ON THE QUANTUM PRODUCT OF SCHUBERT CLASSES

W. FULTON AND C. WOODWARD

ABSTRACT. We give a formula for the smallest powers of the quantum parameters q that occur in a product of Schubert classes in the (small) quantum cohomology of general flag varieties G/P. We also include a complete proof of Peterson's quantum version of Chevalley's formula, also for general G/P's.

1. INTRODUCTION

The Grassmannian was the first variety whose quantum cohomology was studied by physicists [36], and the first whose structure was worked out rigorously by mathematicians [34], [7], [5]. Other homogeneous varieties G/P have been studied (see below), but the story here remains far from complete. Quantum cohomology has gone far beyond these beginnings, with all smooth projective varieties (or compact symplectic manifolds) enjoying a version of quantum cohomology. However, there are still interesting questions to be answered about the case of G/P in general, and Grassmannians in particular. Our aim in this paper is to give an explicit formula for lowest degrees that occur in quantum product of Schubert classes.

The classical cohomology of a Grassmannian $\mathrm{Gr}(k,n)$ of k-planes in $\mathbb{C}^n$ has a basis of Schubert classes σ_λ, as λ varies over partitions whose Young diagram fits in a k by $n-k$ rectangle. The (complex) codimension of σ_λ is $|\lambda| = \sum \lambda_i$, the number of boxes in the Young diagram. The Littlewood-Richardson rule gives the coefficients of a Schubert class σ_ν in a product $\sigma_\lambda \cdot \sigma_\mu$, for $|\nu| = |\lambda| + |\mu|$. It is an easy and well-known fact that the classical product $\sigma_\lambda \cdot \sigma_\mu$ is nonzero precisely when λ and the $180°$ rotation of μ fit in the k by $n-k$ rectangle without overlap; for example, the dual class to σ_λ is the class σ_μ, for $\mu = \lambda^\vee$ the partition such that λ and the rotated μ exactly fill the rectangle without overlap.

The quantum cohomology of the Grassmannian is a free module over the polynomial ring $\mathbb{Z}[q]$, with a basis of Schubert classes; the variable q has (complex) degree n. The quantum product $\sigma_\lambda \star \sigma_\mu$ is a finite sum of terms $q^d \sigma_\nu$, the sum over $d \geq 0$ and $|\nu| = |\lambda| + |\mu| - d\, n$, each occurring with a nonnegative coefficient (a Gromov-Witten invariant); those with $d = 0$ are the classical Littlewood-Richardson coefficients. This ring was studied in [6], where an algorithm involving removing rim hooks was given for calculating

these products. It remains an important open problem to give a combinatorial formula for these coefficients (one that shows them to be nonnegative) when $d > 0$.

A simple argument due to Agnihotri showed that the quantum product $\sigma_\lambda \star \sigma_\mu$ of two Schubert classes in a Grassmannian can never be zero (see [6],§5), so some $q^d \sigma_\nu$ must appear in such a product with positive coefficient. The problem we address here is to find the smallest power of q that occurs in a product $\sigma_\lambda \star \sigma_\mu$. The evidence from small examples, together with the role that rim hooks play in the quantum multiplication, lead one to conjecture that this smallest power of d is the number of rim hooks it takes to cover the overlap of λ and the $180°$ rotation of μ in the k by $n-k$ rectangle. Equivalently, d is the maximum for which there is a diagonal sequence of boxes, from northwest to southeast, in this overlap. Here is an example, for $k = 4$, $n = 9$, $\lambda = (5, 4, 4, 3)$, and $\mu = (5, 4, 4, 1)$:
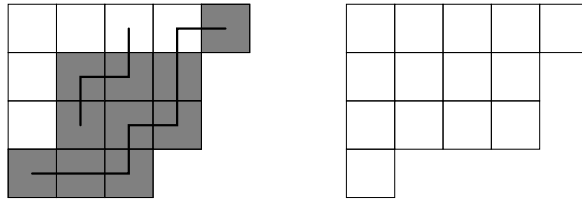
The overlap of λ with the rotation of μ is shaded, and one of the ways of covering the overlap with two rim hooks is indicated. In fact,[1]

$$\sigma_{5\,4\,4\,3} \star \sigma_{5\,4\,4\,1} = q^2(\sigma_{5\,3\,2\,2} + \sigma_{5\,3\,3\,1} + \sigma_{5\,4\,2\,1}) + q^3(\sigma_3 + 2\sigma_{2\,1} + \sigma_{1\,1\,1}).$$

This conjecture is proved in this paper.[2] It would be interesting to give a criterion for exactly which powers of q appear, or even for which coefficients appear; for applications of such criteria, see [1]. In the classical case, our understanding of this has increased dramatically recently, thanks to Klyachko, Knutson, and Tao, see [19], [1]. A. Yong has shown recently [37] that if one of the diagrams of λ or μ has no northwest to southeast sequence of e boxes, then q^e cannot appear in $\sigma_\lambda \star \sigma_\mu$.

In fact, we prove a generalization of this conjecture for any G/P. In general the degree d is an sequence of nonnegative integers, one for each 1-dimensional Schubert class. We give a formula, in terms of the combinatorics of the Bruhat order, for the smallest degrees d such that q^d occurs in a product of Schubert classes. It follows in particular that these quantum products can never be zero.

An understanding of (small) quantum cohomology for a G/P requires first a presentation of the ring $QH^\star(G/P)$, and second, a "quantum Giambelli formula" for the class of a Schubert variety in terms of this presentation. This has been worked out for the variety of complete flags ([21], [16], [18], [15]) and partial flags ([2], [27], [17], [15]), and recently

---

[1]We recommend the program of Anders Buch (www.math.mit.edu/~abuch/lrcalc/) for computing classical and quantum Littlewood-Richardson coefficients.

[2]Buch [11], P. Belkale [4] and A. Yong [37] have recently given proofs of stronger versions of this result for the Grassmannians.

for the Lagrangian Grassmannian [32]. Descriptions of the quantum cohomology ring have been given for general complete flag varieties G/B [26], and partial descriptions for general G/P by Peterson [33], but Giambelli formulas are not yet known in general.

Most of what is known about quantum Giambelli formulas comes from computing formulas for degeneracy loci on certain Quot schemes (although A. Buch [11] has recently given a proof for the Grassmannian, and Buch, Kresch, and Tamvakis [12] for some others, that does not depend on moduli spaces).

Given this limited knowledge about quantum cohomology of general G/P's, it is somewhat surprising that we are able to solve this problem for other G/P's. On the other hand, it indicates that, even in type A, we will not use algebraic formulas for quantum Schubert classes, and we will not use Quot schemes. Rather, we use the spaces $\overline{M}_{0,n}(X,d)$ of stable maps from genus 0 curves with n marked points to $X = G/P$, which were constructed by Kontsevich to prove the associativity of quantum products [31], see [20].

In the next few sections we lay out the necessary notation, recalling the standard facts about the geometry of G/P's and quantum cohomology that are needed to state the theorem precisely. Recall that the Schubert classes σ_u are parametrized by elements u in W/W_P, where W and W_P are the Weyl groups of G and P. The idea behind one implication of this formula can be explained roughly as follows. If a product $\sigma_u \star \sigma_v$ contains a term $q^d \sigma_w$, in the space $\overline{M}_{0,3}(X,d)$ the locus of stable maps of degree d to X that map the first marked point to a Schubert variety for σ_u and the second marked point to an opposite Schubert variety for σ_v must contain a point that is fixed by the maximal torus T of G. This fixed point is a map from a curve C into X, where C is a tree of $\mathbb{P}^1$'s. The images of the intersections of the components of C are fixed points of T in X, which are also indexed by elements of W/W_P. This produces a chain of elements of W/W_P, and this chain forces the elements u and v to be close to each other in a certain way. In the case of the Grassmannian, this closeness translates to the condition that the overlap described above can be covered by d rim hooks.

For the converse, any such chain does arise from a fixed point in such a moduli space, but it is not obvious when a point corresponds to a non-vanishing Gromov-Witten invariant. The key to this is provided by our transversality result in §7, which we deduce from Kleiman's general transversality theorem [28]. Similar ideas can be used to prove Peterson's quantum extension of Chevalley formula for multiplying a general Schubert class by a codimension one Schubert class. We have taken this opportunity to include a complete proof of this formula in §10.

The use of torus action in this setting goes back to Kontsevich [30], who was inspired by Ellingsrud and Strømme. It has been used many times since, see [25], [35]. The idea that Schubert varieties of opposite Borel subgroups are in general position for the purposes of quantum cohomology we learned from Peterson.

We thank Anders Buch and Alex Yong for several helpful conversations.

2. Localization

The following lemma, which is a special case of a theorem of Bott [9], provides simple proofs of the basic facts we need about divisors and curves on homogeneous varieties.

Lemma 2.1 (Localization). *Suppose a torus T acts on a curve $C \cong \mathbb{P}^1$, with fixed points $p \neq q$, and suppose L is a T-equivariant line bundle on C. Let χ_p and χ_q be the weights of T acting on the fibers L_p and L_q, and let ψ_p be the weight of T acting on the tangent space to C at p. Then*

$$\chi_p - \chi_q = n\,\psi_p,$$

where $n = \int_C c_1(L)$ is the degree of L.

Note that $\psi_q = -\psi_p$, so the result is independent of ordering of p and q. Note also that both sides vanish if T acts trivially on C.

3. Schubert varieties in G/P

We recall some basic notions about Schubert varieties and Schubert classes for a variety $X = G/P$, in order to fix our notation. As usual, G denotes a connected, simply connected, semisimple complex Lie group, in which we have fixed a Borel subgroup B and a maximal torus T in B. We use the notation W for the Weyl group $N(T)/T$, $R = R^+ \cup R^-$ for the roots (positive and negative), and Δ for the simple roots; the **reflections** s_α in W are indexed by the positive roots α; they are **simple reflections** if α is in Δ. The **length** $\ell(w)$ of an element w of W is the minimum number of simple reflections whose product is w. The element of longest length is denoted w_o. The **opposite Borel** subgroup is $B^- = w_o B w_o$.

The parabolic subgroups P of G correspond canonically to subsets Δ_P of Δ. Let R_P^+ be the set of positive roots that can be written as sums of roots in Δ_P. If $\mathfrak{g} = \mathfrak{t} \oplus \bigoplus_{\alpha \in R} \mathfrak{g}_\alpha$ is the root space decomposition of the Lie algebra of G, then the Lie algebra $\mathfrak{p}$ of P is the direct sum of $\mathfrak{t}$ and all $\mathfrak{g}_\alpha$ for α in $R^+ \cup (-R_P^+)$. The group W_P, generated by the reflections s_α, for α in Δ_P, is the Weyl group of a Levi subgroup of G corresponding to P; in particular, R_P^+ is the corresponding set of positive roots, which consists of those α in R^+ such that s_α is in W_P.

For an element u in W/W_P, $\ell(u)$ denotes the minimum length of a representative in W. In fact, each u has a unique representative of minimum length; each element w of W can be written uniquely as a product $a \cdot b$, with a the element of minimal length in the coset of w and b in W_P, and with $\ell(w) = \ell(a) + \ell(b)$. (For these facts see [24], §1.10.) The Weyl group acts on the left on W/W_P. For u in W/W_P, we write $u^\vee$ in place of $w_o u$.

For u in W/W_P, we let $X(u) = \overline{BuP/P}$ be the corresponding **Schubert variety**. (The u on the right of this equation should be replaced by a representative first in W, and then by a representative in $N(T)$, but, as the result is independent of these choices, we follow the common convention of omitting them.) This is a subvariety of $X = G/P$ of dimension $\ell(u)$; we denote its cohomology class $[X(u)]$ by $\sigma(u)$. Similarly,

we let $Y(u) = \overline{B^- u P/P}$ be the **opposite Schubert variety**; it is of codimension $\ell(u)$, and we denote its cohomology class by σ_u. Since $Y(u) = w_o X(u^\vee)$, and translations of subvarieties by elements of G have the same cohomology classes, we have

$$\sigma_u = [Y(u)] = \sigma(u^\vee) = [X(u^\vee)] \qquad \text{in} \qquad H^{2\ell(u)}(X).$$

(Cohomology here will always be taken with integer coefficients.) These classes form an additive basis for $H^\star(X)$. For w in W, we sometimes write σ_w for the class (and $X(w)$ and $Y(w)$ for the Schubert varieties) corresponding to the coset wW_P containing w.

For any u in W/W_P, we let $x(u) = uP/P$ be the corresponding point in X. These are the fixed points of the action of T on X. The varieties $X(u)$ and $Y(u)$ meet transversally at the point $x(u)$, and the classes σ_u and $\sigma_{u^\vee} = \sigma(u)$ are dual classes under the intersection pairing: $\int_X \sigma_u \cdot \sigma_v$ is 1 if $v = u^\vee$, and 0 otherwise.

The Schubert classes of dimension one have the form $\sigma(s_\beta)$ as β varies over $\Delta \smallsetminus \Delta_P$. By a **degree** d we mean a nonnegative integral combination $d = \sum d_\beta \, \sigma(s_\beta)$ of these classes; a degree may be identified with a collection of nonnegative integers $(d_\beta)_{\beta \in \Delta \smallsetminus \Delta_P}$. The degrees are the classes of curves on X. If d and d' are degrees, we write $d \leq d'$ to mean that $d_\beta \leq d'_\beta$ for all β.

For any positive root α, write $\alpha = \sum n_{\alpha\beta} \, \beta$ as the nonnegative sum of simple roots β; then define the **degree** $d(\alpha)$ of α by

$$d(\alpha) = \sum_{\beta \in \Delta \smallsetminus \Delta_P} n_{\alpha\beta} \, \frac{(\beta, \beta)}{(\alpha, \alpha)} \, \sigma(s_\beta).$$

Here as usual $(\,,\,)$ is a W-invariant inner product on the real subspace of $\mathfrak{t}^\star$ spanned by R. If $h_\alpha = 2\alpha/(\alpha, \alpha)$, and ω_β is the fundamental weight corresponding to β (so that the weights h_β and ω_β are dual bases, for β in Δ), then $h_\alpha(\omega_\beta) = n_{\alpha\beta}(\beta, \beta)/(\alpha, \alpha)$, so this definition is equivalent to setting

$$d(\alpha) = \sum_{\beta \in \Delta \smallsetminus \Delta_P} h_\alpha(\omega_\beta) \, \sigma(s_\beta).$$

Lemma 3.1. *If w is in W_P, then $d(w(\alpha)) = d(\alpha)$.*

Proof. It suffices to prove this for a generator $w = s_\gamma$, for γ in Δ_P. Since $s_\gamma(\alpha) = \alpha - 2(\alpha, \gamma)/(\gamma, \gamma) \, \gamma$, the coefficients of all β in the expansions of α and $w(\alpha)$ are the same for β in Δ_P. Noting that $(w(\alpha), w(\alpha)) = (\alpha, \alpha)$ for all w and α, the result follows. $\square$

For any positive root α that is not in R_P^+, there is a unique T-invariant curve C_α in X that contains the points $x(1)$ and $x(s_\alpha)$. Indeed, $C_\alpha = Z_\alpha \cdot P/P$, where Z_α is the 3-dimensional subgroup of G whose Lie algebra is $\mathfrak{g}_\alpha \oplus \mathfrak{g}_{-\alpha} \oplus [\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}]$. To see that C_α is unique, by the Bruhat decomposition there is a neighborhood of $x(1)$ that is T-equivariantly isomorphic to $\mathfrak{u}^- = \mathfrak{g}/\mathfrak{p}$ (see [8], §14); the T-invariant curves in $\mathfrak{u}^-$ correspond to weight spaces $\mathfrak{g}_{-\alpha}$ for α in $R^+ \smallsetminus R_P^+$. If α is in $\Delta \smallsetminus \Delta_P$, then $C_\alpha = X(s_\alpha)$ is one of our basic Schubert varieties. If α is in R_P^+, then $Z_\alpha \cdot P/P$ is the point $x(1)$.

If λ is a weight that vanishes on all β in Δ_P, it determines a character on P, and so a line bundle $L(\lambda) = G \times_P \mathbb{C}(\lambda)$ on G/P.

Lemma 3.2. $\int_{C_\alpha} c_1(L(\lambda)) = h_\alpha(\lambda)$.

Proof. Lemma 2.1 gives $\int_{C_\alpha} c_1(L(\lambda)) = (\lambda - s_\alpha(\lambda))/\alpha = h_\alpha(\lambda)$. $\qquad\square$

Applying this to $\lambda = \omega_\beta$ and $\alpha = \beta$ in Δ, we deduce:

Lemma 3.3. *For β in $\Delta \smallsetminus \Delta_P$, $c_1(L(\omega_\beta)) = \sigma_{s_\beta}$.*

Lemma 3.4. *The degree $[C_\alpha]$ of C_α is $d(\alpha)$.*

Proof. This is proved in [14], pp. 14–19. It follows more easily from the preceding two lemmas, since $\sigma_{s_\beta} \cdot [C_\alpha] = h_\alpha(\omega_\beta)$ implies that $[C_\alpha] = \sum h_\alpha(\omega_\beta)\, \sigma(s_\beta)$. $\qquad\square$

Lemma 3.5. *The degree of the first Chern class of X on C_α is $n_\alpha = 4(\rho_P, \alpha)/(\alpha, \alpha)$, where $\rho_P = \frac{1}{2} \sum \gamma$, with the sum over the positive roots γ not in R_P^+. In particular,*

$$c_1(T_X) \;=\; 4 \sum_{\beta \in \Delta \smallsetminus \Delta_P} \frac{(\rho_P, \beta)}{(\beta, \beta)} \sigma_{s_\beta} \;=\; 2 \sum_{\beta \in \Delta \smallsetminus \Delta_P} h_\beta(\rho_P)\, \sigma_{s_\beta}.$$

Proof. This can also be proved by localization. Note that the tangent space to X at $x(1)$ is $\mathfrak{g}/\mathfrak{p} = \bigoplus_{\alpha \in R'} \mathfrak{g}_{-\alpha}$, where the sum is over the set $R' = R^+ \smallsetminus R_P^+$. So $T_{x(u)}X = \bigoplus_{\alpha \in u \cdot R'} \mathfrak{g}_{-\alpha}$. The weight of the tangent space to C_α at $x(1)$ is $-\alpha$. By Lemma 2.1,

$$\int_{C_\alpha} c_1(T_X) = \Big(\sum_{\gamma \in R'} s_\alpha(\gamma) - \gamma\Big)/(-\alpha) = \sum_{\gamma \in R'} 2\frac{(\gamma, \alpha)}{(\alpha, \alpha)}.$$

$\qquad\square$

4. Chains in the Bruhat graph

We need a combinatorial notion corresponding to the notion of a T-invariant curve joining the points $x(u)$ and $x(v)$ in $X = G/P$.

Lemma 4.1. *Let u and v be unequal elements in W/W_P. The following are equivalent:*
 (i) *There is a reflection s in W such that $v = s \cdot u$.*
(ii) *There are representatives $\tilde{u}$ for u and $\tilde{v}$ for v in W, and a reflection t in W such that $\tilde{v} = \tilde{u} \cdot t$.*
(iii) *For any representative $\tilde{u}$ of u in W, there is a reflection s (resp. a reflection t) such that $s \cdot \tilde{u}$ (resp. $\tilde{u} \cdot t$) is a representative of v.*

The reflection s of (i) *is uniquely determined. The reflection t of* (ii) *is determined up to conjugation by an element of W_P.*

Proof. (i) holds when there are representatives $\tilde{u}$ for u and $\tilde{v}$ for v such that $\tilde{v} = s \cdot \tilde{u}$. Equivalently $\tilde{v} = \tilde{u} \cdot t$, with $t = \tilde{u}^{-1} \cdot s \cdot \tilde{u}$, which is (ii). In either case the representative $\tilde{u}$ can be chosen arbitrarily. Both uniqueness assertions will follow from the

Claim. Let s and s' be reflections, not in W_P. If $s' = s \cdot a$ for some a in W_P, then $s' = s$.

Granting the claim, in (i), if $s' \cdot u = s \cdot u \neq u$, then for any representative $\tilde{u}$ of u, $s' \cdot \tilde{u} = s \cdot \tilde{u} \cdot a$ for some a in W_P. Then $\tilde{u}^{-1} \cdot s' \cdot \tilde{u} = \tilde{u}^{-1} \cdot s \cdot \tilde{u} \cdot a$, and since $\tilde{u}^{-1} \cdot s \cdot \tilde{u}$ is not in W_P, the claim implies that $\tilde{u}^{-1} \cdot s' \cdot \tilde{u} = \tilde{u}^{-1} \cdot s \cdot \tilde{u}$, so $s' = s$. Similarly in (ii), if $\tilde{v} = \tilde{u} \cdot t$ and $\tilde{v} \cdot a = \tilde{u} \cdot b \cdot t'$ for some a and b in W_P, then $b \cdot t' = t \cdot a$, so $b \cdot t' \cdot b^{-1} = t \cdot c$, with $c = a \cdot b^{-1}$ in W_P. The claim implies that $t = b \cdot t' \cdot b^{-1}$, as required.

Proof of the claim. Let v be a weight such that $(\beta, v) = 0$ for all β in Δ_P, and $(\beta, v) > 0$ for all β in $\Delta \smallsetminus \Delta_P$. For any w in W, we have $w(v) = v$ if and only if w is in W_P ([10] V, §4.6). In particular $s'(v) = s(v) \neq v$. If $s = s_\alpha$ and $s' = s_\gamma$, for α and γ positive roots, then $v - 2(\alpha, v)/(\alpha, \alpha) \cdot \alpha = v - 2(\gamma, v)/(\gamma, \gamma) \cdot \gamma$. This implies that α and γ are proportional, which cannot happen unless $\alpha = \gamma$. $\square$

The claim amounts to the fact that if α and γ are distinct positive roots that are not sums of roots in Δ_P, then the cosets of s_α and s_γ in W/W_P are distinct.

We will say that two unequal elements u and v in W/W_P are **adjacent** if they are related as in Lemma 4.1. Note that this is a symmetric relation. In this case we define $d(u, v)$ to be the degree $d(\alpha)$, where $t = s_\alpha$ is a reflection relating them as in (ii). If t is replaced by $w \cdot t \cdot w^{-1} = s_{w(\alpha)}$, for w in W_P, the degree does not change (Lemma 3.1), so $d(u, v)$ depends only on u and v. Note that if u and v are adjacent, then for any w in W, $w \cdot u$ and $w \cdot v$ are also adjacent, and $d(w \cdot u, w \cdot v) = d(u, v)$. In particular, $u^\vee$ and $v^\vee$ are also adjacent, with $d(u^\vee, v^\vee) = d(u, v)$.

Lemma 4.2. *Elements u and v in W/W_P are adjacent if and only if $x(u) \neq x(v)$ and there is a T-invariant curve C containing $x(u)$ and $x(v)$. If this is true, the curve C is unique, isomorphic to $\mathbb{P}^1$, and its class $[C]$ in $H_2(X)$ is equal to $d(u, v)$.*

Proof. We have seen that the T-invariant curves containing $x(1)$ are exactly the curves C_α, which also contains $x(s_\alpha)$, for α in $R^+ \smallsetminus R_P^+$. General T-invariant curves in X therefore have the form $w \cdot C_\alpha$, for some α in $R^+ \smallsetminus R_P^+$ and w in W. This curve is the unique T-invariant curve containing $x(w) = w \cdot x(1)$ and $x(w \cdot s_\alpha) = w \cdot x(s_\alpha)$. The result then follows from Lemmas 3.1, 3.4, and 4.1. (For more about T-invariant curves in general, see [13].) $\square$

We use also the **Bruhat order** on W/W_P, which sets $u \prec v$ if $X(u) \subset X(v)$.

Lemma 4.3. *For u and v in W/W_P, the following are equivalent:*

(i) $u \preceq v$;

(ii) *for any sequence of $\ell(v)$ simple transpositions whose product represents v, a representative of u can be obtained by removing some of these transpositions;*

(iii) $x(u) \in X(v)$;

(iv) $v^\vee \preceq u^\vee$;

(v) $x(v) \in Y(u)$.

Proof. For the equivalence of (i), (ii), and (iv), see [24], §5.9, 5.10. The equivalence of (i) and (iii) follows from the fact that $X(u)$ is the closure of $B \cdot x(u)$. Then (iv) is equivalent to $x(v^\vee)$ being in $X(u^\vee)$, or to $x(v) = w_o x(v^\vee)$ being in $w_o X(u^\vee) = Y(u)$, which is (v). □

Now define a **chain** from u to v in W/W_P to be a sequence $u_0, u_1, \ldots, u_r$ in W/W_P such that u_i and u_{i-1} are adjacent for $1 \le i \le r$, and, in addition, $u \preceq u_0$ and $u_r \preceq v^\vee$. For any chain $u_0, u_1, \ldots, u_r$ we define the **degree** of the chain to be the sum of the degrees $d(u_{i-1}, u_i)$, for $1 \le i \le r$. Note that such a chain from u to v determines a chain from v to u, by $v \preceq u_r^\vee, \ldots, u_0^\vee \preceq u^\vee$, and these chains have the same degree. Note also that there is a chain of degree 0 between u and v exactly when $u \preceq v^\vee$.

A chain from u to v determines, and is determined by, a sequence of T-invariant curves $C_1, C_2, \ldots, C_r$ in X, each meeting the next, with C_1 meeting $Y(u)$ and C_r meeting $X(v^\vee)$. Indeed, C_i is the T-invariant curve that connects $x(u_{i-1})$ to $x(u_i)$. The degree of the chain is the sum of the classes $[C_i]$ of the curves.

5. Interpretation for the Grassmannian

For the Grassmannian $\mathrm{Gr}(r, n)$, G is $SL_n(\mathbb{C})$, B is the subgroup of upper triangular matrices, T the diagonal matrices in B, and W is identified with the symmetric group S_n. The simple roots are $\Delta = \{\alpha_i = e_i - e_{i+1}, 1 \le i \le n - 1\}$. The parabolic subgroup P consists of matrices in G that map the subspace of $\mathbb{C}^n$ spanned by the first r basic vectors to itself, and Δ_P consists of all simple roots with the exception of α_r. The Weyl group W_P is identified with $S_r \times S_{n-r}$. The minimal representative of a u in W/W_P is a permutation w such that $w(1) < w(2) < \ldots < w(r)$ and $w(r + 1) < \ldots < w(n)$. From this we form a partition

$$\lambda(u) = (w(r) - r, w(r - 1) - (r - 1), \ldots, w(2) - 2, w(1) - 1),$$

with $n - r \ge \lambda_1(u) \ge \ldots \ge \lambda_r(u) \ge 0$. This sets up natural bijections between: (i) elements of W/W_P; (ii) partitions inside the r by $n - r$ rectangle; and (iii) subsets of $\{1, \ldots, n\}$ with r elements. With this notation, the subset corresponding to u is $\{w(1), w(2), \ldots, w(r)\}$, for any representative w of u in S_n. Note that $u \preceq v$ if and only if $\lambda(u)$ is contained in $\lambda(v)$, i.e., $\lambda_i(u) \le \lambda_i(v)$ for $1 \le i \le r$.

Lemma 5.1. *In the Grassmann case, u is adjacent to v if and only if one of $\lambda(u)$ and $\lambda(v)$ is contained in the other, and the difference is a (connected, nonempty) rim hook. In this case the degree $d(u, v)$ is 1.*

Proof. Let I and J be the subsets of $\{1, \ldots, n\}$ corresponding to u and v. Then u and v are adjacent exactly when I and J differ by one element, i.e., there is a p in $I \setminus J$ and a q in $J \setminus I$ with $I \cup q = J \cup p$; in this case, $v = (p, q) \cdot u$. If $p < q$, then $\lambda(u)$ is obtained from $\lambda(v)$ by removing a rim hook of $q - p$ boxes, starting at the end the kth row, where $k - 1$ is the number of elements in J that are bigger than q. The transposition t has the form (i, j) for $i \le r < j$, and one sees readily that $d(e_i - e_j) = 1$. □

The Bruhat graph for this case is known as the Johnson graph. The figure shows the Johnson graph for $\mathrm{Gr}(2,4)$, labeled by the partitions λ with $2 \geq \lambda_1 \geq \lambda_2 \geq 0$.
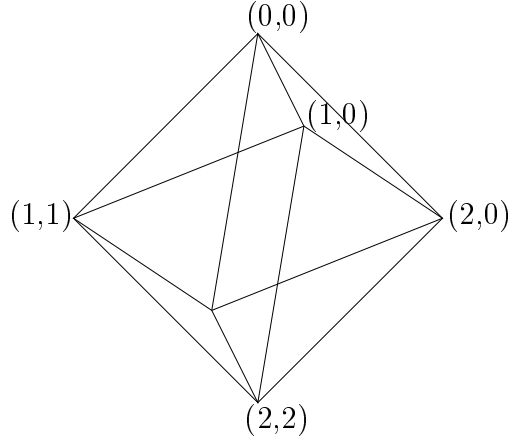
Remark 5.2. In this Grassmann case, it is not hard to see that if there is a chain of degree d from u to v, then there is a *monotone* chain of degree at most d from u to v, i.e., a chain that starts with u_0 equal to u, removes a rim hook at each stage, and ends at a u_r with $\lambda(u_r)$ contained in the $180°$ rotation of $\lambda(v)$.

6. The quantum cohomology of G/P

We next describe the (small) quantum cohomology of X. Take a variable q_β for each β in $\Delta \setminus \Delta_P$, and let $\mathbb{Z}[q]$ be the polynomial ring with these q_β as indeterminants, but giving q_β the degree $2n_\beta$, where $n_\beta = \int_{\sigma(s_\beta)} c_1(T_X)$ from Lemma 3.5. For a degree $d = \sum d_\beta \, \sigma(s_\beta)$, we write q^d for the monomial $\prod_\beta q_\beta^{d_\beta}$. The small quantum cohomology ring $QH^\star(X)$ is, as a $\mathbb{Z}[q]$-module, simply $H^\star(X) \otimes_{\mathbb{Z}} \mathbb{Z}[q]$, so the same Schubert classes $\sigma_u = \sigma_u \otimes 1$ form a basis for $QH^\star(X)$ over $\mathbb{Z}[q]$. The multiplication is a deformation of the classical multiplication:

$$\sigma_u \star \sigma_v = \sum_d q^d \sum_w N_{u,v}^w(d) \, \sigma_w,$$

where the first sum is over all degrees d, and the second is over all w in W/W_P such that $\ell(w) = \ell(u) + \ell(v) - \sum_\beta d_\beta n_\beta$. The coefficient $N_{u,v}^w(d)$ is a Gromov-Witten (GW) invariant: it is the number of morphisms $\varphi : \mathbb{P}^1 \to X$ of degree d (i.e., $\varphi_\star[\mathbb{P}^1] = d$ in $H_2(X)$), such that, for three given distinct points p_1, p_2, p_3 in $\mathbb{P}^1$, and three general $g_1, g_2,$ and g_3 in G, $\varphi(p_1)$ is in $g_1 \cdot Y(u)$, $\varphi(p_2)$ is in $g_2 \cdot Y(v)$, and $\varphi(p_3)$ is in $g_3 \cdot X(w)$. When $d = 0$, this is the usual coefficient of σ_w in the classical product $\sigma_u \cdot \sigma_v$, which is the same as the intersection number $\int_X \sigma_u \cdot \sigma_v \cdot \sigma_{w^\vee}$.

More generally, the (small) GW-invariant $\langle \sigma_{u_1}, \sigma_{u_2}, \dots, \sigma_{u_n} \rangle_d$ can be defined whenever $\sum \ell(u_i) = \dim(X) + \sum_\beta n_\beta d_\beta$. Fix general distinct points $p_1, \dots, p_n$ in $\mathbb{P}^1$. This invariant is the number of maps φ from $\mathbb{P}^1$ to X such that $\varphi(p_i)$ is in $g_i \cdot Y(u_i)$, for $1 \leq i \leq n$,

and $g_1, \ldots, g_n$ general elements of G. This can be interpreted in the cohomology of appropriate moduli spaces. As we will need these spaces in our proofs, we describe them now.

Let $\overline{M}_{0,n}(X, d)$ be the moduli space of stable maps of degree d of n-pointed genus 0 curves into X; a point is written $(C, p_1, \ldots, p_n, \varphi)$, where C is a connected tree of projective lines, meeting in nodes, $p_1, \ldots, p_n$ are distinct nonsingular points of C, and $\varphi : C \to X$ is a morphism with $\varphi_*[C] = d$, with the property that any component of C that is mapped to a point by φ must have at least three points that are either marked points or intersection points with other components. This moduli space is a projective variety of dimension

$$\dim(\overline{M}_{0,n}(X, d)) \,=\, \dim(X) \,+\, \sum_\beta n_\beta \, d_\beta \,+\, n \,-\, 3.$$

It comes equipped with n evaluation maps $e_i : \overline{M}_{0,n}(X, d) \to X$, taking $(C, p_1, \ldots, p_n, \varphi)$ to $\varphi(p_i)$, and a forgetful map $f : \overline{M}_{0,n}(X, d) \to \overline{M}_{0,n}$, where the latter is the space of stable n-pointed curves of genus 0; f takes $(C, p_1, \ldots, p_n, \varphi)$ to $(C, p_1, \ldots, p_n)$, but suitably stabilized by collapsing components of C that have fewer than three markings or intersections with other components. We refer to [20] for construction and basic properties of these spaces and mappings, as well as Kontsevich's proof of the associativity of the quantum product.

The GW-invariant $\langle \sigma_{u_1}, \ldots, \sigma_{u_n} \rangle_d$ is then the intersection number $\int_{\overline{M}_{0,n}(X,d)} f^\star([p]) \cdot e_1{}^\star(\sigma_{u_1}) \cdot e_2{}^\star(\sigma_{u_2}) \cdot \ldots \cdot e_n{}^\star(\sigma_{u_n})$, where p is a point in $\overline{M}_{0,n}$. Equivalently, it is the coefficient of the fundamental class $1 = [\overline{M}_{0,n}]$ in the class

$$f_\star\big(e_1{}^\star(\sigma_{u_1}) \cdot e_2{}^\star(\sigma_{u_2}) \cdot \ldots \cdot e_n{}^\star(\sigma_{u_n})\big) \qquad \text{in} \quad H^0(\overline{M}_{0,n}).$$

In particular, this shows that the Gromov-Witten invariants are the same whether one chooses any distinct points $p_1, \ldots, p_n$ in $\mathbb{P}^1$ instead of general points (see [5]).

The coefficient $N_{u,v}^w(d)$ is equal to $\langle \sigma_u, \sigma_v, \sigma_{w^\vee} \rangle_d$. In fact, these invariants can be used to multiply several Schubert classes directly:

$$\sigma_{u_1} \star \sigma_{u_2} \star \ldots \star \sigma_{u_n} = \sum_d q^d \sum_w \langle \sigma_{u_1}, \ldots, \sigma_{u_n}, \sigma_{w^\vee} \rangle_d \, \sigma_w.$$

7. Transversality

The results of this section are the main tools needed to prove our theorem. We give here a simple proof based on Kleiman's transversality theorem. An alternative proof is sketched briefly at the end of this section.

Lemma 7.1. *Let $U \subset G \times G$ be open, nonempty, and invariant under the left diagonal multiplication by G. Let u_1 and u_2 be in W/W_P. Then for any g_1, g_2 in G such that $g_1 B g_1{}^{-1}$ and $g_2 B g_2{}^{-1}$ intersect in a maximal torus, there is a (h_1, h_2) in U such that*

$$h_1 X(u_1) = g_1 X(u_1) \qquad \text{and} \qquad h_2 X(u_2) = g_2 X(u_2).$$

Proof. Take (h_1, h_2) in the intersection of U with the open set of pairs (h_1, h_2) such that $h_1 B h_1^{-1}$ and $h_2 B h_2^{-1}$ are opposite Borels, i..e., $h_1 B h_1^{-1} \cap h_2 B h_2^{-1}$ is a maximal torus. By [8], §14.1, Cor. 3, there is a g in G such that $g(h_i B h_i^{-1})g^{-1} = g_i B g_i^{-1}$ for $i = 1, 2$. Since B is its own normalizer, this implies that $gh_i B = g_i B$ for $i = 1, 2$. Then (gh_1, gh_2) is in U, and $gh_i X(u_i) = g_i X(u_i)$ for $i = 1, 2$. $\square$

Lemma 7.2. *Let Z be an irreducible G-variety, and let $F : Z \to X \times X$ be a G-equivariant morphism, where G acts diagonally on $X \times X$. Then, for any u and v in W/W_P, the subscheme $F^{-1}(Y(u) \times X(v))$ is reduced, locally irreducible, of codimension $\ell(u) + \ell(v^\vee)$, and nonsingular at any nonsingular point of Z that maps to a nonsingular point of $Y(u) \times X(v)$.*

Proof. Consider the diagram

$$X(u^\vee) \times X(v)$$
$$\downarrow$$
$$Z \longrightarrow X \times X$$

with $G \times G$ acting on $X \times X$. Kleiman's transversality theorem [28] produces a nonempty open set U of pairs (g_1, g_2) in $G \times G$ such that $F^{-1}(g_1 X(u^\vee) \times g_2 X(v))$ satisfies the conclusions of the lemma. (The dimension assertion would be valid in all characteristics; the others use the characteristic zero assumption.) This set U is invariant by the left diagonal action of G because the morphism F is G-equivariant.

Now apply Lemma 7.1 to $(u_1, u_2) = (u^\vee, v)$ and $(g_1, g_2) = (w_o, 1)$. This produces a (h_1, h_2) in U such that $h_1 X(u^\vee) = w_o X(u^\vee) = Y(u)$ and $h_2 X(v) = X(v)$, and Lemma 7.2 follows. $\square$

A point $\zeta = (C, p_1, \ldots, p_n, \varphi)$ in a moduli space $\overline{M}_{0,n}(X, d)$ of stable maps consists of a tree C of $\mathbb{P}^1$'s, and marked points on some of its components, with a stable map from C to X. Such a point lies in a unique locally closed subscheme V, for which the tree of curves has the same topological type (or combinatorial configuration), with marked points on corresponding components (see [3], [20]). The codimension of V is the number of nodes of C.

When n and d are understood, we set

$$E(u, v) = e_1^{-1}(Y(u)) \cap e_2^{-1}(X(v)),$$

a closed subscheme of $\overline{M}_{0,n}(X, d)$.

Lemma 7.3. *If $E(u, v)$ is not empty, then $E(u, v)$ is a reduced, locally irreducible, subscheme of $\overline{M}_{0,n}(X, d)$, of pure codimension $\ell(u) + \ell(v^\vee)$, any component of which which meets any stratum V properly. In particular, each irreducible component of $E(u, v)$ meets the locus $M_{0,n}(X, d)$ consisting of those $(C, p_1, \ldots, p_n, \varphi)$ with $C \cong \mathbb{P}^1$.*

Proof. This follows from Lemma 7.2, and the fact that the strata are locally closed, G-invariant subvarieties in $\overline{M}_{0,n}(X, d)$. $\square$

Lemma 7.4. *For any degree d, for $n > \sum n_\beta d_\beta$, and for any distinct points $p_1, \ldots, p_n$ in $\mathbb{P}^1$ and any points $x_1, \ldots, x_n$ in X, there are only finitely many morphisms $\varphi : \mathbb{P}^1 \to X$ of degree d with $\varphi(p_i) = x_i$ for $1 \leq i \leq n$.*

Proof. Let $\operatorname{Hom}(\mathbb{P}^1, X)_d$ be the space of morphisms of degree d from $\mathbb{P}^1$ to X, and let $e : \operatorname{Hom}(\mathbb{P}^1, X)_d \to X^n$ be the morphism obtained by evaluating at the given points $p_1, \ldots, p_n$. We show that e is unramified, and hence has finite fibers. This will be true if its tangent map [29]

$$\Gamma(\mathbb{P}^1, \varphi^\star(T_X)) \longrightarrow \oplus_{i=1}^n \varphi^\star(T_X)(p_i) = \oplus_{i=1}^n T_{x_i}(X)$$

is injective. Now $\varphi^\star(T_X) = \oplus \mathcal{O}(m_j)$, with $m_j \geq 0$, and $\sum m_j = \sum n_\beta d_\beta < n$, so $m_j < n$ for all j. The lemma follows from the elementary fact that $\Gamma(\mathbb{P}^1, \mathcal{O}(m)) \to \oplus_{i=1}^n \mathcal{O}(m)(p_i)$ is injective for $m < n$ (since a nonzero polynomial of degree m cannot vanish at more than m points). $\square$

The transversality Lemma 7.3 can also be proved by showing that, given a fixed point $\zeta = (C, p_1, \ldots, p_n, \varphi)$ with $\varphi(p_1) = x(u)$ and $\varphi(p_2) = x(v)$, the map from $\operatorname{Hom}(C, X)_d$ to X^2 given by the first two evaluation maps is transversal to the subvariety $Y(u) \times X(v)$ at the point $x(u) \times x(v)$. One shows that the map from the tangent space $\Gamma(C, \varphi^\star(T_X))$ to the normal space to $Y(u) \times X(v)$ at $x(u) \times x(v)$ is surjective. This can be achieved by T-equivariantly decomposing the bundle $\varphi^\star(T_X)$ as a direct sum of line bundles, and using the fact that the weights of the normal spaces of $Y(u)$ at $x(u)$ and $X(v)$ at $x(v)$ are disjoint.

8. CHEVALLEY'S FORMULA IN THE PARABOLIC CASE

Chevalley's formula [14] generalizes Monk's formula from the classical flag variety to an arbitrary G/B, giving a formula for the product of a codimension one Schubert class σ_{s_β} and an arbitrary Schubert class σ_w. We will need the analogous formula on a general G/P. Although it is not hard to deduce such a formula from Chevalley's, by means of the projection $G/B \to G/P$, we include a proof, which combines Chevalley's geometric ideas with our calculations here.

Recall that for a simple root β, and a positive root α, $h_\alpha(\omega_\beta) = n_{\alpha\beta} \frac{(\beta, \beta)}{(\alpha, \alpha)}$, where $n_{\alpha\beta}$ is the coefficient of α in its expansion as a positive linear combination of simple roots (see §3).

Lemma 8.1 (Chevalley's formula). *Let β be in $\Delta \smallsetminus \Delta_P$, let u be in W/W_P, and let $\tilde{u}$ be the minimal length representative of u in W. Then*

$$\sigma_{s_\beta} \cdot \sigma_u = \sum h_\alpha(\omega_\beta)\, \sigma_{\tilde{u}s_\alpha},$$

the sum over all positive roots α such that $\ell(\tilde{u}s_\alpha) = \ell(\tilde{u}) + 1$.

Proof. We must prove that, for v in W/W_P with $\ell(v) = \ell(u) + 1$, $\int_X \sigma_u \cdot \sigma_{v^\vee} \cdot \sigma_{s_\beta} = h_\alpha(\omega_\beta)$ if u and v are adjacent, and u has a representative $\tilde{u}$ such that $\tilde{u} \cdot s_\alpha$ is a representative for v; and that $\int_X \sigma_u \cdot \sigma_{v^\vee} \cdot \sigma_{s_\beta} = 0$ if u and v are not adjacent. Note that if $\sigma_u \cdot \sigma_v^\vee$

is not zero, then $Y(u) \cap X(v)$ is not empty. This locus is fixed by the torus T, and its fixed points consist of those $x(w)$ with $u \preceq w \preceq v$ (Lemma 4.3). Since $\ell(u) = \ell(v) - 1$, the only fixed points are $x(u)$ and $x(v)$. It follows that, set-theoretically at least, this intersection is the curve $C(u,v)$. We claim that $Y(u)$ intersects $X(v)$ properly, with multiplicity 1, in the curve $C(u,v)$. In characteristic zero this follows from Lemma 7.3.[3] Therefore

$$\sigma_u \cdot \sigma_{v^\vee} = [Y(u)] \cdot [X(v)] = [C(u,v)] = d(\alpha) = \sum_\beta h_\alpha(\omega_\beta) \, \sigma(s_\beta),$$

the last by formulas in §3. □

Lemma 8.2. *If $u \preceq v$ in W/W_P, there is a w in W/W_P such that $\sigma_u \cdot \sigma_w$ contains σ_v with positive coefficient.*

Proof. This is trivial if $u = v$. If $\ell(u) = \ell(v) - 1$, and $\tilde{u}$ is the minimal length representative of u, there is an α in R^+ such that $\tilde{u} \cdot s_\alpha$ is a representative of v (see [24],§5.11), and α is not in R_P^+ since $u \neq v$. In this case, if we choose β so that $h_\alpha(\omega_\beta) \neq 0$, then $\sigma_{s_\beta} \cdot \sigma_u$ contains σ_v with positive coefficient by Lemma 8.1. In general, induct on $\ell(v) - \ell(u)$, by choosing u' not equal to u or v with $u \prec u' \prec v$. If $\sigma_{w(1)} \cdot \sigma_u$ contains $\sigma_{u'}$ with positive coefficient, and $\sigma_{w(2)} \cdot \sigma_{u'}$ contains σ_v with positive coefficient, some σ_w that appears in $\sigma_{w(1)} \cdot \sigma_{w(2)}$ must have σ_v occurring in $\sigma_w \cdot \sigma_u$ with positive coefficient. □

9. THE THEOREM

The classes $q^d \sigma_w$, as d varies over degrees, and w varies over W/W_P, form a basis for the quantum cohomology ring $QH^\star(X)$ over $\mathbb{Z}$. Given any element τ in $QH^\star(X)$, we say that q^d **occurs** in τ if the coefficient of $q^d \sigma_w$ is not zero for some w. When τ is a product of Schubert classes, we know that all such coefficients are nonnegative. For example, q^d occurs in $\sigma_u \star \sigma_v$ exactly when there is a $w^\vee$ for which the GW-invariant $\langle \sigma_u, \sigma_v, \sigma_{w^\vee} \rangle_d$ is positive.

We now come to our main result. The theorem gives three equivalent criteria for a degree to be minimal, while the proof shows these are equivalent to eight other related criteria.

Theorem 9.1. *Let u and v be in W/W_P, and let d be a degree. The following are equivalent:*

(1) *There is a degree $c \leq d$ such that q^c occurs in $\sigma_u \star \sigma_v$.*
(2) *There is a chain of degree $c \leq d$ between u and v.*
(3) *There is a morphism $\varphi : \mathbb{P}^1 \to X$ with $\varphi_\star[\mathbb{P}^1] \leq d$ such that $\varphi(\mathbb{P}^1)$ meets $Y(u)$ and $X(v^\vee)$.*

---

[3]In arbitrary characteristic, Chevalley argues as follows. Since $Y(u)$ meets $X(u)$ transversally at the point $x(u)$, and $x(u)$ is a nonsingular point on $X(v)$ (since $X(v)$ is nonsingular in codimension 1 and $\mathrm{codim}(X(u), X(v)) = 1$), it follows that $Y(u)$ meets $X(v)$ transversally at $x(u)$.

Proof. We first state the eight equivalent conditions, and then we construct enough implications to show that each of the eleven implies the others.

(4) There is a degree $c \leq d$, a $u' \succeq u$, and a $v' \succeq v$ such that q^c occurs in $\sigma_{u'} \star \sigma_{v'}$.

(5) There is a τ in $QH^\star(X)$ such that q^d occurs in $\sigma_u \star \sigma_v \star \tau$.

(6) The same as in (5), but with $\tau = \sigma_{w_1} \star \ldots \star \sigma_{w_r}$, for some $w_1, \ldots, w_r$ in W/W_P.

(7) There is a sequence $C_0, \ldots, C_r$ of T-invariant curves on X, with C_0 meeting $Y(u)$ and C_r meeting $X(v^\vee)$, with C_{i-1} meeting C_i for $1 \leq i \leq r$, and with $\sum_{i=0}^r [C_i] \leq d$.

(8) There is a connected curve C in X with $[C] \leq d$, meeting $Y(u)$ and $X(v^\vee)$.

(9) There is an $n \geq 3$ and $w_3, \ldots, w_n$ in W/W_P, and a $c \leq d$, such that, with $e_i : \overline{M}_{0,n}(X, c) \to X$ the evaluation maps, and $f : \overline{M}_{0,n}(X, c) \to \overline{M}_{0,n}$ the forgetful map,

$$f_\star \big(e_1{}^\star(\sigma_u) \cdot e_2{}^\star(\sigma_v) \cdot e_3{}^\star(\sigma_{w_3}) \cdot \ldots \, e_n{}^\star(\sigma_{w_n}) \big) = \kappa \cdot 1$$

in $H^0(\overline{M}_{0,n})$, with $\kappa > 0$.

(10) There is a w in W/W_P and a $c \leq d$ such that

$$\int_{\overline{M}_{0,3}(X,c)} e_1{}^\star(\sigma_u) \cdot e_2{}^\star(\sigma_v) \cdot e_3{}^\star(\sigma_w) \neq 0.$$

(11) There is a $c \leq d$ such that the locus $E(u, v^\vee) = e_1{}^{-1}(Y(u)) \cap e_2{}^{-1}(X(v^\vee))$ is not empty in $\overline{M}_{0,3}(X, c)$.

$(1) \Leftrightarrow (4)$. The implication $(1) \Rightarrow (4)$ is trivial, so assume (4). Since $u \preceq u'$, it follows from Lemma 8.2 that there is a u'' so that $\sigma_{u'}$ occurs in the classical product $\sigma_u \cdot \sigma_{u''}$ with positive coefficient. Take similarly v'' for $v \preceq v'$. The fact that all coefficients of all quantum products of Schubert classes are nonnegative implies that $\sigma_u \star \sigma_{u''} \star \sigma_v \star \sigma_{v''}$ contains all the terms that occur in $\sigma_{u'} \star \sigma_{v'}$, so it must contain some $q^c \cdot \tau$, $\tau \neq 0$. But this is a product of $\sigma_u \star \sigma_v$ and a nonnegative combination of powers of q's times Schubert classes, so $\sigma_u \star \sigma_v$ must contain some q^e, with $e \leq c$.

$(6) \Rightarrow (5)$ and $(5) \Rightarrow (4)$ and $(10) \Rightarrow (9)$ are trivial.

$(1) \Rightarrow (10)$. If $\sigma_u \star \sigma_v$ contains $q^c \sigma_w$ with positive coefficient κ, then $f_\star \big(e_1{}^\star(\sigma_u) \cdot e_2{}^\star(\sigma_v) \cdot e_3{}^\star(\sigma_{w^\vee}) \big) = \kappa \cdot 1$ in $H^0(\overline{M}_{0,3})$, and $\overline{M}_{0,3}$ is a point.

$(9) \Rightarrow (6)$. (9) implies that $q^c \sigma_{w_n^\vee}$ occurs with coefficient κ in $\sigma_u \star \sigma_v \star \sigma_{w_3} \star \ldots \star \sigma_{w_{n-1}}$.

$(7) \Rightarrow (2)$. This follows from the correspondence between T-invariant curves and pairs of adjacent elements of W/W_P (§4).

$(2) \Rightarrow (3)$. A chain of degree c between u and v corresponds to a chain of T-invariant curves between $x(u)$ and $x(v)$. We may assume that no curve appears more than once, since removing duplicates only decreases the degree. This corresponds to a point $\zeta = (C, p_1, p_2, \psi)$ in $\overline{M}_{0,2}(X, c)$, with $\psi : C \to X$ an embedding, $\psi(p_1) = x(u)$, and $\psi(p_2) = x(v)$. If $c = 0$, we are in the classical case, and $u \preceq v^\vee$, so $x(u)$ is in $Y(u) \cap X(v^\vee)$, and the constant map from $\mathbb{P}^1$ to $x(u)$ satisfies the conditions of (3). We may therefore assume that $c > 0$. By Lemma 7.3, $E(u, v^\vee)$ is not contained in any boundary component. It therefore contains a point $(\mathbb{P}^1, p_1, p_2, \varphi)$, and thus we have a map $\varphi : \mathbb{P}^1 \to X$ with $\varphi_\star[\mathbb{P}^1] = c$ and $\varphi(p_1)$ in $Y(u)$ and $\varphi(p_2)$ in $X(v^\vee)$.

(3) $\Rightarrow$ (8). With $\varphi : \mathbb{P}^1 \to X$ as in (3), the image curve $\varphi(\mathbb{P}^1)$ is connected and joins the two Schubert varieties, and its degree is at most $\varphi_\star[\mathbb{P}^1]$.

(8) $\Rightarrow$ (7). This is a consequence of the action of the torus T on a Hilbert scheme (or Chow variety) of curves on X that join the two (T-invariant) Schubert varieties, see [23]. There must be a curve in such a space that is fixed by T, and, being a limit of connected curves, it is connected.

(10) $\Rightarrow$ (11) follows from the fact that loci representing cohomology classes with nonzero product must intersect, and $e_1{}^\star(\sigma_u) \cdot e_2{}^\star(\sigma_v)$ lives on the locus $E(u, v^\vee)$.

(11) $\Rightarrow$ (7). Since the torus preserves $E(u, v^\vee)$, there must be a point ζ in $E(u, v^\vee)$ that is fixed by T. Writing $\zeta = (C, p_1, p_2, p_3, \varphi)$, the image of each irreducible component of C must be a T-invariant curve in X, and the image of each $\varphi(p_i)$ must be a point fixed by T; and we must have $\varphi(p_1)$ in $Y(u)$ and $\varphi(p_2)$ in $X(v^\vee)$. Since the image is connected one can extract from it a chain of T-invariant curves from $\varphi(p_1)$ to $\varphi(p_2)$, and the degree of this chain is at most $\varphi_\star[C] = c$.

(2) $\Rightarrow$ (9). Suppose we have a chain of degree $c \leq d$. Discarding extra curves in the chain, we may assume it is minimal. Take n larger than $\sum n_\beta c_\beta + 2$. The locus $E(u, v^\vee)$ in $\overline{M}_{0,n}(X, c)$ has pure codimension $\ell(u) + \ell(v)$, and it meets the open set $M_{0,n}(X, c)$, by Lemma 7.3. Set

$$e' = e_3 \times \ldots \times e_n \ : \ \overline{M}_{0,n}(X, c) \to X^{n-2}.$$

Take a point p in $M_{0,n}$, i.e., choose n distinct points in $\mathbb{P}^1$. By Lemma 7.4, the restriction of e' to $f^{-1}(p) \cap M_{0,n}(X, c)$ is a finite to one mapping. It follows that $(e')_\star[f^{-1}(p) \cap E(u, v^\vee)] \neq 0$. Hence there are $w_3, \ldots, w_n$ in W/W_P such that

$$(e')_\star[f^{-1}(p) \cap E(u, v^\vee)] \ \cdot \ \big(\sigma_{w_3} \times \ldots \times \sigma_{w_n}\big) = \kappa \cdot [\text{point}],$$

for some $\kappa \neq 0$. This means that

$$f_\star\big(e_1{}^\star(\sigma_u) \cdot e_2{}^\star(\sigma_v) \cdot e_3{}^\star(\sigma_{w_3}) \cdot \ldots \cdot e_n{}^\star(\sigma_{w_n})\big) = \kappa \cdot 1$$

in $H^0(\overline{M}_{0,n})$, as required. $\qquad \square$

10. Peterson's quantum Chevalley formula in the parabolic case

The quantum Chevalley formula in G/P gives the formula for a quantum product $\sigma_{s_\beta} \star \sigma_u$, for β in $\Delta \setminus \Delta_P$ and u in W/W_P. It starts with the classical product $\sigma_{s_\beta} \cdot \sigma_u$ given in Lemma 8.1. The terms with q^d for positive degrees d have a similar combinatorial description. This had been proved for the Grassmannian $\mathrm{Gr}(r, n)$ [5] and the complete flag manifold $\mathrm{Fl}(\mathbb{C}^n)$ [18]. For a positive root α, we use the notation n_α for $\int_{C_\alpha} c_1(T_X)$ as in Lemma 3.5; and, for a simple root β, $h_\alpha(\omega_\beta)$ as before Lemma 3.1. For $P = B$, this formula was stated by Peterson [33].

Theorem 10.1 (Quantum Chevalley Formula). *For β in $\Delta \setminus \Delta_P$, u in W/W_P, with $\tilde{u}$ its minimal length representative in W,*

$$\sigma_{s_\beta} \star \sigma_u = \sum_\alpha h_\alpha(\omega_\beta)\sigma_{\tilde{u}s_\alpha} + \sum_\alpha q^{d(\alpha)} h_\alpha(\omega_\beta)\sigma_{\tilde{u}s_\alpha},$$

the first sum over roots α in $R^+ \setminus R_P^+$ for which $\ell(v) = \ell(u) + 1$, where v is the coset of $\tilde{u}s_\alpha$ in W/W_P, and the second sum over roots α in $R^+ \setminus R_P^+$ for which $\ell(v) = \ell(u) + 1 - n_\alpha$.

Proof. Let $q^d \sigma_v$ be a term appearing in $\sigma_{s_\beta} \star \sigma_u$ with nonzero coefficient κ, i.e.,

$$\kappa = \langle \sigma_u, \sigma_{v^\vee}, \sigma_{s_\beta} \rangle_d = \int_{\overline{M}_{0,3}(X,d)} e_1^\star(\sigma_u) \cdot e_2^\star(\sigma_{v^\vee}) \cdot e_3^\star(\sigma_{s_\beta}) > 0.$$

It suffices to consider the case $d \neq 0$, since the classical case is covered by Chevalley's formula 8.1. Set $E = E(u,v)$ in $\overline{M}_{0,3}(X,d)$. By the transversality Lemma 7.3, E is reduced, locally irreducible, purely 1-dimensional, with each component meeting $M_{0,3}(X,d)$, and with

$$[E] = e_1^\star([Y(u)]) \cdot e_2^\star([X(v)]) = e_1^\star(\sigma_u) \cdot e_2^\star(\sigma_{v^\vee}).$$

So κ is the coefficient of $\sigma(s_\beta)$ in the class $(e_3)_\star[E]$.

We claim that $e_1(E) = x(u)$ and $e_2(E) = x(v)$. Indeed, each of the loci $e_i(E)$ is T-invariant, and if $e_1(E) \neq x(u)$, then $e_1(E)$ would contain some other T-fixed point $x(u')$, with $u \prec u'$, $u \neq u'$. But then E would be contained in $E(u', v)$, which (by transversality again) has larger codimension than E, a contradiction; and similarly if $e_2(E)$ contains $x(v')$, with $v' \prec v$.

Note that $\mathrm{codim}(E) = \ell(u) + \dim(X) - \ell(v) = \dim(\overline{M}_{0,3}(X,d)) - 1 = \dim(X) + \int_d c_1(T_X)$, so

$$\ell(v) = \ell(u) + 1 - \int_d c_1(T_X).$$

In particular, $\ell(v) < \ell(u)$, since, by Lemma 3.5, $\int_d c_1(T_X) \geq 2$.

A point of E not in the boundary has the form $\zeta = (\mathbb{P}^1, p_1, p_2, p_3, \varphi)$, with $\varphi : \mathbb{P}^1 \to X$, $\varphi_\star[\mathbb{P}^1] = d$, and $\varphi(p_1) = x(u)$ and $\varphi(p_2) = x(v)$. Given such a map φ, one can produce a curve in E containing this point by varying where φ maps p_3 in the curve $\varphi(\mathbb{P}^1)$, i.e., varying φ by $\varphi \circ \vartheta$, where ϑ is an automorphism of $\mathbb{P}^1$ that fixes p_1 and p_2. It follows that each irreducible component Z of E must consist generically of such maps. It follows from this that $\varphi(\mathbb{P}^1) = e_3(Z)$. Since $e_3(Z)$ is invariant by T, it follows that $\varphi(\mathbb{P}^1)$ is T-invariant, and, since it contains $x(u)$ and $x(v)$, we must have $\varphi(\mathbb{P}^1) = C(u,v)$. In particular, u and v must be adjacent.

We claim next that φ maps $\mathbb{P}^1$ isomorphically onto $C(u,v)$. If not, $\varphi_\star[\mathbb{P}^1] = k[C(u,v)]$, for $k > 1$. Consider the corresponding locus $E' = E(u,v)$ in

$$\overline{M}' = \overline{M}_{0,3}(C(u,v), k[C(u,v)])$$

consisting of maps to $C(u,v)$ of degree k that take p_1 to $x(u)$ and p_2 to $x(v)$. The codimension of E' in $\overline{M}'$ is at most 2 (in fact, equal to 2 by Lemma 7.3 applied to the variety $C(u,v)$ in place of X). But the dimension of $\overline{M}'$ is $2k + 1$, and since $E' \subset E$, we must have $2k - 1 \leq 1$, i.e., $k = 1$.

It follows that $d = [C(u,v)] = d(u,v) = d(\alpha)$, where $\tilde{u}s_\alpha$ is a representative of v. It also follows that each component Z of E is equal to the locus E' described in the preceding paragraph. In particular, there is only one irreducible component $E = E'$ of

E. (This E can be realized from the blow-up of $C(u,v) \times C(u,v)$ along the two points $x(u) \times x(u)$ and $x(v) \times x(v)$; the exceptional divisors become the extra factors in curves corresponding to the two boundary points of E.) It also follows from this description that e_3 maps E isomorphically onto $C(u,v)$. Therefore $(e_3)_\star[E] = [C(u,v)] = d(\alpha)$, so κ is the coefficient of $\sigma(s_\beta)$ in $d(\alpha)$, which is $h_\alpha(\omega_\beta)$, as we saw in Lemma 3.4. This argument, run backwards, shows that each such v does occur, and completes the proof of the theorem. $\square$

Remark 10.2. The element $\tilde{u}s_\alpha$ is not in general a minimal length representative for v.

For the classical flag manifold $\mathrm{Fl}(\mathbb{C}^n)$, the positive roots and reflections correspond to $\alpha = (a,b)$, $1 \le a < b \le n$, and $h_{(a,b)}(\omega_{(r,r+1)})$ is 1 if $a \le r < b$ and is 0 otherwise. So one recovers the quantum Monk formula of [18].

Peterson's generalization of the Chevalley formula allows the quantum products of classes in the subalgebra generated (using the quantum product) by $H^2(G/P, \mathbb{Z}) \otimes \mathbb{Z}[q]$ to be computed recursively. S. Fomin has pointed out that this subalgebra is the full quantum cohomology ring $QH^*(G/P)$ whenever the classical ring $H^*(G/P)$ is generated by $H^2(G/P)$. (The proof is by induction on the degree.) This holds on the flag varieties G/B.

Peterson has also given an explicit formula for any Gromov-Witten invariant on any G/P as another Gromov-Witten invariant on the corresponding G/B. A proof of this, which uses some of the ideas of this paper, will be given in another paper by the second author.

REFERENCES

[1] S. Agnihotri and C. Woodward, Eigenvalues of products of unitary matrices and quantum Schubert calculus, *Math. Res. Lett.*, 5(6):817–836, 1998.

[2] A. Astashkevich and V. Sadov, Quantum cohomology of partial flag manifolds $F_{n_1 \cdots n_k}$, *Comm. Math. Phys.*, 170:503–528,1995.

[3] K. Behrend and Yu. Manin, Stacks of stable maps and Gromov-Witten invariants, *Duke Math. J.*, 1:1–60, 1996.

[4] P. Belkale. Transformation formulas in Quantum Cohomology, 2001. preprint.

[5] A. Bertram, Quantum schubert calculus, *Adv. Math.*, 128:289–305, 1997,

[6] A. Bertram, I. Ciocan-Fontanine, and W. Fulton, Quantum multiplication of Schur polynomials, *J. Algebra*, 219(2):728–746, 1999.

[7] A. Bertram, G. Daskalopoulos, and R. Wentworth, Gromov invariants for holomorphic maps from Riemann surfaces to Grassmannians, *J. Amer. Math. Soc.*, 9:529–571, 1996.

[8] A. Borel, *Linear algebraic groups*, Springer-Verlag, New York, second edition, 1991.

[9] R. Bott, A residue formula for holomorphic vector-fields, *J. Differential Geometry*, 1:311–330, 1967,

[10] N. Bourbaki, *Éléments de mathématique. Fasc. XXXIV. Groupes et algèbres de Lie. Chapitre IV: Groupes de Coxeter et systèmes de Tits. Chapitre V: Groupes engendrés par des réflexions. Chapitre VI: systèmes de racines*, Hermann, Paris, 1968.

[11] A. Buch, Quantum cohomology of Grassmannians, preprint, 2001.

[12] A. Buch, A. Kresch, and H. Tamvakis, to appear.

[13] J. B. Carrell, The Bruhat graph of a Coxeter group, a conjecture of Deodhar, and rational smoothness of Schubert varieties, In *Algebraic groups and their generalizations: classical methods (University Park, PA, 1991)*, pages 53–61. Amer. Math. Soc., Providence, RI, 1994.

[14] C. Chevalley, Sur les décompositions cellulaires des espaces G/B, In *Algebraic groups and their generalizations: classical methods (University Park, PA, 1991)*, pages 1–23. Amer. Math. Soc., Providence, RI, 1994, With a foreword by Armand Borel.

[15] L. Chen, Quantum cohomology of flag manifolds, math.AG/0010080.

[16] I. Ciocan-Fontanine, The quantum cohomology ring of flag varieties, *Trans. Amer. Math. Soc.*, 351(7):2695–2729, 1999.

[17] I. Ciocan-Fontanine, On quantum cohomology rings of partial flag varieties, *Duke Math. J.*, 98(3):485–524, 1999.

[18] S. Fomin, S. Gelfand, and A. Postnikov, Quantum Schubert polynomials, *J. Amer. Math. Soc.*, 10:565–596, 1997.

[19] W. Fulton, Eigenvalues, invariant factors, highest weights, and Schubert calculus, *Bull. Amer. Math. Soc.*, 37:209–249, 2000.

[20] W. Fulton and R. Pandharipande, Notes on stable maps and quantum cohomology, In *Algebraic geometry—Santa Cruz 1995*, pages 45–96. Amer. Math. Soc., Providence, RI, 1997.

[21] A. Givental and Bumsig Kim, Quantum cohomology of flag manifolds and Toda lattices, *Comm. Math. Phys.*, 168(3):609–641, 1995.

[22] A. Grothendieck. Sur la classification des fibrés holomorphes sur la sphère de Riemann. *Amer. J. Math.*, 79:121–138, 1957.

[23] A. Hirschowitz, Le groupe de Chow équivariant, *C. R. Acad. Sci. Paris Sér. I Math.*, 298(5):87–89, 1984.

[24] J. E. Humphreys, Reflection groups and Coxeter groups, Cambridge University Press, Cambridge, 1990.

[25] B. Kim and R. Pandharipande, The connectedness of the moduli space of maps to homogeneous spaces, math.AG/0003168.

[26] Bumsig Kim, Quantum cohomology of flag manifolds G/B and quantum Toda lattices, *Ann. of Math. (2)*, 149(1):129–148, 1999.

[27] Bumsig Kim, Quantum cohomology of partial flag manifolds and a residue formula for their intersection pairings, *Internat. Math. Res. Notices*, 1:1–15 (electronic), 1995.

[28] S. L. Kleiman, The transversality of a general translate, *Compositio Math.*, 28:287–297, 1974.

[29] J. Kollár, *Rational curves on algebraic varieties*, Springer-Verlag, Berlin, 1996.

[30] M. Kontsevich, Enumeration of rational curves via torus actions, In *The moduli space of curves (Texel Island, 1994)*, pages 335–368. Birkhäuser Boston, Boston, MA, 1995.

[31] M. Kontsevich and Yu. Manin, Gromov-Witten classes, quantum cohomology, and enumerative geometry, *Comm. Math. Phys.*, 164(3):525–562, 1994.

[32] A. Kresch and H. Tamvakis, Quantum cohomology of the Lagrangian Grassmannian, preprint, 2001.

[33] D. Peterson, Lectures on quantum cohomology of G/P, M.I.T., 1996.

[34] B. Siebert and G. Tian, On quantum cohomology rings of Fano manifolds and a formula of Vafa and Intriligator, *Asian J. Math.*, 1:679–695, 1997.

[35] J. F. Thomsen, Irreducibility of $\overline{M}_{0,n}(G/P, \beta)$, *Internat. J. Math.*, 9(3):367–376, 1998.

[36] E. Witten, The Verlinde algebra and the cohomology of the Grassmannian, In *Geometry, topology, and physics*, volume VI of *Conf. Proc. Lecture Notes Geom. Topology*, Berkeley, 1989, 1995. Internat. Press, Cambridge, 1995.

[37] A. Yong, On bounds for quantum multiplication, preprint.