c) $[2\ 2\ 1]$ and $[2\ -1\ 2]$ and
   (d) $[1\ 0\ 1]$ and $[1\ 1\ 0]$.

(3) Write the vector whose components are
   (a) $-1, 3$
   (b) $3, -1$
   (c) $2, 4, 6$.

(4) Draw the following vectors:
   (a) $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$,
   (b) $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$, ,
   (c) $\begin{bmatrix} 1 \\ -2 \end{bmatrix}$,
   (d) $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$.

(5) Find the distance between the points
   (a) $(0, 0), (3, 2)$
   (b) $(1, 1), (3, 2)$
   (c) $(1, -1), (3, 2)$
   (d) $(1, 1, 0), (0, 1, 1), (1, 0, 1)$.

(6) Draw the following vectors: $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ -2 \\ 3 \end{bmatrix}$.

(7) Find the $2\mathbf{v} + 3\mathbf{w}$ for the vectors
   (a) $\mathbf{v} = [5\ 4], \mathbf{w} = [-1\ -2]$.
   (b) $\mathbf{v} = [1\ 0], \mathbf{w} = [0\ 1]$.
   (c) $\mathbf{v} = [5\ 4\ 1],\ \mathbf{w}[-1\ -2\ -3]$.

(8) Write the following systems of equations as a single vector equality, using scalar multiplication to simplify if possible:
   (a) $a = 2, b = 3$
   (b) $2a + b = 1, 3a + 2b = 2$
   (c) $x + y = 0, y = 3$
   (d) $a + b = 5, b - a = 6$

(9) In the pancake-waffle system, find another combination of ingredients (that is, not 10 flour, 10 sugar, 12 eggs) that can be used up without waste.

(10) Show that if $\|\mathbf{v}\| = 0$, then $\mathbf{v}$ is the zero vector. Hint: Write down what it means to have $\|\mathbf{v}\| = 0$, and what it means to be the zero vector in English. Then write it down in equations. Then show why the equation for $\|\mathbf{v}\| = 0$ implies that $\mathbf{v} = 0$.

2. Matrices

A matrix is a table of numbers. Matrices can be used to simplify linear equations even further.

2.1. Types of matrices.

A *matrix* is a table of numbers, called the *entries* of the matrix. If a matrix has m rows and n columns, it is called an $m \times n$ matrix. The entry in the i-th row and j-th column is called the ij-th entry.

Example 2.1. In the last section, we wrote the pancake/waffle system in vector form

$$p \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix} + w \begin{bmatrix} 3/2 \\ 2 \\ 2 \end{bmatrix} = \begin{bmatrix} 10 \\ 10 \\ 12 \end{bmatrix}.$$

The matrix for the pancake-waffle system is the 2×3 matrix

$$A = \begin{bmatrix} 2 & 3/2 \\ 1 & 2 \\ 2 & 2 \end{bmatrix}.$$

One can think of a matrix as a collection of row vectors, or as a collection of column vectors. The row vectors for the matrix A are

$$[2\ 3/2]\,,\ [1\ 2]\,,\ [2\ 2]\,.$$

The column vectors are

$$\begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 3/2 \\ 2 \\ 2 \end{bmatrix}.$$

Here are a few of the many special kinds of special matrices.

(1) The $m \times n$ zero matrix $\mathbf{0}_{mn}$, whose entries are all zero. For example,

$$\mathbf{0}_{32} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

If there is no confusion about the size, we drop the subscripts and write $\mathbf{0}$ for the zero matrix.

(2) A matrix is *square* if it has the same number of rows as columns. For instance,

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

is a square matrix, more precisely 2×2.

(3) A square matrix is *diagonal* if the only non-zero entries are on the northwest-southeast diagonal, for instance,

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

is diagonal.

(4) A square matrix A is *upper triangular* if all of the entries below the diagonal are zero. For instance,

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 0 & 0 & 6 \end{bmatrix}$$

is upper triangular. A is *lower triangular* if all of the entries above the diagonal are zero. A is strictly upper (or lower) triangular if it is upper (or lower)triangular and all of the diagonal entries are zero. For instance,

$$\begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 2 & 3 & 0 \end{bmatrix}$$

is strictly lower triangular.

(5) The *transpose* of an $m \times n$ matrix A is the $n \times m$ matrix A^T whose columns are the rows of A, and whose rows are the columns of A. For example,

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}^T = \begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix}.$$

If a matrix A is equal to its own transpose A^T, it is called *symmetric*. For instance,

$$\begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}$$

is a 3×3 symmetric matrix.

(6) A matrix is a *permutation matrix* if there is exactly one 1 in each row and each column, and otherwise the matrix is zero. For example,

$$P = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

is a permutation matrix.

2.2. **Matrix addition and scalar multiplication.** Matrices are added or subtracted in the same way as vectors, by adding or subtracting entries. For example,

$$\begin{bmatrix} 0 & 1 \\ 2 & 3 \\ 4 & 5 \end{bmatrix} + \begin{bmatrix} 0 & 3 \\ 1 & 4 \\ 2 & 5 \end{bmatrix} = \begin{bmatrix} 0 & 4 \\ 3 & 7 \\ 6 & 10 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 1 \\ 2 & 3 \\ 4 & 5 \end{bmatrix} - \begin{bmatrix} 0 & 3 \\ 1 & 4 \\ 2 & 5 \end{bmatrix} = \begin{bmatrix} 0 & -2 \\ 1 & -1 \\ 2 & 0 \end{bmatrix}.$$

Multiply a scalar times a matrix by multiplying each entry by the scalar.

$$2\begin{bmatrix} 0 & 1 \\ 2 & 3 \\ 4 & 5 \end{bmatrix} = \begin{bmatrix} 0 & 2 \\ 4 & 6 \\ 8 & 10 \end{bmatrix}.$$

2.3. **Matrix products.** Suppose A is a 3×2 matrix with column vectors $\mathbf{v}_1, \mathbf{v}_2$. Suppose $\mathbf{x}$ is the column vector

$$\mathbf{x} = \begin{bmatrix} 5 \\ 2 \end{bmatrix}.$$

Then the product $A\mathbf{x}$ is the sum

$$5\mathbf{v}_1 + 2\mathbf{v}_2.$$

That is, the product of a matrix times a vector is a sum of the column vectors of the matrix, with coefficients given by the components of the vector. For example

$$\begin{bmatrix} 2 & 3/2 \\ 1 & 2 \\ 2 & 2 \end{bmatrix}\begin{bmatrix} 4 \\ 2 \end{bmatrix} = 4\begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix} + 2\begin{bmatrix} 3/2 \\ 2 \\ 2 \end{bmatrix} = \begin{bmatrix} 10 \\ 10 \\ 12 \end{bmatrix}.$$

Here is an example with a 2×2-matrix:

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 3 \\ 4 \end{bmatrix} = 3\begin{bmatrix} 0 \\ 1 \end{bmatrix} + 4\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 4 \\ 3 \end{bmatrix}.$$

Multplying this matrix times a vector has the effect of switching the first and second components!

We can re-write the pancake/waffle equations a second time using this product. The equations (1) are written in matrix form

$$\begin{bmatrix} 2 & 3/2 \\ 1 & 2 \\ 2 & 2 \end{bmatrix}\begin{bmatrix} p \\ w \end{bmatrix} = \begin{bmatrix} 10 \\ 10 \\ 12 \end{bmatrix}.$$

There is another way of looking at the product of a matrix A times a vector $\mathbf{x}$, using products of rows with columns. The components of the product are the products of the rows of A with the vector $\mathbf{x}$. For instance,

$$\begin{bmatrix} 2 & 3/2 \\ 1 & 2 \\ 2 & 2 \end{bmatrix}\begin{bmatrix} 4 \\ 2 \end{bmatrix} = \begin{bmatrix} 4(2) + 2(3/2) \\ 4(1) + 2(2) \\ 4(2) + 2(2) \end{bmatrix}.$$

The first component is the product of $\begin{bmatrix} 4 \\ 2 \end{bmatrix}$ with $[2\ 3/2]$, the second component is the product with $[1\ 2]$ and so on.

More generally, suppose A has column vectors $\mathbf{v}_1, \ldots, \mathbf{v_n}$ and

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \\ x_n \end{bmatrix},$$

Define

$$A\mathbf{x} = x_1\mathbf{v}_1 + \ldots + x_n\mathbf{v_n}.$$

If A has row-vectors $\mathbf{w}_1, \ldots, \mathbf{w}_m$ then

$$A\mathbf{x} = \begin{bmatrix} \mathbf{w}_1 \cdot \mathbf{x} \\ \mathbf{w}_2 \cdot \mathbf{x} \\ \\ \\ \mathbf{w}_m \cdot \mathbf{x} \end{bmatrix}$$

is the vector of products of rows with $\mathbf{x}$.

For each square size, there is a special matrix, called the *identity matrix* I which has 1's along the diagonal and $0'$ everywhere else. For instance, the 3×3 identity is

$$I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

The identity matrix has the property that I times any vector $\mathbf{v}$ is itself:

$$I\mathbf{v} = \mathbf{v}$$

For instance, using the definition we get

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 5 \\ 4 \\ 3 \end{bmatrix} = 5\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + 4\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + 3\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 5 \\ 4 \\ 3 \end{bmatrix}.$$

Matrix-vector products satisfy the properties:

(1)	$A(\mathbf{v}+\mathbf{w}) = A\mathbf{v}+A\mathbf{w}$	(Distributivity over vect. addition)
(2)	$A(c\mathbf{v}) = c(\mathbf{v})$	(Commutes with Scalars)
(3)	$(A+B)\mathbf{v} = A\mathbf{v}+B\mathbf{v}$	(Distrib. over matrix addition)
(4)	$I\mathbf{v} = \mathbf{v}$	(Multiplication by Identity Matrix)
(5)	$0\mathbf{v} = 0$	(Multiplication by Zero Matrix)

These can be proved by writing out the definitions of equality of vectors, that is, by showing that the components of the vectors on both sides are equal. For example, the first component of $A(\mathbf{v}+\mathbf{w})$ is the product of the first row of A, call it $\mathbf{a}_1$, with $\mathbf{v}+\mathbf{w}$. Then $\mathbf{a}_1 \cdot (\mathbf{v}+\mathbf{w}) = \mathbf{a}_1 \cdot \mathbf{v} + \mathbf{a}_1 \cdot \mathbf{w}$ which is the first component of $A\mathbf{v}+A\mathbf{w}$.

Now we define the product of two matrices. The product AB of two matrices A and B is the matrix whose columns are the products of the matrix A with the columns of B. The matrix product is the matrix of products of rows of A with columns of B. For example,

$$\begin{bmatrix} 1 & 2 \\ 2 & 3 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} [1\ 2]\begin{bmatrix} -1 \\ 1 \end{bmatrix} & [1\ 2]\begin{bmatrix} 1 \\ -1 \end{bmatrix} \\ [2\ 3]\begin{bmatrix} -1 \\ 1 \end{bmatrix} & [2\ 3]\begin{bmatrix} 1 \\ -1 \end{bmatrix} \\ [3\ 4]\begin{bmatrix} -1 \\ 1 \end{bmatrix} & [3\ 4]\begin{bmatrix} 1 \\ -1 \end{bmatrix} \end{bmatrix} = \begin{bmatrix} -1+2 & 1+-2 \\ -2+3 & 2+-3 \\ -3+4 & 3+-4 \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 1 & -1 \\ 1 & -1 \end{bmatrix}.$$

Notice that the matrix product only makes sense if the dot products of the rows of A with the columns of B make sense. In other words, the rows of A have to be the same size as the columns of B. To put it one more way, if A is an $m \times n$ matrix and B is a $p \times q$ matrix, the product AB makes sense only if $n = p$. The result is an $m \times q$ matrix.

In general notation let $\mathbf{w}_1, \ldots, \mathbf{w}_m$ be the row-vectors of A, and $\mathbf{v}_1, \ldots, \mathbf{v}_q$ the column vectors of B. The matrix product AB is the matrix whose ij-th entry is the dot product $\mathbf{v}_i \cdot \mathbf{v}_j$.

Matrix multiplication is counter-intuitive in a number of different ways.

(1) Matrix multiplication is *not* commutative, that is AB is not necessarily the same matrix as BA, even if both are defined. For AB to be defined A must have the same number of columns as B has rows. For BA to be defined, B has the same number of columns as A has rows. Here is an example:

$$\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} 3 & 4 \\ 0 & 0 \end{bmatrix},$$

which is not equal to

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 3 \end{bmatrix}.$$

(2) Just because $AB = 0$ doesn't mean that $A = 0$ or $B = 0$. For instance,

$$\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = 0$$

but neither of the matrices (which are equal) are zero. In fact, this property isn't true for vectors either. Suppose that $\mathbf{v}$ and $\mathbf{w}$ are vectors of the same size, and $\mathbf{v} \cdot \mathbf{w} = 0$. As we said before, this means that $\mathbf{v}$ is perpendicular to $\mathbf{w}$, not that either $\mathbf{v}$ or $\mathbf{w}$ is zero.

(3) If $AB = AC$ and A is non-zero, then it is not necessarily true that $B = C$. We can't just divide A from both sides, since the expression $1/A$ doesn't make sense - yet.

(4) If A is a square matrix we can define it's matrix powers

$$A^2 = A\,A,\ A^3 = A\,A\,A,$$

et cetera. If $A^2 = 0$, this doesn't mean that A is zero. For example,

$$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

has $A^2 = 0$, but A is not zero.

Matrix products naturally arise when we have two systems of linear equations, in which one set of variables is the input for one system and the output for the other. For example, in the pancake/waffle example suppose the *cost* and *weight* of the ingredients are

ingredient	cost (cents)	weight (grams)
flour	18	125
sugar	30	200
eggs	10	50

Let B be the matrix

$$B = \begin{bmatrix} 18 & 30 & 10 \\ 125 & 200 & 50 \end{bmatrix}.$$

The matrix B transforms the ingredient vector into the cost/weight vector:

$$B\begin{bmatrix} f \\ s \\ e \end{bmatrix} = \begin{bmatrix} c \\ w \end{bmatrix}$$

where c is the number of cents that the ingredients cost and w is their weight. Since

$$B\begin{bmatrix} 10 \\ 10 \\ 12 \end{bmatrix} = \begin{bmatrix} 600 \\ 3805 \end{bmatrix}$$

the 12 eggs, 10 cups flour and 10 sugar costs 600 cents (6 dollars) and 3850 gram (that is, 3.85 kg).

Naturally one would like a matrix that computes the cost and weight from the number of batches of pancakes and waffles. This is the role of the matrix product:

$$BA\begin{bmatrix} p \\ w \end{bmatrix} = B\begin{bmatrix} e \\ c \\ s \end{bmatrix} = \begin{bmatrix} c \\ w \end{bmatrix}.$$

Given the number of pints of regular and light we want to make, multiplying by BA tells us how much it will cost and weigh. In our case, the matrix-matrix product is

$$\begin{aligned} BA &= \begin{bmatrix} 18 & 30 & 10 \\ 125 & 200 & 50 \end{bmatrix}\begin{bmatrix} 2 & 3/2 \\ 1 & 2 \\ 2 & 2 \end{bmatrix} \\ &= \begin{bmatrix} 18(2)+30(1)+10(2) & 18(3/2)+30(2)+10(2) \\ 125(2)+200(1)+50(2) & 125(3/2)+200(2)+50(2) \end{bmatrix} \\ &= \begin{bmatrix} 86 & 107 \\ 550 & 687.5 \end{bmatrix}. \end{aligned}$$

Matrix-vector products satisfy an associativity property similar for that of numbers. First, if A and B are matrices, and $\mathbf{v}$ is a vector, then

$$B(A\mathbf{v}) = (BA)\mathbf{v}.$$

In our example, this just means that

$$\begin{bmatrix} 18 & 30 & 10 \\ 125 & 200 & 50 \end{bmatrix}\begin{bmatrix} 2 & 3/2 \\ 1 & 2 \\ 2 & 2 \end{bmatrix}\begin{bmatrix} 4 \\ 2 \end{bmatrix} == \begin{bmatrix} 86 & 107 \\ 550 & 687.5 \end{bmatrix}\begin{bmatrix} 4 \\ 2 \end{bmatrix}.$$

In other words, knowing the matrix product gives a "short-cut" towards computing the cost and weight. To prove $A(B\mathbf{v}) = (AB)\mathbf{v}$ in general, let $\mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}$ be an n-vector, let A be a matrix with n columns $\mathbf{w}_1, \ldots, \mathbf{w}_n$. The product $A\mathbf{v}$ is

$$A\mathbf{v} = v_1\mathbf{w}_1 + \ldots v_n\mathbf{w}_n.$$

So

$$B(A\mathbf{v}) = B(v_1\mathbf{w}_1 + \ldots v_n\mathbf{w}_n) = v_1(B\mathbf{w}_1) + \ldots + v_n(B\mathbf{w}_n)$$

by distributivity of matrix multiplication over addition. On the other hand, BA is the matrix with columns

$$B\mathbf{w}_1, \ldots, B\mathbf{w}_n.$$

So

$$(BA)\mathbf{v} = v_1(B\mathbf{w}_1) + \ldots (B\mathbf{w}_n)$$

which equals $B(A\mathbf{v})$.

If we have three matrices A, B, C then associativity says that $(AB)C = A(BC)$. This follows from the version for $AB\mathbf{v}$, applied to the columns of C. If $\mathbf{u}_1, \ldots, \mathbf{u}_p$ are the columns of C, then $A(BC)$ is the matrix with columns

$$A(B\mathbf{u}_1), \ldots, A(B\mathbf{u}_n)$$

and $(AB)C$ is the matrix with columns

$$(AB)\mathbf{u}_1, \ldots, (AB)\mathbf{u}_n.$$

By associativity of matrix-vector products, these are equal.

The following summarizes properties of matrix addition and multiplication:

(1)	$A + B = B + A$	(Commutativity of Addition)
(2)	$A + (B + C) = (A + B) + C$	(Assoc. of Addition)
(3)	$A(BC) = (AB)C$	(Assoc. of Matrix Product)
(4)	$A(B + C) = AB + AC$	(Distrib. of Left Matrix Product)
(5)	$(A + B)C = AC + BC$	(Distrib. of Right Matrix Product)
(6)	$(A^T)^T = A$	(Transpose is an Involution)
(7)	$(AB)^T = B^T A^T$	(Tranpose of a Product Changes Order)

These properties can be justified by writing out the entries of the matrices on both sides. For example, property (1) is justfied by

$$\begin{aligned} ij\text{-th entry of } A+B &= ij\text{-th entry of } A \text{ plus } ij\text{-th entry of } B \\ &= ij\text{-th entry of } B \text{ plus } ij\text{-th entry of } A \\ &= ij\text{-th entry of } B+A \end{aligned}$$

Property (7) is justified as follows:

$$\begin{aligned} ij\text{-th entry of } (AB)^T &= ji\text{-th entry of } AB \\ &= (\text{row } j \text{ of } A)\cdot (\text{column } i \text{ of B}) \\ &= (\text{column } j \text{ of } A^T)\cdot(\text{row } i \text{ of } B^T) \\ &= (\text{row } i \text{ of } B^T)\cdot (\text{column } j \text{ of } A^T) \\ &= ij\text{-th entry of } B^TA^T. \end{aligned}$$

A matrix A is *block diagonal* if it is of the form

$$A = \begin{bmatrix} B & 0 \\ 0 & C \end{bmatrix}$$

for some matrices B, C. For example,

$$A = \begin{bmatrix} 1 & 2 & 3 & 0 \\ 2 & 3 & 4 & 0 \\ 3 & 4 & 5 & 0 \\ 0 & 0 & 0 & 6 \end{bmatrix}$$

is block diagonal with blocks of size $3, 1$. Matrix multiplication of block diagonal matrices is again block diagonal, if the matrix blocks are of the same size. That is, if

$$A_1 = \begin{bmatrix} B_1 & 0 \\ 0 & C_1 \end{bmatrix}, \quad A_2 = \begin{bmatrix} B_2 & 0 \\ 0 & C_2 \end{bmatrix}$$

the

$$A_1A_2 = \begin{bmatrix} B_1B_2 & 0 \\ 0 & C_1C_2 \end{bmatrix}, \quad .$$

2.4. **Matrix form of a linear system.** Using matrix-vector products we can express any linear system in the matrix form $A\mathbf{x} = \mathbf{b}$, where $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$ is the vector of unknowns and $\mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$ is the vector of constants on the right-hand-side of the system. As we already mentioned, the matrix form of the pancake-waffle system

$$\begin{aligned} 2p &+ 3/2w &= 10 \\ p &+ 2w &= 10 \end{aligned}$$

is

$$\begin{bmatrix} 2 & 3/2 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} p \\ w \end{bmatrix} = \begin{bmatrix} 10 \\ 10 \end{bmatrix}.$$

or

$$A\mathbf{x} = \mathbf{b}, \quad A = \begin{bmatrix} 2 & 3/2 \\ 1 & 2 \\ 2 & 2 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} 10 \\ 10 \end{bmatrix}.$$

2.5. **Problems.**

(1) Let $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 0 & -1 \\ -1 & 1 & 0 \end{bmatrix}$. (a) What is the 23 component of A? (b) What is the 13 component of B? (c) What is $A+B$? (d) What is $A-B$? (e) What is $2A-B$?

(2) Give an example of an (a) 3×2 matrix (b) 2×3 matrix (c) 1×3 matrix (d) 3×1 matrix.

(3) Let $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 0 & -1 \\ -1 & 1 & 0 \end{bmatrix}$. In each case, if defined find the products of A and B with the given vectors, or explain why the product is not defined. (a) $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ (b) $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ (c) $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$

(4) Let $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 0 \\ -1 & -1 \\ 1 & 0 \end{bmatrix}$, and $C = \begin{bmatrix} 1 & 0 \\ -1 & -1 \end{bmatrix}$. In each case, find the matrix-matrix product if defined, or explain why the product is not defined. (a) AB (b) BA (c) AA (d) AC (e) CC.

(5) Let $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \end{bmatrix}$, $B = \begin{bmatrix} -1 & -1 \\ 1 & 0 \end{bmatrix}$, and $C = \begin{bmatrix} 1 & 0 \\ -1 & -1 \end{bmatrix}$. Compute the expressions (a) AB (b) AC, (c) $B+C$, (d) $A(B+C)$. (e) Verify that $AB + AC = A(B+C)$. (f) Does $AB + BB$ also equal $(A+B)B$? Why or why not?

(6) True or False? If true, explain; If false, give a counterexample.
(a) If A and B are symmetric, then $A+B$ is symmetric.
(b) If A and B are symmetric matrices, then AB is also symmetric.
(c) If A is square and $A^2 = 0$, then $A = 0$.
(d) If $AA^T = 0$, then $A = 0$.

(7) Let $A = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & 1 & -1 & 2 \\ 1 & 2 & 0 & 1 \\ 2 & 1 & 0 & 3 \end{bmatrix}$.
(a) Compute AB. (b) Compute A^T and B^T.

(8) Compute the matrix product A^2 (A times A) for

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

(9) A matrix A is symmetric iff $A^T = A$. Write the condition for the following matrices to be symmetric: (a) B (c) $B + C$ (d) B^2. (Answer to (a): $B = B^T$.)

(10) Construct a permutation matrix A not equal to I, whose square A^2 is I.

(11) Find the matrix form for the extended pancake-waffle system

$$\begin{array}{rcrcl} 2p & + & 3/2w & = & 10 \\ p & + & 2w & = & 10 \\ 2p & + & 2w & = & 10\,\frac{1}{4}. \end{array} \tag{2}$$

(12) Show that if P is a permutation matrix, then so is P^T.

(13) Prove that $(A^T)^T = A$. Hint: Two matrices are equal iff all their entries are equal. What is the ij-the entry of $(A^T)^T$?

3. Elimination

Elimination is the procedure by which we try to solve a system of linear equations by subtracting multiples of the equations from each other to eliminate the unknowns. When we do these operations in matrix form, they are called row operations.

3.1. **Row operations.**

To explain row operations, we recall the following example.

Example 3.1. To solve the pancake/waffle system we performed the following steps. Subtract twice the second equation from the first in

$$\begin{array}{rcrcl} 2p & + & (3/2)w & = & 10 \\ 1p & + & 2w & = & 10 \;. \\ 2p & + & 2w & = & 12. \end{array}$$

to get

$$\begin{array}{rcrcl} & & -5/2w & = & -10 \\ p & + & 2w & = & 10 \;. \\ 2p & + & 2w & = & 12 \;. \end{array}$$

Multiply the first equation by $-2/5$ to get

$$\begin{array}{rcrcl} & & w & = & 4 \\ p & + & 2w & = & 10 \\ 2p & + & 2w & = & 12 \;. \end{array}$$

Now substitute $w = 4$ into the first and third, and solve for p to obtain $p = 2$.

We can do the same steps in matrix form, using a little less ink. The *augmented matrix* for the ice-cream system (1) is

$$\left[\begin{array}{cc|c} 2 & 3/2 & 10 \\ 1 & 2 & 10 \\ 2 & 2 & 12\;. \end{array}\right].$$

Subtract twice row two from row one to get

$$\left[\begin{array}{cc|c} 0 & -5/2 & -10 \\ 1 & 2 & 10 \\ 2 & 2 & 12\;. \end{array}\right].$$

Multiply the first equation by $-2/5$ to get

$$\left[\begin{array}{cc|c} 0 & 1 & 4 \\ 1 & 2 & 10 \\ 2 & 2 & 12\;. \end{array}\right].$$

The first line means that $0p + 1w = 4$, hence $w = 4$. Substituting into the second and third equations gives $p = 2$, as before.

Example 3.2. Suppose we want to solve the system of three equations with three unknowns

$$\left[\begin{array}{rcrcrcl} x & - & y & & & = & 1 \\ & & y & - & z & = & 2 \\ -x & & & + & z & = & 3 \end{array}\right].$$

The matrix form of the system is

$$\left[\begin{array}{rrr|r} 1 & -1 & 0 & 1 \\ 0 & 1 & -1 & 2 \\ -1 & 0 & 1 & 3 \end{array}\right]$$

In the first step we add equation to equation 1 to equation 3 to get (also written in matrix form on the right)

$$\left[\begin{array}{ccccccc} x & - & y & & & = & 1 \\ & & y & - & z & = & 2 \\ & - & y & + & z & = & 4 \end{array}\right] \qquad \left[\begin{array}{rrr|r} 1 & -1 & 0 & 1 \\ 0 & 1 & -1 & 2 \\ 0 & -1 & 1 & 4 \end{array}\right].$$

Then we add equations 2 and 3 to get

$$\left[\begin{array}{ccccccc} x & - & y & & & = & 1 \\ & & y & - & z & = & 2 \\ & & & & 0 & = & 6 \end{array}\right] \qquad \left[\begin{array}{rrr|r} 1 & -1 & 0 & 1 \\ 0 & 1 & -1 & 2 \\ 0 & 0 & 0 & 6 \end{array}\right].$$

The last equation $0 = 6$ is a contradiction (obviously wrong). This means that the system has no solutions.

Definition 3.3. The three *row operations* used in elimination:

(1) Add a multiple of one row (equation) to another.
(2) Multiply a row (equation) by a non-zero number.
(3) Switch two rows (equations).

Each of these operations does not change the solution set. The goal of elimination is to use these operations repeatedly until the system is solved.

Example 3.4. Let's solve the system corresponding to the augmented matrix

$$\left[\begin{array}{ccc|c} 1 & 2 & 3 & 1 \\ 2 & 4 & 6 & 2 \\ 3 & 6 & 8 & 1 \end{array}\right].$$

We subtract 2 times row 1 from row 2, and 3 times row 1 from row 3 to get

$$\left[\begin{array}{rrr|r} 1 & 2 & 3 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & -2 \end{array}\right].$$

Now we switch rows 2 and 3 and multiply by -1 to get

$$\left[\begin{array}{ccc|c} 1 & 2 & 3 & 1 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{array}\right].$$

The equations are

$$\begin{array}{ccccccc} x & + & 2y & + & 3z & = & 1 \\ & & & & z & = & 2 \\ & & & & 0 & = & 0 \end{array}.$$

Substituting $z = 2$ into the first equation gives $x + 2y = -5$ or $x = -5 - 2y$. The solution set to this system is therefore

$$\left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix}, \quad z = 2 \text{ and } x = -5 - 2y \right\}.$$

In other words, the solution set is

$$\left\{ \begin{bmatrix} -5 - 2y \\ y \\ 2 \end{bmatrix} \right\}.$$

Instead of back substitution, we can do further row operations to get the matrix into reduced row echelon form. For instance, subtracting three times the second row from the first in

$$\left[\begin{array}{ccc|c} 1 & 2 & 3 & 1 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{array}\right].$$

gives

$$\left[\begin{array}{ccc|r} 1 & 2 & 0 & -5 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{array}\right].$$

The equations are

$$x + 2y = -5 \qquad z = 2$$

so the solution set is

$$\left\{ \begin{bmatrix} -5 - 2y \\ y \\ 2 \end{bmatrix} \right\}.$$

as before.

It will be helpful to introduce short-hand for the row operations. Here are some examples:

$$2 \mapsto 2 - 2\,1 \qquad \text{subtract 2 times row 1 from row 2.}$$

$$3 \mapsto 3 - 3\,1 \qquad \text{subtract 3 times row 1 from row 3.}$$

$$2 \leftrightarrow 3 \qquad \text{switch rows 2 and 3}$$

$$2 \mapsto -1\,2 \qquad \text{multiply row 2 by } -1.$$

Note that the row numbers are always circled.

3.2. **Row-echelon form and the general solution to a linear system.** Elimination can stop when the matrix is in *row-echelon form*:

Definition 3.5. A matrix is in *row-echelon form* (or *ref* for short) iff

(1) all rows of zeroes are at the bottom.
(2) The first non-zero entry in any row is a 1, called a *leading* 1 or *pivot*.
(3) The leading 1 in any row is to the right of the leading 1's above it.

If, in addition, the entries above any leading 1 are zero, the matrix is said to be in *reduced row-echelon form* (or *rref* for short).

For instance, the matrix

$$\left[\begin{array}{ccc|c} 1 & 2 & 3 & 1 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{array}\right].$$

is in row echelon form. The matrix

$$\left[\begin{array}{ccc|r} 1 & 2 & 0 & -5 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

is in reduced row-echelon form. Since the equations are

$$x + 2y = -5 \qquad z = 2$$

This system has an infinite number of solutions, one for each value of y. The variables x, z that correspond to columns containing leading 1's are called the *bound variables*; the other variables are called *free variables*. The bound variables can be expressed in terms of the free variables, using the equations corresponding to the rows of the row-echelon form system and back substitution.

The three examples we have done so far show the three possible outcomes of elimination:

Theorem 3.6. (Number of solutions to a system of linear equations)

(1) *There is a unique solution if the system is consistent and there is a leading* 1 *in every column (to the left of the bar) in the rref.*
(2) *There are infinitely many solutions if the system is consistent and there are some columns without leading* 1*'s in the rref.*
(3) *The system is inconsistent if there is a row of zeroes, and a non-zero number to the right of the bar in the rref. In this case there are no solutions.*

Let's do another example with an infinite number of solutions.

Example 3.7. We solve the system

$$\begin{array}{ccccccccc} 2x & + & 6y & - & z & + & 2w & = & 6 \\ x & + & 3y & + & z & + & 10w & = & 15 \end{array}.$$

The augmented matrix for this system is

$$\left[\begin{array}{cccc|c} 2 & 6 & -1 & 2 & 6 \\ 1 & 3 & 1 & 10 & 15 \end{array}\right].$$

Switch the first and second rows to get a leading one in the first column:

$$① \leftrightarrow ② \quad \left[\begin{array}{cccc|c} 1 & 3 & 1 & 10 & 15 \\ 2 & 6 & -1 & 2 & 6 \end{array}\right].$$

Subtract twice the first row from the second to get

$$② - 2① \quad \left[\begin{array}{cccc|c} 1 & 3 & 1 & 10 & 15 \\ 0 & 0 & -3 & -18 & -24 \end{array}\right].$$

Multiply the second row by $-1/3$ to create a leading one in the second row.

$$-(1/3)② \quad \left[\begin{array}{cccc|c} 1 & 3 & 1 & 10 & 15 \\ 0 & 0 & 1 & 6 & 8 \end{array}\right].$$

The matrix is in row-echelon form. At this point we could write out the equations and do back-substitution to find the answer. Instead, we keep going to reduced row-echelon form. Subtract the second row from the first to get

$$① - ② \quad \left[\begin{array}{cccc|c} 1 & 3 & 0 & 4 & 7 \\ 0 & 0 & 1 & 6 & 8 \end{array}\right].$$

Now this is reduced row-echelon form. The equations are

$$\begin{array}{ccccccc} x & + & 3y & & + & 4w & = & 7 \\ & & & z & + & 6w & = & 8 \end{array}.$$

The leading 1's are in the first and third rows. So the bound variables are x and z; the free variables are y and w. Write the bound variables in terms of the free variables

$$x = 7 - 3y - 4w, \quad z = 8 - 6w.$$

The solution set to the system is all vectors satisfying these equations. Since y and w can be anything, the solution set can be written

$$\left\{ \left[\begin{array}{r} 7 - 3y - 4w \\ y \\ 8 - 6w \\ w \end{array}\right] \right\}.$$

Here we have substituted the formulas for the bound variables. Since there are free variables, there are an infinite number of solutions.

The fact that there are always $0, 1$ or infinitely many solutions is a special feature of linear systems of equations. Non-linear equations, can have other numbers of solutions, for example. For example $x^2 = 1$, has two solutions $x = \pm 1$.

Note 3.8. (1) It's not possible to read off the number of solutions from the number of unknowns and the number of equations, without doing the elimination.
(2) If the number of unknowns is greater than the number of equations, some of the unknowns must be free. So there are always infinite or no solutions.
(3) The system is called homogeneous if the numbers to the right of the equals signs/bar are all zero. Since these systems are always consistent, the number of solutions is infinity or one.

3.3. **Geometric interpretation of consistency.** Using what we said about matrix vector products, the following is the geometric interpretation for consistency:

Theorem: A linear system $A\mathbf{x} = \mathbf{b}$ is consistent iff $\mathbf{b}$ is a linear combination of the columns of A iff $\mathbf{b}$ is in the span of the columns of A iff the system $[A|\mathbf{b}]$ has reduced row echelon form which has no row of the form $[0|1]$.

Example 3.9. Determine whether $\left[\begin{array}{cc|c} 1 & 0 & 1 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{array}\right]$ is consistent. Answer: the span of the columns of the matrix on the left is the plane $x+y+z=0$. The vector $\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$ is not in this plane, so the system is inconsistent. It's easy to check this by elimination: the row echelon form has a row of the form $[0\ 0\ 2]$ so the system is inconsistent.

Example 3.10. Is $\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ in the span of the vectors

$$\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 3 \end{bmatrix}?$$

Answer: Note that the span of these vectors could by all of $\mathbb{R}^3$, or a plane, or a line, or a point. We solve the system

$$\left[\begin{array}{ccc|c} 1 & -1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 1 & 3 & 1 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & -1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 2 & 2 & 1 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & -1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right]$$

which is inconsistent. So the given vector is not in the span.

In fact, it's not hard to solve this problem without computing anything. If we denote these three vectors by $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3$, then $\mathbf{u}_3 = 2\mathbf{u}_1 + \mathbf{u}_2$. So the span of the vectors $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3$ is the same as that of $\mathbf{u}_1, \mathbf{u}_2$ which is a plane. It's not hard to see it is the plane $x + 2y = z$. Since $\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ is not in this plane, it is not in the span.

3.4. **Application to polynomial interpolation.** There is a unique line passing through the points $(1, 1)$ and $(-1, 1)$. How many degree two polynomials are there passing through these points? [1] Although this problem seems non-linear (a parabola is a graph of a quadratic function) in fact this is a system of linear equations. Suppose that the function is

$$f(x) = ax^2 + bx + c.$$

The unknowns here are the values of a, b, c, since we are solving for the parabola, not x. Each point gives us an equation for a, b, c:

$$a(1)^2 + b(1) + c = 1$$

$$a(-1)^2 + b(-1) + c = 1.$$

The matrix form of this system is

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & 1 \end{array}\right].$$

Since the number of unknowns is greater than the number of equations, from Theorem 3.8 we know that there are either zero or infinite solutions. To figure out which, we have to do elimination. There is already a leading 1 in the first row, so there's nothing to do there. We subtract the first row from the second to get

$$② - ① \quad \left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & -2 & 0 & 0 \end{array}\right].$$

[1] If you want, imagine you have collected some experimental data and know for some reason that the quantities you are measuring are related by a degree two function.

Divide the second row by -2 to get a leading 1:

$$②/(-2)\quad \left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 \end{array}\right].$$

This matrix is now in row-echelon form. To get reduced row-echelon form, subtract the second row from the first:

$$① - ②\quad \left[\begin{array}{ccc|c} 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 \end{array}\right].$$

The equations are

$$a + c = 1, \quad b = 0.$$

The bound variables are a, b; the free variable is c. Expressing the bound variables in terms of the free variables gives

$$a = 1 - c, \quad b = 0.$$

The solution vectors are

$$\begin{bmatrix} 1-c \\ 0 \\ c \end{bmatrix}.$$

The solution functions are

$$f(x) = (1-c)x^2 + 0x + c = (1-c)x^2 + c.$$

It's easy to check that these all satisfy $f(\pm 1) = 1$. Since there is one solution for each value of c, there are infinite solutions.

Example 3.11. It's easy to come up with a similar example which has *no* solutions: If the data points are $(1,1)$ and $(1,-1)$, there is no function with these values because any function takes only one value at any value of x. If we try to solve this system, we get the matrix form

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & -1 \end{array}\right]$$

which has rref (reduced row-echelon form)

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & -2 \end{array}\right].$$

This is inconsistent.

Example 3.12. Find all polynomials of degree 2 passing through the points $(-1,1), (0,2), (1,1)$.

Pluggin in the data points into the polynomial $ax^2 + bx + c$ gives the system of equations

$$a(-1)^2 + b(-1) + c = 1$$
$$a(0)^2 + b(0) + c = 2$$
$$a(-1)^2 + b(1) + c = 1.$$

This system has augmented matrix

$$\left[\begin{array}{ccc|c} 1 & -1 & 1 & 1 \\ 0 & 0 & 1 & 2 \\ 1 & 1 & 1 & 1 \end{array}\right].$$

The matrix has rref equal to

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 2 \end{array}\right].$$

The equations are

$$a = -1, \quad b = 0, \quad c = 2.$$

There is a unique solution, $\boxed{f(x) = -x^2 + 2}$.

Here is the general result, which we will prove later using determinants:

Theorem 3.13. *Given n points $(x_1, y_1), \ldots, (x_n, y_n)$, with $x_1, \ldots, x_n$ distinct, there is a unique polynomial of degree $n+1$ passing through them. If $d > n+1$, there are infinitely many polynomials passing through these points.*

3.5. **Gauss-Jordan elimination.** Elimination can be done in many different ways. Gauss observed in the 1800s that there is a systematic way of doing elimination. Doing elimination by hand, Gauss' method may not be the best way, for instance, it may involve fractions when a different choice of operations might avoid them.

Gauss' algorithm depends on the following observation: if there is a non-zero entry in a column in a matrix, then that entry can be used to make all the other entries in that column zero, by subtracting multiples of that row from the other rows. We followed Gauss' method when we found the rref of the matrix

$$\begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 3 & 6 & 8 \end{bmatrix}.$$

The steps (in the short-hand we used above) were

$$\begin{array}{c} \\ ② - 2① \\ ③ - 3① \end{array} \begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix} \mapsto \; ② \leftrightarrow ③ \begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{bmatrix} \mapsto \; -② \begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}.$$

This is called the forward pass. The backward pass has just one step:

$$\textcircled{1} - 3\textcircled{2} \quad \left[\begin{array}{ccc|c} 1 & 2 & 0 & -5 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{array}\right].$$

Gauss's algorithm is the following.

Definition 3.14. The *forward pass* in Gauss-Jordan elimination consists of the following steps:

(1) Find the first column containing a non-zero value.
(2) Find the first entry in that column that is non-zero. The place of this entry is a *pivot*.
(3) Switch rows so the pivot is in the first row.
(4) Make the first row have a *leading one*, by dividing the first row by the value in the pivot,
(5) For each non-zero entry below the pivot, multiply the first row by its value, and subtract it to make the entry zero.
(6) Repeat steps (b) - (e) for the next column containing a non-zero value, ignoring rows that already have leading ones.

After the forward pass, the matrix is in row-echelon form. The *backward pass* in Gauss-Jordan elimination consists of the following steps:

(1) Find the last non-zero row, that is, the last row containing a pivot.
(2) For each non-zero entry above the leading one, subtract the value of that entry times the pivot row to make that entry zero.
(3) Repeat step (b) for the row above.

After the backward pass, the matrix is in reduced row-echelon form.

The phrase *Gaussian elimination* is another term for the forward pass in Gauss-Jordan elimination (or alternatively, Gauss-Jordan elimination extends Gaussian elimination by adding the backward pass.)

3.6. **Elementary matrices.** Each row operation can be represented by multiplying by an elementary matrix on the left. The elementary matrix corresponding to the row operation is the result of performing the row operation on the identity matrix.

Example 3.15. The elementary matrix corresponding to $\textcircled{3} \mapsto \textcircled{3} - 3\textcircled{1}$ is

$$\textcircled{3} \mapsto \textcircled{3} - 3\textcircled{1} \quad \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \mapsto \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -3 & 0 & 1 \end{bmatrix}.$$

The elementary matrix corresponding to $\textcircled{1} \leftrightarrow \textcircled{2}$ is

$$\textcircled{1} \leftrightarrow \textcircled{2} \quad \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \mapsto \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

The elementary matrix corresponding to $\textcircled{2} \mapsto -\textcircled{2}$ is

$$\textcircled{2} \mapsto -\textcircled{2} \quad \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \mapsto \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}.$$

Theorem 3.16. *Doing a row operation on a matrix gives the same result as multiplying on the left by the corresponding elementary matrix.*

Example 3.17.

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -3 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & 0 \\ 3 & 6 & 8 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix}.$$

$$\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{bmatrix}.$$

3.7. *LU* **factorization.** In this section we discuss an equivalent formulation of Gaussian elimination in terms of matrix multiplication. This formulation is used in linear algebra software packages such as MATLAB.

To explain the reformulation, note that the row-echelon form of a matrix is always upper triangular, since the leading 1 in each row is to the right of the leading 1 above it. In this section, we call the row-echelon form U, since it is upper triangular. Suppose that A can be put into row-echelon form U by a sequence of row operations correspondiqng to elementary matrices $E_1, \dots, E_k$. Doing the row operation is the same as multiplying by the elementary matrix, by Theorem 3.16. So U is the matrix product

$$U = E_k E_{k-1} \dots E_1 A.$$

Example 3.18. Gaussian elimination on $A = \begin{bmatrix} 2 & 4 & 6 \\ 3 & 6 & 9 \\ 3 & 6 & 11 \end{bmatrix}$ is

$$\begin{bmatrix} \frac{1}{2}\textcircled{1} \\ -3\textcircled{1} \end{bmatrix} \begin{bmatrix} 2 & 4 & 6 \\ 3 & 6 & 9 \\ 3 & 6 & 11 \end{bmatrix} \to \quad \textcircled{2} \leftrightarrow \textcircled{3} \quad \begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

$$\to \quad \textcircled{2}/2 \quad \begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}.$$

The elementary matrices corresponding to these operations are

$$E_1 = \begin{bmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad E_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}, \quad E_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1/2 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Therefore,

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1/2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 4 & 6 \\ 3 & 6 & 9 \\ 3 & 6 & 11 \end{bmatrix}.$$

Check it for yourself.

A *factorization* of a matrix A is a formula for A as a product of matrices. For example,

$$\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$$

is a factorization of $\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$

We can take the inverses of the elementary matrices to get a factorization for A:

(3) $$A = E_1^{-1} E_2^{-1} \dots E_k^{-1} U.$$

Example 3.19. A factorization for $\begin{bmatrix} 2 & 4 & 6 \\ 3 & 6 & 9 \\ 3 & 6 & 11 \end{bmatrix}$ is

$$\begin{bmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}^{-1} \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}.$$

Taking the inverses of these matrices using Proposition 4.8 we get

$$\begin{bmatrix} 2 & 4 & 6 \\ 3 & 6 & 9 \\ 3 & 6 & 11 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}.$$

In general, it's not that easy to take the product of the matrices $E_1^{-1}, \dots, E_k^{-1}$.

However, suppose that *none of the operations are row switches*. The row operations in the forward pass subtract rows from lower rows or multiply rows, so the elementary matrices $E_1, \dots, E_k$ in the forward pass, as well as their inverses, are lower triangular. Since the product of lower triangular matrices is lower triangular, the product L defined by

$$L = E_1^{-1} E_2^{-1} \dots E_k^{-1}$$

is also lower triangular. From this and (3) we get

$$A = LU.$$

This is called an LU factorization of A.

Proposition 3.20. *If the row operations are in the order given by Gauss's algorithm, then each entry in L is the unique non-zero entry in $E_1^{-1} \dots E_k^{-1}$.*

Example 3.21. The row operations used to put $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 4 \\ 3 & 6 & 9 \end{bmatrix}$ into reduced row-echelon form

$$\mathrm{ref}(A) = U = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

are order)

$$② \mapsto ② - 2①, \quad ③ \mapsto ③ - 3①, \quad ② \mapsto - ② /2.$$

The corresponding elementary matrices are

$$E_1 = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad E_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -3 & 0 & 1 \end{bmatrix}, \quad E_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1/2 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Their inverses are

$$E_1^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad E_2^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 3 & 0 & 1 \end{bmatrix}, \quad E_3^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Their product is

$$L = E_1^{-1} E_2^{-1} E_3^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & -2 & 0 \\ 3 & 0 & 1 \end{bmatrix}.$$

Let's summarize the discussion in a theorem:

Theorem 3.22. *If Gaussian elimination on A does not involve any row switches, then there is an LU factorization of A where U is the result of the forward pass in elimination. If elimination involves subtracting a scalar c_{ij} times row i from row j, and dividing row i by a scalar c_{ii}, then L is the lower triangular matrix whose ij-th entry is c_{ij}.*

This simple way of getting L only works if the elementary matrices are in order. For instance the product

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 3 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -4 & -2 & 0 \\ 3 & 0 & 1 \end{bmatrix}$$

which is not equal to L. So in order to get the LU factorization, you have to do the operations in the order that in Gauss' algorithm.

On the other hand, if you are only trying to find the row echelon form by hand, it might be easier to use a different order of operations, for instance in order to avoid fractions.

More generally, if Gaussian elimination on a matrix A involves row switches then all of the row switches can be done after the other row operations. This gives a factorization

$$A = LPU$$

where P is the product of elementary matrices corresponding to the row switches.

Example 3.23. Gaussian elimination on $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 3 & 6 & 8 \end{bmatrix}$ with all the row switches at the end is

$$\begin{matrix} \\ ② - 2① \\ ③ - 3① \end{matrix}\begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix} \mapsto \begin{matrix} \\ \\ -③ \end{matrix}\begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix} \mapsto ② \leftrightarrow ③ \begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{bmatrix}.$$

The upper triangular matrix U is

$$U = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{bmatrix}.$$

The elementary matrices are

$$E_1 = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \ E_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -3 & 0 & 1 \end{bmatrix},$$

$$E_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}, \ E_4 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}.$$

Their inverses are

$$E_1^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \ E_2^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 3 & 0 & 1 \end{bmatrix},$$

$$E_3^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}, \ E_4^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}.$$

To get L we take the product of the matrices corresponding to adding and multiplying row operations:

$$L = E_1^{-1}E_2^{-1}E_3^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & 0 & -1 \end{bmatrix}.$$

To get the P matrix we take the product of the matrices corresponding to the switches. In this case there is only one:

$$P = E_4^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}.$$

The LPU factorization of A is

$$\begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 3 & 6 & 8 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & 0 & -1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}\begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{bmatrix}.$$

3.8. **Problems.**

(1) Write the following systems in matrix form. (a) $a+b=3, a+2b=5$. (b) $x+y=3, x-y=0$. (c) $x_1+x_3=5, x_1-x_2=2$. (d) $x_1+x_3=5, x_3-x_1=5$. (e) $c=0, a+b=1$. (f) $a+b=5, b=2, b=3$.

(2) Write the equations represented by the following matrices, assuming variables x_1, x_2 etc. (a) $\left[\begin{array}{cc|c} 1 & 0 & 1 \\ 0 & 1 & 2 \end{array}\right]$ (b) $\left[\begin{array}{cc|c} 0 & 1 & 1 \\ 1 & 0 & 2 \end{array}\right]$ (c) $\left[\begin{array}{cc|c} 0 & 0 & 1 \\ 1 & 0 & 2 \end{array}\right]$ (d) $\left[\begin{array}{cc|c} 1 & 0 & 1 \\ 0 & 0 & 0 \end{array}\right]$

(3) Find all solutions (if any) to each system by elimination. (You may use matrix form if you wish.) (a) $a+b=3, a+2b=5$. (b) $x+y=3, x-y=0$. (c) $x_1+x_3=5, x_1-x_2=2$. (d) $x_1+x_3=5, x_3-x_1=5$. (e) $c=0, a+b=1$. (f) $a+b=5, b=2, b=3$.

(4) Identify which of the following matrices are in (i) row-echelon form (ii) reduced row-echelon form.

(a) $\begin{bmatrix} 1 & 0 & 1 \\ 0 & 2 & 2 \end{bmatrix}$

(b) $\begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 2 \end{bmatrix}$

(c) $\begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 2 \end{bmatrix}$

(d) $\begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$

(e) $\begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$

(f) $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

(g) $\begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 2 \end{bmatrix}$

(h) $\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 2 \end{bmatrix}$

(i) $\begin{bmatrix} 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$

(5) True or False? Explain your answer.
(a) If the number of columns of A is greater than the number of rows, then $Ax = b$ cannot have a unique solution.
(b) If A has more rows than columns, then the number of solutions of $Ax = b$ is either 0 or 1.
(c) If A has the same number of rows as columns, then any linear system of equations $Ax = b$ has a unique solution.
(d) If A has more columns than rows, then the number of solutions of $Ax = b$ is either 0 or ∞.

(6) Find all solutions to the systems below by finding the reduced row-echelon form (rref) for the associated matrix. Circle the pivots in the rref.
(a) $x_1 + x_2 - 2x_3 + x_4 = 6, 2x_1 - x_2 + x_3 - 3x_4 = 0$
(b) $x_2 - 2x_3 + x_4 = 6, 2x_1 - x_2 + x_3 - 3x_4 = 0$
(c) $x_2 + x_4 = 6, 2x_1 - x_2 - 3x_4 = 0$
(d) $2x_2 + 2x_4 = 6, x_2 + x_4 = 3$
(e) $x_1 + x_2 - 2x_3 + x_4 = 6, 2x_1 - x_2 + x_3 - 3x_4 = 0, x_1 + x_4 = 1$
(f) $x_1 - x_2 + x_3 + 2x_4 = 3, 2x_1 - 2x_2 + 4x_3 + 2x_4 = 6, x_1 - x_2 + 3x_3 = 3$.

(7) (a) a row echelon form and (b) the reduced row echelon form and (c) an LU factorization for the matrix

$$\begin{bmatrix} 1 & -1 & 0 & -8 & -3 \\ -2 & 1 & 0 & 9 & 5 \\ 3 & -3 & 0 & -2 & -11 \end{bmatrix}$$

(8) Using elimination find (a) a row echelon form (b) the reduced row echelon form

$$A = \begin{bmatrix} 1 & 3 & -1 & 1 & 0 \\ 2 & 6 & -1 & 4 & 15 \\ -1 & -3 & 2 & -1 & 15 \end{bmatrix}.$$

Find all solutions to the system whose augmented matrix is A, that is, $\begin{matrix} x_1 + 3x_2 - x_3 + x_4 = 0 \\ 2x_1 + 6x_2 - x_3 + x_4 = 15 \\ -x_1 + -3x_2 + 2x_3 - x_4 = 15 \end{matrix}$.

(9) Using elimination find (a) a row echelon form (b) the reduced row echelon form for the augmented matrix

$$\left[\begin{array}{cccc|c} 2 & 0 & 1 & 2 & -1 \\ 6 & 0 & 3 & 7 & -3 \\ 4 & 0 & 2 & 5 & -2 \end{array}\right].$$

(10) True or False? Explain your answer.
(a) The number of columns containing pivots in rref(A) is the same as the number of rows containing pivots.
(b)

(11) (a) Let $f(t)$ be a function of the form $f(t) = c_0 + c_2t^2 + c_4t^4$. Write the system of equations $f(-1) = 0,\ f(0) = 1,\ f(1) = 0$ in matrix form $Ax = b$.
(b) Using row reduction, find all solutions to this system (i.e. function satisfying these conditions.)
(c) Find an LU factorization for A.

(12) Write a 3 by 3 matrix that
(a) subtracts -5 times row 1 from row 3
(b) exchanges rows 1 and 3
(c) divides row 2 by -2.

(13) Find the unique quadratic (degree 2) polynomial passing through the points $(0, 1), (1, 2), (-1, 3)$.

(14) Find an LU factorization for $A = \begin{bmatrix} 1 & 2 \\ 3 & 2 \end{bmatrix}$.

4. Matrix inverses

We showed above how to multiply matrices. The order of matrix multiplication matters, that is, matrix multiplication is not commutative. In this section we discuss *inverses* of matrices. Every invertible matrix is square, but not every square matrix is invertible. Matrix division can be defined by multiplying by matrix inverses, but the rules for matrix division are more restrictive than for division of numbers.

4.1. **Another example of matrix products.** Here is a second simple example of a linear algebra problem. Suppose that two basketball teams play in the same area, say Rutgers and Seton Hall, and compete for fans. Suppose that each year 20 percent of Seton Hall fans switch to Rutgers, and 10 percent of Rutgers fans switch to Seton Hall, while the total number of fans stays the same. Suppose that each team has ten thousand fans this year.

How many fans will there be with each team next year? It is not really necessary to use linear algebra to solve this problem. Ten percent of ten thousand is one thousand, so one thousand fans switch to Seton Hall while two thousand of Seton Hall's fans switch to Rutgers. Overall Rutgers gains one thousand fans while Seton Hall loses one thousand fans, so that the total next year is eleven thousand for Rutres and nine thousand for Seton Hall.

For other questions, such as how many fans will be with each team many years in the future, or some years in the past, linear algebra is helpful. Here is the linear algebra solution to the problem of how many there will be next year. If we let $r(t)$ and $s(t)$ be the number of Rutgers and Seton Hall fans, then

$$r(t+1) = r(t) - .1r(t) + .2s(t) = .9r(t) + .2s(t)$$

$$s(t+1) = s(t) + .1r(t) - .2s(t) = .1r(t) + .8s(t)$$

or in matrix form

$$\begin{bmatrix} r(t+1) \\ s(t+1) \end{bmatrix} = \begin{bmatrix} .9 & .2 \\ .1 & .8 \end{bmatrix} \begin{bmatrix} r(t) \\ s(t) \end{bmatrix}.$$

If the current year $t = 0$ has ten thousand of each then we do matrix multiplication to find out the number in the next year

$$\begin{bmatrix} r(1) \\ s(1) \end{bmatrix} = \begin{bmatrix} .9 & .2 \\ .1 & .8 \end{bmatrix} \begin{bmatrix} 10,000 \\ 10,000 \end{bmatrix} = \begin{bmatrix} 11,000 \\ 9,000 \end{bmatrix} = .$$

Matrix multiplication is a convenient way to go forward any number of years forward in time, for example, to forward three years in time we compute

$$\begin{bmatrix} r(3) \\ s(3) \end{bmatrix} = \begin{bmatrix} .9 & .2 \\ .1 & .8 \end{bmatrix}^3 \begin{bmatrix} 10,000 \\ 10,000 \end{bmatrix}$$

$$= \begin{bmatrix} .9 & .2 \\ .1 & .8 \end{bmatrix}^2 \begin{bmatrix} 11,000 \\ 9,000 \end{bmatrix} = \begin{bmatrix} .9 & .2 \\ .1 & .8 \end{bmatrix} \begin{bmatrix} 11,700 \\ 8,300 \end{bmatrix} = \begin{bmatrix} 12,190 \\ 7,810 \end{bmatrix}.$$

Now consider the following question. Suppose that each team has ten thousand fans this year. How many fans did each team have *last* year? This is the kind of question you can solve using elimination. But another way of solving this problem is to find the *inverse* of the matrix above and multiply the vector by the inverse matrix. That is, if multiplying by a matrix goes *forward* in time, then multiplying by the inverse matrix goes *backward*.

4.2. **The definition of the inverse.** We said before that $AD = AE$ does not imply $D = E$, even if A is non-zero. This is because it doesn't make any sense to "divide by A on both sides", for arbitrary matrices. The matrices for which it does make sense are called *invertible*.

Definition 4.1. A matrix A is

(1) *left invertible* if there is a matrix B such that $BA = I$
(2) *right invertible* if there is a matrix C such that $AC = I$
(3) *invertible* if it is both left and right invertible.

Note 4.2. (1) If A is both left and right invertible then the left and right inverses are equal:

$$C = IC = (BA)C = B(AC) = BI = B.$$

In this case the inverse is unique, by the same argument. The left/right inverse is called the *inverse* of A and denoted A^{-1}.

(2) If A is left invertible, then $AD = AE$ does imply that $D = E$, since we can multiply both sides by the left inverse B:

$$BAD = BAE \implies ID = IE \implies D = E.$$

(3) Similarly, for right invertible matrices $DA = EA$ implies $D = E$.

Example 4.3. (1) The matrix $A = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$ is invertible with inverse $A^{-1} = \begin{bmatrix} 1/2 & 0 \\ 0 & 1/3 \end{bmatrix}$. More generally, any diagonal matrix with diagonal entries $a_{11}, \ldots, a_{nn}$ is invertible with inverse the diagonal matrix with entries $1/a_{11}, \ldots, 1/a_{nn}$.

(2) The matrix $A = \begin{bmatrix} 1 & n \\ 0 & 1 \end{bmatrix}$ is invertible with inverse $A^{-1} = \begin{bmatrix} 1 & -n \\ 0 & 1 \end{bmatrix}$.

(3) The matrix $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & 1 \end{bmatrix}$ has right inverse $C = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \\ 0 & 0 \end{bmatrix}$.

(4) The inverse of the Rutgers-Seton Hall matrix is

$$\begin{bmatrix} .9 & .2 \\ .1 & .8 \end{bmatrix}^{-1} = (10/7) \begin{bmatrix} .8 & -.2 \\ -.1 & .9 \end{bmatrix}$$

We can check this by multiplying

$$\begin{bmatrix} .9 & .2 \\ .1 & .8 \end{bmatrix} (10/7) \begin{bmatrix} .8 & -.2 \\ -.1 & .9 \end{bmatrix} = .7(10/7)I$$

and the same for the other order. To answer the question in the first section about the number of fans in the previous year, we multiply

$$\begin{bmatrix} .9 & .2 \\ .1 & .8 \end{bmatrix}^{-1} \begin{bmatrix} 10,000 \\ 10,000 \end{bmatrix} = (10/7) \begin{bmatrix} .8 & -.2 \\ -.1 & .9 \end{bmatrix} \begin{bmatrix} 10,000 \\ 10,000 \end{bmatrix}$$

$$= (10/7) \begin{bmatrix} 6,000 \\ 8,000 \end{bmatrix} = \begin{bmatrix} 8571 \\ 11429 \end{bmatrix}.$$

Note we are rounding off to integers since there is no such thing as a "fractional fan". That is, the model is really only approximate since we cannot divide up a single fan into pieces. (Perhaps one could think of fractional fans as fans whose loyalty is divided.)

There is a simple formula for the inverse of a 2×2 matrix. The *determinant* of a 2×2 matrix is

$$\det \begin{bmatrix} a & b \\ c & d \end{bmatrix} = ad - bc.$$

The determinant is non-zero if and only if the matrix is invertible; the a formula for the inverse is (check!)

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}.$$

Later we'll generalize this formula to larger matrices.

If A is invertible then so are its square A^2 and its transpose:

$$(A^2)^{-1} = (A^{-1})^2$$

since $A^2(A^{-1})^2 = AAA^{-1}A^{-1} = AIA^{-1} = I$, and

$$(A^T)^{-1} = (A^{-1})^T.$$

Similarly, the inverse of A^n is $(A^{-1})^n$, for any $n > 0$. If A and B are invertible then

$$(AB)^{-1} = B^{-1}A^{-1}.$$

You can think of the reason for this in the following silly way. Suppose at the beginning of the day you put on your shoes (call this operation A) and tie your shoe laces (call this operation B). What do you do when you come home?

4.3. **Finding inverses via elimination.** The most efficient way of finding the inverse of a square matrix A is via elimination. Consider the vector equation $A\mathbf{x} = \mathbf{y}$. The inverse matrix A^{-1} solves the equation $\mathbf{x} = A^{-1}\mathbf{y}$. So if we can express $\mathbf{x}$ in terms of $\mathbf{y}$, we can read off the coefficients to get the matrix A^{-1}.

Writing out the equations for $A\mathbf{x} = \mathbf{y}$ gives

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n &= y_1 \\ a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n &= y_2 \\ &\vdots \\ a_{n1}x_1 + a_{n2}x_2 + \ldots + a_{nn}x_n &= y_n. \end{aligned}$$

We now have a system of linear equations with variables on the right-hand side. Reading off the coefficients we get the matrix form

$$\left[\begin{array}{cccc|cccc} a_{11} & a_{12} & \ldots & a_{1n} & 1 & 0 & \ldots & 0 \\ a_{21} & a_{22} & \ldots & a_{2n} & 0 & 1 & \ldots & 0 \\ & & & & & & \vdots & \\ a_{n1} & a_{n2} & \ldots & a_{nn} & 0 & 0 & \ldots & 1 \end{array}\right].$$

or, for short, $[A|I]$. To solve for $\mathbf{x}$, we do elimination. If, at the end, we get the identity matrix on the left-hand side, then the right-hand side is the inverse is the matrix on the right.

Example 4.4. To find the inverse of $A = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$ we do elimination on

$$\left[\begin{array}{cc|cc} 2 & 0 & 1 & 0 \\ 0 & 3 & 0 & 2 \end{array}\right].$$

We divide the first row by 2 and the second by 3 to get the rref

$$\left[\begin{array}{cc|cc} 1 & 0 & 1/2 & 0 \\ 0 & 1 & 0 & 1/3 \end{array}\right].$$

The inverse is $A^{-1} = \begin{bmatrix} 1/2 & 0 \\ 0 & 1/3 \end{bmatrix}$.

Example 4.5. Find the inverse of $A = \begin{bmatrix} 2 & 3 \\ 4 & 6 \end{bmatrix}$. We do elimination on

$$\left[\begin{array}{cc|cc} 2 & 3 & 1 & 0 \\ 4 & 6 & 0 & 1 \end{array}\right].$$

We subtract twice the first row from the second to get

$$\left[\begin{array}{cc|cc} 2 & 3 & 1 & 0 \\ 0 & 0 & -2 & 1 \end{array}\right].$$

The second equation is inconsistent: this matrix has no inverse.

Example 4.6. Find the inverse of $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 0 & 1 & 1 \end{bmatrix}$. We do elimination on the augmented matrix

$$\left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 2 & 3 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \end{array}\right].$$

We subtract the first row from the second to get

$$\left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 2 & -1 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \end{array}\right].$$

We subtract the second from the first and third to get

$$\left[\begin{array}{ccc|ccc} 1 & 0 & -1 & 2 & -1 & 0 \\ 0 & 1 & 2 & -1 & 1 & 0 \\ 0 & 0 & -1 & 1 & -1 & 1 \end{array}\right].$$

Multiply the third by -1 to create a leading 1:

$$\left[\begin{array}{ccc|ccc} 1 & 0 & -1 & 2 & -1 & 0 \\ 0 & 1 & 2 & -1 & 1 & 0 \\ 0 & 0 & 1 & -1 & 1 & -1 \end{array}\right].$$

Now add the third to the first, and subtract twice the third from the second to get

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & -1 \\ 0 & 1 & 0 & 1 & -1 & 2 \\ 0 & 0 & 1 & -1 & 1 & -1 \end{array}\right].$$

The inverse is

$$A^{-1} = \begin{bmatrix} 1 & 0 & -1 \\ 1 & -1 & 2 \\ -1 & 1 & -1 \end{bmatrix}.$$

Notice that if A is invertible, then $A\mathbf{x} = \mathbf{y}$ has a unique solution for every $\mathbf{y}$. This implies that $\operatorname{rref}(A) = I$, since (1) if there were a row of zeroes, the system would be inconsistent for some values of y, and (2) if column did not contain a leading 1, there would be free variables, so any solution would not be unique. So *any invertible matrix is automatically square*. Let's summarize what we've shown so far:

Theorem 4.7. *A matrix A is invertible if and only if A is square and $\operatorname{rref}(A) = I$. In this case, the inverse is the right hand side of the matrix $\operatorname{rref}([A|I])$.*

It's easy to check that a 2×2-matrix A has $\operatorname{rref}(A) = I$ if and only if the determinant is non-zero.

4.4. **Application: A formula for the line between two points.** There is a unique line through any two points (x_1, y_1), (x_2, y_2). Let's find a formula for it, using matrices. (Think for a moment about you would find a formula another way.) The equation for a line is $f(x) = ax + b$. The two data points give

$$ax_1 + b = y_1, \quad ax_2 + b = y_2$$

or in matrix form

$$\begin{bmatrix} x_1 & 1 \\ x_2 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}.$$

The solution is

$$\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} x_1 & 1 \\ x_2 & 1 \end{bmatrix}^{-1} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}.$$

Let's find the inverse using the formula for two by two inverses. The determinant is $x_1 - x_2$, so the inverse is

$$\begin{bmatrix} x_1 & 1 \\ x_2 & 1 \end{bmatrix}^{-1} \frac{1}{x_1 - x_2} \begin{bmatrix} x_2 & -1 \\ x_1 & -1 \end{bmatrix}.$$

The solution is

$$\begin{bmatrix} a \\ b \end{bmatrix} = \frac{1}{x_1 - x_2} \begin{bmatrix} y_1 - y_2 \\ -y_1x_2 + x_1y_2 \end{bmatrix}$$

or

$$f(x) = \frac{y_1 - y_2}{x_1 - x_2} x + \frac{x_1y_2 - y_1x_2}{x_1 - x_2}.$$

Check that the slope and y-intercept make sense.

4.5. **Inverse of Elementary Matrices.** You don't have to do elimination to compute the inverse of an elementary matrix:

Proposition 4.8. *If E is an elementary matrix corresponding to a row operation, then E^{-1} is the elementary matrix corresponding to the opposite operation.*

Example 4.9. The elementary matrix corresponding to ③ ↦ ③ − 3① is $E = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -3 & 0 & 1 \end{bmatrix}$. The opposite operation is ③ ↦ ③ + 3①. So the inverse of E is

$$③ \mapsto ③ + 3① \quad \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \mapsto \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 3 & 0 & 1 \end{bmatrix} = E^{-1}.$$

The elementary matrix corresponding to ① ↔ ② is $E = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$. The opposite operation is ① ↔ ② (Switching again undoes the switch.) So the inverse of E is E itself.

The elementary matrix corresponding to ② ↦ −2② is $E = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{bmatrix}$.

The opposite operation is ② ↦ ②/(−2). So the inverse of E is

$$② \mapsto ②/(-2) \quad \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \mapsto \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1/2 \end{bmatrix}.$$

4.6. **The idea of the inverse.** It's important to understand the definition of the inverse informally. A matrix is invertible if it times something is the identity, and something times it is the identity.

Here is an example:

Example 4.10. Show that if a square matrix A is such that A^2 is invertible, then A is also invertible.

Answer: Start out any problem like this by writing out the meaning of the assumption and the conclusion. It is probably best to try it first in English.

A is invertible iff A times something and something times A are the identity.

A^2 is invertible iff A^2 times something and something times A^2 are the identity.

Now let's write out the same thing using symbols. The "somethings" in the previous two sentences might be different, so we use different letters for them.

A is invertible iff A is square and for some B, $AB = I$.

A^2 is invertible iff for some C, $A^2C = I$.

Now assume that A^2 is invertible. So for some C, $A^2C = CA^2 = I$.

Can you find B so that $AB = BA = I$?

Yes! $A^2C = (AA)C = A(AC) = I$. So if we *define* $B = AC$ then $AB = I$. This shows that A is invertible.

4.7. **Problems.**

(1) Compute the matrix product in each case, or explain why the product is undefined.
   (a) $\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} .5 & .5 \\ .5 & -.5 \end{bmatrix}$
   (b) $\begin{bmatrix} 1 & 1 \end{bmatrix} \begin{bmatrix} .5 \\ .5 \end{bmatrix}$
   (c) $\begin{bmatrix} .5 \\ .5 \end{bmatrix} \begin{bmatrix} 1 & 1 \end{bmatrix}$
   (d) $\begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & -2 \\ 0 & 1 \end{bmatrix}$
   (e) $\begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1/2 \\ 0 & 1 \end{bmatrix}$

(2) Find the inverses of the following matrices using elimination.
   (a) $\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$
   (b) $\begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$
   (c) $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

(3) Find the inverses of the following matrices using the formula for two-by-two inverses:
   (a) $\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$
   (b) $\begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$
   (c) $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

(4) Find the inverses of the following matrices using elimination, or explain why the matrix is not invertible.

(a) $\begin{bmatrix} 1 & 1 & -1 \\ 1 & -1 & 1 \\ -1 & 1 & 1 \end{bmatrix}$

(b) $\begin{bmatrix} 1 & 1 & -2 \\ 1 & -2 & 1 \\ -2 & 1 & 1 \end{bmatrix}$

(c) $\begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}$

(d) $\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$

(5) True/False:

(a) If A is invertible then A^T is invertible.

(b) If A and B are matrices such that $AB = I$ then A and B are invertible.

(c) The inverse of an invertible upper triangular matrix is upper triangular.

(d) If A is invertible then $\mathrm{ref}(A)$ is invertible.

(e) The identity matrix I is invertible.

(f) The inverse of a symmetric matrix is symmetric.

(6) Find the inverse of

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 2 & 0 & 0 \end{bmatrix}$$

(7) Find the inverse of

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}.$$

(Matrices of this form are called *upper triangular*.) You might want to check your answer by multiplying A by A^{-1}.

(8) Let $A = \begin{bmatrix} 0 & 0 & 3 \\ 0 & 1 & 0 \\ 1/3 & 0 & 0 \end{bmatrix}$. Compute (a) A^2 (b) A^{-1} and (c) A^T.

(9) Compute (a) A^2 (b) A^{-1} and (c) A^T for $A = \begin{bmatrix} 0 & 0 & 3 \\ 0 & 1 & 0 \\ 1/3 & 0 & 0 \end{bmatrix}$

(or $A = \begin{bmatrix} 0 & 2 & 1 \\ 0 & 1 & 2 \\ 3 & 0 & 0 \end{bmatrix}$. or $A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}$.)

(10) Let $A = \begin{bmatrix} 1 & 2 & 3 & 0 \\ 4 & 5 & 6 & 0 \\ 7 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix}$. Find the inverse of A, by row reduction.

(11) Write out what it means for (i) in English (ii) using symbols for
(a) A to be invertible;
(b) AB to be invertible;
(c) A and B to be invertible;
where A, B are matrices such that AB is defined.

(12) Prove that if a matrix A is invertible, then A^3 is also invertible.

(13) Prove that if a matrix A^3 is invertible, then A is also invertible.

(14) Prove that if A and B are square matrices such that AB is invertible, then A is invertible and B is invertible.

5. Determinants via patterns

The determinant is a number associated to a matrix which "determines" whether the matrix is invertible: The determinant is non-zero if and only if the matrix is invertible.

5.1. The definition of the determinant.

Let A be a square $n \times n$ matrix. A *pattern* in A is a choice of n entries from A, so that one entry is chosen from each row and column. The product of the pattern is the product of chosen entries.

Example 5.1. The patterns in the matrix $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ are

$$\begin{bmatrix} ⓐ & \\ & ⓓ \end{bmatrix}, \quad \begin{bmatrix} & ⓑ \\ ⓒ & \end{bmatrix}.$$

Each pair of entries in the pattern is either oriented southwest-northeast (SW-NE) or southeast-northwest (SE-NW). The pair is said to be an *involution* if it is oriented SW-NE. The *sign* of the pattern is

$$\mathrm{sign}(P) = (-1)^{\#\text{involutions}},$$

that is 1 if the number of involutions is even, and -1 if the number of involutions is odd.

Example 5.2. ad is not an involution in the matrix $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$, but bc is an involution. The sign of the pattern ad is 1, the sign of the pattern bc is -1.

The determinant of A is defined by

$$\det(A) = \sum_{\text{patterns } P} (-1)^{\#\text{involutions}(P)} \text{ product of entries})(P).$$

Example 5.3. The determinant of a 2×2 matrix is $\det(A) = ad - bc$.

We can ignore patterns that contain a zero, since these don't contribute to the determinant.

Example 5.4. Find the determinant of $A = \begin{bmatrix} 1 & 0 & 2 \\ 3 & 4 & 0 \\ 5 & 6 & 7 \end{bmatrix}$. The non-zero patterns are $\begin{bmatrix} 1 & & \\ & 4 & \\ & & 7 \end{bmatrix}$ (no involutions) $\begin{bmatrix} & & 2 \\ & 4 & \\ 5 & & \end{bmatrix}$ (3 involutions) and $\begin{bmatrix} & & 2 \\ 3 & & \\ & 6 & \end{bmatrix}$ (2 involutions). So the determinant is

$$\det(A) = (1)(4)(7) + (-1)^3(5)(4)(2) + (-1)^2(3)(6)(2) = 24.$$

Therefore, the matrix is invertible.

Example 5.5. Find the determinant of the upper triangular matrix

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 2 & 3 & 0 & 0 \\ 4 & 5 & 6 & 0 \\ 7 & 8 & 9 & 10 \end{bmatrix}.$$

The only non-zero choice from the first row is 1. The only non-zero choice from the second row, that is not in the same column as 1, is 3. In the same way, one sees that the only possible non-zero choices from the third and fourth rows are 6 and 10. There are no involutions in this pattern. Therefore, the determinant is

$$\det(A) = (1)(3)(6)(10) = 180$$

and the matrix is invertible.

More generally, the same reasoning shows

Theorem 5.6. *Let A be upper triangular, lower triangular, or diagonal. Then the determinant is the product of diagonal entries. Therefore, A is invertible if none of the diagonal entries are zero.*

5.2. **Determinants of Elementary Matrices.** An elementary matrix E corresponding to a row operation which adds a row to another has 1's along the diagonal, and is either upper or lower triangular. By Theorem 5.6, the determinant of E is 1.

If E is the elementary matrix corresponding to multiplying row i by c, then E is diagonal so by the Theorem $\det(E)$ is the product of diagonal entries

$$\det(E) = c.$$

Example 5.7. Multiplying row 3 by 5 gives

$$5③ \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \mapsto \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 5 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = E.$$

which has determinant 5.

Suppose E is the elementary matrix corresponding to a switch of rows i and j.

Example 5.8. Switching rows 2 and 4 gives

$$② \leftrightarrow ④ \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \mapsto \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} = E.$$

E has exactly one non-zero entry in each column, and so a unique non-zero pattern. The inverted pairs are the ij-th pair, and the ik-th and jk-th pairs for k between i and j. Therefore, the number of inverted pairs is

$$\#2|i-j|+1$$

which implies that

$$\det(E) = (-1)^{2|i-j|+1} = -1.$$

5.3. **Properties of the determinant.**

(1) (Transpose) Let A be a square matrix. Then $\det(A) = \det(A^T)$.

For every pattern in A flips over into a transpose for A^T, and vice-versa. For instance, (2)(3)(6) is a pattern in both

$$\begin{bmatrix} 1 & 0 & 2 \\ 3 & 4 & 0 \\ 5 & 6 & 7 \end{bmatrix}, \quad \begin{bmatrix} 1 & 3 & 5 \\ 0 & 4 & 6 \\ 2 & 0 & 7 \end{bmatrix}.$$

If a pair is oriented SW-NE before in A, the flipped pair is also oriented SW-NE:

$$\begin{bmatrix} & \textcircled{2} \\ \textcircled{3} & \\ & \end{bmatrix}, \quad \begin{bmatrix} & \textcircled{3} \\ & \\ \textcircled{2} & \end{bmatrix}$$

So the number of involutions in both patterns is the same. Since the determinant is the sum over patterns, with sign given by the number of involutions, this shows

(2) (Switching Two Rows) Let B equal the matrix A with two rows switched. Then $\det(B) = -\det(A)$.

Example 5.9.

$$\det \begin{bmatrix} 1 & 0 & 2 \\ 3 & 4 & 0 \\ 5 & 6 & 7 \end{bmatrix} = -\det \begin{bmatrix} 3 & 4 & 0 \\ 1 & 0 & 2 \\ 5 & 6 & 7 \end{bmatrix}.$$

Every pattern in B corresponds to a pattern in A but the number of involutions is different. Say row i is switched with row j. For every row k in between, the pair of entries in row i and row k switches from NE-SW to NW-SE or vice-versa. Similarly for the pair of entries in row j and row k. The pair of entries in rows i,j also switches from NE-SW to NW-SE. As a result the number of involutions changes by 2 times the number of rows in between, plus 1. So the sign $(-1)^{\#\text{involutions}}$ switches from $+$ to $-$, or vice-versa.

(3) (Equal Rows) If A has two rows equal, $\det(A) = 0$.

Example 5.10.

$$\det \begin{bmatrix} 1 & 0 & 2 \\ 3 & 4 & 0 \\ 1 & 0 & 2 \end{bmatrix} = 0$$

because the first and third rows are equal.

Let B be the matrix with the two rows switched, that is, $B = A$. Then $\det(A) = -\det(B) = \det(A)$ which can only happen if $\det(A) = 0$.

(4) (Summing rows or columns) Let $\mathbf{v}, \mathbf{w}$ be n-vectors. Let A, B, C be square matrices so that A, B, C are all equal except that one of the rows is $\mathbf{v}$ for A, $\mathbf{w}$ for B, and $\mathbf{v} + \mathbf{w}$ for C. Then $\det(C) = \det(A) + \det(B)$.

Note it is *not* true that $\det(A+B) = \det(A) + \det(B)$; *it is only true if rows or columns are added*!

This is best proved later, using cofactor expansion.

(5) If B is the matrix obtained by multiplying row i by c, then $\det(B) = c\det(A)$.

Example 5.11.

$$\det \begin{bmatrix} 1 & 0 & 2 \\ 9 & 12 & 0 \\ 5 & 6 & 7 \end{bmatrix} = 3\det \begin{bmatrix} 1 & 0 & 2 \\ 3 & 4 & 0 \\ 5 & 6 & 7 \end{bmatrix}$$

because the second row has been multiplies by 3.

If B is obtained from A by multiplying every row by c, then $\det(B) = c^n \det(A)$. That is,

$$\det(cA) = c^n \det(A).$$

A common mistake is to forget the superscript n. In particular, $\det(-A)$ *is not equal to* $-\det(A)$ unless the size n is odd.

(6) (Adding one row to another) If B is obtained from A by adding a multiple of one row to another, then $\det(B) = \det(A)$.

This is a consequence of the previous two results: Say row i of A is $\mathbf{v}$, and row j is $\mathbf{w}$, and B has row i equal to $\mathbf{v} + c\mathbf{w}$. Then $\det(B) = \det(A) + c\det(C)$, where C is the matrix obtained by substituting $\mathbf{w}$ into row i. But then C has two rows equal, so $\det(C) = 0$.

Example 5.12. $\det \begin{bmatrix} 1 & 0 & 2 \\ 0 & 4 & -6 \\ 5 & 6 & 7 \end{bmatrix} = \det \begin{bmatrix} 1 & 0 & 2 \\ 3 & 4 & 0 \\ 5 & 6 & 7 \end{bmatrix}$ because three times the first row has been subtracted from the second.

Now we're ready to show that the determinant determines whether the matrix is invertible.

Theorem 5.13. *The following are equivalent for a square matrix A:*

(1) $\det(A) \neq 0$
(2) $\operatorname{rref}(A) = I$
(3) *A is invertible.*

Proof. We already proved (2) $\iff$ (3). It remains to prove (1) $\iff$ (2).

Suppose the in the course of reducing A to its reduced row-echelon form, we do a number of row operations, such as adding multiples of rows to other rows, multiplying rows by non-zero numbers $c_1, \dots, c_r$, and switching rows s times.

Under the first type of row operation, the determinant is unchanged. Under the second type, the determinant is multiplied by c_i. Under the third, the determinant changes sign.

$$\det(\mathrm{rref}(A)) = (-1)^s c_1 \dots c_r \det(A).$$

This shows that $\det(A)$ is non-zero if and only if $\det(\mathrm{rref}(A))$ is non-zero.

The $\mathrm{rref}(A)$ is upper triangular, by (3) in its definition (3.5). So its determinant is non-zero if and only if there are non-zero numbers along the diagonal. If $\det(A)$ is non-zero, each of these must be a leading 1. But then the rref must equal the identity. $\square$

Using similar techniques we can show the following.

Theorem 5.14. *For any square matrices A, B, $\det(AB) = \det(A)\det(B)$.*

Proof. Look at the vector equation $A\mathbf{x} = AB\mathbf{y}$. If A is invertible, this equation has solution $\mathbf{x} = B\mathbf{y}$. Suppose in the elimination s rows get switched, and rows get multiplied by non-zero numbers $c_1, \dots, c_r$. Then

$$\det(A)(-1)^s c_1 \dots c_n = \det(I) = 1.$$

Since the same operations happen on the right,

$$\det(AB)(-1)^s c_1 \dots c_r = \det(B).$$

But the left-hand side is $1/\det(A)$. $\square$

Corollary 5.15. (1) *If A is invertible then $\det(A^{-1}) = 1/\det(A)$.*
(2) *AB is invertible if and only if A is invertible and B is invertible.*
(3) *A^n is invertible if and only if A is invertible.*

Proof. (a) $\det(A)\det(A^{-1}) = \det(AA^{-1}) = \det(I) = 1$. Now divide by $\det(A^{-1})$ on both sides. (a) $\det(AB) = \det(A)\det(B)$, so $\det(AB)$ is non-zero exactly if $\det(A)$ and $\det(B)$ are non-zero. (b) is similar. $\square$

Note that multiplication by an elementary matrix is the same as a row operation. We check:

(1) The determinant changes sign when rows are switched. Proof: $\det(P_{ij}A) = \det(P_{ij})\det(A) = -\det(A)$, since $\det(P_{ij}) = -1$.
(2) The determinant is unchanged when a multiple of one row is added to another. Proof: $\det(E_{ij}A) = \det(E_{ij})\det(A) = \det(A)$, since $\det(E_{ij})$ is the product of the diagonal entries which equals 1.
(3) The determinant is multiplied by c, if row i is multiplied by c.

5.4. Problems.

(1) For each matrix (a) list the non-zero patterns (b) identify the involutions in each pattern (c) find the determinant.

(a) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$

(b) $\begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 0 & 0 & 6 \end{bmatrix}$

(c) $\begin{bmatrix} 1 & 0 & 0 \\ 1 & 2 & 0 \\ 1 & 2 & 3 \end{bmatrix}$

(d) $\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$

(e) $\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$

(f) $\begin{bmatrix} 1 & 2 & 0 & 0 \\ 3 & 4 & 0 & 0 \\ 0 & 0 & 5 & 6 \\ 0 & 0 & 7 & 8 \end{bmatrix}$

(g) $\begin{bmatrix} 1 & 0 & 2 \\ 0 & 4 & -6 \\ 5 & 6 & 7 \end{bmatrix}$

(3) For each pair of matrix, compute the determinants and explain the relationship in terms of the properties of determinants under row operations.

(a) $\begin{bmatrix} 1 & 0 & 2 \\ 0 & 4 & -6 \\ 5 & 6 & 7 \end{bmatrix}$ and $\begin{bmatrix} 2 & 0 & 4 \\ 0 & 4 & -6 \\ 5 & 6 & 7 \end{bmatrix}$

(b) $\begin{bmatrix} 1 & 0 & 2 \\ 0 & 4 & -6 \\ 5 & 6 & 7 \end{bmatrix}$ and $\begin{bmatrix} 0 & 4 & -6 \\ 1 & 0 & 2 \\ 5 & 6 & 7 \end{bmatrix}$

(c) $\begin{bmatrix} 1 & 0 & 2 \\ 0 & 4 & -6 \\ 5 & 6 & 7 \end{bmatrix}$ and $\begin{bmatrix} 1 & 0 & 2 \\ 1 & 4 & -4 \\ 5 & 6 & 7 \end{bmatrix}$

(4) True or false:
(a) For any square matrix A, $det(-A) = -det(A)$.
(b) If $\det(A) = 0$, then $Ax = 0$ has infinite solutions.

(5) True/False:
Suppose $A = [v_1|v_2|v_3]$ is the matrix with columns v_1, v_2, v_3. If $\det(A) = 2$, then
(a) the determinant of the matrix $A' = [v_3 - v_1|v_2|v_1 - v_2]$ is -2.
(b) the determinant of the matrix $A' = [v_2|v_3|v_1 - v_2]$ is also 2.
(c) the determinant of the matrix $A' = [v_1 - v_2|v_2 - v_3|v_3 - v_1]$ is also 2.

(6) Find the determinant of the matrix

$$A = \begin{bmatrix} 2 & -1 & 1 \\ -1 & 0 & 3 \\ 2 & 1 & -4 \end{bmatrix}$$

(a) by expanding along the second row;
(b) by row-reducing A to an upper triangular matrices and using the behavior of the determinant under elementary row operations.

(7) (Strang) Compute the determinant of

$$A = \begin{bmatrix} a & a & a \\ a & b & b \\ a & b & c \end{bmatrix}.$$

using row reduction.

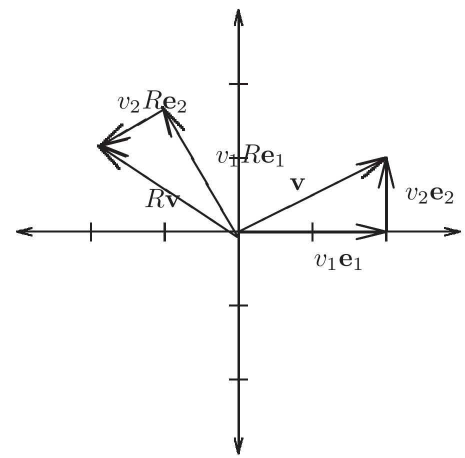
(8) Show that a square matrix A is invertible, if and only if $\det(A) \neq 0$.

5.5. **Geometry of Determinants.** A *rotation matrix* is a matrix of the form

$$R = \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix}.$$

Proposition 5.16. *If* $\mathbf{v}$ *is any 2-vector, then* $R\mathbf{v}$ *is the rotation of* $\mathbf{v}$ *around 0 by angle* θ.

Proof. Write $\mathbf{v} = v_1\mathbf{e}_1 + v_2\mathbf{e}_2$. Linear combinations are preserved under rotation; so the rotation of $\mathbf{v}$ by θ is v_1 times the rotation of $\mathbf{e}_1$ by θ plus v_2 times the rotation of $\mathbf{e}_2$ by θ.



The rotation of $\mathbf{e}_1, \mathbf{e}_2$ by θ are the vectors

$$R\mathbf{e}_1 = \begin{bmatrix} \cos(\theta) \\ \sin(\theta) \end{bmatrix}, \quad R\mathbf{e}_2 = \begin{bmatrix} -\sin(\theta) \\ \cos(\theta) \end{bmatrix}$$

so the rotation of $\mathbf{v}$ is

$$v_1R\mathbf{e}_1 + v_2R\mathbf{e}_2 = R\mathbf{v}.$$

□

Lemma 5.17. *The determinant of* R *is* 1.

Proof. $\det(R) = \cos^2(\theta) + \sin^2(\theta) = 1$. □

Let A be a 2 matrix with columns $\mathbf{v}_1$ and $\mathbf{v}_2$.

Proposition 5.18. *The absolute value* $|\det(A)|$ *of the determinant of* A *is the area of the parallelogram with edge vectors* $\mathbf{v}_1$ *and* $\mathbf{v}_2$.

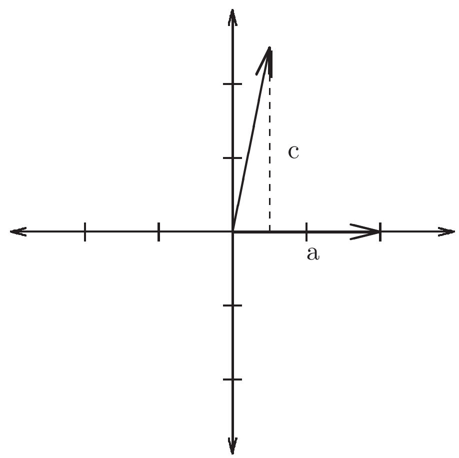
Proof. Choose a rotation matrix R so that $R\mathbf{v}_1$ lies on the x-axis. Since R is either a rotation or a reflection, the area of the parallelogram spanned by the columns of RA

$$\mathbf{w}_1 = R\mathbf{v}_1, \quad \mathbf{w}_2 = R\mathbf{v}_2$$

is the same as that of $\mathbf{v}_1, \mathbf{v}_2$. Since $\mathbf{w}_1$ is on the x-axis,

$$\mathbf{w}_1 = \begin{bmatrix} a \\ 0 \end{bmatrix} \quad \mathbf{w}_2 = \begin{bmatrix} b \\ c \end{bmatrix}$$

the area of the parallelogram is $|ac|$.



On the other hand,

$$\det(RA) = ac.$$

Since $\det(RA) = det(R)\det(A) = \det(A)$, this completes the proof. $\square$

Example 5.19. Find the area of the parallelogram P with vertices at $[1\ 0], [0\ 2], [0\ -1]$, and $[-1\ 1]$.

The edge vectors are

$$\mathbf{v}_1 = [1\ 0] - [0\ 2] = [1\ -2], \quad \mathbf{v}_2 = [0\ -1] - [0\ 2] = [0\ -3]$$

so

$$A = \begin{bmatrix} 1 & -2 \\ 0 & -3 \end{bmatrix}$$

which has determinant -3. Therefore the $\boxed{\text{area(P)} = 3}$.

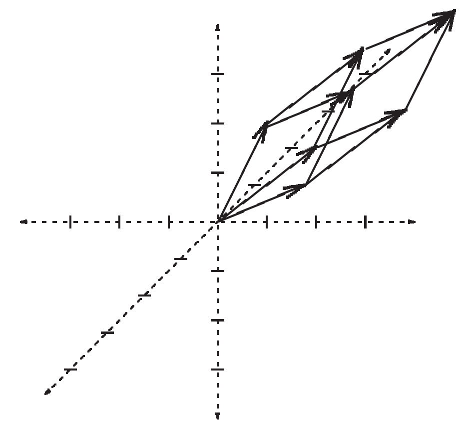
Example 5.20. Find the area of the triangle T with vertices $[1\ 0], [0\ 2], [0\ -1]$.

The area of the triangle is half that of the parallelogram, or $\boxed{\text{area(T)} = 3/2}$.

This formula generalizes to n-vectors (in particular, to $n = 3$) as follows. A *parallelopiped* in $\mathbb{R}^n$ is a set of vectors

$$P = \{\mathbf{v}_0 + c_1\mathbf{v}_1 + \ldots + c_n\mathbf{v}_n, \quad 0 \le c_1, \ldots, c_n \le 1.\}$$

Example 5.21. The parallelopiped with edge vectors $[1\ 1\ 0], [1\ 0\ 2], [0\ 2\ 1]$ is drawn



Proposition 5.22. *If A is the matrix with columns $\mathbf{v}_1, \ldots, \mathbf{v}_n$, and P is the parallopiped with edges $\mathbf{v}_1, \ldots, \mathbf{v}_n$, then*

$$\det(A) = |\operatorname{Vol}(P)|.$$

Proof. We can rotate the vector so that the vector $\mathbf{v}_1$ lies on the x axis, the vector $\mathbf{v}_2$ lies in the xy plane, etc. (See the section on QR factorization.) This doesn't change the volume, nor the determinant of A. Then A is upper triangular, so the determinant is the product of the diagonal entries $a_1, \ldots, a_n$.

Let $P(j)$ denote the parallelopiped in $\mathbb{R}^j$ whose edge vectors are $\mathbf{v}_1, \ldots, \mathbf{v}_j$. Then $P(j)$ is the base of $P(j+1)$ and

$$area(P(j)) = \text{base} \cdot \text{height} = area(P(j-1))|a_j|$$

so

$$area(P(n)) = a_1 \ldots a_n = |\det(A)|.$$

$\square$

Example 5.23. The parallelopiped with edge vectors $[1\ 1\ 0], [1\ 0\ 2], [0\ 2\ 1]$ has volume

$$|\det(\begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 2 \\ 0 & 2 & 1 \end{bmatrix})| = |-4-1| = 5.$$

A *n-simplex* is a set of vectors of the form

$$S = \{\mathbf{v}_0 + c_1\mathbf{v}_1 + \ldots + c_n\mathbf{v}_n, \quad 0 \le c_1, \ldots, c_n \le 1, \quad \sum_{i=1}^{n} c_i \le 1\}.$$

A 2-simplex is just a triangle.

Example 5.24. If $\mathbf{v}_0 = [1\ 1], \mathbf{v}_1 = [1\ 0], \mathbf{v}_2 = [0\ 1]$ S is the triangle with vertices at $(1,1),(1,2),(2,1)$.

Proposition 5.25. *The volume of an n-simplex S with edge vectors $\mathbf{v}_1, \ldots, \mathbf{v}_n$ is*

$$\mathrm{Vol}(S) = \frac{1}{n!}|\det(A)|$$

where A is the matrix with columns $\mathbf{v}_1, \ldots, \mathbf{v}_n$.

Proof. This is the same proof as 5.22, except that the volume of the simplex $S(j)$ with edge vectors $vv_1, \ldots, \mathbf{v}_j$ is related to the volume of $S(j-1)$ by

$$\mathrm{Vol}(S(j)) = \frac{\text{base} \times \text{height}}{\text{dimension}} = \frac{1}{j}\mathrm{Vol}(S(j-1))a_j.$$

□

Example 5.26. The triangle with vertices $(1,1),(3,2),(2,3)$ has edge vectors

$$\mathbf{v}_1 = [3\ 2] - [1\ 1] = [2\ 1]\,, \quad \mathbf{v}_2 = [2\ 3] - [1\ 1] = [1\ 2]$$

and area

$$\mathrm{area}(T) = \frac{1}{2}\left|\det \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}\right| = 3/2.$$

5.6. **Problems.**

(1) Find the area of the parallelogram with vertices
  (a) $[0\ 0], [2\ 1], [1\ 2], [3\ 3]$.
  (b) $[1\ 1], [-2\ 0], [2\ -1]$ and a fourth vertex.
  (c) $[3\ 3], [1\ -1], [-1\ 1]$ and a fourth vertex.
(2) Find the area of the triangle with vertices
  (a) $[1\ 1], [-2\ 0]$ and $[-2\ -1]$
  (b) $[1\ 3], [2\ 4]$ and $[4\ 2]$
  (c) $[3\ 3], [1\ -1], [-1\ 1]$
(3) Find the volume of the parallelopiped with vertices
  (a) $[0\ 0\ 0], [1\ 0\ 0], [1\ 1\ 0], [1\ 1\ 1]$ and four other vertices
  (b) $[2\ 2\ 2], [1\ 0\ 0], [1\ 1\ 0], [1\ 1\ 1]$ and four other vertices
  (c) $[2\ 2\ 2], [1\ 1\ 0], [1\ 0\ 1], [0\ 1\ 1]$ and four other vertices
(4) Find the volume of the tetrahedron with vertices
  (a) $[0\ 0\ 0], [1\ 0\ 0], [1\ 1\ 0], [1\ 1\ 1]$ and four other vertices
  (b) $[2\ 2\ 2], [1\ 0\ 0], [1\ 1\ 0], [1\ 1\ 1]$ and four other vertices
  (c) $[2\ 2\ 2], [1\ 1\ 0], [1\ 0\ 1], [0\ 1\ 1]$ and four other vertices
(5) (a) Give an example of a "degenerate triangle" whose vertices are distinct but which has area zero. Verify the area is zero by computing the determinant of the matrix whose columns are its edges. Sample answer: the "degenerate triangle" with vertices $[1\ 0], [2\ 0], [3\ 0]$. The "edge vectors" are $[1\ 0]$ and $[2\ 0]$ and $\det(\begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix}) = 0$.
  (b) Give an example of a degenerate tetrahedron whose vertices are distinct points in $\mathbb{R}^3$ and whose faces are non-degenerate triangles and whose vertices are distinct but which has volume zero. Verify the volume is zero by computing the determinant of the matrix whose columns are its edges. Give an example of a degenerate tetrahedron whose vertices are distinct points in $\mathbb{R}^3$ whose vertices are distinct but which has a degenerate face. Verify the volume is zero by computing the determinant of the matrix whose columns are its edges.
(6)

6. Determinants via cofactor expansion

In the last section we explained how to define determinants as a sum over patterns in a matrix. In this section we explain how to compute determinants more systematically, using cofactors. In particular we want to explain how the formula for 2×2 matrices

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad - bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

generalizes to bigger size.

6.1. **Cofactors.**

Let A be a square $n \times n$ matrix. Let M_{ij} denote the matrix A with row i and column j deleted. M_{ij} is called the ij-th *minor* of A.

Example 6.1. The 13-th minor of $\begin{bmatrix} 1 & 3 & 5 \\ 0 & 4 & 6 \\ 2 & 0 & 7 \end{bmatrix}$ is $M_{13} = \begin{bmatrix} 0 & 4 \\ 2 & 0 \end{bmatrix}$.

The number

$$A_{ij} = (-1)^{i+j}\det(M_{ij})$$

is the ij-th *cofactor* of A. The signs $(-1)^{i+j}$ are given by the *table of alternating signs*, for example, for 3×3 the table is

$$\begin{bmatrix} + & - & + \\ - & + & - \\ + & - & + \end{bmatrix}.$$

Example 6.2. The 13 cofactor of $\begin{bmatrix} 1 & 3 & 5 \\ 0 & 4 & 6 \\ 2 & 0 & 7 \end{bmatrix}$ is

$$A_{13} = (-1)^{1+3} \det \left(\begin{bmatrix} 0 & 4 \\ 2 & 0 \end{bmatrix} \right) = +(0-8) = -8.$$

Example 6.3. The cofactors of $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ are $d, -c, -b, a$.

Here is the reason we are interested in cofactors:

Theorem 6.4. *For any i, the dot product of the i-th row of A with the cofactors for the i-th row is* $\det(A)$. *If i is not equal to j, the dot product of the i-th row of A with the cofactors for the j-th row is equal to* 0. *That is,*

$$(4) \qquad a_{i1}A_{i1} + \dots a_{in}A_{in} = \det(A), \qquad a_{i1}A_{j1} + \dots a_{jn}A_{jn} = 0.$$

Before we explain the theorem, here is an example.

Example 6.5. The cofactors for the first row of $\begin{bmatrix} 1 & 3 & 5 \\ 0 & 4 & 6 \\ 2 & 0 & 7 \end{bmatrix}$ are

$$+(28-0) = 28, -(0-12) = 12, +(0-8) = -8.$$

The dot product of $[28 \; 12 \; -8]$ with the rows of A are

$$28(1)+12(3)-8(5) = 24, \;\; 28(0)+12(4)-8(6) = 0, \;\; 28(2)+12(0)-8(7) = 0.$$

Proof. Now we prove the theorem. Each pattern in A contains exactly one element in row i, say a_{ik}. The remaining chosen entries form a pattern in M_{ij}. So the product of entries appears in the sum (4). Conversely, any pattern in M_{ij} defines a pattern in A, by adding the entry a_{ij}. So the terms in the sum

$$a_{i1} \det(M_{i1}) + \dots a_{in} \det(M_{in})$$

are the same as those that appear in $\det(A)$.

It remains to explain the sign $(-1)^{i+j}$. The number of involutions in the pattern in A_{ij} is the number of involutions in the pattern in M_{ij}, plus the number v of involutions of pairs containing a_{ij}. Let's compute v. The matrix M_{ij} is naturally broken up into 4 parts: the entries that lie NE, NW, SE, SW of a_{ij}. We have

$$v = \#\text{NE entries} + \text{SW entries}.$$

Since there is only one chosen entries in each row and column

$$\#\text{NE entries} = (i-1) - \#\text{NW entries}, \quad \#\text{SW entries} = (j-1) - \#\text{NW entries}.$$

So

$$v = i + j - 2 - 2\#\text{NW entries}$$

which implies

$$(-1)^v = (-1)^{i+j}.$$

This proves the first part of (4).

Now suppose we take the dot product of row j in A with the cofactors for row i. Let B be the matrix obtained from A by replacing row i with row j. The cofactors for row i are the same for both B and A. The dot product of the j-th row of A with the cofactors, is the same as the dot product of the i-th row of B, with the cofactors. By the first part of (4), applied to B, the result is $\det(B)$. But since B has two rows equal, $\det(B) = 0$. □

6.2. **Cofactor expansion of the determinant.** The first formula in (4) is called the *cofactor expansion* of the determinant along row i. For instance, suppose we want to compute the determinant of

$$A = \begin{bmatrix} 1 & 3 & 5 \\ 0 & 4 & 6 \\ 2 & 0 & 7 \end{bmatrix}.$$

We choose a row along which to expand, say the second. We take each entry in the row, and multiply by the determinant of the corresponding minor, with the appropriate sign from the table of signs:

$$\det(A) = -0 \det \begin{bmatrix} 3 & 5 \\ 0 & 7 \end{bmatrix} + 4 \det \begin{bmatrix} 1 & 5 \\ 2 & 7 \end{bmatrix} - 6 \det \begin{bmatrix} 1 & 3 \\ 2 & 0 \end{bmatrix} = 24.$$

The same thing works for any column. For any i, the dot product of the i-th column of A with the cofactors for the j-th column is $\det(A)$, if $i = j$, and 0 otherwise.

Example 6.6. Suppose we want to find the determinant of the 4×4 matrix

$$A = \begin{bmatrix} 1 & 2 & 0 & 0 \\ 2 & 3 & 4 & 5 \\ 3 & 4 & 0 & 0 \\ 5 & 6 & 7 & 0 \end{bmatrix}.$$

There is only one non-zero entry in the fourth column; therefore it's best to expand along that column. We get

$$\det(A) = -0 + 4\det\begin{bmatrix} 1 & 2 & 0 \\ 3 & 4 & 0 \\ 5 & 6 & 7 \end{bmatrix} - 0 + 0.$$

Now there is only one non-zero entry in the third column, so we expand along it:

$$\det(A) = 4(0 - 0 + 7\det\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = 28(1(4) - 2(3)) = -56.$$

The *adjoint* of A is the transpose of the matrix of cofactors. That is, the ij-the entry of $\mathrm{adj}(A)$ is the ji-th cofactor A_{ji}.

Example 6.7. The adjoint of $A = \begin{bmatrix} 1 & 3 & 5 \\ 0 & 4 & 6 \\ 2 & 0 & 7 \end{bmatrix}$ is the matrix

$$\mathrm{adj}(A) = \begin{bmatrix} 28 & -21 & -2 \\ 12 & -3 & -6 \\ -8 & 6 & 4 \end{bmatrix}.$$

6.3. **The cofactor formula for the inverse.** Here is the promised formula for the inverse:

Theorem 6.8. $A^{-1} = \frac{1}{\det(A)}\,\mathrm{adj}(A)$.

Proof. It suffices to show that $A\,\mathrm{adj}(A) = \det(A)I$. The dot product of the i-th row of A with the j-th column of $\mathrm{adj}(A)$ is the dot product of the i-th row of A with the cofactors for the j-th row. By (4), this equals $\det(A)$ if $i = j$, and 0 otherwise. These are the same as the entries of the matrix $\det(A)I$. $\square$

Example 6.9. The inverse of $A = \begin{bmatrix} 1 & 3 & 5 \\ 0 & 4 & 6 \\ 2 & 0 & 7 \end{bmatrix}$ is

$$A^{-1} = \frac{1}{24}\begin{bmatrix} 28 & -21 & -2 \\ 12 & -3 & -6 \\ -8 & 6 & 4 \end{bmatrix}.$$

The cofactor formula is particularly useful when there are unknowns in the matrix.

Example 6.10. Find the inverse of $A = \begin{bmatrix} a & b & c \\ 0 & a & b \\ 0 & 0 & a \end{bmatrix}$. Since A is upper triangular, the determinant is the product of diagonal entries $\det(A) = a^3$. The adjoint is

$$\mathrm{adj}(A) = \begin{bmatrix} a^2 & ab & b^2 - ac \\ 0 & a^2 & ab \\ 0 & 0 & a^2 \end{bmatrix}.$$

The inverse is

$$A^{-1} = \frac{1}{a^3}\begin{bmatrix} a^2 & ab & b^2 - ac \\ 0 & a^2 & ab \\ 0 & 0 & a^2 \end{bmatrix}.$$

Sometimes when the matrices contain unknowns it's easier to find the determinant using the row operations.

Example 6.11. Suppose we want to the unique polynomial passing through (x_1, y_1), (x_2, y_2) and (x_3, y_3). We write $f(x) = a + bx + cx^2$. The equations

$$a + bx_1 + cx_1^2 = y_1, \quad a + bx_2 + cx_2^2 = y_2, \quad a + bx_3 + cx_3^2 = y_3$$

can be written in matrix form

$$\begin{bmatrix} 1 & x_1 & x_1^2 \\ 1 & x_2 & x_2^2 \\ 1 & x_3 & x_3^2 \end{bmatrix}\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}.$$

The determinant of the matrix is

$$\det\left(\begin{bmatrix} 1 & x_1 & x_1^2 \\ 1 & x_2 & x_2^2 \\ 1 & x_3 & x_3^2 \end{bmatrix}\right) = \det\left(\begin{bmatrix} 1 & x_1 & x_1^2 \\ 0 & x_2 - x_1 & x_2^2 - x_1^2 \\ 0 & x_3 - x_1 & x_3^2 - x_1^2 \end{bmatrix}\right)$$

since subtracting the first row from the second and third does not change the determinant. Multiplying the second by $1/(x_2 - x_1)$ and subtracting $(x_3 - x_1)$ times the second row from the third gives

$$(x_2 - x_1)\det\begin{bmatrix} 1 & x_1 & x_1^2 \\ 0 & 1 & \frac{x_2^2 - x_1^2}{x_2 - x_1} \\ 0 & 0 & (x_3^2 - x_1^2) - \frac{(x_3 - x_1)(x_2^2 - x_1^2)}{x_2 - x_1} \end{bmatrix}.$$

So the determinant is

$$\begin{aligned} & (x_2 - x_1)((x_3^2 - x_1^2) - \frac{(x_3 - x_1)(x_2^2 - x_1^2)}{x_2 - x_1}) \\ = \; & (x_2 - x_1)((x_3 - x_1)(x_3 + x_1) - (x_3 - x_1)(x_2 + x_1)) \\ = \; & (x_2 - x_1)((x_3 - x_1)(x_3 + x_1 - x_2 - x_1)) \\ = \; & (x_2 - x - 1)(x_3 - x_1)(x_3 - x_2). \end{aligned}$$

This is called a *Vandermonde determinant.* You can easily guess how it generalizes to higher size.

6.4. Problems.

(1) For each matrix find the determinant by cofactor expansion along the given row and given column.

(a) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$ by expansion along the first row.

(b) $\begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 0 & 0 & 6 \end{bmatrix}$ by expansion along the first column.

(c) $\begin{bmatrix} 1 & 0 & 0 \\ 1 & 2 & 0 \\ 1 & 2 & 3 \end{bmatrix}$ by expansion along the second row.

(d) $\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$ by expansion along the second column.

(e) $\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$ by expansion along the third row.

(f) $\begin{bmatrix} 1 & 2 & 0 & 0 \\ 3 & 4 & 0 & 0 \\ 0 & 0 & 5 & 6 \\ 0 & 0 & 7 & 8 \end{bmatrix}$ by expansion along the third column.

(g) $\begin{bmatrix} 1 & 0 & 2 \\ 0 & 4 & -6 \\ 5 & 6 & 7 \end{bmatrix}$ by expansion along a row or column of your choice.

(2) (a) Find the cofactor matrix for $A = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$. (b) Find the determinant of A, by expanding along the third column.
(c) Find the inverse of A, using parts (a),(b).
(d) Find the inverse of A, by row reduction.

(3) (Strang)

(4) Compute the determinant of

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 4 & 4 \\ 5 & 6 & 7 \end{bmatrix}$$

using the cofactor formula.

(5) Let $A = \begin{bmatrix} 1 & 2 & 3 & 0 \\ 4 & 5 & 6 & 0 \\ 7 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix}$. Find the determinant of A, by expanding along the third column.

7. Linear transformations

This section explains how to think of matrix multiplication geometrically as a *transformation* of the space of vectors. This is often considered an optional topic in first courses on linear algebra, although for more advanced topics it is essential.

7.1. Definition of a linear transformation.

Definition 7.1. A function from $\mathbb{R}$ to $\mathbb{R}$ assigns to any real number x another real number $f(x)$. A *map* T from $\mathbb{R}^n$ to $\mathbb{R}^m$ is similar, but it assigns to any vector $\mathbf{x}$ in $\mathbb{R}^n$ a vector $T(\mathbf{x})$ in $\mathbb{R}^m$, called the *value* of T at $\mathbf{x}$.

Example 7.2. The following are all maps from $\mathbb{R}^2$ to $\mathbb{R}^2$.

$$T_1[x_1\ x_2] = [5x_1 + 2x_2\ 3x_1 - x_2]$$
$$T_2[x_1\ x_2] = [x_1^2 - x_2^2\ x_1^2 + x_2^2]$$
$$T_3[x_1\ x_2] = [x_1 + 2\ x_2 - 3]$$

Definition 7.3. A map from $\mathbb{R}^n$ to $\mathbb{R}^m$ is a *linear transformation* if it preserves vector addition and scalar multiplication, that is, if

(1) $T(\mathbf{x} + \mathbf{y}) = T(\mathbf{x}) + T(\mathbf{y})$, for all $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$;
(2) $T(c\mathbf{x}) = cT(\mathbf{x})$, for all $\mathbf{x} \in \mathbb{R}^n$, $c \in \mathbb{R}$.

These conditions can be combined into a single condition, that T preserves linear combinations, that is,

$$T(\mathbf{x} + d\mathbf{y}) = cT(\mathbf{x}) + dT(\mathbf{y}).$$

Example 7.4. Of the three maps $\mathbb{R}^2$ to $\mathbb{R}^2$ above, only the T_1 is a linear transformation. In fact, if A is the matrix $\begin{bmatrix} 5 & 2 \\ 3 & -1 \end{bmatrix}$ then $T_1(\mathbf{x}) = A\mathbf{x}$. So

$$T(c\mathbf{x} + d\mathbf{y}) = A(c\mathbf{x} + d\mathbf{y}) = cA\mathbf{x} + dA\mathbf{y} = cT(\mathbf{x}) + dT(\mathbf{y}).$$

More generally, any map $T(\mathbf{x})$ of the form $T(\mathbf{x}) = A\mathbf{x}$ is a linear transformation.

The map T_2 fails because, for example,

$$T(2[3\ 0]) = T([6\ 0]) = [36\ 36]$$

but

$$2T([3\ 0]) = 2[9\ 9] = [18\ 18].$$

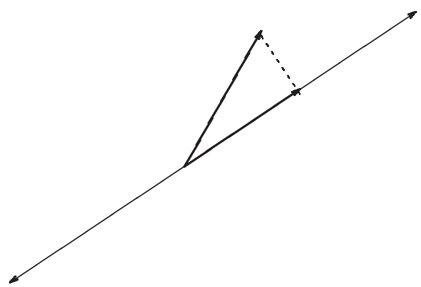
The map T_3 fails because, for example,

$$T_3[1\ 0] + T_3[2\ 0] = [3\ 0] + [4\ 0] = [7\ 0]$$

but $T_3[3\ 0] = [5\ 0$.

7.2. **Examples of linear transformations in two dimensions.** Let's look at some examples of linear transformations in $\mathbb{R}^2$.

(1) Let L be a line in $\mathbb{R}^2$ passing through 0. For any $\mathbf{x} \in \mathbb{R}^2$, define $P(x)$ to be the vector whose head is the closest point to the head of $\mathbf{x}$ in L.



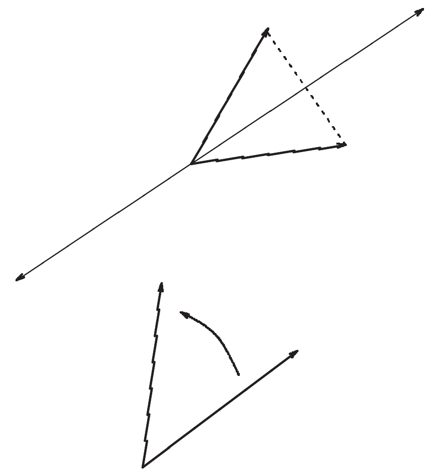
P is *orthogonal projection* onto L. Let's check graphically that it is a linear transformation:

(Figure)

(2) Let L be a line in $\mathbb{R}^2$ passing through 0. For any vector $\mathbf{x} \in \mathbb{R}^2$, define $S(x)$ to be the reflection of $\mathbf{x}$ over L. Then S is a linear transformation $\mathbb{R}^2 \to \mathbb{R}^2$.
(3) Let θ be an angle, and for any vector $\mathbf{x}$ let $R(x)$ be the rotation counterclockwise of $\mathbf{x}$ around 0 by angle θ. Then T is a linear transformation $\mathbb{R}^2 \to \mathbb{R}^2$.

7.3. **Linear transformations are matrices.**

Theorem 7.5. *Any linear transformation $T : \mathbb{R}^n \to \mathbb{R}^m$ is of the form $T(\mathbf{x}) = A\mathbf{x}$ for some $m \times n$ matrix A, called the* matrix for the linear transformation*.*



Proof. Define $\mathbf{e}_1 = [1\ 0 \ldots 0], \mathbf{e}_2 = [0\ 1\ 0 \ldots 0], \ldots, \mathbf{e}_n = [0\ 0 \ldots 0\ 1]$. Define A to be the matrix whose columns are $A\mathbf{e}_1, \ldots, A\mathbf{e}_n$. For any vector $\mathbf{x}$ we have

$$\begin{aligned} \mathbf{x} &= \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} x_1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} + \ldots + \begin{bmatrix} 0 \\ \vdots \\ 0 \\ x_n \end{bmatrix} \\ &= x_1\mathbf{e}_1 + \ldots + x_n\mathbf{e}_n. \end{aligned}$$

Since T preserves linear combinations

$$\begin{aligned} T(\mathbf{x}) &= T(x_1\mathbf{e}_1 + \ldots x_n\mathbf{e}_n) \\ &= x_1T(\mathbf{e}_1) + \ldots x_nT(\mathbf{e}_n) \\ &= [T(\mathbf{e}_1) \ldots T(\mathbf{e}_n)]\mathbf{x} \\ &= A\mathbf{x}. \end{aligned}$$

□

7.4. **Examples in two dimensions.** Let L be the line with slope 1 in $\mathbb{R}^2$, passing through 0. Let's find the matrix for projection onto L.

The closest vector to $\mathbf{e}_1$ in L is $\begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \end{bmatrix}$. The closest vector to $\mathbf{e}_2$ is the same vector. Therefore,

$$P(\mathbf{x}) = A\mathbf{x}, \text{ where} A = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix}.$$

Suppose we want to now find the closest vector to [5 2] in L. We multiply by A to get

$$A\begin{bmatrix}5\\2\end{bmatrix}=\begin{bmatrix}7/2\\7/2\end{bmatrix}.$$

The matrix for reflection is similar. The reflection of $\mathbf{e}_1$ through L is $\mathbf{e}_2$, and the reflection of $\mathbf{e}_2$ through L is $\mathbf{e}_1$. So the matrix for reflection is

$$A=[\mathbf{e}_2\ \mathbf{e}_1]=\begin{bmatrix}0&1\\1&0\end{bmatrix}.$$

Now let R be rotation around 0 by angle θ. We have

$$R(\mathbf{e}_1)=\begin{bmatrix}\cos(\theta)\\ \sin(\theta)\end{bmatrix},\quad R(\mathbf{e}_2)=\begin{bmatrix}-\sin(\theta)\\ \cos(\theta)\end{bmatrix}.$$

So the matrix for R is

$$A=\begin{bmatrix}\cos(\theta)&-\sin(\theta)\\ \sin(\theta)&\cos(\theta)\end{bmatrix}.$$

Which of these matrices are invertible? Let's compute their determinants.

$$\det\begin{bmatrix}\frac{1}{2}&\frac{1}{2}\\ \frac{1}{2}&\frac{1}{2}\end{bmatrix}=1/4-1/4=0.$$

$$\det\begin{bmatrix}0&1\\1&0\end{bmatrix}=0-1=-1.$$

$$\det\begin{bmatrix}\cos(\theta)&-\sin(\theta)\\ \sin(\theta)&\cos(\theta)\end{bmatrix}=\cos^2(\theta)+\sin^2(\theta)=1.$$

In general, reflections and rotations are invertible; projections are not. In fact, the inverse of a reflection is just the same reflection, since $S(S(\mathbf{x}))=\mathbf{x}$. The inverse of a rotation by θ is rotation by $-\theta$.

7.5. **The matrix of a composition is the matrix product.** If $T_1:\mathbb{R}^n\to\mathbb{R}^m$ and $T_2:\mathbb{R}^m\to\mathbb{R}^p$ are maps, the composition is the map

$$T_2\circ T_1:\ \mathbb{R}^n\to\mathbb{R}^p,\quad \mathbf{x}\to T_2(T_1(\mathbf{x})).$$

If T_1 and T_2 are linear, then so is $T_2\circ T_1$:

$$T_2(T_1((c\mathbf{x}+d\mathbf{y}))=T_2(cT_1(\mathbf{x})+dT_2(\mathbf{y}))=cT_2(T_1(\mathbf{x}))+dT_2(T_1(\mathbf{y})).$$

Proposition 7.6. *Let T_1 and T_2 be linear transformations from $\mathbb{R}^n$ to $\mathbb{R}^n$ with matrices A_1 and A_2. Then the matrix for $T_2\circ T_1$ is A_2A_1.*

Example 7.7. Suppose that P is orthogonal projection onto a line L. Since $P(v)$ already has been projected, $P(P(v))=v$, that is, $P\circ P=P$. If A is the matrix for P, then $A^2=A$.

Example 7.8. Suppose S is reflection over a line L in $\mathbb{R}^2$. Then $S(S(v))=v$, that is, the reflection reflects back to the original vector. If A is the matrix for S, then $A^2=I$.

Example 7.9. Suppose R_θ is rotation by θ, and R_φ is rotation by φ. The composition is rotation by $\theta+\varphi$,

$$R_\theta\circ R_\varphi=R_{\theta+\varphi}.$$

We get

$$\begin{bmatrix}\cos(\theta)&-\sin(\theta)\\ \sin(\theta)&\cos(\theta)\end{bmatrix}\begin{bmatrix}\cos(\varphi)&-\sin(\varphi)\\ \sin(\varphi)&\cos(\varphi)\end{bmatrix}=\begin{bmatrix}\cos(\theta+\varphi)&-\sin(\theta+\varphi)\\ \sin(\theta+\varphi)&\cos(\theta+\varphi)\end{bmatrix}.$$

The left hand side is

$$\begin{bmatrix}\cos(\theta)\cos(\varphi)-\sin(\theta)\sin(\varphi)&-\cos(\theta)\sin(\varphi)-\sin(\theta)\cos(\varphi)\\ \cos(\theta)\sin(\varphi)+\sin(\theta)\cos(\varphi)&\cos(\theta)\cos(\varphi)-\sin(\theta)\sin(\varphi)\end{bmatrix}.$$

Equating the entries of the matrices we get the *angle-sum* formulas.

$$\cos(\theta+\varphi)=\cos(\theta)\cos(\varphi)-\sin(\theta)\sin(\varphi)$$

$$\sin(\theta+\varphi)=\cos(\theta)\sin(\varphi)+\sin(\theta)\cos(\varphi).$$

Example 7.10. Derive a formula for the cos of 3θ, using the same method.

7.6. **Problems.**

(1) Determine whether the following maps T are linear transformations. If T is linear, find the matrix A such that $T[x]=Ax$. If T is not linear, explain why.

(a) $T[x_1,x_2,x_3]=[x_1-x_3,x_2-x_3,x_1-x_2]$.
(b) $T[x_1,x_2,x_3,x_4]=[x_1+1,x_2+x_1-3]$.
(c) $T:\ \mathbb{R}^2\to\mathbb{R}^2$ is rotation counterclockwise by 45 degrees, about the point $(0,0)$.
(d) $T:\ \mathbb{R}^3\to\mathbb{R}^3$, where $T[x]$ is the cross-product of x with the vector $[-1,0,1]$.
(e) $T[x_1,x_2]=[x_1,1/x_2]$.

8. Linear independence and span

We already saw that, regardless of the number of variables and equations, a linear system of equations may or may not be solvable, and if it is solvable, then the solution may or may not be unique. In this section we give geometric interpretations to solvability and uniqueness. Namely, a linear system is solvable if the inhomogeneous term is in the *span* of the columns of A, and the solution is unique if the columns of the coefficient matrix are *linearly independent.*

8.1. **Linear independence.** A set of vectors is *linearly dependent* if one is a combination of the others. To give an informal example, a set of two vectors pointing north and east is independent, while a set of two vectors pointing north and south is dependent, since the south-pointing vector is a negative scalar times the north-pointing vector. More formally:

Definition 8.1. (Linear independence)

(1) Vectors $\mathbf{v}_1, \ldots, \mathbf{v}_r$ are *linearly dependent* (or independent, for short) if some vector in the list is a combination of the others, that is, $v_i = c_1\mathbf{v}_1 + \ldots c_{i-1}\mathbf{v}_{i-1} + c_{i+1}\mathbf{v}_{i+1} + \ldots + c_n\mathbf{v}_n$ for some i and constants $c_j, j \neq i$.

(2) If $\mathbf{v}_1, \ldots, \mathbf{v}_r$ are not dependent, they are *independent.*

Example 8.2. (1) Two vectors are independent if and only if they are not proportional. For example, $[-1\ 0\ 1]$ and $[-2\ 0\ 2]$ are dependent, because

$$[-2\ 0\ 2] = 2[-1\ 0\ 1].$$

But $[-1\ 0\ 1]$ and $[2\ 0\ 2]$ are independent.

(2) Three non-zero vectors are independent of none of the vectors lies in the plane or line spanned by the other two. For example, $[1\ -1\ 0], [0\ -1\ 1], [1\ 0\ -1]$ is dependent, because

$$[1\ 0\ -1] = [1\ -1\ 0] - [0\ -1\ 1]$$

is a combination of $[1\ -1\ 0], [0\ -1\ 1]$.

(3) The zero vector is always a linear combination of other vectors, for example, $[0\ 0]$ is a combination of $[1\ 0]$ and $[0\ 1]$ since

$$[0\ 0] = 0[1\ 0] + 0[0\ 1].$$

It follows that any collection of vectors which includes the zero vector is a dependent collection of vectors.

Here are some equivalent definitions:

Theorem 8.3. *Vectors $\mathbf{v}_1, \ldots, \mathbf{v}_r$ are dependent (that is, not independent) iff there is a smaller subset that has the same span as $\mathbf{v}_1, \ldots, \mathbf{v}_r$ iff $\mathbf{v}_1, \ldots, \mathbf{v}_r$ is the minimal generating set for the span.*

Example 8.4. The collection $[1\ 0\ 0], [0\ 1\ 0], [1\ 1\ 0]$ is dependent because its span is the xy-plane, but it is not a *minimal* generating set for the xy plane since $[1\ 0\ 0], [0\ 1\ 0]$ also generate the xy-plane.

The collection $[1\ 0\ 0], [0\ 1\ 0], [1\ 1\ 0]$ is is also dependent because the third vector is a combination of the first two,

$$[1\ 1\ 0] = [1\ 0\ 0] + [0\ 1\ 0].$$

Definition 8.5. A *dependence relation* on $\mathbf{v}_1, \ldots, \mathbf{v}_r$ is a collection of scalars $c_1, \ldots, c_r$ not all zero such that $c_1\mathbf{v}_1 + \ldots c_r\mathbf{v}_r = 0$.

Example 8.6. Consider the vectors $\begin{bmatrix} 1 \\ 1 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}$. Then

$$\begin{bmatrix} 1 \\ 1 \\ 3 \end{bmatrix} = 2\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}, \quad \text{or} \quad -\begin{bmatrix} 1 \\ 1 \\ 3 \end{bmatrix} + 2\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix} = 0$$

is a dependence relation.

Here is the algorithm for checking whether a set of vectors $\mathbf{v}_1, \ldots, \mathbf{v}_r$ is linearly independent. Write the equation

$$c_1\mathbf{v}_1 + \ldots c_r\mathbf{v}_r = 0$$

in matrix form $A\mathbf{c} = 0$, where A is the matrix whose columns are $\mathbf{v}_1, \ldots, \mathbf{v}_r$.

Theorem 8.7. *Vectors $\mathbf{v}_1, \ldots, \mathbf{v}_r$ are dependent iff there is a dependence relation on them iff $A\underset{\sim}{=}0$ has non-trivial solutions, where A is the matrix with columns $\mathbf{v}_1, \ldots, \mathbf{v}_r$, iff* $\mathrm{ref}(A)$ *has free columns.*

Example 8.8. Determine whether $\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$ are independent or dependent. Answer: Elimination gives

$$\begin{bmatrix} 1 & 0 & -1 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & -1 \\ -1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

so the last columns if free, so the vectors are dependent. To find a dependence relation, we find any non-zero solution. The first row gives $x = z$, the second row gives $x = y$, so a solution is $x = y = z = 1$, which gives the dependence relation

$$\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} + \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} = 0.$$

Corollary: If $k > n$ then any collection of n-vectors $\mathbf{u}_1, \ldots, \mathbf{u}_k$ is linearly dependent.

Proof: Since the system $A\mathbf{x} = 0$ has more columns than row, some columns must be free.

Example 8.9. For what values of c are the vectors $[0\ 0\ -1], [1\ 1\ 2], [1\ 1\ c]$ independent?

Answer: the matrix with these vectors as columns has determinant $(-1)(c-2)$ which is non-vanishing iff $c \neq 2$.

To prove that a given subset V of $\mathbb{R}^n$ is a subspace, check that all three conditions hold.

Example 8.10. Show that the set V of vectors of the form $[x\ 0\ ]$ is a subspace of $\mathbb{R}^3$. Answer: (i) contains 0: Setting $x = 0$ shows that $[0\ 0\ 0]$ is in V. (ii) check that it closed under sums: To do this write a pair of *general elements* of V; this means that you need to invent a new variable name. So let $[x_1\ 0\ 0]$ and $[x_2\ 0\ 0]$ be elements of V. Their sum is $[x_1 + x_2\ 0\ 0]$ which is of the form $[x\ 0\ 0]$ with $x = x_1 + x_2$. So V is closed under sums. (iii) check that V is closed under scalar multiplication: Let $[x_1\ 0\ 0]$ be a vector in V and c a scalar. Then $c\,[x_1\ 0\ 0] = [cx_1\ 0\ 0]$ is of the form $[x\ 0\ 0]$ with $x = cx_1$, so V is closed under scalar multiplication.

Proofs involving span and linear independence are similar. For example, to prove that vectors are linearly independent, you can show that there is no dependence relation, by supposing that there is a relation of the form $c_1\mathbf{v}_1 + \ldots + c_k\mathbf{v}_k = 0$, and showing that $c_1 = \ldots c_k = 0$.

Example 8.11. Show that if $\mathbf{u}, \mathbf{v}$ are independent, then so are $\mathbf{u} + 2\mathbf{v}, \mathbf{v}$. Answer: Suppose that there is a dependence relation $c(\mathbf{u} + 2\mathbf{v}) + d\mathbf{v} = 0$. Then $c\mathbf{u} + (2c + d)\mathbf{v} = 0$. Since $\mathbf{u}, \mathbf{v}$ are independent, we must have $c = 0, 2c + d = 0$. But then $c = 0, 2(0) + d = d = 0$, so c, d are both zero. This shows that $\mathbf{u} + 2\mathbf{v}, \mathbf{v}$ do not satisfy a dependence relation.

Example 8.12. Show that if $\mathbf{u}, \mathbf{v}$ span a set V, then so do $\mathbf{u} + 2\mathbf{v}, \mathbf{v}$. Answer: Let $\mathbf{w}$ be in V. We must show that $c(\mathbf{u} + 2\mathbf{v}) + d\mathbf{v} = \mathbf{w}$ for some scalars c, d. Equivalently, $c\mathbf{u} + (2c + d)\mathbf{v} = \mathbf{w}$. We know that $\mathbf{u}, \mathbf{v}$ span V, so $a\mathbf{u} + b\mathbf{v} = \mathbf{w}$ for some a, b. So we want to solve $c = a, (2c + d) = b$. The solution is $c = a, d = b - 2c = b - 2a$.

8.2. **Span.** The span of a collection of vectors is the space of all combinations of them. For an informal example, the span of a vector pointing north would be the line pointing north-south (since southern-pointing vectors are obtained from northern-pointing ones by scalar multiplication by a negative number) while the span of vectors pointing east and north would be a horizontal plane.

Here is the formal definition:

Definition 8.13. (Span) Let $\mathbf{v}_1, \ldots, \mathbf{v}_r$ be vectors in $\mathbb{R}^n$. The *span* of $\mathbf{v}_1, \ldots, \mathbf{v}_r$ is the set of linear combinations

$$c_1\mathbf{v}_1 + \ldots + c_r\mathbf{v}_r.$$

Example 8.14.
(1) The span of a single non-zero vector v is the set of all cv, that is, the line through v.
(2) The span of the vector $[1\ 0\ 0]$ is the x-axis. $V = \{[x\ 0\ 0]\}$ in $\mathbb{R}^3$.
(3) The span of the vectors $[1\ 0\ 0], [0\ 1\ 0]$ is the xy-plane. $V = \{[x\ y\ 0]\}$ in $\mathbb{R}^3$.
(4) The span of the vectors $[1\ 0\ 0], [0\ 1\ 0], [1\ 1\ 0]$ is also the xy-plane. $V = \{[x\ y\ 0]\}$ in $\mathbb{R}^3$. This is because any vector that is a combination of these three vectors is in fact a combination of the first two, since
$$[1\ 1\ 0] = [1\ 0\ 0] + [0\ 1\ ].$$
That is,
$$x[1\ 0\ 0] + y[0\ 1\ 0] + z[1\ 1\ 0] = (x + z)[1\ 0\ 0] + (y + z)[0\ 1\ 0].$$
(5) The span of the vectors $[1\ 0\ 0], [2\ 0\ 0]$ is also the x-axis.

8.3. **Problems.**

(1) Check whether the following vectors are linearly independent or dependent, if necessary using row reduction. In each dependent case, identify one of the vectors that is a combination of the others.
   (a) $\mathbf{v}_1 = [1\ 0], \mathbf{v}_2 = [0\ 1]$.
   (b) $\mathbf{v}_1 = [1\ 0], \mathbf{v}_2 = [0\ 1], \mathbf{v}_3 = [1\ 1]$.
   (c) $\mathbf{v}_1 = [1\ 0], \mathbf{v}_2 = [0\ 1], \mathbf{v}_3 = [0\ 0]$.
   (d) $\mathbf{v}_1 = [0\ 0], \mathbf{v}_2 = [0\ 1], \mathbf{v}_3 = [1\ 0]$.
   (e) $\mathbf{v}_1 = [1\ 1\ 0], \mathbf{v}_2 = [1\ 0\ 1], \mathbf{v}_3 = [0\ 1\ 1]$. $\mathbf{v}_1 = [1\ 1\ 0], \mathbf{v}_2 = [1\ 0\ 1], \mathbf{v}_3 = [0\ 1\ 1]$.
   (f) $\mathbf{v}_1 = [1\ -1\ 0], \mathbf{v}_2 = [1\ 0\ -1], \mathbf{v}_3 = [0\ 1\ -1]$.

(g) $\mathbf{v}_1 = [1\ -1\ 0 0], \mathbf{v}_2 = [0\ 1\ -1\ 0], \mathbf{v}_3 = [0\ 0\ 1\ -1]$.

(2) Check whether the following vectors span the given set.
 (a) $\mathbf{v}_1 = [1\ 0], \mathbf{v}_2 = [0\ 1], V = \mathbb{R}^2$.
 (b) $\mathbf{v}_1 = [1\ 0], \mathbf{v}_2 = [0\ 1], V = \{[x\ 0]\}$.
 (c) $\mathbf{v}_1 = [1\ 0\ 0], \mathbf{v}_2 = [0\ 1\ 0], V = \{[x\ y\ 0]\}$.
 (d) $\mathbf{v}_1 = [1\ 0\ 0], \mathbf{v}_2 = [0\ 1\ 0], \mathbf{v}_3 = [1\ 1\ 0], V = \{[x\ y\ 0]\}$.
 (e) $\mathbf{v}_1 = [1\ -1\ 0 0], \mathbf{v}_2 = [0\ 1\ -1\ 0], \mathbf{v}_3 = [0\ 0\ 1\ -1], V = \{[x\ y\ z\ \ w], x + y + z + w = 0\}$.

(3) For what values of c are the given vectors independent?
 (a) $[1\ 2], [3\ c]$
 (b) $[1\ 2], [0\ c]$
 (c) $[0\ 1\ -1], [1\ 1\ 2], [1\ 1\ c]$
 (d) $[1\ 2\ c], [2\ 3\ c]$.

(4) True/False:
 (a) The vectors $v_1 = [1\ 2\ 3], v_2 = [-1\ 0\ 1],\ v_3 = [2\ 6\ 10]$ are independent.
 (b) The vectors $v_1 = [1\ 0\ -1], v_2 = [1\ -1\ 0],\ v_3 = [0\ 1\ -1]$ are independent.
 (c) The vectors $v_1 = [1\ 0\ 1], v_2 = [1\ 1\ 0],\ v_3 = [0\ 1\ 1]$ are independent.

(5) Find a set of vectors that is as small as possible that has the same span as
 (a) $\{[-1\ 0\ 1], [0\ 1\ 2], [1\ 1\ 1]\}$.
 (b) $\{[-1\ 0\ 1], [-2\ 0\ 2], [-3\ 0\ 3]\}$.
 (c) $\{[-1\ 0\ 1], [0\ 1\ 3], [1\ 1\ 1]\}$.

(6) Prove that
 (a) if $\mathbf{v}_1$ and $\mathbf{v}_2$ are linearly independent, then $\mathbf{v}_1$ and $\mathbf{v}_1 + \mathbf{v}_2$ are linearly independent.
 (b) If $\mathbf{v}_1$ and $\mathbf{v}_2$ span a subset V, then so do $\mathbf{v}_1$ and $\mathbf{v}_1 + \mathbf{v}_2$.
 (c) If $\mathbf{v}_1$ and $\mathbf{v}_2$ are linearly independent, then $\mathbf{v}_1 - \mathbf{v}_2$ and $\mathbf{v}_1 + \mathbf{v}_2$ are linearly independent.
 (d) If $\mathbf{v}_1$ and $\mathbf{v}_2$ span a subset V, then so do $\mathbf{v}_1 - \mathbf{v}_2$ and $\mathbf{v}_1 + \mathbf{v}_2$.

9. Linear subspaces

In three-dimensional geometry *lines* and *planes* play a special role. In linear algebra, we naturally want to do geometry in four dimensional space and higher. What are the analogs of lines and planes? For example, if the variables in four-dimensions are x, y, z, w, then the equation $w = 0$ defines a three-dimensional object, parametrized by variables x, y, z. Sometimes these three-dimensional objects are called *hyperplanes*, but you can see that we go to higher and higher dimensions, we would need more and more new words. A solution is to come up with a word which means something like line or plane, but describes something in any dimension.

These corresponding objects in higher dimensional geometry are called *subspaces*. For example, in four-dimensions with variables x, y, z, w, the equation $w = 0$ would define a three-dimensional subspace, while the equations $w = 0, z = 0$ would describe a two-dimensional subspace.

Here is the formal definition. Recall that for any integer n, $\mathbb{R}^n$ is the set of all n-vectors.

Definition 9.1. (Subspace) A *subspace* V of $\mathbb{R}^n$ is a subset that satisfies the following three properties:

(1) V contains the zero vector 0.
(2) V is closed under vector addition: if v, w are in V then $v + w$ is also in V
(3) V is closed under scalar multiplication: if v is in V and c is a scalar then cv is also in V.

Properties (b) and (c) are equivalent to saying that V is closed under linear combination: if v, w are in V and c, d are scalars then $cv + dw$ is also in V.

Property (a) is equivalent to saying that V is non-empty. This is because if V contains at least one vector v, then it also has to contain $-v$, by (c) and so contain $v + (-v) = 0$, by (b). Usually, when we check that a subset V is a subspace, we will only verify properties (b) and (c), since (a) is usually obvious.

Example 9.2. (1) The set V of all vectors of the form $[x\ x^2]$ is not a subspace, because it is not closed under scalar multiplication. For $v = [2\ 4]$ is in V, but $\frac{1}{2}v = [1\ 2]$ is not.
(2) The set V of all vectors of the form $[x\ 5x]$ is a subspace. In fact, if A is the matrix $[-5\ 1]$, then V is the set of vectors such that $Av = 0$.
(3) The set V of all vectors $[x\ y]$ with $xy = 0$ is not a subspace. It is closed under scalar multiplication but not vector addition, hence not a subspace.
(4) The set V of all vectors $[x\ y]$ with $x \geq 0$ and $y \geq 0$. It is closed under vector addition but not scalar multiplication by negative numbers, hence not a subspace.
(5) The set V of all vectors of the form $[x\ 2x + 1]$ is not a subspace, even though there are no higher order terms. V is closed under

neither scalar multiplication nor vector addition; for instance, [0 1] is in V, but twice it, [0 2] is not.

(6) The set V of all vectors of the form $[x\ y]$ with $x, y \geq 0$ is closed under vector addition, but not scalar multiplication by negative numbers, so it is not a subspace.

(7) The set V of all vectors that are either in the x-axis or the y-axis, that is, $[x\ y]$ such that either x or y is zero, is closed under scalar multiplication but not vector additions, so it is not a subspace.

More generally, we call any subset that is closed under addition and multiplication by scalars a subspace.

Example 9.3. (1) Consider the set M_{nn} of $n \times n$ matrices. Let $V = \{A = A^T\}$ the subset of symmetric matrices. If A, B are symmetric then so is $A + B$; similarly if c is a scalar and A is a symmetric matrix then cA is also symmetric. Hence V is a subspace of M_{nn}.

(2) Consider the set P_n of polynomials in a single variable x of degree at most n. Let V be the set of polynomials such that $p(1) = 0$. Then if p, q are in V, so is $p + q$, and if c is a scalar and p lies in V, then so does cp. Hence V is a subspace.

9.1. **Properties of Subspaces.**

Definition 9.4. (Intersections, unions, and sum of subspaces)

(1) If V and W are subspaces their *intersection* $V \cap W$ is the set of all vectors v that are in both V and in W.

(2) The *union* $V \cup W$ is the set of vectors that are in either V or in W.

(3) The *sum* $V + W$ is the set of vectors of the form $v + w$, for some v in V and w in W.

Example 9.5. If V is the x-axis and W is the y-axis in $\mathbb{R}^3$, then $V \cap W$ is just the origin, a single point; $V \cup W$ is the union of the two axis; $V + W$ is the xy-plane.

Theorem 9.6. (1) *Any subspace V must contain* 0.

(2) *The intersection $V \cap W$ of two subspaces V, W is a subspace.*

(3) *The union $V \cup W$ of two subspaces V, W is not in general a subspace.*

(4) *The sum $V + W$ of two subspaces V, W is a subspace.*

Proof. (1) Take any vector v in V and multiply by $c = 0$. Since V is closed under scalar multiplication, $c\mathbf{v} = 0\mathbf{v} = 0$ is also in V. (2) If v, w are in both V and W, then $c\mathbf{v} + d\mathbf{w}$ is in V and in W, and so in $V \cap W$. (3) See the example above. (4) An element in $V + W$ is of the form $\mathbf{v} + \mathbf{w}$ for some $\mathbf{v}, \mathbf{w}$. Any scalar multiple $c(\mathbf{v} + \mathbf{w}) = c\mathbf{v} + c\mathbf{w}$ is also of this form, so $V + W$is closed under scalar multiplication. We have to show that if we take two elements of $V + W$, they sum to another element. Suppose the second element is $\mathbf{v}' + w'$, Then $\mathbf{v} + \mathbf{w} + (\mathbf{v}' + \mathbf{w}') = (\mathbf{v} + \mathbf{v}') + (\mathbf{w} + \mathbf{w}')$ which is also in $V + W$. □

9.2. **The subspaces associated to a matrix.**

The following three subspaces are associated to a matrix A.

Definition 9.7. (Nullspace, column space, and row space)

(1) The *nullspace* of A is the subspace of all vectors x such that $Ax = 0$, that is, the solution set to the homogeneous system corresponding to A.

(2) The *column space* of A is the span of the columns of A.

(3) The *row space* of A is the span of the rows of A.

Remark 9.8. The meaning of the first two of these spaces is the following:

(1) The *column space* captures whether a linear system $Ax = b$ is *consistent*. Namely, $Ax = b$ is consistent iff b lies in the column space. For example, in the pancake-waffle system elimination gives

$$\left[\begin{array}{cc|c} 2 & 3/2 & f \\ 1 & 2 & s \\ 2 & 2 & e \end{array}\right] \rightarrow \left[\begin{array}{cc|c} 2 & 3/2 & f \\ 0 & 5/4 & s - f/2 \\ 0 & 1/2 & e - f \end{array}\right]$$
$$\rightarrow \left[\begin{array}{cc|c} 2 & 3/2 & f \\ 0 & -5/4 & s - f/2 \\ 0 & 0 & e - f - (2/5)s + (1/5)f \end{array}\right]$$
$$\rightarrow \left[\begin{array}{cc|c} 2 & 3/2 & f \\ 0 & -5/2 & s - 2f \\ 0 & 0 & -(4/5)f - (2/5)s + e \end{array}\right]$$

which is consistent iff $0 = (-4/5)f - (2/5)s + e$ or equivalently

$$e = (4/5)f + (2/5)s.$$

To check, if $f = s = 10$ then we need $(4/5)10 + (2/5)10 = 12$ eggs. So the column space of A is the set of vectors $[f\ \ s\ \ e]$ such that $e = (4/5)f + (2/5)s$, that is, the system is consistent.

(2) The *nullspace* represents the failure of a solution of a linear system $Ax = b$ to be unique. Namely, a solution is unique iff the nullspace of A is the zero vector, and otherwise, any two solutions differ by an element of the null-space. For example, suppose that we have recipes for little (l) pancakes and big (b) pancakes calling for 2 flour

1 sugar 2 eggs and 4 flour 2 sugar 4 eggs respectively. We get a matrix

$$\left[\begin{array}{cc|c} 2 & 4 & f \\ 1 & 2 & s \\ 2 & 4 & e \end{array}\right]$$

This reduces to

$$\left[\begin{array}{cc|c} 2 & 4 & f \\ 1 & 2 & s \\ 2 & 4 & e \end{array}\right] \rightarrow \left[\begin{array}{cc|c} 2 & 4 & f \\ 0 & 0 & s - f/2 \\ 0 & 0 & e - f \end{array}\right]$$

which is consistent if $s - f/2 = 0$ and $e - f = 0$. Let's say that the system is solvable, for example, let's say that $f = e = 10$ and $s = 5$. Then the variable b, the number of batches of big pancakes is free and the solutions are described by $2l + 4b = 10$, or equivalently

$$l + 2b = 5.$$

In other words, we can make 5 batches of little pancakes, 3 batches of little pancakes and 1 batch of little pancakes, 1 batch of little pancakes and 1 batch of big pancakes etc. Note that the solution is not unique, because each big pancake is equivalent to 2 batches of little pancakes.

The nullspace of the matrix is the set of solutions to

$$\left[\begin{array}{cc|c} 2 & 4 & 0 \\ 1 & 2 & 0 \\ 2 & 4 & 0 \end{array}\right]$$

that is, $l + 2b = 0$ or $l = -2b$. Any two solutions to the system differ by a null-space vector. For example, the solutions $l = 5, b = 0$ and $l = 3, b = 1$ differ by the vector $-2, 1$ wihch is in the nullspace.

9.3. **The column space.**

Proposition 9.9. (Interpretation of column space in terms of solving linear systems) *The column space is the space of all vectors b for which $Ax = b$ has a solution.*

In this case, $Ax = b$ has a solution only if the components of b add up to zero. This is because only changes that preserve the total number 90 of students are possible, since we are assuming that no students die from the flu.

Definition 9.10. For any matrix A, the nullspace of A is the set of all vectors $\mathbf{v}$ such that $A\mathbf{v} = 0$.

Lemma 9.11. *For any matrix A, the nullspace of A is a subspace.*

Proof. We have to check that nullspace(A) is closed under linear combinations: Assume that $\mathbf{v}, \mathbf{w}$ are in nullspace(A). By definition $A\mathbf{v} = A\mathbf{w} = 0$, so

$$A(c\mathbf{v} + d\mathbf{w}) = cA\mathbf{v} + dA\mathbf{w} = 0.$$

This implies that $c\mathbf{v} + d\mathbf{w}$ is also in V. □

Also:

Proposition 9.12. (Nullspace as space of dependence relations) *Any vector $\mathbf{w}$ in the null-space gives a dependence relation on the columns $\mathbf{v}_1, \ldots, \mathbf{v}_n$ of A: if $w_1, \ldots, w_n$ are the entries in $\mathbf{w}$ then*

$$w_1\mathbf{v}_1 + \ldots + w_n\mathbf{v}_n = 0.$$

Proof. $Ax = 0$ means $x_1\mathbf{v}_1 + \ldots x_n\mathbf{v}_n = 0$. □

Proposition 9.13. *The span of any set of vectors is a subspace.*

Proof. Closed under $+$: $(c_1\mathbf{v}_1 + \ldots c_r\mathbf{v}_r) + (d_1\mathbf{v}_1 + \ldots d_r\mathbf{v}_r) = (c_1 + d_1)\mathbf{v}_1 + \ldots (c_r + d_r)\mathbf{v}_r$. Closer under $\cdot$: $k(c_1\mathbf{v}_1 + \ldots c_r\mathbf{v}_r) = (kc_1)\mathbf{v}_1 + \ldots (kc_r)\mathbf{v}_r$. □

Here is the algorithm for checking whether a set $\mathbf{v}_1, \ldots, \mathbf{v}_r$ spans $\mathbb{R}^n$: Write the equation

$$c_1\mathbf{v}_1 + \ldots c_n\mathbf{v}_n = v$$

in matrix form. We want to know whether it always has a solution. This is equivalent to showing that the row-echelon form has no rows of zeros.

Example 9.14. Determine whether $[1\ -1\ 0], [1\ 0\ -1], [0\ 1\ -1]$ span $\mathbb{R}^3$.

Answer: We put the vectors in as columns and do elimination:

$$\begin{bmatrix} 1 & 1 & 0 \\ -1 & 0 & 1 \\ 0 & -1 & -1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & -1 & -1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

Since there is a row of zeros in the ref, the system could be inconsistent depending on what is on the other side of equation. So the vectors do not span $\mathbb{R}^3$.

To be even more precise, one can say what the span is:

$$\left[\begin{array}{ccc|c} 1 & 1 & 0 & x \\ -1 & 0 & 1 & y \\ 0 & -1 & -1 & z \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} 1 & 1 & 0 & x \\ 0 & 1 & 1 & x + y \\ 0 & -1 & -1 & z \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} 1 & 1 & 0 & x \\ 0 & 1 & 1 & x + y \\ 0 & 0 & 0 & x + y + z \end{array}\right]$$

which is consisten iff $x+y+z=0$. So the span of the vectors is the plane defined by $x+y+z=0$. An example of a vector which is not in the span of the given three vectors is [1 0 0].

Proposition 9.15. (Number of vectors needed to span $\mathbb{R}^n$) *If $\mathbf{v}_1, \ldots, \mathbf{v}_r$ spans $\mathbb{R}^n$, then r must be at least n.*

Proof. Otherwise, if $r < n$, there will be rows of zeros in the ref of the matrix A with columns $\mathbf{v}_1, \ldots, \mathbf{v}_r$, which means that the system $Ax = b$ will be inconsistent for some choices of b. q ☐

Definition 9.16. (Generating sets) Let V be a subspace of $\mathbb{R}^n$. A set of vectors $S = \{\mathbf{v}_1, \ldots, \mathbf{v}_k\}$ is a *generating set* for V iff $\text{span}(S) = V$, that is, every vector in V is in the span of the vectors in S.

Example 9.17. (1) $\{[1\ 0\ 0], [0\ 1\ 0]$ is a generating set for the plane $z = 0$.
(2) $\{[1\ 0\ 0], [0\ 1\ 0], [1\ 1\ 0]$ is another generating set for the plane $z = 0$.

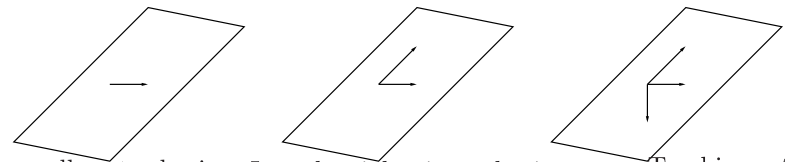
9.4. **Problems.**

(1) For each set of vectors, check whether the set (a) contains 0 (b) is closed under vector addition (c) is closed under scalar multiplication.
  (a) $V = \{0\}$
  (b) $V = \mathbb{R}^n$
  (c) $V = \{[x\ 0\ 0]\}$, the x-axis in $\mathbb{R}^3$.
  (d) $V = \{[x\ y\ 0]\}$, the xy-plane in $\mathbb{R}^3$.
  (e) $V = \{[x\ y\ z], x+y+z=1\}$, a plane in $\mathbb{R}^3$.
  (f) $V = \{[x\ y\ z], x+y+z=0\}$, a plane in $\mathbb{R}^3$.
  (g) $V = \{[x\ y], x \geq 0, y \leq 0\}$, a quadrant in $\mathbb{R}^2$.
  (h) $V = \{[x\ y], xy \leq 0\}$.
  (i) $V = \{[x\ x]\} = \text{span}[1\ 1]$, a line in $\mathbb{R}^2$.
(2) Construct a matrix whose nullspace consists of all combinations of $(1, 1, 1, 0)$ and $(-1, 1, 0, 1)$.
(3) Construct a 2x3 matrix whose column space contains [1 2] and whose null-space contains [1 0 1].
(4) Prove that the span of a single vector $\mathbf{u}$ is a subspace.
(5) Prove that set of vectors $[x\ y\ z]$ with $z = x+y$ is a subspace of $\mathbb{R}^3$.
(6) Prove that set of vectors of the form $[x\ 0\ z]$ is a subspace of $\mathbb{R}^3$.

10. Basis and Dimension

In this section we introduce ways of describing subspaces using finite sets of vectors called *bases*. The number of elements in any basis is called the *dimension* of the subspace.

A basis for a subspace is a minimal generating set. That is, any vector in the subspace is a linear combination of vectors in the basis, and the basis is a smallest set with such a property. Here is a picture:



Too small, not a basis Just the right size, a basis Too big, not a basis

Definition 10.1. A set of vectors $\mathbf{v}_1, \ldots, \mathbf{v}_r$ is a basis for a subspace V if any of the following equivalent conditions are satisfied:

(1) $\mathbf{v}_1, \ldots, \mathbf{v}_r$ is a *minimal generating set* for V
(2) (1) $\mathbf{v}_1, \ldots, \mathbf{v}_r$ is linearly independent and (2) $\mathbf{v}_1, \ldots, \mathbf{v}_r$ spans V.

Example 10.2. $e_1 = [1\ 0\ 0 \ldots\ 0], e_2 = [0\ 1\ 0 \ldots\ 0], \ldots, e_n = [0\ 0\ \ldots\ 0\ 1]$ is the *standard basis* for $\mathbb{R}^n$. We check linear independence: no e_i is a combination of the others, since e_i has a 1 in the i-th entry and the other vectors have i-th entry 0. We check span:

(5)
$$[x_1\ x_2\ \ldots\ x_n] = x_1[1\ 0\ \ldots\ 0] + x_2[0\ 1\ 0 \ldots\ 0] + \ldots + x_n[0\ 0\ \ldots\ 0\ 1]$$
$$x_1 e_1 + \ldots x_n e_n$$

for any $x_1, \ldots, x_n$ which shows every vector in $\mathbb{R}^n$ is a linear combination of the standard basis vectors $e_1, \ldots, e_n$.

The general procedure for finding a basis is the following: Find an expression for the general element of the vector space. Then, express it as a combination of linearly independent elements.

Theorem 10.3. *A set $\mathbf{v}_1, \ldots, \mathbf{v}_r$ of vectors is a basis for a vector space V if and only if any vector in V can be written uniquely as a linear combination of these vectors: $\mathbf{v} = c_1\mathbf{v}_1 + \ldots c_r\mathbf{v}_r$ where $c_1, \ldots, c_r$ are unique.*

Proof. By the "span" part of the definition of basis, any $\mathbf{v}$ can be written as a combination $\mathbf{v} = c_1\mathbf{v}_1 + \ldots c_r\mathbf{v}_r$ of $\mathbf{v}_1, \ldots, \mathbf{v}_r$. To show that the constants $c_1, \dots, c_r$ are unique, we use the "linear independence" part of the definition of basis. Namely, suppose that $\mathbf{v}$ is a linear combiniation in two ways:

$$\mathbf{v} = c_1\mathbf{v}_1 + \dots c_r\mathbf{v}, \quad \mathbf{v} = c_1'\mathbf{v}_1 + \dots c_r'\mathbf{v}.$$

Subtracting the two ways gives

$$0 = (c_1 - c_1')\mathbf{v}_1 + \dots + (c_r - c_r')\mathbf{v}_r.$$

This is a dependence relation unless $c_1 - c_1' = 0, \dots, c_r - c_r' = 0$. Since $\mathbf{v}_1, \dots, \mathbf{v}_r$ are linearly independent, there is no dependence relation, so $c_1 = c_1', \dots, c_r = c_r'$. So the constants $c_1, \dots, c_r$ are unique. □

Theorem 10.4. (Condition for n vectors to be a basis for $\mathbb{R}^n$) *A set of vectors $\mathbf{v}_1, \dots, \mathbf{v}_r$ are a basis for $\mathbb{R}^n$ if and only if the matrix A with columns $\mathbf{v}_1, \dots, \mathbf{v}_r$ is invertible, in particular, A is square so $r = n$.*

Proof. The condition that the vectors span $\mathbb{R}^n$ means that the system $Ax = \mathbf{v}$, is always consistent, so the rref of A has no zero rows. The condition that the vector are independent means that the system $Ax = 0$ has no non-trivial solutions, so the rref of A has a pivot in every column. So $\mathbf{v}_1, \dots, \mathbf{v}_n$ is a basis iff the rref of A is the identity, so A is invertible. □

Example 10.5. Show that if $\mathbf{v}_1, \dots, \mathbf{v}_n$ is a basis, and A is an invertible matrix, that $A\mathbf{v}_1, \dots, A\mathbf{v}_n$ is a also a basis.

Theorem 10.6. (Any two bases have the same number of elements) *Any two bases $\mathbf{v}_1, \dots, \mathbf{v}_r$, $\mathbf{w}_1, \dots, \mathbf{w}_s$ for a finite dimensional vectors space V of $\mathbb{R}^n$ have the same number of elements.*

Proof. By the Theorem above, any $\mathbf{v}_i$ can be written uniquely as a combination of $\mathbf{w}_1, \dots, ww_s$, and any $\mathbf{w}_j$ can be written uniquely as a combination of $\mathbf{v}_1, \dots, \mathbf{v}_r$. That is,

$$\mathbf{v}_1 = c_{11}\mathbf{w}_1 + \dots + c_{1s}\mathbf{w}_s$$

$$\dots$$

$$\mathbf{v}_r = c_{r1}\mathbf{w}_1 + \dots + c_{rs}\mathbf{w}_s$$

and

$$\mathbf{w}_1 = d_{11}\mathbf{v}_1 + \dots + d_{1r}\mathbf{v}_s$$

$$\dots$$

$$\mathbf{w}_s = d_{r1}\mathbf{v}_1 + \dots + d_{sr}\mathbf{w}_s$$

for some constants $c_{11}, \dots, c_{rs}, d_{11}, \dots, d_{rs}$. Let C be the matrix with entries c_{ij} and D is the matrix with entries d_{ij}. Then the above equations can be written

$$\begin{bmatrix} \mathbf{v}_1 \\ \dots \\ \mathbf{v}_r \end{bmatrix} = C \begin{bmatrix} \mathbf{w}_1 \\ \dots \\ \mathbf{w}_s \end{bmatrix}, \begin{bmatrix} \mathbf{w}_1 \\ \dots \\ \mathbf{w}_r \end{bmatrix} = D \begin{bmatrix} \mathbf{v}_1 \\ \dots \\ \mathbf{v}_r \end{bmatrix}.$$

But then by substitution

$$\begin{bmatrix} \mathbf{v}_1 \\ \dots \\ \mathbf{v}_r \end{bmatrix} = DC \begin{bmatrix} \mathbf{v}_1 \\ \dots \\ \mathbf{v}_r \end{bmatrix}, \quad \begin{bmatrix} \mathbf{w}_1 \\ \dots \\ \mathbf{w}_r \end{bmatrix} = CD \begin{bmatrix} \mathbf{w}_1 \\ \dots \\ \mathbf{w}_r \end{bmatrix}.$$

Writing out what this means, if e_{ij} are the coefficients of DC then

$$\mathbf{v}_1 = e_{11}\mathbf{v}_1 + \dots + e_{rr}\mathbf{v}_r$$

$$\dots$$

$$\mathbf{v}_1 = e_{11}\mathbf{v}_1 + \dots + e_{rr}\mathbf{v}_r$$

But since each $\mathbf{v}_1$ is a combination of $\mathbf{v}_1, \dots, \mathbf{v}_r$ is *a unique way,*

$$\mathbf{v}_1 = 1\mathbf{v}_1 + \dots + 0\mathbf{v}_r$$

$$\dots$$

$$\mathbf{v}_1 = 0\mathbf{v}_1 + \dots + 1\mathbf{v}_r$$

we get $e_{11} = e_{22} = \dots e_{rr} = 1$ and the remaining coefficients are 0, and similarly for the coefficients of CD. That is,

$$DC = I, \quad DC = I.$$

But then C, D are invertible which implies C, D are square so $r = s$.

□

Definition 10.7. (Dimension of a subspace) Let V be a subspace. The *dimension* of V is the number of elements in any basis.

Example 10.8.
(1) Let $V = \mathbb{R}^n$. Then $e_1 = [1\ 0\ \dots 0\ 0], \dots e_n = [0\ \ 0\ \dots 0\ 1]$ is a basis, the standard basis, with n vectors. So the dimension of V is n.
(2) Let $V = \{[x\ 0\ ]\}$ the x-axis in $\mathbb{R}^3$. Then a basis is $[1\ 0\ 0]$ so the dimension of V is 1.
(3) Let $V = \{[x\ y\ z], x = 0\}$ be the yz-plane in $\mathbb{R}^3$. A basis is $[0\ 1\ 0], [0\ 0\ 1]$, which has 2 elements so the dimension of V is 2.
(4) Let $V = \{[x\ y\ z], x + y + z = 0\}$. This is a plane in $\mathbb{R}^3$. A basis is $[1\ -1\ 0], [1\ 0\ -1]$ which has size 2. So the dimension is 2.

Definition 10.9. By our convention, the span of the empty set of vector is $\{0\}$, so $V = \{0\}$ has basis given by the empty set and V has dimension 0.

10.1. **Rank and Nullity.** In this section, we call a column of A bound (resp. free) if the corresponding column in $\mathrm{ref}(A)$ contains (resp. does not contain) a leading 1.

Theorem 10.10. (Basis and dimension of subspaces associated to a matrix) *Let A be any matrix.*

(1) *A basis for the null-space is obtained by solving the homogeneous system $Ax = 0$. The dimension of the null-space is the number of free variables.*
(2) *A basis for the column-space is given by the bound columns in A. The dimension of the column-space is the number of leading 1's.*
(3) *A basis for the row-space is given by the non-zero rows in the* $\mathrm{ref}(A)$.

Example 10.11. Find a basis for the nullspace, the row-space, and the column space of the matrix

$$A = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 4 & 7 & 9 & 11 \\ 3 & 6 & 10 & 13 & 16 \end{bmatrix}.$$

Gaussian elimination gives

$$\begin{array}{r} \\ -2\textcircled{1} \\ -3\textcircled{3} \end{array}\begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 \end{bmatrix} \mapsto \begin{array}{r} \\ \\ -\textcircled{2} \end{array}\begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\mapsto \begin{array}{r} -3\textcircled{2} \\ \\ -\textcircled{2} \end{array}\begin{bmatrix} 1 & 2 & 0 & 1 & 2 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

which is the rref of A. The pivots are in columns 1 and 3, so taking the first and third columns from the first matrix gives a basis for the column space

$$\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 3 \\ 7 \\ 10 \end{bmatrix}.$$

Taking the non-zero rows in the rref (one could use the ref as well) gives a basis for the row-space

$$[1\ 2\ 0\ 1\ 2]\,[0\ 0\ 1\ 1\ 1].$$

To find the basis for the null-space, we write out the equations for the rref

$$a = -2b - d - e, \quad c = -d - e$$

which imply the solution set to $A\mathbf{x} = 0$ is

$$\left\{ \begin{bmatrix} -2b - d - e \\ b \\ -d - e \\ d \\ e \end{bmatrix} = b \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + d \begin{bmatrix} -1 \\ 0 \\ -1 \\ 1 \\ 0 \end{bmatrix} + e \begin{bmatrix} -1 \\ 0 \\ -1 \\ 0 \\ 1 \end{bmatrix}. \right\}$$

So a basis for the null-space is

$$\begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ -1 \\ 0 \\ 1 \end{bmatrix}.$$

Proof of Theorem: The null-space is the set of vectors $\mathbf{x}$ such that $A\mathbf{x} = 0$. For any solution $\mathbf{x}$, we call an entry in $\mathbf{x}$ *bound* if it corresponds to a pivot column in A, and otherwise we say it is *free*. By Section 3, the bound entries in $\mathbf{x}$ are linear functions of the free variables, and free entries in $\mathbf{x}$ are equal to the free variables. So $\mathbf{x}$ is linear combination of vectors, one for each free variable, and each vector has one 1 and the rest 0's in its free entries.

The non-zero rows in the ref or rref are linearly independent, because the pivot columns contain exactly one non-zero entry, and span the row-space of A since the span of the rows is unchanged by row operations.

Each basis vector for the null-space gives a dependence relation on the columns of A, containing just one free column. Therefore, the free columns can be expressed in terms of the bound columns, using the basis for the null-space. There are no dependence relations on the free columns in A, since there are no null-space vectors with free variables all zero.

Definition 10.12. (Rank and nullity)

(1) The *rank* of a matrix is the dimension of the column space, which is the same by the theorem above as the dimension of the row-space, and the same as the number of leading 1's.
(2) The *nullity* of a matrix is the dimension of the null-space, which by the theorem above is the same as the number of columns without leading 1's.

An $m \times n$ matrix has rank between 0 and the minimum of m and n, since there can be at most one leading 1 in each row and column.

Example 10.13. Find the rank and nullity of $\begin{bmatrix} 1 & 2 & 3 \\ -1 & -2 & 0 \\ 4 & 6 & 1 \end{bmatrix}$.

Theorem 10.14. *A matrix has rank 0 if and only if it is the zero matrix. A matrix A has rank n if and only if A is invertible.*

Proof. If there are no leading 1's, $\text{ref}(A) = 0$, but then $A = 0$. If every column has a leading 1, $\text{ref}(A) = I$, so A is invertible. $\square$

Theorem 10.15. *(Rank-Nullity Theorem) The dimension of the column space (the rank) plus the dimension of the null-space (the nullity) is equal to the number of columns.*

For instance, if A is a 5×3 matrix with column space 2 dimensional, then the null-space is one-dimensional, so there are homogeneous solutions.

Corollary 10.16. *If A is an $m \times n$ matrix, and $m > n$ then the rows of A are dependent. If $n < m$ then the columns are dependent.*

Proof. If $m > n$ then the rank is at most n, so the dimension of the row-space is at most n. Since there are m vectors in an n-dimensional space, they are dependent. Similar for the case $n < m$ $\square$

Theorem 10.17. *The rank of A is equal to the rank of A^T.*

Proof. The rank of A^T equals the dimension of the column-space of A^T equals the dimension of the row-space of A, which equals the dimension of the column-space of A, which equals the rank of A. $\square$

10.2. **Uniqueness of Reduced Row-Echelon Form.**

Theorem 10.18. *The reduced row echelon form* $\text{rref}(A)$ *of a matrix A is unique.*

Proof. Let W denote the row-space of A, $\mathbf{w}_1, \ldots, \mathbf{w}_r$ the non-zero rows of the $\text{rref}(A)$ and i_j the column number of the leading 1 in $\mathbf{w}_j$. Let $e_1, \ldots, e_n$ be the standard basis for R^n and

$$V_n = \text{span}(e_n), V_{n-1} = \text{span}(e_{n-1}, e_n), \ldots, V_1 = \text{span}(e_1, \ldots, e_n).$$

By induction on $i = r - j + 1$, we show that $\mathbf{w}_j, \ldots, \mathbf{w}_r$ is the unique basis for the intersection $V_{i_j} \cap W$ such that the matrix with rows $\mathbf{w}_j, \ldots, \mathbf{w}_r$ is in reduced row-echelon form.

Case $i = 1$: Then $j = r$. $\mathbf{w}_r$ is the unique vector in the row-space in $V_{i_r} \cap W$ with leading coefficient 1.

Case i implies $i + 1$: Assume $\mathbf{w}_{j+1}, \ldots, \mathbf{w}_r$ is unique. Then there is a unique choice of $\mathbf{w}_j$ so that $\mathbf{w}_j, \ldots, \mathbf{w}_r$ is in reduced row-echelon form, since the entries above the leading 1's must be zero. $\square$

10.3. **Problems.**

(1) Make the following generating sets for the given subspaces into bases by removing vectors.
  (a) $\{[1\ 0], [0\ 1], [1\ 1]\}, V = \mathbb{R}^2$.
  (b) $\{[1\ 0], [2\ 0], [1\ 1]\}, V = \mathbb{R}^2$.
  (c) $\{[0\ 0], [2\ 0], [1\ 1]\}, V = \mathbb{R}^2$.
  (d) $\{[1\ 0\ -1], [0\ 1\ -1], [1\ -1\ 0]\}, V = \{[x\ y\ z], x + y + z = 0\}$.
  (e) $\{[1\ 0\ -1], [2\ 0\ -2], [1\ -1\ 0]\}, V = \{[x\ y\ z], x + y + z = 0\}$.
(2) Find a basis for the given subspaces.
  (a) The subspace of R^4 defined by $x_1 + 2x_3 + x_4 = 0$.
  (b) The space V of solutions to the equation $x - z - w = 0$ in R^4.
  (c) he space V of vectors perpendicular to $[1\ 1\ 1\ 1]$ and $[1\ 2\ 3\ 4]$.
(3) Make the following linearly independent sets for the given subspaces into bases by adding vectors.
  (a) $\{[1\ 0]\}, V = \mathbb{R}^2$.
  (b) $\{[2\ 2]\}, V = \mathbb{R}^2$.
  (c) $\{[1\ 1\ -2]\}, V = \{[x\ y\ z], x + y + z = 0\}$.
  (d) $\{[0\ 0\ 1\ 0]\}, V = \{[x\ y\ z\ w], x + y + w = 0\}$.
  (e) $\{[-1\ 0\ 0\ 1]\}, V = \{[x\ y\ z\ w], x + y + w = 0\}$.
(4) Find a basis for the (a) column-space (b) row-space and (c) null-space of the following matrices.
  (a) $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$.
  (b) $A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$.
  (c) $A = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$.
  (d) $A = \begin{bmatrix} 1 & 2 \\ 1 & 2 \\ 2 & 4 \\ 2 & 4 \end{bmatrix}$.

(e) $A = \begin{bmatrix} 1 & 2 \\ 1 & 3 \\ 2 & 4 \\ 2 & 5 \end{bmatrix}$.

(f) $A = \begin{bmatrix} 1 & -1 & 1 & 2 \\ 2 & -2 & 4 & 2 \\ 1 & -1 & 3 & 0 \end{bmatrix}$.

(5) For each of the matrices in the previous problem, find the rank and nullity of A? Write down a dependence relation on the columns of A, if one exists.

(6) (a) Find a basis for the subspace $V = \{[a\ b\ c\ d],\ a = d,\ b = c\}$.
(b) Find a basis for the subspace $V = \{[a\ b\ c\ d],\ a = b = c\}$.
(c) (See Section 9.) Find a basis for the subspace $V = \{A \in M_{33}, A = A^T\}$.
(d) (See Section 9.) Find a basis for the subspace $V = \{A \in M_{33}, A = -A^T\}$.
(e) (See Section 9.) Find a basis for the subspace V of polynomials $p(x)$ of degree at most 6 such that $p(x) = p(-x)$.

(7) True/False:
(a) The row-space of rref(A) is the same as the row-space of A.
(b) The row space is the orthogonal complement of the nullspace.

11. Geometry of vectors

A vector has a *length* and a *direction*. This means that we can talk about the *angle* between any two non-zero vectors. In this section we relate these geometric ideas to the algebraic notion of *dot product*.

Definition 11.1. If two vectors $\mathbf{v}$ and $\mathbf{w}$ are the same size, define the dot product of $\mathbf{v}$ and $\mathbf{w}$ by taking the sum of the product of their components:

$$\begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix} \cdot \begin{bmatrix} w_1 \\ w_2 \\ \vdots \\ w_n \end{bmatrix} = v_1 w_1 + v_2 w_2 + \dots v_n w_n.$$

For example,

$$\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \cdot \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix} = 1(4) + 2(5) + 3(6) = 32.$$

The dot product can also be thought of as a matrix product between a matrix with one row and a matrix with one column:

$$\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \cdot \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 \end{bmatrix} \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix} = 32.$$

Proposition 11.2. *The dot product of a vector with itself is the square of its length:*

$$\mathbf{u} \cdot \mathbf{u} = u_1^2 + u_2^2 + \ldots + u_n^2 = \|\mathbf{u}\|^2$$

The dot product is related to the angle θ between the two vectors as follows.

Theorem 11.3. *For any two vectors $\mathbf{u}, \mathbf{v}$ of the same size,*

$$\mathbf{u} \cdot \mathbf{v} = \|\mathbf{u}\| \|\mathbf{v}\| \cos(\theta)$$

.

Proof. This formula can be proved using the law of cosines. Look at the triangle with edge vectors $\mathbf{u}, \mathbf{v}$, and $\mathbf{u} - \mathbf{v}$. The law of cosines says

$$\|\mathbf{u} - \mathbf{v}\|^2 = \|\mathbf{u}\|^2 + \|\mathbf{v}\|^2 - 2\|\mathbf{u}\|\|\mathbf{v}\| \cos(\theta).$$

The left-hand-side is

$$(\mathbf{u} - \mathbf{v}) \cdot (\mathbf{u} - \mathbf{v}) = \mathbf{u} \cdot \mathbf{u} - 2\mathbf{u} \cdot \mathbf{v} + \mathbf{v} \cdot \mathbf{v}.$$

Subtracting the quantity

$$\mathbf{u} \cdot \mathbf{u} + \mathbf{v} \cdot \mathbf{v} = \|\mathbf{u}\|^2 + \|\mathbf{v}\|^2.$$

from both sides leads to the formula. □

The equation (11.3) can be used to compute the angle between vectors:

$$\theta = \arccos\left(\frac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{u}\| \|\mathbf{v}\|}\right).$$

Example 11.4. The angle between the vectors $\mathbf{u} = [1\ 1\ 0]$, $\mathbf{v} = [0\ 1\ 1]$ is

$$\begin{aligned} \theta &= \arccos\left(\frac{1(0) + 1(1) + 1(1)}{\sqrt{1^2 + 1^2 + 0^2}\sqrt{0^2 + 1^2 + 1^2}}\right) \\ &= \arccos\left(\frac{1}{2}\right) = \frac{\pi}{3}. \end{aligned}$$

The formula (11.3) has an important special case.

Definition 11.5. Two vectors are *perpendicular* or *orthogonal* if the angle between them is $\pi/2$, that is, 90 degrees. This is the case if and only if

$$\theta = \frac{\pi}{2} \iff \cos(\theta) = 0 \iff \mathbf{u} \cdot \mathbf{v} = 0.$$

That is, *two vectors are perpendicular if and only if their dot product is zero.* By convention, the zero vector is perpendicular to every vector.

Example 11.6. Suppose we want to find a vector $\mathbf{v} = [v_1\ v_2\ v_3]$ perpendicular to $\mathbf{u} = [1\ 1\ 1]$. The condition that the dot product is zero is

$$1(v_1) + 1(v_2) + 1(v_3) = v_1 + v_2 + v_3 = 0.$$

It's easy to find solutions. For example, $v_1 = 1, v_2 = -1, v_3 = 0$, or $v_1 = 0, v_2 = 1, v_3 = -1$ are both solutions. These give the vectors

$$\mathbf{v} = [1\ -1\ 0]\,, \text{ or } \mathbf{v} = [0\ 1\ -1]\,.$$

11.1. Problems.

(1) Find the dot product of the vectors
  (a) $[1\ 0]$ and $[0\ 1]$
  (b) $[1\ 0]$ and $[-1\ -1]$
  (c) $[2\ 2\ 1]$ and $[2\ -1\ 2]$
  (d) $[1\ 0\ 1]$ and $[1\ 1\ 0]$.
(2) Find the angle between the vectors
  (a) $[1\ 0]$ and $[0\ 1]$
  (b) $[1\ 0]$ and $[-1\ -1]$
  (c) $[2\ 2\ 1]$ and $[2\ -1\ 2]$
  (d) $[1\ 0\ 1]$ and $[1\ 1\ 0]$.
(3) In each case, find a non-zero vector perpendicular to the given vector.
  (a) $[1\ 0]$
  (b) $[-1\ -1]$
  (c) $[2\ 2\ 1]$
  (d) $[a\ b\ c]$, where a, b, c are arbitrary scalars.
(4) Find a unit vector $\mathbf{u}$ perpendicular to
  (a) $\mathbf{w} = [1\ 0]$.
  (b) $\mathbf{w} = [1\ 1]$.
  (c) $\mathbf{w} = [2\ 1\ 2]$.
  (d) Prove $(c\mathbf{u}) \cdot \mathbf{v} = c(\mathbf{u} \cdot \mathbf{v})$ for two-vectors $\mathbf{u}, \mathbf{v}$.
  (e) Prove $(c\mathbf{u}) \cdot \mathbf{v} = c(\mathbf{u} \cdot \mathbf{v})$ for n-vectors $\mathbf{u}, \mathbf{v}$.

12. Orthogonality and Gram-Schmidt

In this section we discuss *collections* of orthogonal vectors. A collection of vectors is orthogonal if each pair is orthogonal. Then we discuss a way of creating orthogonal collections, called the *Gram-Schmidt* algorithm.

12.1. Orthogonality.

We begin with orthogonal vectors. Two vectors are orthogonal iff they are perpendicular. For example, given two vectors, if one points north and the other east in the plane then they are orthogonal. If one vector points north and the other points northeast then the two vectors are not orthogonal.

Definition 12.1. (Orthogonal vectors) Vectors $\mathbf{v}_1, \ldots, \mathbf{v}_r$ are *orthogonal* if any two vectors $\mathbf{v}_i, \mathbf{v}_j$ with $i \neq j$ are perpendicular, that is, $\mathbf{v}_i \cdot \mathbf{v}_j = 0$

Example 12.2.
(1) The vectors $[1\ 0\ 0], [0\ 1\ 0], [0\ 0\ 1]$ are orthogonal in $\mathbb{R}^3$.
(2) The vectors $[1\ 0\ 0], [0\ 1\ 0], [0\ 0\ 0]$ are orthogonal in $\mathbb{R}^3$.
(3) The vectors $[1\ 1\ 1\ 1], [1\ -1\ 0\ 0], [0\ 0\ 1\ -1]$ are orthogonal in $\mathbb{R}^4$.

Example 12.3. The standard basis $e_1, \ldots, e_n$ for R^n is orthogonal.

Example 12.4. $[1\ 0\ 0], [0\ 2\ 0], [0\ 0\ 3]$ are orthogonal.

Example 12.5. $[1\ 1], [1\ -1]$ are orthogonal.

Orthogonal vectors are particularly nice for a number of reasons. For instance,

Theorem 12.6. (Orthogonality implies linear independence) *Any orthogonal set of vectors is linearly independent.*

Proof. First proof: Suppose that $\mathbf{v}_1, \ldots, \mathbf{v}_r$ is an orthogonal set of vectors. Suppose one vector, say $\mathbf{v}_r$, is a combination of the others:

$$\mathbf{v}_r = c_1\mathbf{v}_1 + \ldots + c_{r-1}\mathbf{v}_{r-1}.$$

Dot with $\mathbf{v}_r$ on both sides to get

$$\mathbf{v}_r \cdot \mathbf{v}_r = 0 \implies \mathbf{v}_r = 0$$

which is a contradiction.

Second, more symmetric proof: Suppose that

$$c_1\mathbf{v}_1 + \ldots c_r\mathbf{v}_r = 0.$$

Dot with $\mathbf{v}_1$ to get

$$c_1\mathbf{v}_1 \cdot \mathbf{v}_1 + c_1\mathbf{v}_2 \cdot \mathbf{v}_1 + \ldots = c_1\mathbf{v}_1 \cdot \mathbf{v}_1 = 0$$

which implies $c_1 = 0$. Dotting with $\mathbf{v}_2, \mathbf{v}_3$ etc. gives $c_2 = c_3 = 0$. So there are no dependence relations. □

A basis is *orthogonal* if it consists of orthogonal vectors. One of the nice things about orthogonal vectors is that if a vector is in the span of some orthogonal vectors then it is easy to find the coefficients:

Theorem 12.7. (Formula for linear combinations of orthogonal vectors) *Suppose that $\mathbf{v}_1, \ldots, \mathbf{v}_r$ is a basis for V, so that any v can be written uniquely*

$$\mathbf{v} = c_1\mathbf{v}_1 + \ldots + c_r\mathbf{v}_r. \tag{6}$$

If the basis is orthogonal, there is a simple expression for the coefficients $c_1, \ldots, c_r$:

$$c_1 = \frac{\mathbf{v}\cdot\mathbf{v}_j}{\mathbf{v}_j\cdot\mathbf{v}_j}.$$

Proof. Dot both sides of (6) with $\mathbf{v}_j$ to get

$$\begin{aligned}
\mathbf{v}\cdot\mathbf{v}_j &= (c_1\mathbf{v}_1 + \ldots + c_r\mathbf{v}_r)\cdot\mathbf{v}_j \\
&= c_1\mathbf{v}_1\cdot\mathbf{v}_j + \ldots + c_{j-1}\mathbf{v}_{j-1}\cdot\mathbf{v}_j + c_j\mathbf{v}_j\cdot\mathbf{v}_j + c_j\mathbf{v}_{j-1}\cdot\mathbf{v}_j + \ldots c_r\mathbf{v}_r\cdot\mathbf{v}_j \\
&= c_1 0 + c_{j-1}0 + c_j\mathbf{v}_j\cdot\mathbf{v}_j + c_{j+1}0 + \ldots c_r 0 \\
&= c_j\mathbf{v}_j\cdot\mathbf{v}_j.
\end{aligned}$$

Now divide both sides by $\mathbf{v}_j\cdot\mathbf{v}_j$. □

Example 12.8. Suppose we want to express [3 2] as a combination of [1 1] and [1 -1]. One way would be to solve the system

$$\left[\begin{array}{cc|c} 1 & 1 & 3 \\ 1 & -1 & 2 \end{array}\right].$$

But since [1 1], [1 -1] is an orthogonal basis, there is an easier way:

$$c_1 = \frac{[3\ 2]\cdot[1\ 1]}{[1\ 1]\cdot[1\ 1]} = \frac{5}{2}.$$

$$c_2 = \frac{[3\ 2]\cdot[1\ -1]}{[1\ -1]\cdot[1\ -1]} = \frac{1}{2}.$$

Example 12.9. Express [3 2 1] as a combination of

$$\mathbf{v}_1 = [1\ 1\ 1], \mathbf{v}_2 = [1\ -1\ 0], \mathbf{v}_3 = [1\ 1\ -2].$$

Step 1: check that $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ forms an orthogonal basis. $\mathbf{v}_1\cdot\mathbf{v}_2 = 1-1 = 0, \mathbf{v}_2\cdot\mathbf{v}_3 = 1-1 = 0, \mathbf{v}_1\cdot\mathbf{v}_3 = 1+1-2 = 0$. Step 2: Compute the coefficients c_1, c_2, c_3: $c_1 = (3+2+1)/(1+1+1) = 2, c_2 = (3-2)/(1+1), c_3 = (3+2-2)/(1+1+4) = \frac{1}{2}$.

12.2. **Orthonormality.** Orthonormality is similar to orthogonality but in addition to orthogonality one requires that each vector has lengthone.

Definition 12.10. (Orthonormal vectors) A set of vectors $\mathbf{v}_1, \ldots, \mathbf{v}_r$ is *orthonormal* if

(1) $\mathbf{v}_1, \ldots, \mathbf{v}_r$ are orthogonal and
(2) each of the vectors $\mathbf{v}_1, \ldots, \mathbf{v}_r$ is a unit vector.

Proposition 12.11. (Making orthogonal vector orthonormal) *Any orthogonal set of vectors $\mathbf{v}_1, \ldots, \mathbf{v}_r$ can be made into an orthonormal set by dividing by the lengths*

$$\mathbf{u}_1 = \frac{\mathbf{v}_1}{\|\mathbf{v}_1\|}, \ldots, \mathbf{u}_r = \frac{\mathbf{v}_r}{\|\mathbf{v}_r\|}.$$

Example 12.12. The vectors [1 1 1], [1 -1 0], [1 1 -2] is orthogonal, but not orthonormal. To make it orthonormal we divide by the lengths to get

$$\frac{[1\ 1\ 1]}{\sqrt{3}}, \frac{[1\ -1\ 0]}{\sqrt{2}}, \frac{[1\ 1\ -2]}{\sqrt{6}}.$$

A basis $\mathbf{v}_1, \ldots, \mathbf{v}_r$ for a subspace V of R^n is called orthonormal if the vectors are orthonormal.

Proposition 12.13. (Formula for linear combinations of orthonormal vectors) *Suppose $\mathbf{u}_1, \ldots, \mathbf{u}_r$ is an orthonormal basis. Then any vector v can be written*

$$v = c_1\mathbf{u}_1 + \ldots + c_r\mathbf{u}_r$$

where

$$c_j = \mathbf{v}_j\cdot\mathbf{v}.$$

Example 12.14. (Silly example) Let $e_1, \ldots, e_n$ be the standard basis for R^n. Then for any vector x, the formula gives $c_j = e_j\cdot x = x_j$ so that $x = x_1e_1 + \ldots x_ne_n = [x_1\ 0\ 0 \ldots\ 0] + \ldots [0\ \ldots\ 0\ x_n] = [x_1\ \ldots\ x_n] = x$.

Example 12.15. Let's express [3 2 1] in terms of $\mathbf{u}_1 = \frac{[1\ 1\ 1]}{\sqrt{3}}$, $\mathbf{u}_2 = \frac{[1\ -1\ 0]}{\sqrt{2}}$, $\mathbf{u}_3 = \frac{[1\ 1\ -2]}{\sqrt{6}}$. We get

$$c_1 = 6/\sqrt{3}, \quad c_2 = 1/\sqrt{2}, \ c_3 = 3/\sqrt{6}.$$

12.3. **Gram-Schmidt.** Any basis can be made into an orthonormal basis, by a procedure call the *Gram-Schmidt* process. Let's start with just two vectors. We define $\mathbf{u}_1$ by making $\mathbf{v}_1$ into a unit vector: $\boxed{\mathbf{u}_1 = \frac{\mathbf{v}_1}{\|\mathbf{v}_1\|}}$. We want to define $\mathbf{u}_2$ to be a unit vector perpendicular to $\mathbf{u}_1$. It's easier to first construct a vector perpendicular to $\mathbf{u}_1$, and then make it a unit vector, since changing the length doesn't change any angles. Let's try

$$\mathbf{w}_2 = \mathbf{v}_2 - c\mathbf{u}_1.$$

In order to get $\mathbf{u}_1 \cdot \mathbf{u}_2 = 0$, we need

$$(\mathbf{v}_2 - c\mathbf{u}_1) \cdot \mathbf{u}_1 = 0 \implies \mathbf{v}_2 \cdot \mathbf{u}_1 = c\mathbf{u}_1 \cdot \mathbf{u}_1 \implies c = \mathbf{v}_2 \cdot \mathbf{u}_1.$$

Hence $\boxed{\mathbf{w}_2 = \mathbf{v}_2 - (\mathbf{v}_2 \cdot \mathbf{u}_1)\mathbf{u}_1, \quad \mathbf{u}_2 = \frac{\mathbf{v}_2}{\|\mathbf{v}_2\|}.}$

Example 12.16. Make the vector $[3\ 2], [2\ 3]$ into an orthonormal basis using Gram-Schmidt.

$$\mathbf{u}_1 = [3\ 2]/\sqrt{13}.$$

$$\mathbf{w}_2 = [2\ 3] - \frac{12}{13}[3\ 2] = [-10/13\ 15/13].$$

$$\mathbf{u}_2 = \mathbf{w}_2/\|\mathbf{w}_2\| = [-2\ 3]/\sqrt{13}.$$

One can continue the process for more than two vectors:

$\boxed{\mathbf{w}_3 = \mathbf{w}_3 - (\mathbf{w}_3 \cdot \mathbf{u}_1)\mathbf{u}_1 - (\mathbf{w}_3 \cdot \mathbf{u}_2)\mathbf{u}_2, \quad \mathbf{v}_3 = \frac{\mathbf{w}_3}{\|\mathbf{w}_3\|}}$ and so on.

Example 12.17. Make the basis $[1\ 1\ 0], [0\ 1\ 1], [1\ 0\ 1]$ into an orthonormal basis using Gram-Schmidt.

$$\mathbf{u}_1 = \mathbf{v}_1/\|\mathbf{v}_1\| = [1\ 1\ 0]/\sqrt{2}.$$

$$\begin{aligned}
\mathbf{w}_2 &= \mathbf{v}_2 - (\mathbf{v}_2 \cdot \mathbf{u}_1)\mathbf{u}_1) \\
&= ([0\ 1\ 1] - \frac{1}{2}[1\ 1\ 0]) \\
&= [-\frac{1}{2}\ \frac{1}{2}\ 1]
\end{aligned}$$

$$\mathbf{u}_2 = \mathbf{w}_2/\|\mathbf{w}_2\| = [-1 1\ 2]/\sqrt{6}$$

$$\begin{aligned}
\mathbf{w}_3 &= \mathbf{v}_3 - (\mathbf{v}_3 \cdot \mathbf{u}_1) \cdot \mathbf{u}_1 - (\mathbf{v}_3 \cdot \mathbf{u}_2)\mathbf{u}_2 \\
&= [1\ 0\ 1] - \frac{1}{2}[1\ 1\ 0] - \frac{1}{6}[-1\ 1\ 2] \\
&= [\frac{2}{3}\ \ -\frac{2}{3}\ \frac{2}{3}]
\end{aligned}$$

$$\mathbf{u}_3 = \mathbf{v}_3/\|\mathbf{w}_3\| = [1\ \ -1\ 1]/\sqrt{3}.$$

Theorem 12.18. *Let $\mathbf{v}_1, \ldots, \mathbf{v}_r$ be linearly independent. Then the formulas*

$$\mathbf{u}_1 = \mathbf{v}_1/\|\mathbf{v}_1\|$$
$$\mathbf{w}_2 = \mathbf{v}_2 - (\mathbf{v}_2 \cdot \mathbf{u}_1)\mathbf{u}_1, \quad \mathbf{u}_2 = \mathbf{w}_2/\|\mathbf{w}_2\|, \quad \ldots$$
$$\mathbf{v}_r' = \mathbf{v}_r - (\mathbf{v}_r \cdot \mathbf{u}_1)\mathbf{u}_1 - \ldots - (\mathbf{v}_r \cdot \mathbf{u}_{r-1})\mathbf{u}_{r-1}, \quad \mathbf{u}_r = \mathbf{v}_r/\|\mathbf{v}_r'\|$$

define an orthonormal basis for the span of $\mathbf{v}_1, \ldots, \mathbf{v}_r$.

Proof. By induction on r. Step $r = 1$: Clearly $\mathbf{u}_1$ is a unit vector, with the same span as $\mathbf{v}_1$. Step $r - 1 \implies r$. Suppose we have shown that $\mathbf{u}_1, \ldots, \mathbf{u}_{r-1}$ are orthonormal with the same span as $\mathbf{v}_1, \ldots, \mathbf{v}_{r-1}$. Since $\mathbf{v}_r$ is not a combination of $\mathbf{v}_1, \ldots, \mathbf{v}_{r-1}$,

$$\mathbf{v}_r' = (\mathbf{v}_r - (\mathbf{v}_r \cdot \mathbf{u}_1)\mathbf{u}_1 - \ldots - (\mathbf{v}_r \cdot \mathbf{u}_{r-1})\mathbf{u}_{r-1})$$

is non-zero. So $\mathbf{u}_r$ is also non-zero. Therefore, the formula makes sense, and it clearly defines a unit vector. It remains to check $\mathbf{u}_r \cdot \mathbf{u}_j = 0, j < r$. This follows from the formula above, since

$$\mathbf{v}_r' \cdot \mathbf{u}_j = (\mathbf{v}_r \cdot \mathbf{u}_j - 0 - \ldots - (\mathbf{v}_r \cdot \mathbf{u}_j)\mathbf{u}_j \cdot \mathbf{u}_j - 0 \ldots) = \mathbf{v}_r \cdot \mathbf{u}_j - \mathbf{v}_r \cdot \mathbf{u}_j = 0.$$

□

12.4. **Problems.**

(1) Check whether the following sets of vectors are orthogonal, by checking whether the dot products between distinct vectors are zero.
  (a) $[1\ 0\ 0], [0\ 1\ 0], [0\ 0\ 1]$.
  (b) $[1\ 0\ 0\ 0], [0\ 1\ 0\ 0], [0\ 0\ 1\ 1]$.
  (c) $[1\ 0\ 1\ 0], [0\ 1\ 0\ 0], [0\ 0\ 1\ 1]$.
  (d) $[1\ 0\ 0\ 0], [0\ 1\ 0\ 0], [0\ 0\ 0\ 0]$.
(2) Express each vector as a combination of the given vectors, if possible, using the formula for the constants above.
  (a) $[1\ 2\ 3]$ as a combination of $[1\ 0\ 0], [0\ 1\ 0], [0\ 0\ 1]$.
  (b) $[1\ 2\ 3\ 3]$ as a combination of $[1\ 0\ 0\ 0], [0\ 1\ 0\ 0], [0\ 0\ 1\ 1]$.
  (c) $[1\ 2\ 3\ 4]$ as a combination of $[1\ 0\ 0\ 0], [0\ 1\ 0\ 0], [0\ 0\ 1\ 1]$.
  (d) $[3\ \ -4\ 1]$ as a combination of $[1\ \ -1\ 0]$ and $[1\ 1\ \ -2]$.
(3) Use Gram-Schmidt to make the following independent sets (a) orthogonal (b) orthonormal.
  (a) $[1\ 0], [1\ 1]$.
  (b) $[2\ 1], [2\ 2]$.
  (c) $[1\ 0\ 0], [0\ 1\ 0], [1\ 1\ 1]$.
  (d) $[1\ 0\ 0], [1\ 1\ 0], [1\ 1\ 1]$.
  (e) $[1\ \ -1\ 0\ 0], [0\ 1\ \ -1\ 0], [0\ 0\ 1\ \ -1]$.

(f) $[1\ -1\ 0\ 0], [0\ 0\ -1\ 1], [0\ 0\ 1\ -1]$.

(4) Find an orthonormal basis for the following subspaces by (a) finding a basis and (b) making it orthonormal using Gram-Schmidt.
 (a) $V = \{x = 0\}$ in $\mathbb{R}^2$.
 (b) $V = \{x + y = 0\}$ in $\mathbb{R}^2$.
 (c) $V = \{y = 0\}$ in $\mathbb{R}^3$.
 (d) $V = \{x + y + z = 0\}$ in $\mathbb{R}^3$.
 (e) $V = \{x + z + w = 0\}$ in $\mathbb{R}^4$.
 (f) $V = \{x + z + w = 0, y + w = 0\}$ in $\mathbb{R}^4$.

(5) Apply the Gram-Schmidt process to make $\{(1, 1), (2, 0)\}$ into an orthonormal basis.

(6) Find an orthonormal basis for the space V of solutions to the equation $x - z - w = 0$ in R^4. Find the projection of the vector $v = [0, 0, 0, 1]$ onto V. Find the distance of v from V. Find the matrix for the orthogonal projection of V.

13. Orthogonal Matrices and applications

Invertible matrices are matrices whose columns form a basis. Orthogonal matrices are matrices whose columns form an orthonormal basis. It would have been better to call these orthonormal matrices, but the terminology has been established for a long time. It turns out that orthogonal matrices are especially nice, for example, it is easy to find their inverses.

13.1. Orthogonal matrices.

The definition of an orthonormal basis can be written in matrix form. Let Q be the matrix with columns $\mathbf{v}_1, \ldots, \mathbf{v}_r$. Note that the rows of Q^T are $\mathbf{v}_1, \ldots, \mathbf{v}_r$. So Q^TQ is the matrix whose entries are $\mathbf{v}_i \cdot \mathbf{v}_j$, that is, the rows of Q^T dotted with the columns of Q.

Proposition 13.1. *The following conditions are equivalent:*

(1) $\mathbf{v}_1, \ldots, \mathbf{v}_r$ *is orthonormal;*
(2) $\mathbf{v}_i \cdot \mathbf{v}_j = 1$, *if* $i = j$, *and* 0 *otherwise;*
(3) *The matrix whose entries are* $\mathbf{v}_i \cdot \mathbf{v}_j$ *is the identity matrix ;*
(4) $Q^TQ = I$.

Proof. The equivalence of (1) and (2) is just the definition of orthonormal. The equivalence of (2) and (3) follows since matrices are equal iff all their entries are equal. The equivalence of (3) and (4) is just the definition of matrix products Q^TQ whose ij-th entry is the dot product $\mathbf{v}_i \cdot \mathbf{v}_j$ of the i-th row $\mathbf{v}_i$ of Q^T (hence columns of Q) with the j-th column $\mathbf{v}_j$ of Q. □

Proposition 13.2. *The following conditions are equivalent. If any of them hold then the matrix is called* orthogonal*:*

(1) Q *is square and* $Q^TQ = I$.
(2) Q *is square and* $Q^{-1} = Q^T$.
(3) *The columns of* Q *form an orthonormal basis for* R^n*, where* n *is the number of columns of* Q.

Thus orthogonal matrices are those for which the inverse operation is the same as the transpose operation. In particular, inverses of orthogonal matrices are much easier to find than inverses of arbitrary matrices.

Example 13.3. (1) The identity matrix I is orthogonal. Indeed, $I^{-1} = I = I^T$. The columns of I form the standard basis for R^n, which is orthonormal.
(2) Any permutation matrix is orthogonal, for example, $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$.
(3) Any rotation matrix $\begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix}$ is orthogonal.

Orthogonal matrices have a number of nice properties:

Proposition 13.4. (1) *If* Q *is orthogonal, then so is* Q^{-1}.
(2) *If* Q_1 *and* Q_2 *are orthgonal, then so is* Q_1Q_2.
(3) *If* Q *is orthogonal, then* $\det(Q) = \pm 1$.

Proof. (1) $Q^T = Q^{-1}$ implies $(Q^{-1})^T = (Q^T)^{-1} = (Q^{-1})^{-1}$. (2) is left to you. (3) If $Q^TQ = I$ then applying det to both sides we get

$$1 = \det(I) = \det(Q^TQ) = \det(Q^T)\det(Q) = \det(Q)^2$$

so $\det(Q) = \pm 1$. □

Example 13.5. Classify orthonormal bases for R^2. The first vector $\mathbf{u}_1$ can be any unit vector. This means $\mathbf{u}_1 = [\cos(\theta)\ \ \sin(\theta)]$ for some angle θ. The vector $\mathbf{u}_2$ must be a unit vector perpendicular to $\mathbf{u}_1$. There are only two possibilities: $\mathbf{u}_2 = \pm[-\sin(\theta)\ \ \cos(\theta)]$.

Classify orthogonal 2×2 matrices. By what we have just said, the only possibilities are

$$Q_\theta = \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix}, \quad Q'_\theta = \begin{bmatrix} \cos(\theta) & \sin(\theta) \\ \sin(\theta) & +\cos(\theta) \end{bmatrix}.$$

The matrix Q_θ has determinant $\cos^2(\theta) + \sin^2(\theta) = 1$; it is the matrix for the linear transformation given by counter-clockwise rotation by angle θ. The matrix Q'_θ has determinant $-\cos^2(\theta) - \sin^2(\theta) = -1$.

13.2. QR **Factorization.** Suppose that A is the matrix with columns $\mathbf{v}_1, \dots, \mathbf{v}_n$, and let Q be the matrix whose columns are the result of Gram-Schmidt. Each of the column operations in Gram-Schmidt can be realized as multiplication on the right by an elementary matrix:

$$Q = AE_1E_2 \dots E_k.$$

Example 13.6. Let's apply Gram-Schmidt to the three vectors

$$\mathbf{v}_1 = [1\ 1\ 0]\,, \mathbf{v}_2 = [1\ 0\ 1]\,,\ \mathbf{v}_3 = [0\ 1\ 1]\,.$$

Then

$$\mathbf{u}_1 = [1\ 1\ 0]\,/\sqrt{2},$$

$$\mathbf{w}_2 = \mathbf{v}_2 \mapsto \mathbf{v}_2 - (\mathbf{v}_2 \cdot \mathbf{u}_1)\mathbf{u}_1 = [1\ 0\ 1] - \frac{1}{2}[1\ 1\ 0] = \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} & 1 \end{bmatrix} = [1\ \ -1\ 2]$$

$$\mathbf{u}_2 = [1\ \ -1\ 2]\,/\sqrt{6}$$

(7) $$\mathbf{w}_3 = (\mathbf{v}_3 \cdot \mathbf{u}_1)\mathbf{u}_1 - (\mathbf{v}_3 \cdot \mathbf{u}_2)\mathbf{u}_2 = [0\ 1\ 1] - \frac{1}{2}[1\ 1\ 0] - \frac{1}{6}[1\ \ -1\ 2] = \begin{bmatrix} -\frac{2}{3} & \frac{2}{3} & \frac{2}{3} \end{bmatrix} \mapsto [-1\ 1\ 1]\,.$$

$$\mathbf{u}_3 = [-1\ 1\ 1]\,/\sqrt{3}.$$

Each of these operations is equivalent to multiplication of the matrix

$$A = [\mathbf{v}_1\ \mathbf{v}_2\ \mathbf{v}_3] = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}.$$

by an elementary matrix on the *right*. The elementary matrices are

$$E_1 = \begin{bmatrix} 1/\sqrt{2} & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}, \quad E_2 = \begin{bmatrix} 1 & -1/\sqrt{2} & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad E_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \sqrt{\frac{2}{3}} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$E_4 = \begin{bmatrix} 1 & 0 & -1/\sqrt{2} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad E_5 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -1/\sqrt{6} \\ 0 & 0 & 1 \end{bmatrix}, \quad E_6 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1/\sqrt{3} \end{bmatrix}.$$

Switching the elementary matrices over to the other side gives

$$A = QE_k^{-1} \dots E_1^{-1}.$$

Define

$$R = E_k^{-1} \dots E_1^{-1}$$

so that

$$A = QR.$$

Since the E's are upper triangular, so is R. The ij-th entry of R is $\mathbf{u}_i \cdot \mathbf{v}_j$, and the ii-th entry of R is $\|\mathbf{w}_i\|$.

Example 13.7. In the example above

$$R = E_6^{-1}E_5^{-1}E_4^{-1}E_3^{-1}E_2^{-1}E_1^{-1} = \begin{bmatrix} \sqrt{2} & 1/\sqrt{2} & 1/\sqrt{2} \\ 0 & \sqrt{\frac{3}{2}} & 1/\sqrt{6} \\ 0 & 0 & 2/\sqrt{3}/2 \end{bmatrix}.$$

Therefore, the QR factorization of A is

$$\begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{6} & -\frac{2}{3} \\ 1/\sqrt{2} & -1/\sqrt{6} & \frac{2}{3} \\ 0 & 2/\sqrt{6} & \frac{2}{3} \end{bmatrix} \begin{bmatrix} \sqrt{2} & 1/\sqrt{2} & 1/\sqrt{2} \\ 0 & \sqrt{\frac{3}{2}} & 1/\sqrt{6} \\ 0 & 0 & \sqrt{3} \end{bmatrix}.$$

Let's summarize the discussion in a theorem:

Theorem 13.8. *Any matrix A can be factored $A = QR$, where the columns of Q are orthonormal vectors and R is the upper triangular matrix whose ij-th entry is $\mathbf{u}_i \cdot \mathbf{v}_j$ and whose ii-th entry is $\|\mathbf{w}_i\|$.*

13.3. **Problems.**

(1) Determine whether the following matrices Q are orthogonal by checking whether $Q^TQ = I$.
 (a) $\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$
 (b) $\begin{bmatrix} 2 & 0 \\ 0 & 1/2 \end{bmatrix}$
 (c) $\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}/\sqrt{2}$
 (d) $\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$

(2) Find the inverse of each of the following orthogonal matrices in the easiest way possible.
 (a) $\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$
 (b) $\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}/\sqrt{2}$
 (c) $\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$

(3) True/False: (a) The rows of an orthogonal matrix form an orthonormal basis. (b) If Q_1 and Q_2 are orthogonal, then so is $Q_1 Q_2$. (c) If Q is orthogonal, then $\det(Q) = \pm 1$.
(4) True or false:
   (a) If Q is orthogonal, then so is Q^T.
   (b) If Q is an orthogonal matrix, then any eigenvalue of Q has $|\lambda| = 1$.
   (c) If Q is an orthogonal matrix, then so is Q^2.
   (d) If Q^2 is an orthogonal matrix then so is Q.

14. Orthogonal complements and projections

In this section we discuss applications of orthogonality, in particular, orthogonal projections which allow one to find the "closest point" to a given point in a line, plane, etc.

14.1. Orthogonal complements.

The *orthogonal complement* of a subspace is the subspace perpendicular to it. For example, the orthogonal complement of the east-west line through the origin is the north-south line, and the orthogonal complement of the horizontal plane is the vertical line. More formally:

Definition 14.1. (Orthogonal complements) Let V be a subspace of R^n. The *orthogonal complement* of V, denoted $V^\perp$, is the set of all vectors w such that w is perpendicular to v for all vectors v in V. Equivalently, $w \cdot v = 0$ for all v in V.

Example 14.2. (1) The orthogonal complement of $V = \{[x\ 0]\}$ is $V^\perp = \{[0\ y]\}$.
(2) The orthogonal complement of $V = \text{span}[1\ 1]$ is $V^\perp = \text{span}[1\ -1]$.
(3) The orthogonal complement of $V = \text{span}[1\ 1\ 1]$ is $V^\perp = \{[x\ y\ z], x+y+z=0\} = \text{span}[1\ -1\ 0], [0\ 1\ -1]$.
(4) The orthogonal complement of $V = \{[x\ y\ 0\ 0]\}$ in $\mathbb{R}^4$ is $V^\perp = \{[0\ 0\ z\ w]\}$.
(5) The orthogonal complement of $V = \{x+y+z+w\}$ in $\mathbb{R}^4$ is $V^\perp = \{[x\ x\ x\ x]\}$.

Example 14.3. Describe the orthogonal complement of the span of $[2\ 1\ 0\ 1]$ and $[0\ 1\ 1\ 0]$. The equations for V are

$$[2\ 1\ 0\ 1] \cdot \mathbf{v} = [0\ 1\ 1\ 0] \cdot \mathbf{v} = 0$$

which is equivalent to

$$\begin{bmatrix} 2 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix} \mathbf{v} = 0.$$

So we are trying to find the null-space of the matrix $\begin{bmatrix} 2 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix}$. We do this by the nullspace algorithm:

$$\begin{bmatrix} 2 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix} \mapsto [\ -2\ ] \begin{bmatrix} 2 & 0 & -1 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix}$$

which has equations

$$a = c - d, \quad b = -c$$

and solution set

$$\begin{bmatrix} c-d \\ -c \\ c \\ d \end{bmatrix} = c \begin{bmatrix} 1 \\ -1 \\ 1 \\ 0 \end{bmatrix} + d \begin{bmatrix} -1 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

so the space has basis

$$\{ \begin{bmatrix} 1 \\ -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 0 \\ 1 \end{bmatrix}.$$

Note that

Lemma 14.4. *If $\mathbf{v}_1, \ldots, \mathbf{v}_r$ is a basis for V, then w is in $V^\perp$ if and only if w is perpendicular to $\mathbf{v}_1, \ldots, \mathbf{v}_r$.*

Proof. $w\mathbf{v}_j = 0$ for $j = 1, \ldots, r$ implies $w(c_1\mathbf{v}_1 + \ldots c_r\mathbf{v}_r) = 0$ for any scalars c_j, which implies $wv = 0$ for all v in V. □

Example 14.5. Let $v = [1\ 2\ 3]$ in R^3 and let V be the span of V. The orthogonal complement is the set of all vectors perpendicular to v, that is the set of $w = [x\ y\ z]$ such that $x + 2y + 3z = 0$. $V^\perp$ is the plane perpendicular to (or with normal equal to) $[1\ 2\ 3]$.

Example 14.6. Let V be the span of $\mathbf{v}_1 = [1\ 2\ 3]$ and $\mathbf{v}_2 = [3\ 2\ 1]$. Then V is a plane and $V^\perp$ is the perpendicular to this plane, and so is a line. To compute the equation for $V^\perp$, we do elimination:

$$\begin{aligned} V^\perp &= \{[x\ y\ z],\ x + 2y + 3z = 0\} \\ &= \{[-2y - 3z\ y\ z]\} \\ &= \text{span}[-2\ 1\ 0], [-3\ 0\ 1]. \end{aligned}$$

Note we have been using the null-space algorithm to find a basis for $V^\perp$. We can always do this because of the following:

Proposition 14.7. *For any matrix A, the nullspace of A is the orthogonal complement of the rowspace of A.*

Proof. w is in the null-space of A if and only if $Aw = 0$ if and only if each row $\mathbf{v}_j$ dotted with w gives 0. □

14.2. Properties of Orthogonal Complements.

Theorem 14.8.
1. *V and $V^\perp$ intersect in the zero vector.*
2. *If V has dimension r, then $V^\perp$ is a subspace of dimension $n-r$.*
3. *Any vector u in $\mathbb{R}^n$ may be written uniquely as a combination of a vector v in V and a vector w in $V^\perp$.*
4. *For any subspace V, $(V^\perp)^\perp = V$.*

Proof. (1) If u is in V and u is in $V^\perp$, then $u \cdot u = 0$, so $u = 0$.

(2) Since $\mathbf{v}_1, \ldots, \mathbf{v}_r$ are linearly independent, there is a leading one in every row. So there are r leading 1's. Therefore, $\dim V^\perp$ is the number of free variables, which is the number of columns without leading ones, which is $n-r$.

(3) Pick a basis $\mathbf{v}_1, \ldots, \mathbf{v}_r$ for V, and a basis $\mathbf{w}_1, \ldots, \mathbf{w}_{n-r}$ for $V^\perp$. Then $\mathbf{v}_1, \ldots, \mathbf{v}_r, \mathbf{w}_1, \ldots, \mathbf{w}_{n-r}$ is orthonormal, so linearly independent, so a basis for R^n. Hence any vector can be written uniquely

$$\mathbf{u} = c_1\mathbf{v}_1 + \ldots + c_n\mathbf{w}_{n-r}.$$

Let $\mathbf{v} = c_1\mathbf{v}_1 + \ldots + c_r\mathbf{v}_r$ and $\mathbf{w} = c_{r+1}\mathbf{w}_1 + \ldots c_n\mathbf{w}_{n-r}$. Then $\mathbf{v}$ is in V and $\mathbf{w}$ is in W.

We prove that $\mathbf{v}$ and $\mathbf{w}$ are unique. Suppose $\mathbf{u} = v' + w'$ with $\mathbf{v}' \in V, \mathbf{w}' \in W$. Then

$$\mathbf{v} + \mathbf{w} = \mathbf{v}' + \mathbf{w}' \implies \mathbf{v} - \mathbf{v}' = \mathbf{w}' - \mathbf{w}.$$

So $\mathbf{v} - \mathbf{v}' \in W$ and $\mathbf{w}' - \mathbf{w} \in V$. But this is a contradiction, by (1).

(4) $(V^\perp)^\perp$ is the set of vectors u such that u is perpendicular to any vector in $V^\perp$. Given any such vector, we may write it $\mathbf{u} = \mathbf{v} + \mathbf{w}$ by (3). But then $\mathbf{u}$ is perpendicular to $\mathbf{w}$ so

$$\mathbf{u} \cdot \mathbf{w} = 0 + \mathbf{w} \cdot \mathbf{w} = 0$$

which implies $\mathbf{w} = 0$. Hence $\mathbf{u}$ is in V. Conversely, any vector v in V is perpendicular to $V^\perp$, and so lies in $(V^\perp)^\perp$. We have shown that V is contained in $(V^\perp)^\perp$ and vice-versa, so the two subspaces must be equal. □

14.3. Orthogonal Projections.

The orthogonal projection of a vector on a subspace is its "shadow at high noon", that is, its image under rays moving perpendicular to the subspace. In particular, if the vector already lies in the subspace then its orthogonal projection is itself, while if the vector is already orthogonal to the subspace then its orthogonal projection is zero. (A tall thin person at high noon has practically no shadow.)

Definition 14.9. (Orthogonal projections of vectors) Let V be a subspace of R^n, and $\mathbf{u}$ a vector, and $\mathbf{u} = \mathbf{v} + \mathbf{w}$ the decomposition given by (3) above so that $\mathbf{v} \in V$ and $\mathbf{w} \in V^\perp$. The *orthogonal projection* of u onto V is the vector v.

Example 14.10. Suppose V is the xy-plane and $u = [1\ 2\ 3]$. Then $V^\perp$ is the z-axis and the decomposition of u is

$$[1\ 2\ 3] = [1\ 2\ 0] + [0\ 0\ 3].$$

So $v = [1\ 2\ 0]$ is the projection of u onto V and $w = [0\ 0\ 3]$ is the projection of u onto $V^\perp$.

Orthogonal projections can be computed easily given an orthogonal basis for the subspace. Let's start with the case that the subspace V is a line:

Theorem 14.11. (Formula for the orthogonal projection of a vector onto a line) *Let V be the span of a single vector $\mathbf{v}_1$.*

1. *The projection of u onto V is $v = \frac{u \cdot \mathbf{v}_1}{\mathbf{v}_1 \cdot \mathbf{v}_1} v$.*
2. *The projection of u onto $V^\perp$ is $w = v - \frac{u \cdot \mathbf{v}_1}{\mathbf{v}_1 \cdot \mathbf{v}_1}$.*

Proof. Let's write $\mathbf{u}$ as a sum of vectors in V and an orthogonal vector

$$\mathbf{u} = c\mathbf{v}_1 + (\mathbf{u} - c\mathbf{v}_1).$$

Now we solve for c so that

$$c\mathbf{v}_1 \cdot (\mathbf{u} - c\mathbf{v}_1) = 0.$$

We get

$$c\mathbf{v}_1 \cdot \mathbf{u} = c^2\mathbf{v}_1 \cdot \mathbf{v}_1 \implies c = \frac{\mathbf{v}_1 \cdot \mathbf{u}}{\mathbf{v}_1 \cdot \mathbf{v}_1}.$$

□

Example 14.12. Find the projection of the vector $\mathbf{u} = [1\ 0\ 0]$ onto the span V of $\mathbf{v}_1 = [1\ 2\ 3]$. Find the projection of $\mathbf{u}$ onto $V^\perp$.

Now we prove the formula in general. Note that the hard part about the formula is not the formula itself, but actually the assumption that the basis for the subspace is orthogonal. Finding such a basis (which lets you apply the formula) usually requires much more work than applying the formula itself.

Theorem 14.13. (Projection of a vector onto a subspace with an orthogonal basis) *Suppose that V is a subspace with orthogonal basis $\mathbf{v}_1, \dots, \mathbf{v}_r$. Then the projection of u onto V is*

$$v = (u \cdot \mathbf{v}_1)\mathbf{v}_1 + \dots (u \cdot \mathbf{v}_r)\mathbf{v}_r$$

and the projection of u onto $V^\perp$ is

$$w = v - (u \cdot \mathbf{v}_1)\mathbf{v}_1 + \dots (u \cdot \mathbf{v}_r)\mathbf{v}_r.$$

Proof. We write

$$v = c_1\mathbf{v}_1 + \dots c_r\mathbf{v}_r, \quad w = u - c_1\mathbf{v}_1 + \dots c_r\mathbf{v}_r$$

and solve for $c_1, \dots, c_r$ so that $w \cdot \mathbf{v}_j = 0$ for $j = 1, \dots, r$. □

Example 14.14. Find the projection of the vector $\mathbf{u} = [1\ 2\ 3]$ onto the subspace V spanned by $\mathbf{v}_1 = [1\ 1\ 0]$ and $\mathbf{v}_2 = [0\ 1\ 1]$.

14.4. Projection Matrices.

Theorem 14.15. *The map T that sends u to its projection v is a linear transformation. If $\mathbf{v}_1, \dots, \mathbf{v}_r$ is an orthonormal basis, the matrix P for T is*

$$P = \mathbf{v}_1\mathbf{v}_1^T + \dots + \mathbf{v}_r\mathbf{v}_r^T.$$

If $\mathbf{v}_1, \dots, \mathbf{v}_r$ is an orthogonal basis, the formula for the matrix T is

$$P = \frac{\mathbf{v}_1\mathbf{v}_1^T}{\mathbf{v}_1^T\mathbf{v}_1} + \dots + \frac{\mathbf{v}_r\mathbf{v}_r^T}{\mathbf{v}_r^T\mathbf{v}_r}.$$

If $\mathbf{v}_1, \dots, \mathbf{v}_r$ is an arbitrary basis, the formula for the matrix is

$$P = A(A^TA)^{-1}A^T$$

where A is the matrix with columns $\mathbf{v}_1, \dots, \mathbf{v}_r$.

Proof. In the case $\mathbf{v}_1, \dots, \mathbf{v}_r$ is an arbitrary basis for V, Pu is the unique point in V such that Pu is a combination of $\mathbf{v}_1, \dots, \mathbf{v}_r$, and $Pu \cdot \mathbf{v}_j = u \cdot \mathbf{v}_j$ for each $\mathbf{v}_j$. In matrix form, this means that

$$A^TPu = A^Tu.$$

If $P = A(A^TA)^{-1}A^T$ then Pu is A times something, and so a combination of $\mathbf{v}_i'$s. Also

$$A^TPu = A^TA(A^TA)^{-1}A^Tu = A^Tu$$

so $A^TP = A^T$. This shows the formula. □

Example 14.16. Find the matrix for projection onto the xy-plane.

Example 14.17. Find the matrix for projection onto the span of $[1\ 1\ 0]$ and $[0\ 1\ 1]$.

14.5. Problems.

(1) Identify the orthogonal complement to the following spaces.
  (a) $V = \{[x\ 0], x \in \mathbb{R}\}$
  (b) $V = \{[x\ x], x \in \mathbb{R}\}$
  (c) $V = \{[x\ y], x + y = 0\}$
  (d) $V = \mathrm{span}[1\ 1\ 1]$
  (e) $V = \{[x\ y\ z], x + z = 0\}$

(2) In each case, find a basis for V and a basis for $V^\perp$.
  (a) $V = \{[x\ 0], x \in \mathbb{R}\}$
  (b) $V = \{[x\ x], x \in \mathbb{R}\}$
  (c) $V = \{[x\ y], x + y = 0\}$
  (d) $V = \mathrm{span}[1\ 1\ 1]$
  (e) $V = \{[x\ y\ z], x + z = 0\}$

(3) In each case, find an orthonormal basis for V and for $V^\perp$.
  (a) $V = \{[x\ 0], x \in \mathbb{R}\}$
  (b) $V = \{[x\ x], x \in \mathbb{R}\}$
  (c) $V = \{[x\ y], x + y = 0\}$
  (d) $V = \mathrm{span}[1\ 1\ 1]$
  (e) $V = \{[x\ y\ z], x + z = 0\}$

(4) Find the projection of the given vector on the given subspace.
  (a) $\mathbf{v} = [2\ 3], V = \{[x\ 0], x \in \mathbb{R}\}$
  (b) $\mathbf{v} = [2\ 3], V = \{[x\ x], x \in \mathbb{R}\}$
  (c) $\mathbf{v} = [2\ 3], V = \{[x\ y], x + y = 0\}$
  (d) $\mathbf{v} = [1\ 1\ 1], V = \mathrm{span}[1\ 1\ 1]$
  (e) $\mathbf{v} = [1\ 2\ 3], V = \mathrm{span}[1\ 1\ 1]$
  (f) $\mathbf{v} = [1\ 1\ 1], V = \{[x\ y\ z], x + z = 0\}$

(5) Find an orthonormal basis for the subspace V that is the span of the vectors $[1\ 1\ 1]$ and $[1\ 0\ 1]$. Find the matrix for orthogonal projection onto V. Find the projection of the vector $[1\ 0\ 2]$ onto V. Find the closest point to $[1\ 0\ 2]$ in $V^\perp$.

(6) Find a non-zero vector perpendicular to $v = [3\ 1\ 2\ 4]$ in R^4.

(7) Find a basis for the plane V in $\mathbb{R}^4$ *perpendicular* to $[1\ 0\ 0\ 1]$ and $[1\,1\,0\,0]$. (b) Make the basis you found in part (a) into an orthonormal basis, using Gram-Schmidt. (c) Find the matrix for projection onto V. Find the projection of $b = [0\ -1\ 0\ 1]$ onto V.

(8) If V is the plane of vectors in R^4 satisfying $x_1 + 2x_2 + x_3 = 0$, find a basis for $V^\perp$.

(9) (a) Find the projection of the vector $\mathbf{v} = [101]$ onto the line V through $[-10-1]$. (b) Find the matrix P for projection onto V. Check your answer to (a) by computing the product $P\mathbf{v}$.

(10) True/False:
   (a) If P is the matrix for orthogonal projection onto a subspace V, then $P^3 = P$.
   (b) If P is the matrix for orthogonal projection onto a subspace V, then V is an eigenspace for P.
   (c) If P is the matrix for orthogonal projection onto a subspace V, then $V^\perp$ is an eigenspace for P.
   (d) If P is the matrix for orthogonal projection onto a subspace V, then P is diagonalizable.
   (e) If P is the matrix for orthogonal projection onto a subspace V, then the only eigenvalues for P are $0, 1$.

15. Least Squares Approximation

In this section we discuss another application of orthogonality called *least squares approximation*. Least squares approximations arise whenever one is trying to fit a function to a set of data points that have experimental error. The least squares technique allows one to find the function that "best fits" the data points in a precise sense.

Let's start with a simple example in which the function that we are trying to fit is linear. Suppose we want to find the line that best fits the data points $(0,0), (1,0)$ and $(2,3)$. Before we saw how to set this problem up as a system of linear equations: We write $f(x) = c_1x + c_0$ and solve for c_1, c_0

$$c_1(0) + c_0 = 0$$

$$c_1(1) + c_0 = 0$$

$$c_1(2) + c_0 = 3.$$

Since the three points are not colinear, there is no solution. The problem is that the vector $b = \begin{bmatrix} 0 \\ 0 \\ 3 \end{bmatrix}$ is not in the column space of the matrix $A = \begin{bmatrix} 0 & 1 \\ 1 & 1 \\ 2 & 1 \end{bmatrix}$.

Definition 15.1. (Least squares system) Given a linear system $A\mathbf{x} = \mathbf{b}$ that is not solvable, we create a new system that is solvable by orthogonally *projecting* the vector b onto the column space of A. This gives a vector Pb which is a close to b as possible, yet now has a solution. The equation

$$Ax = Pb$$

is called the *least squares equation.*

Here is a shortcut to solving it. Since $b - Pb$ is in the perp of the column space,

$$A^T(b - Pb) = 0.$$

Hence $A^T(Ax - Pb) = 0$ which implies that

$$A^TAx = A^Tb.$$

Any solution is called a least square solution.

Example 15.2. Suppose we want to find the line that best fits the data points $(0,0), (1,0)$ and $(2,3)$. The least squares solution is

$$\begin{bmatrix} 0 & 1 \\ 1 & 1 \\ 2 & 1 \end{bmatrix}^T \begin{bmatrix} 0 & 1 \\ 1 & 1 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 1 \\ 2 & 1 \end{bmatrix}^T \begin{bmatrix} 0 \\ 0 \\ 3 \end{bmatrix}$$

which becomes

$$\begin{bmatrix} 5 & 3 \\ 3 & 3 \end{bmatrix} \begin{bmatrix} c_1 \\ c_0 \end{bmatrix} = \begin{bmatrix} 6 \\ 3 \end{bmatrix}.$$

The solution is $c_1 = \frac{3}{2}, \quad c_0 = -\frac{1}{2}$.

Example 15.3. Find all functions of the form $f(t) = c_0 + c_1|t|$ that best fit the data points $(0,0), (1,1), (-1,2)$.

Answer: The equations are

$$f(0) = c_0 + c_1|0| = c_0 = 0$$
$$f(1) = c_0 + c_1|1| = c_0 + c_1 = 1$$
$$f(-1) = c_0 + c_1|-1| = c_0 + c_1 = 2.$$

Note that since $|-1|=1$, the second and third equations are the same on the left-hand-side, but different on the right, so the system is inconsistent. The matrix form is

$$\left[\begin{array}{cc|c} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 1 & 1 & 2 \end{array}\right].$$

The least squares equation is

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} c_0 \\ c_1 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}$$

or

$$\begin{bmatrix} 3 & 2 \\ 2 & 2 \end{bmatrix}\begin{bmatrix} c_0 \\ c_1 \end{bmatrix} = \begin{bmatrix} 3 \\ 3 \end{bmatrix}.$$

This system reduces

$$\left[\begin{array}{cc|c} 3 & 2 & 3 \\ 2 & 2 & 3 \end{array}\right] \to \left[\begin{array}{cc|c} 1 & 0 & 0 \\ 2 & 2 & 3 \end{array}\right] \to \left[\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 2 & 3 \end{array}\right] \to \left[\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 3/2 \end{array}\right]$$

so the least squares solution is $c_0 = 0, c_1 = 3/2$. The least squares function is

$$f(t) = (3/2)|t|.$$

Example 15.4. Find the functions $f(t) = a + bt + ct^2$ that best fit the data points $(0,0)(1,1),(1,2)$.

Answer: The equations are

$$f(0) = a + b(0) + c(0)^2 = 0$$

$$f(1) = a + b(1) + c(1)^2 = 1$$

$$f(1) = a + b(1) + c(1)^2 = 2.$$

Note that since $|-1|=1$, the second and third equations are the same on the left-hand-side, but different on the right, so the system is inconsistent. The matrix form is

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 11 & & 2 \end{array}\right].$$

The least squares equation is

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}$$

or

$$\begin{bmatrix} 3 & 2 & 2 \\ 2 & 2 & 2 \\ 2 & 2 & 2 \end{bmatrix}\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 3 \\ 3 \\ 3 \end{bmatrix}.$$

This system reduces

$$\left[\begin{array}{ccc|c} 3 & 2 & 2 & 3 \\ 2 & 2 & 2 & 3 \\ 2 & 2 & 2 & 3 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 2 & 2 & 2 & 3 \end{array}\right]$$

$$\to \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 2 & 2 & 3 \\ 0 & 0 & 0 & 0 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 3/2 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

so the least squares solutions are $a = 0, b = 3/2 - c$. The least squares functions are

$$f(t) = (3/2 - c)t + ct^2.$$

15.1. **Problems.**

(1) Find the closest line, through the origin, to the points $(0,8)(1,8),(3,5),(4,10)$.
(2) Using the least squares method, find the function of the form
   (a) $f(x) = c_0 + c_2x^2$ that best fits the data points $(-1,1),(0,0),(1,2)$.
   (b) $f(x) = a + bx^3$ that best fits the data points $(-1,-1),(0,0),(1,3)$.
   (c) $f(x) = ax + bx^2$ that best fits the data points $(-1,0),(1,0),(2,2)$.
   Compute the values of the function $f(x)$ that you found at $x = -1, x = 0$ and $x = 1$. Draw a rough sketch of the function $f(x)$.
(3) Using least-squares approximation find all the functions of the form $f(t) = c_0 + c_1t$. which are best fits for the data points $(-1,0),(0,0),(0,2)$.
(4) Apply the least squares method to find the closest line(s) to the data points $(0,0),(2,0),(2,3)$.
(5) Apply the least squares method to find the curve of the form $c_0 + c_1x + c_2x^2$ best fitting the points $(-1,1),(0,0),(1,1),(2,1)$.
(6) Find all the functions of the form $f(t) = c_0 + c_1\cos(\pi t) + c_2\cos(2\pi t)$. that are best fits for the data points $(-\frac{1}{2},1),(0,0),(\frac{1}{2},0)$.

16. Eigenvectors and eigenvalues

Eigenvectors and eigevalues are often a method of computing large matrix powers in models where time evolution is given by matrix multiplication.

Here is a motivating example. Consider the following mathematical model for the market for cola. Suppose $c(t)$ (resp. $p(t)$) is the number of Coke (resp. Pepsi) drinkers at time t months. Suppose each month, 10 percent of the Coke drinkers switch to become Pepsi drinkers, and 20 percent of the Pepsi drinkers switch to Coke. If we start with 100 Pepsi drinkers and no Coke drinkers, what happens as t goes to infinity?

t	p	c
0	0	100
1	20	80
2	34	66

To set this up as a linear algebra problem we write

$$c(t+1) = .9c(t) + .2p(t)$$

$$p(t+1) = .1c(t) + .8p(t)$$

or in matrix form

$$\mathbf{x}(t+1) = A\mathbf{x}(t) \text{ where } A = \begin{bmatrix} .9 & .2 \\ .1 & .8 \end{bmatrix} \text{ and } \mathbf{x}(t) = \begin{bmatrix} c(t) \\ p(t) \end{bmatrix}.$$

This implies that

$$\mathbf{x}(t) = A\mathbf{x}(t-1) = A^2\mathbf{x}(t-2) = \ldots = A^t\mathbf{x}(0)$$

for any time t. The best method for solving this for large t is *eigenvectors/eigenvalues*.

Definition 16.1. An *eigenvector* of a square matrix A is a vector $\mathbf{x}$ such that $A\mathbf{x} = \lambda\mathbf{x}$ for some number λ, called the eigenvalue of $\mathbf{x}$. An *eigenvalue* of a square matrix A is a number λ such that $A\mathbf{x} = \lambda\mathbf{x}$ for some vector $\mathbf{x}$, called an eigenvector for λ.

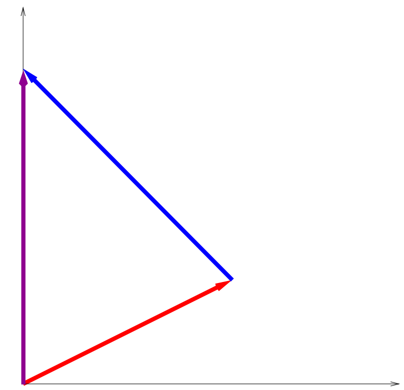
Geometrically, an eigenvector is a vector $\mathbf{x}$ such that $A\mathbf{x}$ lies in the same direction (or opposite direction) as the original vector. The eigenvalue λ is the "stretch factor".

Example 16.2. Say $A = \begin{bmatrix} .9 & .2 \\ .1 & .8 \end{bmatrix}$ as above. Then $\mathbf{x}_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ is an eigenvector with eigenvalue .7. Also $\mathbf{x}_2 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ is an eigenvector with eigenvalue 1.

Let's use these eigenvectors to solve the coke/pepsi problem described above. To begin, we write the *initial state vector* $\mathbf{x}_0 = \begin{bmatrix} 0 \\ 100 \end{bmatrix}$ in terms of the eigenvectors:

$$\mathbf{x}_0 = \begin{bmatrix} 0 \\ 100 \end{bmatrix} = -(200/3)\begin{bmatrix} 1 \\ -1 \end{bmatrix} + (100/3)\begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} -66 \\ 66 \end{bmatrix} + \begin{bmatrix} 66 \\ 33 \end{bmatrix}.$$

Graphically, the vector $\mathbf{x}_0$ is the purple vector while the blue and red vectors are the second and first components, respectively. (The second is drawn first, for reasons that will become clear in a moment.)

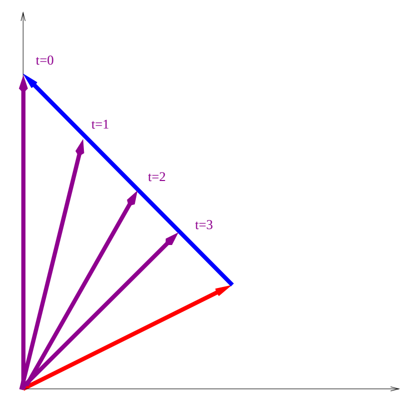


Now

$$\mathbf{x}_t = A\mathbf{x}_{t-1} = A^2\mathbf{x}_{t-2} = \ldots = A^t\mathbf{x}_0 = (200/3)A^t\begin{bmatrix} 1 \\ -1 \end{bmatrix} + (100/3)A^t\begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

$$= (200/3)(.7)^t\begin{bmatrix} 1 \\ -1 \end{bmatrix} + (100/3)(1)^t\begin{bmatrix} 2 \\ 1 \end{bmatrix}.$$

Each time step, the first component of $\mathbf{x}_t$ shrinks by 30 percent, while the second factor stays the same. This is shown in the figure below:



For t very large, $(.7)^t$ is approximately zero. (That is, the blue component shrinks to zero.) So

$$\mathbf{x}_t \to (100/3)\begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 66.7 \\ 33.3 \end{bmatrix}.$$

That is, in the long run $\frac{2}{3}$ of the customers are with Coke, and 1/3 with Pepsi.

16.1. **Finding eigenvalues.** First we find the eigenvalues. The following theorem gives a practical method for doing so.

Theorem 16.3. *The following are equivalent:*

(1) λ *is an eigenvalue of* A.
(2) $A\mathbf{v} = \lambda\mathbf{v}$ *for some vector* $\mathbf{v} \neq 0$.
(3) $(A - \lambda I)\mathbf{v} = 0$ *for some vector* $\mathbf{v} \neq 0$.
(4) $\text{nullspace}(A - \lambda I) \neq 0$.
(5) $A - \lambda I$ *is not invertible.*
(6) $\det(A - \lambda I) = 0$.

Proof. (i) $\iff$ (ii) is the definition of eigenvalue. (iii) is obtained from (ii) by subtracting $\lambda\mathbf{v}$ from both sides. (iii) $\iff$ (iv) is the definition of null-space. The null-space contains a non-zero vector iff there is a free variable iff rref(A) is not the identity iff A is invertible, hence (iv) $\iff$ (v). Since a matrix is invertible iff it has non-zero determinant, (v) $\iff$ (vi). $\square$

By the Theorem, to find the eigenvalues we have to solve $\det(A - \lambda I) = 0$. The polynomial $\det(A - \lambda I)$ is the *characteristic polynomial* of A.

Example 16.4. If $A = \begin{bmatrix} .9 & .2 \\ .1 & .8 \end{bmatrix}$ then

$$0 = \det(A - \lambda I) = \det(\begin{bmatrix} .9 & .2 \\ .1 & .8 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} = (.9 - \lambda)(.8 - \lambda) - .02 = .7 - 1.7\lambda + \lambda^2.$$

Solving such an equation is equivalent to factoring

$$.7 - 1.7\lambda + \lambda^2 = (\lambda - \lambda_1)(\lambda - \lambda_2).$$

We want to find numbers λ_1, λ_2 so that $\lambda_1 + \lambda_2 = 1.7$ and $\lambda_1\lambda_2 = .7$. The solution is

$$\lambda_1 = 1, \quad \lambda_2 = .7.$$

This gives the eigenvalues above.

Example 16.5. Let $A = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}$. To find the eigenvalues we set

$$0 = \det(A - \lambda I) = \begin{bmatrix} 2-\lambda & -1 & 0 \\ -1 & 2-\lambda & -1 \\ 0 & -1 & 2-\lambda \end{bmatrix}.$$

By expanding along the first row this equals

$$(2-\lambda)((2-\lambda)^2 - 1) - (-1)(-1)(2-\lambda) = (2-\lambda)(\lambda^2 - 4\lambda + 3) - (2-\lambda)$$

$$= (2-\lambda)(\lambda^2 - 4\lambda + 2) = (2-\lambda)(\lambda - (2+\sqrt{2}))(\lambda - (2-\sqrt{2})).$$

So the eigenvalues are

$$\lambda_1 = 2, \lambda_2 = (2+\sqrt{2}), \lambda_3 = (2-\sqrt{2}).$$

Example 16.6. Let $A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}$. To find the eigenvalues we set

$$0 = \det(A - \lambda I) = (1-\lambda)(1-\lambda)(1-\lambda).$$

So the eigenvalues are $\lambda = 1, 1, 1$.

More generally, if A is upper or lower triangular or diagonal, then $A - \lambda I$ is also upper or lower triangular, so that $\det(A - \lambda I)$ is the product $(a_{11} - \lambda)\dots(a_{nn} - \lambda)$. This shows

Theorem 16.7. *If A is upper or lower triangular or diagonal then the eigenvalues of A are the diagonal entries $a_{11}, \dots, a_{nn}$.*

16.2. **Finding eigenvectors.** Once we have found the eigenvalues, we can find the eigenvectors. The following are equivalent:

(1) v is an eigenvector of A with eigenvalues λ;
(2) $Av = \lambda v$
(3) $(A - \lambda I)v = 0$
(4) v is in the nullspace of $A - \lambda I$.

So to find the eigenvectors we have to find the nullspace of $A - \lambda I$, for each eigenvalue λ.

Example 16.8. Let $A = \begin{bmatrix} .9 & .2 \\ .1 & .8 \end{bmatrix}$. The eigenvalues are $\lambda = .7$ and $\lambda = 1$. We compute

$$\begin{aligned} \text{nullspace}\, A - .7I &= \text{nullspace} \begin{bmatrix} .9 & .2 \\ .1 & .8 \end{bmatrix} - \begin{bmatrix} .7 & 0 \\ 0 & .7 \end{bmatrix} \\ &= \text{nullspace} \begin{bmatrix} .2 & .2 \\ .1 & .1 \end{bmatrix} \\ &= \text{nullspace} \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \\ &= \text{span} \begin{bmatrix} 1 \\ -1 \end{bmatrix}. \end{aligned}$$

$$\begin{aligned} \text{nullspace}\, A - (1)I &= \text{nullspace} \begin{bmatrix} .9 & .2 \\ .1 & .8 \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \\ &= \text{nullspace} \begin{bmatrix} -.1 & .2 \\ .1 & -.2 \end{bmatrix} \\ &= \text{nullspace} \begin{bmatrix} 1 & -2 \\ 0 & 0 \end{bmatrix} \\ &= \text{span} \begin{bmatrix} 2 \\ 1 \end{bmatrix}. \end{aligned}$$

So "the" eigenvectors are

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \ \mathbf{v}_2 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

as we claimed above. Note these vectors are not unique: any multiples of $\mathbf{v}_1, \mathbf{v}_2$ are also eigenvectors.

Example 16.9. Let $A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 3 \end{bmatrix}$ so the eigenvalues are $1, 1, 3$. Then

$$\begin{aligned} \text{nullspace}\, A - (1)I &= \text{nullspace} \begin{bmatrix} 0 & 0 & 3 \\ 0 & 0 & 2 \\ 0 & 0 & 2 \end{bmatrix} \\ &= \text{nullspace} \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} = \text{span} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}. \end{aligned}$$

and

$$\begin{aligned} \text{nullspace}\, A - (3)I &= \text{nullspace} \begin{bmatrix} -2 & 2 & 0 \\ 0 & -2 & 3 \\ 0 & 0 & 0 \end{bmatrix} \\ &= \text{nullspace} \begin{bmatrix} 1 & 0 & -\frac{3}{2} \\ 0 & 1 & -\frac{3}{2} \\ 0 & 0 & 0 \end{bmatrix} = \text{span} \begin{bmatrix} \frac{3}{2} \\ \frac{3}{2} \\ 1 \end{bmatrix}. \end{aligned}$$

So in this case there are two eigenvectors

$$\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} \frac{3}{2} \\ \frac{3}{2} \\ 1 \end{bmatrix}.$$

Example 16.10. Every vector is an eigenvector for the identity matrix I with eigenvalue 1.

16.3. **Properties of the eigenvalues.** The characteristic polynomial $\det(A - \lambda I)$ has degree n, so there are at most n solutions to $\det(A - \lambda I) = 0$. If there are exactly n solutions $\lambda_1, \ldots, \lambda_n$ to the equation $\det(A - \lambda I) = 0$ in the real numbers, so that

$$\det(A - \lambda I) = (\lambda_1 - \lambda) \ldots (\lambda_n - \lambda),$$

we say that the eigenvalues of A are *all real*. This terminology will be explained later; for the moment we assume that all eigenvalues are real. The number of times a factor $(\lambda_i - \lambda)$ appears is the *algebraic multiplicity* of λ_i.

Example 16.11. Suppose $A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 3 \end{bmatrix}$. Then $\det(A - \lambda I) = (1 - \lambda)^2(3 - \lambda)$ so $\lambda = 1$ is an eigenvalue with algebraic multiplicity 2 and $\lambda = 3$ is an eigenvalue with algebraic multiplcity 1.

The *trace* of a square matrix A is the sum of the diagonal entries:

$$\text{Tr}(A) = a_{11} + \ldots + a_{nn}.$$

For example, the trace of the $n \times n$ identity matrix is $1 + \ldots + 1 = n$.

Theorem 16.12. (Properties of eigenvalues)

(1) *The number of real eigenvalues, counted with algebraic multiplicity, is at most n.*
(2) *The number of complex eigenvalues, counted with algebraic multiplicity, is exactly n.*

(3) *The determinant of A is the product of the complex eigenvalues taken with algebraic multiplicity:* $\det(A) = \lambda_1\lambda_2\dots\lambda_n$.
(4) *The trace of A is the sum of the eigenvalues, with algebraic multiplicity:* $\mathrm{tr}(A) = \lambda_1 + \lambda_2 + \dots + \lambda_n$.
(5) *The tranpose A^T of a square matrix A has the same eigenvalues as A.*
(6) *Suppose that the columns of A sum up to 1. Then $\lambda = 1$ is an eigenvalue for A.*

Proof. (c) We plug $\lambda = 0$ into the characteristic polynomial to get $\det(A) = (\lambda_1)(\lambda_2)\dots(\lambda_n)$. (d) Note that the characteristic polynomial is

$$(\lambda_1 - \lambda)\dots(\lambda_n - \lambda) = (-1)^n\lambda^n + (\lambda_1 + \dots + \lambda_n)(-1)^{n-1} + O(\lambda^{n-2})$$

where $O(\lambda^{n-2})$ means terms of order at most $n-2$ in λ. On the other hand, the only term in $\det(A - \lambda I)$ involving at least $n-1$ λ's is

$$(a_{11}-\lambda)\dots(a_{nn}-\lambda) = (-1)^n\lambda^n + (a_{11}+\dots+a_{nn})(-1)^{n-1}\lambda^{n-1} + O(\lambda^{n-2}).$$

Equating the coefficients of λ^{n-1} finishes the proof. (e) The characteristic polynomial

$$\det(A^T - \lambda I) = \det(A^T - \lambda I^T) = \lambda((A - \lambda I)^T) = \det(A - \lambda I).$$

So the eigenvalues, which are the roots of the characteristic polynomial, are also the same. (f) If the columns of A sum up to 1 then

$$[1\ 1\ 1\dots 1]\,A = [1\ 1\ 1\dots 1]\,A$$

which implies

$$A^T \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix}$$

which implies that $\lambda = 1$ is an eigenvalue of A^T. By the Theorem, this implies implies that $\lambda = 1$ is an eigenvalue of A. □

Corollary 16.13. *A matrix is invertible only if* 0 *is not an eigenvalue.*

Proof. A is invertible, iff $\det(A) \neq 0$, iff none of the λ_i's is zero. □

Suppose A is a matrix which represents a physical system in which the total number is preserved, e.g. the matrix in the Coke/Pepsi example

$$A = \begin{bmatrix} .9 & .2 \\ .1 & .8 \end{bmatrix}.$$

Any such matrix has $\lambda = 1$ as an eigenvalue. This means that there is a vector v that is an equilibrium for the system, that is $Av = v$.

16.4. **Properties of the eigenvectors.** For any eigenvalue λ define

$$E_\lambda = \mathrm{nullspace}(A - \lambda I).$$

This is the λ-*eigenspace for A*. The dimension of E_λ is called the *geometric multiplicity* of λ.

Theorem 16.14. (Properties of eigenvectors)

(1) *The number of independent eigenvectors with eigenvalue λ is between 1 and the algebraic multiplicity of λ.*
(2) *Eigenvectors from different eigenspaces are independent: If $\mathbf{v}_1, \dots, \mathbf{v}_r$ is a collection of vectors from different eigenspaces $E_{\lambda_1}, \dots, E_{\lambda_r}$ then $\mathbf{v}_1, \dots, \mathbf{v}_r$ are linearly independent.*
(3) *The total number of independent eigenvectors of an $n \times n$ matrix is between 1 and n.*

Proof. (b) Suppose one, say $\mathbf{v}_r$ is a combination of the others

$$\mathbf{v}_r = c_1\mathbf{v}_1 + \dots c_{r-1}\mathbf{v}_{r-1}.$$

Applying A to both sides we get

$$\lambda_r\mathbf{v}_r = c_1\lambda_1\mathbf{v}_1 + \dots + c_{r-1}\lambda_{r-1}\mathbf{v}_{r-1}.$$

Subtracting λ_r times the first equation we get

$$0 = c_1(\lambda_1 - \lambda_r)\mathbf{v}_1 + \dots + c_{r-1}(\lambda_{r-1} - \lambda_r)\mathbf{v}_{r-1}.$$

By the inductive hypothesis, $\mathbf{v}_1, \dots, \mathbf{v}_{r-1}$ are independent so

$$c_1(\lambda_1 - \lambda_r) = \dots = c_{r-1}(\lambda_{r-1} - \lambda_r) = 0.$$

Since all the eigenvalues $\lambda_1, \dots, \lambda_r$ are distinct, this implies that

$$c_1 = \dots = c_{r-1} = 0$$

which shows that $\mathbf{v}_1, \dots, \mathbf{v}_r$ are independent. □

Example 16.15. Show that if v is an eigenvector for A and B, then it is an eigenvector for $A + B$.

Answer: Suppose v is an eigenvector for A and B. Then Av is in the same direction as v, and so is Bv. So $Av = \lambda v$ for some λ. Bv is also λv for some λ, but we cannot use the same notation because they might be different scalars. So suppose $Av = \lambda_A v$ and $Bv = \lambda_B v$. Now we ask, is $(A + B)v$ the same direction as v? To answer this we distribute

$(A+B)v = Av + Bv = \lambda_A v + \lambda_B v = (\lambda_A + \lambda_B)v$ which is in the same direction as v. So, yes.

16.5. **Problems.**

(1) In each case, determine whether the given vector is an eigenvector for the given matrix and, if so, find the eigenvalue.
   (a) $A = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$.
   (b) $A = \begin{bmatrix} 3 & 0 \\ 0 & 4 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$.
   (c) $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$.
   (d) $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$.
   (e) $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$.
   (f) $A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$.
   (g) $A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$.
   (h) $A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$.

(2) In each case, find the characteristic polynomial of the matrix and factor it to find the eigenvalues.
   (a) $A = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$.
   (b) $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$.
   (c) $A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$.
   (d) $A = \begin{bmatrix} .9 & .2 \\ .1 & .8 \end{bmatrix}$.
   (e) $A = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix}$
   (f) $A = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$
   (g) $A = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 2 & 3 \\ 0 & 0 & 4 \end{bmatrix}$.

(3) True or false? Explain.
   (a) A square matrix is not invertible, if and only if 0 is an eigenvalue.

(4) Show that λ is an eigenvalue of A, if and only if $\det(A - \lambda I) = 0$.

(5) Show that if v_1, v_2 are eigenvectors of a matrix A with different eigenvalues λ_1, λ_2 then v_1, v_2 are linearly independent.

(6) Show that if v is an eigenvector for A and B, then v is an eigenvector for AB.

17. Diagonalization

In this section we rephrase the method of finding large matrix powers using eigenvalues and eigenvectors in a different, more efficient way.

Definition 17.1. If an $n \times n$ matrix A has n independent eigenvectors $\mathbf{v}_1, \ldots, \mathbf{v}_n$, A is called *diagonalizable*. In this case the eigenvectors $\mathbf{v}_1, \ldots, \mathbf{v}_n$ form a basis for R^n called an *eigenbasis*.

Example 17.2. (1) Any diagonal matrix is diagonalizable.

(2) In particular, the identity matrix is diagonal, so diagonalizable. Any vector is an eigenvector for the identity matrix, so any basis is an eigenbasis for the identity matrix.

(3) If $A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$ then $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $\mathbf{v}_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ are eigenvectors with eigenvalues $2, 0$ respectively.

(4) If $A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ then A is rotation by 90 degrees. So there are no (real) eigenvectors, and so no eigenbasis.

Theorem 17.3. *If A is diagonalizable, then $A = SDS^{-1}$ where S is the matrix whose columns are the eigenvectors of A, and D is the diagonal matrix of eigenvalues.*

Example 17.4. Find the diagonalization of $A == \begin{bmatrix} .9 & .2 \\ .1 & .8 \end{bmatrix}$, if it exists.

Answer: $\det(A - \lambda I) = \lambda^2 - 1.7\lambda + (.72 - .02) = (\lambda - 1)(\lambda - .7)$. So the eigenvalues are $\lambda = 1, -.7$. (Alternatively just note that A is Markov so $\lambda = 1$ is an eigenvalue and the sum of the eigenvalues is the trace 1.7 so the other eigenvalue is .7.) The eigenvectors are

$$E_{\lambda=1} = \text{nullspace}(A - I) = \text{nullspace}\begin{bmatrix} -.1 & .2 \\ .1 & -.2 \end{bmatrix} = \text{span}\begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

and

$$E_{\lambda=-.7} = \text{nullspace}(A - .7I) = \text{nullspace}\begin{bmatrix} .2 & .2 \\ .1 & .1 \end{bmatrix} = \text{span}\begin{bmatrix} 1 \\ -1 \end{bmatrix}.$$

So

$$S = \begin{bmatrix} -1 & 2 \\ 1 & 1 \end{bmatrix}, \quad D = \begin{bmatrix} .7 & 0 \\ 0 & 1 \end{bmatrix}$$

then $A = SDS^{-1}$.

Remark 17.5. The order that you choose for the eigenvalues doesn't matter as long as you choose the same order for the eigenvectors.

Example 17.6. The matrix $A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 3 \end{bmatrix}$ with eigenvalues $1, 1, 3$ is *not* diagonalizable because there are only *two* independent eigenvectors $\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} \frac{3}{2} \\ \frac{3}{2} \\ 1 \end{bmatrix}$.

Example 17.7. The identity matrix, or more generally any diagonal matrix is diagonalizable since it is already diagonal! In this case $S = I$ and $D = A$.

Finding the matrices S and D is called *diagonalizing A*.

Theorem 17.8. *If a matrix A has n distinct eigenvalues $\lambda_1, \dots, \lambda_n$, then A is diagonalizable.*

Proof. Choose an eigenvector $\mathbf{v}_i$ for each eigenvalue λ_i. Since there are n-eigenvalues, there are n eigenvectors, independent by the theorem above. □

17.1. **Application to matrix powers.** Suppose A is a square matrix, and we want to find a large power of A, say A^t. The best way to do this is using diagonalization:

$$A^t = (SDS^{-1})^t = SDS^{-1}SDS^{-1} \dots SDS^{-1} = SD^tS^{-1}.$$

Since D is diagonal, its matrix powers are easy to compute:

$$D^t = \text{diag}(\lambda_1^t, \dots, \lambda_n^t).$$

Example 17.9. Find A^t, where $A == \begin{bmatrix} .9 & .2 \\ .1 & .8 \end{bmatrix}$. Then

$$\begin{aligned} A^t &= SD^tS^{-1} \\ &= \begin{bmatrix} -1 & 2 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} .7^t & 0 \\ 0 & 1^t \end{bmatrix}\begin{bmatrix} -1 & 2 \\ 1 & 1 \end{bmatrix}^{-1} \\ &= \begin{bmatrix} -1(.7)^t & 2 \\ 1(.7)^t & 1 \end{bmatrix}\frac{-1}{3}\begin{bmatrix} 1 & -2 \\ -1 & -1 \end{bmatrix} \\ &= \frac{1}{3}\begin{bmatrix} (.7)^t + 2 & -2(.7)^t + 2 \\ -(.7)^t + 1 & 2(.7)^t + 1 \end{bmatrix}. \end{aligned}$$

As t becomes very large this matrix approaches

$$A^\infty := \frac{1}{3}\begin{bmatrix} 2 & 2 \\ 1 & 1 \end{bmatrix}.$$

17.2. **Similarity.** Two matrices A, B are said to be *similar* if there exists an invertible matrix S such that $A = SBS^{-1}$.

Proposition 17.10. *(1) A is similar to itself. (2) If A is similar to B, then B is similar to A. (3) If A is similar to B and B is similar to C then A is similar to C.*

Theorem 17.11. *If A and B are similar, then they have the same characteristic polyonmial. As a result, they have the same eigenvalues, with the same algebraic multiplicities.*

A matrix is diagonalizable if and only if it is similar to a diagonal matrix. Let's apply this to prove the following statement:

Proposition 17.12. *The geometric multiplicity of an eigenvalue λ_i is between 1 and the algebraic multiplicity of λ_i.*

Proof. First

$$\text{geommult}\,\lambda_i = \dim(\text{nullspace}(A - \lambda_i I)) \geq 1$$

since $\det(A - \lambda_i I) = 0$.

Second, let $\mathbf{v}_1, \dots, \mathbf{v}_r$ be a basis for E_{λ_i} and extend it to a basis $\mathbf{v}_1, \dots, \mathbf{v}_n$ for R^n. Let S bd the matrix whose columns are $\mathbf{v}_1, \dots, \mathbf{v}_n$. Note that

$$SAS^{-1}e_j = SA\mathbf{v}_j = S\lambda_i\mathbf{v}_j = \lambda_i e_j$$

for $j \leq r$, so SAS^{-1} is of the form

$$SAS^{-1} = \begin{bmatrix} \lambda_i I_r & * \\ 0 & * \end{bmatrix}$$

that is, block upper-triangular with $\lambda_i I_r$ in the upper-left corner. Hence $\det(A - \lambda I) = \det(SAS^{-1} - \lambda I)$ has at least r copies of $\lambda - \lambda_i$. So the algebraic multiplicity is at least r. $\square$

17.3. **Problems.**

(1) True or false? Explain.
  (a) The identity matrix is diagonalizable.
  (b) An $n \times n$ matrix is diagonalizable if and only if the sum of the geometric multiplicities of the eigenvalues is n.
  (c) The sum of the algebraic multiplicities of the eigenvalues of an $n \times n$ matrix is n, if complex eigenvalues are included.
  (d) Any projection matrix P is diagonalizable.
  (e) If A is diagonalizable, then so is A^2.
  (f) Any diagonalizable matrix has n distinct eigenvalues.
  (g) If an invertible square matrix A is diagonalizable, then so is A^{-1}.
  (h) If A is diagonalizable, then so is A^T.

18. Complex eigenvalues

Complex numbers are obtained from real numbers by supposing the existence of square roots of negative numbers. One of the basic facts about polynomials is that, using complex numbers, any polynomial has number of roots counted with multiplicity equal to the degree of the polynomial. In linear algebra, this means that we can always find as many eigenvalues as the size of the matrix.

18.1. **Imaginary and complex numbers.**

To define complex numbers, we suppose that -1 has a square root, called i, the *imaginary unit*, so

$$i^2 = -1.$$

This is similar to how negative numbers are introduced: $-x$ is the number which satisfies the equation $-x + x = 0$.

An imaginary number is any real multiple bi of the imaginary unit i. A complex number is the sum of a real number plus an imaginary number. The sum of complex numbers is defined by summing the real and imaginary parts

$$(5 + 2i) + (3 - 4i) = 8 - 2i.$$

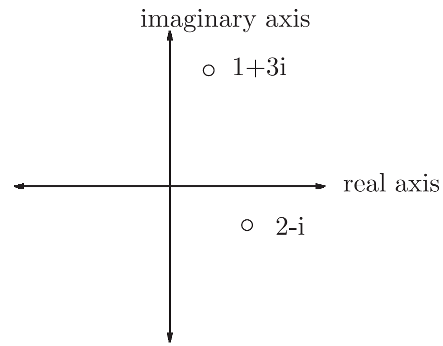
Differences are similar:

$$(5 + 2i) - (3 - 4i) = 2 + 6i.$$

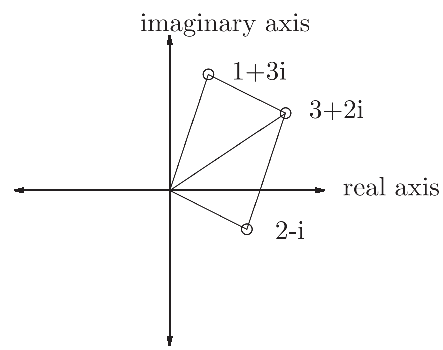
The product of complex numbers is again a complex number, using that $i^2 = -1$:

$$(5 + 2i)(3 - 4i) = 15 - 20i + 6i - 8i^2 = 15 - 14i - 8(-1) = 23 - 14i.$$

Geometrically, complex numbers are represented as points in the complex plane, which has horizontal axis the *real axis* and vertical axis the *imaginary axis*.



Sum and subtraction of complex numbers is the same as addition and subtraction of two-vectors.



The *complex conjugate* of a complex number $z = a + bi$ is the reflection of that number over the real axis,

$$\overline{z} = a - bi.$$

The *norm* $|z|$ of a complex number $z = a + bi$ is the length of the corresponding 2-vector,

$$|z| = \sqrt{a^2 + b^2}.$$

The norm can also be defined using the conjugate:

$$z\overline{z} = (a + ib)(a - ib) = a^2 + b^2 \quad \text{so} \quad |z| = \sqrt{z\overline{z}}.$$

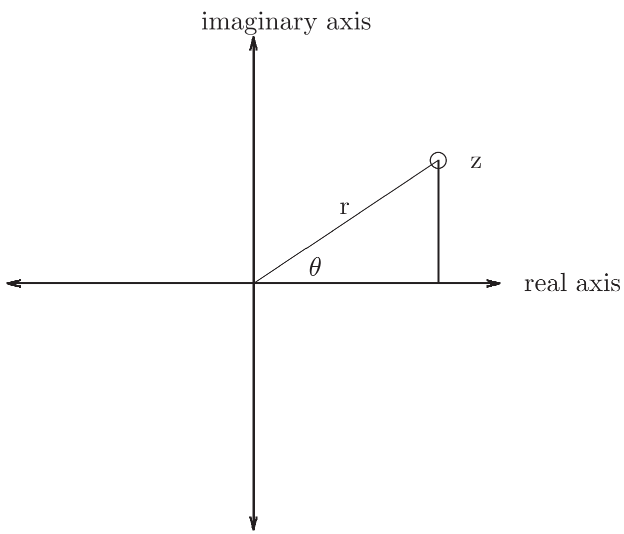
This gives us a way to define inverses of complex numbers: We have

$$z\frac{\overline{z}}{z\overline{z}} = 1 \text{ so } z^{-1} = \frac{\overline{z}}{z\overline{z}}.$$

Example 18.1. Find the inverse of $5 + 2i$.

$$(5+2i)^{-1} = \frac{5-2i}{(5+2i)(5-2i)} = \frac{5-2i}{25+4} = \frac{5}{29} - \frac{2}{29}i.$$

18.2. **Polar form.** The geometric meaning of multiplication is best explained by the *polar form* of a complex number. Define the *argument* $\arg z$ to be the angle θ between z and the positive real axis. Define $r = |z|$ the *modulus* of z.



Then the adjacent (resp.) side in the picture is

$$a = r\cos(\theta), b = r\sin(\theta).$$

The polar form of z is

$$z = r\cos(\theta) + r\sin(\theta)i.$$

The old way $z = a + bi$ is called *Cartesian form*.

Example 18.2. Find the polar form of $1, i, 1+i, -1, -1-i$.

From the picture: For $z = 1$ we have $r = 1, \theta = 1$. For $z = i$ we have $r = 1, \theta = \pi/2$. For $z = 1+i$ we have $r = \sqrt{2}, \theta = \pi/4$. etc.

Proposition 18.3. *Complex conjugation in polar form reverses the sign of the angle θ: If $z = r\cos(\theta) + r\sin(\theta)i$ then $\overline{z} = r\cos(-\theta) + r\sin(-\theta)i$.*

Proof. $\overline{z} = r\cos(\theta) - r\sin(\theta)i = r\cos(-\theta) + r\sin(-\theta)i$. □

Multiplication in polar form is simpler than in Cartesian form. Recall the Taylor series expansions

$$e^x = 1 + x + x^2/2! + x^3/3! + \ldots.$$

$$\cos(x) = 1 - x^2/2! + x^4/4! - x^6/6! \ldots$$

$$\sin(x) = x - x^3/3! + x^5/5! - x^7/7! \ldots$$

From the Taylor series for e^x we get

$$\begin{aligned} e^{i\theta} &= 1 + i\theta + (i\theta)^2/2! + (i\theta)^3/3! + \ldots \\ &= (1 - \theta^2/2! + \theta^4/4! - \theta^6/6! + \ldots) + (\theta - \theta^3/3! + \theta^5/5! + \ldots)i \\ &= \cos(\theta) + i\sin(\theta). \end{aligned}$$

This shows

Theorem 18.4 (Euler)**.** $e^{i\theta} = \cos(\theta) + i\sin(\theta)$.

Now we can re-write the polar form

$$z = r\cos(\theta) + r\sin(\theta)i = re^{i\theta}.$$

This implies the following geometric interpretation of multiplcation of complex numbers.

Proposition 18.5. *Multiplication of complex numbers is given by multiplying the lengths and adding the angles. That is, if $z_1 = r_1e^{i\theta_1}$ and $z_2 = r_2e^{i\theta_2}$ then*

$$z_1z_2 = r_1r_2e^{i(\theta_1+\theta_2)}.$$

18.3. **Application: Angle-sum formulas.** It's easy to derive from Euler's formula the formulas for the cosine or sine of the sum of two, three, or more angles. For instance,

$$e^{i(\theta_1+\theta_2)} = \cos(\theta_1+\theta_2) + i\sin(\theta_1+\theta_2).$$

On the other hand,

$$\begin{aligned} e^{i(\theta_1+\theta_2)} &= e^{i\theta_1}e^{i\theta_2} \\ &= \cos(\theta_1) + i\sin(\theta_1))(\cos(\theta_2) + i\sin(\theta_2)) \\ &= (\cos(\theta_1)\cos(\theta_2) - \sin(\theta_1)\sin(\theta_2)) + i(\cos(\theta_1)\sin(\theta_2) + \sin(\theta_1)\cos(\theta_2)) \end{aligned}$$

Matching up the real and imaginary parts, we get

Proposition 18.6. $\cos(\theta_1+\theta_2) = \cos(\theta_1)\cos(\theta_2) - \sin(\theta_1)\sin(\theta_2))$ *and* $\sin(\theta_1+\theta_2) = \cos(\theta_1)\sin(\theta_2) + \sin(\theta_1)\cos(\theta_2)$.

18.4. **Application: Powers of complex numbers.** The best way to find a large power of a complex number is to first write it in polar form.

Example 18.7. Find z^{20} where $z = 1 + i$. Since the length $r = \sqrt{2}$ and the angle $\theta = \pi/4$, the polar form is $z = \sqrt{2}e^{i(\pi/4)}$. So

$$z^{20} = \sqrt{2}^{20}(e^{i(\pi/4)})^{20} = 2^{10}e^{i5\pi} = 1024(\cos(5\pi)+i\sin(5\pi)) = 1024(-1) = -1024.$$

18.5. **Complex vectors and matrices.** A complex vector or matrix is a vector or matrix with complex entries. For example,

$$\mathbf{v} = \begin{bmatrix} 1+i \\ 1-i \end{bmatrix}$$

is a complex column vector. Similarly

$$A = \begin{bmatrix} 1 & i \\ i & 1 \end{bmatrix}$$

is a complex matrix. We can multiply matrices and vectors just as before:

$$A\mathbf{v} = \begin{bmatrix} 1(1+i) + i(1-i) \\ i(1+i) + 1(1-i) \end{bmatrix} = \begin{bmatrix} 2+2i \\ 0 \end{bmatrix}.$$

For complex vectors v, the norm $\|v\|$ is defined by

$$\|\mathbf{v}\| = \sqrt{\mathbf{v} \cdot \overline{\mathbf{v}}}.$$

If $\mathbf{v} = [a_1 + b_1 i \; a_2 + b_2 i \; \ldots \; a_n + b_n i]$ then

$$\|\mathbf{v}\| = \sqrt{a_1^2 + b_1^2 + \ldots a_n^2 + b_n^2}.$$

For any complex number z and complex vector v, $\|z\mathbf{v}\| = |z|\|\mathbf{v}\|$. The only vector with norm 0 is the zero vector.

18.6. **The fundamental theorem of algebra.** If we allow complex numbers, then any quadratic polynomial now has exactly two roots (counted with multiplicity) since $\sqrt{b^2 - 4ac}$ always makes sense as a complex number.

Example 18.8. Find the roots of $f(x) = x^2+x+3$. Ans $x = (-1\pm\sqrt{-8})/2$.

In fact, we have the following theorem.

Theorem 18.9 (Fundamental Theorem of Algebra)**.** *If $f(x) = c_0 + c_1 x + \ldots + c_n x^n$ is a polynomial of degree n, then f can be factored $f(x) = c_n(x - z_1)(x - z_2)\ldots(x - z_n)$ where $z_1, \ldots, z_n$ are complex numbers. That is, any degree n polynomial has exactly n roots, counted with multiplicity.*

Corollary 18.10. *Any $n \times n$ matrix has exactly n eigenvalues, counted with algebraic multiplicity, if we allow complex eigenvalues.*

Proof. Apply the fundamental theorem of algebra to the characteristic polynomial $\det(A - \lambda I)$. □

If we have a complex eigenvalues, we can define complex eigenvectors just as before.

Example 18.11. Find the complex eigevalues and eigenvectors of the matrix $A = \begin{bmatrix} 0 & 1 \\ -1 & 1 \end{bmatrix}$. Ans. The characteristic polynomial is $\det(A - \lambda I) = \lambda^2 + 1$ which has roots $\lambda_\pm = \pm i$. The eigenspaces are: $\lambda = i$.

$$E_i = \text{nullspace}(A - iI) = \text{nullspace} \begin{bmatrix} -i1 & \\ -1 & i \end{bmatrix} = \text{span} \begin{bmatrix} 1 \\ i \end{bmatrix}.$$

Similarly, the eigenspace E_{-i} is the span of $\begin{bmatrix} 1 \\ -i \end{bmatrix}$.

The geometric multiplicity of a complex eigenvector λ is the dimension of the (complex) subspace E_λ. It is at least 1 and at most the algebraic multiplicity.

Corollary 18.12. *A matrix is diagonalizable over the complex numbers if and only if the geometric multiplicity equals the algebraic multiplicity of λ, for each complex eigenvalue λ.*

Example 18.13. Find the eigenvalues and eigenvectors for the "shift matrix"

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix}.$$

Ans. $\det(A - \lambda I) = \lambda^4 - 1 = 0$ implies that $\lambda^4 = 1$, so writing $\lambda = re^{i\theta}$ we get

$$r^4 e^{4i\theta} = 1 \implies r = 1, 4\theta = 2\pi k$$

so $\lambda = \exp(2k\pi i/4)$ for $k = 0, 1, 2, 3$. Using $e^{i\pi/2} = i$ we get

$$\lambda = 1, i, -1, -i.$$

The eigenvectors are

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} 1 \\ i \\ -1 \\ -i \end{bmatrix}, \mathbf{v}_3 = \begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \end{bmatrix}, \mathbf{v}_4 = \begin{bmatrix} 1 \\ -i \\ -1 \\ i \end{bmatrix}.$$

So the diagonalization is

$$A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & i & -1 & -i \\ 1 & -1 & 1 & -1 \\ 1 & -i & -1 & i \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & i & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -i \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & i & -1 & -i \\ 1 & -1 & 1 & -1 \\ 1 & -i & -1 & i \end{bmatrix}^{-1}.$$

18.7. **Problems.**

(1) Compute the following expressions involving complex numbers.
 (a) $(1+i)+(2+5i)$.
 (b) $(1+i)-(2+5i)$.
 (c) $\overline{2+5i}$.
 (d) $(2+5i)\overline{2+5i}$.
 (e) $(1+i)(2+5i)$.
 (f) $(2+5i)^{-1}$.
 (g) $(1+i)/(2+5i)$.
(2) Factor the given polynomials and find all complex or real roots.
 (a) λ^2-1
 (b) λ^2+1
 (c) $\lambda^2+\lambda+1$
 (d) $\lambda^3+\lambda^2+\lambda+1$. (Hint: $\lambda=-1$ is a root.)
 (e) λ^4-1
(3) Find polar form for each given complex number
 (a) $1+i$
 (b) $2i$
 (c) $(1/2)+(\sqrt{3}/2)i$
 (d) $-\sqrt{3}/2+(1/2)i$.
(4) Find the Cartesian form of each of the following complex numbers in polar form.
 (a) e^0
 (b) $e^{\pi i}$
 (c) $e^{\pi i/2}$
 (d) $10e^{\pi i/3}$
 (e) $50e^{9\pi i/2}$.
 (f) $e^{\pi(1+i)/2}$
(5) Find the real and complex eigenvalues for the following matrices.
 (a) $\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$
 (b) $\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$
 (c) $\begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 2 \end{bmatrix}$
 (d) $\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$
 (e) $\begin{bmatrix} .8 & 0 & .2 \\ .2 & .8 & 0 \\ 0 & .2 & .8 \end{bmatrix}$
(6) True/False:
 (a) An $n\times n$ matrix with n even has at least one real
 (b) An $n\times n$ matrix with n odd has at least one complex eigenvalue.
 (c) An $n\times n$ matrix with n odd has at least one real eigenvector.
(7) Let $z=1+i$.

 (a) Find the polar form of z.
 (b) Find the complex conjugate $\bar{z}$ of z.
 (c) Find the inverse of z.
 (d) Find z^{20}. (Hint: use your answer to (a).)
(8) Let $z=-1+i$. Find (a) The polar form of z.
 (b) The conjugate of z.
 (c) The inverse of z.
 (d) The power z^{10} of z.
(9) (a) Find the (real and complex) eigenvalues for the matrix $A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$ (b) Find the diagonalization of A. (c) Find A^{15}, using the diagonalization from (b).

19. Diagonalizability of symmetric matrices and quadratic forms

19.1. **Eigenvalues/vectors for Symmetric matrices.** One class of matrices for which complex eigenvectors and eigenvalues never appear is symmetric matrices.

Proposition 19.1. *If A is symmetric and real (that is, $A=A^T=\overline{A}$) then all the eigenvalues are real.*

Proof. If v is a (possibly complex) eigenvector with (possibly complex) eigenvalue λ then either $\lambda = 0$, which is real, or

$$\begin{aligned} \|v\| &= v^T\overline{v} &&= (\frac{Av}{\lambda})^T\overline{v} \\ &= \tfrac{1}{\lambda}v^TA^T\overline{v} &&= \frac{1}{\lambda}v^T\overline{A}\overline{v} \\ &= \tfrac{1}{\lambda}v^T\overline{Av} &&= \frac{1}{\lambda}v^T\overline{\lambda v} \\ &= \tfrac{1}{\lambda}v^T\overline{\lambda}\overline{v} &&= \frac{\overline{\lambda}}{\lambda}v^T\overline{v} \\ &= \tfrac{\overline{\lambda}}{\lambda}\|v\|^2 \end{aligned}$$

Since $\|v\| \neq 0$, we must have $\lambda = \overline{\lambda}$, which means that λ is on the real axis. □

Proposition 19.2. *If A is a symmetric real matrix, and Av and w are perpendicular vectors, then so are v and Aw.*

Proof. $0 = (Av) \cdot w = (Av)^T w = v^T A^T w = v^T A w = v \cdot (Aw)$. □

Theorem 19.3. *If A is a symmetric $n \times n$ matrix then A is diagonalizable, and there exists an orthonormal basis of eigenvectors $\mathbf{v}_1, \ldots, \mathbf{v}_n$.*

Proof. Let $\mathbf{v}_1$ be an eigenvector normalized to have length one, and $\mathbf{v}_2, \ldots, \mathbf{v}_n$ vectors so that $\mathbf{v}_1, \ldots, \mathbf{v}_n$ is an orthonormal basis. Then $\lambda_1\mathbf{v}_1 = A\mathbf{v}_1$ is perpendicular to $\mathbf{v}_j, j > 1$, which implies that $A\mathbf{v}_j$ is perpendicular to $\mathbf{v}_1$. Hence $A\mathbf{v}_j$ is a combination of the vectors $\mathbf{v}_2, \ldots, \mathbf{v}_n$.

Let S be the matrix with columns $\mathbf{v}_1, \ldots, \mathbf{v}_n$. Since these form an orthonormal basis, S is orthogonal, $S^{-1} = S^T$. Then

$$S^TASe_1 = S^TA\mathbf{v}_1 = \lambda_1 e_1$$

and

$$S^TASe_j = S^TA\mathbf{v}_j$$

is a combination of the vectors $e_2, \ldots, e_n$. Therefore, S^TAS has block diagonal form

$$S^TAS = \begin{bmatrix} \lambda_1 & 0 \\ 0 & A_1 \end{bmatrix}$$

for some $n-1 \times n-1$-matrix A_1. Since A is symmetric and S is orthogonal, S^TAS is symmetric. So A_1 is symmetric as well. Now choose an eigenvector for A_1 and continue with smaller and smaller matrices. In this way, we obtain an orthonormal eigenbasis for A. □

Example 19.4. ATT,MCI,Sprint are competing for customers. Each loses 20 per cent of its customers to each of its competitors, each month. If $a(t), m(t), s(t)$ denote the number of customers in month t, then

$$\begin{bmatrix} a(t+1) \\ m(t+1) \\ s(t+1) \end{bmatrix} = \begin{bmatrix} .6 & .2 & .2 \\ .2 & .6 & .2 \\ .2 & .2 & .6 \end{bmatrix}.$$

Because the columns of this matrix sum up to one, we know that $\lambda = 1$ is an eigenvalue. To find the others, we long divide $(\lambda - 1)$ into the characteristic polynomial ....

19.2. **Quadratic Forms.** Let $x_1, \ldots, x_n$ be coordinates on $\mathbb{R}^n$. We have the following kinds of functions in the variables $x_1, \ldots, x_n$:

(1) A *constant function* is a function of the form $f(x_1, \ldots, x_n) = c$ for some constant c.
(2) A *linear function* is a function of the form $f(x_1, \ldots, x_n) = c_1x_1 + \ldots + c_nx_n$.
(3) A *quadratic function*, also called a *quadratic form*, is a function of the form $f(x_1, \ldots, x_n) = c_{11}x_1^2 + c_{12}x_1x_2 + c_2^2x_2^2 + \ldots + c_{nn}x_n^2$, that is, a linear combination of functions $x_1^2, x_1x_2, x_2^2, \ldots$.

By Taylor's theorem in multivariable calculus, any function can be written as a sum of a constant function, linear function, quadratic function, plus higher order terms.

Any quadratic form can be written as a vector-matrix-vector matrix product using a symmetric matrix. For example,

$$q(x_1, x_2) = x_1^2 - 4x_1x_2 + x_2^2 = [x_1 \ x_2] \begin{bmatrix} 1 & -2 \\ -2 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}.$$

In general,

Proposition 19.5. *A quadratic form $q(x_1, \ldots, x_n)$ with diagonal terms $c_ix_i^2$ and off-diagonal terms $d_{ij}x_ix_j$ is equal to*

$$[x_1 \ldots x_2]\, A \begin{bmatrix} x_1 \\ \ldots \\ x_n \end{bmatrix}.$$

where A is the matrix whose diagonal entries are c_i and whose off-diagonal entries are $d_{ij}/2$.

The factor of $1/2$ is because the ij-th and ji-th entries contribute the same sort of terms to $q(x_1, \ldots, x_n)$.

In this section we use diagonalization to understand what the graphs of quadratic functions look like.

Example 19.6. (1) $q(x_1, x_2) = x_1^2 + x_2^2$ is a quadratic form in two variables. It has a graph in $\mathbb{R}^3$, given by setting the vertical coordinate (height) equal to the value of q given by $x_1^2 + x_2^2$. Its shape is a *paraboloid* (three-dimensional version of parabola.)

(2) $q(x_1, x_2) = x_1^2 - x_2^2$ is a parabola point up in the x_1 direction and a parabola pointing down in the x_2 direction. Its graph has a saddle shape. (Hyperbolic paraboloid if you want to be technical.)

(3) $q(x_1, x_2) = x_1^2 - 4x_1x_2 + x_2^2$ is a quadratic form in two variables x_1 and x_2. We can try to graph the function q in three dimensions. First we can graph the function when $x_1 = 0$, and then when $x_2 = 0$.

To figure out the shape of the graph, let's write q in matrix form. Let Q be the matrix whose diagonal entries are the coefficients of x_1^2 and x_2^2, and whose off-diagonal entries are $\frac{1}{2}$ the coefficient of x_1, x_2:

$$Q = \begin{bmatrix} 1 & -2 \\ -2 & 1 \end{bmatrix}.$$

Let's find the eigenvectors and eigenvalues for Q. The characteristic polynomial is

$$\det(Q - \lambda I) = (1-\lambda)^2 - 4 = \lambda^2 - 2\lambda - 3 = (\lambda+1)(\lambda-3)$$

so the eigenvalues are

$$\lambda = -1, 3.$$

We can find the eigenvalues: For $\lambda = 1$ the eigenspace is

$$E_{-1} = \text{nullspace} \begin{bmatrix} 2 & -2 \\ -2 & 2 \end{bmatrix} = \text{span} \begin{bmatrix} 1 \\ 1 \end{bmatrix} / \sqrt{5}.$$

$$E_3 = \text{nullspace} \begin{bmatrix} -2 & -2 \\ -2 & -2 \end{bmatrix} = \text{span} \begin{bmatrix} 1 \\ -1 \end{bmatrix} / \sqrt{5}.$$

So Q can be diagonalized

$$Q = SDS^T, \quad D = \begin{bmatrix} -1 & 0 \\ 0 & 3 \end{bmatrix}, \quad S = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}.$$

This means that if we define new coordinates

$$\mathbf{y} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = S^T \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

Then

$$q(y_1, y_2) = \mathbf{y}^T D \mathbf{y} = -y_1^2 + 3y_2^2.$$

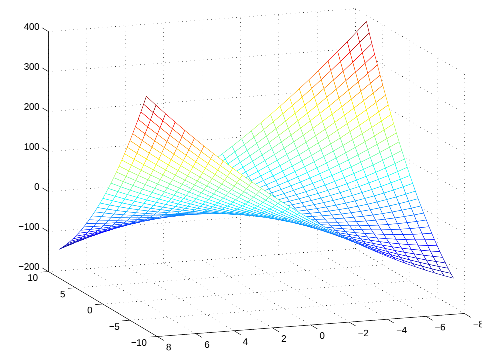


FIGURE 6. Saddle

This quadratic form is easy to understand; it is a "saddle". See Figure **??**

Another way of graphing the quadratic form is to draw it's level sets

$$q(y_1, y_2) = c.$$

For the example above, these are hyperbolas.

Example 19.7. Graph the quadratic form $q(x_1, x_2) = 2x^2 + 2xy + 2y^2$. Describe the level set $q(x_1, x_2) = 4$.

There is a similar story with three or more variables.

Example 19.8. Find the diagonalization of the quadratic form $q(x_1, x_2, x_3) = 6x_1x_2 + 8x_2x_3$.

The matrix Q is

$$\begin{bmatrix} 0 & 3 & 0 \\ 3 & 0 & 4 \\ 0 & 4 & 0 \end{bmatrix}.$$

The eigenvalues of Q are found by setting

$$\begin{aligned} 0 &= \det(Q-\lambda I) \\ &= \det\left(\begin{bmatrix} -\lambda & 3 & 0 \\ 3 & -\lambda & 4 \\ 0 & 4 & -\lambda \end{bmatrix}\right) \\ &= -\lambda^3+4\lambda \\ &= -\lambda(\lambda^2-25) \\ &= -\lambda(\lambda-5)(\lambda+5). \end{aligned}$$

Therefore, the eigenvalues are

$$\lambda = 0, 5, -5.$$

The eigenvectors are

$$\begin{aligned} E_0 &= \text{nullspace } Q \\ &= \text{nullspace} \begin{array}{r} ② \leftrightarrow ① \\ \\ ③ - (4/3)① \end{array} \begin{bmatrix} 3 & 0 & 4 \\ 0 & 4 & 0 \\ 0 & 0 & 0 \end{bmatrix} \\ &= \text{span} \begin{bmatrix} -4 \\ 0 \\ 3 \end{bmatrix}. \end{aligned}$$

The other eigenvectors are

$$\begin{aligned} E_{\pm 2} &= \text{nullspace } Q \mp 5I \\ &= \text{nullspace} \begin{bmatrix} \mp 5 & 3 & 0 \\ 3 & \mp 5 & 4 \\ 0 & 4 & \mp 5 \end{bmatrix} \\ &= \text{nullspace} \begin{array}{r} ①/5 \\ -(3/5)① \\ \\ \end{array} \begin{bmatrix} \mp 1 & 3/5 & 0 \\ 0 & \mp 16/5 & 4 \\ 0 & 4 & \mp 5 \end{bmatrix} \\ &= \text{nullspace} \begin{array}{r} \\ (5/16)② \\ \\ \end{array} \begin{bmatrix} \mp 1 & 3/5 & 0 \\ 0 & \mp 1 & 5/4 \\ 0 & 0 & 0 \end{bmatrix} \\ &= \text{span} \begin{bmatrix} 3 \\ \pm 5 \\ 4 \end{bmatrix}. \end{aligned}$$

An orthonormal eigenbasis is obtained by dividing by the lengths:

$$\frac{1}{5}\begin{bmatrix} -4 \\ 0 \\ 3 \end{bmatrix}, \quad \frac{1}{\sqrt{25}}\begin{bmatrix} 3 \\ 5 \\ 4 \end{bmatrix} \quad \frac{1}{\sqrt{25}}\begin{bmatrix} 3 \\ -5 \\ 4 \end{bmatrix}.$$

Therefore,

$$Q = S^T D S = \begin{bmatrix} -4/5 & 0 & 3/5 \\ 3/\sqrt{25} & 1/\sqrt{2} & 4/\sqrt{25} \\ 3/\sqrt{25} & -1/\sqrt{2} & 4/\sqrt{25} \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 5 \end{bmatrix} \begin{bmatrix} -4/5 & 3/\sqrt{25} & 3/\sqrt{25} \\ 0 & 1/\sqrt{2} & -1/\sqrt{2} \\ 3/5 & 4/\sqrt{25} & 4/\sqrt{25} \end{bmatrix}.$$

Setting

$$\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = S \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -4x_1/5 + 3x_2/\sqrt{25} + 3x_3/\sqrt{25} \\ x_2/\sqrt{2} - x_3/\sqrt{2} \\ 3x_1/5 + 4x_2/\sqrt{25} + 4x_3/\sqrt{25} \end{bmatrix}$$

we get $\boxed{q(y_1, y_2, y_3) = 5y_2^2 - 5y_3^2.}$

19.3. **Problems.**

(1) Let $q(x_1, x_2) = 2x_1^2 - 4x_1x_2 - x_2^2$. Find coordinates y_1, y_2 and numbers a, b such that $q(y_1, y_2) = ay_1^2 + by_2^2$.

(2) (a) Find the eigenvalues for the matrix $A = \frac{1}{3}\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$. (b) Find the diagonalization of A. (c) Using diagonalization, find A^5. (d) Using diagonalization, find the eigenvalues of $I - A$.

20. Linear dynamical systems

By dynamical system we will mean a mathematical model for the time evolution of a system.

20.1. **An economic example.**

Example 20.1. A typical example is the Coke/Pepsi example discussed earlier. Recall: Suppose $c(t)$ (resp. $p(t)$) is the number of Coke (resp. Pepsi) drinkers at time t months. Suppose each month, 10 percent of the Coke drinkers switch to become Pepsi drinkers, and 20 percent of the Pepsi drinkers switch to Coke. If we start with 100 Pepsi drinkers and no Coke drinkers, what happens as t goes to infinity?

t	p	c
0	0	100
1	20	80
2	34	66

To set this up as a linear algebra problem we write

$$c(t+1) = .9c(t) + .2p(t)$$

$$p(t+1) = .1c(t) + .8p(t)$$

or in matrix form

$$\mathbf{x}(t+1) = A\mathbf{x}(t) \text{ where } A = \begin{bmatrix} .9 & .2 \\ .1 & .8 \end{bmatrix} \text{ and } \mathbf{x}(t) = \begin{bmatrix} c(t) \\ p(t) \end{bmatrix}.$$

This implies that

$$\mathbf{x}(t) = A\mathbf{x}(t-1) = A^2\mathbf{x}(t-2) = \ldots = A^t\mathbf{x}(0)$$

The *state* of the system in any month t is described by the vector of Coke/Pepsi drinkers

$$\mathbf{x}(t) = \begin{bmatrix} c(t) \\ p(t) \end{bmatrix}.$$

The *time evolution* of the system is an equation for $\mathbf{x}(t+1)$ in terms of $\mathbf{x}(t)$:

$$\mathbf{x}(t+1) = \begin{bmatrix} .9 & .2 \\ .1 & .8 \end{bmatrix} \mathbf{x}(t).$$

The matrix in this equation is the *time evolution matrix.*

Example 20.2. Two companies are competing for customers. Each year, company A loses 60 percent of its customers to company B, while company B each year loses 70 percent of its customers to company A. (Clearly neither of the company's produce a very high quality product!) (a) Write down the state vector and time evolution matrix for this system. That is, represent the system in the form $x(t+1) = Ax(t)$, for some matrix A. (b) Find the diagonalization of the matrix A. (c) Suppose that initially, 100 customers are with company A and none with company B. Find a formula for the number of customers with company A at time t. (d) How many customers does A have, for t very large. (e) Show the evolution of the system on the graph with axes A,B.

Example 20.3. A party is going on in two rooms, the living room and the kitchen, in someone's apartment. Every hour, ninety percent of the people in the living room move to the kitchen, and sixty percent of the people in the kitchen move to the living room. (a) Find the transition matrix for this problem, that is, the matrix A such that

$$x(t+1) = Ax(t), \qquad \text{where } x(t) = \begin{bmatrix} l(t) \\ k(t) \end{bmatrix}$$

and $l(t), k(t)$ are the number of people in the living room, kitchen at time t. (b) Find the eigenvectors and eigenvalues for A. (c) Find matrices S and D such that $A = SDS^{-1}$. (d) Suppose that initially, 100 party-goers are distributed equally among the two rooms. Find a formula for the number of people in the kitchen at time t. (e) How many people are in the kitchen, in the limit $t \to \infty$? (f) Show the evolution of the system on the graph with axes l, k. (t is not an axis!)

Answer: (a) $\begin{bmatrix} l(t+1) \\ k(t+1) \end{bmatrix} = \begin{bmatrix} l(t) - .9l(t) + .6k(t) \\ k(t) + .9l(t) - .6k(t) \end{bmatrix} = \begin{bmatrix} .1l(t) + .6k(t) \\ .9l(t) + .4k(t) \end{bmatrix} = \begin{bmatrix} .1 & .6 \\ .9 & .4 \end{bmatrix} \begin{bmatrix} l(t) \\ k(t) \end{bmatrix}$. (b) Columns sum to 1, so $\lambda_1 = 1$ and $\lambda_1 + \lambda_2 = -1$ implies $\lambda_2 = -1/2$. The eigenspace for $\lambda_1 = 1$ is nullspace$(A - 1I) =$ nullspace $\begin{bmatrix} -.9 & .6 \\ .9 & .6 \end{bmatrix} = \begin{bmatrix} 1 & -2/3 \\ 0 & 0 \end{bmatrix} =$ span $\begin{bmatrix} 2 \\ 3 \end{bmatrix}$. The eigenspace for $\lambda_1 = -.5$ is nullspace$(A - -.5I) =$ nullspace $\begin{bmatrix} .6 & .6 \\ .9 & .9 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} =$ span $\begin{bmatrix} -1 \\ 1 \end{bmatrix}$. (c) S is the matrix of eigenvectors and D the matrix of eigenvalues so $S = \begin{bmatrix} 2 & -1 \\ 3 & 1 \end{bmatrix}$ and $D = \begin{bmatrix} 1 & 0 \\ 0 & -1/2 \end{bmatrix}$. (d) $\begin{bmatrix} l(t) \\ k(t) \end{bmatrix} = A^t \begin{bmatrix} l(0) \\ k(0) \end{bmatrix} = SD^tS^{-1} \begin{bmatrix} 50 \\ 50 \end{bmatrix} = \begin{bmatrix} 2 & -1 \\ 3 & 1 \end{bmatrix} \begin{bmatrix} 1^t & 0 \\ 0 & (-1/2)^t \end{bmatrix} \begin{bmatrix} 2 & -1 \\ 3 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 50 \\ 50 \end{bmatrix} = \begin{bmatrix} 2 & -1 \\ 3 & 1 \end{bmatrix} \begin{bmatrix} 1^t & 0 \\ 0 & (-1/2)^t \end{bmatrix} \begin{bmatrix} 1 & 1 \\ -3 & 2 \end{bmatrix} /5 \begin{bmatrix} 50 \\ 50 \end{bmatrix} = \begin{bmatrix} 2 & -1 \\ 3 & 1 \end{bmatrix} \begin{bmatrix} 1^t & 0 \\ 0 & (-1/2)^t \end{bmatrix} \begin{bmatrix} 20 \\ -10 \end{bmatrix} = \begin{bmatrix} 2 & -1 \\ 3 & 1 \end{bmatrix} \begin{bmatrix} 20 \\ -(-1/2)^t 10 \end{bmatrix} = \begin{bmatrix} 2 & -1 \\ 3 & 1 \end{bmatrix} \begin{bmatrix} 20 \\ -(-1/2)^t 10 \end{bmatrix} = \begin{bmatrix} 40 + (-1/2)^t 10 \\ 60 - (-1/2)^t 10 \end{bmatrix}$ (e) As $t \to \infty$ we get $\begin{bmatrix} 40 \\ 60 \end{bmatrix}$. On the l, k axis, the time t vector is given by adding the vectors $\begin{bmatrix} 40 \\ 60 \end{bmatrix}$ and $(-1/2)^t \begin{bmatrix} 10 \\ -10 \end{bmatrix}$.

20.2. **A biological example.** Let's look at the following model of flu epidemic.

Example 20.4. Suppose that in a population of 80 students, at any point in time there are w well students, s sick students, and i students who have already been sick and developed immunity. Suppose each week 20 percent of the well students get sick, 50 percent of the sick students get better and develop immunity, but after one week the immunity wears off. Find the matrix A that expresses the change $\Delta w, \Delta s, \Delta i$ in the numbers of well, sick, and immune students in terms of w, s, i.

$$\begin{bmatrix} \Delta w \\ \Delta s \\ \Delta i \end{bmatrix} = A \begin{bmatrix} w \\ s \\ i \end{bmatrix}, \quad A = \begin{bmatrix} -.2 & 0 & 1 \\ +.2 & -.5 & 0 \\ 0 & +.5 & -1 \end{bmatrix}.$$

What is the practical meaning of the nullspace? It is the set of all vectors $[w\ s\ i]$ such that the change to the next week is zero. That is, the population stays the same. The vectors $[w\ s\ i]$ for which this happens are called *equilibrium* vectors. Let's find the null-space, by elimination.

$$\text{nullspace}(A) = \text{span} \begin{bmatrix} 5 \\ 2 \\ 1 \end{bmatrix}.$$

For instance, 50 well, 20 sick, and 10 immune is an equilibrium population.

Example 20.5. Continuing the model for the flu epidemic we discussed earlier, suppose that in a population of 80 students, at any point in time there are w well students, s sick students, and i students who have already been sick and developed immunity. Suppose each week 20 percent of the well students get sick, 50 percent of the sick students get better and develop immunity, but after one week the immunity wears off.

The state vector is the number of well, sick, and immune students

$$\mathbf{x}(t) = \begin{bmatrix} w(t) \\ s(t) \\ i(t) \end{bmatrix}$$

and the time evolution matrix is

$$A = \begin{bmatrix} .8 & 0 & 1 \\ .2 & .5 & 0 \\ 0 & +.5 & 0 \end{bmatrix}.$$

To solve for $\mathbf{x}(t)$, we find the eigenvectors and eigenvalues .....

Example 20.6. Let's consider a new example, a population model. Suppose that we consider a population of baby, adult, and retired rabbits. The baby rabbits (resp. adult rabbits) mature into adult (resp. retired) rabbits after one year; life span is three years. Each adult rabbit has $\frac{3}{4}$ baby rabbit, on average, each year. Each retired rabbit, has on average, $1/4$ a baby each year. Find the state vector and the time evolution matrix for this situation. Find the ratio of baby, adult, and retired rabbits in the limit $t \to \infty$.

20.3. **A physics example.** A party is going on in two rooms. Each hour, 70 percent of the people in room 1 move to room 2, and 80 percent of the people in room 2 move to room 1.
(a) Write the system in matrix form, i.e. find a matrix A such that $Ax(t+1) = x(t)$, where $x(t)$ is the distribution vector. (b) Find the percent of customers with each company, in the long run. (c) Suppose that initially, 100 people are in room 1.

20.4. **A computer science example.** The Google PageRank algorithm is based on matrix algebra. Consider for simplicity that the web has only three pages, A, B, C, with A linking to B, C, B linking to C, and C linking to A, B. Suppose that people surfing the web click randomly on the next link, once per minute. The number $a(t), b(t), c(t)$ of people looking at each page changes according to the equations

$$\begin{aligned} a(t+1) &= (1/2)c(t) \\ b(t+1) &= (1/2)a(t) + (1/2)c(t) \\ c(t+1) &= (1/2)a(t) + b(t) \end{aligned}$$

or

$$\begin{bmatrix} a(t+1) \\ b(t+1) \\ c(t+1) \end{bmatrix} = \begin{bmatrix} 0 & 0 & .5 \\ .5 & 0 & .5 \\ .5 & 1 & 0 \end{bmatrix} \begin{bmatrix} a(t) \\ b(t) \\ c(t) \end{bmatrix}.$$

We write this as

$$\mathbf{x}(t+1) = A\mathbf{x}(t)$$

where A is the matrix above. Google ranks the web-sites containing the search term according to popularity in the following sense: the most popular web-site should be the web-site which most people are visiting at even given time, assuming that people surf randomly.

To find which web-site this is, we find the diagonalization of the matrix A above. The eigenvalues $\lambda_1, \lambda_2, \lambda_3$ satisfy

$$\lambda_1 + \lambda_2 + \lambda_3 = \text{Tr}(A) = 0$$

$$\lambda_1\lambda_2\lambda_3 = \det(A) = .25$$

Since A is Markov, one of the eigenvalues, let's say the first, is $\lambda_1 = 1$. So $\lambda_2 + \lambda_3 = -1$ and $\lambda_2(-1-\lambda_2) = 1/4$ implies that $\lambda_2^2 + \lambda_2 + 1/4 = 0$ so the other eigenvalues $\lambda_2 = \lambda_3 = -1/2$. The equilibrium for the system is a multiple of the eigenvector for $\lambda_1 = 1$,

$$E_{\lambda_1=1} = \text{nullspace}(A-I) = \text{nullspace}(\begin{bmatrix} -1 & 0 & .5 \\ .5 & -1 & .5 \\ .5 & 1 & -1 \end{bmatrix}) = \text{span} \begin{bmatrix} 1 \\ 1.5 \\ 2 \end{bmatrix} = \text{span} \begin{bmatrix} 2/9 \\ 3/9 \\ 4/9 \end{bmatrix}$$

In the equilibrium state 2/9 of the people will be on page A, 3/9 of the people on page B, and 4/9 of the people on page C.

Suppose that you search for a search term, and pages A, C both contain the term. Google will list page C first, on the logic that it is more popular.

When this approach was introduced in the 90's it made it more difficult for companies to trick a search engine such as google into listing its results first, since in order to influence the ranking one has to create a whole network of pages which "flow traffic" to the given page, and these are easily detected by the search engine.

Of course now Google's search algorithm is much more complicated (and secret.)

20.5. **Markov processes.** A square matrix A is called a *Markov* or *probability matrix* if

(1) The sum of each column is equal to 1.

(2) All of the entries in A are between 1 and 0.

The first condition is equivalent to

$$\mathbf{u}A = \mathbf{u}, \quad \mathbf{u} = [1\ 1\ \dots\ 1].$$

The second condition, assuming the first, is equivalent to saying that all of the entries in A are non-negative (since if the sum of non-negative numbers is equal to 1, each number must be at most 1).

Proposition 20.7. *If A and B are Markov matrices, then so is AB.*

Proof. Clearly if A and B are non-negative, then so is AB. So it suffices to check $\mathbf{u}(AB) = (\mathbf{u}A)B = \mathbf{u}B = \mathbf{u}$ since A and B are Markov. $\square$

The following is sometimes called the Perron-Frobenius theorem:

Theorem 20.8. *If A is a Markov matrix, then 1 is an eigenvalue for A.*

Proof. Since $\mathbf{u}A = \mathbf{u}$, we have $A^T\mathbf{u}^T = \mathbf{u}^T$ so $\mathbf{u}^T$ is an eigenvector of A^T with eigenvalue 1. Since A and A^T have the same eigenvalues, 1 is also an eigenvalue for A. $\square$

Knowing that 1 is an eigenvalue of A helps to find its other eigenvalues. For instance, to find the eigenvalues of a three-by-three Markov matrix you could compute its characteristic polynomial $\det(A - \lambda I)$, factor out $(\lambda - 1)$ using long division, and then factor the remaining degree two polynomial using the quadratic formula.

Some other properties of Markov matrices that we will not show are

(1) all the other eigevalues have norm $\|\lambda\|$ at most 1

(2) if all the entries of A are positive, then A has a unique eigenvector with eigenvalue 1.

20.6. **Recursive sequences.** A *recursive sequence* is a sequence of numbers where the n-th number is defined by a formula involving previous numbers. The most famous example, the Fibonacci sequence, is

$$1, 1, 2, 3, 5, 8, 13, \dots.$$

Each number is the sum of the two previous numbers. We will find a closed formula for the n-th Fibonacci number, using eigenvalues and eigenvectors.

To express this as a linear system, we do the following trick which is very important not only for linear algebra but also differential equations.

$$f(n+1) = f(n) + f(n-1)$$

$$f(n) = f(n)$$

What's the point of writing the second equation, which is obvious? The point is that the vector $\begin{bmatrix} f(n+1) \\ f(n) \end{bmatrix}$ can now be written as a matrix times $\begin{bmatrix} f(n) \\ f(n-1) \end{bmatrix}$:

$$\begin{bmatrix} f(n+1) \\ f(n) \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} f(n) \\ f(n-1) \end{bmatrix}.$$

This means that

$$\begin{bmatrix} f(n+1) \\ f(n) \end{bmatrix} = A^n \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

where $A = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$. To find A^n, we diagonalize A.

To find the eigenvalues, we set

$$\begin{aligned} 0 &= \det(A - \lambda I) \\ &= (1-\lambda)(-\lambda) - 1 \\ &= \lambda^2 - \lambda - 1 \\ &= (\lambda - (1+\sqrt{5})/2)(\lambda - \frac{1-\sqrt{5}}{2}). \end{aligned}$$

The eigenvectors are

$$\mathbf{v}_\pm = \begin{bmatrix} \frac{1\pm\sqrt{5}}{2} \\ 1 \end{bmatrix}.$$

This means that the diagonalization of A is

$$\begin{aligned} A &= SDS^{-1} \\ &= \begin{bmatrix} \frac{1+\sqrt{5}}{2} & \frac{1-\sqrt{5}}{2} \\ 1 & 1 \end{bmatrix} \begin{bmatrix} \frac{1+\sqrt{5}}{2} & 0 \\ 0 & \frac{1-\sqrt{5}}{2} \end{bmatrix} \begin{bmatrix} \frac{1+\sqrt{5}}{2} & \frac{1-\sqrt{5}}{2} \\ 1 & 1 \end{bmatrix}^{-1}. \end{aligned}$$

The inverse of S is

$$S^{-1} = \frac{1}{\sqrt{5}} \begin{bmatrix} 1 & -\frac{1-\sqrt{5}}{2} \\ -1 & \frac{1+\sqrt{5}}{2} \end{bmatrix}$$

so that the Fibonacci numbers are

$$\begin{aligned} \begin{bmatrix} f(t+2) \\ f(t+1) \end{bmatrix} &= A^t \begin{bmatrix} 1 \\ 1 \end{bmatrix} \\ &= S^{-1}D^tS \begin{bmatrix} 1 \\ 1 \end{bmatrix} \\ &= \begin{bmatrix} \frac{1+\sqrt{5}}{2} & \frac{1-\sqrt{5}}{2} \\ 1 & 1 \end{bmatrix} \begin{bmatrix} (\frac{1+\sqrt{5}}{2})^t & 0 \\ 0 & (\frac{1-\sqrt{5}}{2})^t \end{bmatrix} \\ & \quad \frac{1}{\sqrt{5}} \begin{bmatrix} 1 & -\frac{1-\sqrt{5}}{2} \\ -1 & \frac{1+\sqrt{5}}{2} \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} \\ &= \frac{1}{\sqrt{5}} \begin{bmatrix} \frac{1+\sqrt{5}}{2} & \frac{1-\sqrt{5}}{2} \\ 1 & 1 \end{bmatrix} \begin{bmatrix} \frac{1+\sqrt{5}}{2}^{t+1} \\ -\frac{1-\sqrt{5}}{2}^{t+1} \end{bmatrix} \\ &= \frac{1}{\sqrt{5}} \begin{bmatrix} \frac{1+\sqrt{5}}{2}^{t+2} - \frac{1-\sqrt{5}}{2}^{t+2} \\ \frac{1+\sqrt{5}}{2}^{t+1} - \frac{1-\sqrt{5}}{2}^{t+1} \end{bmatrix}. \end{aligned}$$

So the Fibonacci number

$$f(t) = \frac{1}{\sqrt{5}} \left((\frac{1+\sqrt{5}}{2})^t - (\frac{1-\sqrt{5}}{2})^t \right).$$

Example 20.9. Let's look now at some other recursive formula, for instance

$$f(n+1) = f(n) - f(n-1)$$

which gives the sequence

$$1, 1, 0, -1, -1, 0, 1, 1, \ldots$$

Find a closed formula for $f(n)$.

To solve the equation we introduce a second equation

$$f(n) = f(n)$$

so that we get a system of linear equations

$$\begin{bmatrix} f(n+1) \\ f(n) \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} f(n) \\ f(n-1) \end{bmatrix}.$$

The characteristic polynomial is

$$\det(A - \lambda I) = (1-\lambda)(-\lambda) + 1 = \lambda^2 - \lambda + 1$$

which has roots

$$\lambda_\pm = \frac{1 \pm \sqrt{-3}}{2} = e^{\pm 2\pi i/3}.$$

This means that the matrix A is diagonalizable

$$A = SDS^{-1}, \quad D = \begin{bmatrix} e^{2\pi i/3} & 0 \\ 0 & e^{-2\pi i/3}. \end{bmatrix}$$

If n is a multiple of 3 then

$$A^n = SD^nS^{-1} = SS^{-1} = I.$$

This explains why the sequence is periodic with period 3.

Example 20.10. Define a sequence $f(n)$ by $f(n+1) = 2f(n) + f(n-1)$. Find a closed formula for $f(n)$ using linear algebra.

20.7. **Problems.**

(1) Two companies are competing for customers. Each year, company A loses 40 percent of its customers to company B, while company B each year loses 30 percent of its customers to company A.

(a) Write down the state vector and time evolution matrix for this system.
(b) Find the percent of customers with each company, in the long run.
(c) Suppose that initially, 100 customers are with company A. Show the evolution of the system on the graph with axes A,B.

(2) GM and Ford are competing for customers. Suppose that each year, thirty percent of GM's customers leave for Ford, while ten percent of Ford's customers leave for GM. The remaining customers remain loyal. Let $g[t], f[t]$ be the number of customers with GM and Ford in year T.

(a) Find a matrix A such that

$$\begin{bmatrix} g[t+1] \\ f[t+1] \end{bmatrix} = A \begin{bmatrix} g[t] \\ f[t] \end{bmatrix}.$$

(b) Find the eigenvalues and eigenvectors for the matrix A you found in part (a).

(c) Let $g[0] = 100,\ f[0] = 0$. Compute $[g[t], f[t]]$ for $t = 1, 2$.

(d) Find the number of customers with GM and with Ford, in the limit $t \to \infty$.

(e) Draw a sketch showing the sequence of points $[g[t], f[t]]$.

(3) Fibonacci considered a model for a population of rabbits where each pair of adult rabbits produces a pair of juvenile rabbits each month, and each pair of juvenile rabbits becomes adult after one month. A more realistic model takes into account death of the adult rabbits. Suppose 1/2 of the adult rabbits die each month. Then the equations for the model are

$$j(t+1) = a(t), \quad a(t+1) = \frac{1}{2}a(t) + j(t),$$

where $j(t), a(t)$ are the number of pairs of juvenile and adult rabbits at time t.

(a) Find the time evolution matrix A for this system.
(b) Find the eigenvalues and eigenvectors for A.
(c) Find the ratio of adult rabbits to juvenile rabbits, as t approaches infinity. (Hint: express the initial state as a linear combination $c_1\mathbf{v}_1 + c_2\mathbf{v}_2$ of the eigenvectors. You do not need to find c_1 and c_2.)
(d) If you did (b) correctly, the eigenvectors of A are perpendicular. What property does A have which guarantees this is so?
(e) Write a recursive formula for $a(t)$ in terms of $a(t-1)$ and $a(t-2)$, similar to the Fibonacci formula $F(t) = F(t-1) + F(t-2)$.
(f) Draw a graph showing the trajectory (time evolution) of the system. Indicate the eigenspaces on your graph by dotted lines.

(4) Consider the recursive sequence defined by $f(n) = f(n-1) - f(n-2)$ with $f(1) = f(2) = 1$. Call these the Ibonacci numbers. The first few are $1, 1, 0, -1, -1, 0, \ldots$.

(a) Find a matrix A so that

$$\begin{bmatrix} f(n+1) \\ f(n) \end{bmatrix} = A \begin{bmatrix} f(n) \\ f(n-1) \end{bmatrix}.$$

(b) Find the (complex) eigenvalues and eigenvectors of A.

(c) Find a formula for the n-th Ibonacci number. It should be clear from your formula that the sequence of Ibonacci numbers is repeating.

21. Singular value decomposition

Let A be a matrix.

Proposition 21.1. *The matrix A^TA has n non-negative real eigenvalues.*

Proof. A^TA is symmetric: $(A^TA)^T = A^T(A^T)^T = A^TA$. Therefore the eigenvalues are real. Let v be an eigenvector of A^TA with eigenvalue λ. Then

$$\lambda(v \cdot v) = v \cdot A^TAv = (Av) \cdot (Av) \geq 0.$$

Since $v \cdot v > 0$, this implies $\lambda \geq 0$. □

Definition 21.2. The *singular values* of a matrix A are the square roots of the eigenvalues of A^TA.

Proposition 21.3. *If A is a matrix and Q is an orthogonal matrix, then the singular values of A are the same as those of QA or AQ, if these products are defined.*

Proof. The singular values of QA are the square roots of eigenvalues of $(QA)^TQA = A^TQ^TQA = A^TA$ since Q is orthogonal. The proof for AQ is similar. □

Theorem 21.4. *Let A be any matrix. There exist orthogonal matrices Q_1, Q_2 and a diagonal matrix D such that $A = Q_1DQ_2$. The diagonal elements of D are the singular values of A, and D is unique up to ordering of the diagonal elements. If A is invertible, then given a choice of D the matrices Q_1 and Q_2 are also unique.*

Proof. By diagonalization of symmetric matrices, the diagonalization of A^TA is $Q_1^TD_1Q_1$ for some orthogonal Q_1 and diagonal D_1. Since the eigenvalues are non-negative, $D_1 = D^TD$ where D is the diagonal matrix of square roots of D_1. Hence

$$A^T = Q_1^TD^TDQ_1. \tag{8}$$

Choose a basis $\mathbf{v}_1, \ldots, \mathbf{v}_k$ for nullspace$(A)^\perp$. Then the vectors

$$\mathbf{w}_1 = A\mathbf{v}_1, \ldots, \mathbf{w}_p = A\mathbf{v}_k$$

are an orthornormal basis for colspace(A). Let $\mathbf{w}_{p+1}, \ldots, \mathbf{w}_n$ be an orthonormal basis for colspace$(A)^\perp$. Let

$$\mathbf{u}_1 = DQ_1\mathbf{v}_1, \ldots, \mathbf{u}_p = DQ_1\mathbf{v}_p$$

and $\mathbf{u}_{p+1}, \ldots \mathbf{u}_n$ an orthornormal basis for colspace$(DQ_1)^\perp$. By (8)

(9) $\mathbf{w}_j \cdot \mathbf{w}_k = (A\mathbf{v}_j)\cdot(A\mathbf{v}_k) = (DQ_1\mathbf{v}_j)\cdot(DQ_1\mathbf{v}_k) = \mathbf{u}_j \cdot \mathbf{u}_k \;\; j,k = 1,\ldots,p.$

Since $\mathbf{w}_j$ and $\mathbf{u}_j$, $j \geq p+1$ are orthonormal,

$$\mathbf{w}_j \cdot \mathbf{w}_k = \mathbf{u}_j \cdot \mathbf{u}_k$$

for any $j,k = 1,\ldots,n$. Let S_1 be the matrix whose columns are $\mathbf{w}_1, \ldots, \mathbf{w}_n$ and let S_2 be the matrix whose columns are $\mathbf{u}_1, \ldots, \mathbf{u}_n$. Let $Q_2 = S_1 S_2^{-1}$. Then

$$Q_2 DQ_1\mathbf{v}_j = S_1 e_1 = A\mathbf{v}_j, \quad j = 1,\ldots,p.$$

so $Q_2DQ_1 = A$. By (9),

$$(Q_2\mathbf{w}_j)\cdot(Q_2\mathbf{w}_k) = \mathbf{u}_j \cdot \mathbf{u}_k = \mathbf{w}_j \cdot \mathbf{w}_k, \quad \forall j,k = 1,\ldots,n$$

so Q_2 is orthogonal. □

The decomposition

$$A = Q_1 D Q_2$$

is called a *singular value decomposition* of A.

Here is the geometric meaning of singular values. Assume A is invertible, and let S be the unit sphere in $\mathbb{R}^n$:

$$S = \{v_1^2 + \ldots + v_n^2 = 1\}.$$

Multiplication by Q_2 maps S to itself:

$$\mathbf{v} \in S \implies Q_2\mathbf{v} = 1.$$

Multiplication by D maps S to an ellipsoid

$$E = DS = \{\lambda_1^{-2}v_1^2 + \ldots + \lambda_n^{-2}v_n^2 = 1\}.$$

The numbers $\lambda_1, \ldots, \lambda_n$ are the *axis lengths* of E. Multiplication by Q_1 rotations E into another ellipsoid. See Figure 7.

Let's summarize the discussion in a theorem:

Theorem 21.5. *Let S be the unit sphere in $\mathbb{R}^n$, and A a square matrix. The set of vectors $A\mathbf{v}$ for $\mathbf{v} \in S$ form an ellipsoid in $\mathbb{R}^n$, whose semiaxis lengths are the singular values of A.*

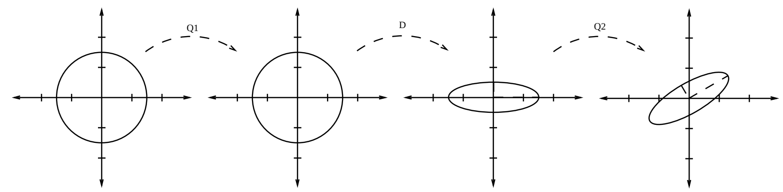


FIGURE 7. Geometric intepretation of singular values

22. Jordan normal form

In the previous sections, we saw that many matrices can be diagonalized, that is, transformed into diagonal matrices by multiplying by an invertible matrix on the left and its inverse on the right. This makes finding matrix powers easier.

In this section, we extend this to all matrices, by show that any matrix has a *Jordan normal form*. This is like a diagonal matrix, but possibly with some 1's above the diagonal.

Definition 22.1. (1) Let λ be a number. A *Jordan block* with eigenvalue λ is a matrix

$$\begin{bmatrix} \lambda & 1 & 0 & \ldots & 0 \\ 0 & \lambda & 1 & \ldots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \lambda & 1 \\ 0 & 0 & 0 & \ldots & \lambda \end{bmatrix}$$

with λ's on the diagonal and 1's above the λ's.

(2) A matrix is in *Jordan normal form* if J is a block diagonal matrix and each block is a Jordan block.

Theorem 22.2. *Any square matrix A is similar to a matrix J in Jordan normal form, which is unique up to ordering of the blocks.*

Proof. Let $m_1, \ldots, m_k$ be the algebraic multiplicities of the eigenvalues so that

$$\det(A - \lambda I) = (\lambda - \lambda_1)^{m_1} \ldots (\lambda - \lambda_k)^{m_k}.$$

For each j let
$$\overline{E}_{\lambda_j} = \{v|(A-\lambda_j I)^m v),\ m \gg 0\}$$
denote the *j-th generalized eigenspace* which contains the *j-th eigenspace*
$$E_{\lambda_j} = \{v|(A-\lambda_j I)v)\}.$$
First note that if $v \in \overline{E}_{\lambda_j}$ then for $m \gg 0$ we have
$$(A-\lambda_i I)^m v_j = ((A-\lambda_i I)+(\lambda_i-\lambda_j)I)^m v_j = (\lambda_i-\lambda_j)^m v_j.$$
It follows that vectors from different generalized eigenspaces are linearly independent. Indeed, suppose $v_j \in \overline{E}_{\lambda_j}, j = 1,\dots,k$ and
$$c_1 v_1 + \ldots + c_k v_k = 0.$$
Then
$$(A-\lambda_j I)^m (c_1 v_1+\ldots+c_k v_k) = (c_1(\lambda_1-\lambda_j)^m v_1+\ldots+c_k(\lambda_k-\lambda_j)^m v_k) = 0$$
for all $m \gg 0$ which can happen only if $c_1 = \ldots = c_k = 0$. Similarly, if $v \in \overline{E}_{\lambda_j}$ then the non-zero vectors on the list
$$v, (A-\lambda_j I)v, (A-\lambda_j I)^2 v, \ldots \tag{10}$$
are all independent, multiplying any dependence relation by powers of $A-\lambda_j I$. Indeed suppose that $m+1$ is the first power of $(A-\lambda_j I)$ so that $(A-\lambda_j I)^{m+1}v = 0$ and suppose
$$c_1 v + c_2(A-\lambda_j I)v + \ldots + c_m(A-\lambda_j I)^m v = 0.$$
Then multiplying by $(A-\lambda_j I)^m$ gives $c_1(A-\lambda_j I)^m v = 0$, so $c_1 = 0$. Multiplying by $(A-\lambda_j I)^{m-1}$ gives $c_2 = 0$ etc. In particular, this shows that $m \le n$, that is,
$$(A-\lambda_j I)^n v = 0, \forall v \in \overline{E}_{\lambda_j}.$$
For $v, w \in \overline{E}_{\lambda_j}$ we say that v is an *ancestor* of w if $w = (A-\lambda_j)^i v$ for some integer $i \ge 0$, and a chain such as (10) a *chain of ancestors*. The discussion above shows that each vector has a chain of ancestors of length at most n. Now let m be the length of the longest chain of ancestors, that is, the largest power of $(A-\lambda_j I)$ so that $(A-\lambda_j I)^m \overline{E}_{\lambda_j}$ is non-zero. Inside the eigenspace E_{λ_j} consider the subspaces
$$(A-\lambda_j I)^m \overline{E}_{\lambda_j}, (A-\lambda_j I)^{m-1}\overline{E}_{\lambda_j} \cap E_{\lambda_j}, \ldots, E_{\lambda_j};$$
these are the subspaces with ancestor chains of length m, $m-1$ etc. Choose a basis for E_{λ_j}, first by choosing a basis for $(A-\lambda_j I)^m \overline{E}_{\lambda_j}$ then $(A-\lambda_j I)^{m-1}\overline{E}_{\lambda_j} \cap E_{\lambda_j}$ etc. For each of these basis vectors choose a chain of ancestors of maximal length. Putting all these vectors together gives a linearly independent set $v_1,\ldots,v_l$, which we claim has size n. Indeed, extend this set to a basis $v_1,\ldots,v_n$ and let S be the matrix whose columns are $v_1,\ldots,v_n$. For the first l vectors, either $(A-\lambda_j)v_i = v_{i-1}$, if v_i is an ancestor of v_{i-1}, or $(A-\lambda_j)v_i = 0$, otherwise, the matrix SAS^{-1} is in Jordan normal form in the first l columns, that is,
$$SAS^{-1} = \begin{bmatrix} J & B \\ 0 & C \end{bmatrix}$$
for some Jordan matrix J and other matrices B, C.

We claim that C must have size zero, so that SAS^{-1} is in Jordan form. Otherwise, C has at least one eigenvalue, say λ_i, and eigenvector, say v. Then $(A-\lambda_i I)v$ is a linear combination of vectors in $\overline{E}_{\lambda_j}, j = 1,\ldots,k$, that is,
$$v = v_1 + \ldots + v_k.$$
For each $v_j, j \ne i$ choose a vector v_j' so that $(A-\lambda_i I)v_j' = v_j$. Then
$$(A-\lambda_i I)(v - \sum_{j\ne i} v_j') = v_i.$$
But then $v - \sum_{j\ne i} v_j' \in \overline{E}_{\lambda_i}$, which is a contradiction. $\square$

Example 22.3. Find the Jordan normal form for $A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$. The eigenvalue is 1 with algebraic multiplicity 3. The nullspaces are
$$E_1 = \text{nullspace}(A-1I) = \text{nullspace}\begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \text{span}\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}.$$
$$\overline{E}_1 = \text{nullspace}(A-1I)^2 = \text{nullspace}(0) = \mathbb{R}^3.$$
The vector $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ has an ancestor, the vector $\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ does not. Thus our basis is
$$\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\}.$$
The matrices S, J are
$$S = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}, \quad J = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$