AUGMENTATION VARIETIES AND DISK POTENTIALS II

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Abstract. This is the second in a sequence of papers in which we construct Chekanov-Eliashberg algebras for Legendrians in circle-fibered contact manifolds and study the associated augmentation varieties. In this part, we first define the Chekanov-Eliashberg algebra and its Legendrian contact homology. For a tame Lagrangian cobordism between Legendrians, we define a chain map between their Chekanov-Eliashberg algebras.

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1. Introduction

Contact homology as introduced by Eliashberg, Givental, and Hofer [19] is a theory whose differential counts holomorphic curves in the symplectization of a contact manifold. Its relative version, Legendrian contact homology, has been developed by Ekholm-Etnyre-Ng-Sullivan [17] and related works in situations where the contact manifold fibers over, for example, an exact symplectic manifold. This paper develops a version of Legendrian contact homology in case that the contact manifold fibers over a monotone symplectic manifold, extending work of Sabloff [32], Asplund [3], and Dimitroglou-Rizell-Golovko [12]. We consider a Legendrian Λ in a compact contact manifold Z that is a negatively-curved circle-fibration over a symplectic base Y; we have in mind especially the example that Z is the unit canonical bundle over a Fano projective toric variety Y, and Λ is the horizontal lift of a Lagrangian in the base. A typical example is the horizontal lift of the Clifford torus in complex projective space, which is a Legendrian torus in the standard contact sphere. We prove the following:

Theorem 1.1. *Suppose that Λ is a tame Legendrian in a circle-fibered stable Hamiltonian manifold (Z, α, ω). Counting punctured holomorphic disks $u : C \to \mathbb{R} \times Z$ bounding $\mathbb{R} \times \Lambda$ defines Legendrian contact homology groups*

$$HE(\Lambda) = \frac{\ker(\delta)}{\mathrm{im}(\delta)}, \quad \delta : CE(\Lambda) \to CE(\Lambda)$$

independent of the choice of almost complex structures and perturbations, and is an invariant of the Legendrian isotopy class of Λ.

The construction is a special case of symplectic field theory with Lagrangian boundary conditions, where the contact manifold is circle-fibered. The simplifications in this case are comparable to that of genus-zero relative Gromov-Witten theory compared to full symplectic field theory with contact boundary. In the first paper [4] in this series, we used Cieliebak-Mohnke [9] perturbations to regularize the moduli spaces of treed punctured disks, and proved compactness and orientability of the moduli spaces. The natural chain complex $CE(\Lambda)$ corresponding to limits of such disks is generated by words in chains on the Reeb chords and chains on the Legendrian, both are realized by critical points of Morse functions; the addition of these *classical generators* does not occur in the version of Legendrian contact homology over exact symplectic manifolds in Ekholm-Etnyre-Sullivan [16]. These classical generators arise because of nodes developing along points in the interior of the cobordism, which then lead to trajectories as in Ekholm-Ng [18]; in particular [18] assigns augmentations to the Harvey-Lawson filling in the three-dimensional setting of knot contact homology while the construction here is intended to work in all dimensions, under some restrictions.

The maps between homology groups associated to different choices of almost complex structures are special cases of cobordism maps associated to Lagrangian cobordisms between Legendrians. By a symplectic cobordism from Z_- to Z_+ we mean a symplectic manifold $\tilde{X}$ with boundary

$$\partial \tilde{X} = Z_-^{\mathrm{op}} \cup Z_+,$$

where Z_-^{op} indicates Z_- with the opposite orientation. The complement $X = \tilde{X} - \partial\tilde{X}$ is then a manifold with cylindrical ends. Cobordisms can be composed in the standard way using the coisotropic embedding theorem. However, our cobordisms do not form the morphisms of any category as the trivial cobordism does not admit a symplectic structure satisfying the conditions we require.

Our cobordisms will often arise by removing symplectic hypersurfaces from compact symplectic manifolds as follows. Let $\overline{X}$ be a compact symplectic manifold and

$$Y_-, Y_+ \subset \overline{X}$$

be codimension two symplectic submanifolds of $\overline{X}$. Let $\widetilde{X}$ be the real blow-up of $\overline{X}$ along Y_-, Y_+, obtained by replacing $Y_\pm$ with their unit normal bundles.

$$Z_\pm^{\pm 1} \subset \widetilde{X}$$

where Z_-^{-1} denotes the principal circle bundle obtained by reversing the direction of the action on Z_-. We assume that the restriction of the symplectic form on $\overline{X}$ restricts to two-forms on $Y_\pm$ that are positive multiples of the curvature of $Z_\pm$. We call the interior X of $\widetilde{X}$ a symplectic cobordism from Z_- to Z_+. This definition of cobordism is essentially equivalent to the definition in Etnyre-Honda [20]; there are many other definitions of symplectic cobordism in the literature. Conversely, given any such X one may obtain a compact symplectic manifold without boundary $\overline{X}$ by Lerman's symplectic cut construction [26].

Given a symplectic cobordism, a Lagrangian cobordism is a Lagrangian submanifold that has the desired Legendrian boundary. That is, a Lagrangian cobordism between Legendrians Λ_-, Λ_+ is a Lagrangian $L \subset X$ whose closure $\overline{L} \subset \widetilde{X}$ has intersection with the boundary $\partial\widetilde{X}$ given by

$$L \cap Z_\pm = \Lambda_\pm.$$

A punctured disk or sphere to X of finite energy extends to an unpunctured disk or sphere mapping to the compactification $\overline{X}$, by removal of singularities. We assume that the Lagrangian satisfies *tameness* conditions detailed in [4, Definition 3.18], which imply that rigid holomorphic spheres and disks with only outgoing punctures do not appear in the boundary of our moduli spaces, and there is no breaking of holomorphic disks along closed Reeb orbits. However, the Lagrangian is not necessarily exact. This enables us to discuss the relation between fillings and augmentations in a general setting.

The construction of cobordism maps depends on the existence of bounding chain similar to the definition in Fukaya-Oh-Ohta-Ono [21]. By definition such a chain is an element

$$b \in CE(L, \partial L)$$

(where $CE(L, \partial L)$ denotes relative chains with coefficients in some completed group ring, see Equation (39)) that solves a *Maurer-Cartan equation*: Let

$$m_d : CE(L, \partial L)^{\otimes d} \to CE(L)$$

denote the counts (defined precisely in Definition 41) of treed holomorphic disks with d inputs at critical points of the Morse function in the interior and one output

either in the interior or the boundary as in (39). The Maurer-Cartan equation

$$\sum_{d\geq 0} m_d(b,\dots,b) = 0 \in CE(L).$$

has a space of solutions denoted $MC(L)$. (The maps m_d do not form an A_∞ algebra, because of the additional terms arising from the cylindrical ends of the Legendrian.)

Theorem 1.2. *Let (X,L) be a tame cobordism from (Z_-,Λ_-) to (Z_+,Λ_+) as in Theorem 1.1 equipped with a bounding chain $b \in MC(L)$. Counting punctured holomorphic disks defines a chain map*

$$\varphi(L,b) : CE(\Lambda_-,\hat{G}(L)) \to CE(\Lambda_+,\hat{G}(L)), \quad \varphi\delta_- = \delta_+\varphi$$

between the complexes associated to the ends over a coefficient ring $\hat{G}(L)$ given by a completed group ring on $H_2(\overline{X},\overline{L})$, and independent of the choice of almost complex structure, Morse functions, and perturbations for disks bounding L up to chain homotopy and a change in the bounding chain $b \in MC(L)$.

A special case of the theorem, for trivial cobordisms, may be used to prove invariance of contact homology under Legendrian isotopy in Theorem 4.28 below. As a second case of Theorem 1.2, fillings of a Legendrian with good properties (such as the Harvey-Lawson filling below) define augmentations: chain maps

$$\varphi(L,b) :\ CE(\Lambda) \to \hat{G}(L).$$

In the third part [5] of this series we study the augmentations associated to these fillings, and apply the resulting augmentation varieties to distinguish Legendrian isotopy classes.

2. Classification results

In this section, we carry out a classification of disks in certain situations that allow us to make partial computations of the differential in Legendrian contact homology introduced in the next section. For example, we wish to compute the image of the differential on degree one generators in the case of Legendrian lifts of Lagrangian torus orbits in toric varieties.

2.1. Examples of punctured holomorphic disks.

We begin with a few examples. Let Z be a circle-fibered contact manifold over a symplectic base Y with a projection $p : Z \to Y$, and X a symplectic cobordism as defined in [4, I-sympcob] When $X = \mathbb{R}\times Z$, there is the projection $p_Z : X \to Z$.

Example 2.1. Recall from [4, (1)] the Clifford Legendrian Λ_{Cliff} is the horizontal lift of the Clifford Lagrangian defined as

$$\Lambda_{\text{Cliff}} := \left\{ (z_1,\dots,z_n) \in \mathbb{C}^n \;\middle|\; \begin{array}{l} |z_1|^2 = \dots = |z_n|^2 = \frac{1}{n} \\ z_1\dots z_n \in (0,\infty) \end{array} \right\} \cong T^{n-1} \subset S^{2n-1}. \tag{1}$$

The Clifford Lagrangian in complex projective space is denoted

$$\Pi_{\text{Cliff}} := \left\{ [z_1,\dots,z_n] \;\middle|\; \sum_{i=1}^n |z_i|^2 = 1 \right\} \cong T^{n-1} \subset \mathbb{C}P^{n-1}.$$

The corresponding Lagrangian in the symplectization is

$$\mathbb{R} \times \Lambda_{\text{Cliff}} \cong \left\{ (z_1, z_2, \ldots, z_n) \;\middle|\; \begin{array}{l} |z_1| = |z_2| = \ldots = |z_n| \\ z_1 z_2 \ldots z_n \in (0, \infty) \end{array} \right\} \subset \mathbb{C}^n - \{0\}.$$

Example 2.2. An example of a Lagrangian filling of Λ_{Cliff} is given by the Harvey-Lawson type filling. Let

$$a_1, \ldots, a_n \in \mathbb{R}$$

be real numbers with exactly two zero. Define as in Joyce [23, (37)]

$$L_{(1)} := \left\{ (z_1, \ldots, z_n) \in \mathbb{C}^n \;\middle|\; \begin{array}{l} |z_1|^2 - a_1^2 = |z_2|^2 - a_2^2 = \ldots = |z_n|^2 - a_n^2 \\ z_1 \ldots z_n \in [0, \infty) \end{array} \right\}. \tag{2}$$

The parameters a_i are often omitted in the notation.

In the following Section 2.2, we describe more generally how holomorphic disks in the base lift to punctured disks in the symplectization. A particular example of this phenomenon is the following description of lifts of Maslov-index-two-disks bounding the Clifford Lagrangian:

Proposition 2.3. *Lifts of the holomorphic Maslov index two disks in complex projective space $\mathbb{C}P^{n-1}$ bounding the cylinder on the Clifford Lagrangian torus $\Lambda_{\text{Cliff}} \subset S^{2n-1}$ are the holomorphic maps on the once-punctured disks defined by*

$$\begin{aligned} u &: \mathbb{H} \to \mathbb{C}^n, \\ z &\mapsto (e^{i\theta_1}(cz - ai)(cz + ai)^{(2/n)-1}, e^{i\theta_2}(cz + ai)^{2/n}, \ldots, e^{i\theta_n}(cz + ai)^{2/n}) \end{aligned} \tag{3}$$

for $\text{Im}(z) \geq 0, c > 0, a > 0$ and phases $e^{i\theta_1}, \ldots, e^{i\theta_n}$ with unit product

$$e^{i\theta_1} \cdot \ldots \cdot e^{i\theta_k} = 1.$$

Each such disk has a unique strip-like end asymptotic to a minimal length Reeb chord γ at infinity.

Proof. The product of entries is a real quadratic polynomial which is positive on the real line. Thus, we can find $c > 0, a > 0$ such that the product is the real polynomial

$$u_1 u_2 \ldots u_n = (cz - ai)(cz + ai).$$

Furthermore,

$$|u_k| = |cz \pm ai|^{2/n}, \quad k = 1, \ldots, n.$$

The composition $\overline{u}(z)$ of $u(z)$ with the projection to $\mathbb{P}^{n-1}$ is

$$\overline{u}(z) = \left[\frac{cz - ai}{cz + ai}, 1, \ldots, 1 \right]$$

which represents a disk with one intersection with the anticanonical divisor, and so has Maslov index 2. On the other hand, the projection $p_Z u(re^{i\theta})$ is asymptotic as $r \to \infty$ to

$$\lim_{r \to \infty} p_Z u(re^{\pi i\theta}) = \frac{1}{\sqrt{n}}(ce^{2\pi i\theta/n}, \ldots, ce^{2\pi i\theta/n}).$$

The limit is a Reeb chord $\gamma \in \mathcal{R}(\Lambda_{\text{Cliff}})$ with angle change $2\pi/n$ at infinity. □

We will see in Section 2.2 that this is a complete list of punctured disks with one incoming Reeb chord of minimal length. One obtains examples of treed disks by adding Morse trajectories limiting to index one critical points.

Example 2.4. Identify Λ_{Cliff} with T^{n-1} by projection on the first $n-1$ components. Let

$$f_\bullet : \Lambda_{\text{Cliff}} \to \mathbb{R}, \quad (z_1, \ldots, z_{n-1}) \mapsto \sum_{i=1}^{n-1} \text{Re}(z_i)$$

be the standard Morse function on Λ_{Cliff}, given by the sum of height functions on the factors. The index one critical points of $f_\bullet$ are of the form

$$\mathfrak{c}_i = \{(1, \ldots, 1, -1, 1, \ldots, 1, -1)\} \in \text{crit}(f_\bullet) \subset \Lambda_{\text{Cliff}} \cong T^{n-1}$$

where the first -1 is in the i-th component. The stable manifold

$$\Sigma_i^s \subset \Lambda_{\text{Cliff}}$$

of the critical point $\mathfrak{c}_i$ is the locus of points

$$\Sigma_i^s = \{(\lambda_1, \ldots, \lambda_i = 1, \ldots, \lambda_{n-1}) \in \Lambda_{\text{Cliff}} \tag{4}$$

with i-th coordinate λ_i equal to -1. The map of (3) intersects Σ_i^s once transversally. Therefore there exists a treed disk u' whose domain C is the union of the once-punctured disk S and a single segment T.

In Section 3.4 we will see that such examples lead to infinite sequences of treed disks.

Example 2.5. We next describe the disks that bound a particular connected filling of a disconnected Legendrian. Let $\Lambda_{\text{Cliff}} \subset S^{2n-1}$ be the Clifford Legendrian in Example 2.1. Define the *Hopf Legendrian*

$$\Lambda_{\text{Hopf}} = \Lambda_{\text{Cliff}} \cup \exp\left(\frac{\pi i}{n}\right) \Lambda_{\text{Cliff}} \tag{5}$$

$$= \left\{ (z_1, \ldots, z_n) \in \mathbb{C}^n \;\middle|\; \begin{array}{l} |z_1|^2 = \ldots = |z_n|^2 = \frac{1}{n} \\ z_1 \ldots z_n \in \mathbb{R} \end{array} \right\} \tag{6}$$

$$\cong T^{n-1} \sqcup T^{n-1} \subset S^{2n-1} \tag{7}$$

the union of Λ_{Cliff} with its image $\exp\left(\frac{\pi i}{n}\right) \Lambda_{\text{Cliff}}$ under multiplication by $\exp\left(\frac{\pi i}{n}\right)$; the terminology is based on the analogy with the Hopf link in the three-sphere. An immersed filling of Λ_{Hopf} is given by the union of the Harvey-Lawson fillings:

$$L_{(1,1)} := L_{(1)} \cup \exp\left(\frac{\pi i}{n}\right) L_{(1)} \cong ((S^1)^{n-2} \times \mathbb{R}^2) \sqcup ((S^1)^{n-2} \times \mathbb{R}^2) \subset \mathbb{C}^n.$$

The intersection of the two components can be made transverse after perturbation. On the other hand, a connected filling is given by the set

$$L_{(2)} := \{(z_1, \ldots, z_n) | z_1 \ldots z_n \in \mathbb{R} + i\epsilon, \; |z_1|^2 = \ldots = |z_n|^2.\} \tag{8}$$

The exact filling described in (8) can be described in terms of symplectic parallel transport as follows. Let

$$\pi : \mathbb{C}^n \to \mathbb{C}, \quad (z_1, \ldots, z_n) \mapsto z_1 \ldots z_n$$

be the map given by the product of components. Define the *symplectic connection* on the fibration $\pi : \mathbb{C}^n - \{0\} \to \mathbb{C}$ as the symplectic perpendicular to the vertical subspace

$$T^v\mathbb{C}^n - \{0\} = \ker(D\pi)$$

of the differential $D\pi$, with respect to the standard symplectic form. Consider the path

$$\vartheta : \mathbb{R} \to \mathbb{C}, \quad t \mapsto t + i\epsilon$$

or more generally any path avoiding the critical value $0 \in \mathbb{C}$. Symplectic parallel transport $\rho_t^{t'}$ of the Lagrangian torus T^{n-1} in any fiber

$$T^{n-1} \cong \{ \ | \ z_1| = \ldots = |z_n| \ | \ \pi(z_1, \ldots, z_n) = t \ \}, \quad t \gg 0$$

along the path ϑ defines the Lagrangian filling

$$L_{(2)} := \bigcup_{t' \in \mathbb{R}} \gamma_t^{t'} T^{n-1} \cong T^{n-1} \times \mathbb{R}$$

of Λ.

The holomorphic disks bounding this filling now have the possibility of limiting to Reeb chords connecting the two sheets of the Legendrian at infinity. For each $k = 1, \ldots, n$ define a holomorphic map

$$u_{01}^k : \mathbb{H} \to \mathbb{C}^n, \tag{9}$$

$$z \mapsto \left((-z - i\epsilon)^{\frac{1}{n}}, \ldots, \underbrace{\frac{-z + i\epsilon}{-z - i\epsilon}(-z - i\epsilon)^{\frac{1}{n}}}_{k^{th} \text{ coordinate}}, \ldots, (-z - i\epsilon)^{\frac{1}{n}} \right).$$

This map is asymptotic along the unique strip-like end to a Reeb chord γ_{12} of angle change π/n. Second, consider the holomorphic map

$$u_{10} = ((z + i\epsilon)^{\frac{1}{n}}, \ldots, (z + i\epsilon)^{\frac{1}{n}}). \tag{10}$$

This map asymptotic to a Reeb chord γ_{21} of angle change π/n connecting the sheets in the reverse order.

2.2. **Lifting disks from the base.** In this section, we prove a bijection between disks bounding the Lagrangian in the symplectization and disks bounding the Lagrangian projection of the Legendrian. This bijection is a version of a result in Dimitriglou-Rizell [11, Theorem 2.1] adapted to our setting. Later this lifting result will be used to prove that the augmentation variety is contained in the zero level set of the Landau-Ginzburg potential. To state the result, let $\mathcal{M}(\Lambda)$ denote the moduli space of holomorphic punctured disks of finite Hofer energy bounding $\mathbb{R} \times \Lambda$ as in [4] and $\mathcal{M}(\Pi)$ the moduli space of (unpunctured) holomorphic disks bounding Π. Here we mean disks with irreducible domain, not treed disks, although an extension to treed disks will be used later.

Lemma 2.6. *Let Z be a circle-fibered contact manifold equipped with a cylindrical almost complex structure as in Definition I-3.3. Composition with the projection $p : Z \to Y$ defines a map of moduli spaces*

$$\mathfrak{P} : \mathcal{M}(\Lambda) \to \mathcal{M}(\Pi), \quad u \mapsto p \circ u. \tag{11}$$

Proof. Given any punctured disk u mapping to $\mathbb{R} \times Z$, composition with the projection from $\mathbb{R} \times Z$ to Y defines a punctured disk mapping to Y, since the projection is almost complex. Now the fact that u is finite Hofer energy implies that the projection is finite energy, and so extends over the punctures by removal of singularities. □

We consider the problem of lifting a particular map to the base.

Definition 2.7. Consider a map bounding Π

$$u_Y : S \to Y, \quad u_Y(\partial S) \subset \Pi.$$

A lift u of u_Y to a map to $\overline{X}$ is equivalent to a section

$$\sigma : S \to u_Y^*(Z \times_{S^1} \mathbb{P}^1 \to Y)$$

of the pull-back $u_Y^*(Z \times_{S^1} \mathbb{P}^1 \to Y)$ of the $\mathbb{P}^1$-bundle $Z \times_{S^1} \mathbb{P}^1 \to Y$ associated to Z that we call the *section associated to the lift*.

Since the domains of our maps are disks, these sections are given by their zero-and-pole-structure as we now explain.

Definition 2.8. The *angle map* assigns to any punctured disk $u : S \to X$ the collection of angles of Reeb chords at the punctures

$$\mathcal{M}_{\mathbb{\Gamma}}(\Lambda) \to \mathbb{R}_{>0}^{\#\,\mathrm{Edge}_{\to,\circ}(\mathbb{\Gamma})}, \quad u \mapsto (\theta_e(u))_{e \in \mathcal{E}(S)}. \tag{12}$$

Introduce the following notation for the set of possible angle changes. Let

$$\mathcal{A}(u_Y) = \{(\theta_e(u))_{e \in \mathcal{E}(S)} | p \circ u = u_Y\} \subset \mathbb{R}_{>0}^{\#\,\mathrm{Edge}_{\to,\circ}(\mathbb{\Gamma})}$$

be the set of tuples such which satisfy the condition in [4, Lemma 3.15].

The angle map is a bijection on the space of lifts, similar to the exact case by an observation of Dimitriglou-Rizell [11, Theorem 2.1].

Theorem 2.9. *Suppose the projection Π of Λ is embedded in Y and all Reeb chords from Λ to Λ have angles that are multiples of $2\pi/k$ for some $k \in \mathbb{Z}_{>0}$. The map (12) defines a bijection between equivalence classes of lifts $u : S \to X$ of a map $u_Y : \overline{S} \to Y$ and length collections $\theta_e, e \in \mathcal{E}(S)$ satisfying the angle change formula in [4, Lemma 3.15] where two lifts are considered equivalent if they are equal up to a scalar multiplication preserving Λ.*

Proof. We claim that the angle map (12) is bijective on the space of lifts of a given map, up to multiplication by scalars. In particular, we will show that for any $u_Y \in \mathcal{M}_{\mathbb{\Gamma}}(\Pi)$, the restriction of the angle map (12) to $\mathfrak{P}^{-1}(u_Y)$ is bijective onto $\mathcal{A}(u_Y)$.

Step 1: The angle map is injective. Consider two lifts u, u' of u_Y. Since $\mathbb{C}^\times$ acts transitively on the fibers of p, there exists a function f so that the lifts u, u' are related by multiplication by f, using the $\mathbb{C}^\times$ action on $\mathbb{R} \times Z$:

$$f : S \to \mathbb{C}, \quad u' = fu.$$

Since both u and u' are holomorphic, the function f is holomorphic as well. By exponential convergence of punctured holomorphic disks to Reeb chords (see for example [6, Theorem 3.1]) the maps u, u' are asymptotic to maps of the form

$$(s,t) \mapsto (\theta_e s, t^{\theta_e})(c,z) \text{ resp. } (s,t) \mapsto (\theta'_e s, t^{\theta'_e})$$

for some $(c,z) \in Z \times \mathbb{R}$ and constants θ_e, θ'_e. So f is asymptotic to

$$f(s,t) \sim ((\theta_e - \theta'_e)s, t^{\theta_e - \theta'_e})$$

on each end $e \in \mathcal{E}(S)$. If the angles θ_e, θ'_e are equal then f is asymptotically constant along the ends, so bounded on S. Any such bounded holomorphic function is constant, which corresponds to the scalar multiplication in the fiber direction.

Step 2: The angle map is surjective in the case of no outgoing punctures and the Lagrangian in the base is embedded. The bundle $u_Y^*(\mathbb{R} \times Z \to Y)$ has totally real boundary condition that is a union of k real subspaces $u_Y^*(\mathbb{R} \times \Lambda)$. Consider the $-k$-fold tensor product

$$u_Y^*(\mathbb{R} \times Z^{\otimes -k} \to Y).$$

A boundary condition is given by the real sub-bundle

$$(\partial u_Y^*)(\mathbb{R} \times \Lambda^{\otimes -k}) \subset u_Y^*(\mathbb{R} \times Z^{\otimes -k} \to Y)$$

where $\Lambda^{\otimes -k}$ denotes the image of Λ under $Z \mapsto Z^{\otimes -k}$. (The branches combine under tensor product.) The Maslov index of the pair

$$(u_Y^*(\mathbb{R} \times Z^{\otimes -k} \to Y), \partial u_Y^*(\mathbb{R} \times \Lambda^{\otimes -k}))$$

is the winding number with respect to some trivialization over the interior. Since the boundary condition is orientable, the Maslov index is even and so equal to $2d$ for some integer d. Negativity of the bundle $Z \to Y$ implies that d is non-negative. By Lemma 2.10 below, the bundle pair above is isomorphic to the pair $(\mathcal{O}(d), \mathcal{O}_{\mathbb{R}}(d))$ whose global sections are real polynomials

$$\{c(z - z_1) \dots (z - z_d) \mid c, z_1, \dots, z_d \in \mathbb{R}\} \subset \Gamma(\mathcal{O}(d), \mathcal{O}_{\mathbb{R}}(d))$$

whose zeroes all occur on the boundary, and the location of these zeroes may be arbitrarily specified.

We obtain sections of the original boundary value problem by lifting. Since S is simply connected and the map

$$u_Y^*(\mathbb{R} \times Z \to Y) \to u_Y^*(\mathbb{R} \times Z^{\otimes k} \to Y)$$

is a k-fold cover, the section $\tilde{s}$ lifts to a section

$$s : S \to u_Y^*(\mathbb{R} \times Z \to Y)$$

with boundary values in $\partial u_Y^*(\mathbb{R} \times \Lambda^{\otimes k})$ as claimed with poles at $z_1, \dots, z_d$. This shows that the map (12) is surjective in the case that S has no outgoing punctures.

Step 3: The angle map is surjective in the case of both incoming and outgoing punctures. Suppose $\theta'_e \in 2\pi\mathbb{Z}/k$ is a collection of angles at the same punctures z_i satisfying the condition in [4, Lemma 3.15]. Consider the function

$$f : S \to \mathbb{C}, \quad z \mapsto \prod_{e \in \mathcal{E}(S)} (z - z_i)^{\pm(\theta'_e - \theta_e)} \tag{13}$$

(with sign depending on whether the puncture is incoming or outgoing). Because the angle sums are equal, the function f is bounded at infinity. The section fs has angle changes θ'_e and real boundary conditions as desired. □

In the proof, we used the fact that real Cauchy-Riemann operators on a given rank one bundle E on the disk S with real boundary condition $F \subset E|\partial S$ are all isomorphic. That is,

Lemma 2.10. *Let S be a disk, $E \to S$ a complex line bundle, and $F \subset E|_{\partial S}$ a totally real boundary condition. Let $D'_{E,F}$ and $D''_{E,F}$ be real Cauchy-Riemann operators on E with boundary values in F. Then there exists a real gauge transformation $g : S \to \mathrm{Aut}_{\mathbb{R}}(E)$ equal to the identity on the boundary so that*

$$g D'_{E,F} g^{-1} = D''_{E,F}. \tag{14}$$

Proof. We remark that in the integrable, higher rank case, Cauchy-Riemann operators are classified by their splitting type as in Oh [29]. In the rank one case the argument is straight-forward: It suffices to show that the orbit through every real Cauchy-Riemann operator is open. By definition $D'_{E,F}, D''_{E,F}$ are of the form

$$\xi \mapsto \overline{\partial}\xi + A'\xi, \quad \xi \mapsto \overline{\partial}\xi + A''\xi$$

for some real-matrix-valued $0,1$-forms A', A''. A real gauge transformation $g : E \to E$ acts on the space of such operators by

$$A \mapsto g\overline{\partial}g^{-1} + \mathrm{Ad}(g)A.$$

The tangent space to the orbit is the image of $\Omega^0(S, \mathrm{End}_{\mathbb{R}}(E))$ under the map

$$\xi \mapsto \overline{\partial}_{-A} := \overline{\partial}\xi + [\xi, A] = \overline{\partial}\xi - [A, \xi].$$

The cokernel of this operator is the kernel of the adjoint $\overline{\partial}^*_{-A}$, and equal to the kernel of the generalized Laplacian $\overline{\partial}_{-A}\overline{\partial}^*_{-A}$. On the other hand, the domain of this Laplacian consists of functions that vanish on the boundary. The kernel of $\overline{\partial}^*_{-A}$ is trivial by the unique continuation principle as in [28, Section 2.3]. □

In the case that the Lagrangian in the base is immersed, the map extends continuously but not holomorphically to the boundary. We will need the following lemma which allows us to assume that operator is trivial on the strip-like ends:

Lemma 2.11. *Let S be a surface with boundary and strip-like ends, $E \to S$ a line bundle trivialized along the strip-like ends $E_{\kappa_e(s,t)} \cong \mathbb{C}$ with connection $\alpha \in \Omega^1(S, \mathrm{End}(E))$ that decays exponentially along the strip-like ends in the sense that there exist constants $c, \mu > 0$ so that*

$$|\alpha(s,t)| < ce^{-\mu s}$$

in coordinates (s,t) along each end. Let $F \to \partial S$ be a totally real boundary condition constant along the strip-like ends

$$F_{(s,0)} = F_{(s,1)} = F_e \in \mathbb{R}, \quad s \gg 0.$$

Then the Cauchy-Riemann operator is equivalent (by an isomorphism of bundles that is the identity on the boundary, as in (14)) to one that is trivial on the strip-like ends.

Proof. The proof is similar to that in the previous Lemma 2.10. For small positive decay constant $\lambda > 0$ the Laplacian

$$\overline{\partial}_{-A}\overline{\partial}^*_{-A} : \Omega^{0,1}(S, \partial S)_{k,p,-\lambda} \to \Omega^{0,1}(S)_{k-2,p,-\lambda}$$

on S acting on the Sobolev space of functions of class k, p with exponential decay constant $\lambda > 0$ is invertible, since the kernel is the trivial space of constant functions and the operator being Fredholm. The statement of the Lemma now follows from the implicit function theorem: Any orbit of the group of complexified gauge transformations is both open and closed. Since the space of connection is connected, there is a unique orbit. □

Corollary 2.12. *Let $D_{E,F}$ be a Cauchy-Riemann operator on a rank one bundle E with boundary condition F on a surface with strip-like or cylindrical ends S. Then $D_{E,F}$ is equivalent to an operator that extends to the compactification $\overline{S}$ of S obtained by adding a point at infinity along each strip-like or cylindrical end $e \in \mathcal{E}(S)$.*

Proof of Theorem 2.9 in the immersed case. Given the Lemma, the Theorem 2.9 extends to the case that Π is only immersed by taking (13) as the definition of the lift, using the trivialization in Lemma 2.11 to prove surjectivity. The rest of the proof is the same as in the embedded case. □

Rigidity and regularity of a map to the base is equivalent to rigidity and regularity of any lift to the symplectization:

Proposition 2.13. *Let C be a treed disk, Z a circle-fibered contact manifold, $J \in \mathcal{J}(\mathbb{R} \times Z)$ a (possibly domain-dependent) cylindrical almost complex structure in the sense of [4, 3.3], and $\Lambda \subset Z$ a compact Legendrian.*

- *(a) A holomorphic treed disk $u : C \to \mathbb{R} \times Z$ bounding $\mathbb{R} \times \Lambda$ is regular if and only if $p \circ u : C \to Y$ is regular.*
- *(b) A holomorphic treed disk $u : C \to \mathbb{R} \times Z$ bounding $\mathbb{R} \times \Lambda$ is rigid if and only if $p \circ u : C \to Y$ is rigid.*

We first introduce a long exact sequence that will be used in the proof. Let $\tilde{D}_u, \tilde{D}_{u_Y}$ be the linearized Cauchy-Riemann operators of u and u_Y from [4, Equation (31)]. The tangent space of TX splits into the kernel of Dp, denoed $T^{\mathrm{v}}X$ and a bundle isomorphic to p^*TY.

Lemma 2.14. *Let $u : C \to X : \mathbb{R} \times Z$ be a treed disk and $u_Y = p \circ u$ its projection to the base. The short exact sequence of bundles*

$$0 \to T^{\mathrm{v}}X \to TX \to p^*TY \to 0$$

induces a long exact sequence of kernels and cokernels:

$$0 \to \ker(D_u^{\mathbb{v}}) \to \ker(\tilde{D}_u) \to \ker(\tilde{D}_{u_Y}) \to \operatorname{coker}(D_u^{\mathbb{v}}) \to \dots$$

Proof. The short exact sequence induces a short exact sequence of two-term complexes. The first term of this complex is the short exact sequence of one forms with Lagrangian boundary values

$$\begin{aligned}(15)\quad 0 \to \Omega^0(C, u^*T^{\mathbb{v}}X, (\partial u)^*T^{\mathbb{v}}L) \to T_C\mathcal{M}_\Gamma \oplus \Omega^0(C, u^*TX, (\partial u)^*TL) \\ \to T_C\mathcal{M}_\Gamma \oplus \Omega^0(C, u_Y^*Y, (\partial u_Y^*)T\Pi) \to 0.\end{aligned}$$

Here $\Omega^0(C, \dots)$ includes both the surface S and tree T components, with matching condition at the intersection $S \cap T$. The second term in the short exact sequence is the corresponding exact sequence of one-forms. The differentials of the complex are the vertical linearized operator $D_u^{\mathbb{v}}$, the parametrized linear operator $\tilde{D}_u$ and the parametrized linear operator for the projection $\tilde{D}_{u_Y}$ respectively. The domains of these operators on spaces with finite Banach norms of [4, Section 4.2] which we omit to simplify notation. The claim follows from standard homological algebra. □

Proof of Proposition 2.13. For the first claim, by Lemma 2.14, it suffices to show that the horizontal and vertical parts of the cokernel vanish. The kernel of the vertical part $\ker(D_u^{\mathbb{v}})$ has a section given by the infinitesimal action of $\mathbb{R}^\times$:

$$\xi(z) = \frac{\mathrm{d}}{\mathrm{d}t}(tu(z))|_{t=0} \in u^*T^{\mathbb{v}}Z.$$

As a rank one boundary value problem, either the kernel or the cokernel vanishes as explained in Oh [29]. Since the kernel is non-vanishing, $\operatorname{coker}(\tilde{D}_u^{\mathbb{v}})$ vanishes. The argument for the second claim is similar.

For the second claim about rigidity, we show that the kernel of the linearized operator has the same dimension as the horizontal part up to a factor generated by translation. The pull-back bundle $(u|S)^*T(\mathbb{R} \times Z)^{\mathbb{v}}$ is isomorphic to the trivial bundle via the identification

$$(u|S)^*T(\mathbb{R} \times Z)^{\mathbb{v}} \mapsto S \times \mathbb{C}, \quad \xi \mapsto D\exp_u \xi.$$

The map $D\exp_u$ is multiplication by u after identifying any particular fiber with $\mathbb{C}^\times$. The kernel $\ker(D_{u|S}^{\mathbb{v}})$ is the space of bounded holomorphic functions. The map $D\exp_u$ identifies the kernel with complex valued functions with the same order at the poles and zeros of the map $u|S$.

We claim that the vertical part of the kernel is one-dimensional. Since S is a union of disks, $\ker(D_{u|S}^{\mathbb{v}})$ consists of constant functions. The matching conditions at the edges T_e connecting components S_{v_-}, S_{v_+} impose a codimension one set of constraints, so that $\ker(D_u^{\mathbb{v}})$ is one-dimensional. Since translations act on the space of buildings, a disk u is rigid if and only if $\ker(\tilde{D}_{u_Y})$ vanishes, which is to say that u_Y is rigid. □

Because the equation on the segments is invariant, the projection map above extends naturally to a map of treed disks:

Lemma 2.15. *Composing with the projection $\mathbb{R} \times Z \to Y$ defines a map of treed disks $\mathcal{M}_{\mathbb{\Gamma}}(\Lambda)$ of type $\mathbb{\Gamma}$ to treed disks $\mathcal{M}_{p(\mathbb{\Gamma})}(\Pi)$, where $p(\mathbb{\Gamma})$ is the map type for treed disks obtained by the corresponding map on homotopy groups on each component. Given a rigid map $u \in \mathcal{M}_{\mathbb{\Gamma}}(\Lambda)$ the corresponding map $p \circ u \in \mathcal{M}_{p(\mathbb{\Gamma})}(\Pi)$ is rigid, and the converse holds if the vertical part of the index vanishes. The projection map $\mathcal{M}(\Lambda)_0 \to \mathcal{M}(\Pi)_0$ is orientation preserving.*

Proof. The first few statements are immediate from the definitions. For the statement on orientations, recall from [4, Section 4.6] that the the orientations on $\mathcal{M}(\Lambda), \mathcal{M}(\Pi)$ are induced by deformations of spin structure. The Cauchy-Riemann operator for $\mathcal{M}(\Lambda)$ splits into vertical and horizontal parts, and these may be deformed separately. Namely, let $u : S \to \mathbb{R} \times Z$ be a rigid holomorphic map bounding $\mathbb{R} \times \Lambda$. The kernel of the linearized map D_u is naturally isomorphic to the direct sum

$$\ker(D_u) \cong \ker(D_u^{\mathbb{v}}) \oplus \ker(D_u^{\mathbb{h}}) \cong \mathbb{R} \oplus \ker(D_{p \circ u}).$$

Hence

$$\ker(D_u)/\mathbb{R} \cong \ker(\tilde{D}_{p \circ u}).$$

To see that this isomorphism is orientation preserving, note that the orientations on both sides are obtained by capping off the strip-like ends to obtain a surface without strip-like ends $\overline{S}_v$ and deforming to the operator obtained by gluing together a trivial Cauchy-Riemann operator on a disk $\overline{S}_v'$ with a Cauchy-Riemann operator on a sphere $\overline{S}_v''$. Then the operator on the vertical part is the standard Cauchy-Riemann operator $D_u^{\mathbb{v}}$ on a trivial bundle with trivial boundary conditions. Thus $D_u^{\mathbb{v}}$ has trivial cokernel and one-dimensional kernel canonically identified with $\mathbb{R}$. The capping operation induces the standard orientation on this factor, and so the orientation is that induced by the horizontal part. □

We apply the above results to the disks that contribute to the disk potential of the Lagrangian projection. Let Y be a monotone symplectic manifold and $\Pi \subset Y$ a compact, connected, relatively spin, monotone Lagrangian submanifold. Recall the *disk potential* defined as follows, as in for example Cho-Oh [8]. Choose a generic almost complex structure J, a generic point $\mathrm{pt} \in \Pi$ and let $\mathcal{M}(\Pi, \mathrm{pt})_0$ denote the moduli space of rigid (necessarily Maslov index two) J-holomorphic disks passing through $\mathrm{pt} \in \Pi$.

Definition 2.16. The disk potential of a compact monotone relativel spin Lagrangian $\Pi \subset Y$ is the function on the space of local systems $\mathrm{Rep}(\Pi)$ obtained by counting Maslov-index-two-disks weighted by holonomy:

$$W_\Pi : \mathrm{Rep}(\Pi) \to \mathbb{C}^\times, \quad \rho \mapsto \sum_{u \in \mathcal{M}(\Pi, \mathrm{pt})_0} \rho([\partial u]).$$

By Proposition 2.13, disks contributing to the potential in the base lift to punctured disks in the symplectization of a circle bundle contributing to the contact differential. Let Z denote a circle bundle with connection α whose curvature is $-\omega$. Since $c_1(Z) = -[\omega]$, we have

$$c_1(Y) = -\tau c_1(Z)$$

where τ is the monotonicity constant. Let $\Lambda \subset Z$ denote the horizontal lift of Π.

Corollary 2.17. *Suppose Y is monotone with monotonicity constant τ and $\Pi \subset Y$ a monotone Lagrangian. Any Maslov index two disk $(u_Y : \overline{S} \to Y) \in \mathcal{M}(\Pi, \mathrm{pt})_0$ lifts to a rigid once-punctured disk $(u : S \to \mathbb{R} \times Z) \in \mathcal{M}(\Lambda, \mathfrak{a})_0$ asymptotic to a Reeb chord with angle change $2\pi/\tau$ at the puncture.*

Proof. Let $u_Y : \overline{S} \to Y$ be a disk of Maslov index two. By the monotonicity assumption, the symplectic area of u_Y is $1/\tau$. The result then follows from the angle change formula [4, Lemma 3.15]. □

Example 2.18. In the case of the Clifford torus, continuing Example 2.1, the Maslov index two disks have simple formulas given in (3).

Finally we discuss lifts of constant disks. The following Proposition shows in particular that lifts of such disks are rigid only if they have three punctures:

Proposition 2.19. *Let $Z \to Y$ be a circle-fibered contact manifold and $\Lambda \subset Z$ a compact Legendrian submanifold. Any rigid stable holomorphic disk $u : C \to \mathbb{R} \times Z$ bounding $\mathbb{R} \times \Lambda$ with only boundary punctures whose image in Y is constant is a three-punctured disk with incoming Reeb chord γ_{02} and incoming Reeb chords γ_{01}, γ_{12} so that γ_{02} is the concatenation of γ_{01} and γ_{12}, as in Figure 1.*

Proof. Such maps are discussed in more detail in the context of symplectic field theory in Ekholm [13, Section 6.1.E]. Any rigid such map has exactly three punctures by an index computation. The bundles

$$u^*T(\mathbb{R} \times Z) \to S, \quad (\partial u)^*T(\mathbb{R} \times \Lambda) \to \partial S$$

are trivial in this case, and the reader may check that the kernel of D_u is one-dimensional and the cokernel vanishes; in particular these maps are regular. Assuming only boundary punctures, The index of the trivial Cauchy-Riemann operator $\mathrm{rank}(F) + d(\circ) - 3 = \mathrm{d}(\circ) - 2$, and after quotienting the action of $\mathbb{R}$ by translation, the dimension is $d(\circ) - 3$.

The existence of such a map u follows, for example, from the Cartheodory-Torhorst extension of the Riemann mapping theorem, as explained in Rempe-Gillen [31]. Torhorst [34] shows that the biholomorphism on the interior of a domain has a continuous extension if the boundary is locally connected. The map u is a diffeomorphism on the interior, and from this it follows that u is unique up to automorphism of the domain. □

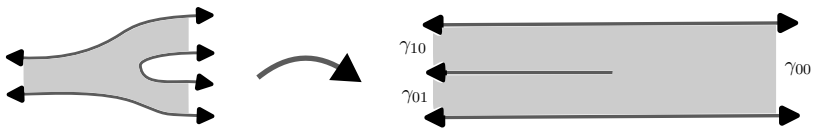

FIGURE 1. A zero-area map with three punctures

Example 2.20. Let Λ_{Hopf} be the generalized Hopf Legendrian as in the previous example. Identify a fiber of $\mathbb{R} \times Z \to Y$ with $\mathbb{C}^\times$. The intersection of the product Λ with each circle fiber is finite of order $2n$. In particular, let $\gamma_{00} \in \mathcal{R}(\Lambda_{\text{Hopf}})$ represent a Reeb chord of angle change $2\pi/n$ connecting the first branch Λ_0 to itself, and let $\gamma_{01}, \gamma_{10} \in \mathcal{R}(\Lambda_{\text{Hopf}})$ represent Reeb chords of angle change π/n connecting Λ_0 to Λ_1 and Λ_1 to Λ_0.

2.3. **Disks bounding toric Legendrians.** We describe some classification results in the case that the Legendrian is a horizontal lift of a Lagrangian torus orbit in a toric variety. By the equivalence with geometric invariant theory quotients, one may obtain a classification of disks in toric varieties as in Cho-Oh [8]. Recall from Delzant [10] that any projective toric variety is a symplectic quotient of a vector space.

Definition 2.21. A *symplectic toric manifold* is a compact connected symplectic manifold Y with an effective action of a torus T of half the dimension of of Y; by [10] such actions are classified by their moment polytope P and generic stabilizer.

The polytopes occurring are called *Delzant polytopes*, and have the property that the primitive lattice vectors to the facets at any vertex form a lattice basis. Let Y_P be a symplectic toric manifold correponding to a Delzant polytope P as in [10].

Theorem 2.22. *Any compact toric symplectic manifold Y_P is the symplectic quotient $\mathbb{C}^k /\!\!/ H$ for some Hamiltonian action of a torus H with moment map $\Psi : \mathbb{C}^k \to \mathfrak{h}^\vee$ so that the torus action on Y_P is that of the residual action of $U(1)^k/H$.*

This description was used by Cho-Oh [8] to obtain a description of the holomorphic disks bounding toric moment fibers. As explained in Kirwan's thesis [25], the toric variety may be obtained alternatively as the Kähler quotient of the semistable locus by the complexified group action. Denote the semistable locus, given as the union of orbits of the complex orbit through the zero level set, by

$$\mathbb{C}^{k,\mathfrak{s}} = H_{\mathbb{C}}\Psi^{-1}(0) \subset \mathbb{C}^k$$

The *Kähler quotient* is the quotient of the stable locus by the complexified group action

$$Y_P = \mathbb{C}^{k,\mathfrak{s}}/H_{\mathbb{C}}$$

and is diffeomorphic to the symplectic quotient as explained in [25]. Here we use the assumption that $H_{\mathbb{C}}$ acts with only finite stabilizers on the semistable locus. Let Π be a Lagrangian orbit of the residual group $T = U(1)^k/H$ acting on Y_P.

Lemma 2.23. *(Cho-Oh [8]) Let $Y_P = \mathbb{C}^k /\!\!/ H$ as above. Any holomorphic disk $u : S \to Y_P$ bounding a Lagrangian torus fiber Π is given by a tuple of Blaschke products*

$$u : S \to Y_P, \quad z \mapsto \left(c_j \prod_{i=1}^{d(j)} \frac{z - a_i}{1 - \overline{a_i} z} \right)_{j=1}^{k} . \tag{16}$$

The proof uses the fact disks in a git quotient lift to the original space: In this case the pull-back of the bundle $\mathbb{C}^{k,\mathfrak{s}} \to Y_P$ to any disk is trivial. Any disk to Y_P bounding Π lifts to a disk in $\mathbb{C}^{k,\mathfrak{s}}$ bounding the product of circles in the factors. Each such factor is necessarily a Blaschke product of the form in (16).

Example 2.24. For example, complex projective space $\mathbb{C}P^{n-1}$ is the symplectic quotient of $\mathbb{C}^n$. The Maslov index two disks in $\mathbb{C}P^{n-1}$ have lifts to $\mathbb{C}^n$ of the form

$$u : S \to \mathbb{C}^n, \quad z \mapsto [e^{i\theta_1}, \dots, e^{i\theta_{k-1}}, e^{i\theta_k} z, e^{i\theta_{k+1}}, \dots, e^{i\theta_n}]$$

for some constants $\theta_1, \dots, \theta_n \in \mathbb{R}$.

2.4. **Disks in the Harvey-Lawson filling.** In this section, we classify some of the disks bounding the Harvey-Lawson filling in Example 2.2.

Lemma 2.25. *Any collection of Blaschke products in the first $n-2$ coordinates as in (16) defines a disk bounding $L_{(1)}$.*

Proof. The statement follows from the ffact that the intersection of the Harvey-Lawson Lagrangian with $\mathbb{C}^{n-2} \times \{(0,0)\}$ is a standard Lagrangian torus $(S^1)^{n-2}$ in $\mathbb{C}^{n-2}$. □

Example 2.26. In particular, for $n = 3$ the *basic disk* is defined by

$$u : S \to X, \quad z \mapsto (z, 0, 0). \tag{17}$$

Generalizations of these basic disks to other fillings are considered in Song Yu [36].

Lemma 2.27. *For the standard complex structure on $\mathbb{C}^n$ all unpunctured holomorphic disks in $\mathbb{C}^n$ bounding the Harvey-Lawson filling $L_{(1)}$ have image contained in $\mathbb{C}^{n-2}$. In particular, for $n = 3$ any such disk is a multiple cover of (a scalar multiple of) the basic disk of (17).*

Proof. We examine the composition of any such disk with the map taking the product of components, which must be a holomorphic disk in the complex line bounding the reals. Let S be an (unpunctured) disk and $u : S \to \mathbb{C}^n$ a holomorphic map bounding L. The composition of u with the product map

$$\pi : \mathbb{C}^n \to \mathbb{C}, \quad (z_1, \dots, z_n) \mapsto z_1 \dots z_n$$

must have boundary $\pi \circ u|_{\partial S}$ contained in $[0, \infty)$. Since $\pi \circ u$ is bounded, $\pi \circ u$ must be a constant map equal to some $c \in \mathbb{R}$. If this constant c is non-zero then the boundary $\pi \circ u|\partial S$ must lie on an $(n-1)$-torus

$$\{|z_1|^2 - a_1^2 = |z_2|^2 - a_2^2 = \dots = |z_n|^2 - a_n^2 = c\} \subset L_{(1)}.$$

The winding number

$$[\pi \circ u|\partial S \cong S^1] \in \pi_1(S^1) \cong \mathbb{Z}$$

of $\pi \circ u$ around 0 is the sum of the winding numbers

$$[\partial u_j] \in \pi_1(S^1) \cong \mathbb{Z}$$

of the components $u_1, \dots, u_n$ around 0 and equal to 0. Since u_j is holomorphic, each winding number $[\partial u_j]$ is non-negative, and at least one is non-zero. This is a contradiction. □

3. Circle-fibered Legendrian contact homology

In this section, we define the Legendrian contact homology, which will be generated by words in certain Reeb orbits as well as critical points of a Morse function. We assume that Λ is a compact, spin Legendrian in a fibered contact manifold satisfying the following tameness condition.

Definition 3.1 (Definition 3.1 in [4])**.** A fibered contact manifold (Z, α) with base $(Y, -\operatorname{curv}(\alpha))$ is *tame* if the symplectic form on the base $\omega_Y = -\operatorname{curv}(\alpha)$ is integral,

$$[\omega_Y] \in H^2(Y, \mathbb{Z})$$

and monotone in the sense that there exists a *monotonicity constant* $\tau > 1$ so that

$$c_1(Y) = \tau[\omega_Y] \in H^2(Y, \mathbb{Z}).$$

If $\Lambda \subset Z$ is a Legendrian we call (Z, Λ) a *tame pair*. Henceforth we will assume that Λ is equipped with a relative spin structure.

The definition guarantees, in particular, that each rigid disk in $\mathbb{R} \times Z$ bounding $\mathbb{R} \times \Lambda$ has at least one incoming puncture, and no punctured spheres are rigid. See subsection 3.4 in [4] for a detailed discussion.

3.1. Coefficient rings.

Our coefficient ring is the group ring in relative homology over the rationals completed with the filtration defined by area. Recall that our contact manifold fibers over a monotone symplectic manifold Y, and our Legendrian Λ is a finite cover over a (not necessarily monotone) Lagrangian in the base.

Definition 3.2. The *group ring on second homology* is the space of compactly supported functions

$$\mathbb{C}[H_2(Y, \Pi)] := \{g : H_2(Y, \Pi) \to \mathbb{C}, \operatorname{supp}(g) \text{ compact}\}.$$

The *area function* is

$$A : H_2(Y, \Pi) \to \mathbb{R}, \quad [u_Y] \mapsto \langle [u_Y], [\omega_Y] \rangle.$$

The *area filtration* is

$$\mathbb{C}[H_2(Y, \Pi)] = \bigcup_{A \in \mathbb{R}} \mathbb{C}[H_2(Y, \Pi)]_A$$

where

$$\mathbb{C}[H_2(Y, \Pi)]_A = \{g \in \mathbb{C}[H_2(Y, \Pi)], A(\beta) < A \implies g(\beta) = 0\}$$

is the subgroup of functions supported on classes with area at least A.

The *group ring* for Λ is

$$G(\Lambda) = \{\, g : H_2(Y, \Pi) \to \mathbb{C} \mid \#\operatorname{supp}(g) < \infty \,\} \tag{18}$$

The *completed group ring* is

$$\begin{aligned} \hat{G}(\Lambda) &= \lim_{\infty \leftarrow A} \mathbb{C}[H_2(Y,\Pi)]/\mathbb{C}[H_2(Y,\Pi)]_A \\ &= \{\, g : H_2(Y,\Pi) \to \mathbb{C} \mid \forall A, \#\operatorname{supp}(g)_A < \infty \,\} \end{aligned} \tag{19}$$

where

$$\operatorname{supp}(g)_A := \operatorname{supp}(g) \cap \{c | A(c) < A\}.$$

Remark 3.3. If (Y,Π) is monotone then the completion with respect to the area filtration will not be necessary for the definition of the differential and the uncompleted group ring will suffice. However, even in the simplest examples our augmentations will require a completed coefficient ring. In the monotone case we often choose to work with the group ring $\mathbb{C}[H_1(\Lambda)]$ over the first homology $H_1(\Lambda)$, to keep with standard conventions in the literature.

The group ring has the natural convolution product. For any homology class μ we follow standard abuse of notation in the field and denote by

$$[\mu] \in \hat{G}(\Lambda)$$

the delta function at μ, so that

$$[\mu_1][\mu_2] = [\mu_1 + \mu_2].$$

The product extends in the obvious way to the completion $\hat{G}(\Lambda)$.

3.2. **The chain group.** Chains are generated by words in the generators. We first recall the definition of a Morse datum from [4].

Definition 3.4. A *Morse datum* for (Z,Λ) consists of a pair of vector fields on the space of Reeb chords and on the cylinder on the Legendrian

$$\zeta_\circ \in \operatorname{Vect}(\mathcal{R}(\Lambda)), \quad \zeta_\bullet \in \operatorname{Vect}(\mathbb{R} \times \Lambda)^{\mathbb{R}}$$

arising as follows:

(a) There exists a Morse function on the space of Reeb chords

$$f_\circ : \mathcal{R}(\Lambda) \to \mathbb{R};$$

so that $\zeta_\circ$ is the gradient vector field:

$$\zeta_\circ := \operatorname{grad}(f_\circ) \in \operatorname{Vect}(\mathcal{R}(\Lambda)). \tag{20}$$

(b) There exists a Morse function

$$f_\bullet : \Lambda \to \mathbb{R};$$

with gradient vector field

$$\operatorname{grad}(f_\bullet) \in \operatorname{Vect}(\Lambda)$$

so that $\zeta_\bullet$ is a translation-invariant lift of $\operatorname{grad}(f_\bullet)$.

Since each component of the space of Reeb chords is isomorphic to the Legendrian, we could take the vector fields to be equal under the identification of the various components; however, this assumption is not necessary.

Definition 3.5. A vector field $\zeta_\bullet \in \mathrm{Vect}(\mathbb{R}\times\Lambda)$ is *positive* if in coordinates (s,λ) on $\mathbb{R}\times\Lambda$ there exists a function

$$a : \Lambda \to \mathbb{R}_{>0}$$

so that

$$\zeta_\bullet = a(\lambda)\partial_s + p^* \operatorname{grad}(f_\bullet) \tag{21}$$

where

$$p^* : \mathrm{Vect}(\Lambda) \to \mathrm{Vect}(\mathbb{R}\times\Lambda)^{\mathbb{R}}$$

is the obvious identification of translationally-invariant vector fields trivial in the $\mathbb{R}$-direction with vector fields on Λ.

The limit of any Morse trajectory along any infinite length trajectory is a zero of the gradient vector field. We introduce labels for the possible limits of the trajectories above as follows. Denote by

$$\underline{\mathbb{R}} \cong T\mathbb{R}\times\Lambda$$

the translational factor in $T(\mathbb{R}\times\Lambda) = T\mathbb{R}\oplus T\Lambda$. The zeroes of the vector field $p_*(\zeta_\bullet)$ correspond to tangencies of $\zeta_\bullet$ with the translational factor:

$$p_*(\zeta_\bullet)^{-1}(0) = \zeta_\bullet^{-1}(\underline{\mathbb{R}}) \subset \mathbb{R}\times\Lambda.$$

Definition 3.6. The union of the zeroes of the vectors fields is denoted

$$\mathcal{I}(\Lambda) := \mathcal{I}_\circ(\Lambda)\cup\mathcal{I}_\bullet(\Lambda), \quad \mathcal{I}_\circ(\Lambda) := \zeta_\circ^{-1}(0), \quad \mathcal{I}_\circ(\Lambda) := \zeta_\bullet^{-1}(0). \tag{22}$$

Let $\mathcal{W}(\Lambda)$ denote the space of ordered words in the generators $\mathcal{I}(\Lambda)$ defined as above:

$$\mathcal{W}(\Lambda) = \bigcup_{d\geq 0} \mathcal{I}(\Lambda)^d. \tag{23}$$

For any word $w = \gamma_1\dots\gamma_k \in \mathcal{W}(\Lambda)$ denote by

$$\ell(w) = k \in \mathbb{Z}_{\geq 0}$$

the length of w and by $\ell_\bullet(w)$ the number of *classical generators* $\gamma_i \in \mathcal{I}_\bullet(\Lambda)$. The space of *contact chains* is the completion

$$CE(\Lambda) = \left\{ \sum_{i=1}^{\infty} c_i\Sigma_i \;\middle|\; \Sigma_i \in \mathcal{W}(\Lambda), c_i \in \hat{G}(\Lambda), \quad \lim_{i\to\infty}(\ell(\Sigma_i)) = \infty \right\} \tag{24}$$

of $\hat{G}(\Lambda)$-valued functions on $\mathcal{W}(\Lambda)$ with respect to the filtration given by the classical length $\ell_\bullet$. We also call $CE(\Lambda)$ the Chekanov-Eliashberg algebra of the Legendrian Λ. Denote by

$$CE_\ell(\Lambda) = \bigoplus_{\ell(w)=\ell} \hat{G}(\Lambda)w \subset CE(\Lambda)$$

the subspace generated by words w of length ℓ. This ends the Definition.

The definition of Chekanov-Eliashber chains is similar to that used for immersed Lagrangian Floer theory in Akaho-Joyce [2] and for Legendrian contact homology using contact instantons in Oh [30]. More generally, for any $G(\Lambda)$-module G we denote by $CE(\Lambda, G) = CE(\Lambda) \otimes_{G(\Lambda)} G$ the chains with coefficients in G.

For the construction of Legendrian contact homology we also need a $\mathbb{Z}_2$-grading. Similar to the case of orbifold quantum cohomology, in the monotone case Legendrian contact homology admits a natural grading by the reals, given by a shifted sum of angles of the Reeb chords.

Definition 3.7. (Gradings) Let $\Lambda \subset Z$ be a Legendrian with projection $\Pi \subset Y$.

(a) ($\mathbb{Z}_2$-grading) If Π is embedded then any Reeb orbit γ over a critical point $x \in \mathcal{I}_\circ(\Lambda)$ define the *$\mathbb{Z}_2$-grading*

$$\deg_{\mathbb{Z}_2} : \mathcal{I}_\circ(\Lambda) \to \mathbb{Z}_2, \quad \gamma \mapsto \deg_{\mathbb{Z}_2} x + 1$$

where $\deg_{\mathbb{Z}_2} x = \operatorname{ind}_x(f_\circ) \bmod 2\mathbb{Z}$. In the case Π is immersed and

$$x = (x_-, x_+) \in \Pi \times_X \Pi$$

is an ordered transverse self-intersection point define

$$\deg_{\mathbb{Z}_2}(\gamma) = \deg_{\mathbb{Z}_2} x + 1$$

where $\deg_{\mathbb{Z}_2} x = 1$ resp. -1 if the isomorphism

$$T_{x_-}\Pi \oplus T_{x_+}\Pi \cong T_x X$$

is orientation preserving resp. reversing.

(b) ($\mathbb{R}$-grading) Suppose that $\Pi \subset (Y, -\operatorname{curv}(\alpha))$ is monotone with monotonicity constant $\tau \in \mathbb{R}$. Define the *real grading*

$$\mathcal{I}_\circ(\Lambda) \to \mathbb{R}, \quad \gamma \mapsto |\gamma| = (\operatorname{ind}(f_\circ))(\gamma) + \frac{\tau\theta}{\pi} - 1$$

where θ is the angle change of the Reeb chord γ. Define

$$\mathcal{I}_\bullet(\Lambda) \to \mathbb{R}, \quad \gamma \mapsto \deg_{\mathbb{R}}(\gamma) := \operatorname{ind}(f_\bullet)(\gamma).$$

For words define the degree map as the sum of the degrees of the factors:

$$\mathcal{W}(\Lambda) \to \mathbb{R}, \quad \gamma_1 \otimes \ldots \otimes \gamma_k \mapsto \deg_{\mathbb{R}}(\gamma_1 \otimes \ldots \otimes \gamma_k) := \sum_{i=1}^{k} \deg_{\mathbb{R}}(\gamma_i).$$

If Λ is connected then the Reeb chords are multiples of π/τ and the real grading defines an $\mathbb{Z}$-grading.

Example 3.8. Since the real degree of a classical generator is its Morse degree shifted by one we have embeddings of the classical generators of Morse-degree 0 resp. 1

$$CM_0(\Lambda) \subset CE_{-1}(\Lambda), \quad CM_1(\Lambda) \subset CE_0(\Lambda).$$

Suppose Π is monotone with monotonicity constant τ, and

$$\mathcal{R}(\Lambda)_{2\pi/\tau} \subset \mathcal{R}(\Lambda)$$

is the locus of Reeb chords of angle change $2\pi/\tau$. We have by definition inclusions with a degree shift of one

$$CM_0(\mathcal{R}(\Lambda)_{2\pi/\tau}) \subset CE_1(\Lambda), \quad CM_1(\mathcal{R}(\Lambda)_{2\pi/\tau}) \subset CE_2(\Lambda).$$

The remaining components of $\mathcal{R}(\Lambda)$ contribute generators in higher degrees. This ends the Example.

For later use, we introduce notation for the subspaces generated by the classical and Reeb generators. Define

$$\mathcal{W}_\bullet(\Lambda) = \bigcup_{d\geq 0} \mathcal{I}_\bullet(\Lambda)^d, \quad \mathcal{W}_\circ(\Lambda) = \bigcup_{d\geq 0} \mathcal{I}_\circ(\Lambda)^d.$$

and let

$$CE_\circ(\Lambda) \subset CE(\Lambda), \quad CE_\bullet(\Lambda) \subset CE(\Lambda) \tag{25}$$

denote the subspaces generated by words in $\mathcal{W}_\bullet(\Lambda)$ resp. $\mathcal{W}_\circ(\Lambda)$. We call $CE_\bullet(\Lambda)$ the *classical sector*.

3.3. **The contact differential.** As above, suppose that (Z, Λ) is a tame pair. Assume $\underline{P}$ is a collection of perturbation data that are coherent, stabilizing, and regular as in [4, Section 4].

Definition 3.9. The contact differential $\delta \in \mathrm{End}(CE(\Lambda))$ is the weighted count of punctured holomorphic surfaces, with all components disks and with one incoming puncture on each component:

$$\delta : CE(\Lambda) \to CE(\Lambda),\ \underline{\gamma}_- \mapsto \sum_{u\in\mathcal{M}(L,\underline{\gamma}_-,\underline{\gamma}_+)_0} (-1)^{\heartsuit}\, \mathrm{wt}(u)\underline{\gamma}_+ \tag{26}$$

where

$$\heartsuit = \sum_{i=1}^{d_-} i|\gamma_{-,i}| \tag{27}$$

d_- is the length of d_- and the weight $\mathrm{wt}(u)$ is defined as the product of factors

$$\mathrm{wt}(u) := c_i d(\circ)^{-1} \epsilon(u)[p\circ u] \tag{28}$$

with $u : C \to X$ being a map from a curve C of some type Γ, c_i being the coefficient of the perturbation $P_{\Gamma,i}$ in the multivalued perturbation P_Γ, $d(\circ)$ is the number of interior edges mapping to the Donaldson hypersurface and $\epsilon(u)$ is the orientation sign. Here the domain type Γ ranges over rigid types of disconnected punctured disks with one incoming puncture on each component, and $[u]$ is the homology class of the capped off boundary defined in [4, Equation (19)]. This ends the Definition.

There are many variations using other coefficient rings. In case that (Y, Π) is monotone, one can replace $[p\circ u] \in H_2(Y, \Pi)$ in (28)] with $[\partial u]$ as defined in [4, Equation (19)], or work over the complex numbers as coefficient ring.

Proposition 3.10. *The map $\delta : CE(\Lambda) \to CE(\Lambda)$ is well-defined.*

Proof. We show that the completion with respect to word length makes the infinite sum in the Definition (26) well-defined. Let

$$\underline{\gamma}_- \in \mathcal{I}(\Lambda)^d$$

be a collection of incoming generators. It suffices to check that for any particular word length $\ell \in \mathbb{Z}_{\geq 0}$ there exist finitely many terms in $\delta(\underline{\gamma}_-)$ bounded by the given word length ℓ, and for each such possible output $\underline{\gamma}_+$ and energy $A > 0$, there exist finitely many disks of energy at most A contributing to the coefficient of $\underline{\gamma}_+$ in $\delta(\underline{\gamma}_-)$. By [4, Lemma 3.15], the total angle change in $\underline{\gamma}_+$ is bounded. Thus only finitely many Reeb chords in $\mathcal{R}(\Lambda)$ are possible, for the given input $\underline{\gamma}_-$. It follows that the possibly outputs $\underline{\gamma}_+$ of a rigid configuration are finite in number. By Gromov compactness for maps to (Y, Π), the number of homology classes in (Y, Π) represented by holomorphic disks of area less than A is finite. For each such disk there exist finitely many possible lifts to punctured disks mapping to $(\mathbb{R} \times Z, \mathbb{R} \times \Lambda)$, by Theorem 2.9. It follows that the number of disks contributing to each $\underline{\gamma}_+$ of area below the given constant A is finite. □

The differential can be understood in terms of disk counts, rather than counts of disconnected surfaces.

Definition 3.11. A *trivial strip* is a map $u : \mathbb{R} \times [0, 1] \to \mathbb{R} \times Z$ that is an open immersion in some fiber $p^{-1}(u)$ of the projection $\mathbb{R} \times Z \to Y$ of the form

$$u(s, t) = e^{\theta(s+it)} z$$

for some constant θ and point $z \in Z$.

Lemma 3.12. *Any disconnected stable rigid configuration $u : C \to X$ has exactly one connected component $u_i : C_i \to X$ that is not a trivial strip.*

Proof. The statement of the Lemma follows from the fact that the components of the union may be translated separately. Suppose that $u : C \to \mathbb{R} \times Z$ is a configuration given as the disjoint union of u_1 and u_2 with connected components and $\lambda \in \mathbb{R}$ is scalar. The union of u_1 and λu_2 is a union of punctured disks, necessarily isomorphic to u since u is rigid. Thus there exists a scalar $\mu(\lambda)$ and an automorphism $\psi : C \to C$ restricting to automorphisms

$$\psi_1 := \psi|_{C_1} : C_1 \to C_1, \quad \psi_2 := \psi|_{C_2} : C_2 \to C_2$$

such that

$$\mu(\lambda) u_1 \circ \psi_1 \sqcup (\mu(\lambda) + \lambda) u_2 \circ \psi_2 = u_1 \sqcup u_2.$$

Suppose $\lambda \neq 0$, and without loss of generality $\lambda + \mu(\lambda)$ is non-zero. Then

$$(\lambda + \mu(\lambda)) u_2 = u_2 \circ \psi_2^{-1}.$$

This equality implies that u_2 has an automorphism group of positive dimension. Necessarily, C_2 is a disk with two punctures, one incoming and one outgoing, and u_2 maps C_2 to a single fiber of the projection $\mathbb{R} \times Z \to Y$. Thus u_2 is a trivial strip. By the stability condition, not all the components are trivial strips, so u_1 must not

be a trivial strip. The case of maps with more than two connected components in the domain is similar. □

Proposition 3.13. *The map δ satisfies the Leibniz rule*

$$\delta(\gamma_1\gamma_2) = \delta(\gamma_1)\gamma_2 + (-1)^{\deg_{\mathbb{Z}_2}(\gamma_1)}\gamma_1\delta(\gamma_2).$$

Proof. Since each connected component C_i of the domain C is assumed to have one incoming end $e \in \mathcal{E}(S_i)$, any configuration $u : C \to X$ with two inputs γ_1, γ_2 is the disjoint union of two components $u_1 : C_1 \to X$ and $u_2 : C \to X$ each with a single incoming end. Exactly one of these is a trivial strip by Lemma 3.12. The statement of the Proposition follows. □

Theorem 3.14. *With (Z, Λ) and $\underline{P}$ as above, $\delta^2 = 0$.*

Proof. By the description of the true boundary components of the moduli space of buildings $u \in \mathcal{M}_\Gamma(\Lambda)$ in [4, Theorem 4.20], boundary points in the one-dimensional moduli spaces correspond to treed configurations with two levels $u = (u_1, u_2)$. The assumptions imply that $\mathbb{R} \times Z$ is tame by [4, Lemma 3.22]. Moreover, [4, Lemma 3.23] and [4, Lemma 3.24] imply that each component $u_v : C_v \to \mathbb{X}$ of u is a disk with at least one incoming Reeb chord. Rigidity then implies that in each level u_i, exactly one connected component u_v is not a trivial strip, by Lemma 3.12. Thus u_v has exactly one incoming puncture labelled by a Reeb chord γ_- and some non-negative number of outgoing punctures. To check signs, (the sign computation for the case of non-degenerate Reeb chords was carried out in Ekholm-Etnyre-Sullivan [15]) let

$$\delta_d(\gamma_0; \gamma_1, \ldots, \gamma_d) \in \hat{G}(\Lambda)$$

be the coefficient of $\gamma_1 \ldots \gamma_d$ in $\delta_d(\gamma_0)$. The coefficient of $\gamma_1 \ldots \gamma_d$ in $\delta^2(\gamma_0)$ is

$$\sum_{e+f=d, \gamma_0'} (-1)^{\deg_{\mathbb{Z}_2}(\gamma_1)+\ldots+\deg_{\mathbb{Z}_2}(\gamma_e)} \delta_{d-e+1}(\gamma_0, \gamma_1, \ldots, \gamma_e, \gamma_0', \gamma_{e+f+1}, \ldots, \gamma_d) \delta_e(\gamma_0'; \gamma_{e+1}, \ldots \gamma_{e+f}) = 0. \tag{29}$$

This identity is the A_∞ relation for the Lagrangian projection $\Pi \subset Y$ and follows by the same computation; the most standard reference would be Seidel's book [33] while our particular conventions are detailed in Charest-Woodward [7]. □

Definition 3.15. The contact homology of a compact, spin Legendrian Λ in a compact circle-fibered contact manifold (Z, α) is

$$HE(\Lambda) = \frac{\ker(\delta)}{\mathrm{im}(\delta)}.$$

Lemma 3.16. *The differential δ has degree -1 with respect to any of the gradings (over $\mathbb{Z}_2$, $\mathbb{Z}$, or $\mathbb{R}$) in Definition 3.7 that may be defined. In particular, in the case that $\Pi \subset (Y, -\mathrm{d}\alpha)$ is monotone, the homology $HE(\Lambda)$ inherits an $\mathbb{R}$-grading and $\mathbb{Z}_2$-grading from $CE(\Lambda)$.*

Proof. Suppose $u : C \to X$ is a rigid punctured disk. By [4, Lemma 3.15] the index is given by

$$\begin{aligned}(\pi)^{-1} \sum_{e \in \mathcal{E}_+(S)} (\tau\theta_e - 1) - (\pi)^{-1} \sum_{e \in \mathcal{E}_-(S)} (\tau\theta_e - 1) &= (d-2) + 2\tau \int_S u_Y^* \omega_Y \\ &= (d-2) + I(u_Y) = 1.\end{aligned}$$

It follows that the degrees of the generators are related by

$$\sum_{e \in \mathcal{E}_+(S)} \deg_{\mathbb{R}} \gamma_e = \sum_{e \in \mathcal{E}_-(S)} \deg_{\mathbb{R}} \gamma_e - 1.$$

The claim for the real degree follows. The claim for the $\mathbb{Z}_2$-degree is the same as in the Fukaya algebra case, and left to the reader. □

Lemma 3.17. *Let (Z, Λ) be a tame pair as above. If $CE(\Lambda)$ is defined over the group ring $H_1(\Lambda)$ resp. $H_2(Z, \Lambda)$ then the contact differential δ is independent, up to isomorphism, of the choice of capping paths resp. capping disks.*

Proof. Given two choices of capping paths

$$\hat{\gamma}_b : [0,1] \to \Lambda, \quad \hat{\gamma}'_b : [0,1] \to \Lambda, \quad b \in \{0,1\}$$

let

$$\delta : CE(\Lambda) \to CE(\Lambda), \quad \delta' : CE(\Lambda) \to CE(\Lambda)$$

be the corresponding differentials. The difference between the capping paths is a collection of loops

$$i_b(\gamma) : S^1 \to \Lambda, \quad i := \hat{\gamma}_b^{-1} \# \hat{\gamma}'_b.$$

Given a disk $u : C \to X$ contributing to δ and δ', the homology classes $[\partial u]$ and $[\partial u']$ are related by

$$[\partial u]' = [\partial u] \prod_{\gamma} i(\gamma)$$

where the product is over Reeb chords at the punctures of u. It follows that the map

$$CE(\Lambda) \to CE(\Lambda), \quad \gamma \mapsto i_0(\gamma)^{-1} i_1(\gamma) \gamma$$

(with products taken as delta functions in the group ring) intertwines the differentials δ and δ'. The case of coefficient ring the group ring on $H_2(Z, \Lambda)$ is similar. □

We next compute some of the lowest order contributions to the differential. In particular, there are contributions to the differential from zero area disks with no punctures, which already arise in the Morse A_∞ algebra of the Legendrian. Recall from (25) that the *classical sector* $CE_\bullet(\Lambda) \subset CE(\Lambda)$ is generated by words $\mathcal{W}_\bullet(\Lambda)$ on critical points of the Morse function on the Legendrian.

Lemma 3.18. *The differential δ preserves the classical sector $CE_\bullet(\Lambda)$. Furthermore, for any generator $\gamma \in \mathcal{I}_\bullet(\Lambda)$ the image $\delta(\gamma) \in CE(\Lambda)$ is equal to the Morse boundary $\delta_{\mathrm{Morse}}(\gamma)$ plus multiples of words $w \in \mathcal{W}(\Lambda)$ of length at least two.*

Proof. By [4, Corollary 3.16], any non-constant punctured disk has at least one incoming puncture. Therefore the disks $u : C \to \mathbb{R} \times Z$ contributing to $\delta|CE_\bullet(\Lambda)$ are all constant, and have no outgoing punctures. Because of stability, each disk component $u_v : S_v \to \mathbb{R} \times Z$ must have at least three adjacent edges $T_e \subset C$, so u must have at least one outgoing edge. If there is a single outgoing edge, then u consists of a Morse trajectory $u : C \cong \mathbb{R} \to \mathbb{R} \times Z$ and no disks. Thus u contributes to the Morse differential δ_{Morse}. Otherwise, there are at least two outgoing edges T_{e_1}, T_{e_2} and the output $\underline{\gamma}_+ \in \mathcal{W}(\Lambda)$ is a word of length at least two. □

Example 3.19. Suppose that $\Lambda = T^2$ is the two-torus with its standard Morse function given by the sum of the height functions on each of its circle factors. Let $\mathfrak{b}_\bullet \in \mathcal{I}_\bullet(\Lambda)$ be the unique degree one (Morse degree two) generator. The Morse differential $\delta_{\text{Morse}}(\mathfrak{b}_\bullet) = 0$. On the other hand, let

$$\mathfrak{c}_{\bullet,1}, \mathfrak{c}_{\bullet,2} \in \mathcal{I}_\bullet(\Lambda)$$

be the degree zero (Morse degree one) generators. The unstable manifolds for $\mathfrak{c}_{\bullet,1}, \mathfrak{c}_{\bullet,2}$ intersect in a unique point, contained in the (dimension two) stable manifold of $\mathfrak{b}_\bullet$ for generic choices of Morse datum. As a result, the leading order terms in $\delta(\mathfrak{b}_\bullet)$ (in the sense of lowest area and fewest outputs) are

$$\delta(\mathfrak{b}_\bullet) = \pm\mathfrak{c}_{\bullet,1}\mathfrak{c}_{\bullet,2} \mp \mathfrak{c}_{\bullet,2}\mathfrak{c}_{\bullet,1} + \dots.$$

Let $\mathfrak{a}_\bullet$ denote the degree -1 (Morse degree zero) generator in $\mathcal{I}_\bullet(\Lambda)$. Constant disks to $\mathfrak{b}_\bullet$ with two outputs labelled $\mathfrak{a}_\bullet, \mathfrak{b}_\bullet$ and input labelled $\mathfrak{b}$ are rigid and contribute

$$\delta(\mathfrak{b}_\bullet) = \dots + \mathfrak{b}_\bullet\mathfrak{a}_\bullet + \mathfrak{a}_\bullet\mathfrak{b}_\bullet + \dots.$$

The higher order terms depend on choices of orientations and the additional terms depend on the particular perturbation scheme (for example, $\mathfrak{a}_\bullet$ may be dual to a strict unit in the Morse A_∞ algebra of Λ, or not).

The structure coefficients of the differential on the classical sector for Λ are equal to structure coefficients of the Morse A_∞ algebra for the Lagrangian Π. Since self-intersections of cycles are not transverse, these structure coefficients depend on the perturbations chosen.

Remark 3.20. The differential induces a *reduced differential* $\delta_\circ$ on the space of chains $CE_\circ(\Lambda)$ generated by words on $\mathcal{I}_\circ(\Lambda)$ corresponding to Reeb chords by omitting the classical generators in the definition of $\mathcal{W}(\Lambda)$ in (23). Define

$$HE_\circ(\Lambda) = \frac{\ker(\delta_\circ)}{\operatorname{im}(\delta_\circ)}.$$

Although there is a well-defined homology theory corresponding to allowing Reeb generators only, cobordisms such as the Harvey-Lawson filling do not in general define chain maps for this theory, at least directly and we will not have any use for the homology theory $HE_\circ(\Lambda)$ in this paper.

3.4. **Divisor insertions.** In this section, we study disks with edges mapping to degree one critical points. The basic phenomenon is the following: Consider the situation that all holomorphic disks are made regular by some domain-independent almost complex structure.

Definition 3.21. Let $u : C \to X$ be a rigid configuration with such a leaf T_e. For any integer $k \geq 1$, define a new configuration $u_{(k)} : C_{(k)} \to X$ by replacing T_e with the union of a disk S_v and a collection $T_{e_1}, \ldots, T_{e_k}$ of k copies of T_e:

$$C_{(k)} = C_{(k),z_1,\ldots,z_k} = (C - T_e) \cup (T_{e_1} \cup \ldots \cup T_{e_k})$$

with attaching points chosen at some points $z_1, \ldots, z_k$ on the boundary of S_v. There is A configuration $u_{(k)}$ obtained from a configuration u by replacing an edge T_e by k-copies, attached to a constant disk S_v is said to be *obtained repeating inputs.*

The configurations described by the Definition are typically not regular. Indeed for $k \geq 2$ the configuration $u_{(k)}$ lies in a stratum of dimension at least $k-2$ more than that containing u, which is not equal to the expected dimension.

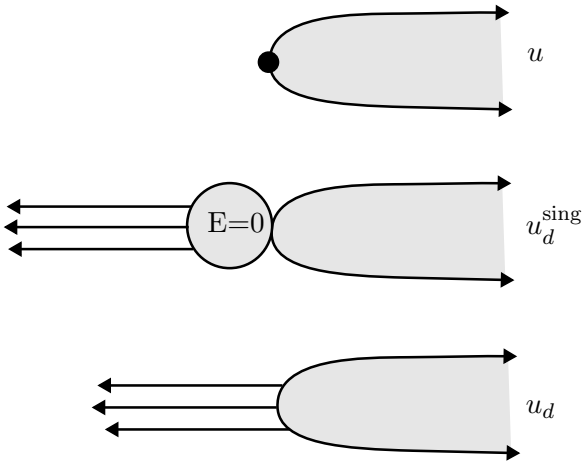

FIGURE 2. Adding multiple trajectories at an intersection point

The "canonical" way of constructing perturbations, as in the divisor equation in the theory of Fukaya algebras, is to construct the perturbations in a symmetric way so that the contribution of such configurations is an inverse factorial. The simplest case of the Theorem which we wish to prove is the following:

Theorem 3.22. *Suppose (X, L) is a tame cobordism pair, $u : C \to X$ is a rigid holomorphic treed disk bounding L with weight $\mathrm{wt}(u)$, one incoming edge labelled $\mathfrak{a}$ and no outgoing edges as above. There exist coherent perturbations so that, if Σ intersects ∂u once positively, then the coefficient of $\mathfrak{c}^d$ in $\delta(\mathfrak{a})$ is $1/d!$.*

More generally we consider the following situation where various intersections with codimension one cycles are repeated Let

$$\Sigma_1^s, \ldots, \Sigma_l^s \subset \Lambda$$

be the stable manifolds of the index one critical points $x_1, \ldots, x_l$ corresponding to one generators

$$\mathfrak{c}_1, \ldots, \mathfrak{c}_l \in \mathcal{I}(\Lambda).$$

Suppose $u : C \to X$ is a reduced configuration meeting these codimension one manifolds transversally in cyclic order $\Sigma^s_{i_1}, \ldots, \Sigma^s_{i_r}$ around the boundary. Let

$$d = (d_1, \ldots, d_r) \in \mathbb{Z}^r$$

a sequence of integers representing the number of repetitions of the labels $\mathfrak{c}_{i_1}, \ldots, \mathfrak{c}_{i_r}$ corresponding to the intersection points. Suppose u' has d_j leaves labelled x_j nearby these intersection points for $j = 1, \ldots, r$. The more general version of Theorem 3.22 which allows for several intersections is the following:

Theorem 3.23. *Let (X, L) be a tame cobordism pair. Suppose that for vanishing perturbations, every reduced configuration is regular and meets the codimension one stable manifolds $\Sigma^s_{x_1}, \ldots, \Sigma^s_{x_l}$ corresponding to degree zero generators $x_1, \ldots, x_l$ of $\mathcal{I}(\Lambda_+)$ transversally. There exist regular perturbations P_Γ satisfying the conditions in [4, Theorem 4.13] (coherent, stabilizing, and regular) for which for all collections $\underline{\gamma}_+$ of degree one there exists a forgetful map*

$$\bigcup_{\Gamma \to \Gamma_0} \mathcal{M}_\Gamma(L, \gamma_-, \underline{\gamma}_+) \to \mathcal{M}^{\mathrm{red}}_{\Gamma_\circ}(L, \gamma_-), \quad u \mapsto u_\circ$$

for the corresponding type $\Gamma_\circ$ with no outputs which has weighted count over each fiber given by

$$\mathrm{wt}(u_{(d)}) = \prod_{j=1}^{r} (\pm 1)^{d_j} (d_j!)^{-1} \, \mathrm{wt}(u) \tag{30}$$

with sign $+1$ resp. -1 if ∂u intersects $\Sigma^s_{x_j}$ positively resp. negatively.

Proof. Consider the following averaging procedure. Let $\underline{P} = (P_\Gamma)$ be a generic perturbation obtained by perturbing the Morse functions on the edges T_{e_j} that are semi-infinite. The stable manifold of x_j intersects each boundary $u|\partial C$ at a collection of distinct points

$$z_1, \ldots, z_{d_j} \in \partial C$$

which may not be in cyclic order around the boundary. From such a perturbation and a choice of permutation $g \in \Sigma_{d_j}$ of d_j letters, one obtains a new perturbation gP_Γ by pulling back P_Γ.

For a perturbation system given by such an averaging procedure, the weight of each configuration is given by an inverse factorial. Indeed, the intersection points $z_1, \ldots, z_{d_k} \in \partial S$ will lie in cyclic order around the boundary for exactly one of these perturbations, hence the factorial $(d_j!)^{-1}$ in the statement of the Proposition. The determinant lines are isomorphic except from the sign $(\pm 1)^{d_j}$ arising from the intersection of ∂u with the oriented stable manifold for x_j. The sign is obtained by definition of the orientation as induced from enforcing the matching condition at the intersection of T_{e_j} with the surface part S; since the matching condition is cut out by the diagonal, the sign is positive exactly if the intersection is positive.

The fibers of the forgetful map consist of ways of ordering the intersection points with the hypersurfaces. For each intersection z of $u_\circ(\partial S_\circ)$ with $\Sigma^s_{x_j}$, there may be a collection of boundary leaves mapping to perturbations of $\Sigma^s_{x_j}$; or similarly for the intersections of $u_\circ(T_\circ)$ as in Figure 3.

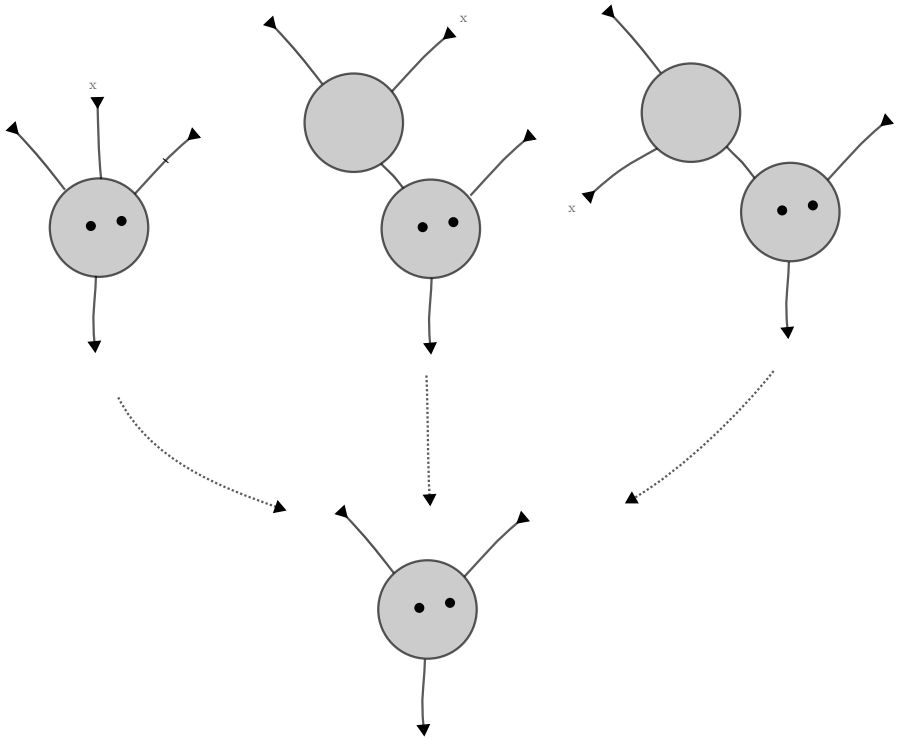

FIGURE 3. Examples of the forgetful map

In the latter case, the forgetful map collapses a constant disk S_v at the intersection point with S_v adjacent to exactly three edges of T. Configurations of the latter type come in pairs, with two configurations $u_+, u_- : C \to X$ related by switching the order of the incoming edges T_e at S_v. The intersection number of the corresponding unstable manifold $\Sigma^s_{x_j}$ with the edge $u(T_e)$ of these configurations is opposite, and so these configurations cancel. □

Example 3.24. We partially compute the differential of the Clifford Legendrian Λ_{Cliff}. The projections of the disks $u_1, \dots, u_n$ contributing to the differential are of the form (3) and permutations. The disk potential is

$$W(\tilde{y}_1, \dots, \tilde{y}_n) = \tilde{y}_1 + \dots + \tilde{y}_n$$

subject to the relation $\tilde{y}_1 \dots \tilde{y}_n = 1$ in coordinates $y_1, \dots, y_n$ on $\text{Rep}(\Pi)$. The tori Π, Λ may be identified with the lattice quotients

$$\Pi = \Delta / \left(\Delta \cap \{(\lambda_1, \dots, \lambda_n) \in (\mathbb{Z}/n)^n, \lambda_i - \lambda_j \in \mathbb{Z} \ \forall i, j\}\right)$$

where Δ is the subgroup given by

$$\Delta = \{(\lambda_1, \ldots, \lambda_n) \in \mathbb{R}^n, \lambda_1 + \ldots + \lambda_n = 0\}$$

and

$$\Lambda = \Delta/\Delta_0, \quad \Delta_0 := \{\lambda \in \mathbb{Z}^n, \lambda_1 + \ldots + \lambda_n = 0\}.$$

The boundaries of the disks $u_1, \ldots, u_n$ give paths of the form

$$\partial u_1, \ldots \partial u_n : [0,1] \to \Lambda, \quad \begin{matrix} t \mapsto t((n-1)/n, -1/n, \ldots, -1/n) \\ \ldots \\ t \mapsto t(-1/n, \ldots, -1/n, (n-1)/n) \end{matrix} \mod \Delta_0.$$

Let the capping path $\hat{\gamma}$ be the reverse of the first path ∂u_1 above. The capped-off boundaries $\partial u_1^{-1} \# \partial u_i$ are then homotopic to the loops given by

$$\partial u_1^{-1} \# \partial u_i : \ S^1 \to \Lambda, \quad t \mapsto 1 + t(-1, 1, 0, \ldots, 0) + \ldots + t(-1, 0, \ldots, 1).$$

The non-trivial paths in this expression give a basis $\mu_1, \ldots, \mu_{n-1}$ for the homology $H_1(\Lambda)$. Let

$$\mathfrak{a} \in \mathcal{R}(\Lambda), \quad \deg_{\mathbb{R}}(\mathfrak{a}) = \frac{n}{\pi}\frac{2\pi}{n} - 1 = 1$$

denote the Reeb chord of length $2\pi/n$ that is a top degree critical point of $f_\circ$. Let

$$\mathfrak{c}_i \in CE(\Lambda), \quad \deg_{\mathbb{R}}(\mathfrak{c}_i) = 0$$

denote the critical point of $f_\circ$ representing the Morse cycle corresponding to μ_i, for $i = 1, \ldots, n-1$. Denote the stable and unstable manifolds

$$\Sigma_i^s, \Sigma_i^u \subset \Lambda, \quad \dim(\Sigma_i^s) = \dim(\Lambda) - 1, \quad \dim(\Sigma_i^u) = 1$$

using the standard Morse function and metric. Each Σ_i^s is the product of factors in Λ except the i-th factor, as in (4). The i-th loop in Λ has intersection number with the codimension one submanifold Σ_i^s equal to one, and none of the other submanifolds $W_j^s, j \neq i$. As in Theorem 3.35 below, the repeating inputs Theorem 3.23 gives

$$\delta(\mathfrak{a}) = \left(\pm 1 + \sum_{i=1}^n \pm y_i, \right) + \ldots \in CE(\Lambda), \quad y_i = [\mu_i] \exp(\mathfrak{c}_i) \tag{31}$$

where the additional contributions ... arise from configurations lifting some disks in the base that are not Maslov index two; these could be constant disks lifting to punctured disks in a fiber, or non-constant disks of higher Maslov index with higher-codimension constraints at the leaves. See Dimitriglou-Rizell-Golovko [12, Section 6] for computations using Legendrian isotopy and Morse flow trees. We do not discuss here any computation of linearized or bilinearized contact homology groups; it's natural to wonder whether there is a relationship between the contact homology groups and the homology of the fillings as in Ekholm [14].

3.5. **Abelianization.** In this section we show that many of the zero-area terms in the differential may be arranged to vanish after passing to the abelianization.

Definition 3.25. Denote by

$$\delta^{\mathrm{ab}} : CE^{\mathrm{ab}}(\Lambda) \to CE^{\mathrm{ab}}(\Lambda)$$

the map induced by identifying words equal up to ordering. Let

$$\delta^{\mathrm{ab},0} : CE_0^{\mathrm{ab}}(\Lambda) \to CE_0^{\mathrm{ab}}(\Lambda)$$

denote the composition of δ^{ab} with the projection on the space of words in the degree zero generators (that is, projecting out words whose total degree is zero but whose individual letters may have non-zero degree.)

We construct perturbations which reflect the symmetry groups of the underlying trees. Given a treed disk C of some type Γ containing an surface component S_v and incoming edges $T_{e_1}, \ldots, T_{e_k}$ incident to S_v, consider the homeomorphism from T to itself given by re-ordering $T_{e_1}, \ldots, T_{e_d}$. This operation induces an isomorphism of the one-dimensional part of the universal curve.

$$\sigma_\Gamma : \ \mathcal{T}_\Gamma \to \mathcal{T}_\Gamma$$

defined in [4, Section 4.3]. Let σ_Γ be the identity on $\mathcal{S}_\Gamma$.

Definition 3.26. A multivalued perturbation system $\underline{P} = (P_\Gamma)$ will be called *invariant* if for each pair of types Γ', Γ as above, P_Γ is pulled back from $P_{\Gamma'}$ under $\sigma_{\Gamma,\Gamma'}$.

Lemma 3.27. *The conclusion of [4, Theorem 4.13] holds, that is, coherent, regular perturbations exist with the added condition that the perturbations are invariant.*

The proof is a repeat of the previous arguments with the symmetry condition imposed, and omitted.

Lemma 3.28. *Let (Z, Λ) be a tame pair. For invariant perturbations $\underline{P}$, configurations $u : C \to X$ with a leaf T_{e_i} labelled by a degree one generator $\mathfrak{c}_i$ attached to a constant disk S_v do not contribute to the abelianized differential $\delta^{\mathrm{ab}} : CE^{\mathrm{ab}}(\Lambda) \to CE^{\mathrm{ab}}(\Lambda)$.*

Proof. Any such configuration produces another configuration $u' : C' \to X$ of type Γ', with opposite sign, by changing the order of leaves at the S_v. Such contributions cancel by the invariance assumption. □

We introduce notation for the coefficients of the differential. In [4, Section 3.3] we defined domain type Γ and map type $\mathbb{\Gamma}$ for treed maps.

Definition 3.29. Let $\delta_{\mathbb{\Gamma}}(\gamma)$ is the contribution to δ from maps with domain type Γ so that

$$\delta(\gamma) = \sum_{\mathbb{\Gamma}} \delta_{\mathbb{\Gamma}}(\gamma).$$

For $\gamma_1, \ldots, \gamma_d$ let

$$\delta_d(\gamma_0; \gamma_1, \ldots, \gamma_d) \in \hat{G}(\Lambda)$$

denote the coefficient of $\gamma_1 \dots \gamma_d$ in $\delta_d(\gamma_0)$. For any permutation σ let

$$(-1)^{\deg_{\mathbb{Z}_2}(\sigma,\gamma_1,\dots,\gamma_d)} \in \{+1,-1\}$$

denote the Koszul sign of the expression $\gamma_{\sigma(1)} \dots \gamma_{\sigma(d)}$ with respect to $\gamma_1 \dots \gamma_d$, that is the product of the signs $(-1)^{\deg_{\mathbb{Z}_2}(\gamma_i)\deg_{\mathbb{Z}_2}(\gamma_{i+1})}$ in a decomposition of σ into transpositions. We say that δ is *classically commutative* for any γ a degree two generator in the classical sector $CE_\bullet(\Lambda)$, For any type Γ any automorphism $\sigma \in \mathrm{Aut}(\Gamma)$ and any classical generators $\gamma_1, \dots, \gamma_d$

$$\delta_{\mathbb{\Gamma}}(\gamma; \gamma_{\sigma(1)}, \dots, \gamma_{\sigma(d)}) = (-1)^{\deg_{\mathbb{Z}_2}(\sigma,\gamma_1,\dots,\gamma_d)} \delta_{\mathbb{\Gamma}}(\gamma; \gamma_1, \dots, \gamma_d).$$

Proposition 3.30. *Let (Z,Λ) be a tame pair, $\underline{P} = (P_\Gamma)$ a coherent, regular system of invariant perturbations. Then δ is classically commutative.*

Proof. Given a type Γ, any tree automorphism $\sigma \in \mathrm{Aut}(\Gamma)$ induces an automorphism

$$\mathcal{T}_\Gamma(\sigma) : \overline{\mathcal{T}}_\Gamma \to \overline{\mathcal{T}}_\Gamma$$

of the universal tree $\overline{\mathcal{T}}_\Gamma \to \overline{\mathcal{M}}_\Gamma$. By assumption σ then induces a bijection between zero-area moduli spaces[1]

$$\mathcal{M}(\Lambda, \gamma_0, \gamma_{\sigma(1)}, \dots, \gamma_{\sigma(d)}) \to \mathcal{M}(\Lambda, \gamma_0, \gamma_1, \dots, \gamma_d).$$

For Γ so that each vertex has valence three, the induced action of $\sigma_{i(i+1)}$ on $\mathcal{M}_\Gamma$ is trivial since it preserves the edge lengths. The orientation $\mathcal{M}_\Gamma(\Lambda)$ permutes the determinant lines associated to the leaves and root edge, and so changes the orientation by the claimed Koszul sign. □

Corollary 3.31. *For $\gamma \in \mathcal{I}_\bullet(\Lambda)$ of degree two, the projection of $\delta(\gamma)$ on $CE^{\mathrm{ab}}(\Lambda)$ (identifying words equal up to permutation) is the classical boundary $\delta_{\mathrm{Morse}}(\gamma) = \delta(\gamma)$ plus terms involving words with at least one Morse-degree-zero generator.*

Proof. Since the moduli space of trees with d leaves and 1 root has dimension $d-2$, the constrained moduli space can be rigid only if the output is of the form $\gamma_1 \dots \gamma_d \in \mathcal{I}(\Lambda)^d$ for some $\gamma_1, \dots, \gamma_d$ of total Morse degree d. If all of $\gamma_1, \dots, \gamma_d$ are not Morse degree 1, then at least one has Morse degree zero, since the average Morse degree is one. For each such output arising from a tree Γ, choose edges e_i, e_{i+1} adjacent to the same vertex. By the previous Proposition 3.30, the contribution of maps with domain type Γ is skew-symmetric under the transposition σ_i, σ_{i+1}. So the contribution of such maps disappears after abelianization. □

Example 3.32. We continue the discussion of the differential for the Clifford Legendrian in Example 3.24. For degree reasons, the output of $\delta^{\mathrm{ab},0}$ consists of a linear

[1] We remark that this action cannot in general extend to the positive-area moduli spaces, because in that case the matching conditions at the intersection of the tree and surface parts is not preserved. Such an action, if it existed, would imply that Lagrangian Floer homology is always graded commutative. However, by mirror symmetry it is expected that there are Lagrangians corresponding to higher rank bundles, and these typically have non-commutative endomorphism algebras.

combination of words $\underline{\gamma}_-$ in the degree one generators $\mathfrak{c}_1, \ldots, \mathfrak{c}_{n-1}$. The coefficients of these are determined by Theorem 3.23 to be inverse factorials. Hence

$$\delta^{\mathrm{ab},0}(\mathfrak{a}) = \left(\pm 1 + \sum_{i=1}^{n} \pm y_i, \quad y_i = [\mu_i] \exp(\mathfrak{c}_i) \right) \in CE^{\mathrm{ab}}(\Lambda) \tag{32}$$

in coordinates $y_1, \ldots, y_n$ on $\mathrm{Rep}(\Lambda)$.

In the case of the Clifford Legendrian $\Lambda \cong T^2$ in $Z = S^5$, the disks in Example 2.1 lifting the Maslov index two disks in $Y = \mathbb{C}P^2$ give rise to the leading order terms in the differential as in (31)

$$\delta^{\mathrm{ab},0}(\mathfrak{a}) = 1 \pm y_1 \pm y_2.$$

On the other hand, one may also take Z to be the unit anti-canonical bundle. The projection from Λ to Π is an isomorphism obtains using the capping path corresponding to the monomial $y_1 y_2$

$$\delta^{\mathrm{ab},0}(\mathfrak{a}) = 1 \pm y_1^2 y_2 \pm y_1 y_2^2 + \ldots.$$

corresponding to the shifted disk potential for the Clifford torus.

We now compute the leading order term in the abelianized differential for connected horizontal lifts of monotone tori. Let Z be a negative bundle over a monotone symplectic manifold Y and Λ a connected lift of a monotone Lagrangian torus Π. The natural map $\Lambda \to \Pi$ induces a map $\pi_1(\Lambda) \to \pi_1(\Pi)$. By pull-back, the map on fundamental groups induces a map on representation varieties

$$\mathrm{Rep}(p) : \mathrm{Rep}(\Pi) \to \mathrm{Rep}(\Lambda).$$

If $W : \mathrm{Rep}(\Pi) \to \mathbb{C}$ is a function that is the pull-back of a function W' on $\mathrm{Rep}(\Lambda)$ then we write

$$W' = \mathrm{Rep}(p)_* W : \mathrm{Rep}(\Lambda) \to \mathbb{C}.$$

Let W_Π be the disk potential of Π and choose $v \in H_1(\Pi)$ such that x^v corresponds to a vertex in the Newton polytope of W. Choose a capping path to be equal to the lift of a path representing v, and consider the Laurent polynomial

$$x^{-v} W_\Pi \in \mathbb{C}[\mathrm{Rep}(\Pi)].$$

This choice of capping path ensures that the Newton polytope of W_Π has 0 as a vertex. Since the Newton polytope is convex, we can choose a basis of $H_1(\Lambda)$ to ensure that the Newton polytope of W_Π lies in the positive cone generated by the basis, see [5, Figure 1] for the case when H_1 is two-dimensional, and so in the corresponding coordinates $x^{-v} W_\Pi$ is a polynomial.

Lemma 3.33. *If u_1, u_2 are Maslov index two disks in Y bounding Π, then the boundary difference $[\partial u_1] - [\partial u_2] \in \pi_1(\Pi)$ lies in the image of $\pi_1(\Lambda)$.*

Proof. Let u_1, u_2 be as in the statement of the Lemma. The monodromy of Z around $[\partial u_1], [\partial u_2]$ is equal, by the monotonicity relation, hence $[\partial u_1] - [\partial u_2]$ lifts to an element in $\pi_1(\Lambda)$. □

Corollary 3.34. *Let $\Lambda \to \Pi$ be a Legendrian covering a monotone Lagrangian Π as above. The function $x^{-v}W_\Pi$ descends to a function on $\mathrm{Rep}(\Lambda)$ denoted*

$$W_\Lambda := \mathrm{Rep}(p)_*(x^{-v}W_\Pi) \in \mathbb{C}[\mathrm{Rep}(\Lambda)]. \tag{33}$$

We call W_Λ the augmentation polynomial for the Legendrian Λ.

We now relate the potential to the leading order term in the differential. Let

$$\mathfrak{a} \in \mathcal{I}_\circ(\Lambda), \quad \deg_{\mathbb{R}}(\mathfrak{a}) = 1$$

denote the degree one generator representing a point in the space of Reeb chords $\mathcal{R}(\Lambda)$ with minimal angle change $2\pi/\tau$, where τ is the monotonicity constant.

Theorem 3.35. *Let (Z, Λ) be a tame pair as above. The differential of the degree one generator $\mathfrak{a} \in CE_1(\Lambda)$ is*

$$\delta^{\mathrm{ab},0}(\mathfrak{a}) = W_\Lambda([\mu_1]\exp(\mathfrak{c}_1), \dots, [\mu_k]\exp(\mathfrak{c}_k)). \tag{34}$$

Proof. We will show that the disks contributing to the abelianized, words-in-degree-zero part of the differential arise from lifts of Maslov index two disks with a single puncture in the lift. Since the incoming angle from $\mathfrak{a}$ is minimal, [4, Lemma 3.15] implies that any contribution to $\delta(\mathfrak{a})$ has a single puncture. The projection to Y must be a disk bounding Π with Maslov index two by monotonicity. Conversely, by Theorem 2.9 any Maslov two disk $u_Y : S \to Y$ bounding Π lifts to a punctured disk in $\mathbb{R} \times Z$ bounding $\mathbb{R} \times \Lambda$ with incoming punctured labelled by $\mathfrak{a} \in \mathcal{I}_\circ(\Lambda)$. The contribution of each disk is now governed by Theorem 3.23 which gives an appearance of coefficient $\exp(\langle \mathfrak{c}, [\partial u]\rangle)$ from the intersection with the divisor cycles. □

Example 3.36. For the standard lift Λ of the torus Π in $\mathbb{P}^1 \times \mathbb{P}^1$ (also arising as conormal for the unknot in [17]) there is a single Reeb chord $\mathfrak{a}$ of index one with

$$\delta^{\mathrm{ab},0}(\mathfrak{a}) = (1 \pm y_1 \pm y_2 \pm y_1 y_2)$$

which is the knot contact differential for the unknot consider in, for example, Aganagic-Ekholm-Ng-Vafa [1, p. 39].

Example 3.37. We partially compute the differential for the generalized Hopf Legendrian $\Lambda \subset S^{2n-1}$ in (5) and Example 2.5. Let

$$\mathfrak{a}_{11}, \mathfrak{a}_{22} \in \mathcal{I}(\Lambda), \quad \deg_{\mathbb{R}}(\mathfrak{a}_{11}) = \deg_{\mathbb{R}}(\mathfrak{a}_{22}) = 1$$

denote the point generators in the components of $\mathcal{R}(\Lambda)$ connecting Λ_1 resp. Λ_2 with angle change $2\pi/n$. Introduce coordinates by the projection given by

$$\Lambda \to T^{n-1}, \quad (z_1, \dots, z_n) \mapsto (z_1, \dots, z_{n-1}).$$

Denote the standard Morse function on $\mathcal{R}(\Lambda)$ (the sum of the height functions on each torus component) by

$$f_\circ : \mathcal{R}(\Lambda) \to \mathbb{R}.$$

This is the function that on each component of $\mathcal{R}(\Lambda)$ diffeomorphic to $\Lambda \cong T^{n-1}$ by evaluation at the starting point of the Reeb chord is given by the sum of the heights on each circle component. Denote by

$$\mathfrak{a}_{12}, \mathfrak{a}_{21} \in \mathcal{I}_\circ(\Lambda), \quad \deg_{\mathbb{R}}(\mathfrak{a}_{12}) = \deg_{\mathbb{R}}(\mathfrak{a}_{21}) = (n/\pi)(\pi/n) - 1 = 0$$

degree-zero generators in the component of $\mathcal{R}(\Lambda)$ connecting Λ_1 to Λ_2 or vice versa with angle change π/n. Denote by

$$\mathfrak{c}_{12,k}, \mathfrak{c}_{21,k} \in \mathcal{I}_\circ(\Lambda), \quad \deg_{\mathbb{R}}(\mathfrak{c}_{12,k}) = \deg_{\mathbb{R}}(\mathfrak{c}_{21,k}) = (n/\pi)(\pi/n) + 1 - 1 = 1$$

for $k = 1, \dots, n-1$ the generators corresponding to Morse-index one critical points for the standard Morse function on the k-th factor in $\Lambda \cong T^{n-1}$

We compute the leading order terms in the differential arising from disks of area zero.

Lemma 3.38. *The image of the degree one generator $\mathfrak{c}_{12,k} \in \mathcal{I}_\circ(\Lambda)$ under the abelianized differential is given by*

$$\delta^{\mathrm{ab}}(\mathfrak{c}_{12,k}) = (1 - y_{k,1}y_{k,2}^{-1})\mathfrak{a}_{12}, \quad \delta^{\mathrm{ab}}(\mathfrak{c}_{21,k}) = (1 - y_{k,2}y_{k,1}^{-1})\mathfrak{a}_{21}. \tag{35}$$

Proof. Any configuration $u : C \to \mathbb{R} \times Z$ contributing to $\delta(\mathfrak{c}_{12,k})$ has no disk components, since the angle of $\mathfrak{c}_{12,k}$ is minimal and [4, Lemma 3.15] implies that the angles at the outgoing punctures would be less than the angle at the incoming puncture. Thus u must be a Morse trajectory. There are two Morse trajectories of $f_\circ$ in $\mathcal{R}(\Lambda)$ connecting $\mathfrak{c}_{12,k}$ to $\mathfrak{a}_{12}$, corresponding to the two different ways of descending along the k-th circle factor in $\Lambda \cong T^{n-1}$ from the maximum to the minimum. To calculate the contribution of these trajectories, we must associate to each a homology class in $H_1(\Lambda)$. We may assume that the capping paths are chosen so that the homology class of the first gradient trajectory is trivial: Choose the capping path for $\mathfrak{a}_{12}$ by concatenating the capping path for $\mathfrak{c}_{12,k}$ with the gradient trajectories images in Λ_1 and Λ_2. The homology class of the second gradient trajectory then differs from the first by the difference in homology classes $\mu_{k,1} - \mu_{k,2}$. After including the contributions with outgoing edges limiting to the support of $\mathfrak{c}_i$ as in Lemma 3.23, we obtain (35).

On the other hand, the differential of $\mathfrak{a}_{11}$ includes a term

$$\delta(\mathfrak{a}_{11}) = \mathfrak{a}_{12} \otimes \mathfrak{a}_{21} + \dots$$

which corresponds to a constant map to the base $\mathbb{P}^{n-1}$ but with Reeb chord from Λ_1 to Λ_1 breaking into two pieces as in Example 2.20. Let

$$\mu_{1,b}, \dots, \mu_{n-1,b} \in H_1(\Lambda_b)$$

be the standard basis for $H_1(\Lambda_b)$ and

$$y_{i,b} = [\mu_{i,b}] \exp(\mathfrak{c}_{i,b}) \in CE^{\mathrm{ab}}(\Lambda).$$

The differential of $\mathfrak{a}_{11}, \mathfrak{a}_{22}$ is given by

$$\delta^{\mathrm{ab}}(\mathfrak{a}_{11}) = (1 - y_{1,1} - y_{2,1}) + \mathfrak{a}_{12} \otimes \mathfrak{a}_{21} + \dots \tag{36}$$

where the additional terms disappear after projection onto words in the degree zero generators. Similarly,

$$\delta^{\mathrm{ab}}(\mathfrak{a}_{22}) = (1 - y_{2,1} - y_{2,2}) + \mathfrak{a}_{21} \otimes \mathfrak{a}_{12} + \dots.$$

□

This ends the Example.

4. Chain maps from cobordisms

In this section, we use the moduli space of holomorphic maps to cobordisms defined in [4, Section 3] to associate to any cobordism L from Λ_- to Λ_+ satisfying certain "no-cap" conditions a map of differential algebras

$$CE(\Lambda_-, \hat{G}(L)) \to CE(\Lambda_+, \hat{G}(L)).$$

In particular, for tame fillings we obtain augmentations.[2]

4.1. **Cobordism maps.** Recall the following definitions of symplectic cobordism and Lagrangian cobordism as in [4, Section 2].

Definition 4.1. Fix two circle-fibered stable Hamiltonian manifolds $(Z_\pm, Y_\pm, \alpha_\pm, \omega_\pm)$. A *symplectic cobordism* from $(Z_+, Y_+, \alpha_+, \omega_+)$ to $(Z_-, Y_-, \alpha_-, \omega_-)$ is a symplectic manifold with boundary $(\tilde{X}, \partial\tilde{X}, \omega)$ whose boundary is equipped with a partition

$$\partial\tilde{X} = \partial_+\tilde{X} \cup \partial_-\tilde{X}$$

and diffeomorphisms

$$\iota_\pm : Z_\pm \to \partial_\pm\tilde{X}$$

so that

(a) the two forms satisfy the proportionality relation

$$\iota_\pm^*\omega \,|_{\partial_\pm\tilde{X}} = \lambda_\pm\omega_\pm$$

for some positive constants $\lambda_\pm \in \mathbb{R}$, and

(b) the natural isomorphism between the normal bundle of $Z_\pm$ in $\tilde{X}$ and the kernel of $Dp : TZ_\pm \to TY_\pm$ is orientation preserving resp. reversing for the positive resp. negative boundary.

Denote by X the interior of $\widetilde{X}$, viewed as a manifold with cylindrical ends, and, abusing terminology, call X the cobordism.

Definition 4.2. For a symplectic cobordism X from Z_+ to Z_- and two Legendrian submanifolds $\Lambda_\pm \subset Z_\pm$ so that X is equipped with cylindrical ends, a *Lagrangian cobordism cylindrical near infinity* from Λ_+ to Λ_- is a Lagrangian submanifold $L \subset X$ such that

$$L \cap ((\sigma_{\pm,0}, \sigma_{\pm,1}) \times Z_\pm) = (\sigma_{\pm,0}, \sigma_{\pm,1}) \times \Lambda_\pm$$

for some intervals $(\sigma_{\pm,0}, \sigma_{\pm,1}) \subset \mathbb{R}$.

We define cobordism maps as follows. Assume that $L \subset X$ is a tame compact Lagrangian cobordism from $\Lambda_- \subset Z_-$ to $\Lambda_+ \subset Z_+$ in the sense of [4, Definition 3.18] equipped with a spin structure. We assume that $\Lambda_\pm$ are equipped with Morse data as in Definition 3.4, and in particular with Morse functions

$$f_{\bullet,\pm} : \Lambda_\pm \to \mathbb{R}.$$

[2] In Ekholm-Ng [18] and similar papers, the augmentation associated to a filling is phrased in terms of a open Gromov-Witten invariants counting disks without outputs. We make no claim about existence of such invariants, but rather consider the chain map as an invariant up to chain homotopy.

To simplify notation define

$$H_2 := H_2(\tilde{X}, \tilde{L} \cup p^{-1}(\Pi), \mathbb{Z}).$$

Any punctured disk in X bounding L extends to a map from a manifold with corners $\tilde{S}$ mapping to $\tilde{X}$ and bounding $\tilde{L} \cup p^{-1}(\Pi)$, by adding in the Reeb chords at infinity, and so defines a homology class in H_2. The symplectic form $\tilde{\omega}$ vanishes on $\tilde{L} \cup p^{-1}(\Pi)$ and so induces a natural area map

$$A : H_2 \to \mathbb{R}.$$

In our examples, $H_2(\tilde{X})$ will often vanish and the long exact sequence gives an identification

$$H_2(\tilde{X}, \tilde{L} \cup p^{-1}(\Pi), \mathbb{Z}) \cong H_1(\tilde{L} \cup p^{-1}(\Pi), \mathbb{Z})).$$

The *completed group ring* $\hat{G}(L)$, similar to $\hat{G}(\Lambda)$ in (19), is

$$\hat{G}(L) = \{\, g : H_2 \to \mathbb{C}, \quad \forall A, \#\{c \in H_2 \mid g(c) \neq 0, A(c) < A\} \text{ finite} \,\}. \tag{37}$$

It admits a natural filtration

$$\hat{G}(L) = \bigcup_A \hat{G}(L)_A, \quad \hat{G}(L)_A = \{g | g(c) = 0, \forall A(c) < A\}. \tag{38}$$

Definition 4.3. A Morse datum for L is a Morse function

$$f_L : L \to \mathbb{R}, \quad f_L|_{\Pi_\pm} = f_{\bullet,\pm}$$

extending the given Morse functions $f_{\bullet,\pm} : \Pi_\pm \to \mathbb{R}$.

The set of *relative* resp. *absolute generators* is

$$\mathcal{I}(L, \partial L) := \mathrm{crit}(f_L), \quad \mathcal{I}(L) := \mathcal{I}(L, \partial L) \cup \mathcal{I}(\Lambda_-) \cup \mathcal{I}(\Lambda_+).$$

The space of *absolute* resp. *relative chains* is

$$CE(L) := \sum_{x \in \mathcal{I}(L)} \hat{G}(L) x \tag{39}$$

$$CE(L, \partial L) := \sum_{x \in \mathcal{I}(L, \partial L)} \hat{G}(L) x \tag{40}$$

given by the formal sum of generators over the ring $\hat{G}(L)$, with length filtrations going to infinity.

Define maps between Chekanov-Eliashberg algebras by counts of holomorphic disks bounding the Lagrangian cobordism; we will show that this count defines a chain map if a certain Maurer-Cartan equation is satisfied.

Definition 4.4. For integers $d_-, d_+ \geq 0$ let

$$\mathcal{M}_{d_-,d_+}(L) = \{u : C \to X | \overline{\partial} u = 0, \quad u(\partial C) \subset L\}$$

denote the moduli space of perturbed holomorphic treed disks in X bounding L with

- 1 incoming edge labelled by a generator of $\mathcal{I}(\Lambda_-)$
- d_- incoming edges labelled generators $\mathcal{I}(L, \partial L)$
- d_+ outgoing edges labelled by generators $\mathcal{I}(\Lambda_+)$.

By [4, Section 4], for generic systems of domain-dependent perturbations the components of the moduli spaces $\mathcal{M}_{d_-,d_+}(L)$ of dimension at most one are transversally cut out, equipped with orientations, and compact for any given energy bound.

Definition 4.5. Given a cobordism pair (X, L) from (Z_-, Λ_-) to (Z_+, Λ_+) and an element $b \in CE(L, \partial L)$, the *cobordism map*

$$\varphi(L, b) : CE(\Lambda_-, \hat{G}(L)) \to CE(\Lambda_+, \hat{G}(L))$$

is defined counting holomorphic treed curves $u : S \to X$ bounding L with arbitrarily many insertions of b at boundary edges, and exactly one incoming puncture in each component, as follows: For a single incoming generator

$$\gamma = \sum_{x \in \mathcal{I}(\Lambda_-)} \gamma(x) x$$

we have for

$$x_0 \in \mathcal{I}(\Lambda_-), \quad \underline{x}_+ = (x_1, \dots, x_{d_+}) \in \mathcal{I}(\Lambda_+)^{d_+}, \quad \underline{x}_\bullet = (x_{\bullet,1}, \dots, x_{\bullet,d_-}) \in \mathcal{I}(L, \partial L)^{d_-}$$

$$(\varphi(L, b))(\gamma) := \sum_{\underline{x}, u \in \mathcal{M}(L, x_0, \underline{x}_+, \underline{x}_\bullet)_0} (-1)^{\sum_{i=1}^{d_+} i|x_i|} \left(\prod b(x_{\bullet,i}) \right) \gamma(x_0) x_1 \otimes \dots \otimes x_{d_-}.$$

See Figure 4 and (27). For words

$$w = \gamma_1 \dots \gamma_d$$

define $(\varphi(L, b))(w)$ by its action on generators

$$(\varphi(L, b))(\gamma_1 \dots \gamma_d) = \varphi(L, b)(\gamma_1) \dots \varphi(L, b)(\gamma_d).$$

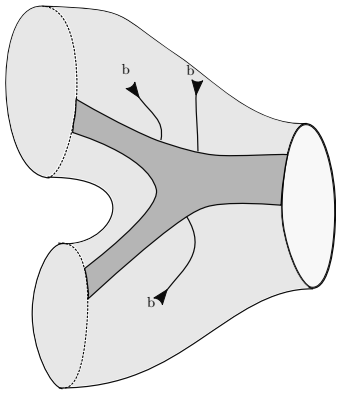

FIGURE 4. Disks defining the chain map

In the next section, we discuss what properties of the bounding cochain guarantee the cobordism map is a chain map. We first discuss the grading properties of the cobordism maps. The cobordism map

$$\varphi_L : CE(\Lambda_-) \to CE(\Lambda_+)$$

defined by a Lagrangian cobordism is $\mathbb{Z}_2$-graded, by the standard computation from Fukaya categories. The cobordism map φ_L is $\mathbb{R}$-graded under a suitable trivial-Maslov-class condition.

Corollary 4.6. *Let (X, L) be a tame cobordism from (Z_-, Λ_-) to (Z_+, Λ_+). Suppose that the trivialization of the canonical bundle of $\mathbb{R} \times Z_\pm$, agreeing with the given trivialization over $\mathbb{R} \times \Lambda_\pm$ extends over the cobordism X and b has degree one. Then the real grading on $CE(\Lambda_\pm)$ is preserved by the cobordism map $\varphi(L, b) : CE(\Lambda_-) \to CE(\Lambda_+)$.*

Proof. For strata $\mathcal{M}_\Gamma$ of maximal dimension, the dimension is

$$\dim \mathcal{M}_\Gamma = d - 2 = \sum_{e \in \mathcal{E}_+(S)} 1 - \sum_{e \in \mathcal{E}_-(S)} 1 - 1.$$

Rigidity implies that

$$\begin{aligned}
\sum_{e \in \mathcal{E}_+(S)} (\mathrm{ind}(f_{\circ,+}))(\gamma_e) + \frac{\tau\theta_e}{2\pi} \quad &- \quad 1 - \sum_{e \in \mathcal{E}_-(S)} (\mathrm{ind}(f_{\circ,-}))(\gamma_e) + \frac{\tau\theta_e}{2\pi} - 1 \\
&= \sum_{e \in \mathcal{E}_+(S)} (\mathrm{ind}(f_{\circ,+}))(\gamma_e) - \sum_{e \in \mathcal{E}_-(S)} (\mathrm{ind}(f_{\circ,-}))(\gamma_e) + (d-1) \\
&= \sum_{e \in \mathcal{E}_+(S)} (\mathrm{ind}(f_{\circ,+}))(\gamma_e) + \frac{\tau\theta_e}{2\pi} - \sum_{e \in \mathcal{E}_-(S)} (\mathrm{ind}(f_{\circ,-}))(\gamma_e) + \frac{\tau\theta_e}{2\pi} \\
&= \mathrm{Ind}(D_u) - \dim(\Lambda) - 1 \\
&= 0.
\end{aligned}$$

Thus the difference is real gradings between incoming and outgoing generators vanishes. □

Example 4.7. In our motivating examples, the canonical bundle is not typically trivial in the sense above. For example, take $Z_\pm \cong S^1$, let Λ_- be a disjoint union of two points and Λ_+ the emptyset. Let $L \cong \mathbb{R}$ be a cobordism from Λ_- to Λ_+, that is, a path joining the two points of Λ_-. One sees easily the trivialization of the canonical bundle on $\mathbb{R} \times Z_\pm$ (agreeing with the trivialization determined by the orientation on $\mathbb{R} \times \Lambda_\pm$). Clearly, there is no trivialization of the canonical bundle on $(\mathbb{R} \times Z_\pm, L)$ which extends the given trivializations on $(\mathbb{R} \times Z_\pm, \mathbb{R} \times \Lambda_\pm)$.

Example 4.8. We discuss the grading of the augmentation map associated to the exact filling of the Hopf Legendrian with a single component discussed earlier in (2.5). Let $\Lambda_{\mathrm{Hopf}} \subset S^{2n-1}$ denote the Hopf Legendrian and $L_{(2)}$ the filling in (2.5) containing punctured holomorphic disks asymptotic to Reeb chords with angle change π/n. The degrees of the corresponding generator $\mathfrak{c}_{12} \in \mathcal{R}(\Lambda), \mathfrak{c}_{21} \in \mathcal{R}(\Lambda)$ are

$$\deg_{\mathbb{R}}(\mathfrak{c}_{12}) = \deg_{\mathbb{R}}(\mathfrak{c}_{21}) = 0.$$

Depending on whether the path defining the filling $L_{(2)}$ passes below or above the critical value in the Lefschetz fibration, the augmentation $\varphi(\mathfrak{c}_{12})$ resp. $\varphi(\mathfrak{c}_{21})$ has either 1 or n terms corresponding to the disks in (9) or (10).

Example 4.9. We give an example of a filling, arising naturally from a union of paths of a symplectic fibration, which cannot preserve the grading in any natural way. Suppose $\Lambda' \subset S^{2n-1}$ is the Legendrian obtained by taking the disjoint union of Λ with $e^{i\theta}\Lambda$, where θ is close, but not equal, to $\pi/6$, for simplicity with $n = 3$. Consider the filling obtained by a path from Λ to Λ' as in Lemma 4.24. Let φ denote the corresponding augmentation. Since the filling is exact, the element $1+y_{1,1}+y_{2,2}$ maps to $1+y_1+y_2$ under φ, so we must have $\varphi(\mathfrak{c}_{12})\varphi(\mathfrak{c}_{21}) \neq 0$. On the other hand, the elements $\mathfrak{c}_{12}, \mathfrak{c}_{21}$ do not necessarily have zero grading. There seems to be no way of fixing this issue: Even if the angle φ is chosen so that $\mathfrak{c}_{12}$ and $\mathfrak{c}_{21}$ have degree zero, we could add another copy of the Clifford Legendrian at angle θ' between θ and $2\pi/3$. The differential of the Reeb generator $\mathfrak{c}_{21}$ connecting the second sheet to the first sheet passes through the third sheet, so that

$$\delta(\mathfrak{c}_{21}) = \mathfrak{c}_{23} \otimes \mathfrak{c}_{31}.$$

For degree reasons, we must have

$$\deg(\mathfrak{c}_{23}\mathfrak{c}_{31}) = \deg(\mathfrak{c}_{23}) + \deg(\mathfrak{c}_{31}) = \deg(\mathfrak{c}_{21}) - 1 = 0 - 1 = -1.$$

So not both of $\mathfrak{c}_{23}$ and $\mathfrak{c}_{31}$ can have degree zero. But we can easily find a filling, using Lemma 4.24 again, whose associated augmentation φ maps $\mathfrak{c}_{23}$ or $\mathfrak{c}_{31}$ to a non-zero element of $G(\varphi')$. This shows that there is no grading for which all of the augmentations associated to fillings of the type considered in Lemma 4.24 are graded.

Remark 4.10. In good cases, one can define a map associated to a cobordism over group ring on first homology, rather than relative second homology; this better matches conventions in earlier papers in the field. Suppose for example that (X, L) is a filling of (Z, Λ), and only holomorphic disks in (X, L) with no punctures contribute to the definition of $\varphi(L, b)$. (This could be the case for grading reasons, for example, since the Reeb chords all have positive $\mathbb{R}$-degree, so $\varphi(L, b)$ is non-vanishing only on the classical generators $\mathcal{I}_\bullet(\Lambda)$). Suppose furthermore, that X is simply-connected, so that

$$H_2(X, L) \to H_1(L)$$

is a surjection whose kernel is equal to the image $H_2(X)$ in $H_2(X, L)$. From the monotonicity condition [4, **(P2)**] of tame fillings, we have that for each $[\gamma] \in H_1(L)$ there is an upper bound on the area of rigid disks u such that $[\partial u] = [\gamma]$. Thus we can apply the usual Gromov compactness argument to show that $\varphi(L, b)$ can be defined using $H_1(L)$ coefficients.

4.2. **Bounding chains.** In this section, we study under what conditions the cobordism maps from the previous section are chain maps. Recall from [4, Theorem 4.21] that the boundary configurations in the one-dimensional components of the moduli space of curves bounding the Lagrangian include buildings with two levels and buildings with a single level with an infinite length edge. In order to obtain a chain map, contributions from the second type of map must vanish.

Following Fukaya-Oh-Ohta-Ono [21] we define a notion of Maurer-Cartan solutions as follows. The Maurer-Cartan equation counts curves with inputs arising from

chains in the interior of the cobordism, and outputs either in the interior or on the boundary, as in Figure 5. As such, it does not arise from an A_∞ algebra because of the additional breakings into levels. We first define the combinatorial types of curves that occur.

Definition 4.11. A treed building $u : C = S \cup T \to X$ bounding L is *of obstruction type* if all of the input edges $T_{e_i} \subset T$ have limits that are critical points $x(e) \in \mathcal{I}_\bullet(L)$ in the interior of L, rather than critical points on Λ or $\mathcal{R}(\Lambda)$, and one output edges T_{e_0} either asymptotic to a critical point $x(e_0)$ in the interior of L or in Λ or $\mathcal{R}(\Lambda)$.

We introduce notation for moduli spaces of holomorphic disks of obstruction type. Denote by

$$\mathcal{M}_d(L, \partial L) = \bigcup_{\mathbb{\Gamma}} \mathcal{M}_{\mathbb{\Gamma}}(L, \partial L)$$

the moduli space of treed buildings in X bounding L of obstruction types with arbitrary number d of inputs labelled by $\mathcal{I}(L, \partial L)$, and one output labelled by $\mathcal{I}(L)$ as in Figure 5. By definition, configurations $u : C \to X$ in $\mathcal{M}(L, \partial L)$ are treed disks in X bounding L not meeting Z_- and meeting Z_+ at most once. As a special kind of building, the moduli spaces of $\mathcal{M}_d(L, \partial L)$ of expected dimension at most one are compact and regular by Theorems 4.13 and 4.18 in [4].

Definition 4.12. Let (X, L) be a tame cobordism. The *higher relative composition maps*

$$m_d : CE(L, \partial L)^{\otimes d} \to CE(L) \tag{41}$$

are defined by

$$(x_d, \dots, x_1) \mapsto \sum_{u \in \mathcal{M}_e(L, x_0, \dots, x_d)_0} (-1)^\heartsuit \operatorname{wt}(u) x_d$$

where the weight $\operatorname{wt}(u)$ is defined by the expression in (28) and the sign is from (27). Let

$$CE(L, \partial L)_{>0} := \left\{ b = \sum_{x \in \mathcal{I}(L, \partial L)} b(x) x \;\middle|\; \forall x \in \mathcal{I}(L),\ \exists A > 0, b(x) \in \hat{G}(L)_A \right\}$$

be the subset of critical points with positive part of the filtration (38).

Given $b \in CE(L, \partial L)_{>0}$ the *curvature* of b is the element

$$m(b) := \sum_{d \geq 0} \frac{1}{d!} m_d(b, \dots, b) \in CE(L)$$

obtained by all possible insertions of b in m_d, well defined by definition of the completion in (37).

The chain b is a *bounding chain* or a *Maurer-Cartan solution* if $m(b) = 0$. In particular, if $m_0(1) = 0$ then the element $b = 0$ is a bounding chain.

The *Maurer-Cartan space* of bounding chains is

$$MC(L) = \{b \in CE(L, \partial L)_{>0} | m(b) = 0\}.$$

The cobordism is L *unobstructed* if $MC(L)$ is non-empty, that is, there exists a bounding chain. This ends the Definition.

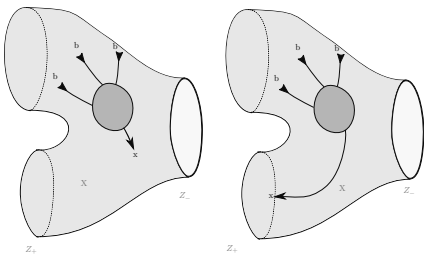

FIGURE 5. Holomorphic curves defining the curvature

Theorem 4.13. *Let (X, L) be a tame pair and suppose $b \in MC(L)$ is a Maurer-Cartan solution. Then the map $\varphi(L, b)$ is a map of differential algebras (chain map and algebra homomorphis) and so induces maps in contact homology*

$$H(\varphi(L,b)) : HE(\Lambda_-) \to HE(\Lambda_+), \quad H(\varphi_\circ(L,b)) : HE_\circ(\Lambda_-) \to HE_\circ(\Lambda_+).$$

Proof. The map $\varphi(L, b)$ is an algebra map by definition, since the product is given by concatenation. To prove it is a chain map under the hypotheses given in the Theorem, we classify the types of breakings in the true boundary configurations. Any true boundary configuration $u : C \to X$ of the one-dimensional component of the moduli space $\mathcal{M}(L)$ of treed disks bounding L is a configuration with some collection of trajectories $T_{e_1}, \dots, T_{e_k}$ of infinite length. The broken trajectories $T_{e_1}, \dots, T_{e_k}$ may either connect two different levels C_i, C_{i+1} of C, or two disks S_{v_1}, S_{v_2} within the same level C_i as shown in Figure 6. In case there are two different levels, one of the levels maps to either the incoming or outgoing cylindrical end.

We analyze the count of boundary configurations in each case. First consider configurations $u : C \to X$ with a single level with two components C_1, C_2 separated by a broken trajectory T_e, say passing through a critical point x of f_L and one of the components C_1 without punctures. The count of the configurations $u_1 = u|S_1$ is the coefficient of x in $m(b)$ which vanishes;

Second, consider configurations $u : C \to X$ with two levels C_1, C_2 in X and $\mathbb{R} \times Z_+$, with the level in X having two connected components $C_{1,1}, C_{1,2}$, one $C_{1,2}$ with no incoming puncture and the other $C_{1,2}$ with one incoming puncture. Suppose $C_{1,1}$ is joined to C_2 by an edge passing through a generator $x \in \mathcal{I}(\Lambda_+)$. The count of configurations $C_{1,2}$ is the coefficient of x in $m(b)$ which vanishes by assumption.

Finally, consider onfigurations $u : C \to X$ with two levels C_1, C_2 either in $\mathbb{R} \times Z_-$ and X or in X and $\mathbb{R} \times Z_+$, with exactly one incoming puncture on each component. These configurations contribute to the expressions

$$\delta_+ \circ \varphi(L,b), \quad \varphi(L,b) \circ \delta_-$$

respectively.

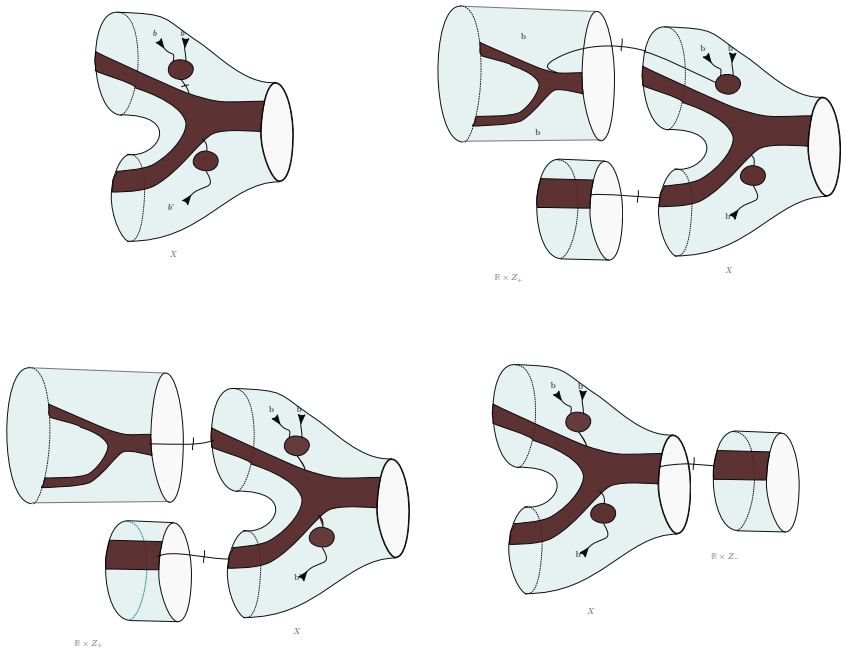

FIGURE 6. Boundaries of moduli spaces bounding the Lagrangian

The chain property follows the fact that the boundary of the one-dimensional moduli space of configurations has vanishing signed count; with the sign computation being that in Karlsson [24]. □

4.3. **Independence of the cobordism maps from choices.** In this section, we show that the chain maps associated to cobordisms are independent of all choices, up to chain homotopy and a suitable change in the bounding chain, as in the theory of Fukaya-Oh-Ohta-Ono [21, 22]. To state the theorem, we recall from Charest-Woodward [7] the notion of treed quilted disk.

Definition 4.14. A *quilted disk* is a disk S in the complex plane $\mathbb{C}$, equipped with a *seam* Q which is circle in S tangent to the boundary ∂S at a single point. Up to isomorphism we may assume that S is the unit disk centered at 0.

A *nodal pre-stable quilted disk* is a connected union (S, Q) of disks, spheres, and quilted disks $S_v \subset S$ corresponding to vertices $v \in \mathrm{Vert}(\Gamma)$ of some tree Γ, together with a collection of boundary markings

$$\underline{z}_\circ = (z_{\circ,0}, \ldots, z_{\circ,d}) \in \partial S$$

disjoint from the nodes, and a collection of interior markings

$$\underline{z}_\bullet = (z_{\bullet,0}, \ldots, z_{\bullet,e}) \in \mathrm{int}(S)$$

with the following property:

On any path of components S_v from the *root marking* z_0 to a *leaf marking* $z_i, i > 0$ there is exactly one quilted component $S_v \subset S$.

A pre-stable quilted disk (S, Q) is *stable* if each unquilted resp. quilted component has at least three resp. two marked or nodal points.

A *stable treed quilted disk* (C, Q) is obtained from a stable quilted disk (S, Q) by replacing each node $S_{v_-} \cap S_{v_+}$ with an edge T_e which is an interval of length $\ell(e) \in [0, \infty]$, and each marking with a semi-infinite edge T_e with the following property:

If $T_{e'}$ and $T_{e''}$ are two boundary leaves, and $T_{e_1}, \ldots, T_{e_k}$ are the edges in a path connecting them, then

$$\sum_{i=1}^{k} \pm \ell(e_i) = 0$$

where the sign is positive if the edges T_{e_i} in the path points towards the root edge T_{e_0} and negative otherwise.

Each disk component $S_{v_1}, \ldots, S_{v_k}$ of S is equipped with a natural (possibly infinite) *distance to the seam*

$$d(v) = \pm \sum_{e \in \gamma} \ell(e) \in [-\infty, \infty] \tag{42}$$

where γ is the path of edges to the component S_{v_0} connecting to the quilted vertices, and the sign is negative if S_v lies closer resp. further away from the root vertex than the seam. Define the *normalized distance* as

$$\tilde{d}(v) = \frac{d(v)}{\sqrt{d(v)^2 + 1}} \in [-1, 1] \tag{43}$$

taking values in $\{-1, 1\}$ whenever the distance $d(v)$ is infinite.

The boundary edges T_e are partitioned into those that are closer to the root edge T_{e_0} than the quilted disk components, and those that are further away; we call these groups of edges *root-close* and *root-distant* respectively and denote the subsets of such edges $\mathrm{Edge}_{\pm}(\Gamma)$ respectively. See Figure 7. This ends the Definition.

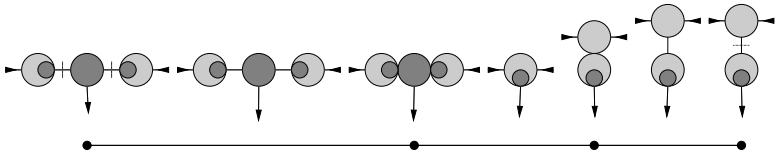

FIGURE 7. Moduli space of treed quilted disks

We define quilted disks associated to homotopies of almost complex structures and Morse functions. Let $(J_\pm, f_\pm)$ be two choices of cylindrical almost complex structure and Morse data and $J_t, t \in [-1, 1]$ a homotopy from J_- to J_+ resp. f_t an isotopy of Morse data from f_- to f_+.

Definition 4.15. A *quilted holomorphic disk* of domain type Γ is a continuous map $u : C \to X$ that is $J_{\tilde{d}(v)}$ holomorphic on each disk component $S_v, v \in \mathrm{Vert}(\Gamma)$, where $\tilde{d}(v)$ was defined in (43) and on each edge the map u is an $f_{\tilde{d}(e)}$-gradient trajectoriy. In particular, satisfies the f_+ gradient vector field equation on root-distant edges $T_e, e \in \mathrm{Edge}_+(\Gamma)$ and the f_- gradient vector field on root-close edges $T_e, e \in \mathrm{Edge}_-(\Gamma)$. See Figure 8.

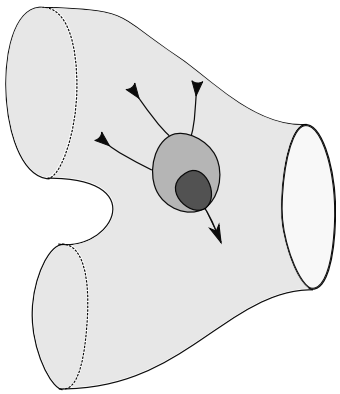

FIGURE 8. A quilted disk in a cobordism

Map types for quilted tree disks are graphs decorated with homology classes as before. In particular, a map type $\mathbb{\Gamma}$ consists of a tree Γ with vertices $\mathrm{Vert}(\Gamma)$ and edges $\mathrm{Edge}(\Gamma)$, a subset $\mathrm{Vert}^Q(\Gamma) \subset \mathrm{Vert}(\Gamma)$ of vertices corresponding to the quilted components, and a labelling of vertices v by homology classes $\beta(v) \in H_2(\overline{X}, \overline{L}) \cup H_2(\overline{X})$. For any type $\mathbb{\Gamma}$ of quilted treed disk denote by $\mathcal{M}^q_{\mathbb{\Gamma}}(L)$ the moduli space of quilted treed disks of type $\mathbb{\Gamma}$ and

$$\mathcal{M}^q_d(L) = \bigcup_{\mathbb{\Gamma}} \mathcal{M}^q_{\mathbb{\Gamma}}(L)$$

the union over combinatorial types with d incoming leaves, $\mathcal{M}^q_{d,1}(L, \underline{x}_-, x_+)$ the locus with limits $\underline{x}_-, x_+$ along the incoming and outgoing edges and $\mathcal{M}^q_{d,1}(L, \underline{x}_-, x_+)_0$ the locus of rigid maps in the sense of [4, Equation (33)]. Define

$$q_d : CE(L, \partial L; f_0)^{\otimes d} \to CE(L, \partial L; f_1),\ \underline{x}_- \mapsto \sum_{u \in \mathcal{M}_{d,1}(L, \underline{x}_-, x_+)_0} (-1)^{\heartsuit} \mathrm{wt}(u) x_+.$$

The use of quilts here is a purely combinatorial device, and the seams of the quilts have no geometric meaning as opposed to their use for Lagrangian correspondences in [27].

Theorem 4.16. *Let $(J_\pm, f_\pm)$ be two choices of cylindrical almost complex structure and (J_t, f_t) a homotopy between the as above. Counting quilted holomorphic disks defines a map*

$$MC(J_\bullet, f_\bullet) : MC(J_-, f_-) \to MC(J_+, f_+).$$

In particular, non-emptiness of $MC(J, f)$ is independent of the choice of almost complex structure J on X and Morse function f_L on L.

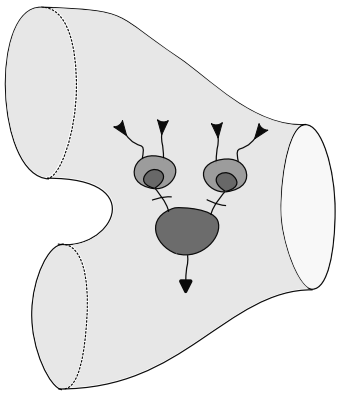

FIGURE 9. Contributions giving $m(b_1)$

Proof. The map between Maurer Cartan spaces is defined as follows. The count of quilted disks with incoming edges labelled $b_- \in MC(J_-, f_-)$ defines a formal sum

$$b_+ = \sum_{d \geq 0} q_d(b_-, \ldots, b_-).$$

To see that this sum defines a Maurer-Cartan solution, we consider the boundary components of the one-dimensional moduli spaces. For some $d \geq 0$ consider the one-dimensional boundary component $\mathcal{M}_d^q(L)_1$ of the moduli space of quilted disks with d incoming leaves. Boundary configurations $u \in \partial \mathcal{M}_d^q(L)_1$ are of the following types: either there is a single edge of infinite length connecting components in same level, or a configuration of infinite length edges connecting different levels. We treat the different cases in turn and their contributions to the map of Maurer-Cartan spaces.

First consider a configuration u with one level and an unquilted disk u_{v_0} attached to a collection of quilted disks $u_{v_1}, \ldots, u_{v_r}$ via a collection of broken edges $T_{e_1}, \ldots, T_{e_r}$ passing through critical points of f_L. The sum of such configurations is

$$\begin{aligned} \sum_{r, j_1, \ldots, j_r \geq 0} m_r(q_{j_1}(b_-, \ldots, b_-), \ldots, q_{j_r}(b_-, \ldots, b_-)) &= \sum_{r \geq 0} m_r(b_+, \ldots, b_+) \\ &= m(b_+). \end{aligned}$$

Next consider a configurations u with one level and a quilted disk u_{v_1} attached to an unquilted disk u_{v_2} via a broken edge. These configurations contribute zero because of the Maurer-Cartan condition:

$$\begin{aligned} &\textstyle\sum_{r,d} q_{r-d+1}(b_-, \ldots, b_-, m_d(b_-, \ldots, b_-), b_-, \ldots, b_-) \\ &\quad = \textstyle\sum_{r \geq 0} q_{r-d+1}(b_-, \ldots, b_-, 0, b_-, \ldots, b_-) = 0. \end{aligned}$$

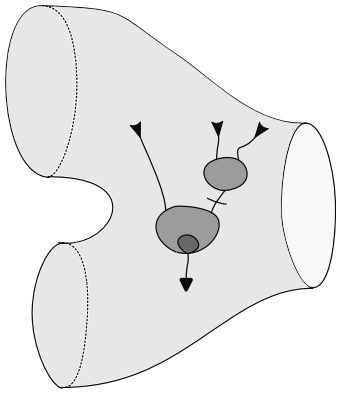

FIGURE 10. Contributions involving $m(b_-)$

The last two cases involve breaking off a level onto a cylindrical end of the configuration: One the one hand, there are configurations u with two levels with one level u_1 mapping to X and one level u_2 mapping to $\mathbb{R} \times Z_+$ attached separated by a broken trajectory $u|T_e$. On the other hand, there are configurations u with two levels u_1, u_2 with one level u_1 in $\mathbb{R} \times Z_-$ and one level u_2 in X separated by a broken trajectory $u|T_e$ of $\zeta_\circ$. Such configurations are impossible since the level u_1 in $\mathbb{R} \times Z_-$ necessarily is collection of disks $u_v : S_v \to \mathbb{R} \times Z_-$, at least one of which has positive area $A(u_v) > 0$ and so at least one puncture. This contradicts [4, Lemma 3.23]. The same argument also excludes configurations in (c).

The signed count of the boundary points in the one-dimensional component of the moduli space vanishes, for any energy bound. It follows that

$$0 = \sum_{u \in M_d^q(L,x_0,\ldots,x_d)} (-1)^\heartsuit \left(\prod_{i=1}^d b_+(x_i) \right) \mathrm{wt}(u) x_0 = m(b_+)$$

as desired. □

Theorem 4.17. *If $b_+ \in MC(J_+, f_+)$ is the image of $b_- \in MC(J_-, f_-)$ then the induced maps*

$$\varphi(L, b_-), \varphi(L, b_+) : CE(\Lambda_-, \hat{G}(L)) \to CE(\Lambda_+, \hat{G}(L))$$

are chain homotopic algebra maps, and similarly for the maps on the complex generated by Reeb chords only the maps

$$\varphi_\circ(L, b_-), \varphi_\circ(L, b_+) : CE_\circ(\Lambda_-, \hat{G}(L)) \to CE_\circ(\Lambda_+, \hat{G}(L))$$

are chain homotopic.

Proof. The proof is an argument using parametrized moduli spaces, in which the rigid points in the parametrized moduli space define a homotopy operator for the difference between cobordism maps. Given an isotopy (J_t, f_t) as above we consider the moduli space of quilted punctured disks in X bounding L with a single incoming

puncture. Let $\mathcal{M}^q_{d,e}(\underline{x}_-, \underline{x}_\bullet)_0$ denote the moduli space of rigid quilted disks with limits

$$\underline{x} = (x_0 \in \mathcal{I}(\Lambda_-), x_1, \ldots, x_{d_+} \in \mathcal{I}(\Lambda_+))$$

and a collection of odd generators

$$\underline{x}_\bullet = (x_{\bullet,1}, \ldots, x_{\bullet,d_\bullet}) \in \mathcal{I}(L, \partial L)^{d_\bullet}.$$

Counts of pairs $(t, u : C \to \mathbb{X})$ consisting of rigid quilted punctured disks u with respect to the datum (J_t, f_t) define a map

$$h(b_-) : CE(\Lambda_-, \hat{G}(L)) \to CE(\Lambda_+, \hat{G}(L)).$$

On words of length one, the definition is

$$(h(b_-))(\gamma) = \sum_{\underline{x}, u \in \mathcal{M}^q_{d,e}(\underline{x};\underline{y})_0} (-1)^\heartsuit \left(\prod_{i=1}^{e} b_-(x_{\bullet,i}) \right) \gamma(x_0) x_1 \otimes \ldots \otimes x_d$$

of degree -1, with sign from (27), as in Figure 11.

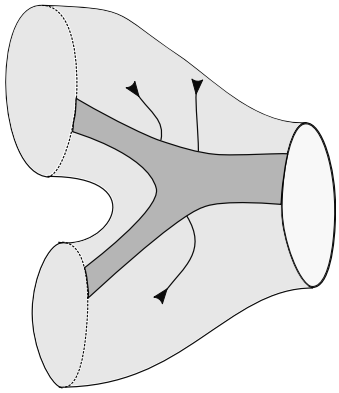

FIGURE 11. Disks defining the homotopy operator

Boundary configurations of the one-dimensional components of the moduli spaces defining the homotopy operator are those configurations with some collection of edges with infinite length.

The first possibility is that of configurations with two levels, connected by infinite length trajectories, where the first level is quilted. Let $u = (u_1, u_2)$ where the first level (C_1, u_1) is a quilted disk mapping to $\mathbb{R} \times Z_-$ and the second level (C_2, u_2) maps to X as in Figure 12. Because the almost complex structure on $\mathbb{R} \times Z_-$ is translation-invariant, forgetting the quilting produces a stable unquilted treed disk $u : C \to \mathbb{R} \times Z$ in a moduli space of negative expected dimension, unless the unquilted disk is a constant strip in which case it is unstable. It follows that these configurations define the cobordism map $\varphi(L, b_-)$.

The secod possibility is that the second level is quilted. Let (C, u) be a configuration with two levels (C_1, u_1) and (C_2, u_2) where (C_1, u_1) is a (J_1, f_1)-holomorphic map to X and (C_2, u_2) is a collection of quilted disks and spheres in $\mathbb{R} \times Z_+$ as

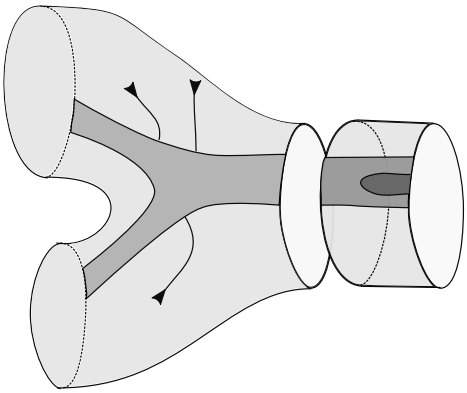

FIGURE 12. Contributions to $\varphi(L, b_-)$

in Figure 13. Again, forgetting the quiltings produces a configuration in a moduli space of lower expected dimension unless the quilted disks are constant. Thus these configurations contribute to $\varphi(L, b_+)$.

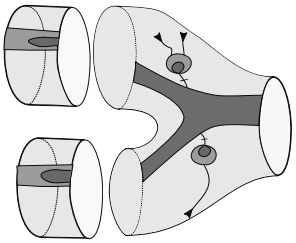

FIGURE 13. Contributions to $\varphi(L, b_+)$

A third possibility is that of a configuration with a single level, separated by an infinite length trajectory. Configurations (C, u) consisting of a quilted disk $(C_0.u_0)$ mapping to X connecting to a unquilted disk (C_1, u_1) mapping to X with edges labelled b_-, as in Figure 14. These contributions cancel by the assumption that $m(b_-)$ vanishes.

Finally, there are configurations with unquilted disks bubbling off the ends. Configurations (C, u) consisting of a quilted disk (C_0, u_0) mapping to X connected to an unquilted disk (C_1, u_1) mapping to $\mathbb{R} \times Z_\pm$, contributing to the graded commutator of the homotopy operator with the differential

$$h(b_-)\delta_- + \delta_+ h(b_-).$$

See Figure 15 and Figure 16. The justification of signs is similar to that for the square-zero property of the differential.

Putting everything together, we obtain

$$\varphi(L, b_+) - \varphi(L, b_-) = h(b_-)\delta_- + \delta_+ h(b_-).$$

This equality implies that $\varphi(L, b_1)$ and $\varphi(L, b_0)$ are chain homotopic. The case of the cobordism maps $CE_\circ(\Lambda_-) \to CE_\circ(\Lambda_+)$ is similar; in this case, all the outgoing

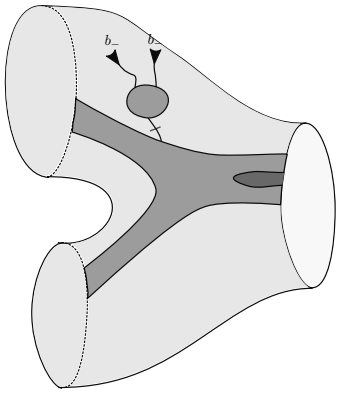

FIGURE 14. Contributions involving $m(b_-)$

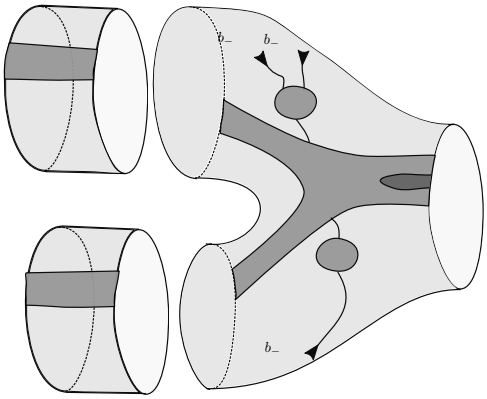

FIGURE 15. Contributions to $\delta_+ h(b_-)$

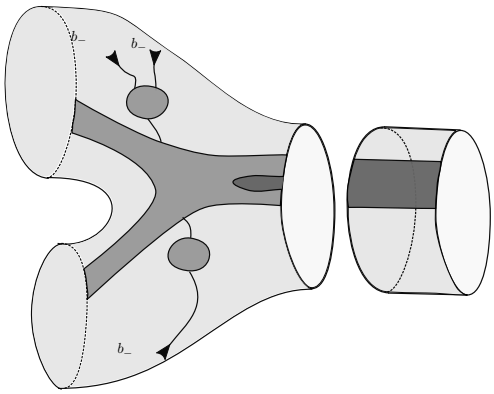

FIGURE 16. Contributions to $h(b_-)\delta_-$

edges correspond to punctures and so there can be no breaking between levels that correspond to classical generators. □

Remark 4.18. Suppose that $\Pi = p(\Lambda)$ is a monotone Lagrangian in Y. For any fixed number of punctures, the number of punctured disks $u : C \to \mathbb{R} \times Z bounding \mathrm{R}\times\Lambda$ is finite as the completion in (19) is not necessary. Now Let Λ_ρ be a family of Legendrians in a family of stable Hamiltonian manifolds (Z, α_ρ, ω) for $\rho \in [0, 1]$. Given an almost complex structure J taming ω_0, J also tames ω_ρ for ρ sufficiently small. The holomorphic disks bounding Λ_ρ which define the differential on $CF(\Lambda_\rho)$ are in bijection with disks bounding Λ_0, which define the differential on $CF(\Lambda_0)$, by transversality. Thus $CE(\Lambda_\rho) = CE(\Lambda)$ for ρ sufficiently small. In particular, any tame filling of Λ_ρ defines an augmentation for $CE(\Lambda)$. We have in mind the example especially of the Clifford Legendrian Λ. In this case, any filling of a perturbation of Λ (for example, the intersection $L \cap Z$ of the Harvey-Lawson filling L with $Z = S^{2n-1}$) defines an augmentation of $CE(\Lambda)$.

Example 4.19. Continuing Example 2.5, the two fillings $L_{(2),\pm} \subset \mathbb{C}^n$ obtained from paths $\gamma_\pm : \mathbb{R} \to \mathbb{R}^2$ above respectively below the critical value of the projection are not isotopic through exact fillings. Indeed, since the generators $\mathfrak{c}_{12}, \mathfrak{c}_{21}$ are closed, existence of an isotopy would imply that the values of the augmentations $\varphi(L_{(2),\pm})$ on $\mathfrak{c}_{12}, \mathfrak{c}_{21}$ are equal, which is not the case (having values $n, 1$ and $1, n$ respectively.) In the case $n = 2$, the non-existence of an isotopy is the local version of the statement that the Chekanov and Clifford tori are not Hamiltonian isotopic as in, for example, Vianna [35]. That is, existence of a local isotopy would imply existence of a global isotopy.

4.4. **The composition theorem.** According to the philosophy of topological quantum field theory, chain maps associated to cobordisms should satisfy a composition-equals-gluing axiom. For cobordism maps associated to tame cobordisms equipped with bounding chains we show that for a natural operation on bounding chains (the formal sum in the limit of long neck length), the composition of the cobordism maps is the cobordism map for the composition.

Theorem 4.20. *Suppose (X', L') and (X'', L'') are positive composable tame cobordism pairs equipped with bounding chains $b' \in MC(L'), b'' \in MC(L'')$. Denote the cobordism maps*

$$\varphi(L', b') : HE(\Lambda'_-) \to HE(\Lambda'_+), \quad \varphi(L'', b'') : HE(\Lambda''_-) \to HE(\Lambda''_+)$$

and $X = X'' \circ X'$ is the composed cobordism then there exists a map

$$\circ : MC(L') \times MC(L'') \to MC(L) \tag{44}$$

such that

$$[\varphi(L'' \circ L', b'' \circ b')] = [\varphi(L'', b'')] \circ [\varphi(L', b')]. \tag{45}$$

Furthermore, the cobordism map associated to the trivial cobordism is the identity:

$$\varphi(\mathbb{R} \times \Lambda, 0) = \mathrm{Id}_{CE(\Lambda)}\,.$$

Proof. We first construct the composition map (44). It suffices to construct the map for the complex structure on X in the neck stretching limit in which X approaches the broken symplectic manifold $X = X' \cup X''$ glued along $Y'_+ = Y''_-$. In this limit,

holomorphic curves in X are replaced by buildings in $X = X' \cup X''$, and the Morse function f_L on L is equal to the Morse functions $f_{L'}, f_{L''}$ on L', L''. The critical locus is the union

$$\mathrm{crit}(f_L) = \mathrm{crit}(f_{L'}) \cup \mathrm{crit}(f_{L''}).$$

With respect to this splitting, define $b'' \circ b'$ as the formal sum of b'' and b'.

We prove the composition law (45). Let $u : C \to X$ be a building contributing to $\varphi(L'' \circ L', b'' \circ b')$, necessarily consisting of two levels u', u''. Each of u', u'' have a single incoming puncture. Indeed, otherwise there would be a component of u' with no incoming puncture, which is impossible by [4, Lemma 3.23]. Let $v = (v', v'')$ denote the component of u which corresponds to the bounding chain insertions. The sum over all treed disks u' with x as output and remaining leaves labelled by u' vanishes, being the coefficient of x in $m(b')$. Thus, the only contributions arise from configurations $u = (u', u'')$ where both u', u'' have outputs in the interior of their corresponding levels as in Figure 17. Furthermore, the tameness assumptions

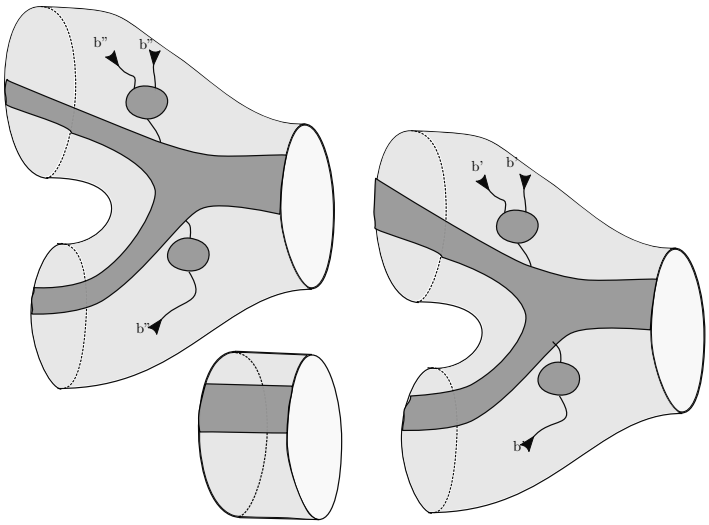

FIGURE 17. Composition of bounding chains

guarantee that each disk has at least one incoming puncture as in [4, Corollary 3.16]. Thus if $b = b'' \circ b'$ then

$$m(b) = m(b') + m(b'') = 0.$$

The same arguments as before show that the map defined by counts of buildings in X and counts of maps in X for some modified b define chain-homotopic maps from $CE(\Lambda, \hat{G}(L))$ to $CE(\Lambda'', \hat{G}(L))$, and the claim follows. □

4.5. **Examples of unobstructed fillings.** We wish to show the the fillings of Harvey-Lawson type are unobstructed and so define augmentations of the corresponding Chekanov-Eliashberg algebras. The existence of these fillings will be used to compute the augmentation varieties in various examples. Specifically, the computations will be used to give lower bounds for dimensions of augmentation varieties in the examples where there exist unobstructed fillings.

Lemma 4.21. *The Harvey-Lawson filling $L \cong \mathbb{R}^2 \times (S^1)^{n-2} \subset \mathbb{C}^n$ in Example 2.2 is unobstructed.*

Proof. We claim that the zero chain is a bounding chain for the standard complex structure and a judicious choice of Morse function. Consider a Morse function $f : L \to \mathbb{R}$ with the property that any gradient trajectory $\gamma : [0, \infty) \to L$ starting from the boundary of a holomorphic disk $u : C \to X$ flows to infinity in the sense that $\lim_{t\to\infty} |\gamma(t)| = \infty$. Fix a coordinate z_1 on the $\mathbb{R}^2$ factor in L so that the basic disks map to $z_1 = 0$, by the classification in Lemma 2.27. A particular function with the required property is

$$f(z_1) = \frac{(|z_1|^2 - c^2)^2}{(|z_1|^2 - c^2)^2 + 1}.$$

Any generic perturbation of f is Morse and still has the property that gradient trajectories meeting $z_1 = 0$ flow to infinity. By the choice of Morse function, there are no configurations connecting such a disk to a critical point. □

Next we generalize the filling of the Hopf Legendrian in (8) to fillings of Legendrians with two components. First, we consider the following one-dimensional situation.

Definition 4.22. A *matching* is a locally compact one-manifold $\Upsilon \subset \mathbb{R} \times S^1$ which is cylindrical near infinity in the sense that

$$\begin{aligned} \Upsilon \cap ([T - \epsilon, T] \times S^1) &= [T - \epsilon, T] \times \Upsilon_+ \\ \Upsilon \cap ([-T, -T + \epsilon] \times S^1) &= [-T, -T + \epsilon] \times \Upsilon_- \end{aligned}$$

for some finite sets $\Upsilon_\pm$ and any large enough $T > 0$, as shown in Figure 18.

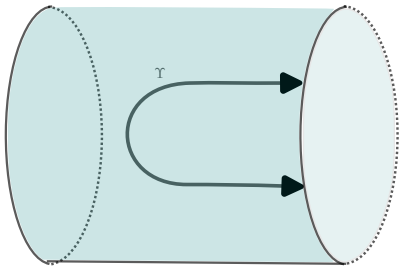

FIGURE 18. A one-dimensional Lagrangian with cylindrical ends

Now let Z be a tame circle-fibered contact manifold and $\mathbb{R} \times Z$ the symplectization equipped with symplectic form $\mathrm{d}(e^s\alpha)$.

Definition 4.23. Given a compact Legendrian $\Lambda \subset Z$ and a matching Υ, the *matching cobordism* is the image of $\Upsilon \times \Lambda$ under the action map $(\mathbb{R} \times S^1) \times Z \to Z$, denoted

$$\Upsilon\Lambda = \left\{ (r, e^{i\theta} \cdot z) \middle| z \in \Lambda, \ \ (r, e^{i\theta}) \in \Upsilon \right\}.$$

Lemma 4.24. *The submanifold $\Upsilon\Lambda \subset X$ is a Lagrangian cobordism from $\Upsilon_-\Lambda$ to $\Upsilon_+\Lambda$. If Y has integral symplectic class and Λ_+ is empty then L is tame. If Υ is simply connected then L is unobstructed.*

Proof. We first check the Lagrangian condition. The tangent space $T(\Upsilon\Lambda)$ is spanned by the images of $T\Upsilon$ and $T\Lambda$. Let $v, w \in \mathrm{Vect}(L)$ be vector fields in the images of $\mathrm{Vect}(\Upsilon)$ and $\mathrm{Vect}(\Lambda)$ respectively. The contraction of v, w with the symplectic form is

$$\mathrm{d}(e^s\alpha)(v,w) = e^s\mathrm{d}\alpha(v,w) + e^s\mathrm{d}s\alpha(v,w) = 0$$

since $\iota(v)\mathrm{d}\alpha = \iota(w)(\mathrm{d}s \wedge \alpha) = 0$. It follows that L is Lagrangian.

Next we check the tameness conditions. By assumption Y has integral symplectic class and Λ_+ is empty. The condition [4, **(P2)**] holds since

$$c_1(\overline{X}) - [Y_-]^\vee = p^*c_1(Y).$$

The conditions **(P1)** and **(P3)** in [4] hold trivially. Suppose Υ is simply connected, that is, each component is an interval.

Finally we check unobstructedness. We must show that for the standard complex structure J the element $b = 0$ is a bounding chain. In this case, the only possible contributions to $m(b)$ arise from disks $u : C \to X$ with no inputs. By the stability condition, any such disk u must have at least one non-constant component $S_v \subset C$. The first homology $H_1(\Lambda)$ surjects onto $H_1(L)$, and it follows that L is exact and bounds no holomorphic disks. Therefore, L is unobstructed. □

The following construction generalizes the Harvey-Lawson filling to other toric Fano varieties.

Theorem 4.25. *Let $Z \to Y$ be the unit canonical bundle over a toric Fano variety Y of dimension $n-1$ with polytope P, p the barycenter of P, F a codimension two face of P, and $\mathrm{hull}(p,F)$ the convex hull of p and F. For any point λ in the interior of $\mathrm{hull}(p,F)$ let Π denote the Lagrangian torus over λ and Λ a section of the bundle $Z|_\Pi \to \Pi$. Let $L \cong \mathbb{R}^2 \times T^{n-2}$ be the Harvey-Lawson-type filling constructed in [4, Lemma 2.32]. Then L is unobstructed.*

Proof. We will show that the filling is unobstructed by classifying holomorphic disks with no punctures. Let $u : S \to X$ be a holomorphic disk bounding L. With notation from [4, Lemma 2.32], since L is contained in the zero level set of $\Phi_1 - \lambda$, by removal of singularities u descends to a holomorphic map $u_1 : S \to Y_1$ bounding the Lagrangian L_1, and Y_1 admits a morphism to $\mathbb{C}^2$ so that L_1 is the pull-back of the antidiagonal. Since there are no non-constant holomorphic disks in $\mathbb{C}^2$ bounding the antidiagonal (such a disk would correspond to a non-constant holomorphic sphere) u_1 maps to the fiber of $Y_1 \to \mathbb{C}^2$, and in particular the boundary of u_1 is trivial. It follows that the boundary of u is contained in the locus $T^{n-2} \times \{0\}$ in $L \cong T^{n-2} \times \mathbb{R}^2$. The same proof of unobstructedness used in for unobstructedness of the Harvey-Lawson filling in Example 4.21 applies (choose a Morse function on $\mathbb{R}^2$ so that 0 flows to infinity; then there are no gradient trajectories $u_e : T_e \to L$ from the boundaries of the holomorphic disks $u_v : S_v \to X$ to a critical point of f_L.) □

4.6. **Invariance of Legendrian contact homology.** The results of the previous section allow us to prove that $HE(\Lambda)$ is independent of the choice of almost complex structure and vector fields.

Lemma 4.26. *For any tame pair (Z, Λ) the trivial cobordism pair $(\mathbb{R} \times Z, \mathbb{R} \times \Lambda)$ is unobstructed.*

Proof. Indeed, $L = \mathbb{R} \times \Lambda$ bounds no holomorphic disks without punctures, by the angle change formula [4, Lemma 3.15]. It follows that the curvature $m(b)$ of the trivial chain $b = 0$ vanishes. □

Lemma 4.27. *Let $\Lambda_- = \Lambda_+$, and suppose that $\mathbb{R} \times Z$ is equipped with a translation-invariant almost complex structure and a Morse datum so that the vector field ζ_L is translation-invariant. Then the chain map $\varphi : CE(\Lambda_-) \to CE(\Lambda_+)$ is the identity map.*

Proof. We check that in this case the only rigid maps contributing to the cobordism map are trivial cylinder. By assumption $\Lambda_- = \Lambda_+$ and each moduli space has the action of translations since the isotopy from J_- to J_+ and $\zeta_{\bullet,-}$ to $\zeta_{\bullet,+}$ is invariant under the $\mathbb{R}$-action. It follows that the only rigid configurations are those that are invariant under the $\mathbb{R}$-action, so are trivial cylinders with the same incoming and outgoing Reeb chord. The induced map $\varphi : CE(\Lambda_\pm) \to CE(\Lambda_\pm)$ is therefore the identity map. □

Theorem 4.28. *Suppose that (Z, Λ) is a tame pair. Given two choices of cylindrical almost complex structures, Morse data and perturbations there exists a homotopy equivalence between the corresponding chain complexes $CE(\Lambda)$ and $CE(\Lambda)'$. Furthermore, this homotopy equivalence may be taken to preserve classical sectors, that is, map $CE_\bullet(\Lambda)$ to $CE_\bullet(\Lambda)'$ and intertwine with the projections of $CE_\bullet(\Lambda) \to \hat{G}(\Lambda)$ and $CE_\bullet(\Lambda)' \to \hat{G}(\Lambda)$ given by taking the coefficient of the empty word. As a result, the contact homology $HE(\Lambda)$ is independent, up to isomorphism, of all such choices.*

Proof. For convenience we recall the choices involved. Let $J_\pm, \zeta_{\bullet,\pm}, \zeta_{\circ,\pm}$ be almost complex structures resp. vector fields and $\underline{P}_\pm = (P_{\Gamma,\pm})$ two choices of perturbations making all moduli spaces of expected dimension at most one regular. Let $CE(\Lambda)_\pm$ denote the corresponding Chekanov-Eliashberg algebras.

We wish to extend the given data over the symplectization, with the given limits on the positive and negative ends, so that the corresponding cobordism maps induce isomorphisms of contact homology. First, extend the almost complex structures and vector fields

$$J_\pm \in \mathcal{J}(\mathbb{R} \times Z_\pm), \quad \zeta_{\bullet,\pm} \in \mathrm{Vect}(\mathbb{R} \times \Lambda_\pm)$$

over $X = \mathbb{R} \times Z$ to an almost complex structure and vector field

$$J \in \mathcal{J}(X), \quad \zeta_\bullet \in \mathrm{Vect}(L)$$

that agree with the given ones on the cylindrical ends. Since $\zeta_{\bullet,\pm}$ is positive, we may assume that its extension $\zeta_\bullet$ has no zeroes on $\mathbb{R} \times Z$.

Given two choices of almost complex structures, Morse data, and perturbation data we obtain from the trivial cobordism equipped with almost complex structures,

Morse data and perturbations extending the given ones on the ends a chain map

$$\varphi : CE(\Lambda)_- \to CE(\Lambda)_+.$$

By Theorem 4.17, any homotopy between two different choices $J', \zeta'_\bullet$ and $J'', \zeta''_\bullet$ (again with no zeroes, and defining chain maps φ', φ'') defines a chain homotopy

$$h : CE(\Lambda)_- \to CE(\Lambda)_+$$

satisfying

$$\varphi'' - \varphi' = \delta_+ h + h\delta_-.$$

Similarly, we have a map

$$\varphi'' : CE(\Lambda_+) \to CE(\Lambda_-)$$

inducing a map in homology independent of all choices up to chain homotopy. By the composition Theorem 4.20, we have

$$[\varphi''] \circ [\varphi'] = \mathrm{Id}_{HE(\Lambda_-)}, \quad [\varphi'] \circ [\varphi''] = \mathrm{Id}_{HE(\Lambda_+)}$$

so that $[\varphi']$ and $[\varphi'']$ are inverses.

By the angle-change formula in [4, Lemma 3.15] one sees that if the disk has no incoming puncture then it has no outgoing punctures either, so the classical sector $CE_\bullet(\Lambda)$ maps to $CE_\bullet(\Lambda)'$. Furthermore, if the input is from the classical generators $\mathcal{I}_\bullet(\Lambda)$ then all disks are constant and so the map $CE_\bullet(\Lambda) \to CE_\bullet(\Lambda)'$ preserves the length filtration, and in particular, maps words of positive length to words of positive length. It follows that the chain map intertwines with projection to the span of the zero length word, as claimed. □

Remark 4.29. Naturally one expects the version of Legendrian contact homology here to be related to the one described in Ekholm-Etnyre-Sullivan [16], in cases where both are defined. The natural expectation would be that the complexes on $CE_\circ(\Lambda)$ (that is, projecting out the classical generators) would be chain homotopic. However, this would require a perturbation scheme allowing generic contact forms, which is certainly outside the scope of this paper.

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