AUGMENTATION VARIETIES AND DISK POTENTIALS III

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ABSTRACT. This is the third in a series of papers in which we construct Chekanov-Eliashberg algebras for Legendrians in circle-fibered contact manifolds and study the associated augmentation varieties. In this part, we prove that for connected Legendrian covers of monotone Lagrangian tori, the augmentation variety in this model is equal to the image of the zero level set of the disk potential, as suggested by Dimitroglou-Rizell-Golovko [16]. In particular, we show that Legendrian lifts of Vianna's exotic tori are not Legendrian isotopic, as conjectured in [16]. Using related ideas, we show that the Legendrian lift of the Clifford torus admits no exact fillings, extending results of Dimitroglou-Rizell [14] and Treumann-Zaslow [44] in dimension two. We consider certain disconnected Legendrians, and show, similar to another suggestion of Aganagic-Ekholm-Ng-Vafa [3], that the components of the augmentation variety correspond to certain partitions and each component is defined by a (not necessarily exact) Lagrangian filling. An adaptation of the theory of holomorphic quilts shows that the cobordism maps associated to bounding chains are independent of all choices up to chain homotopy.

CONTENTS

1. Introduction

In this third part in the series, we construct an augmentation variety associated to Legendrians in circle-fibered contact manifolds, and prove that it is a Legendrian isotopy invariant. The augmentation variety is closely related to the space of fillings of the Legendrian [28, 30, 22, 34, 39, 20, 13, 15, 10, 2, 17, 8, 27, 43, 7] and as such plays a role in the possible desingularizations of a singular Lagrangian in a symplectic manifold. In particular, any tamed filling defines a subset of the augmentation variety defined by polynomials that are killed by the corresponding augmentation. We denote by

$$\mathrm{Aug}_{\mathrm{geom}}(\Lambda) \subset \mathrm{Aug}(\Lambda)$$

the locus of the augmentation variety defined by tamed fillings; it is also a Legendrian isotopy invariant. In examples, each tamed filling L defines an irreducible component

$$\mathrm{Aug}_L(\Lambda) \subset \mathrm{Aug}_{\mathrm{geom}}(\Lambda) \subset \mathrm{Aug}(\Lambda);$$

it would be interesting to know whether this is always the case. In particular, the Harvey-Lawson Lagrangian considered as a filling has augmentation variety equal to that of the Legendrian

$$\mathrm{Aug}_L(\Lambda) = \mathrm{Aug}_{\mathrm{geom}}(\Lambda) = \mathrm{Aug}(\Lambda).$$

The same phenomenon repeats itself for Legendrians associated to other Fano toric varieties. The study of fillings and augmentation varieties is related to the following question in mirror symmetry. The mirror symmetry conjectures suggest that the space of deformations of a Lagrangian brane should have a complex analytic structure, and examples show that such a deformation space is only well-behaved if one includes certain surgery operations. In this surgery operation, the Lagrangian is modified locally by replacing one Lagrangian filling of a Legendrian by another. Thus the space of fillings describes the possible surgery operations.

The main result is a relationship between augmentation varieties and potentials suggested by Dimitroglou-Rizell-Golovko in [16, Conjecture 9.1]. Let

$$\Pi \subset Y$$

be a compact, connected, monotone, relatively spin Lagrangian. The disk potential is a polynomial function

$$W_\Pi : \mathrm{Rep}(\Pi) \to \mathbb{C}^\times$$

defined by a count of Maslov two holomorphic disks passing through a generic point in Π. In the language of A_∞ algebras, it has the following interpretation: Let

$$m_d : CF(\Pi)^{\otimes d} \to CF(\Pi)$$

be the structure maps of the Fukaya algebra $CF(\Pi)$ of the Lagrangian $\Pi \subset Y$, defined for simplicity over $\mathbb{C}$. The zero-th structure map

$$m_0 : CF(\Pi)^{\otimes 0} := \mathbb{C} \to CF(\Pi)$$

has image $m_0(1) \in CF(\Pi)$ the *curvature* of the Fukaya algebra $CF(\Pi)$. The projective version of the Maurer-Cartan equation requires that $m_0(1)$ is a multiple w of the unit $1_\Pi \in CF(\Pi)$, in which case one says that $CF(\Pi)$ is projectively flat. We

say that a projectively flat $CF(\Pi)$ is *flat* if w vanishes. Under suitable monotonicity assumptions $CF(\Pi)$ is automatically projective flat and viewing $w(y)$ as a function of the local system y on Π defines the potential $W_\Pi : \mathrm{Rep}(\Pi) \to \mathbb{C}$. For example, in the case $\Pi \subset \mathbb{C}P^2$ is the Clifford torus then one obtains

$$W_\Pi : \mathrm{Rep}(\Pi) \to \mathbb{C}, \quad (\hat{y}_1, \hat{y}_2) \mapsto \hat{y}_1 + \hat{y}_2 - \hat{y}_1^{-1}\hat{y}_2^{-1}$$

in coordinates $\hat{y}_1, \hat{y}_2$ on $\mathrm{Rep}(\Pi)$, with each term corresponding to one of the three Maslov-index-two disks in the complex projective plan bounding the Clifford torus. The zero level set of the potential is then the space of absolute, rather than projective, Maurer-Cartan solutions.

In the version of Legendrian contact homology considered in this paper, we have the following relationship between the disk potential of the Lagrangian projection and the augmentation variety. Such a relationship was conjectured in Dimitroglou-Rizell-Golovko [16] in the Ekholm-Entyre-Sullivan model [19] on the basis of computations for the Clifford and Chekanov tori. Let $\Lambda \subset Z$ be a horizontal lift of Π and denote the map on representation varieties induced by p

$$\mathrm{Rep}(p) : \mathrm{Rep}(\Pi) \to \mathrm{Rep}(\Lambda).$$

Theorem 1.1. *Suppose that Y is a monotone symplectic manifold with minimal Chern number at least two and Z a contact manifold whose curvature is a negative multiple of the symplectic form. For a Legendrian lift $\Lambda \subset Z$ of a connected compact monotone Lagrangian $\Pi \subset Y$ with minimal Maslov number two, the augmentation variety $\mathrm{Aug}(\Lambda)$ is equal to the image $\mathrm{Rep}(p)(W_\Pi^{-1}(0))$ of the zero level set of the disk potential W_Π.*

As a corollary of Theorem 1.1 we show the existence of infinitely many Legendrian tori in odd dimensional spheres with the standard contact structure which are pairwise non-isotopic. In the case of the lifts of Vianna tori and the tori in Chanda-Hirschi-Wang [9], invariance of the augmentation variety under Legendrian isotopy implies that these Legendrian tori are non-isotopic.

Corollary 1.2. *(Corollary 2.16 below) An odd-dimensional sphere with standard contact structure, (S^{2n-1}, ξ_{std}), has infinitely many Legendrian tori which are not Legendrian isotopic to each other.*

The examples of disconnected Legendrians in Theorem 2.32 show that for disconnected Legendrians the augmentation variety may be reducible, and so not directly related to the Maurer-Cartan space for the Lagrangian projection. In some cases, we may also compute the geometric augmentation variety:

Theorem 1.3. *Suppose that either Y is a monotone toric variety and Π is a monotone Lagrangian torus orbit with a non-trivial spin structure, or $Y = \mathbb{C}P^2$ and Π is one of Vianna's exotic tori associated to a Markov triple of the form $(1, b, c)$ for some integers b, c, in which case b, c are Fibonacci numbers. Then the geometric augmentation variety $\mathrm{Aug}_{\mathrm{geom}}(\Lambda)$ is equal to $\mathrm{Aug}(\Lambda)$.*

Using related considerations we generalize a result of Dimitroglou-Rizell [14] and Treumann-Zaslow [44] ruling out exact fillings of the Clifford Legendrian, see Dimitroglou-Rizell-Golovko [16, p.3], to arbitrary dimension:

Theorem 1.4. *(Theorem 2.33 below) The Clifford Legendrian $\Lambda_{\mathrm{Cliff}} \cong T^{n-1} \subset S^{2n-1}$ has no exact Lagrangian filling for $n > 2$. It has no spin Lagrangian filling for the trivial spin structure, and for any non-trivial spin structure its augmentation variety $\mathrm{Aug}(\Lambda_{\mathrm{Cliff}})$ is defined by the augmentation for the Harvey-Lawson filling $L_{(1)}$ of I-(2) for some choice of spin structure on $L_{(1)}$.*

This result does not use augmentations themselves, but rather a study of the moduli spaces involved in the construction of augmentations. The proof proceeds by showing that a filling would imply that the image of the restriction map to the boundary in cohomology is too large for the image to be a maximally isotropic subspace.

Remark 1.5. We frequently use the results in [4, 5] and cite them with prefixes I- and II- respectively.

2. Augmentations and fillings

In this section, we define an analog of the augmentation variety of Ng [33]; see also Aganagic-Ekholm-Ng-Vafa [3], Diogo-Ekholm [18] and Gao-Shen-Weng [25] for definitions in other contexts. We make various computations of augmentation varieties, and in particular prove Theorem 1.1 from the introduction.

2.1. **Algebraic augmentation varieties.** For the remainder of the paper, we assume that Λ covers a monotone Lagrangian $\Pi \subset Y$ and $CE(\Lambda)$ is defined over the uncompleted group ring $G(\Lambda)$.

An *augmentation* is a chain algebra map

$$\varphi : CE(\Lambda) \to G(\varphi)$$

for some abelian ring $G(\varphi)$, considered as a trivial complex. In particular, any augmentation must vanish on the image of the differential δ.

A *graded augmentation* is defined similar, requiring that φ is a dga map and $G(\varphi)$ is concentrated in degree zero.

Example 2.1. Consider the contact dga $CE(\Lambda)$ for the Clifford Legendrian $\Lambda \subset S^{2n-1}$ where Λ has the standard spin structure. The generators $\mathfrak{c}_1, \ldots, \mathfrak{c}_{n-1} \in \mathcal{I}_\bullet(\Lambda)$ are degree zero, and so may be mapped to non-zero elements in $G(\varphi)$ under the augmentation. The degree one generators are the classical generator $\mathfrak{b} \in \mathcal{I}_\bullet(\Lambda)$ of Morse degree two, and the Reeb chord $\mathfrak{a} \in \mathcal{I}_\bullet(\Lambda)$ of Morse degree zero and length $2\pi/n$, with differentials with $\mathbb{C}$-coefficients

(1) $\qquad \delta^{\mathrm{ab},0}(\mathfrak{b}) = 0, \quad \delta^{\mathrm{ab},0}(\mathfrak{a}) = \pm 1 \pm [\mu_1] \exp(\mathfrak{c}_1) \pm \ldots \pm [\mu_{n-1}] \exp(\mathfrak{c}_{n-1}).$

Define as coefficient ring

$$G(\varphi) = \mathbb{C}[[\mu_1, \ldots, \mu_{n-1}]].$$

Define a map $\varphi : CE(\Lambda) \to G(\varphi)$ by

$$\varphi(\mu_n) = \pm 1$$

where the sign is chosen so that the relation given by (1) becomes

$$\varphi(\exp(\mathfrak{c}_{n-1})) = \varphi(1 \pm [\mu_1] \pm \ldots \pm [\mu_{n-2}]);$$

this guarantees that the logarithm

$$\varphi(\mathfrak{c}_{n-1}) := \ln(1 \pm [\mu_1] \pm \ldots \pm [\mu_{n-2}])$$

has zero constant term and is so well-defined in the completed coefficient ring. Then φ defines a graded augmentation.

Lemma 2.2. *Any augmentation φ induces a map $H(\varphi) : HE(\Lambda) \to G(\varphi)$, and if φ is graded then $H(\varphi)$ is trivial except in degree zero. Conversely, if $CE(\Lambda)$ is concentrated in non-negative degree then any map $HE_\bullet(\Lambda) \to G(\varphi)$ defines an augmentation.*

Proof. Given a map $HE_0(\Lambda) \to G(\varphi)$, one obtains a lift to a map $CE_0(\Lambda) \to G(\varphi)$ by composition with $CE_0(\Lambda) = \ker(\delta) \to HE_0(\Lambda)$. $\square$

It possible that there is some version of the Lemma above even in the case that there are generators of negative degree, but we will not need the Lemma and so do not pursue this question further.

By the previous section, any tamed filling L gives rise to an augmentation with target $G(\varphi) = \hat{G}(L)$. We wish to extract from the space of augmentations a subvariety of the abelian representation variety.

Definition 2.3. (Augmentation variety) The *extended augmentation ideal*

$$\tilde{I}(\Lambda) = \bigcap_\varphi (\iota \circ \varphi)^{-1}(0_{G(\varphi)}) \subset CE(\Lambda)$$

is the set of elements in $CE(\Lambda)$ in the kernel $\ker(\iota \circ \varphi)$ of every augmentation

$$\varphi : CE(\Lambda) \otimes G(\varphi) \to G(\varphi)$$

where the coefficient ring $G(\varphi)$ is some abelian ring and

$$\iota : CE(\Lambda) \to CE(\Lambda) \otimes G(\varphi), \ a \mapsto a \otimes 1$$

is the tensoring-by-one morphism.

Let

$$CE^{\mathrm{ab}}(\Lambda) = CE(\Lambda)/\sim$$

denote the quotient of $CE(\Lambda)$ obtained by identifying two words that are equivalent up to re-ordering up to a sign determined by the grading of the letter. More precisely, we quotient by the relation $ab\tilde{(} - 1)^{|a||b|}ba$. We denote by

$$\delta^{\mathrm{ab}} : CE^{\mathrm{ab}}(\Lambda) \to CE^{\mathrm{ab}}(\Lambda)$$

the map induced by the differential δ. Since each ring $G(\varphi)$ is abelian, $\tilde{I}(\Lambda)$ is invariant under permutation of any elements in the constituent words, and so is the inverse image of an abelianized ideal

$$\widetilde{I}^{\mathrm{ab}}(\Lambda) = \widetilde{I}(\Lambda)/\sim.$$

Suppose that Λ is monotone, and $CE(\Lambda)$ is defined using the group ring $\hat{G}(\Lambda)$ on $H_1(\Lambda)$. Choose a basis

$$\mathfrak{c}_1, \ldots, \mathfrak{c}_k \in CE(\Lambda)$$

in the span of the generators $\mathcal{I}_\bullet(\Lambda)$ representing a basis of codimension one cycles given by $\mu_1, \ldots, \mu_k \in H_1(\Lambda)$. We embed functions on $\mathrm{Rep}(\Lambda)$ in $CE^{\mathrm{ab}}(\Lambda)$ by assigning to each weight

$$\lambda = (\lambda_1, \ldots, \lambda_k)$$

the monomial

$$y^\lambda = y_1^{\lambda_1} \cdots y_k^{\lambda_k}$$

where

$$y_i = [\mu_i] \exp(\mathfrak{c}_i).$$

The *augmentation ideal* is

$$I(\Lambda) = \pi(\widetilde{I}(\Lambda)) \cap \mathbb{C}[\mathrm{Rep}(\Lambda)].$$

Let

$$\mathrm{Aug}(\Lambda) \subset \mathrm{Rep}(\Lambda)$$

be the variety defined by $I(\Lambda)$. (One could also take the *augmentation scheme* defined by $I(\Lambda)$, which, the arguments below will show, is also a Legendrian isotopy invariant, but we avoid schemes since we have no application for this more refined invariant at the moment.) The $\mathbb{R}$-graded augmentation variety $\mathrm{Aug}_\mathbb{R}(\Lambda) \subset \mathrm{Rep}(\Lambda)$ and $\mathbb{Z}_2$-graded augmentation variety $\mathrm{Aug}_{\mathbb{Z}_2}(\Lambda) \subset \mathrm{Rep}(\Lambda)$ is defined similarly, by allowing only $\mathbb{R}$ or $\mathbb{Z}_2$-graded augmentations. We have a natural inclusion

$$\mathrm{Aug}_\mathbb{R}(\Lambda) \subset \mathrm{Aug}(\Lambda)$$

induced by the reverse inclusion of ideals. This ends the Definition.

Example 2.4. In the case of the Clifford Legendrian with trivial spin structure, the differential

$$\delta(\mathfrak{a}) = 1 + [\mu_1] \exp(\mathfrak{c}_1) + \mu_2 \exp(\mathfrak{c}_2)$$

must map to zero under any augmentation, and so

(2) $$\mathrm{Aug}(\Lambda) \subset \{1 + y_1 + y_2\}.$$

On the other hand, by Example 2.1 there exists an augmentation which does not vanish on any polynomial in y_1, so $\mathrm{Aug}_{L_{(1)}}(\Lambda)$ is a hypersurface. It follows that the inclusion (2) is an equality.

Remark 2.5. Multiple definitions of the augmentation variety appear in the literature. Ng [33], working with Legendrian two-tori, defines the augmentation variety as the space of points in the representation variety for which there exists an augmentation corresponding to that specialization of variables. Gao-Shen-Weng [25] define the augmentation variety as the moduli space of augmentations. Aganagic-Ekholm-Ng-Vafa [3] and Diogo-Ekholm [18] define the augmentation variety as a subvariety of the quantized torus in the case of contact knot homology.

A simpler version of the augmentation variety in our case would be to consider the sub-complex $CE_\circ(\Lambda)$ generated by Reeb chords. The homology $HE_\circ(\Lambda)$ has

degree zero part $HE_{\circ,0}(\Lambda)$ (with apologies for the notation) which is a graded ring. We can therefore define the *lch spectrum*

$$\text{(3)} \qquad \text{Aug}_\circ(\Lambda) = \text{Spec}(HE^{\text{ab}}_{\circ,0}(\Lambda)).$$

This variety will also be shown to be a Legendrian isotopy invariant, but Lagrangian fillings do not necessarily define components of this augmentation variety. There is a natural map

$$\text{Aug}_\circ(\Lambda) \to \text{Rep}(\Lambda)$$

obtained from the embedding of functions in $HE_{\circ,0}(\Lambda)$ as the coefficients of the zero length word.

Proposition 2.6. *The augmentation varieties* $\text{Aug}(\Lambda)$ *resp.* $\text{Aug}_\mathbb{R}(\Lambda)$ *resp.* $\text{Aug}_{\mathbb{Z}_2}$ *are independent of the choice of embedding* $\mathbb{C}[\text{Rep}(\Lambda)] \to CE^{\text{ab}}(\Lambda, \hat{G}(\Lambda))$, *that is, independent of the choice of cycles* $\mathfrak{c}_1, \ldots, \mathfrak{c}_k \in H_1(\Lambda, \mathbb{Z})_{\text{free}}$.

Proof. We will check that any two choices have difference that is a coboundary of the Chekanov-Eliashberg differential, up to terms that vanish under abelianization or projection to degree zero words. Let $\mathfrak{c}'_1, \ldots, \mathfrak{c}'_k \in \mathcal{I}_\bullet(\Lambda)$ be another choice of cycle representatives for $H_1(\Lambda)_{\text{free}}$ so that $[\mathfrak{c}_i] = [\mathfrak{c}'_i]$ for all i. By definition, there exist chains

$$\mathfrak{b}_i \in CE_\bullet(\Lambda)$$

of Morse degree two, and so real degree one, with

$$\delta_{\text{Morse}}(\mathfrak{b}_i) = \mathfrak{c}_i - \mathfrak{c}'_i.$$

The contact differential applied to $\mathfrak{b}_i$ gives these terms plus additional terms involving (constant or non-constant) holomorphic disks:

$$\delta(\mathfrak{b}_i) = \mathfrak{c}_i - \mathfrak{c}'_i + \sum_{j_1, \ldots, j_k} \delta_d(\mathfrak{b}_i, \mathfrak{c}_{j_1}, \ldots, \mathfrak{c}_{j_k}) \mathfrak{c}_{j_1} \ldots \mathfrak{c}_{j_k}$$

where $\delta_d(\mathfrak{b}_i, \mathfrak{c}_{j_1}, \ldots, \mathfrak{c}_{j_k})$ is the coefficient of $\mathfrak{c}_{j_1} \ldots \mathfrak{c}_{j_k}$ in $\delta_d(\mathfrak{b}_i)$, with notation from Proposition II-3.19. The additional terms in $\delta(\mathfrak{b}_i)$ are of two types: all $\mathfrak{c}_{j_1}, \ldots, \mathfrak{c}_{j_k}$ are degree zero or at least one of them has negative degree. By Proposition 3.19, we may assume that the structure coefficients

$$\delta_d(\mathfrak{b}_i, \mathfrak{c}_{j_1}, \ldots, \mathfrak{c}_{j_k}) \in \mathbb{C} \subset \hat{G}(\Lambda)$$

of the Morse A_∞ algebra of Λ make the degree zero terms skew-symmetric. The skew-symmetric words lie in the kernel of φ since φ is an algebra homomorphism and $\hat{G}(L)$ is abelian. On the other hand, φ is graded and so vanishes on negative degree generators. Thus

$$0 = \varphi(\delta(\mathfrak{b}_i)) = \varphi(\mathfrak{c}_i) - \varphi(\mathfrak{c}'_i)$$

as desired. $\qquad \square$

Lemma 2.7. *Let* $\varphi : CE(\Lambda, G(\varphi)) \to G(\varphi)$ *be an algebra map such that* $\varphi(\delta(a)) = 0$ *for all single letter generators* $a \in \mathcal{I}(\Lambda)$, *then* φ *defines an augmentation.*

Proof. We need to check $\varphi \circ \delta = 0$. Since from the Leibniz rule we have,

$$\varphi(\delta(a_1 a_2)) = \varphi(\delta(a_1)a_2 \pm a_1\delta(a_2)),$$

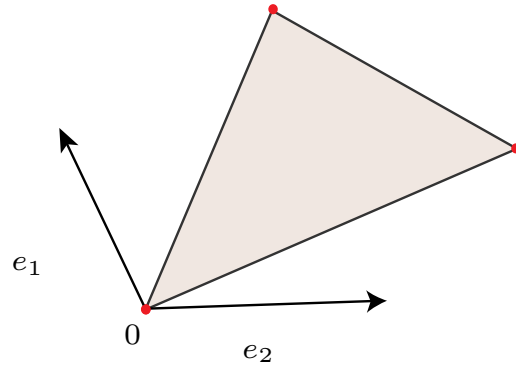the lemma follows from an inductive argument on word length. $\square$

FIGURE 1. Basis to ensure non-negative exponents

Theorem 2.8. *Let Z be a negative circle bundle over a monotone symplectic manifold Y with minimal Chern number at least two. Suppose $\Lambda \subset Z$ a connected Legendrian lift of a connected compact monotone Lagrangian Π with minimal Maslov number two and equipped with a relative spin structure. The augmentation variety of Λ satisfies*

$$\mathrm{Aug}(\Lambda) = \mathrm{Aug}_\circ(\Lambda) = \mathrm{Rep}(p)((W)^{-1}(0)).$$

Proof. One direction of containment follows from the partial computation of the differential in Example II-3.5: If $\mathfrak{a} \in \mathcal{I}_\circ(\Lambda)$ is the Morse-degree-zero generator of minimal Reeb length then

(4) $$\varphi(\delta^{\mathrm{ab}}(\mathfrak{a})) = W_\Lambda(\varphi([\mu_1])e^{\varphi(\mathfrak{c}_1)}, \ldots, \varphi([\mu_k])e^{\varphi(\mathfrak{c}_k)}).$$

Thus

$$\mathrm{Aug}(\Lambda) \subseteq \mathrm{Rep}(p)((W)^{-1}(0)).$$

To show the reverse inclusion, we explicitly construct an augmentation valued in a certain formal power series ring which takes the desired values given by any point in the augmentation variety. We may without loss of generality assume that a basis for $H_1(\Lambda)$ has been chosen so that the exponents in (4) are non-negative.

We find an augmentation by a formal expansion around a transversally-cut-out zero of the disk potential. Let

$$W_\Lambda(y_k) = W(0, \ldots, 0, y_k)$$

denote the polynomial obtained by setting the first $k-1$ coordinates to zero. Then $W_\Lambda(y_k)$ is a polynomial in y_k, and as such has at least one solution over the complex numbers, call it $\kappa \in \mathbb{C}$, so that

$$W_\Lambda(\kappa) = 0.$$

As a simple case of Sard's theorem, for a generic linear transformation in $GL(k, \mathbb{C})$ in the coordinates $y_1, \ldots, y_k$, each non-zero solution κ is transversally cut out. Equivalently, that is, the roots of the polynomial $W_\Lambda(\kappa)$ have multiplicity one. Indeed, this condition is equivalent to the condition that a generic line intersects $W^{-1}(0)$ transversally. The projection

$$\pi : W_\Lambda^{-1}(0) - \{0\} \to \mathbb{C}P^{k-1}$$

onto complex projective space $\mathbb{C}P^{k-1}$ has finite fiber over any line $\ell \in \mathbb{C}P^{k-1}$. Indeed, if not the fiber $\pi^{-1}(\ell)$ would be a line and so contain 0, which violates the condition that the constant term in W is non-vanishing. So the image $\pi(W_\Lambda^{-1}(0) - \{0\})$ is a quasiprojective variety of dimension $k-1$ and so dense. The fiber over a generic line in $\mathbb{C}P^{k-1}$ has the desired property, by Sard's theorem. Let

$$G(\varphi) = \mathbb{C}[[\mu_1, \ldots, \mu_{k-1}]].$$

Define an augmentation with values in $G(\varphi)$ by setting

$$\varphi([\mu_{n-1}]) = \kappa$$
$$\varphi(\mathfrak{c}_1) = \ldots = \varphi(\mathfrak{c}_{k-1}) = 0$$

and

$$\varphi(\mathfrak{c}_k) \in G(\varphi)$$

so that

(5) $$W_\Lambda([\mu_1], \ldots, [\mu_{k-1}], \kappa \exp(\varphi(\mathfrak{c}_k))) = 0.$$

The existence of such a formal solution follows from a formal version of the implicit function theorem, which is to say an order-by-order analysis: The leading order term vanishes by assumption. Consider the linearization

$$D_{y_k} W_\Lambda(\kappa) : \mathbb{C}[\mu_1, \ldots, \mu_{k-1}] \to \mathbb{C}[\mu_1, \ldots, \mu_{k-1}]$$

given by multiplication by the number $D_{y_k} W_\Lambda(\kappa) \in \mathbb{C}^\times$ of (5). This map is an isomorphism, as the solution $(0, \ldots, 0, \kappa)$ is transversally cut out. [1] Given a solution $\mathfrak{s}_d$ of (5) to order d define

$$\mathfrak{s}_{d+1} = \mathfrak{s}_d + (D_\kappa W_\Lambda)^{-1}(W_\Lambda([\mu_1], \ldots, [\mu_{k-1}], \kappa \exp(\mathfrak{s}_d))$$

Then $\mathfrak{s}_{d+1}$ is a solution to order $d+1$ and agrees with $\mathfrak{s}_d$ to order d. Taking the limit gives the desired solution $\mathfrak{s} = \varphi(\mathfrak{c}_k)$. By construction, φ is non-vanishing

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[1]Here we use the fact that $CE(\Lambda)$ is defined over $G(\Lambda)$ rather than the completion $\hat{G}(\Lambda)$; if we used $\hat{G}(\Lambda)$ then we would have to justify that φ is well-defined on the completion which is unclear to us.

on the ring generated by the first $k - 1$ coordinates $y_1, \ldots, y_{k-1}$, so the subvariety $\mathrm{Aug}_\varphi(\Lambda)$ defined by the ideal $\varphi^{-1}(0)$ is a hypersurface containing $(0, \ldots, 0, \kappa)$. Thus the irreducible component of $\mathrm{Rep}(p)(W^{-1}(0))$ containing $(0, \ldots, 0, \kappa)$ is contained in $\mathrm{Aug}(\Lambda)$.

Repeating this procedure for each irreducible component of $\mathrm{Rep}(p)(W^{-1}(0))$ proves that all irreducible components are so contained. More precisely, suppose that

$$W_\Lambda = W_{\Lambda,1}^{d_1} W_{\Lambda,2}^{d_2} \ldots W_{\Lambda,l}^{d_l}$$

is the decomposition of W_Λ into irreducible factors with multiplicities $d_1, \ldots, d_l$. For each $i = 1, \ldots, l$ choose coordinates and $\kappa \in \mathbb{C}$ so that $(0, \ldots, 0, \kappa)$ is a transversally cut out solution to $W_{\Lambda,i} = 0$ not contained in any other $W_{\Lambda,j}^{-1}(0)$. The construction of the previous paragraph gives an augmentation such that $\mathrm{Aug}_{\varphi_i}(\Lambda) \subset \mathrm{Aug}(\Lambda)$ is a hypersurface containing $W_{\Lambda,i}^{-1}(0)$. Thus $\mathrm{Aug}(\Lambda)$ contains each of the irreducible components of $\mathrm{Rep}(p)(W^{-1}(0))$, and this proves the equality claimed in the Theorem.

It remains to show that lch spectrum is also given by the zero level set of the disk potential. For degree reasons the elements $\delta_\circ(\mathfrak{a})$ generate the image of $\delta_\circ$, as $\mathfrak{a}$ is the only Reeb generator of degree one. It follows that $\mathrm{Aug}_\circ(\Lambda)$ is the variety defined by $\delta_\circ(\mathfrak{a})$. □

Remark 2.9. We can modify the Definition 2.3 by considering the augmentations to $G(\varphi)$ where $G(\varphi)$ is an integral domain. We call this ideal the *extended domain-augmentation ideal* and denote it with $\widetilde{I}_D$. We can similarly define the domain-augmentation ideal,I_D, by projecting to the image of $\mathbb{C}[\mathrm{Rep}(\Lambda)]$. The proof of Theorem 2.8 shows that under the same hypothesis as the Theorem,

$$I_D = \sqrt{\langle W_\Lambda \rangle}.$$

Remark 2.10. One can also show that under the same hypothesis, Theorem 2.8 can be extended to show equality of the augmentation schemes instead of varieties, i.e.

$$I = \langle W_\Lambda \rangle.$$

We give a quick sketch of the proof. Assume that the we have the following irreducible factorization as before,

$$W_\Lambda = W_{\Lambda,1}^{d_1} W_{\Lambda,2}^{d_2} \ldots W_{\Lambda,l}^{d_l}.$$

The idea is similar to that of Theorem 2.8, but instead of constructing an augmentation φ which vanishes on one of the irreducible factor $W_{\Lambda,i}$ of W_Λ, we can construct an augmentation ϕ such that $\varphi(W_{\Lambda,i})$ is a nilpotent element of order d_i. Let

$$G(\varphi) = R[[\mu_1, \ldots, \mu_{k-1}]], \quad R = \mathbb{C}[\alpha]/\langle \alpha^{d_i} \rangle.$$

After a change in basis as in the proof of Theorem 2.8, we may assume that $W_{\Lambda,i}(0, .., y_k)$ is a polynomial with only transverse roots and let 1 be a root. Thus $W'_{\Lambda,i}(0, .., 1) \neq 0$. Then, by viewing $W_{\Lambda,i}$ as a polynomial with R coefficients, we have

$$W_{\Lambda,i}(0, .., 1 + \alpha) = c\alpha + \text{higher order terms in } \alpha, \quad c \in \mathbb{C}^\star.$$

Note that $W_{\Lambda,i}(0,..,1+\alpha)$ is a nilpotent element of order d_i. By performing a implicit function theorem inspired induction as before, we can create an augmentation

$$\varphi : CE(\Lambda) \to R[[\mu_1, \ldots, \mu_{k-1}]]$$

such that

$$\varphi(\mathfrak{c}_i) = 0 \ \forall i < k$$
$$\varphi([\mu_i]) = [\mu_i] \ \forall i < k$$
$$\varphi([\mu_k]) = 1$$

and $\varphi(\mathfrak{c}_k)$ satisfies the equation

$$W_{\Lambda,i}(\mu_1, \ldots, \mu_{k-1}, \exp(\varphi(\mathfrak{c}_k))) = W_{\Lambda,i}(0,..,1+\alpha).$$

Thus we have that the ideal, I_φ, corresponding to the augmentation φ contains $W_{\Lambda,i}^{d_i}$ but not any lower order exponents. Thus, by repeating this argument for each irreducible factor, we can conclude that

$$I = \langle W_\Lambda \rangle.$$

The benefit of such a result is that, since $W_\Lambda = x^{-v}W_\Pi$, the equality of ideals shows that the augmentation ideal can be completely recovered from the disk-potential and vice-versa. This ends the Remark.

Example 2.11. We construct an augmentation for the Clifford Legendrian with the trivial (unfillable) spin structure. We have

$$\delta^{\mathrm{ab}}(\mathfrak{a})) = 1 + [\mu_1]e^{\mathfrak{c}_1} + [\mu_2]e^{\mathfrak{c}_2}.$$

We choose as solution the element

(6) $$\varphi([\mu_1]) = 0, \varphi([\mu_2]) = \kappa = -1.$$

A formal solution is then given by

$$\varphi(\mathfrak{c}_2) = \ln(1 + [\mu_1]).$$

On the other hand, we could expand around the solution (6)

$$\varphi(\mathfrak{c}_1) = \ln(1 + [\mu_2]).$$

If the spin structure was fillable, we obtain similar augmentations with different sign choices; these different augmentations correspond to the different choices of smoothing of the Harvey-Lawson cone as we will explain in the next section.

We wish to show that the augmentation variety is an invariant of Legendrian isotopy. For this, we will show that the cobordism maps induce maps on augmentation varieties. We first achieve a partial skew-symmetry for the cobordism maps. Write

$$\varphi(\gamma) = \sum_\Gamma \varphi_\Gamma(\gamma)$$

where $\varphi_\Gamma(\gamma)$ is the contribution from maps with domain type Γ.

Lemma 2.12. *For invariant perturbations in the sense of Remark II-3.5, the contributions to the cobordism map $\varphi = \varphi(L, b)$ applied to a degree one (Morse-degree two) generator $\mathfrak{b} \in \mathcal{I}_\bullet(\Lambda)$ have the following skew-symmetry property: Suppose that $e_i, e_{i+1} \in \mathrm{Edge}_\rightarrow(\Gamma)$ are leaves of a tree Γ incident on the same vertex $v \in \mathrm{Vert}(\Gamma)$. Then the transposition $\sigma_{i(i+1)}$ satisfies*

$$\sigma_{i(i+1)} \varphi_\Gamma(\mathfrak{b}) = (-1)^{\deg_{\mathbb{Z}_2}(\gamma_i) \deg_{\mathbb{Z}_2}(\gamma_{i+1})} \varphi_\Gamma(\gamma).$$

That is, the output of δ is graded-commutative with respect to the transposition of the edges e_i, e_{i+1}. In particular, after abelianization the output of φ consists of length one words, and φ is the classical Morse continuation map.

The proof is the same as that of Lemma II-3.21. For any map of dga's

$$\varphi : CE(\Lambda_-, G(\varphi)) \to CE(\Lambda_+, G(\varphi))$$

denote the abelianization

$$\varphi^{\mathrm{ab}} : CE^{\mathrm{ab}}(\Lambda_-, G(\varphi)) \to CE^{\mathrm{ab}}(\Lambda_+, G(\varphi)).$$

Lemma 2.13. *Let $X = \mathbb{R} \times Z$ is a symplectization and L is the cobordism constructed from an isotopy of Legendrians in Example I-2.21. The abelianized cobordism map*

$$\varphi(L, b)^{\mathrm{ab}} : CE^{\mathrm{ab}}(\Lambda_-, G(\Lambda_-)) \to CE^{\mathrm{ab}}(\Lambda_+, G(\Lambda_+))$$

preserves the sub-rings $\mathbb{C}[\mathrm{Rep}(\Lambda_-)] \cong \mathbb{C}[\mathrm{Rep}(\Lambda_+)]$ up to elements of the augmentation ideal $\tilde{I}^{\mathrm{ab}}(\Lambda_+)$, that is,

$$\varphi(L, b)^{\mathrm{ab}}(\mathbb{C}[\mathrm{Rep}(\Lambda_-)]) \subset \mathbb{C}[\mathrm{Rep}(\Lambda_+)] + \tilde{I}^{\mathrm{ab}}(\Lambda_+).$$

Proof. First note that we have an identification of coefficient rings. Indeed since L is diffeomorphic to $\mathbb{R} \times \Lambda$ we have natural identifications

$$\hat{G}(L) \cong G(\Lambda_-) \cong G(\Lambda_+).$$

Next we check that the coordinate rings on the representative variety are preserved. Let $u : S \to X$ be a disk bounding L with no punctures. The number of intersections of the boundary ∂u with the geometric cycle given by the union of stable manifolds Σ_i^s corresponding to the Morse cycle $\mathfrak{c}_i$ is the intersection number of ∂u and the closure of Σ_i^s. The intersection number is topological and independent of the choice of representative of $[\mathfrak{c}_i]$. The augmentation $\varphi(L, b)$ is defined on the generators $\mathcal{I}_\bullet(\Lambda)$ by counts of parametrized trajectories without disk components. Indeed, since the disks have no incoming strip-like ends, the positivity conditions imply that there are no non-constant disks, so that the treed disks in X bounding L have zero area. By Lemma II-3.21, any collection of cycles $\mathfrak{c}_{i,-}, i = 1, \ldots, k$ maps under $\varphi(L, b)$ to a collection of cycles $\mathfrak{c}_{i,+}, i = 1, \ldots, k$, up to terms vanishing under every augmentation. It follows that the subspace $\mathbb{C}[\mathrm{Rep}(\Lambda)_-]$ in $CE(\Lambda_-, \hat{G}(\Lambda))$ generated by polynomials y^λ is mapped to $\mathbb{C}[\mathrm{Rep}(\Lambda_+)]$ in $CE(\Lambda_+, \hat{G}(\Lambda))$, up to terms in $\tilde{I}^{\mathrm{ab}}(\Lambda_+)$, as claimed. $\square$

Theorem 2.14. *Let $X = \mathbb{R} \times Z$ is a symplectization and L is the cobordism constructed from an isotopy of Legendrians in Example I-2.21. The cobordism map*

$$\varphi(L,b) : CE(\Lambda_-, G(\Lambda_-)) \to CE(\Lambda_+, G(\Lambda_+))$$

induces isomorphisms of augmentation varieties and lch spectra

$$\mathrm{Aug}(L)\,\mathrm{Aug}(\Lambda_-) \to \mathrm{Aug}(\Lambda_+), \quad \mathrm{Aug}_\circ(L) : \mathrm{Aug}_\circ(\Lambda_-) \to \mathrm{Aug}_\circ(\Lambda_+).$$

Proof. We must show that the cobordism map induces a map of augmentation ideals. Since the cobordism maps $\varphi(L,b)$ (in this case $b = 0$) are chain maps, if φ' is an augmentation of $CE(\Lambda_+, G(\Lambda_+))$ then the composition $\varphi' \circ \varphi(L,b)$ is an augmentation of $CE(\Lambda_-, G(\Lambda_-)$ By Lemma 2.13, the subring $\mathbb{C}[\mathrm{Rep}(\Lambda)]$ and so the augmentation ideal $I(\Lambda_\pm)$ is preserved by $\varphi(L,b)$:

$$\varphi(L,b)(I(\Lambda_-)) = I(\Lambda_+).$$

So the chain map $\varphi(L,b)$ induces a map of augmentation varieties from $\mathrm{Aug}(\Lambda_+)$ to $\mathrm{Aug}(\Lambda_-)$. The reverse isotopy induces a map $I(\Lambda_+) \to I(\Lambda_-)$. The composed maps $CE^{\mathrm{ab}}(\Lambda_\pm) \to CE^{\mathrm{ab}}(\Lambda_\pm)$ are necessarily chain homotopic to the identity, and so equal to the identity on $\mathbb{C}[\mathrm{Rep}(\Lambda)_\pm]$ up to boundaries, which lie in $I(\Lambda_\pm)$. Thus the map of augmentation varieties $\mathrm{Aug}(\Lambda_+) \to \mathrm{Aug}(\Lambda_-)$ is an isomorphism. $\square$

Chanda-Hirschi-Wang [9] extend Vianna's construction of monotone tori to higher dimensional projective spaces. In particular, for every Markov triple (a, b, c), they construct a monotone Lagrangian torus $\overline{T}_{abc}$ in $\mathbb{P}^n$ and show that their disk potentials have distinct Newton polytope. By taking the Bohr-Sommerfeld lifts of these monotone Lagrangian tori, we obtain embedded Legendrian tori in S^{2n-1}. We denote the Bohr-Sommerfeld lift of the n-dimensional lifted Vianna torus corresponding to the Markov triple (a, b, c) as Λ_n^{abc}.

Lemma 2.15. *The augmentation polynomial, $W_{\Lambda_{n-1}^{abc}}$, of the n-dimensional Legendrian torus is an irreducible polynomial.*

Proof. The augmentation polynomial is defined as the polynomial $x^{-v} W_{\overline{T}_{abc}}$ where v is a vertex of the Newton polytope of $W_{\overline{T}_{abc}}$. Thus, the augmentation polynomial has the same Newton polytope as that of $W_{\overline{T}_{abc}}$, up to a translation and change of basis. When $n = 3$, since the Newton polytope of $W_{\overline{T}_{abc}}$ is a triangle, from the Irreducibility Criterion of [26], we have that $W_{\Lambda_2^{abc}}$ is an irreducible polynomial. For $n > 3$ we use induction to finish the proof. Assume that $W_{\Lambda_{d-1}^{abc}}$ is irreducible for $d > 3$. From Proposition 4.6 of [9], we see $\mathrm{Newt}(W_{\Lambda_{d-1}^{abc}})$ is a $d - 1$ simplex obtained from a suspension of $\mathrm{Newt}(W_{\Lambda_{d-2}^{abc}})$. Pick a basis of $\mathbb{Z}^{d-1}$ such that the face $\mathrm{Newt}(W_{\Lambda_{d-2}^{abc}})$ lies in the cone generated by the first $d - 2$ coordinates and $\mathrm{Newt}(W_{\Lambda_{d-1}^{abc}})$ lies in the positive cone corresponding to the basis. Call the polynomial variables corresponding to this choice of basis $(x_1, \ldots, x_{d-2}, y)$. From the choice of our basis, we see that setting $y = 0$ in $W_{\Lambda_{d-1}^{abc}}(x_1, \ldots, x_{d-2}, y)$ recovers the augmentation polynomial $W_{\Lambda_{d-2}^{abc}}$. See Figure 2. If the polynomial $W_{\Lambda_{d-1}^{abc}}(x_1, \ldots, x_{d-2}, y)$ was reducible, then
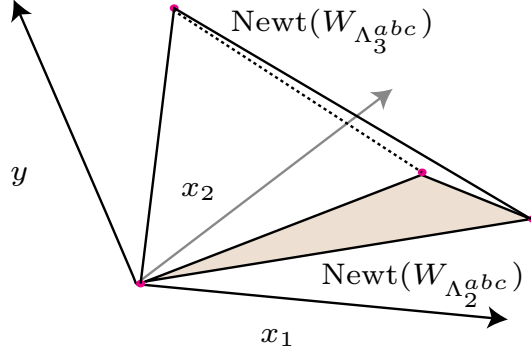
FIGURE 2. Newton polytope for two and three-dimensional tori corresponding to to (a, b, c).

we will have

$$W_{\Lambda_{d-1}^{abc}}(x_1, \ldots, x_{d-2}, y) = f(x_1, \ldots, x_{d-2}, y)g(x_1, \ldots, x_{d-2}, y).$$

Here f, g are non-constant polynomials. Then, from [38] (or Lemma 2.1 in [26]) we have $\mathrm{Newt}(W_{\Lambda_{d-1}^{abc}}) = \mathrm{Newt}(f) + \mathrm{Newt}(g)$, where '+' denotes Minkowski sum. By setting $y = 0$, we get a factorization of $W_{\Lambda_{d-2}^{abc}}$, which we know is irreducible. This implies either f or g is a polynomial solely consisting of the variable y. Without loss of generality, say g is a polynomial in y. Then $\mathrm{Newt}(g)$ is a line-segment along the y-axis. Since $\mathrm{Newt}(W_{\Lambda_{d-1}^{abc}}) = \mathrm{Newt}(f) + \mathrm{Newt}(g)$, thus $\mathrm{Newt}(W_{\Lambda_{d-2}^{abc}}) \subset \mathrm{Newt}(f)$. From Proposition 4.6 of [9] we have that $\mathrm{Newt}(W_{\Lambda_k^{abc}})$ is a k-simplex, which forces $\mathrm{Newt}(g)$ to be a point, i.e. g is a constant polynomial. This is a contradiction, thus $W_{\Lambda_{d-1}^{abc}}$ is irreducible. □

Corollary 2.16. *The Legendrian tori, Λ_{n-1}^{abc} and $\Lambda_{n-1}^{a'b'c'}$ in S^{2n-1} corresponding to Markov triples (a, b, c) and (a', b', c') for $(a, b, c) \neq (a', b', c')$ are not Legendrian isotopic.*

Proof. The argument is essentially the same as Vianna's argument that tori corresponding to distinct Markov triples are not Hamiltonian isotopic [45]. Suppose that the Legendrians Λ_n^{abc} and $\Lambda_n^{a'b'c'}$ are Legendrian isotopic. After identification of first homology groups, we obtain an isomorphism of augmentation varieties

$$\mathrm{Aug}(\Lambda_n^{abc}) \to \mathrm{Aug}(\Lambda_n^{a'b'c'}), \quad \mathrm{Aug}_{\circ}(\Lambda_n^{abc}) \to \mathrm{Aug}_{\circ}(\Lambda_n^{a'b'c'}).$$

The corresponding augmentation polynomials are irreducible from Lemma 2.15. Thus, from Theorem 2.8, we have that the Newton polytope of $W_{\Lambda_{n-1}^{abc}}$ and $W_{\Lambda_{n-1}^{a'b'c'}}$ are the same up to change of basis. As explained in Vianna [45], and Chanda-Hirschi-Wang [9], this equality is impossible because, for example, their Newton polygons cannot be related by any $GL(n-1, \mathbb{Z})$-transformation, as their edges have different lattice lengths. $\qquad\square$

Remark 2.17. We could circumvent the need to prove the irreducibility of $W_{\Lambda_{n-1}^{abc}}$ if we used the equality of the augmentation schemes as discussed in Remark 2.10 instead of the equality of augmentation varieties.

2.2. **Geometric augmentation varieties.** The augmentation variety has distinguished subvarieties corresponding to fillings. For a Lagrangian filling L of Λ, there is an inclusion $H_2(Y, \Pi) \to H_2(\overline{X}, \overline{L})$ which induces a map from $\hat{G}(\Lambda)$ to $\hat{G}(L)$. So we can change coefficients from $CE(\Lambda, \hat{G}(\Lambda))$ to $CE(\Lambda, \hat{G}(L))$. In the following, we write $CE(\Lambda, \hat{G}(\Lambda))$ as $CE(\Lambda)$ to simplify notation.

Definition 2.18. (Geometric augmentation variety) For any tamed filling L equipped with a bounding cochain $b \in MC(L)$ let

$$\varphi(L, b) : CE(\Lambda) \to \hat{G}(L)$$

denote the corresponding augmentation. Let

$$\mathcal{I}_{(L,b)}(\Lambda) = \ker(\varphi(L, b))$$

the corresponding augmentation ideal. Let

$$\mathrm{Aug}_{L,b}(\Lambda) \subset \mathrm{Aug}(\Lambda)$$

denote the variety defined by $\mathcal{I}_{(L,b)}(\Lambda)$. Denote

$$\mathrm{Aug}_{\mathrm{geom}}(\Lambda) = \bigcup_{(L,b)} \mathrm{Aug}_{(L,b)}(\Lambda)$$

the union over tamed fillings with bounding chains (L, b).

We compute the augmentation variety corresponding to the Harvey-Lawson filling. Recall from I-(2) that the Harvey-Lawson filling is

$$L_\epsilon = \{|z_1|^2 = |z_2|^2 + \epsilon = |z_3|^2 + \epsilon, \quad z_1 z_2 z_3 \in (0, \infty)\}$$

and fills the Legendrian $\Lambda_\epsilon = L_\epsilon \cap S^{2n-1}$. Fillings of the perturbed and unperturbed Legendrians define augmentations of the samme Chekanov-Eliashberg algebra. Indeed, for ϵ sufficiently small, the moduli spaces $\mathcal{M}(\Lambda)$ and $\mathcal{M}(\Lambda_\epsilon)$ are in bijection assuming the same almost complex structure and Morse function is used for both Legendrians. The Chekanov-Eliasberg algebra $CE(\Lambda_\epsilon)$ has completed coefficient ring $\hat{G}(\Lambda_\epsilon)$ which is a completion of $G(\Lambda) \cong G(\Lambda_\epsilon)$. Therefore, we have a chain map

$$CE(\Lambda) \to CE(\Lambda_\epsilon).$$

In particular, any augmentation of $CE(\Lambda_\epsilon)$ defines an augmentation of $CE(\Lambda)$.

Proposition 2.19. *The geometric augmentation variety of Λ_ϵ equipped with the spin structure corresponding to the element $(0,1) \in H_1(L_\epsilon, \mathbb{Z}_2)$ and trivial chain $b = 0$ is*

$$\mathrm{Aug}_{(L_\epsilon, 0)}(\Lambda_\epsilon) = \mathrm{Aug}(\Lambda_\epsilon) = \{1 + y_1 - y_2\}.$$

We first compute the images of the degree one Morse generators.

Lemma 2.20. *The augmentation $\varphi(L_\epsilon, 0)$ for the Harvey-Lawson filling L_ϵ satisfies*

(7) $$\varphi(L_\epsilon, 0)(\mathfrak{c}_1) = 0, \quad \varphi(L_\epsilon, 0)(\mathfrak{c}_2) = \ln(1 + \mu_1).$$

For any of the other generators $\mathfrak{c}_3, \mathfrak{c}_4$ in $\mathcal{I}_\bullet(\Lambda)$, the augmentation $\varphi(L_\epsilon, 0))$ vanishes for reasons of degree.

Proof. First we note that the images of the Morse generators under the augmentation must satisfy a relation. By definition of augmentation,

$$\varphi(L_\epsilon, 0)(\delta^{\mathrm{ab}}(\mathfrak{a})) = \varphi(L_\epsilon, 0)(1 + [\mu_1] \exp(\mathfrak{c}_1) - [\mu_2] \exp(\mathfrak{c}_2)) = 0.$$

Here the sign of the $[\mu_1]$-term is positive, matching the sign of the Maslov index two disk in [11], while the sign of the $[\mu_2]$-term is negative since the spin structure on that generator is negative by the discussion in Section I-4.6, and the change in spin structure reverses the orientation of the moduli space of disks, by the discussion in [24, 8.1.2].

Now we claim that the image of the first generator vanishes. The image of each $\mathfrak{c}_i$ under $\varphi(L_\epsilon, 0)$ is a count of disks in L_ϵ with an incoming trajectory of $f_\bullet$ limiting to $\mathfrak{c}_i$. We may assume that

$$f_\bullet : L_\epsilon \cong S^1 \times \mathbb{R}^2 \to \mathbb{R}$$

is the sum of the standard height function on S^1 and a quadratic function on $\mathbb{R}^2$, and after a generic perturbation the images of the holomorphic disks are disjoint from $S^1 \times \{0\}$. Thus, if $u_v : S_v \to \mathbb{C}^3$ is a disk connecting to a trajectory $u_e : T_e \to L_\epsilon$, the limiting point of the trajectory at infinity is the limit of the flow of the projection of $u_e(T_e \cap S_v)$ under the map $S^1 \times \mathbb{R}^2 - \{0\} \to S^1 \times S^1$ induced by the gradient flow. Thus, $u_e(T_e \cap S_v)$ must map to the inverse image of the unstable manifold of $\mathfrak{c}_i$ under the projection from $S^1 \times \mathbb{R}^2$. We may assume that the unstable manifold for $\mathfrak{c}_i$ is a small translate of $S^1 \times \{1\}$ in the second coordinate. Then there are no possibilities for $u_e(T_e \cap S_v)$ if $i = 1$, corresponding to the self-intersections of the cycle $S^1 \times \{1\}$ and exactly one possibility if $i = 2$, corresponding to the intersections of the cycles $S^1 \times \{1\}$ and $\{1\} \times S^1$. Thus

$$\varphi(L_\epsilon, 0)(\mathfrak{c}_1) = 0.$$

The relation in the first paragraph now determines the value of the augmentation on the second generator. We have from the fact that $\varphi(L_\epsilon, 0)\delta(\mathfrak{a}) = 0$ that

(8) $$\varphi(L_\epsilon, 0)(\mathfrak{c}_2) = \ln(1 + [\mu_1])$$

as claimed. $\square$

We will give an alternative justification of (8) by an explicit count of holomorphic disks in Section 3.

Lemma 2.21. *The projection of $\mathrm{Aug}_{(L_\epsilon,0)}(\Lambda)$ onto the first factor in $\mathrm{Rep}(\Lambda) \cong (\mathbb{C}^\times)^2$ is surjective.*

Proof. We claim that there is no polynomial in the first coordinate that vanishes on the augmentation variety. Since

$$\varphi(L_\epsilon,0)([\mu_1]) = [\mu_1], \quad \varphi(L_\epsilon,0)(\mathfrak{c}_1) = 0,$$

any polynomial in y_i maps to the corresponding polynomial in $[\mu_i]$ under $\varphi(L_\epsilon,0)$. So there is no polynomial in y_i that vanishes under $\varphi(L_\epsilon,0)$. Thus, the projection of the augmentation variety on the first factor of $(\mathbb{C}^\times)^2$ is surjective. $\square$

Proof of Proposition 2.19. By Lemma 2.21, $\mathrm{Aug}(\Lambda)$ is a curve and so the containment in the zero locus of the potential is an equality:

$$\mathrm{Aug}_{(L,b)}(\Lambda) = \{1 + y_1 - y_2 = 0\}.$$

$\square$

Lemma 2.22. *The geometric augmentation variety of the Clifford Legendrian Λ with the trivial spin structure is empty.*

Proof. Any geometric augmentation $\varphi(L,b) : CE(\Lambda) \to \hat{G}(L)$ must satisfy

(9) $$\varphi(L,b)(W_\Lambda([\mu_1]\exp(\mathfrak{c}_1),\ldots,[\mu_k](\exp(\mathfrak{c}_k)))) = 0$$

using II-(25). On the other hand, the minimal area terms in $(\mathrm{Rep}(p))_*W$ with respect to the filtration in II-(29) all have positive coefficient one. Indeed the areas of the corresponding disks in Y are equal. Thus, contributions to

$$\varphi(\mathfrak{c}_i), i = 1,\ldots,n-1$$

must involve either positive area disks in L or insertions from the bounding chain b. The latter contributions lie in a non-trivial subspace in the energy filtration of $CE(\Lambda)$ by assumption on the positive q-valuation of the bounding chain b. This contradicts (9). $\square$

2.3. **Toric examples.** In this section, we make various computations and in particular, prove our main result Theorem 1.1.

Example 2.23. We first continue the study of the Harvey-Lawson Legendrian in Example II-4.17. The perturbed Clifford Legendrian Λ_ϵ is Legendrian for a contact structure α_ϵ that is a perturbation of the standard contact structure, and fibers over a non-monotone Lagrangian torus orbit in $\mathbb{C}P^{n-1}$. By Gray stability, (Z,α) is contactomorphic to (Z,α_ϵ), and so the Legendrian dga's $CF(\Lambda_0)$ and $CF(\Lambda_\epsilon)$ may be taken to be equal. The Harvey-Lawson filling $L_{(1)}$ defines an augmentation

$$\varphi : CE(\Lambda_\epsilon) \to G(\varphi) := G(L_{(1)})$$

given by (with signs depending on a suitable choice of spin structure)

$$\varphi([\mu_i]) = [\mu_i], \ i < n-1 \quad \varphi(\mu_{n-1}) = 1$$
$$\varphi(\mathfrak{c}_i) = 0, \ i < n-1 \quad \varphi(\mathfrak{c}_{n-1}) = \ln(1 + \pm[\mu_1] + \ldots + \pm[\mu_{n-2}]).$$

We obtain an augmentation

$$\varphi : CE(\Lambda_{\text{Cliff}}) \to G(\varphi)$$

with the property that for any polynomial $f(y_1, \ldots, y_{n-2})$, the image is

$$\varphi(f(y_1, \ldots, y_{n-2})) = f([\mu_1], \ldots, [\mu_{n-2}]).$$

It follows that

$$(10) \qquad \operatorname{Aug}(\Lambda) \subset \left\{ 1 - \sum_{i=1}^{n-1} \pm y_i = 0 \right\}$$

is a hypersurface. Since this hypersurface is irreducible, the inclusion in (10) is an equality. This ends the Example.

We describe the augmentation variety of a toric Legendrian, as claimed in Theorem 1.1 from the introduction.

Theorem 2.24. *Let Z be a negative circle bundle over a Fano toric variety Y with minimal Chern number at least two and so that the variety $\operatorname{Rep}(p)(W)^{-1}(0))$ is reduced. If the spin structure extends over the filling constructed in Theorem II-4.20, then the augmentation variety is equal to the geometric augmentation variety.*

Proof. Since the augmentation variety is an irreducible hypersurface, it suffices to show that the locus defined by the given filling is also a hypersurface. Let $\varphi(L, 0)$ be the augmentation for the filling constructed in Theorem II-4.20. We have

$$\varphi(y_i) = [\mu_i], \quad i = 1, \ldots, n - 2.$$

Indeed, as in the proof of Theorem II-4.20, the holomorphic disks have boundary contained in the locus

$$\{z_{n-2} = z_{n-1} = 0\} \cap L \cong T^{n-2}.$$

For the standard metric and Morse function, the set of points connecting to $\mathfrak{c}_k$ by an infinite trajectory $u_e : T_e \to L$ is the dimension two cycle obtained as the closure

$$\overline{A}_k \subset L, \quad A_k = \{(1, \ldots, e^{i\theta}, 1, \ldots, 1)\} \times \mathbb{R}_{>0}$$

of the k-th factor in

$$T^{n-1} \times \mathbb{R}_{>0} \subset T^{n-2} \times \mathbb{R}^2$$

induced by the map $e^{i\theta}, r \mapsto re^{i\theta}$ on the last two factors and the identity map on T^{n-2}. For generic perturbations of these cycles, the holomorphic disks are disjoint from $\overline{A}_k$ unless $k = n - 2$.

We claim that the projection of the augmentation variety onto the torus corresponding to the first $n - 2$ coordinates is surjective. Since

$$\varphi(y_i) = \mu_k, k = 1, \ldots, n - 2$$

each polynomial in $y_1, \ldots, y_{n-2}$ is mapped by φ to the corresponding polynomial in $[\mu_1], \ldots, [\mu_{n-1}]$ and so there are no polynomials in $y_1, \ldots, y_{n-2}$ vanishing on $\operatorname{Aug}(\Lambda)$. Since the polynomial in (4) vanishes on $\operatorname{Aug}(\Lambda)$ and is reduced, equality holds. $\square$

Theorem 2.25. *If $\Lambda = \Lambda(1, b, c)$ is a Legendrian lift of a Vianna torus corresponding to a Markov triple of the form $(1, b, c)$ and the spin structure on Λ is non-trivial then*

$$\text{(11)} \qquad \text{Aug}_{\text{geom}}(\Lambda) = \text{Aug}(\Lambda) = \text{Rep}(p)(W)^{-1}(0).$$

Proof. As in the toric case, we will show that the filling constructed above in Lemma II-4.18 has an augmentation variety that is an irreducible hypersurface and so the containment of Theorem 2.8 must be an equality. let $L \subset \mathbb{C}^3 - \{0\}$ denote the filling constructed in Lemma II-4.18 and $\varphi(L, 0)$ the corresponding augmentation. From the description of disks in the proof of Lemma II-4.18, the augmentation $\varphi(L, 0)$ vanishes on the one-cycles $\mathfrak{c}_1, \ldots, \mathfrak{c}_{n-1}$, and so

$$\varphi(L, 0): \ y_i := [\mu_i] \exp(\mathfrak{c}_i) \mapsto [\mu_i], \quad i = 1, \ldots, n - 1.$$

It follows that the projection of $\text{Aug}(\Lambda)$ onto $(\mathbb{C}^\times)^{n-2}$ is a hypersurface contained in $\text{Rep}(p)(W)^{-1}(0))$. Since $\text{Rep}(p)(W)^{-1}(0))$ is isomorphic via mutation to the zero level set of the augmentation variety of the Clifford Legendrian (see Pascaleff-Tonkonog [42]) the augmentation variety $\text{Aug}(\Lambda)$ is irreducible. Hence the equality (11) holds. $\square$

2.4. **Exact augmentations.** In this section we investigate augmentations associated to exact fillings and several examples.

Proposition 2.26. *Any exact filling L with trivial bounding chain, $b = 0$, defines an augmentation $\varphi(L, b)$ that maps all Morse-degree-one generators $\mathfrak{c} \in \mathcal{I}_\bullet(\Lambda)$ to zero.*

Proof. Since L has no holomorphic disks, $b = 0$ is a Maurer-Cartan solution. Let $\mathfrak{c} \in \mathcal{I}_\bullet(\Lambda)$ have Morse degree one. The image $\varphi(L, b)(\mathfrak{c})$ is a count of holomorphic disks in L with no punctures. Any such disk is necessarily constant, and the stability condition of at least three special points on any such disk implies that such configurations do not exist. Hence $\varphi(L, b)(\mathfrak{c}) = 0$. $\square$

Definition 2.27. An augmentation $\varphi : CE(\Lambda) \to G(\varphi)$ is *exact* if $\varphi(\mathfrak{c}) = 0$ for all classical generators $\mathfrak{c} \in \mathcal{I}_\bullet(\Gamma)$.

Example 2.28. We consider the Legendrian lift of the product of equators in the product of two-spheres. That is, let $\Lambda \cong T^2$ denote the Legendrian in the unit hyperplane bundle $Z = S^3 \times_{S^1} S^3 \cong S^3 \times S^2$ given by lifting the product $\Pi = S^1 \times S^1$ in $Y = \mathbb{C}P^1 \times \mathbb{C}P^1$. The disks lifting the Maslov index two disks in Y give rise to the leading order terms in the differential as in II-(22)

$$\delta^{\text{ab},0}(\mathfrak{a}) = 1 - y_1 - y_2 + y_1 y_2.$$

In this example, one sees that the augmentation variety has two components defined by $y_1 - 1$ and $y_2 - 1$ respectively. There is an obvious filling which produces these components: Write Z as the unit sphere bundle in T^*S^3, and let Λ be the unit conormal bundle of the unknot $S^1 \subset S^3$. Then the unit disk provides an exact filling. Taking the two different ways of writing Z as the unit sphere bundle gives the two components of the augmentation variety, so that $\text{Aug}_{\text{geom}}(\Lambda) = \text{Aug}(\Lambda)$. On

the other hand, one may also take Z to be the unit anti-canonical bundle, in which case the projection from Λ to Π is an isomorphism obtains using the capping path corresponding to the monomial y_1 for non-trivial spin structures on both factors is

$$\delta^{\mathrm{ab},0}(\mathfrak{a}) = 1 - y_1^2 + y_1 y_2 - y_1/y_2.$$

The augmentation variety in this case is

$$\mathrm{Aug}(\Lambda) = \{1 - y_1^2 + y_1 y_2 - y_1/y_2 = 0\}$$

by Theorem 1.1.

Claim: There is no augmentation defined over a polynomial ring of the form

$$(12) \qquad\qquad \hat{G}(L) = \hat{G}(\Lambda)/(a_1[\mu_1] - a_2[\mu_2])$$

for some constants $a_1, a_2 \in \mathbb{Z}$ which maps both $\mathfrak{c}_1$ and $\mathfrak{c}_2$ to zero. Indeed, any such augmentation would map

$$W([\mu_1], [\mu_2]) := 1 - [\mu_1]^2 + [\mu_1][\mu_2] - [\mu_1]/[\mu_2]$$

to zero. This would imply that $W([\mu_1], [\mu_2])$ has a linear factor, which contradicts irreducibility.

To show that Λ has no exact filling, suppose that L is a filling of Λ. The inclusion $H_1(\Lambda) \to H_1(L)$ has one-dimensional kernel, since the image of $H^1(L) \to H^1(\Lambda)$ is Lagrangian (See [47, Lemma 3.2.4].) Hence $\hat{G}(L)$ is of the form (12), and the claim above shows that such an augmentation does not exist.

Example 2.29. We continue Example II-4.3 regarding the exact filling $L_{(2)}$ of the disconnected Hopf Legendrian $\Lambda_{\mathrm{Hopf}} \subset S^{2n-1}$. Depending on whether the path defining the filling $L_{(2)}$ passes below or above the critical value in the Lefschetz fibration, the augmentation $\varphi(\mathfrak{c}_{12})$ resp. $\varphi(\mathfrak{c}_{21})$ has either 1 or n terms.

2.5. **Fillings and partitions.** We compute some examples of augmentation varieties in the case that the Legendrian is disconnected. A conjecture of Aganagic-Ekholm-Ng-Vafa [2] describes the augmentation varieties of Legendrians associated to links in terms of certain partitions; here we obtain a simplified version of this conjecture, albeit in arbitrary dimension, for unions of translations of Legendrian lifts of Lagrangian tori in toric varieties.

We begin with the example of the *Hopf Legendrian* from [5, Equation (2)] which is a disjoint of union of two copies of the Clifford Legendrian, related by a translation by a small angle using the circle action.

Lemma 2.30. *Suppose Λ is the Hopf Legendrian in II-(2). If the spin structures on the two components are identical (under the isomorphism provided by translation) then the augmentation variety $\mathrm{Aug}(\Lambda)$ is a union of two irreducible components*

$$(13) \quad \mathrm{Aug}(\Lambda) = \{\pm 1 \pm y_{1,b} \pm \ldots \pm y_{n,b} = 0, \quad b = 1, 2\}$$
$$\cup \{y_{i,1} = y_{i,2}, \quad i = 1, \ldots, n\}.$$

If the spin structures on both components are identical and non-trivial, then the geometric augmentation variety is equal to the algebraic augmentation variety:

$$\mathrm{Aug}_{\mathrm{geom}}(\Lambda) = \mathrm{Aug}(\Lambda).$$

Otherwise, if the spin structure is trivial, then the geometric augmentation variety is empty.

The lch spectrum $\mathrm{Aug}_\circ(\Lambda)$ from (3) is

$$(14) \quad \mathrm{Aug}_\circ(\Lambda) = \{1 - [\mu_{1,1}] - [\mu_{2,1}] + \mathfrak{a}_{12}\mathfrak{a}_{21} = 1 - [\mu_{1,2}] - [\mu_{2,2}] + \mathfrak{a}_{21}\mathfrak{a}_{12} =$$

$$\mathfrak{a}_{12}([\mu_{1,2}] - [\mu_{1,1}])\mathfrak{a}_{21}([\mu_{2,2}] - [\mu_{2,1}]) = 0\} \subset \mathbb{C}^6.$$

Proof. We break down the possibilities for augmentations based on their values on the Reeb chords connecting the two components of the Legendrian. With notation from Example II-3.27, suppose that an augmentation φ satisfies either

$$(15) \qquad\qquad \varphi(\mathfrak{a}_{12}) = 0 \quad \text{or} \quad \varphi(\mathfrak{a}_{21}) = 0.$$

Then

$$\varphi^{\mathrm{ab}}(1 \pm y_{1,b} \pm \ldots \pm y_{n,b}) = 0, \quad b = 1, 2.$$

In the non-vanishing case that (15) does not hold,

$$\varphi(\mathfrak{c}_{12}) = \varphi(\mathfrak{c}_{21}) = 0$$

using II-(26). Hence

$$\varphi^{\mathrm{ab}}(1 - y_{1,k}y_{2,k}^{-1}) = 0, \quad k = 1, \ldots, n.$$

Thus, we see that the augmentation variety $\mathrm{Aug}(\Lambda)$ is contained in the union of two irreducible components in the statement of the Lemma. On the other hand, we may define an augmentation over $G(\varphi) = \hat{G}(\Lambda)$ by setting

$$\varphi(\mathfrak{c}_{i,1}) = \varphi(\mathfrak{c}_{i,2}) = 0, \quad i = 1, \ldots, n-1$$

and choose

$$\varphi(\mathfrak{a}_{12}), \varphi(\mathfrak{a}_{21})$$

so that

$$1 \pm [\mu_{1,k}]e^{\varphi(\mathfrak{c}_{1,k})} \pm \ldots \pm [\mu_{n-1,k}]e^{\varphi(\mathfrak{c}_{n-1,k})} + \varphi(\mathfrak{a}_{12})\varphi(\mathfrak{a}_{21}) = 0.$$

Then φ defines an augmentation.

Suppose the spin structures on the two connected components are non-trivial and isomorphic; we will show that the components of the augmentation variety are geometrically realizable. The first component in (13) is geometrically realized by the union of the two Harvey-Lawson fillings as in the proof of Theorem 2.24. We claim that the second component is that of the filling constructed in II-(5). Since the filling is exact, the augmentation vanishes on the generators corresponding to the critical points of $f_\bullet$:

$$\varphi(\mathfrak{c}_i^a) = 0, \quad a \in \{1, 2\}.$$

On the other hand, the map $H_1(\Lambda) \to H_1(L)$ identifies μ_i^1 with μ_i^2 for each i and so

$$\varphi([\mu_i^1]) = \varphi([\mu_i^1]\exp(\mathfrak{c}_{i,1})) = \varphi([\mu_i^2]) = \varphi([\mu_i^2]\exp(\mathfrak{c}_{i,2})).$$

Therefore the augmentation component corresponding to this filling lies in the diagonal component of the augmentation variety is

$$\mathrm{Aug}(\Lambda)_L = \{(y_1, y_2) \in \mathcal{R}(\Lambda_1) \times \mathcal{R}(\Lambda_2), \quad y_1 = y_2\}.$$

The lch spectrum is defined by the equations corresponding to the differentials of the degree one generators. These generators are $\mathfrak{a}, \mathfrak{c}_{1,1}, \mathfrak{c}_{1,2}, \mathfrak{c}_{2,1}, \mathfrak{c}_{2,2}$ as in Lemma II-26. It follows that the lch spectrum is as stated. $\square$

The Lemma above gives an example of a Legendrian where the augmentation variety and the lch spectrum disagree.

We compute the geometric augmentation variety of a Legendrian with three components and show that the augmentation variety has four irreducible components. Consider the Legendrian

$$\Lambda = \Lambda_0 \cup \beta \Lambda_0 \cup \beta^2 \Lambda_0 \subset S^{2n-1}$$

where

$$\beta = \exp(i\theta)$$

is a root of unity with $\theta > 0$ small. For any partition $\{\{i, j\}, \{k\}\}$ of $\{1, 2, 3\}$ define Then

$$\mathrm{Aug}(\Lambda)_{\{\{i,j\},\{k\}\}} = \{y_{i,b} = y_{j,b}, \quad \forall b, \pm 1 \pm y_{k,1} \pm y_{k,2} = 0\}$$

with signs depending on the choice of relative spin structure. On the other hand, define

$$\mathrm{Aug}(\Lambda)_{\{\{1\},\{2\},\{3\}\}} = \{\pm 1 \pm y_{k,1} \pm y_{k,2} = 0, k \in \{1, 2, 3\}.\}$$

Theorem 2.31. *For any collection of spin structures so that the spin structures on each component are isomorphic via translation, the augmentation variety* $\mathrm{Aug}(\Lambda)$ *is the union*

$$(16) \quad \mathrm{Aug}(\Lambda) = \mathrm{Aug}(\Lambda)_{\{\{1\},\{2\},\{3\}\}} \cup \mathrm{Aug}(\Lambda)_{\{\{1,2\},\{3\}\}}$$
$$\cup \mathrm{Aug}(\Lambda)_{\{\{1\},\{2,3\}\}} \cup \mathrm{Aug}(\Lambda)_{\{\{1,3\},\{2\}\}}.$$

If the spin structure is non-trivial, then the augmentation variety $\mathrm{Aug}(\Lambda)$ *is equal to the geometric augmentation variety* $\mathrm{Aug}_{\mathrm{geom}}(\Lambda)$*, and each irreducible component corresponds to a filling.*

Thus, in total there are four components of the augmentation variety, corresponding to partitions of $\{1, 2, 3\}$ into subsets of size at most 2.

Proof. We prove the Lemma for the case $n = 3$; the general case is similar. Let

$$\Lambda_j = \exp(ji\theta)\Lambda_{\mathrm{Cliff}}$$

be the j-th sheet of Λ and $\mu_{b,j}, b = 1, 2$ the standard basis for $H_1(\Lambda_j)$ and

$$y_{j,b} = [\mu_{j,b}] \exp(\mathfrak{c}_{i,b}) \in CE(\Lambda), \quad b = 1, 2, \ j = 1, 2, 3.$$

Let $\mathfrak{a}_{jj}$ denote the Reeb chords of minimal length connection each sheet to itself, of real degree one, over the degree zero critical point in $\mathcal{R}(\Lambda)$ and let $\mathfrak{a}_{jk}$ be the Reeb chords for $j < k$ connecting the j-th sheet to the k-th sheet. The real degree is

$$\deg_{\mathbb{R}}(\mathfrak{a}_{jk}) = \frac{3}{\pi}(k - j)\frac{2\pi}{9} - 1 = \frac{2}{3}(k - j) - 1.$$

In particular,

$$\deg_{\mathbb{R}}(\mathfrak{a}_{12}) = \deg_{\mathbb{R}}(\mathfrak{a}_{23}) = \deg_{\mathbb{R}}(\mathfrak{a}_{31}) = -\frac{1}{3}$$

while

$$\deg_{\mathbb{R}}(\mathfrak{a}_{13}) = \deg_{\mathbb{R}}(\mathfrak{a}_{21}) = \deg_{\mathbb{R}}(\mathfrak{a}_{32}) = \frac{1}{3}.$$

The $\mathbb{Z}_2$ grading of all these elements would be 1 mod 2 so that

$$\deg_{\mathbb{Z}_2}(\mathfrak{a}_{ik}) = \deg_{\mathbb{Z}_2}(\mathfrak{a}_{ik}) = 1, \quad \deg_{\mathbb{Z}_2}(\mathfrak{a}_{ij}) = \deg_{\mathbb{Z}_2}(\mathfrak{a}_{jk}) = 1.$$

We have

$$\delta(\mathfrak{a}_{ii}) = \pm 1 \pm y_{i,1} \pm y_{i,2} + \sum_{k \neq i} \mathfrak{a}_{ik} \otimes \mathfrak{a}_{ki}.$$

On the other hand, for the real-degree $\frac{1}{3}$ generators we have

$$\delta(\mathfrak{a}_{ik}) = \mathfrak{a}_{ij} \otimes \mathfrak{a}_{jk}, \quad \forall (i,j,k) \in \{(1,2,3),(2,3,1),(3,1,2)\}$$

arising from the cover of the constant disk in $\mathbb{C}P^2$, with no other outputs possible since any outgoing collection of Reeb chords with smaller total angle change for a punctured disk would force the projection into Y of the disk to be non-constant, in which case the outgoing angle change would be negative. Thus, we have for any augmentation φ

$$\varphi(\mathfrak{a}_{ij}) \otimes \varphi(\mathfrak{a}_{ji}) \neq 0 \quad \implies \quad \varphi(\mathfrak{c}_{ij}) = \varphi(\mathfrak{c}_{ji}) = 0$$

in which case $y_{i,k} = y_{j,k}$ for all k. Also, since φ is an augmentation,

$$\varphi(\delta(\mathfrak{a}_{ik})) = \varphi(\mathfrak{a}_{ij}) \otimes \varphi(\mathfrak{a}_{jk}) = 0$$

so either $\varphi(\mathfrak{a}_{ij})$ or $\varphi(\mathfrak{a}_{jk})$ vanishes. Without loss of generality assume that $\varphi(\mathfrak{a}_{jk})$ vanishes. Then

$$\varphi(\pm 1 \pm y_{i,1} \pm y_{i,2} + \mathfrak{a}_{ij} \otimes \mathfrak{a}_{ji}) = 0.$$

If $\varphi(c_{ij})$ also vanishes then

$$\pm 1 \pm y_{k,1} \pm \ldots \pm y_{k,2} = 0, k \in \{1,2,3\}.$$

If $\varphi(c_{ij})$ is non-zero then the relation

$$y_{b,i} = y_{b,j}, \quad b \in \{1,2\}$$

must hold. Putting everything together, $\mathrm{Aug}(\Lambda)$ is contained in the union of the four irreducible components in Theorem 2.31.

On the other hand, we may construct augmentations as follows. For the partition $\{\{1\},\{2\},\{3\}\}$ we may choose $\varphi(\mathfrak{c}_{j,b})$ so that

$$\varphi(\mathfrak{c}_{j,1}) = 0, \quad \pm\varphi(\mathfrak{c}_{j,2}) = \ln(1 \pm [\mu_{j,1}])$$

for all j, and

$$\varphi(\mathfrak{a}_{12}) = \varphi(\mathfrak{a}_{21}) = 0.$$

For the partition $\{\{1,2\},\{3\}\}$, for any values $\varphi(\mathfrak{c}_{j,1}) = \varphi(\mathfrak{c}_{j,2})$, for $b \in \{1,2\}$, choose $\varphi(\mathfrak{a}_{12}), \varphi(\mathfrak{a}_{21})$ so that

$$1 \pm \varphi(\mathfrak{c}_{1,1}) \pm \varphi(\mathfrak{c}_{1,2}) + \varphi(\mathfrak{a}_{12})\varphi(\mathfrak{a}_{21}) = 0.$$

Choose $\varphi(\mathfrak{c}_{j,3})$ so that

$$1 \pm \varphi(\mathfrak{c}_{3,1}) \pm \varphi(\mathfrak{c}_{3,2}) = 0$$

We claim that φ vanishes on the image of δ. Indeed, we have taken the standard Morse function on Λ, so for each Morse-degree-two generator $\mathfrak{b} \in \mathcal{I}_\circ(\Lambda)$, the image $\delta(\mathfrak{b})$ is the Morse differential, which vanishes, plus terms that have at least one skew-symmetry as in Proposition II-3.19. Thus, $\varphi(\delta(\mathfrak{b})) = 0$. Similarly, if $\mathfrak{a}$ is a minimal length Reeb orbit representing a degree-zero generator in $\mathcal{I}(\bullet)$ then the formula (23) implies that $\varphi(\delta(\mathfrak{a})) = 0$. Thus, φ vanishes on $\mathrm{im}(\delta)$ and defines an augmentation.

For non-trivial spin structures. one has a symplectic filling by taking the union of the filling of II-(5) and a copy of the Harvey-Lawson filling. One also has a filling given by the union of three translates of the Harvey-Lawson filling. The holomorphic disks were computed in Example 2.29. Taking these together one sees that each component in the statement of the Theorem 2.31 actually appears. $\qquad\square$

A similar computation holds for any union of translates of the Clifford Legendrian:

Theorem 2.32. *For a disjoint union $\Lambda \cong T_{\mathrm{Cliff}}^{\sqcup \ell}$ of ℓ copies of the Clifford Legendrian, the augmentation variety $\mathrm{Aug}(\Lambda)$ is the union of irreducible components $\mathrm{Aug}_P(\Lambda)$ indexed by partitions P of $\{1, \ldots, \ell\}$ into subsets of size one or two so that the spin structures on the components Λ_j, Λ_k agree for each pair $\{j, k\} \in P$. Each component $\mathrm{Aug}_P(\Lambda)$ is the product of products of $\mathrm{Rep}(p)(W^{-1}(0))$ for each singleton in the partition P, assuming irreducibility, with varieties of the form*

$$\{y_i^a = y_i^b\}$$

for each pair of integers a, b such that $\{a, b\}$ lies in the partition P. If the spin structures in the singletons in P are non-trivial, then $\mathrm{Aug}_P(\Lambda)$ corresponds to a geometric filling.

We leave the proof to the reader, as it is similar to Theorem 2.31. It would be interesting to know whether the linearized contact homology (viewed as a "sheaf" over the augmentation variety) also has an interpretation in terms of the base data.

2.6. **Ruling out exact fillings.** In this section, we use the moduli spaces of buildings to rule out the existence of exact fillings of the Clifford Legendrian.

Theorem 2.33. *(Dimitroglou-Rizell [14] and Treumann-Zaslow [44] for $n = 3$) For $n > 2$ the Clifford Legendrian $T^{n-1} \subset S^{2n-1}$ has no exact Lagrangian filling.*

Proof for n odd. In odd dimensions, the proof of the Theorem follows from the fact that the number of boundary components of the one-dimensional moduli space is necessarily even, which contradicts the disk count. In more detail, let

$$\iota_* : H_1(\Lambda) \to H_1(L)$$

denote the inclusion map. Suppose that L is an exact relative spin filling. By the results of the previous sections, L defines an augmentation

$$\varphi : CF(\Lambda) \to \hat{G}(L).$$

Necessarily φ vanishes on the image of the differential and so

$$
\begin{aligned}
(17) \qquad 0 = \varphi(\delta^{\mathrm{ab}}(\mathfrak{a})) \quad &= \quad \varphi(\pm 1 \pm y_1 \pm \ldots \pm y_n) \\
(18) \qquad &= \quad \pm 1 \pm \iota_* \mu_1 \pm \ldots \pm \iota_* \mu_n.
\end{aligned}
$$

Here φ is defined over the group ring $G(L)$; integrating the identity (17) over L gives

$$
0 = \pm 1 \pm 1 \pm \ldots \pm 1
$$

with an odd number of terms on the right-hand side. Since zero is even, this is a contradiction.

We generalize the argument to the case that the filling is not necessarily relatively spin by examining the one-dimensional component in the moduli space of buildings directly. The one-dimensional component of the moduli space of buildings $\mathcal{M}(L)_1$ contains $\mathcal{M}(\Lambda)_0$ as a collection of boundary components, where a building $u = (u_0, u_1)$ in L is obtained from a building u_1 in Λ by considering the first level u_0 to be empty. Since the Lagrangian $\Pi \subset Y$ is monotone in this case and L bounds no holomorphic disks, there are no boundary components of $\mathcal{M}(L)_1$ other than those arising from $\mathcal{M}(\Lambda)_0$:

$$
\partial \mathcal{M}(L)_1 = \mathcal{M}(\Lambda)_0.
$$

As a result, the number of rigid buildings in $\mathcal{M}(\Lambda)_0$ asymptotic to a given incoming Reeb orbit $\gamma \in \mathcal{R}(\Lambda)$ must be even. Each rigid building in Λ with Reeb orbit $\mathfrak{a}$ correspond to a Maslov index two disk in $\mathbb{C}P^{n-1}$. The number of such is equal to n and so odd, which is a contradiction. $\qquad \square$

In the case of even dimension we examine the moduli space of buildings more closely. We show relations in the homology arising from the moduli spaces of buildings in both degree one and codegree one. We begin with the degree one relations:

Lemma 2.34. *Let L be a compact, oriented exact filling of a compact Legendrian Λ. After re-ordering, the image of $H_1(\Lambda)$ in $H_1(L)$ is described by the relations*

$$
\iota_* \mu_1 = 0, \quad \iota_* \mu_{2i} = \iota_* \mu_{2i+1}, \quad i = 1, \ldots, \frac{1}{2}(n-1).
$$

Proof. The statement of the Lemma follows from a matching of the ends of the one-dimensional components of the moduli space of buildings bounding the filling. Let $\mathcal{M}_i(\Lambda)_0$ denote the moduli space of rigid buildings in $\mathbb{R} \times Z$ of class $\iota_* \mu_i$. Each building in $\mathcal{M}_i(\Lambda)_0$ represents the boundary of a one-dimensional component $\mathcal{M}(L)_1$ of the moduli space of buildings $u : C \to \mathbb{X}$ bounding the filling L. Since each such component of $\mathcal{M}(L)$ has two ends and any one-manifold is oriented, without loss of generality $\mathcal{M}_{2i}(\Lambda)$ is connected to $\mathcal{M}_{2i+1}(\Lambda)$ by such a component for $i \geq 1$. $\qquad \square$

To obtain relations in higher degree homology groups, we study more general moduli spaces of buildings as follows. We introduce the following moduli space with constraints in a submanifold of Reeb chords.

Definition 2.35. Let $A \subset \mathcal{R}(\Lambda)$ be a closed submanifold of the component $\mathcal{R}(\Lambda)_{min} \cong \Lambda$ of $\mathcal{R}(\Lambda)$ consisting of Reeb chords of angle change $2\pi/n$. Let

$$
\overline{\mathcal{M}}(L, A) = \{u : S \to \mathbb{X}, \mathrm{ev}_e(u) \in A\}
$$

the moduli space of holomorphic maps $u : C \to X$ with domain a disk S with a single strip-like end like end e asymptotic along that end to a Reeb chord in A, lifting the Maslov index two disks bounding Π in (22).

We investigate the boundary of the moduli space of once-punctured disks bounding the filling with evaluation in the given cycle on the Legendrian. The moduli of disks $\overline{\mathcal{M}}(L, A)$ bounding the Lagrangian contains the subset $\mathcal{M}(\Lambda, A)$ of disks in $\mathbb{R} \times A$ bounding $\mathbb{R} \times \Lambda$ with a limiting Reeb chord in A, considered as buildings (u_0, u_1) in X with the first component u_0 empty. Recall that a *pseudocycle-with-boundary* is a submanifold with boundary whose closure is equal to the union of its image and a finite collection of images of manifolds of dimension at least two less; see for example Zinger [48].

Lemma 2.36. *Let $Z = S^{2n-1}$ and $\Lambda = T^{n-1}$ the Clifford Legendrian. For the standard complex structure on $\mathbb{R} \times Z$, the moduli space $\mathcal{M}(\Lambda, A)_{\dim(A)}$ is smooth and compact. For generic perturbations, the component $\overline{\mathcal{M}}(L, A)_{\dim(A)+1}$ of dimension $\dim(A) + 1$ is a pseudocycle-with-boundary whose boundary is diffeomorphic to $\mathcal{M}(\Lambda, A)_{\dim(A)}$.*

Proof. The regularity statement on $\mathcal{M}(\Lambda, A)_{\dim(A)}$ follows from the lifting properties in Section II-2.2 and regularity of disks in $\mathbb{C}P^{n-1}$ bounding the Clifford torus of Maslov index two; these are given by Blaschke products with a single factor and so regular. Since there is a single boundary-puncture going to a Reeb chord γ with the smallest possible a ngle at infinity, any codimension one bubbling in $\mathcal{M}(L, A)_{\dim(A)+1}$ must involve bubbling off a positive-area disk $u_v : S_v \to X$ without ends. Such bubbling is impossible by the exactness assumption. Therefore, the moduli space $\mathcal{M}(\Lambda, A)_{\dim(A)}$ considered as a subset of the moduli space of buildings $\mathcal{M}(L, A)_{\dim(A)+1}$ (with empty first level) is the only boundary stratum. The Cieliebak-Mohnke regularization from [12] equips $\mathcal{M}(L, A)$ with the structure of a pseudocycle; the only possible bubbling (besides a level going off to infinity along the cylindrical end) appears when markings constrained to map to the Donaldson hypersurface come together to form a ghost bubble with more than one marking. As in [12], these configurations are not cut out transversally but represent a tangency with the Donaldson hypersurface. Such tangencies are real codimension at least two and their image is covered by configurations obtained by removing the ghost bubble and enforcing a tangency of the required order at an interior marking. To end the argument, one needs charts for $\overline{\mathcal{M}}(L)$ considered as a manifold with boundary near $\mathcal{M}(\Lambda)$. These are produced by a gluing argument as in [41, Lemma 5.12], in which the gluing parameter (representing the translation which the map to $\mathbb{R} \times Z$ is glued into the neck of X) and the coordinates on $\mathcal{M}(\Lambda)$ locally produce coordinates on $\mathcal{M}(L)$. $\square$

Proposition 2.37. *Let K be a compact oriented manifold with boundary ∂K. The image of $H^\bullet(K)$ in $H^\bullet(\partial K)$ is a maximally isotropic subspace with respect to the Poincaré pairing. In particular, for $n \geq 4$ the image of the restriction map*

$$H^1(K) \oplus H^{n-2}(K) \to H^1(\partial K) \oplus H^{n-2}(\partial K)$$

is of dimension $\dim(H^1(\partial K))$.

Proof. The first claim is [47, Lemma 3.2.4], although it must be more widely known. The second claim follows since the pairing between $H^1(\partial K)$ and $H^{n-2}(\partial K)$ is perfect. Thus, any maximally isotropic subspace of split form is of the form $V \oplus V^{\mathrm{ann}}$ where $V \subset H^1(K)$ and $V^{\mathrm{ann}} \subset H^{n-2}(\partial K)$ is its annihilator. $\square$

We obtain relations in codimension one by considering moduli spaces constrained to pass through codimension one cycles. Consider the Pontrjagin product on $H_\bullet(\Lambda, \mathbb{Z}_2)$ generated by the group multiplication $\Lambda \times \Lambda \to \Lambda$ using the diffeomorphism $\Lambda \cong T^{n-1}$. For each i define a class $\check{\mu}_i$ by taking the Pontrjagin product of the classes μ_j with $j \neq i$;

$$\check{\mu}_i = \mu_1\mu_2 \ldots \mu_{i-1}\mu_{i+1} \ldots \mu_{n-1} \in H_{n-2}(\Lambda, \mathbb{Z}_2).$$

Define subspaces of dimension and codimension one classes corresponding to the relations in Lemma 2.34:

$$\begin{aligned}
I &= \mathrm{span}(\mu_1, \mu_2 - \mu_3, \mu_4 - \mu_5, \ldots, \mu_{n-1} - \mu_n) \subset H_1(\Lambda, \mathbb{Z}_2) \\
\hat{I} &= \mathrm{span}(\check{\mu}_1 - \check{\mu}_2, \check{\mu}_3 - \check{\mu}_4, \ldots, \check{\mu}_{n-1}, \check{\mu}_n) \subset H_{n-2}(\Lambda, \mathbb{Z}_2).
\end{aligned}$$

Lemma 2.38. $\dim(I) + \dim(\hat{I}) > n - 1$.

Proof. There are $n/2$ relations in I and $n/2$ in $\hat{I}$ for a total of n relations. $\square$

Proposition 2.39. *Suppose that* $\Lambda = T^{n-1}$ *is the Clifford Legendrian in* $Z = S^{2n-1}$. *The kernel of the inclusion map* $H_\bullet(\Lambda, \mathbb{Z}_2) \to H_\bullet(L, \mathbb{Z}_2)$ *contains* $I \oplus \hat{I}$.

Proof. The relations in degree one were proved in Lemma 2.34. To prove the relations in codegree one, let $A_i \subset \Lambda$ be a submanifold that is a representative of μ_i. Consider the once-marked moduli space $\mathcal{M}_1(\Lambda, A)_{\dim(A)+1}$ for the codimension two cycle

$$(19) \qquad\qquad A = A_1 \ldots A_{i-1}A_{i+2} \ldots A_{n-1} \subset \Lambda.$$

skipping the i and $i + 1$-st factors. Let $\mathcal{M}_1(L, A, i)_{\dim(A)+2}$ be the component of $\mathcal{M}_1(L, A)_{\dim(A)+2}$ containing the marked i-th disk u_i, obtained by lifting the i-th Maslov two disk in $\mathbb{C}P^{n-1}$, with a single marking on the boundary evaluating to L. The moduli space $\mathcal{M}_1(\Lambda, A, i)_{\dim(A)+1}$ may be viewed as part of the boundary of $\mathcal{M}_1(L, A, i)_{\dim(A)+2}$ in which the disk has gone off to infinity along the neck. That is, any boundary component of $\mathcal{M}(L, A, i)_{\dim(A)+2}$ consists of buildings in X with first level empty and second level mapping to $\mathbb{R} \times Z$; the boundary of $\mathcal{M}_1(L, A, i)_{\dim(A)+2}$ is the union of components of $\mathcal{M}_1(\Lambda, A)_{\dim(A)+1}$ together with configurations where the marking has gone to infinity onto a trivial strip, diffeomorphic to a component of $\mathcal{M}(L, A)_{\dim(A)+1}$

An addition of capping paths for each element in the moduli space gives a pseudocycle in the Lagrangian filling. Define

$$\widehat{\mathcal{M}}_1(\Lambda, A, i)_{\dim(A)+1} = \mathcal{M}_1(\Lambda, A, i)_{\dim(A)+1} \cup \left(\mathcal{M}(\Lambda, A, i)_{\dim(A)} \times [0, 1]\right).$$

A map

$$\phi_{\Lambda,i} : \widehat{\mathcal{M}}_1(\Lambda, A, i)_{\dim(A)+1} \to \Lambda$$

is defined by evaluation of the map on the first component

$$\mathcal{M}_1(\Lambda, A, i)_{\dim(A)+1} \to \Lambda, \quad (u, z) \mapsto u(z)$$

and evaluation of the capping path $(\mathrm{ev}_e(u))(0)\hat{\gamma}$ for the limit $\mathrm{ev}_e(u)$ on the second:

$$\mathcal{M}(\Lambda, A, i)_{\dim(A)} \times [0, 1] \to \Lambda, \quad (u, t) \mapsto (\mathrm{ev}_e(u))(0)\hat{\gamma}(t).$$

Here by translation, a capping path $\hat{\gamma}$ for one element of $\mathcal{M}(\Lambda, A, i)$ limiting to a Reeb chord $\mathrm{ev}_e(u)$ starting at the identity induces capping paths

$$(\mathrm{ev}_e(u))(0)\hat{\gamma} : [0, 1] \to \Lambda$$

for all elements in the same component by translation by $\mathrm{ev}_e(u)(0) \in \Lambda \cong T^{n-1}$. Similarly, via evaluation the moduli space $\mathcal{M}_1(L, A, i)_{\dim(A)+2}$ gives rise to a pseudocycle in L

$$\phi_{L,i} : \widehat{\mathcal{M}}_1(L, A, i)_{\dim(A)+2} \to L.$$

bounding the components of $\widehat{\mathcal{M}}_1(\Lambda, A, i)$. Note that $\mathcal{M}_1(L, A, i)$ may also contain possibly other lifts of Blaschke products u_k for k in some subset I_i of $\{1, \ldots, n\}$, as shown in Figure 3. Consider the fibration
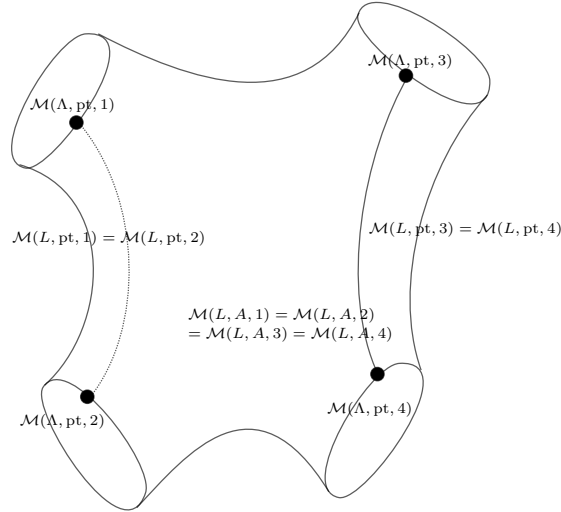

FIGURE 3. Homological relations between components of moduli spaces

$$\mathcal{M}_1(\Lambda, A, k)_{\dim(A)+1} \to \mathcal{M}(\Lambda, A, k)_{\dim(A)}.$$

Each fiber evaluates to the translation of the boundary ∂u_k by an element of A, the moduli space being invariant under multiplication by A. It follows that the homology class of the image in $\Lambda = \partial L$ is

$$[\phi_{\Lambda,k}(\widehat{\mathcal{M}}_1(\Lambda, A, k)] = \mu_k[A] \in H_{n-2}(L, \mathbb{Z}_2).$$

We obtain from the pseudocycle $\mathcal{M}_1(L, A, i)_{\dim(A)+2}$ the relation

$$
\begin{aligned}
0 &= [\partial \widehat{\mathcal{M}}_1(L, A, i)_{\dim(A)+2}] \\
&= \sum_{k \in I_i} \mu_k[A] \in H_{n-2}(L, \mathbb{Z}_2).
\end{aligned}
$$

By definition of A in (19),

$$
\mu_k[A] = 0 \quad \text{unless} \quad k \in \{i, i+1\}.
$$

We obtain the relation

$$
\check{\mu}_i + \check{\mu}_{i+1} = \mu_i[A] + \mu_{i+1}[A] = 0
$$

if both $i, i+1$ lie in I_i and

$$
\check{\mu}_i = \mu_i[A] = 0
$$

otherwise, corresponding to the case that the i-th disk pairs with the disk used to construct the capping path. The coefficient ring is $\mathbb{Z}_2$ and so we obtain the desired relations. □

Proof of Theorem 2.33. It remains to show Theorem 2.33 in the case $n = 2k$ is even. The dimension of the subspaces I and $\hat{I}$ are both half of $\dim(\Lambda) + 1$, and so larger than $\frac{1}{2} \dim(H_1(\Lambda, \mathbb{Z}_2))$. On the other hand, the intersection pairing

$$
H_1(\Lambda, \mathbb{Z}_2) \times H_{n-2}(\Lambda, \mathbb{Z}_2) \to \mathbb{Z}_2
$$

is trivial on $\ker(\iota_*)$. Indeed, suppose that $A \subset \Lambda, B \subset \Lambda$ are pseudocycles that extend to pseudocycles with boundary $\hat{A}, \hat{B}$ in L bounding A, B. The intersection $\hat{A} \cap \hat{B}$ is, after generic perturbation, a one-dimensional submanifold with boundary $A \cap B$, necessarily representing the zero homology class in $H_0(\Lambda, \mathbb{Z}_2)$. Since the pairing $H_1(\Lambda, \mathbb{Z}_2) \times H_{n-2}(\Lambda, \mathbb{Z}_2) \to \mathbb{Z}_2$ is non-degenerate, the dimension of any isotropic subspace is at most dimension $(n-1)$. This contradicts the estimate $\dim(I \oplus \hat{I}) > n - 1$ from Lemma 2.38. □

3. The multiple cover formula

In this section we justify the formula for the augmentation associated to the Harvey-Lawson filling and other similar fillings by localization techniques. The localization technique is not used in the proofs of the earlier results, but serves as a sanity check to explain why the formula for the augmentation holds by an explicit disk count (whose signs appeal to the gluing results already used earlier.) The computations in Aganagic-Ekholm-Ng-Vafa [2] and Ekholm-Ng [21] in the case of the Harvey-Lawson filling rely on the Ooguri-Vafa multiple cover formula [36], similar to Pandharipande-Solomon-Walcher [40]. The localization computation here is similar to that in Katz-Liu [29].

3.1. **Localization for Euler numbers.** The contribution from disks in the filling is given by a dilogarithm, up to a sign. The localization computation here is essentially the same as in Lin [31]. In general, localization computations on a manifold with boundary, such as the manifolds of disks considered here, only make sense under some additional assumption on the perturbation data on the boundary. We have in mind the following situation, similar to that in Metzler [32].

Definition 3.1. Let M be a manifold with boundary. Let $E \to M$ be a vector bundle with $\mathrm{rank}(E) = \dim(M)$, and $T \cong S^1$ a circle acting on M that is free on ∂M with a lifting of the action to E. The Euler number of E is well-defined as follows: Let

$$\overline{M} = M \cup (\partial M/T), \quad \overline{E} = E \cup (\partial E)/T$$

be the *cut spaces* obtained by collapsing the boundary using the circle action, as in Metzler [32]. Let '

$$s : M \to E$$

be a section whose restriction to ∂M is T-invariant. Then s defines a section $\tilde{s} := \partial s/T$ of $\partial E/T$. Conversely, given any such section $\tilde{s}$ there exist a section s inducing $\tilde{s}$, by a patching argument. Since $\partial E/T$ has rank

$$\mathrm{rank}(\partial E/T) = \dim(\partial M/T) + 1,$$

for generic choices of section s there are no zeroes $s^{-1}(0)$ on the boundary ∂M. Define

$$\mathrm{Eul}(E) := \mathrm{Eul}(\overline{E}) := \sum_{x \in s^{-1}(0)} \pm 1 \in \mathbb{Z}$$

with the sign ± 1 arising from a comparison of orientations in the map

$$\mathrm{d}s_x : T_x X \to T E_x/T_x X.$$

In this context we have an analog of the usual Lefschetz fixed point formula, found in for example Bott [6]: The action of T on E induces an action of T on $\overline{E}$ and the usual Lefschetz fixed point formula holds: For each fixed point $p \in M^T$ let

$$\mu_1(p), \ldots, \mu_n(p) \in \mathbb{Z} \quad \lambda(p), \ldots, \lambda_n(p) \in \mathbb{Z}$$

denote the weights of T on E_p resp. $T_p M$. The localization formula for Euler numbers reads:

Proposition 3.2. *The Euler number $\mathrm{Eul}(E)$ is equal to the sum over fixed points of the ratio of products of weights:*

$$\mathrm{Eul}(E) = \sum_{p \in M^T} \frac{\mu_1(p) \ldots \mu_n(p)}{\lambda(p), \ldots \lambda_n(p)}.$$

Proof. This special case of the Bott residue formula [6], which is also a version of the Lefschetz fixed point formula, may be proved as follows. Let

$$\{U_p \subset M, \quad p \in M^T\}$$

be a collection of T-invariant open balls around the fixed points. Since G acts freely on $M - \cup_p U_p$, there exists a section s that has zeroes only in $\cup_p U_p$. Suppose that

the quotients μ_j/λ_j are non-negative integers; in this case we may assume that $s|U_p$ is given in local coordinates as

$$(s|U_p)(z_0, \ldots, z_{d-1}) = (z_1^{\mu_1/\lambda_1}, \ldots, z_{d-1}^{\mu_{d-1}/\lambda_{d-1}}).$$

More generally, if μ_j/λ_j is a negative integer then we obtain an invariant section by replacing $z_j^{\mu_j/\lambda_j}$ with $\overline{z}_j^{-\mu_j/\lambda_j}$. By construction

$$
\begin{aligned}
(gs)(z_0, \ldots, z_{d-1}) &= g(s(g^{-1}(z_0, \ldots, z_{d-1}))) \\
&= g(s(g^{-\lambda_1} z_0, \ldots, g^{-\lambda_{d-1}} z_{d-1})) \\
&= (g^{\mu_1}(g^{-\lambda_1} z_1)^{\mu_1/\lambda_1}, \ldots, g^{\mu_{d-1}}(g^{-\lambda_{d-1}} z_{d-1})^{\mu_{d-1}/\lambda_{d-1}}) \\
&= s(z_0, \ldots, z_{d-1}).
\end{aligned}
$$

After a generic perturbation $s|U_p$ has

$$\#(s|U_p)^{-1}(0) = \frac{\mu_1(p) \ldots \mu_n(p)}{\lambda_1(p), \ldots \lambda_n(p)}$$

which proves the localization formula. The same formula holds even in the case that $\mu_j(p)/\lambda_j(p)$ are not integers, by passing to a ramified cover and quotienting by the degree of the cover. $\square$

Example 3.3. Let $E = TM$ be the tangent bundle. Then the Euler number $\mathrm{Eul}(E)$ is the number of fixed points of the action, by taking the section to be the vector field $\xi_M \in \mathrm{Vect}(M)$ generated by a generic Lie algebra vector $\xi \in \mathfrak{t}$ for the action of T on M. For example, $\mathrm{Eul}(TS^2) = 2$ using the standard action of S^1 by rotation.

Remark 3.4. If M is a compact orbifold (Deligne-Mumford stack over the category of smooth manifolds) with a torus action and $E \to M$ an orbi-vector bundle with a torus action then

$$\mathrm{Eul}(E) = \sum_{p \in M^T} (\# \mathrm{Aut}(p))^{-1} \frac{\mu_1(p) \ldots \mu_n(p)}{\lambda(p), \ldots \lambda_n(p)}$$

by a similar argument, using the definition of Euler number of an orbi-bundle as a *weighted* count of the number of zeros of a generic orbi-section. We will apply the result only in the case of an orbifold that is a quotient of a manifold by a finite group action, in which case the formula follows from the formula for the covering manifold.

3.2. **Localization for disk counts.** We apply these localization considerations to moduli spaces of holomorphic disks as follows. Invariant almost complex structures are generally not sufficiently generic for the moduli spaces to be cut out transversally, but in good situations the moduli spaces are smooth and one may identify the moduli space for a generic almost complex structure with the zero set of a section of the so-called *obstruction bundle*. In the present situation, things are somewhat more complicated as the moduli space for the invariant almost complex structure does not have constant rank obstruction bundle, but rather only a certain subset as we now explain. Let $\mathcal{M}_\Gamma(L)$ be the moduli space of holomorphic treed disks of type Γ bounding a Lagrangian L.

Definition 3.5. A subset of U of $\mathcal{M}_{\Gamma}(L)$ has *constant rank obstruction* if

$$\dim(\mathcal{E}_u), \mathcal{E}_u := \operatorname{coker}(D_u)$$

is independent of $u \in U$.

For example, if $U = \mathcal{M}_{\Gamma}(L)$ then the constant rank assumption implies that the spaces $\mathcal{E}_u$ fit into a smooth vector bundle $\mathcal{E}_{\Gamma}(L)$ over $\mathcal{M}_{\Gamma}(L)$.

Lemma 3.6. *Suppose that the moduli space* $\overline{\mathcal{M}}_{\Gamma}(L) = \bigcup_{\Gamma} \mathcal{M}_{\Gamma}(L)$ *has constant rank obstruction. Then the vector bundles* $\mathcal{E}_{\Gamma}(L)$ *fit together to a topological vector bundle* $\overline{\mathcal{E}}_{\Gamma}(L)$.

Proof. The long exact sequence in Wehrheim-Woodward [46, Theorem 2.4.5] shows the cokernels are isomorphic. More precisely, suppose that $u : C \to X$ is a stable disk bounding L of combinatorial type Γ, and $\tilde{u} : \tilde{C} \to X$ is a nearby map of type $\tilde{\Gamma}$, necessarily obtained by gluing. There is an isomorphism

(20) $$\operatorname{coker}(D_u) \to \operatorname{coker}(D_{\tilde{u}}).$$

defined as follows. The proof [46, Theorem 2.4.5] shows that such an isomorphism exists for between the cokernel of $D_{\tilde{u}}$ and an operator $D_{\tilde{u}}^{\mathrm{red}}$ (called the reduced operator corresponding to $\tilde{u}$, constructed on [46, page 21]). On the other hand, the same Theorem shows that $D_{\tilde{u}}^{\mathrm{red}}$ is a perturbation of D_u.

If the cokernel of $D_{\tilde{u}}$ is the same dimension then the natural projection $\operatorname{coker}(D_u) \to \operatorname{coker}(D_{\tilde{u}}^{\mathrm{red}})$ with respect to the L^2 metric is an isomorphism, and provides a local trivialization of the obstruction bundle. □

We make a number of similar definitions for moduli spaces with varying type. It is not reasonable to expect a constant rank condition to hold in case of varying edge lengths. Given a type Γ, there will typically be strata $\Gamma \prec \Gamma'$ corresponding to allowing some subset of the edges to acquire non-zero lengths. The cokernel of $u \in \mathcal{M}_{\Gamma}(L)$ will typically be larger dimension than that of $u \in \mathcal{M}_{\Gamma'}(L)$ because of the additional constraints. On the other hand, one may view $\overline{\mathcal{M}}_{\Gamma}(L)$ as a smooth manifold with corners at u, and then there is a linearized operator $\tilde{D}'_u$ which allows deformation of the edge lengths. Naturally we have inclusions

$$\ker(\tilde{D}_u) \subset \ker(\tilde{D}'_u), \quad \operatorname{coker}(\tilde{D}'_u) \subset \operatorname{coker}(\tilde{D}_u).$$

We say that $\overline{\mathcal{M}}_{\Gamma}(L)$ has *constant rank obstruction* if the dimension of the cokernel of the operators $\tilde{D}'_u$ is equal,

$$\dim \operatorname{coker} \tilde{D}'_{u_1} = \dim \operatorname{coker} \tilde{D}'_{u_2}$$

for any $u_1 \in \mathcal{M}_{\Gamma_1}(L), u_2 \in \mathcal{M}_{\Gamma_2}(L)$ with $\Gamma_1, \Gamma_2 \prec \Gamma$.

We compute the number of zeroes of an equivariant section of the obstruction bundle. Let $T = S^1$ and suppose that X admits a T-action preserving L. Let $\overline{\mathcal{M}}(L)$ be the moduli space of stable disks bounding L. The circle action induces a circle action on $\overline{\mathcal{M}}(L)$ by translation. The action of T naturally extends to the action of $\mathcal{E}(L)$, via the identification of cokernels of maps related by the action. Denote

$$\tilde{\mathcal{M}}(L) = \overline{\mathcal{M}}(L) \cup \partial \overline{\mathcal{M}}(L)/T$$

the quotient of the boundary by the circle action; since $\overline{\mathcal{M}}_\Gamma(L)$ is topologically a manifold with boundary, the quotient operation induces on $\tilde{\mathcal{M}}_\Gamma \mid (L)$ a topological manifold structure, with a smooth structure away from a codimension two locus. The bundle $\mathcal{E}(L)$ descends to a bundle $\tilde{\mathcal{E}}_\Gamma(L)$ on the quotient $\tilde{\mathcal{M}}_\Gamma(L)$, as in Metzler [32]. Define the Euler number of $\tilde{\mathcal{E}}_\Gamma(L)$ as the sign count of zeros of a generic section. Inductively we construct a collection of sections s_Γ of $\tilde{\mathcal{E}}_\Gamma(L)$ satisfying the compatibility condition given by (20). The contribution of each zero of the section is given by (3.2).

The result for disks without markings imply a similar result for treed disks with a single edge. Let $\overline{\mathcal{M}}_1(L)$ denote the moduli space of stable tree disks with a single boundary edge. We have an evaluation map

$$\mathrm{ev} : \overline{\mathcal{M}}_1(L) \to L$$

obtained by evaluating the limit of the trajectory at infinity. For a cycle $\Sigma \subset L$ denote by

$$\overline{\mathcal{M}}_1(L, \Sigma) := \mathrm{ev}^{-1}(\Sigma).$$

We suppose that Σ is a slice for the T-action in the sense that Σ meets each T-orbit transversally. More precisely consider the forgetful map

$$f : \overline{\mathcal{M}}_1(L) \to \overline{\mathcal{M}}(L)$$

obtained by forgetting the marking, denoted $w(e)$, and stabilizing. The fiber over $f(u)$ may be identified with the universal curve, so that $w(e) \in S$ is the attaching point of the edge.

Lemma 3.7. *The obstruction bundle $\overline{\mathcal{E}}_1(L)$ is isomorphic to the pull-back $f^*\mathcal{E}(L)$ under the forgetful map f, and in particular, any section of the obstruction bundle over $\mathcal{M}(L)$ pulls back to a section of the obstruction bundle over $\mathcal{M}_1(L)$.*

Proof. The condition $\mathrm{ev}(u) \in \Sigma$ is transversally cut out if u is regular as a map without marking. The operator $\tilde{D}_u$ is related to that for $f(u)$ by

$$\tilde{D}_u(\xi, \zeta) = (\tilde{D}_{f(u)}(\xi), \pi(D_\zeta(f(u)) + \xi(w(e))))$$

where

$$\xi \in \Omega^0(C, u^*TX), \quad \zeta \in T_{w(e)}\partial S,$$

the vector $\xi(w(e))$ is the evaluation of ξ at $w(e)$, $\zeta \in T_{w(e)}S$ represents a variation of the marking $w(e)$, $D_\zeta f(u)$ is the derivative of $f(u)$ in the direction of ζ, and π is the projection $TX|L \to NL$ to the normal bundle NL. We have a natural isomorphism of cokernels

$$\mathrm{coker}(D_u) \cong \mathrm{coker}(D_{f(u)})$$

and so an isomorphism of the obstruction bundle $\overline{\mathcal{E}}_1(L)$ with the pull-back $f^*\mathcal{E}(L)$. $\square$

The isomorphism of cokernels implies that we can also compute the contribution of constrained maps, even if the constraint is not invariant. Pullback gives a section

$$s_\Gamma := f^* s_{f(\Gamma)}$$

for any type Γ of treed disk with one leaf. The zeros of s_Γ map to the zeroes of $s_{f(\Gamma)}$ under f. Thus we may compute the Euler number for s_Γ by summing the zeroes of $s_{f(\Gamma)}$ with multiplicity.

We now apply this general discussion to justify the computation (8) of the Harvey-Lawson augmentation via localization, starting in dimension $\dim(L) = 3$. Consider the filling of S^5 by $\mathbb{C}^3$, and $\Lambda = T^2$ the Clifford Legendrian of I-(1) with its Harvey-Lawson filling $L_{(1)} \cong S^1 \times \mathbb{R}^2$ of I-(2). The spin structure depends on a specific such identification such as

$$(21) \qquad \phi : S^1 \times \mathbb{R}^2 \to L_{(1)}, \quad (z, s, t) \mapsto (z(1 + s^2 + t^2)^{\frac{1}{2}}, z^{-1}(s + it), (s - it))$$

We leave it to the reader to check that ϕ is indeed a diffeomorphism. The tangent bundle of $S^1 \times \mathbb{R}^2$ is canonically trivial and so induces a spin structure on $L_{(1)}$. Let $\overline{\mathcal{M}}_d$ denote the moduli space of stable disks with d interior markings and one outgoing root edge. Each holomorphic disk u bounding L is a Blaschke product

$$(22) \qquad u : S \to \mathbb{C}^n, \quad z \mapsto \left(c \prod_{i=1}^d \frac{z - a_i}{1 - \overline{a}_i z}, 0, \ldots, 0 \right), \quad c \in S^1$$

in the first component.

We apply localization to compute the contribution of these multiple covers to the Euler number. Consider the action of $T = S^1$ on $\mathbb{C}^3$ with weights $1, -1, 0$ given by

$$e^{i\theta}(z_1, z_2, z_3) = (e^{i\theta}z_1, e^{-i\theta}z_2, z_3).$$

This action preserves the Harvey-Lawson Lagrangian $L \subset \mathbb{C}^3$ and leaves the spin structure invariant. Thus the action on X induces an action on the moduli space $\overline{\mathcal{M}}(L)$ of disks bounding L. The fixed points are given by maps of the form

$$u_{(d)} : S \to \mathbb{C}^3, \quad z \mapsto (z^d, 0, 0).$$

The Lie algebra $\mathrm{aut}(S) \cong \mathfrak{sl}(2, \mathbb{R})$ of automorphisms of the disk embeds in the space of infinitesimal automorphisms of the map $H^0(u^*TX, u^*TL)$ via the infinitesimal action

$$\mathrm{aut}(S) \to H^0(u^*TX, u^*TL), \quad z \mapsto D_{\phi(z)}u.$$

Denote by $H^0(u^*TX, u^*TL)/\mathrm{aut}(S)$ the quotient. Denote by

$$NX \cong \mathbb{C}^3/\mathbb{C} \cong \mathbb{C}^2, \quad NL_1 = TL/TL_1$$

the normal bundles of $\mathbb{C}$ resp. L_1 in $\mathbb{C}^3$ resp. L. The fibers of the normal bundle are given by

$$N_z = \mathrm{span}\{v_1(z), v_2(z)\}$$

where

$$v_1(z) = z^{-1/2}(0, 1, 1), \quad v_2(z) = z^{-1/2}(0, i, -i)$$

for either choice of square root. Indeed, the paths induced by (c.f. (21))

$$(z(1 + t^2)^{\frac{1}{2}}, z^{-1/2}t, z^{-1/2}t), \quad (z(1 + s^2)^{\frac{1}{2}}, z^{-1/2}is, -z^{-1/2}is)$$

lie in $L_{(1)}$ and have the given tangent vectors at $t = 0, s = 0$ where they are linearly independent. The complex span of v_1 resp. v_2 is given by the subspaces

$$\operatorname{span}_{\mathbb{C}} v_1(z) = \operatorname{span}_{\mathbb{C}}(0, 1, 1), \quad \operatorname{span}_{\mathbb{C}} v_2(z) = \operatorname{span}_{\mathbb{C}}(0, 1, -1).$$

We next compute the weights of the action at the fixed point. For any real vector space V with an action of the circle T, denote by $\operatorname{wt}(V) \subset \mathbb{Z}/(\pm 1)$ the set of weights μ, that is, the set of integers up to sign so that V contains a two-dimension subspace on which any $e^{i\theta} \in T$ acts by rotation by angle $\mu\theta$.[2]

Lemma 3.8. *The only fixed point of the action of the circle T on the moduli space $\mathcal{M}(L)$ of disks bounding the Harvey-Lawson Lagrangian L is the d-fold cover $u_{(d)}(z) = (z^d, 0, 0)$ of II-(13). The weights of $T/\operatorname{Aut}(u_{(d)})$ on $H^1(u_{(d)}^* TX, u_{(d)}^* TL)$ are*

$$\operatorname{wt}(H^1(u_{(d)}^* TX, u_{(d)}^* TL)) = \{\pm 1, \ldots, \pm(d-1)\}$$

while the weights of $T/\operatorname{Aut}(u_{(d)})$ on

$$T_{[C, u_{(d)}]} \mathcal{M} \cong H^0(u_{(d)}^* TX, u_{(d)}^* TL)/\operatorname{aut}(S)$$

are

$$\operatorname{wt}(H^0(u_{(d)}^* TX, u_{(d)}^* TL)/\operatorname{aut}(S)) = \{\pm 2, \ldots, \pm d\}.$$

Proof. We begin by finding the fixed points of the action. Any fixed point of the action must satisfy

$$gu(z) = u(\phi_g(z))$$

for some automorphism of the domain ϕ_g depending on $g \in T$. In particular, it follows that ϕ_g permutes the zeros of u. Since T is connected, ϕ_g must fixed the zeros of u. So u has a single zero $z_1 \in S$ (with multiplicity) which we may take to be the origin $z_1 = 0$. Thus, in degree d the map $u_{(d)}(z) = z^d$ is the unique fixed point with irreducible domain. There are no fixed points with reducible domain. Indeed, the action is free on the boundary, and so any map with a boundary node cannot be fixed.

To identify the weights of the action at the fixed points, we first identify the kernel of the linearized operator with a space of holomorphic functions. Write $u = u_{(d)}$ to simplify notation. Since the almost complex structure is standard, the kernel $\ker(D_u)$ consists of sections $\xi : S \to u^* TX$ satisfying the boundary condition $\xi(\partial S) \subset u^* TL$. Explicitly any element of the kernel is of the form

$$\xi = (\xi_1, \xi_2, \xi_3) : S \to \mathbb{C}^3$$

where

$$\xi_1(z) \in T_{u(z)} L \cap (\mathbb{C} \times \{0\} \times \{0\}) \cong z^d i\mathbb{R}, \quad \forall z \in \partial S.$$

Multiplying by z^{-d} gives a function

$$\xi_1'(z) = -iz^{-d}\xi_1(z)$$

---

[2]We thank Chindu Mohanakumar for helpful discussions on the following Lemma.

with real boundary values and a pole of order at most d at $z = 0$, so that $\ker(D_u)$ is identified with such functions. As in Oh [35, Section 5], the Fourier expansion of such a function is

$$\xi_1(z) = \sum_{j=0}^{2d} c_j z^j$$

with Fourier coefficients satisfying

$$c_{d+j} = \bar{c}_{d-j}$$

for $j = 0, \ldots, d$. The action of $g \in T$ is given by

$$(g\xi_1)(z) = g\xi_1(g^{-1/d}z).$$

In particular, if $\xi_1(z) = z^{d-j} + z^{d+j}$ then

$$(g\xi_1)(z) = z^{d-j}g^{1-(d-j)/d} + z^{d+j}g^{1-(d+j)/d} = g^{j/d}z^{d-j} + g^{2j/d}z^{d+j}.$$

Thus, g acts on the orbispace

$$V_j := \operatorname{span}(z^{d-j} + z^{d+j}, i(z^{d-j} - z^{d+j}))/\mathbb{Z}_d \subset H^0(u^*TX, u^*TL)/\mathbb{Z}_d$$

with fractional weight j/d for $j = 0, \ldots, d$. The orbi-spaces V_j have only real structures, so the weights are only defined up to signs, with the overall product of signs fixed by the orientation on the bundle. Since the map $u_{(d)}$ has a d-fold orbifold singularity, $T/\operatorname{Aut}(u_{(d)})$ acts with weights $2j$ for $j = 0, \ldots, d$. The Lie algebra $\operatorname{aut}(S) \cong \mathfrak{sl}(2, \mathbb{R})$ has weights 0 and 2. The weights on $H^0(u^*T\mathbb{C}, u^*TL_1)/\operatorname{aut}(S)$, with $L_1 = L \cap (\mathbb{C} \times \{0\} \times \{0\})$ are therefore

$$\operatorname{wt}(H^0(u^*T\mathbb{C}, u^*TL_1)/\operatorname{aut}(S)) = \pm\{0, 1, \ldots, d\} - \pm\{0, 1\} = \pm\{2, 3, \ldots, d\}.$$

Since the kernel of D_u acting on sections ξ_1 is non-vanishing, the cokernel vanishes.

The cokernel arises from the sections taking values in the normal bundle. The vectors v_1, v_2 span totally real sub-bundles of Maslov indices $-1, -1$ in these sub-bundles, so that one obtains a splitting

$$u_{(d)}^*(TX, TL) = \mathcal{O}(2) \oplus \mathcal{O}(-1) \oplus \mathcal{O}(-1).$$

We compute the weights for the circle action on the kernel and cokernel for the linearized operator acting on sections of the normal bundle. It follows from the discussion above that

$$H^1(u_{(d)}^*NX, u_{(d)}^*NL) = \{0\}$$

and

$$H^1(u_{(d)}^*NX, u_{(d)}^*NL) = \operatorname{span}\ \eta_{1,j}, \eta_{2,j}, \eta_{3,j}, \eta_{4,j}, \quad j \in [0, d/2 - 1] \ \text{ with } j - d/2 \in \mathbb{Z}$$

where

$$\begin{aligned}
\eta_{1,j} &:= (z^{-d/2+j} + z^{-d/2-j}, z^{-d/2+j} + z^{-d/2-j})d\bar{z} \\
\eta_{2,j} &:= (iz^{-d/2+j} - iz^{-d/2-j}, iz^{-d/2+j} - iz^{-d/2-j})d\bar{z} \\
\eta_{3,j} &:= (iz^{-d/2+j} + iz^{-d/2-j}, -iz^{-d/2+j} - iz^{-d/2-j})d\bar{z} \\
\eta_{4,j} &:= (-z^{-d/2+j} + z^{-d/2-j}, z^{-d/2+j} - z^{-d/2-j})d\bar{z}
\end{aligned}$$

One sees from the generators that the weights are in the range $\pm(-1, 1 - d)$, do not repeat (that is, each j produces a different pair of weights) and must lie in the interval

$$\pm \{-1, \ldots, 1 - d\}.$$

The claim follows. $\qquad \square$

We now translate the localization computation above into a count of treed disks which contribute to the augmentation. The basic disk of II-(13) has boundary intersecting $\{1\} \times \mathbb{R}^2$ in a unique point, which is connected to the index one critical point corresponding to infinity labelled $\mathfrak{c}_2$ by a unique gradient trajectory $v : [0, \infty) \to L$. Recall that $\mathfrak{c}_2 \in CE(\Lambda)$ is the corresponding degree zero generator. Let $\overline{\mathcal{M}}_1(L, d)$ be the moduli space of disks of degree d with a constraint in $\{1\} \times \mathbb{R}^2$, whose Euler number $\mathrm{Eul}(\overline{\mathcal{E}}_1(L, d))$ is the contribution of $\overline{\mathcal{M}}_1(L, d)$ to the coefficient of μ_1^d in $\delta(\mathfrak{c}_2)$ as in (7).Consider the cover of the basic disk given by

$$u_{(d)}(z) = z^d.$$

Since the basic disk has Maslov index zero, so does $u_{(d)}$, and since $u_{(1)}$ has area ϵ (for the symplectic form normalized so that the area of a disk of radius r is r^2) the area of $u_{(d)}$ is $d\epsilon$.

Proposition 3.9. *(c.f. Ooguri-Vafa [37], Lin [31, Section 5.7], [40]) For any perturbation system constructed as above, each basic disk $u_{(d)}(z) = z^d$ counts towards the Euler number of $\overline{\mathcal{E}}_1(L)$ with factor $(-1)^{d-1}/d^2$, and towards the Euler number of $\overline{\mathcal{E}}_1(L)$ with factor $(-1)^{d-1}/d$.*

Proof of Proposition 3.9. We claim that $u_{(d)}$ is the unique fixed point of the circle action. Indeed any configuration $u : C \to X$ with more than one disk components S_v must be joined at a boundary node $w \in C$ mapping to a fixed point in L, and the fixed point set of the action on L is empty.

Using Lemma 3.8, the localization computation gives a contribution from $u_{(d)}$

$$\mathrm{Eul}(\mathcal{E}(L)) = (\# \mathrm{Aut}(u))^{-1} \frac{\mu_1(p) \ldots \mu_n(p)}{\lambda_1(p), \ldots \lambda_n(p)} = \frac{1}{d} \frac{(-1) \ldots (1 - d)}{2(3) \ldots (d)} = (-1)^{d-1} \frac{1}{d^2}.$$

For a type Γ of map with a single marking mapping to $\mathbb{R}^2$, the pull-back section s_Γ has a zero only if Γ is a type of map with a single disk as domain. Forgetting the marking produces multiple cover of the basic disk $u_{(d)}(z) = (z^d, 0, 0)$. There are d such zeros of s_Γ lying over corresponding to the d choices of edge mapping to $\{1\} \times \mathbb{R}^2 \subset S^1 \times \mathbb{R}^2 \cong L$.

We make some brief remarks on the sign of the contributions from the multiple covers. First of all, the sign of the d-fold cover $u_{(d)}$ must be $(-1)^{d-1}$ in order to satisfy the relation (7). One may justify the sign using the definition of the orientations in [23], [46] as follows. The defintion of the orientation of the determinant line involves a deformation of the linearized operator $D_{E,F}$ for the given boundary value problem E, F to one obtained by gluing together a disk with trivial problem and a sphere, on which the determinant line is oriented because of the deformation to a complex linear operator. The given bundle here is a sum of bundles with Maslov indices $2d, -d, -d$,

denoted $\mathcal{O}_{\mathbb{R}}(2d) \oplus \mathcal{O}_{\mathbb{R}}(-d) \oplus \mathcal{O}_{\mathbb{R}}(-d)$. The recipe for constructions orientations in [23] requires deforming the Cauchy-Riemann operator on $\mathcal{O}_{\mathbb{R}}(-d) \oplus \mathcal{O}_{\mathbb{R}}(-d)$ with its given action (which does not preserve the splitting) to the Cauchy-Riemann operator obtained by gluing the bundle on the sphere $\mathcal{O}(d_1) \oplus \mathcal{O}(d_2 2)$ to the trivial bundle on the disk for some integers d_1, d_2 with $d_1 + d_2 = -d$.

We claim that one of the factors in the above decomposition must be trivial. For this, note that the rotation action of S^1 on the bundle over the disk may be assumed to extends over the family used in the deformation, to include the action on the nodal disk C with a sphere attached at the node. Indeed, the latter Cauchy-Riemann operator may be assumed to be S^1-equivariant, and then the family of operators in the deformation may be averaged. By definition the action of S^1 on the fiber at the node of the configuration C has weight 0. A simple case of the localization formula implies that the weights of the action on the cohomology of $\mathcal{O}(d_1)$ and $\mathcal{O}(d_2)$ lie in $[0, d_1]$ and $[0, d_2]$ respectively, with d_1, d_2 positive resp. negative if and only if the cohomology is concentrated in degree 0 resp. 1. Since the original Cauchy-Riemann operator $D_{E,F}$ on the normal bundle to the d-fold cover has trivial kernel (and only cokernel) and there is no repetition in the weights, we must have $d_1 = -d$ and $b = d_2$. Thus the bundle on the sphere component on C must be isomorphic to $\mathcal{O}(-d) \oplus \mathcal{O}(0)$. Thus, the weights of the circle action on the index are $-1, \ldots, 1 - d$. (There is a $\mathbb{R}$-factor with trivial action in the both the kernel and the cokernel.) The spin structure is compatible with the action, by definition, and so the sign of the weights is that claimed. $\qquad\square$

We partially compute the augmentation associated to the Harvey-Lawson filling in higher dimensions using localization. As in (7), denote the standard Morse function

$$f_L : L \cong (S^1)^{n-2} \times \mathbb{R}^2 \to \mathbb{R}.$$

The two-cycles consisting of points that flow to $\mathfrak{c}_1, \ldots, \mathfrak{c}_{n-2}$ at infinity are the inverse images of the unstable manifolds

$$\Sigma_1^-, \ldots, \Sigma_{n-2}^- \subset \Lambda$$

of $\mathfrak{c}_1, \ldots, \mathfrak{c}_{n-2}$ under the projection

$$(S^1)^{n-2} \times (\mathbb{R}^2 - \{0\}) \to (S^1)^{n-1}.$$

In particular, the projection of the unstable manifolds $\Sigma_1^-, \ldots, \Sigma_{n-2}^-$ onto $\mathbb{R}^2 - \{0\}$ are paths $\gamma_1, \ldots, \gamma_{n-2} \subset \mathbb{R}^2 - \{0\}$. For generic choices of perturbation data, these paths are disjoint from the point image $\{0\}$ of the boundaries of the holomorphic disks $u : S \to X$ bounding L. Indeed, the latter for the unperturbed almost complex structure are contained in some torus $(S^1)^{n-2} \times \{0\}$.

On the other hand, by stability any configuration contributing to the augmentation involves at least one positive area disk. Indeed let $u : C \to X$ be such a configuration. Since there are no outgoing edges, the disk components S_v furthest away from the incoming edge have a single attaching boundary edge T_e. Such disk components S_v are unstable unless there are interior edges $T_{e'}$ corresponding to intersections with the Donaldson hypersurface D. In that case, the area of such a disk component $A(S_v)$ must be positive by stability.

By the previous paragraph, any configuration contributing to the augmentation has an incoming puncture rather than an incoming edge labelled by a critical point on the Legendrian. It follows that the values of the augmentation on the incoming classical generators are

$$\varphi(\mathfrak{c}_1) = \ldots = \varphi(\mathfrak{c}_{n-2}) = 0.$$

Consider the spin structure on the filling which is non-trivial on the $n-1$-st factor and trivial on the remaining factors. Since the sign of the i-th disk is positive if and only if the spin structure on that factor is trivial (see for example [11, Section 9])

(23) $$\delta^{\mathrm{ab},0}(\mathfrak{a}) = 1 + y_1 + \ldots + y_{n-2} - y_{n-1}.$$

The definition of augmentation implies

$$\varphi(y_{n-1}) = \ln(1 + [\mu_1] + \ldots + [\mu_{n-1}]).$$

By power series expansion the contribution of degree $(d_1, \ldots, d_{n-2})$ maps bounding to the Clifford torus in $\mathbb{C}^{n-2}$ (in the sense described in 3.1) to the contact differential is

$$\frac{(-1)^{d_1 + \ldots + d_{n-2} - 1} [\mu_1]^{d_1} \cdots [\mu_{n-2}]^{d_{n-2}}}{(d_1 + \ldots + d_{n-2})^2}.$$

To see this fact via localization, let $\mathcal{M}(L_{(3)})$ be the moduli space of disks bounding the three-dimensional Harvey-Lawson filling $L_{(3)} \subset \mathbb{C}^3$. We have a natural map

$$\phi : \mathcal{M}_{(n)} \to \mathcal{M}_{(3)}, \quad u = (u_1, \ldots, u_{n-2}, u_{n-1}, u_n) \mapsto \left(\prod_{j=1}^{n-2} u_j, u_{n-1}, u_n \right).$$

Lemma 3.10. *The obstruction bundle $\mathcal{E}_{(n)} \to \mathcal{M}_{(n)}$ is isomorphic to the pull-back of $\mathcal{E}_{(3)}$ under ϕ.*

Proof. We must show that the cokernels of the linearized operator are isomorphic for any map $u : S \to \mathbb{C}^n$ with boundary on $L_{(1)}$:

$$\operatorname{coker}(D_u) \cong \operatorname{coker}(D_{\phi(u)}).$$

The cokernel of D_u arises from the Cauchy-Riemann operator on sections of the normal bundle of $\mathbb{C}^{n-2}$ in $\mathbb{C}^n$, which canonically isomorphic to as the pull-back of the normal bundle of $\mathbb{C}$ in $\mathbb{C}^3$ for $\phi(u)$. $\square$

We relate the moduli spaces of maps bounding the Harvey-Lawson fillings in different dimensions. The map ϕ from $\mathcal{M}_{(n)}$ to $\mathcal{M}_{(3)}$ is finite to one with degree

$$\deg_{\mathbb{R}}(\phi) = \binom{d}{d_1 \ldots d_{n-2}}$$

given by the number of ways of distributing the factors of u into Blaschke products in the $n-2$ factors. To compute the Euler number of $\mathcal{E}_{(n)}$, choose a section $s_{(3)}$ of the bundle $\mathcal{E}_{(3)} \to \mathcal{M}_{(3)}$ and pull-back to obtain a section $s_{(n)}$ of $\mathcal{E}_{(n)} \to \mathcal{M}_{(n)}$. Associated to any zero u of $s_{(3)}$ is a collection of zeros of $s_{(n)}$ of order

(24) $$\#\phi^{-1}(u) = \binom{d}{d_1 \ldots d_{n-2}}.$$

The contribution from $\mathcal{E}_{(3)}$ from the multiple cover of the basic disk $(z^d, 0, 0)$ was already computed in Example 2.23. The augmentation φ has

$$\begin{aligned}
\varphi(\mathfrak{c}_i) &= 0 & i < n-1 \\
\varphi([\mu_i]) &= [\mu_i] & i < n \\
\varphi([\mu_{n-1}]) &= 1
\end{aligned} \qquad .$$

By the computation of the multiple cover contributions in 2.23 and (24),

$$\begin{aligned}
\varphi(\mathfrak{c}_{n-1}) &= \sum_{d_1,\ldots,d_{n-2}} \binom{d}{d_1 \ldots d_{n-2}} \frac{(-1)^{d_1+\ldots+d_{n-2}-1}[\mu_1]^{d_1} \ldots [\mu_{n-2}]^{d_{n-2}}}{(d_1 + \ldots + d_{n-2})} \\
&= \ln(1 + [\mu_1] + \ldots + [\mu_{n-2}]).
\end{aligned}$$

The computation for the other spin structures on the filling is similar.

3.3. **Comparison of perturbation schemes.** It remains to show that the count of zeros of a section of the obstruction bundle is given by the number of perturbed solutions to the Cauchy-Riemann equation.

Lemma 3.11. *Any section s of $\mathcal{E}_\Gamma(L)$ is equal to the pull-back of a domain-dependent Hamiltonian perturbation $H_\Gamma : \mathcal{S}_\Gamma \to \Omega^1(\mathcal{S}_\Gamma, \mathrm{Vect}_h(X))$ in the sense that for any map u with domain type Γ the section is given by*

$$s(u) = (u^* H_\Gamma)^{0,1}.$$

Furthermore, any such section s is equal mod $\mathrm{Im}(D_u)$ to the pull-back $(u^* H_\Gamma)^{0,1}$ of a Hamiltonian perturbation H_Γ vanishing in an open neighborhood of the Donaldson hypersurface D.

Proof. Any one form η with values in $u^* TX$ is the restriction of a Hamiltonian perturbation $H_\Gamma \in \Omega^1(S, \mathrm{Vect}_h(X))$, since any tangent vector $u(z) \in T_{u(z)} X$ extends to a Hamiltonian vector field. Since the equation $D_{J_\Gamma, u} \xi = \eta$ is locally solvable (via the associated Dirichlet equation) we may assume that H_Γ vanishes on an open neighborhood of a finite set of points. In paritcular H_Γ may be taken to vanish in an open neighborhood of the points $z \in S$ that map to the Donaldson hypersurface D. $\square$

Lemma 3.12. *Suppose that $\mathcal{M}(L)$ has constant-rank obstruction bundle in an open neighborhood of $u \in \mathcal{M}(L)$. Then $\mathcal{M}(L)$ is smooth near u with tangent space $T_u \mathcal{M}(L) = \ker(D_u)$. In particular, there exists a homeomorphism from an open neighborhood U of 0 in $\ker(D_{u_0})$ to an open neighborhood of u in the solution set $\overline{\partial}_{J_\Gamma}^{-1}(0)$.*

Proof. The statement of the Lemma follows from the constant rank mapping theorem for Banach manifolds, see [1, Theorem 2.5.15], applied to the map $\mathcal{F}$ cutting out the moduli space from (26). $\square$

We put ourselves In the following, situation where the moduli spaces are smooth but not transversally cut out. Suppose that $\overline{\mathcal{M}}_\Gamma(L)$ is a smooth manifold with corners with chart at any (C, u) given by a map

$$T_{[C]}\mathcal{M}_\Gamma \times \ker(D_u) \to \overline{\mathcal{M}}_\Gamma(L)$$

whose linearization is the identity. Suppose furthermore that the cokernel of D_u is constant rank on $\overline{\mathcal{M}}_\Gamma(L)$.

Theorem 3.13. *Under the assumptions above, let H_Γ is a domain-dependent Hamiltonian perturbation whose zeroes are transverse and disjoint from the boundary of $\overline{\mathcal{M}}_\Gamma(L)$. There exists an ϵ_0 so that if $\epsilon < \epsilon_0$ then solutions to the Hamiltonian-perturbed equation*

$$\overline{\partial}_{J_\Gamma, \epsilon H_\Gamma}(C, u) = 0$$

are in bijection with zeroes of H_Γ.

Proof. We first prove a compactness statement, namely that a family of solutions to the perturbed equation converges to a zero of the section of the obstruction bundle in the limit. Let u_ϵ be a family of solutions to the $(J_\Gamma, \epsilon H_\Gamma)$-perturbed equation converges to a zero of H_Γ. Suppose that u_ϵ is such a family of solutions

(25)
$$\overline{\partial}_{J_\Gamma, \epsilon H_\Gamma}(u_\epsilon) = 0$$

for all ϵ sufficiently small with domains C_ϵ and bounded energy. By Gromov compactness, we obtain in the limit a solution u_0 to

$$\overline{\partial}_{J_\Gamma}(u_0) = 0.$$

with possibly nodal domain C_0. Conversely, given a solution to the limiting equation

$$\overline{\partial}_{J_\Gamma}(u_0) = 0, \quad H_\Gamma^{0,1}(u_0) = 0$$

we have

$$\overline{\partial}_{J_\Gamma, \epsilon H_\Gamma} u_0 = 0$$

for any ϵ.

To see that these constructions give a bijection, for the moment, we assume that the domain is without nodes. By assumption, the space of holomorphic maps is a smooth manifold parametrized locally by elements of $\xi_0 \in \ker(D_{u_0})$. We denote by $u_0(\xi_0)$ the holomorphic map given by the chart. Consider the splitting

$$\Omega^0(u_0^* TX) = \ker(D_{u_0}) \oplus \text{im}(D_{u_0}^*).$$

Let p_0 resp. p_1 denote projection onto the first resp. second factor. By the constant rank embedding theorem, the image of the map

(26)
$$\mathcal{F} : \Omega^0(u_0^* TX) \to \Omega^{0,1}(u_0^* TX), \quad \xi \mapsto \mathcal{T}_\xi^{-1} \overline{\partial}_{J_\Gamma} \exp_{u_0}(\xi)$$

is equal to the image of the restriction of $\mathcal{F}$ to $\text{im}(D_{u_0}^*)$ in a neighborhood of 0. Since

(27)
$$\overline{\partial}_{J_\Gamma, \epsilon H_\Gamma} u_\epsilon = \overline{\partial}_{J_\Gamma} u_\epsilon + \epsilon H_\Gamma^{0,1}(u_0) = 0$$

it follows that

$$H_\Gamma^{0,1}(u_0) = 0.$$

We write maps u_1 near u_0 in terms of geodesic exponentiation

$$u_1 = \exp_{u_0(\xi_0)} \xi_1$$

where $\xi_0 \in \ker(D_{u_0})$, $u_0(\xi_0)$ is holomorphic, and $\xi_1 \in \text{im}(D_{u_0}^*)$. Using parallel transport and projection, we identify

$$\mathcal{T}_{\xi_0} : \text{im}(D_{u_0}^*) \to \text{im}(D_{u_0(\xi_0)}^*).$$

Since D_{u_0} restricted to $\mathrm{im}(D_{u_0}^*)$ is an isomorphism onto its image,

$$
\begin{aligned}
\|\xi_1\|_{k,p} \leq K_1 \|\mathcal{T}_{\xi_0}\xi_1\|_{k-1,p} &\leq K_2 \|D_{u_0(\xi_0)}\mathcal{T}_{\xi_0}\xi_1\|_{k-1,p} \\
&\leq \epsilon K_3 \|L_{\xi_1}H_\Gamma(u_0(\xi_0))\|_{k-1,p} + K_3 \|\xi_1\|_{k,p}^2 \\
&\leq K_4 \epsilon \|\xi_1\|_{k,p}
\end{aligned}
$$

for $\xi_{\epsilon,1}$ sufficiently small, where K_1, K_2, K_3, K_4 are positive constants depending only on u_0, the subscript k, p denotes the $W^{k,p}$-norm used in the Sobolev completions described above, and

$$
\|\epsilon L_{\xi_1}H_\Gamma(u_0(\xi_0)) + D_{u_0(\xi_0)}\mathcal{T}_{\xi_0}\xi_1\|_{k-1,p} \leq K_5 \|\xi_1\|_{k,p}^2
$$

by linearizing (27). It follows that for $\epsilon < 1/K_4$, the vector ξ_1 vanishes. Thus, any solution to the perturbed equation is already holomorphic.

In general, each domain in the sequence is obtained from the limiting domain by gluing with some gluing parameters associated to the nodes. For simplicity, we assume that the nodes are boundary nodes. Let

$$
\delta(\epsilon) = (\delta_1(\epsilon), \ldots, \delta_k(\epsilon)) \in \mathbb{R}_{\geq 0}^k.
$$

The curve C_ϵ is obtained from C_0 via identifications

$$
z \mapsto \delta_k(\epsilon)/z
$$

in a local coordinate near the k-th node. Let $u_0(\epsilon, \xi_0)$ denote the operator obtained by "pregluing" $u_0(\xi_0)$ as in [46, Theorem 2.4.5] using gluing parameters $\delta(\epsilon)$. The kernels and cokernels of the operator $D_{u_0(\epsilon,\xi_0)}$ are then identified with those of $u_0(\xi_0)$. We may write

$$
u_\epsilon = \exp_{u_0(\epsilon,\xi_0)} \xi_1, \quad \xi_1 \in \mathrm{im}(D_{u_0(\epsilon,\xi_0)}^*).
$$

A similar argument as before for the case of no nodes, using the convergence of linearized operators for $u_0(\epsilon, \xi_0)$ to that for u_0, shows that ξ_1 vanishes, so that any solution to the perturbed equation is already holomorphic. $\square$

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