FLOER COHOMOLOGY AND GEOMETRIC COMPOSITION OF LAGRANGIAN CORRESPONDENCES

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ABSTRACT. We prove an isomorphism of Floer cohomologies under geometric composition of Lagrangian correspondences in exact and monotone settings.

1. INTRODUCTION

Lagrangian correspondences were described by Weinstein [27, 26] as generalizations of symplectomorphisms, in an attempt to build a symplectic category with composable morphisms between non-symplectomorphic manifolds. By definition a Lagrangian correspondence from M_0 to M_1 is a Lagrangian submanifold in the product, $L_{01} \subset M_0^- \times M_1$, with respect to the symplectic structure $(-\omega_{M_0}) \times \omega_{M_1}$. The basic examples are graphs of symplectomorphisms. Composition of symplectomorphisms generalizes to *geometric composition* of Lagrangian correspondences $L_{01} \subset M_0^- \times M_1$, $L_{12} \subset M_1^- \times M_2$, defined by

$$L_{01} \circ L_{12} := \big\{ (x_0, x_2) \in M_0 \times M_2 \,\big|\, \exists x_1 : (x_0, x_1) \in L_{01}, (x_1, x_2) \in L_{12} \big\}. \tag{1}$$

In general this will be a singular subset of $M_0^- \times M_2$ with isotropic tangent spaces. However, if we assume transversality of the intersection $L_{01} \times_{M_1} L_{12} := \big(L_{01} \times L_{12}\big) \cap \big(M_0^- \times \Delta_{M_1} \times M_2\big)$, then the restriction of the projection $\pi_{02} : M_0^- \times M_1 \times M_1^- \times M_2 \to M_0^- \times M_2$ to $L_{01} \times_{M_1} L_{12}$ is an immersion [4, 20], and hence $L_{01} \circ L_{12} \subset M_0^- \times M_2$ is an immersed Lagrangian correspondence. We will study the class of *embedded* geometric compositions, for which in addition π_{02} is injective, and hence $L_{01} \circ L_{12}$ is a smooth Lagrangian correspondence.

Lagrangian correspondences arise naturally in in various contexts. Perutz [10, 11] proposed a construction of three and four-manifold invariants, defined by Floer cohomology of Lagrangian correspondences in symmetric products. Seidel proposed a generalized version of his exact triangle in Floer cohomology [15] for fibered versions of symplectic Dehn twists, whose vanishing cycle is a spherically fibered Lagrangian correspondence. Seidel and Smith [17] proposed a symplectic definition of Khovanov homology, using Lagrangians constructed as geometric compositions of the fibered vanishing cycles. Finally, moduli spaces of flat bundles on three-dimensional cobordisms define Lagrangian correspondences [24] between the moduli spaces of bundles on the boundary surfaces, such that composition of cobordisms corresponds to geometric composition. The corresponding Floer cohomology groups may be viewed as symplectic versions of instanton Floer homology for three manifolds.

Naturally the question arises of how composition of correspondences affects Floer cohomology. In this paper we prove that Floer cohomology is isomorphic under embedded geometric composition. For a precise general statement, it is best to use the language of quilted Floer cohomology developed in [20] which defines $HF(L_{01}, L_{12}, \ldots, L_{(k-1)k})$ for a cyclic sequence of Lagrangian correspondences $L_{(\ell-1)\ell} \subset M_{\ell-1}^- \times M_\ell$ between symplectic manifolds $M_0, M_1, \ldots, M_k = M_0$. If the composition $L_{(\ell-1)\ell} \circ L_{\ell(\ell+1)}$ is embedded, then we obtain under suitable monotonicity assumptions a canonical isomorphism

$$HF(\ldots, L_{(\ell-1)\ell}, L_{\ell(\ell+1)}, \ldots) \cong HF(\ldots, L_{(\ell-1)\ell} \circ L_{\ell(\ell+1)}, \ldots). \tag{2}$$

Here the quilted Floer cohomology on the left hand side counts k-tuples of holomorphic strips $(u_j : \mathbb{R} \times [0,1] \to M_j)_{j=0,\ldots,k-1}$, whose boundaries match up via the Lagrangian correspondences, $(u_{j-1}(s,1), u_j(s,0)) \in L_{(j-1)j}$. On the right hand side of (2), no strip in M_ℓ is taken into account, and the strips $M_{\ell-1}$ and $M_{\ell+1}$ match up directly via $(u_{\ell-1}(s,1), u_{\ell+1}(s,0)) \in L_{(\ell-1)\ell} \circ L_{\ell(\ell+1)}$. Rather than going through the general definition in detail, we will prove in detail the following representative example in the familiar notation of Floer cohomology for pairs of Lagrangians in the same symplectic manifold.

Theorem 1.0.1. *Let M_0, M_1, M_2 be symplectic manifolds and let*

$$L_0 \subset M_0, \quad L_{01} \subset M_0^- \times M_1, \quad L_{12} \subset M_1^- \times M_2, \quad L_2 \subset M_2^-$$

be compact Lagrangian submanifolds such that the geometric composition $L_{01} \circ L_{12}$ is embedded. Then the canonical bijection $(L_0 \times L_{12}) \cap (L_{01} \times L_2) \cong (L_0 \times L_2) \cap (L_{01} \circ L_{12})$ induces an isomorphism

$$HF(L_0 \times L_{12}, L_{01} \times L_2) \xrightarrow{\sim} HF(L_0 \times L_2, L_{01} \circ L_{12}), \tag{3}$$

provided the following assumptions hold:

(a) *Each of M_0, M_1, M_2 is monotone with the same monotonicity constant $\tau \geq 0$, that is $[\omega_{M_i}] = \tau \cdot c_1(TM_i)$ for $i = 0,1,2$. Note that $\tau = 0$ is the case of exact symplectic manifolds, which are necessarily noncompact. We thus require that each of M_0, M_1, M_2 is compact or satisfies the "bounded geometry" assumptions as in* [16].

(b) *The pair $(L_0 \times L_{12},\ L_{01} \times L_2)$ of Lagrangian submanifolds in $M_0 \times M_1^- \times M_2$ is monotone for Floer theory, that is with the $\tau \geq 0$ from* (a) *we have*

$$2\textstyle\int v^*\omega_N = \tau \cdot I_{\mathrm{Maslov}}(v^*T(L_0 \times L_{12}), v^*T(L_{01} \times L_2))$$

for all maps from the annulus $v : S^1 \times [0,1] \to M_0 \times M_1^- \times M_2$ with Lagrangian boundary conditions $v(S^1 \times \{0\}) \subset L_0 \times L_{12}$ and $v(S^1 \times \{1\}) \subset L_{01} \times L_2$. The Maslov index is defined by choosing a trivialization $v^*T(M_0 \times M_1^- \times M_2) \cong S^1 \times [0,1] \times \mathbb{C}^n$, then $I_{\mathrm{Maslov}}(v^*T(L_0 \times L_{12}), v^*T(L_{01} \times L_2))$ is the difference of Maslov indices of the two loops in the Lagrangian Grassmannian of $\mathbb{C}^n$.

(c) *The minimal positive Maslov index in* (b) *is 2, that is there exists no annulus v with $I_{\mathrm{Maslov}}(v^*T(L_0 \times L_{12}), v^*T(L_{01} \times L_2)) = 1$.*

(d) *Each of the L_0, L_{01}, L_{12}, L_2 has minimal Maslov index ≥ 3. (Here the minimal Maslov index of $L \subset M$ is the positive generator of $I_{\mathrm{Maslov}}(\pi_2(M,L)) \subset \mathbb{Z}$.)*

Assumptions (b) and (c) are met, for example, if all Lagrangians are orientable and exact, or if they are orientable, monotone, and the image of either $\pi_1(L_0 \times L_{12})$ or $\pi_1(L_{01} \times L_2)$ in $\pi_1(M_0 \times M_1 \times M_2)$ is torsion. In this paper, the isomorphism (3) of Floer cohomology groups is completely proven only with $\mathbb{Z}_2$-coefficients; to reduce the length, we banished the discussion of coherent orientations – in the presence of orientations and relative spin structures on the Lagrangians – to a separate paper [23]. There should also be versions of this result for Floer cohomology with gradings, coefficients in flat vector bundles, and Novikov rings. We give a detailed statement and proof for the gradings in [20]. Below we explain the necessity of the monotonicity and Maslov index assumptions. In [21] we give some alternative assumptions and generalizations, and in [25] generalize Theorem 1.0.1 to an isomorphism in the derived category of matrix factorization, allowing to drop assumption (d).

Throughout we will use the construction of Floer cohomology, mainly due to Floer [2], Oh [9], and Floer-Hofer-Salamon [3]. The Floer differential for $(L_0 \times L_{12}, L_{01} \times L_2)$ counts triples of holomorphic strips in M_0, M_1^-, M_2 (see Figure 1 below). In the standard definition,

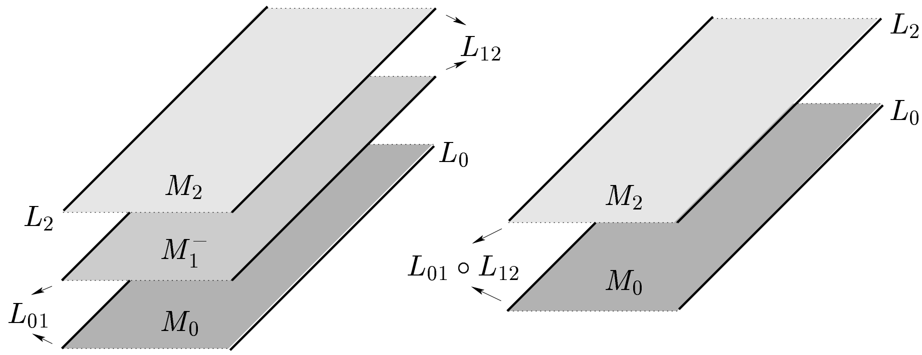


Figure 1. Tuples of holomorphic strips that are counted for $HF(L_0 \times L_{12}, L_{01} \times L_2)$ and for $HF(L_0 \times L_2, L_{01} \circ L_{12})$

one would take the width of all three strips to be equal, but in fact one can allow the widths of the strips to differ. (These domains are not conformally equivalent due to the identification between boundary components.) The main difficulty then is to prove that under the stated assumptions and with the width of the middle strip sufficiently close to zero, the triples of holomorphic strips in M_0, M_1^-, M_2 are in one-to-one correspondence with the pairs of holomorphic strips in M_0, M_2 that are counted in the Floer differential for $(L_0 \times L_2, L_{01} \circ L_{12})$. As in similar situations in Floer theory, the proof is an application of the implicit function theorem, on one hand, and compactness results for certain J-holomorphic strips, on the other. In the limit various kinds of bubbling can occur: sphere bubbles in M_0, M_1, or M_2; disk bubbles in $(M_0 \times M_1, L_{01})$, $(M_1 \times M_2, L_{12})$, or $(M_0 \times M_2, L_{01} \circ L_{12})$; and a novel type of bubble which we call a *figure eight bubble*. The latter is a triple of J-holomorphic maps $v_0 : \mathbb{R} \times (-\infty, -1] \to M_0, v_1 : \mathbb{R} \times [-1, 1] \to M_1, v_2 : \mathbb{R} \times [1, \infty) \to M_2$ such that $(v_0(\tau, -1), v_1(\tau, -1)) \in L_{01}$, $(v_1(\tau, 1), v_2(\tau, 1)) \in L_{12}$.

Viewed from $z = \infty$ the lines $\mathrm{Im}(z) = \pm 1$ appear as a figure eight, as in Figure 2. We conjecture that the maps (v_0, v_1, v_2) can be extended continuously to S^2 by a point $(v_0(\infty), v_1(\infty), v_2(\infty))$ that lies in both $L_{01} \times M_2$ and $M_0 \times L_{12}$.

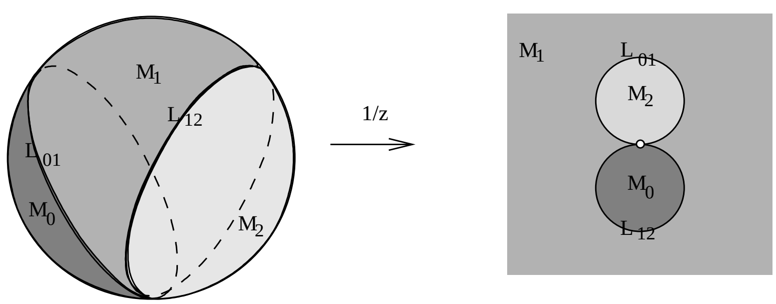


Figure 2. Figure Eight bubble

However, we cannot in general prove this removal of singularities at $z = \infty$ for figure eight bubbles and thus are lacking the construction of a moduli space of figure eight bubbles. Instead, as in [19] we exclude bubbling by energy quantization without giving a geometric description of the bubble. However, this method hinges on strict monotonicity with nonnegative constant $\tau \geq 0$ as well as the 2-grading assumption (d).

A few of the applications of the result of this paper are the following. First, there are various applications to symplectic topology: Using the result we prove in [21] the nondisplaceability of a Lagrangian 3-sphere $\Sigma \subset (\mathbb{CP}^2)^- \times \mathbb{CP}^1$, whose projection to $\mathbb{CP}^2$ contains the nondisplaceable Clifford torus. An application to non-triviality of symplectic mapping class groups is given in [25]. Second, our isomorphism is key to proving the topological invariance of various groups defined using Floer cohomology and decomposition in low-dimensional topology; for example, the symplectic version of instanton knot homology constructed in [25], Seidel-Smith homology and Heegard-Floer homology, for which it provides alternative constructions [12], [7].

From a more conceptual point of view, the results of this paper are used in [20] to give a solution to the problem in Weinstein's construction that composition of Lagrangian correspondences is not always defined. Using the result here, one may construct a symplectic 2-category, in which all Lagrangian correspondences are composable morphisms and Floer cohomology groups (as 2-morphism spaces) are well defined. Thus one removes the quotes in Weinstein's "category" by promoting the construction to a 2-category, using Floer theory.

We thank Paul Seidel and Ivan Smith for encouragement and helpful discussions.

2. Floer cohomology for monotone Lagrangian correspondences

In this section we first explain why both Floer cohomologies in Theorem 1.0.1 are well defined. Then we give a specific "quilted" setup and choice of perturbations for both that reduce the isomorphism of Floer cohomologies to a bijection of moduli spaces that is proven in Section 3.

2.1. Monotonicity assumptions and index identities.

The significance of the monotonicity and Maslov index assumptions in Theorem 1.0.1 is the following energy-index relation and relative grading.

Proposition 2.1.1. *Suppose that the pair (L_0, L_1) of Lagrangians in M is monotone, transverse, and has minimal annulus Maslov index $N \geq 2$. (That is, N is the positive generator of $\{I_{\text{Maslov}}(v^*TL_0, v^*TL_1) \,|\, v : S^1 \times [0,1] \to M, v(S^1 \times \{j\}) \subset L_j\} \subset \mathbb{Z}$.)*

Then for any $x_\pm \in L_0 \cap L_1$ there exist constants $c(x_-, x_+) \in \mathbb{R}$ and $\mu(x_-, x_+) \in \mathbb{Z}$ such that for all strips $u : \mathbb{R} \times [0,1] \to M$ with boundary values in (L_0, L_1) and limits $u(\pm\infty, \cdot) = x_\pm$ we have

$$2E(u) = \tau \cdot \operatorname{Ind}(D_u) + c(x_-, x_+), \qquad \operatorname{Ind}(D_u) \equiv \mu(x_-, x_+) \quad \text{mod } N. \tag{4}$$

Here $E(u) = \int u^*\omega$ is the energy and D_u the linearized Cauchy-Riemann operator at u.

Proof. Given two strips $u_1, u_2 : \mathbb{R} \times [0,1] \to M$ glue them together (reversing the orientation of u_2) to an annulus $v : S^1 \times [0,1] \to M$, then $\int v^*\omega = E(u_1) - E(u_2)$ and $I_{\text{Maslov}}(v^*TL_0, v^*TL_1) = \operatorname{Ind}(D_{u_1}) - \operatorname{Ind}(D_{u_2})$. So the energy-index relation follows from monotonicity, and the index identity follows from $I_{\text{Maslov}}(v^*TL_0, v^*TL_1) \subset N\mathbb{Z}$. □

The energy-index relation ensures energy bounds for the moduli spaces of fixed index and thus compactness up to bubbling and breaking of trajectories. Together with the index identity it excludes bubbling in moduli spaces of index less than N as follows: Any bubbling leads to a new (possibly broken) trajectory connecting the same points but with less energy. By monotonicity, less energy means strictly less index. By the index identity mod N that means negative index. By transversality (previously established for moduli spaces of negative index) that means an empty set: The new trajectory doesn't exist, so the bubbling didn't happen. We spelled out this argument because we will use it again to exclude figure eight bubbling – by only proving energy loss, not actually giving a geometric description of the bubble.

Working with $N = 2$ there is just one point in the construction of Floer cohomology where this argument fails: The 1-dimensional moduli spaces of self-connecting Floer trajectories have index 2, so bubbling could lead to an index 0 solution (which are always constant due to the $\mathbb{R}$-action). Assumption (d) serves to exclude this scenario by index additivity arguments: Any holomorphic disk bubble with boundary on L will reduce the index by at least N_L, the minimal Maslov index on $\pi_2(M, L)$. So $N_L \geq 3$ ensures that the remaining solution would have negative index (and the same holds for sphere bubbles whose Chern number would be at least $\frac{1}{2}N_L$). Note that this argument, unlike the previous bubbling exclusion by energy loss, requires an identification of the bubbles as spheres and disks. In our case it also requires that we work with a split almost complex structure (preserving the factors of $M_0 \times M_1^- \times M_2$), otherwise holomorphic disks in the product manifold don't necessarily have the minimal index of a disk in one of the factors. We will show in Section 2.2 that we can achieve transversality with a split almost complex structure, and hence our assumptions indeed ensure that the Floer cohomology $HF(L_0 \times L_{12}, L_{01} \times L_2)$ is well defined.

The next Lemma shows that the Floer cohomology $HF(L_0 \times L_2, L_{01} \circ L_{12})$ for the composed Lagrangian correspondence is also well defined.

Lemma 2.1.2. *In the setting of Theorem 1.0.1, the assumptions* (b) *and* (c) *imply the analogous assumptions for the pair* $(L_0 \times L_2, L_{01} \circ L_{12})$ *of Lagrangians in* $M_0 \times M_2^-$. *Assumption* (d) *implies that* $\partial^2 = 0$ *on* $CF(L_0 \times L_2, L_{01} \circ L_{12})$, *and hence the Floer cohomology is well defined.*

Proof. Consider any annulus $(u_0 \times u_2) : S^1 \times [0,1] \to M_0 \times M_2^-$ with Lagrangian boundary conditions $(u_0 \times u_2)(S^1 \times \{0\}) \subset L_0 \times L_2$ and $(u_0 \times u_2)(S^1 \times \{1\}) \subset L_{01} \circ L_{12}$. By the embedded composition there exists a unique lift $u_1 : S^1 \to M_1^-$ such that $(u_0|_{t=1} \times u_1)(S^1) \subset L_{01}$ and $(u_1 \times u_2|_{t=1})(S^1) \subset L_{12}$. Now we can reverse the parametrization in $\overline{u}_2(s,t) := u_2(s, 1-t)$ and extend u_1 constant along $[0,1]$ to define an annulus $(u_0 \times u_1 \times \overline{u}_2) : S^1 \times [0,1] \to M_0 \times M_1^- \times M_2$ as in (b). Here $u_1^*\omega_1 = 0$, hence $\int (u_0 \times u_1 \times \overline{u}_2)^*(\omega_0 \times (-\omega_1) \times \omega_2) = \int (u_0 \times u_2)^*(\omega_0 \times (-\omega_2))$. To identify the Maslov indices, pick the same trivializations $u_j^* TM_j \cong S^1 \times [0,1] \times V_j$ for $j = 0, 2$ in both cases, then equality follows from the identity

$$I(\gamma_{01}) + I(\gamma_{12}) = I(\gamma_{01} \times \gamma_{12}) = I(\gamma_{02}) \tag{5}$$

for loops of Lagrangians $\gamma_{01} : S^1 \to \mathrm{Lag}(V_0^- \times V_1)$, $\gamma_{12} : S^1 \to \mathrm{Lag}(V_1^- \times V_2)$, and $\gamma_{02} : S^1 \to \mathrm{Lag}(V_0^- \times V_2)$ given by $\gamma_{02}(s) = (\gamma_{01}(s) \times \gamma_{12}(s)) \cap (V_0 \times \Delta_{V_1} \times V_2)$. The first equality is simply additivity of the Maslov index. To see the second equality we fix Lagrangians $\Lambda_j \subset V_j$ for $j = 0, 2$, then the Maslov indices can be expressed as the intersection number with $\Lambda_0 \times \Delta_{V_1} \times \Lambda_2$ resp. $\Lambda_0 \times \Lambda_2$. With this choice the intersections are identified,

$$K(s) := \big(\gamma_{01}(s) \times \gamma_{12}(s)\big) \cap \big(\Lambda_0 \times \Delta_{V_1} \times \Lambda_2\big) \cong \gamma_{02}(s) \cap \big(\Lambda_0 \times \Lambda_2\big).$$

Now we need to compare the crossing forms $\Gamma_{0112}(s), \Gamma_{02}(s) : K(s) \to \mathbb{R}$ at regular crossings $s \in S^1$. Fix a Lagrangian complement $\gamma_{02}(s)^c \subset V_0 \times V_2^-$, then $\gamma_{02}(s)^c \times \Delta_{V_1}$, after appropriate transposition of factors, is a Lagrangian complement for $\gamma_{01}(s) \times \gamma_{12}(s)$, due to the assumption of transversality $(L_{01} \times L_{12}) \pitchfork (M_0 \times \Delta_{M_1} \times M_2)$. So for $v_{0112} = (v_0, v_1, v_1, v_2) \in K(s)$ one finds $(w_0, w_2)(t) \in \gamma_{02}(s)^c$ and $w_1 \in V_1$ such that $v + (w_0, w_1, w_1, w_2)(t) \in$

$(\gamma_{01}\times\gamma_{12})(s+t)$. For the corresponding vector $v_{02}=(v_0,v_2)\in K(s)$ this automatically gives $v_{02}+(w_0,w_2)(t)\in\gamma_{02}(s+t)$. With this we identify the crossing forms

$$\begin{aligned}\Gamma_{0112}(s)v_{0112} &= \tfrac{d}{dt}\big|_{t=0}(\omega_0\oplus-\omega_1\oplus\omega_1\oplus-\omega_2)\big(v_{0112},(w_0,w_1,w_1,w_2)(t)\big)\\ &= \tfrac{d}{dt}\big|_{t=0}\big(-\omega_0(v_0,w_0)+\omega_1(v_1,w_1)-\omega_1(v_1,w_1)+\omega_2(v_2,w_2)\big)\\ &= \tfrac{d}{dt}\big|_{t=0}(\omega_0\oplus-\omega_2)\big(v_{02},(w_0,w_2)(t)\big)=\Gamma_{02}(s)v_{02}.\end{aligned}$$

This proves equality of the Maslov indices in (5) and this finishes the proof of (b) and (c).

In the absence of assumption (d) we have $\partial^2=w\mathrm{Id}$ a multiple of the identity in both Floer theories, see [9] and [25]. A derived version of Theorem 1.0.1 implies that the value of w is the same for both theories, see Remark 2.2.3. Assuming (d) for the pair $(L_0\times L_{12},L_{01}\times L_2)$ we obtain $w=0$ and thus also $\partial^2=0$ on $CF(L_0\times L_2,L_{01}\circ L_{12})$. □

The index calculation in (5) analogously holds for strips. This identifies the index on the two complexes in Theorem 1.0.1. Recall here from [2] that the index of the linearized Cauchy-Riemann operator D_u at a map $u:\mathbb{R}\times[0,1]\to M$ with Lagrangian boundary conditions $u(\mathbb{R}\times\{i\})\subset L_i$ for $i=0,1$ and limits $u(s,\cdot)\underset{s\to\pm\infty}{\longrightarrow} L_0\pitchfork L_1$ at transverse intersection points is given by the Maslov-Viterbo index,

$$\mathrm{Ind}(D_u)=I_{MV}(u):=I(\gamma_0,\gamma_1),\qquad \gamma_i(s)=T_{u(s,i)}L_i.$$

Here the Maslov index of the pair of paths is defined by choosing a trivialization $u^*TM\cong\mathbb{R}\times[0,1]\times V$ (independent of $t\in[0,1]$ for $s\to\pm\infty$) so that γ_i becomes a path of Lagrangian subspaces in the symplectic vector space V.

Lemma 2.1.3. *Let $L_0\subset M_0$, $L_{01}\subset M_0^-\times M_1$, $L_{12}\subset M_1^-\times M_2$, and $L_2\subset M_2^-$ be Lagrangians such that the composition $L_{01}\circ L_{12}=:L_{02}$ is embedded. Suppose that the intersection $L_0\times L_{12}\cap L_{01}\times L_2$ (and hence also $L_0\times L_2\cap L_{01}\circ L_{12}$) is transverse and consider a map $(u_0,u_2):\mathbb{R}\times[0,1]\to M_0\times M_2$ taking boundary values in $(L_0\times L_2,L_{01}\circ L_{12})$, and limiting to intersection points as $s\to\pm\infty$. Let $(u_0,u_1,\overline{u}_2):\mathbb{R}\times[0,1]\to M_0\times M_1\times M_2$ be the corresponding map which takes boundary values in $(L_0\times L_{12},L_{01}\times L_2)$ and satisfies $\partial_t u_1=0$. (Here $\overline{u}_2$ reverses the $[0,1]$-parametrization of u_2.) Then the indices of the linearized operators and the energies are equal,*

$$\mathrm{Ind}(D_{(u_0,u_2)})=\mathrm{Ind}(D_{(u_0,u_1,\overline{u}_2)}),\qquad E((u_0,u_2))=E((u_0,u_1,\overline{u}_2)).$$

Proof. The identity of Maslov indices follows as in Lemma 2.1.2. Alternatively, it could be deduced from a more general result of Viterbo [18, Proposition 3]. For the energies just note that $\int u_2^*\omega_2=\int\overline{u}_2^*(-\omega_2)$ and $\int u_1^*\omega_1=0$. □

2.2. **Quilted setup for Floer cohomology.** As in Theorem 1.0.1 let M_0,M_1,M_2 be symplectic manifolds and let

$$L_0\subset M_0,\quad L_{01}\subset M_0^-\times M_1,\quad L_{12}\subset M_1^-\times M_2,\quad L_2\subset M_2^-$$

be Lagrangian submanifolds such that the geometric composition $L_{02}:=L_{01}\circ L_{12}$ is embedded. The aim of this section is to introduce the "quilted" setup and give compatible choices of perturbation data for the two Floer cohomologies $HF(L_0\times L_{12},L_{01}\times L_2)$ and $HF(L_0\times L_2,L_{02})$.

First, we need to fix Hamiltonians such that the perturbed intersection points are finite and nondegenerate. In fact, the following Proposition shows that we can pick a Hamiltonian of split type which achieves simultaneous transversality for the intersection points in both Floer theories. Given a pair of time-dependent Hamiltonian functions $(H_0, H_2) \in C^\infty([0,1] \times M_0) \times C^\infty([0,1] \times M_2)$ consider the Hamiltonians $H_{02}(t, x_0, x_2) = H_0(t, x_0) - H_2(1-t, x_2)$ on $M_0 \times M_2$ and $H_{012}(t, x_0, x_1, x_2) = H_0(t, x_0) + H_2(t, x_2)$ on $M_0 \times M_1 \times M_2$ and denote their time 1 flows by $\phi^{H_{02}}$ and $\phi^{H_{012}}$. Then the perturbed intersection points $\phi^{H_{02}}(L_0 \times L_2) \cap L_{02}$ can be identified with

$$L_0 \times_{\phi^{H_0}} L_{02} \times_{\phi^{H_2}} L_2 = \left\{ (m_0, m_2) \in M_0 \times M_2 \,\middle|\, m_0 \in L_0, (\phi^{H_0}(m_0), m_2) \in L_{02}, \phi^{H_2}(m_2) \in L_2 \right\}$$

and analogously

$$\phi^{H_{012}}(L_0 \times L_{12}) \cap (L_{01} \times L_2) \cong L_0 \times_{\phi^{H_0}} L_{01} \times_{\phi^{H_1}} L_{12} \times_{\phi^{H_2}} L_2,$$

where ϕ^{H_j} is the time 1 flow of the Hamiltonian H_j and we use the trivial function $H_1 \equiv 0$ on M_1. Note that the Hamiltonians are constructed such that the perturbed intersection points for the two Floer theories are still canonically identified. Indeed, by assumption every point in $L_{02} = L_{01} \circ L_{12}$ has a unique lift to $L_{01} \times_{\mathrm{Id}_{M_1}} L_{12}$.

Proposition 2.2.1. *There is a dense open subset* $\mathrm{Ham}(L_0, L_{02}, L_2) \subset C^\infty([0,1] \times M_0) \times C^\infty([0,1] \times M_2)$ *such that for every* $(H_0, H_2) \in \mathrm{Ham}(L_0, L_{02}, L_2)$ *and* $H_1 \equiv 0$ *the defining equations for both sets* $L_0 \times_{\phi^{H_0}} L_{02} \times_{\phi^{H_2}} L_2$ *and* $L_0 \times_{\phi^{H_0}} L_{01} \times_{\phi^{H_1}} L_{12} \times_{\phi_1^{H_2}} L_2$ *are transversal.*

Proof. By assumption L_0, L_{02}, L_2 are embedded submanifolds and so locally they are the zero sets of submersions $\psi_0 : M_0 \to \mathbb{R}^{n_0}$, $\psi_{02} : M_0 \times M_2 \to \mathbb{R}^{n_0+n_2}$, $\psi_2 : M_2 \to \mathbb{R}^{n_2}$. Then the defining equations for $L_0 \times_{\phi^{H_0}} L_{02} \times_{\phi^{H_2}} L_2$ are

$$\psi_0(m_0) = 0, \quad \psi_{02}\big(\phi^{H_0}(m_0), m_2\big) = 0, \quad \psi_2\big(\phi^{H_2}(m_2)\big) = 0. \tag{6}$$

Consider the universal moduli $\mathcal{U}$ space of data (H_0, H_2, m_0, m_2) satisfying (6), where now each H_j has class C^ℓ for some $\ell \geq 2$. The linearized equations for $\mathcal{U}$ are

$$D\psi_0(v_0) = 0, \quad D\psi_{02}(D\phi^{H_0}(h_0, v_0), v_2) = 0, \quad D\psi_2\big(D\phi^{H_2}(h_2, v_2)\big) = 0. \tag{7}$$

for $v_j \in T_{m_j} M_j$ and $h_j \in C^\ell([0,1] \times M_j)$. The product of the operators on the left-hand sides of (7) are surjective since each of the maps $C^\ell([0,1] \times M_j) \to T_{\phi^{H_j}(m_j)} M_j$, $h_j \mapsto D\phi^{H_j}(h_j, 0)$ is surjective. So by the implicit function theorem $\mathcal{U}$ is a smooth Banach manifold, and we consider its projection to $C^\ell([0,1] \times M_0) \times C^\ell([0,1] \times M_2)$. By the Sard-Smale theorem, the set of regular values is dense. On the other hand, the set of regular values is clearly open. Hence the set of smooth functions that are regular values is open and dense. This is exactly the set of functions $H = (H_0, H_2)$ such that the perturbed intersection $L_0 \times_{\phi^{H_0}} L_{02} \times_{\phi^{H_2}} L_2$ is transversal.

Moreover, the perturbed intersection $L_0 \times_{\phi^{H_0}} L_{01} \times_{\phi^{H_1}} L_{12} \times_{\phi_1^{H_2}} L_2$ is also transversal, since by assumption $L_{01} \times L_{12}$ is transverse to the diagonal $M_0 \times \Delta_{M_1} \times M_2$. □

In the following, instead of working with perturbed intersection points, we will apply the Hamiltonian diffeomorphisms to the Lagrangians to achieve transversality. Replacing L_0 with $L_0' = \phi^{H_0}(L_0)$ and L_2 with $L_2' = (\phi^{H_2})^{-1}(L_2)$ the generators of the two Floer chain groups are the transverse intersections

$$(L_0' \times L_2') \cap L_{01} \cong L_0' \times_{\mathrm{Id}_{M_0}} L_{02} \times_{\mathrm{Id}_{M_2}} L_2',$$
$$(L_0' \times L_{12}) \cap (L_{01} \times L_2') \cong L_0' \times_{\mathrm{Id}_{M_0}} L_{01} \times_{\mathrm{Id}_{M_1}} L_{12} \times_{\mathrm{Id}_{M_2}} L_2'.$$

The forgetful map $(m_0, m_1, m_2) \mapsto (m_0, m_2)$ is a bijection from $\mathcal{I}$ to $(L_0' \times L_2') \cap L_{01}$ since by assumption $L_{01} \times_{\mathrm{Id}_{M_1}} L_{12} \to L_{02}$ is bijective. So, after a generic Hamiltonian perturbation, we have a natural isomorphism of the Floer chain groups

$$CF(L_0 \times L_{12}, L_{01} \times L_2) \overset{\sim}{\to} CF(L_0 \times L_2, L_{02}) \tag{8}$$

and it remains to identify the Floer differentials. For that purpose we now drop the Hamiltonian from the notation: By abuse of notation we can assume to start out with unperturbed transverse intersections and a natural bijection

$$\mathcal{I} := (L_0 \times L_2) \pitchfork L_{01} \cong (L_0 \times L_{12}) \pitchfork (L_{01} \times L_2).$$

To investigate the Floer trajectories note that we consider $(L_0 \times L_2, L_{02})$ as a pair of Lagrangians in $M_0 \times M_2^-$ and $(L_0 \times L_{12}, L_{01} \times L_2)$ as a pair of Lagrangians in $M_0 \times M_1^- \times M_2$. For any symplectic manifold M let $\mathcal{J}(M)$ be the space of almost complex structures on M that are compatible with the symplectic structure ω_M. We pick time-dependent almost complex structures $J_0 \in \mathcal{C}^\infty([0,1], \mathcal{J}(M_0))$ and $J_2 \in \mathcal{C}^\infty([0,1], \mathcal{J}(M_2))$, then $J_0(t, m_0) \times (-J_2(1-t, m_2))$ defines a compatible almost complex structure on $M_0 \times M_2^-$. Now any pseudoholomorphic strip $w_{02} : \mathbb{R} \times [0,1] \to M_0 \times M_2^-$ with boundary values on $(L_0 \times L_2, L_{02})$ corresponds by "unfolding" $w_{02}(s,t) = (u_0(s,t), u_2(s, 1-t))$ to a pair of strips $\big(u_i : \mathbb{R} \times [0,1] \to M_i\big)_{i=0,2}$ satisfying

$$\begin{aligned} &\partial_s u_0 + J_0(t, u_0)\partial_t u_0 = 0, \qquad \partial_s u_2 + J_2(t, u_2)\partial_t u_2 = 0, \\ &u_0(s,0) \in L_0, \qquad (u_0(s,1), u_2(s,0)) \in L_{02}, \qquad u_2(s,1) \in L_2. \end{aligned} \tag{9}$$

Similarly, pick an almost complex structure $J_1 \in \mathcal{J}(M_1)$, then $J_0 \times (-J_1) \times J_2$ defines a compatible almost complex structure on $M_0 \times M_1^- \times M_2$ and any pseudoholomorphic strip with boundary values on $(L_0 \times L_{12}, L_{01} \times L_2)$ corresponds by "unfolding" to a triple of strips $\big(v_i : \mathbb{R} \times [0,1] \to M_i\big)_{i=0,1,2}$ satisfying

$$\begin{aligned} &\partial_s v_0 + J_0(t, v_0)\partial_t v_0 = 0, \quad \partial_s v_1 + J_1(v_1)\partial_t v_1 = 0, \quad \partial_s v_2 + J_2(t, v_2)\partial_t v_2 = 0, \\ &v_0(s,0) \in L_0, \quad (v_0(s,1), v_1(s,0)) \in L_{01}, \quad (v_1(s,1), v_2(s,0)) \in L_{12}, \quad v_2(s,1) \in L_2. \end{aligned} \tag{10}$$

In both cases, the trajectories have finite energy $\sum_i \int |\partial_s u_i|^2$ resp. $\sum_i \int |\partial_s v_i|^2$ iff they converge uniformly to intersection points

$$\lim_{s \to \pm\infty} (u_0, u_2)(s, \cdot) = (x_0^\pm, x_2^\pm) \in \mathcal{I} \quad \text{resp.} \quad \lim_{s \to \pm\infty} (v_0, v_1, v_2)(s, \cdot) = (x_0^\pm, x_1^\pm, x_2^\pm) \in \mathcal{I}. \tag{11}$$

For any $x^-, x^+ \in \mathcal{I}$ let us denote by

$$\widetilde{\mathcal{M}}_0^1(x^-, x^+) = \big\{ (u_0, u_2) \,\big|\, (9), (11),\ \mathrm{Ind}(D_{(u_0,u_2)}) = 1 \big\}$$

the one dimensional (i.e. index 1) component of the moduli space of Floer trajectories for $(L_0 \times L_2, L_{02})$. One can achieve transversality of these moduli spaces (of any index ≤ 1) by choosing t-dependent almost complex structures J_0 and J_2 that are constant near $t = 0$ and $t = 1$.[1] Note that we cannot expect a bijection with the moduli spaces of Floer

[1] Indeed, note that the unique continuation theorem [3, Thm.4.3] applies to the interior of each non-constant strip $u_i : \mathbb{R} \times (0,1) \to M_i$. It implies that the set of regular points, $(s_0, t_0) \in \mathbb{R} \times (0,1)$ with $\partial_s u_i(s_0, t_0) \neq 0$ and $u_i^{-1}(u_j(\mathbb{R} \cup \{\pm\infty\}), t_0) = \{(s_0, t_0)\}$, is open and dense. These points can be used to prove surjectivity of the linearized operator for a universal moduli space of solutions with respect to split almost complex structures (J_0, J_2). (The constant solutions are automatically transverse due to the previously ensured transversality of the intersection points.) Note that it suffices to work with almost complex structures that are t-independent outside of $[\frac{1}{3}, \frac{2}{3}]$. The existence of a Baire second category set of regular (J_0, J_2) then follows from the usual Sard-Smale argument as in [8].

trajectories for $(L_0\times L_{12}, L_{01}\times L_2)$ as in (10). However, by the independence theorem in [21], the cohomology defined from the above Floer differential is isomorphic to the cohomology defined by the "quilted Floer differential" arising from the moduli spaces

$$\widetilde{\mathcal{M}}^1_\delta(x^-,x^+) = \big\{(v_0,v_1,v_2)\,\big|\,(10)_\delta,(11),\ \mathrm{Ind}(D_{(v_0,v_1,\overline{v}_2)})=1\big\}$$

for any choice of $\delta>0$. Here we consider strips v_0,v_2 of width 1 as before but middle strips $v_1:\mathbb{R}\times[0,\delta]\to M_1$ of width $\delta>0$, and $(10)_\delta$ denotes the same boundary value problem as above except for the seam condition $(v_1(s,\delta),v_2(s,0))\in L_{12}$. Moreover, we use almost complex structures $J_{0,\delta},J_{2,\delta}$ that converge to J_0,J_2 in the $\mathcal{C}^\infty$-topology as $\delta\to 0$. The specific choice follows from the constructions in the proof [2] and will also ensure that the moduli spaces $\widetilde{\mathcal{M}}^1_\delta(x^-,x^+)$ are cut out transversely for $\delta>0$ sufficiently small.

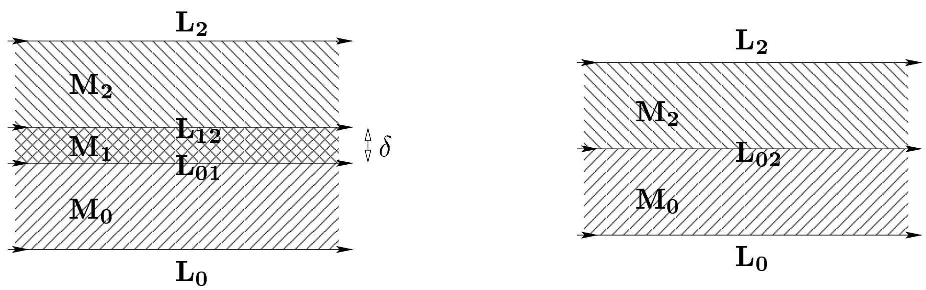


FIGURE 3. Shrinking the middle strip

In order to prove Theorem 1.0.1 it now suffices to show that the isomorphism (8) of chain groups descends to cohomology for an appropriate choice of $\delta>0$. We will prove this by establishing a bijection between the Floer trajectories for (L_0,L_{02},L_2) on strips of width $(1,1)$ and those for (L_0,L_{01},L_{12},L_2) on strips of width $(1,\delta,1)$ for sufficiently small width $\delta>0$ of the middle strip. These Floer trajectories are holomorphic quilts associated to the pictures in Figure 3. More precisely, we will consider the (zero dimensional, compact) moduli spaces of Floer trajectories modulo $\mathbb{R}$-translation and prove the following.

Theorem 2.2.2. *Under the assumptions* (a), (b), (c) *of Theorem 1.0.1 and for all sufficiently small $\delta>0$, the moduli spaces $\widetilde{\mathcal{M}}^1_\delta(x^-,x^+)$ are regular and there is a bijection*

$$\mathcal{T}_\delta:\ \mathcal{M}^1_0(x^-,x^+):=\widetilde{\mathcal{M}}^1_0(x^-,x^+)/\mathbb{R}\longrightarrow\widetilde{\mathcal{M}}^1_\delta(x^-,x^+)/\mathbb{R}=:\mathcal{M}^1_\delta(x^-,x^+).$$

Remark 2.2.3. In the situation of Theorem 1.0.1 except for assumption (d), the constructions in this section provide naturally isomorphic chain groups $CF(L_0\times L_1,L_{02})$ and $CF(L_0\times L_{12},L_{01}\times L_2)$ and well defined differentials ∂_0 resp. ∂_δ on them, defined from the moduli spaces $\mathcal{M}^1_0(x^-,x^+)$ and $\mathcal{M}^1_\delta(x^-,x^+)$. As discussed in Section 2.1, due to obstructions from disks of minimal Maslov index 2, both differentials square to a multiple of the identity, see [9] and [25]. So we have $\partial_0^2=w_0\mathrm{Id}$ and $\partial_\delta^2=w_\delta\mathrm{Id}$ for any $\delta>0$ (as long as the moduli spaces $\mathcal{M}^1_\delta(x^-,x^+)$ are regular). Now Theorem 2.2.2 implies that for sufficiently small $\delta>0$ and any $x\in\mathcal{I}$ (viewed as generator in both chain groups) we have $w_0\langle x\rangle=\partial_0^2\langle x\rangle=\partial_\delta^2\langle x\rangle=w_\delta\langle x\rangle$, and hence $w_0=w_\delta$. (If $\mathcal{I}$ is empty then both theories are trivial.)

[2] Due to more technical folding, $J_{0,\delta},J_{2,\delta}$ are given by rescaling J_0 to $[0,1-\delta/2]$ and J_2 to $[\delta/2,1]$, and extending them constantly by $J_0(1)$ and $J_2(0)$ respectively. The convergence holds since each J_i is smooth and constant near $t=0,1$.

If $w_\delta = 0$ (e.g. by assumption (d)) or $w_0 = 0$ for some other reason, then this proves that both Floer cohomologies are well defined and (again by Theorem 2.2.2) are isomorphic.

For any value of $w_0 = w_\delta$ this proves that there exists a canonical isomorphism

$$\big(CF(L_0 \times L_{12}, L_{01} \times L_2), \partial_0\big) \xrightarrow{\sim} \big(CF(L_0 \times L_2, L_{01} \circ L_{12}), \partial_\delta\big)$$

in the derived category of factorizations of $w_0 \mathrm{Id}$.

3. Bijection of moduli spaces under strip shrinking

In this section we prove Theorem 2.2.2. We start by describing the strategy of proof and introducing the relevant notations. First we use the assumption that $L_{01} \circ L_{12}$ is embedded by π_{02}. Consider a solution $u = (u_0, u_2) \in \widetilde{\mathcal{M}}^1_0(x^-, x^+)$, that is a pair $u_0 : \mathbb{R} \times [0,1] \to M_0$, $u_2 : \mathbb{R} \times [0,1] \to M_2$ of index 1, with limits $\lim_{s\to\pm\infty}(u_0, u_2)(s, \cdot) = x^\pm$, and satisfying

$$\begin{aligned}
&\overline{\partial}_{J_0} u_0 = 0, \qquad \overline{\partial}_{J_2} u_2 = 0,\\
&u_0|_{t=0} \in L_0, \qquad (u_0|_{t=1}, u_2|_{t=0}) \in L_{02}, \qquad u_2|_{t=1} \in L_2.
\end{aligned}$$

We can identify (u_0, u_2) with the map $u_{02} : \mathbb{R} \times [0,1] \to M_0 \times M_2$ given by $u_{02}(s,t) = (u_0(s, 1-t), u_2(s,t))$, which satisfies $\lim_{s\to\pm\infty} u_{02}(s, \cdot) = x^\pm$ and

$$\overline{\partial}_{J_{02}} u_{02} = 0, \qquad u_{02}|_{t=0} \in L_{02}, \qquad u_{02}|_{t=1} \in L_0 \times L_2.$$

Here we denoted $J_{02}(s,t) := (-J_0(s, 1-t), J_2(s,t))$. We will also denote $\bar{J}_{02} := J_{02}|_{t=0}$ and $\bar{u}_{02} := u_{02}|_{t=0} : \mathbb{R} \to L_{02}$. Finally, we will denote by $(L_{01} \times L_{12})^T \subset M_0 \times M_2 \times M_1 \times M_1$ the obvious transposition of factors. Since $\pi_{02} : L_{01} \times_{M_1} L_{12} \to L_{02} \subset M_0 \times M_2$ is transversal and embedded, there is a unique smooth map $\ell_1 : L_{02} \to M_1$ such that

$$(x_{02}, \ell_1(x_{02}), \ell_1(x_{02})) \in (L_{01} \times L_{12})^T \qquad \forall x_{02} \in L_{02}. \tag{12}$$

This provides the lift $\bar{u}_1 := \ell_1 \circ \bar{u}_{02} : \mathbb{R} \to M_1$. We also denote by $\bar{u} := (\bar{u}_{02}, \bar{u}_1, \bar{u}_1)$ the extension $\mathbb{R} \times [0, \delta] \to M_0 \times M_2 \times M_1 \times M_1$ that is constant along $[0, \delta]$. Given δ these choices are unique, so we can identify u with the pair $(u_{02}, \bar{u})$. In the same spirit we find unique points $x_1^\pm \in M_1$ such that $(x^\pm, x_1^\pm) \in (L_0 \times L_{12}) \cap (L_{01} \times L_2) \subset M_0 \times M_1 \times M_2$. In this notation we have the limit $\lim_{s\to\pm\infty} \bar{u}_1(s) = x_1^\pm$. Given $u \in \widetilde{\mathcal{M}}^1_0(x^-, x^+)$ as above and $\delta > 0$ we wish to find a corresponding $(v_0, v_1, v_2) \in \widetilde{\mathcal{M}}^1_\delta(x^-, x^+)$, that is a triple $v_0 : \mathbb{R} \times [0,1] \to M_0$, $v_1 : \mathbb{R} \times [0, \delta] \to M_1$, $v_2 : \mathbb{R} \times [0,1] \to M_2$ with limits $\lim_{s\to\pm\infty}(v_0, v_2)(s, \cdot) = x^\pm$, $\lim_{s\to\pm\infty} v_1(s, \cdot) = x_1^\pm$, and satisfying

$$\begin{aligned}
&\overline{\partial}_{J_{0,\delta}} v_0 = 0, \qquad \overline{\partial}_{J_1} v_1 = 0, \qquad \overline{\partial}_{J_{2,\delta}} v_2 = 0,\\
&v_0(s,0) \in L_0, \quad (v_0(s,1), v_1(s,0)) \in L_{01}, \quad (v_1(s,\delta), v_2(s,0)) \in L_{12}, \quad v_2(s,1) \in L_2.
\end{aligned}$$

Here $J_{0,\delta}, J_{2,\delta}$ are given by linearly rescaling J_0 to $[0, 1-\delta/2]$ and J_2 to $[\delta/2, 1]$, and extending them constantly by $J_0(1)$ and $J_2(0)$ respectively. This choice of almost complex structures is more natural in the following reformulation of the δ-moduli spaces.

Let $\bar{\delta} := \delta/(2-\delta)$ (or equivalently $\delta = 2\bar{\delta}/(1+\bar{\delta})$). Instead of the triple strip we consider a quadruple of maps $v = (v_{02}, v'_{02}, v_1, v'_1)$ with $v_{02} \in \mathcal{C}^\infty(\mathbb{R} \times [0,1], M_0 \times M_2)$, $v'_{02} \in \mathcal{C}^\infty(\mathbb{R} \times [0, \bar{\delta}], M_0 \times M_2)$, $v_1, v'_1 \in \mathcal{C}^\infty(\mathbb{R} \times [0, \bar{\delta}], M_1)$ that have limits $\lim_{s\to\pm\infty} v_{02}(s, \cdot) =$

$\lim_{s\to\pm\infty} v_{02}'(s,\cdot) = x^\pm$, $\lim_{s\to\pm\infty} v_1(s,\cdot) = \lim_{s\to\pm\infty} v_1(s,\cdot) = x_1^\pm$, and satisfy

$$\begin{aligned}
&\overline{\partial}_{J_{02}} v_{02} = 0, \qquad \overline{\partial}_{-\bar J_{02}} v_{02}' = 0, \qquad \overline{\partial}_{-J_1} v_1' = 0, \qquad \overline{\partial}_{J_1} v_1 = 0,\\
&(v_{02}', v_{02})|_{t=0} \in \Delta_0 \times \Delta_2, \qquad (v_1', v_1)|_{t=0} \in \Delta_1,\\
&(v_{02}', v_1', v_1)|_{t=\bar\delta} \in (L_{01}\times L_{12})^T, \qquad v_{02}|_{t=1} \in L_0 \times L_2.
\end{aligned} \tag{13}$$

For notational convenience we will also group these quadruples of maps as $v = (v_{02}, \hat v)$ with $\hat v = (v_{02}', v_1, v_1')$. Then we can abbreviate $J = (J_{02}, \hat J)$ with $\hat J := (-\bar J_{02}, -J_1, J_1)$, and reformulate (13) as

$$\begin{aligned}
&\overline{\partial}_J v := \big(\overline{\partial}_{J_{02}} v_{02}\,,\, \overline{\partial}_{\hat J} \hat v\big) = 0,\\
&(v_{02}, \hat v)|_{t=0} \in \Delta_0 \times \Delta_2 \times \Delta_1, \qquad \hat v_{t=\bar\delta} \in (L_{01}\times L_{12})^T, \qquad v_{02}|_{t=1} \in L_0 \times L_2.
\end{aligned}$$

We denote the moduli space of such solutions $v = (v_{02}, \hat v)$ by $\widehat{\mathcal M}^1_{\bar\delta}(x^-, x^+)$. It is in one-to-one correspondence to $\widetilde{\mathcal M}^1_\delta(x^-, x^+)$ as follows: Given $v = (v_{02}, v_{02}', v_1', v_1) \in \widehat{\mathcal M}^1_{\bar\delta}(x^-, x^+)$ we obtain $\bar v = (v_0, v_1, v_2) \in \widetilde{\mathcal M}^1_\delta(x^-, x^+)$ from

$$\begin{aligned}
\big(v_0(s,1-t), v_2(s,t)\big) &= \begin{cases} v_{02}'((1+\bar\delta)s, \bar\delta - (1+\bar\delta)t) & \text{for } 0 \le t \le \tfrac12\delta,\\ v_{02}((1+\bar\delta)s, (1+\bar\delta)t - \bar\delta) & \text{for } \tfrac12\delta \le t \le 1,\end{cases}\\
v_1(s,t) &= \begin{cases} v_1'((1+\bar\delta)s, \bar\delta - (1+\bar\delta)t) & \text{for } 0 \le t \le \tfrac12\delta,\\ v_1((1+\bar\delta)s, (1+\bar\delta)t - \bar\delta) & \text{for } \tfrac12\delta \le t \le \delta.\end{cases}
\end{aligned}$$

The two different formulations for double and triple strips each are indicated in Figure 4. The bijection $\mathcal T_\delta$ to the moduli space $\mathcal M^1_0(x^-, x^+)$ can then be established via a bijection

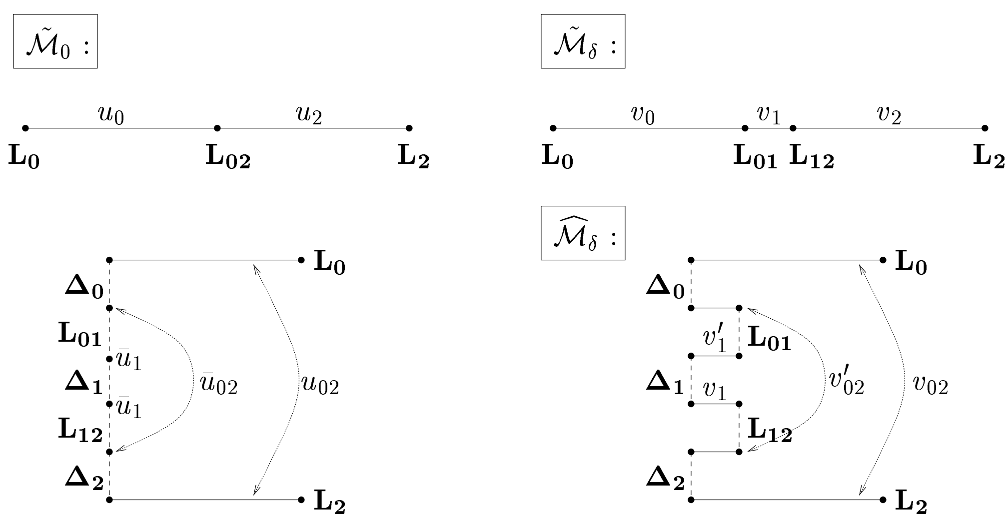


FIGURE 4. Double and triple strips

$$\mathcal T_{\bar\delta} : \mathcal M^1_0(x^-, x^+) \to \mathcal M^1_{\bar\delta}(x^-, x^+) := \widehat{\mathcal M}^1_{\bar\delta}(x^-, x^+)/\mathbb R. \tag{14}$$

This map will be constructed by the implicit function theorem 3.1.1. We prove injectivity in corollary 3.1.6, and the surjectivity will follow from the compactness theorem 3.3.1.

3.1. **Implicit function theorem.** The purpose of this section is to construct the map $\mathcal{T}_\delta : \mathcal{M}^1_0(x^-,x^+) \to \mathcal{M}^1_\delta(x^-,x^+)$ of Theorem 2.2.2. We will do this by constructing the map (14), with $\bar\delta$ replaced by δ, from the following implicit function theorem.

Theorem 3.1.1. *There exist constants C_0, $\epsilon > 0$, and $\delta_0 > 0$ such that the following holds for every $\delta \in (0,\delta_0]$. For every $u \in \widetilde{\mathcal{M}}^1_0(x^-,x^+)$ there exists a unique $v_u \in \widehat{\mathcal{M}}^1_\delta(x^-,x^+)$ such that $v_u = e_u(\xi)$ with $\xi \in \Gamma_{1,\delta}(\epsilon) \cap K_0$. The solution moreover satisfies*

$$\|\xi\|_{H^2_{1,\delta}} \le C_0\sqrt{\delta}. \tag{15}$$

Here $e_u(\xi) := (v_{02}, v'_{02}, v'_1, v_1)$ is given in terms of $u = (u_{02}, \bar u)$ and $\xi = (\xi_{02}, \hat\xi)$ with $\xi_{02} \in \Gamma(u_{02}^*T(M_0 \times M_2))$ and $\hat\xi = (\xi'_{02}, \xi'_1, \xi_1) \in \Gamma(\bar u^*T(M_0 \times M_2 \times M_1 \times M_1))$. The precise definitions of the exponential map e_u, the ϵ-ball $\Gamma_{1,\delta}(\epsilon)$, the $H^2_{1,\delta}$-norm, and the local slice K_0 of the $\mathbb{R}$-shift symmetry will be given in the process of the proof.

To prove the theorem we fix a solution $u \in \widetilde{\mathcal{M}}^1_0(x^-,x^+)$, and in the following will allow all constants to depend on u up to translation in $\mathbb{R}$. (Since $\mathcal{M}^1_0(x^-,x^+)$ is finite we can then easily find uniform constants C_0 and $\delta_0 > 0$.) We will then roughly solve $\overline{\partial}_J e_u(\xi) = 0$ for sections $\xi = (\xi_{02}, \hat\xi)$, $\hat\xi = (\xi'_{02}, \xi'_1, \xi_1)$ satisfying the boundary conditions

$$\begin{aligned} (\xi'_{02}, \xi_{02})|_{t=0} &\in T_{(\bar u_{02}, \bar u_{02})}\Delta_{M_0\times M_2}, \qquad (\xi'_1, \xi_1)|_{t=0} \in T_{(\bar u_1, \bar u_1)}\Delta_1, \\ \hat\xi|_{t=\delta} = (\xi'_{02}, \xi'_1, \xi_1)|_{t=\delta} &\in T_{\bar u}(L_{01} \times L_{12})^T, \qquad \xi_{02}|_{t=1} \in T_{u_{02}}(L_0 \times L_2). \end{aligned} \tag{16}$$

The exponential map $e_u(\xi)$ will then be constructed such that the nonlinear Lagrangian boundary conditions are satisfied automatically. The index of the new solution v_u will coincide with that of the given solution u due to Lemma 2.1.3. Here we identified v_u with a solution $\bar v_u \in \widetilde{\mathcal{M}}^1_{\tilde\delta}(x^-,x^+)$, $\tilde\delta = 2\delta/(1+\delta)$. Then the homotopy between $v_u = e_u(\xi)$ and $(u_{02}, \bar u)$ induces a homotopy $\bar v_u \cong (u_0, \bar u_1, u_2)$.

To set up the implicit function theorem we introduce the space of H^k-sections over $(u_{02}, \bar u)$ for $k \in \mathbb{N}_0$,

$$H^k_{1,\delta} := \left\{ (\eta_{02}, \eta'_{02}, \eta'_1, \eta_1) \;\middle|\; \begin{array}{l} \eta_{02} \in H^k(\mathbb{R}\times[0,1], u_{02}^*T(M_0\times M_2)), \\ \eta'_{02} \in H^k(\mathbb{R}\times[0,\delta], \bar u_{02}^*T(M_0\times M_2)), \\ \eta'_1, \eta_1 \in H^k(\mathbb{R}\times[0,\delta], \bar u_1^*TM_1) \end{array} \right\}.$$

We also write these sections as $\eta = (\eta_{02}, \hat\eta) \in H^k_{1,\delta}$, where the subscripts indicate the width of the domains of η_{02} and $\hat\eta = (\eta'_{02}, \eta'_1, \eta_1) \in H^k(\mathbb{R}\times[0,\delta], \bar u^*T(M_0\times M_2\times M_1\times M_1))$. The corresponding H^k-norm on this space is

$$\begin{aligned} \big\|(\eta_{02}, \eta'_{02}, \eta'_1, \eta_1)\big\|^2_{H^k_{1,\delta}} &:= \|\eta_{02}\|^2_{H^k(\mathbb{R}\times[0,1])} + \|\hat\eta\|^2_{H^k(\mathbb{R}\times[0,\delta])} \\ &= \|\eta_{02}\|^2_{H^k(\mathbb{R}\times[0,1])} + \|\eta'_{02}\|^2_{H^k(\mathbb{R}\times[0,\delta])} + \|\eta'_1\|^2_{H^k(\mathbb{R}\times[0,\delta])} + \|\eta_1\|^2_{H^k(\mathbb{R}\times[0,\delta])}. \end{aligned}$$

We denote the space of H^2-sections satisfying the boundary conditions by

$$\Gamma_{1,\delta} := \big\{ \xi \in H^2_{1,\delta} \,\big|\, (16) \big\}$$

and equip this space with the norm

$$\|\xi\|_{\Gamma_{1,\delta}} := \|\xi\|_{H^2_{1,\delta}} + \|\nabla\xi\|_{L^4_{1,\delta}},$$

with the L^4-norm $\|\nabla(\xi_{02},\hat\xi)\|_{L^4_{1,\delta}} := \big(\|\nabla\xi_{02}\|^4_{L^4_{1,\delta}(\mathbb{R}\times[0,1])} + \|\nabla\hat\xi\|^4_{L^4_{1,\delta}(\mathbb{R}\times[0,\delta])}\big)^{1/4}$ on the multi-strip. We denote the ϵ-ball in $\Gamma_{1,\delta}$ by

$$\Gamma_{1,\delta}(\epsilon) := \big\{\xi\in H^2_{1,\delta}\,\big|\,\|\xi\|_{\Gamma_{1,\delta}}<\epsilon, (16)\big\}.$$

We equip the target space $\Omega_{1,\delta} := H^1_{1,\delta}$ with the norm

$$\|\eta\|_{\Omega_{1,\delta}} := \|\eta\|_{H^1_{1,\delta}} + \|\eta\|_{L^4_{1,\delta}}.$$

The reason for adding the L^4-norms in domain and target is that we do not have uniform Sobolev embeddings on the strips of varying width. Instead, we build the necessary Sobolev multiplication properties into the norms.

Next, we make some preparations for defining an exponential map that is compatible with the boundary conditions (16).

Lemma 3.1.2. *(Existence of compatible quadratic corrections) There exists $\epsilon_0>0$ and smooth families of maps (defined on the ϵ_0-balls)*

$$Q_s : T_{\bar u(s)}\big(M_0\times M_2\times M_1\times M_1\big)\supset B_{\epsilon_0}\to T_{\bar u(s)}\big(M_0\times M_2\times M_1\times M_1\big),\qquad \forall s\in\mathbb{R},$$

$$Q^{02}_{s,t} : T_{u_{02}(s,t)}(M_0\times M_2)\supset B^{02}_{\epsilon_0}\to T_{u_{02}(s,t)}(M_0\times M_2)\qquad \forall (s,t)\in\mathbb{R}\times[0,1],$$

that are a diffeomorphism onto their image and have the following properties:

(Quadratic): *$Q_s(0)=0$, $dQ_s(0)\equiv 0$, $Q^{02}_{s,t}(0)=0$, and $dQ^{02}_{s,t}(0)\equiv 0$ for all $(s,t)\in\mathbb{R}\times[0,1]$. In particular, there is a constant C_Q such that for all $\hat\xi\in B_{\epsilon_0}$ and $\xi_{02}\in B^{02}_{\epsilon_0}$*

$$|Q_s(\hat\xi)|\le C_Q|\hat\xi|^2,\qquad |Q^{02}_{s,t}(\xi_{02})|\le C_Q|\xi_{02}|^2. \tag{17}$$

(Linearizing $\mathbf{L_{01}\times L_{12}}$): *$\exp_{\bar u(s)}\circ(1+Q_s)$ maps $T_{\bar u(s)}(L_{01}\times L_{12})^T\cap B_{\epsilon_0}$ to $(L_{01}\times L_{12})^T$.*

(Linearizing $\mathbf{M_0\times M_2\times \Delta_1}$): *$\exp_{\bar u(s)}\circ(1+Q_s)$ maps $T_{\bar u(s)}(M_0\times M_2\times\Delta_1)\cap B_{\epsilon_0}$ to $M_0\times M_2\times\Delta_1$.*

(Linearizing $\mathbf{L_{02}}$): *$\exp_{u_{02}(s,1)}\circ(1+Q^{02}_{s,1})$ maps $T_{u_{02}(s,1)}L_{02}\cap B^{02}_{\epsilon_0}$ to L_{02}.*

(Compatible): *Restricting Q_s to $T_{\bar u}(M_0\times M_2\times\Delta_1)$ and composing it with the projection $\mathrm{Pr}_{02}: T_{(\bar u_{02},\bar u_1,\bar u_1)}(M_0\times M_2\times M_1\times M_1)\to T_{\bar u_{02}}(M_0\times M_2)$ yields a map that is independent of the $T_{(\bar u_1,\bar u_1)}\Delta_1$-component. The resulting family*

$$Q^{02}_s : T_{\bar u_{02}(s)}(M_0\times M_2)\supset B^{02}_{\epsilon_0}\to T_{\bar u_{02}(s)}(M_0\times M_2)$$

coincides with $Q^{02}_{s,0}$.

Proof. We fix $s\in\mathbb{R}$ and restrict the exponential map $\exp_{\bar u(s)}$ to a geodesic ball around 0. The subsequent constructions will depend smoothly on $s\in\mathbb{R}$, which we drop from now on. By assumption the submanifold $\mathcal{L}_{0211} := \exp_{\bar u}^{-1}(L_{01}\times L_{12})^T$ in the vector space $X := T_{\bar u}(M_0\times M_2\times M_1\times M_1)$ is transverse to the subspace $\Delta := T_{\bar u}(M_0\times M_2\times\Delta_1)$. Their intersection $\hat{\mathcal{L}}_{02} := \mathcal{L}_{0211}\cap\Delta$ is diffeomorphic to the submanifold $\mathcal{L}_{02} := \exp^{-1}_{\bar u_{02}}(L_{02})\subset T_{\bar u_{02}}(M_0\times M_2)$ by a map $(m_0,m_2)\mapsto(m_0,m_2,m_1,m_1)$ with uniquely determined $m_1 = m_1(m_0,m_2)$. So we have a direct sum decomposition

$$\Delta = T_{\bar u_{02}}(M_0\times M_2)\times T_{(\bar u_1,\bar u_1)}\Delta_1 = T_0\hat{\mathcal{L}}_{02}\oplus\big((T_0\mathcal{L}_{02})^\perp\times\{0\}\big)\oplus\big(\{0\}\times T_{(\bar u_1,\bar u_1)}\Delta_1\big).$$

As a submanifold we can now write $\hat{\mathcal{L}}_{02}\subset\Delta$ as the graph of a map ψ over a sufficiently small ϵ-ball,

$$\psi = \psi_{02}^\perp\times\psi_{11} : T_0\hat{\mathcal{L}}_{02}\supset B_\epsilon\to\big(T_0\mathcal{L}_{02}\big)^\perp\times T_{(\bar u_1,\bar u_1)}\Delta_1$$

with $\psi(0)=0$ and $d\psi(0)\equiv 0$. We moreover pick a complement C of $T_0\hat{\mathcal{L}}_{02}\subset T_0\mathcal{L}_{0211}$,

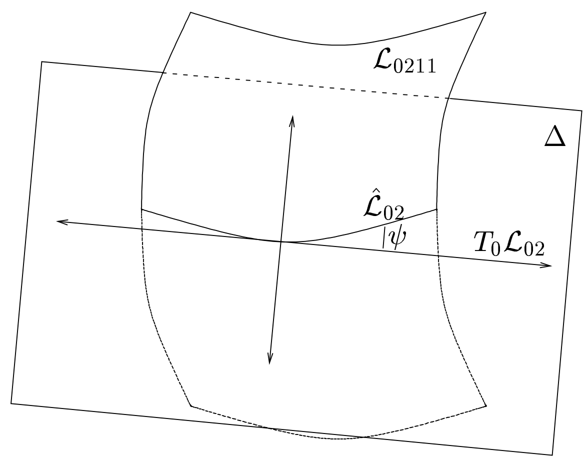
$$T_0\mathcal{L}_{0211}=C\oplus T_0\hat{\mathcal{L}}_{02},$$

then the transversality $X=T_0\mathcal{L}_{0211}+\Delta$ implies the splitting

$$X=C\oplus T_0\hat{\mathcal{L}}_{02}\oplus\bigl(T_0\mathcal{L}_{02}\bigr)^{\perp}\times\{0\}\oplus\{0\}\times T_{(\bar{u}_1,\bar{u}_1)}\Delta_1. \tag{18}$$

We write $X\ni x=x_C+x_{02}+(x_{02}^{\perp},0)+(0,x_{11})$ in this splitting and define a map $\Psi:X\supset B_\epsilon\to X$ by

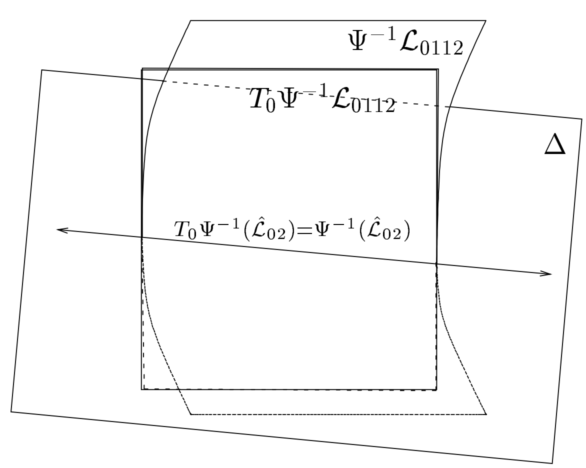
$$\begin{aligned}\Psi(x)&:=x+(\psi_{02}^{\perp}(x_{02}),0)+(0,\psi_{11}(x_{02}))\\&=x_C+x_{02}+(x_{02}^{\perp}+\psi_{02}^{\perp}(x_{02}),0)+(0,x_{11}+\psi_{11}(x_{02})).\end{aligned}$$



This map linearizes the intersection, $\Psi(T_0\hat{\mathcal{L}}_{02})=\hat{\mathcal{L}}_{02}$, and we have $\Psi(0)=0$ and $d\Psi(0)=\mathrm{Id}$. In order to linearize the entire Lagrangian $\mathcal{L}_{0211}$ we remark that $T_0\bigl(\Psi^{-1}(\mathcal{L}_{0211})\bigr)=d\Psi(0)^{-1}T_0\mathcal{L}_{0211}=T_0\mathcal{L}_{0211}$. So we can write $\Psi^{-1}(\mathcal{L}_{0211})$ as graph of a map

$$\phi=\phi_{02}^{\perp}\times\phi_{11}:T_0\mathcal{L}_{0211}\supset B_\epsilon\to\bigl(T_{\bar{u}_{02}}\mathcal{L}_{02}\bigr)^{\perp}\times T_{(\bar{u}_1,\bar{u}_1)}\Delta_1$$

with $\phi(0)=0$, $d\phi(0)\equiv 0$, and by the previous construction $\phi|_{T_0\hat{\mathcal{L}}_{02}}\equiv 0$.



Finally we define the entire linearization $\Phi : X \supset B_\epsilon \to X$ by

$$\Phi(x) := \Psi\big(x + (\phi_{02}^\perp(x_C + x_{02}), 0) + (0, \phi_{11}(x_C + x_{02}))\big)$$

for $x = x_C + x_{02} + (x_{02}^\perp, 0) + (0, x_{11})$ in the splitting (18). Now $Q_s := \Phi - \mathrm{Id}$ is quadratic and linearized $(L_{01} \times L_{12})^T$ by construction. Explicitly, we have

$$Q_s(x) = \big(\psi_{02}^\perp(x_{02}) + \phi_{02}^\perp(x_C + x_{02}), \psi_{11}(x_{02}) + \phi_{11}(x_C + x_{02})\big). \tag{19}$$

The construction moreover ensures that Q_s linearizes $M_0 \times M_2 \times \Delta_1$, that is $\Phi(\Delta) \subset \Delta$, since $x \in \Delta = \{x_C = 0\}$ is mapped to $\Phi(x) = x + \big(\psi_{02}^\perp(x_{02}), \psi_{11}(x_{02})\big) \in \Delta$.

To construct Q_s^{02} compatible with Q_s note that for $x = (m_0, m_2, m_1, m_1) \in T_{\bar{u}}(M_0 \times M_2 \times \Delta_1) \subset X$ we have a splitting

$$x = (m_0, m_2, 0, 0) + (0, 0, m_1, m_1) = x_C + x_{02} + (x_{02}^\perp, 0) + (0, x_{11} + (m_1, m_1)),$$

where $x_C, x_{02}, x_{02}^\perp, x_{11}$ only depend on (m_0, m_2). With this we can see in (19) that indeed $Q_s(m_0, m_2, m_1, m_1)$ is independent of m_1. We then simply define $Q_{s,0}^{02}(m_0, m_2) := \mathrm{Pr}_{02} Q_s(m_0, m_2, 0, 0)$. Moreover, a graph construction as above provides a map $Q_{s,1}^{02} : T_{u_{02}(s,1)}(M_0 \times M_2) \supset B_\epsilon^{02} \to T_{u_{02}(s,1)}(M_0 \times M_2)$ that is quadratic and linearizes L_{02}. Now the two families $Q_{s,0}^{02}$ and $Q_{s,1}^{02}$ can easily be interpolated by the smooth family $Q_{s,t}^{02} := (1-t)Q_{s,0}^{02} + tQ_{s,1}^{02}$ of quadratic maps. □

With these quadratic corrections we can now define the exponential map e_u by $e_u(\xi) := (e_{u_{02}}(\xi_{02}), e_{\bar{u}}(\hat{\xi}))$ for $\xi = (\xi_{02}, \hat{\xi}) \in \Gamma_{1,\delta}(\epsilon)$, where

$$e_{u_{02}}(\xi_{02}) := \exp_{u_{02}} \circ (1 + Q^{02})(\xi_{02}), \qquad e_{\bar{u}}(\hat{\xi}) := \exp_{\bar{u}} \circ (1 + Q)(\hat{\xi}). \tag{20}$$

Note that we have the usual properties of an exponential map,

$$e_u(0) = (u_{02}, \bar{u}), \qquad de_u(0) = \mathrm{Id}.$$

To define e_u on $\Gamma_{1,\delta}(\epsilon)$ the $\epsilon > 0$ should be chosen such that $\|\xi_{02}\|_{\mathcal{C}^0}$ and $\|\hat{\xi}\|_{\mathcal{C}^0}$ are sufficiently small for the quadratic corrections in Lemma 3.1.2 to be defined. Lemma 3.1.4 below ensures that we can pick a uniform $\epsilon > 0$ for all $\delta > 0$. Now solutions $v_u \in \widehat{\mathcal{M}}_\delta^1(x^-, x^+)$ in a neighborhood of u correspond to zeroes of the map $\mathcal{F}_u : \Gamma_{1,\delta}(\epsilon) \to \Omega_{1,\delta}$ given by

$$\mathcal{F}_u(\xi) := \big(\, \Phi_{u_{02}}(\xi_{02})^{-1}(\overline{\partial}_{J_{02}} e_{u_{02}}(\xi_{02}))\,,\, \Phi_{\bar{u}}(\hat{\xi})^{-1}(\overline{\partial}_{\hat{J}} e_{\bar{u}}(\hat{\xi}))\,\big).$$

Here $\Phi_u(\xi)$ denotes the parallel transport $T_u M \to T_{e_u(\xi)} M$ along the path $\tau \mapsto e_u(\tau\xi)$. For $\Phi_{u_{02}}$ this parallel transport on $T(M_0 \times M_2)$ can simply use the Levi-Civita connection. In the definition of $\Phi_{\bar{u}}$ we however use a Hermitian connection $\tilde{\nabla}$ on the tangent bundle $T(M_0 \times M_2 \times M_1 \times M_1)$ that leaves $\hat{J}$ invariant. This can be done by the same construction as in [8, Proposition 3.1.1], which brings the linearized operator into simple form.

Next, we introduce projections related to the various Lagrangians:

$$\pi_{0211}^\perp \in \mathrm{Aut}\big(\mathcal{C}^\infty(\mathbb{R}, \bar{u}^* T(M_0 \times M_2 \times M_1 \times M_1))\big),$$

$$\pi_{02}\,,\, \pi_{02}^\perp \;\in \mathrm{Aut}\big(\mathcal{C}^\infty(\mathbb{R}, \bar{u}_{02}^* T(M_0 \times M_2))\big)$$

are linear operators, given by pointwise orthogonal projection onto the subspaces $(T(L_{01} \times L_{12})^T)^\perp \subset T(M_0 \times M_2 \times M_1 \times M_1)$ resp. $TL_{02}, (TL_{02})^\perp \subset T(M_0 \times M_2)$. The following lemma contains the estimates resulting from the transversality assumption.

Lemma 3.1.3. *(Quantitative transversality) There exists a constant C such that the following holds.*

(a) *For every* $s \in \mathbb{R}$ *and* $\hat{\xi} = (\xi'_{02}, \xi'_1, \xi_1) \in T_{\bar{u}(s)}(M_0 \times M_2 \times M_1 \times M_1)$

$$|\hat{\xi}| \le C\big(|\pi_{02}\xi'_{02}| + \big|\xi'_1 - \xi_1\big| + \big|\pi^{\perp}_{0211}\hat{\xi}\big|\big),$$
$$|\pi^{\perp}_{02}\xi'_{02}| \le C\big(|\pi^{\perp}_{0211}\hat{\xi}| + |\xi'_1 - \xi_1|\big).$$

(b) *For every* $\hat{\xi} \in \mathcal{C}^\infty(\mathbb{R}, \bar{u}^* T(M_0 \times M_2 \times M_1 \times M_1))$

$$\|\hat{\xi}\|_{H^1(\mathbb{R})} \le C\big(\|\pi_{02}\xi'_{02}\|_{H^1(\mathbb{R})} + \|\xi'_1 - \xi_1\|_{H^1(\mathbb{R})} + \|\pi^{\perp}_{0211}\hat{\xi}\|_{H^1(\mathbb{R})}\big),$$

and the same holds with H^1 *replaced by* $\mathcal{C}^1$ *or* L^p *for any* $p \ge 1$. *Moreover,*

$$\|\pi^{\perp}_{02}\xi'_{02}\|_{L^2(\mathbb{R})} \le C\big(\|\pi^{\perp}_{0211}\hat{\xi}\|_{L^2(\mathbb{R})} + \|\xi'_1 - \xi_1\|_{L^2(\mathbb{R})}\big),$$
$$\|\pi^{\perp}_{02}\xi'_{02}\|_{H^1(\mathbb{R})} \le C\big(\|\pi^{\perp}_{0211}\hat{\xi}\|_{H^1(\mathbb{R})} + \|\xi'_1 - \xi_1\|_{H^1(\mathbb{R})} + \big\||\partial_s\bar{u}| \cdot |\hat{\xi}|\big\|_{L^2(\mathbb{R})}\big).$$

Proof. The Lagrangian $L_{01} \times L_{12}$ intersects $M_0 \times \Delta_1 \times M_2$ transversally in $\hat{L}_{02}$, which injects to $L_{02} \subset M_0 \times M_2$. So at every point of $\hat{L}_{02}$ we have a decomposition $T(M_0 \times M_2 \times M_1 \times M_1) = T\hat{L}_{02} \oplus (T\hat{L}_{02})^\perp$, where we can change the first factor to $TL_{02} \times \{0\}$. On the other hand, the transverse intersection implies

$$(T\hat{L}_{02})^\perp = \big(\{0\} \times (T\Delta_1)^\perp\big) \oplus T(L_{01} \times L_{12})^\perp, \tag{21}$$

so we obtain a splitting

$$T(M_0 \times M_2 \times M_1 \times M_1) = \big(TL_{02} \times \{0\}\big) \oplus \big(\{0\} \times (T\Delta_1)^\perp\big) \oplus T(L_{01} \times L_{12})^\perp. \tag{22}$$

This means that the product of the three orthogonal projections onto the factors defines an isomorphism. The norm of this isomorphism is bounded at each $\bar{u}(s) \in \hat{L}_{02}$, so for every $\hat{\xi} = (\xi'_{02}, \xi'_1, \xi_1) \in T_{\bar{u}(s)}(M_0 \times M_2 \times M_1 \times M_1)$ we have

$$|\hat{\xi}| \le C\big(|\pi_{02}\xi'_{02}| + \big|\xi'_1 - \xi_1\big| + \big|\pi^{\perp}_{0211}\hat{\xi}\big|\big)$$

with a uniform constant C as claimed in (a). (Here the projection onto $(T\Delta_1)^\perp$ is given by $(\xi'_{02}, \xi'_1, \xi_1) \mapsto \frac{1}{2}(\xi'_1 - \xi_1, \xi_1 - \xi'_1)$.) Moreover, the splitting (22) commutes with

$$T(M_0 \times M_2) = TL_{02} \oplus (TL_{02})^\perp$$

via the canonical projection on the left hand side, and on the right hand side the identity on TL_{02} combined with a bounded map $\big(\{0\} \times (T\Delta_1)^\perp\big) \oplus T(L_{01} \times L_{12})^\perp \to TL_{02} \oplus (TL_{02})^\perp$. This implies that

$$|\pi^{\perp}_{02}\xi'_{02}| \le C\big(|\pi^{\perp}_{0211}\hat{\xi}| + |\xi'_1 - \xi_1|\big)$$

with another uniform constant C. This proves (a). For $\hat{\xi} \in \mathcal{C}^\infty(\mathbb{R}, \bar{u}^* T(M_0 \times M_2 \times M_1 \times M_1))$ we can then apply the pointwise estimates to $\hat{\xi}(s)$ and integrate over $s \in \mathbb{R}$ to obtain for any $p \ge 1$ including $p = \infty$

$$\|\hat{\xi}\|_{L^p(\mathbb{R})} \le C\big(\|\pi_{02}\xi'_{02}\|_{L^p(\mathbb{R})} + \|\xi'_1 - \xi_1\|_{L^p(\mathbb{R})} + \|\pi^{\perp}_{0211}\hat{\xi}\|_{L^p(\mathbb{R})}\big), \tag{23}$$
$$\|\pi^{\perp}_{02}\xi'_{02}\|_{L^p(\mathbb{R})} \le C\big(\|\pi^{\perp}_{0211}\hat{\xi}\|_{L^p(\mathbb{R})} + \|\xi'_1 - \xi_1\|_{L^p(\mathbb{R})}\big).$$

In order to prove the H^1- and $\mathcal{C}^1$-estimates we also apply the pointwise estimates to $\nabla_s\hat{\xi}(s)$,

$$|\nabla_s\hat{\xi}| \le C\big(|\pi_{02}(\nabla_s\xi'_{02})| + \big|\nabla_s\xi'_1 - \nabla_s\xi_1\big| + \big|\pi^{\perp}_{0211}(\nabla_s\hat{\xi})\big|\big),$$
$$|\pi^{\perp}_{02}(\nabla_s\xi'_{02})| \le C\big(|\pi^{\perp}_{0211}(\nabla_s\hat{\xi})| + |\nabla_s\xi'_1 - \nabla_s\xi_1|\big).$$

Here we will need the inequalities

$$\begin{aligned}|\pi_{02}(\nabla_s\xi'_{02})| &\leq C\big(|\nabla_s(\pi_{02}(\xi'_{02}))| + |\hat{\xi}|\big),\\ |\pi_{0211}^{\perp}(\nabla_s\hat{\xi})| &\leq C\big(|\nabla_s(\pi_{0211}^{\perp}(\hat{\xi}))| + |\partial_s\bar{u}|\cdot|\hat{\xi}|\big),\\ |\nabla_s(\pi_{02}^{\perp}(\xi'_{02}))| &\leq C\big(|\pi_{02}^{\perp}(\nabla_s\xi'_{02})| + |\partial_s\bar{u}|\cdot|\hat{\xi}|\big).\end{aligned}$$

The first inequality (and similarly the others) can be seen by writing ξ'_{02} in a local orthonormal frame given by $(\gamma_i(s))_{i=1,\dots,k} \in \bar{u}_{02}(s)^*TL_{02}$ and $(\eta_i(s))_{i=1,\dots,K} \in \bar{u}_{02}(s)^*(TL_{02})^{\perp}$. Writing $\hat{\xi} = \sum \lambda^i\gamma_i + \sum \mu^i\eta_i$ we have

$$\begin{aligned}\big|\pi_{02}(\nabla_s\xi'_{02}) - \nabla_s(\pi_{02}(\xi'_{02}))\big| &= \Big|\sum \lambda^i\big(\pi_{02}(\nabla_s\gamma_i) - \nabla_s\gamma_i\big) + \sum \mu^i\pi_{02}\nabla_s(\eta_i)\Big|\\ &\leq C|\partial_s\bar{u}_{02}|\cdot|\xi'_{02}|.\end{aligned}$$

Note here that $\nabla_s\gamma_i = \nabla_{\partial_s\bar{u}_{02}}\gamma_i$ and $\nabla_s\eta_i = \nabla_{\partial_s\bar{u}_{02}}\eta_i$ are uniformly bounded. Putting things together we obtain the first estimate in (b) with an extra $\|\hat{\xi}\|_{L^2(\mathbb{R})}$ or $\|\hat{\xi}\|_{\mathcal{C}^0(\mathbb{R})}$ on the right hand side, for which we can use (23). For the last estimate in (b) we obtain

$$\|\nabla_s(\pi_{02}^{\perp}\xi'_{02})\|_{L^2(\mathbb{R})} \leq C\big(\|\nabla_s(\pi_{0211}^{\perp}\hat{\xi})\|_{L^2(\mathbb{R})} + \|\nabla_s\xi'_1 - \nabla_s\xi_1\|_{L^2(\mathbb{R})} + \big\||\partial_s\bar{u}|\cdot|\hat{\xi}|\big\|_{L^2(\mathbb{R})}\big).$$

This finishes the proof of (b). □

The following lemma contains a Sobolev estimate with a constant independent of the width δ of the middle strip; here the transversality assumption is used in a crucial way.

Lemma 3.1.4. *(Uniform Sobolev Estimate) There is a constant C_S such that for all $\delta \in (0,1]$ and $\xi = (\xi_{02},\hat{\xi}) \in H^2_{1,\delta}$*

$$\begin{aligned}&\|\xi_{02}\|_{\mathcal{C}^0([0,1],H^1(\mathbb{R}))} + \|\hat{\xi}\|_{\mathcal{C}^0([0,\delta],H^1(\mathbb{R}))}\\ &\leq C_S\big(\|\xi\|_{H^2_{1,\delta}} + \|(\xi_{02}-\xi'_{02})|_{t=0}\|_{H^1(\mathbb{R})} + \|(\xi_1-\xi'_1)|_{t=0}\|_{H^1(\mathbb{R})} + \|\pi_{0211}^{\perp}\hat{\xi}|_{t=\delta}\|_{H^1(\mathbb{R})}\big).\end{aligned}$$

In particular, for all $p > 2$ including $p = \infty$ and for $\xi \in \Gamma_{1,\delta}$ satisfying the boundary conditions (16),

$$\|\xi_{02}\|_{L^p(\mathbb{R}\times[0,1])} + \|\hat{\xi}\|_{L^p(\mathbb{R}\times[0,\delta])} \leq C_S\|\xi\|_{H^2_{1,\delta}}.$$

Proof. The $\mathcal{C}^0$- and L^p-estimates will follow from the continuous embeddings $H^1(\mathbb{R}) \hookrightarrow \mathcal{C}^0(\mathbb{R})$ and $H^1(\mathbb{R}) \hookrightarrow L^p(\mathbb{R})$ for $p \geq 2$. So it suffices to suppose by contradiction that there are sequences $\delta^\nu > 0$ and $\xi^\nu \in H^2_{1,\delta^\nu}$ with $\|\xi^\nu_{02}\|_{\mathcal{C}^0([0,1],H^1(\mathbb{R}))} + \|\hat{\xi}^\nu\|_{\mathcal{C}^0([0,\delta^\nu],H^1(\mathbb{R}))} = 1$ but $\|\xi^\nu\|_{H^2_{1,\delta^\nu}} + \|(\xi^\nu_{02}-\xi'^\nu_{02})|_{t=0}\|_{H^1(\mathbb{R})} + \|(\xi^\nu_1-\xi'^\nu_1)|_{t=0}\|_{H^1(\mathbb{R})} + \|\pi_{0211}^{\perp}\hat{\xi}^\nu|_{t=\delta^\nu}\|_{H^1(\mathbb{R})} \to 0$. By the standard Sobolev embedding

$$H^2([0,1]\times\mathbb{R}) \subset H^1([0,1],X) \hookrightarrow \mathcal{C}^0([0,1],X) \qquad \text{with } X = H^1(\mathbb{R})$$

this implies $\|\xi^\nu_{02}\|_{\mathcal{C}^0([0,1],H^1(\mathbb{R}))} \to 0$, and so

$$\|\xi'^\nu_{02}|_{t=0}\|_{H^1(\mathbb{R})} \leq \|\xi^\nu_{02}|_{t=0}\|_{H^1(\mathbb{R})} + \|(\xi^\nu_{02}-\xi'^\nu_{02})|_{t=0}\|_{H^1(\mathbb{R})} \to 0. \tag{24}$$

We can moreover integrate for all $t_0 \in [0,\delta^\nu]$ to obtain

$$\|\hat{\xi}^\nu|_{t=t_0} - \hat{\xi}^\nu|_{t=\delta^\nu}\|^2_{H^1(\mathbb{R})} \leq \delta^\nu\int_0^{\delta^\nu}\|\nabla_t\hat{\xi}^\nu\|^2_{H^1(\mathbb{R})} \leq \delta^\nu\|\hat{\xi}^\nu\|^2_{H^2(\mathbb{R}\times[0,\delta^\nu])} \to 0. \tag{25}$$

Using Lemma 3.1.3 we then obtain

$$\begin{aligned}
&\|\hat\xi^\nu|_{t=\delta^\nu}\|_{H^1(\mathbb{R})}\\
&\le C\big(\|\pi_{02}\xi'^\nu_{02}|_{t=\delta^\nu}\|_{H^1(\mathbb{R})} + \|(\xi_1^\nu-\xi_1'^\nu)|_{t=\delta^\nu}\|_{H^1(\mathbb{R})} + \|\pi^\perp_{0211}\hat\xi^\nu|_{t=\delta^\nu}\|_{H^1(\mathbb{R})}\big)\\
&\le C\big(\|\pi_{02}(\xi'^\nu_{02}|_{t=\delta^\nu} - \xi'^\nu_{02}|_{t=0})\|_{H^1(\mathbb{R})} + \|\pi_{02}(\xi'^\nu_{02})|_{t=0}\|_{H^1(\mathbb{R})}\\
&\qquad + \|(\xi_1^\nu-\xi_1'^\nu)|_{t=0}\|_{H^1(\mathbb{R})} + 2\|\hat\xi^\nu|_{t=\delta^\nu} - \hat\xi^\nu|_{t=0}\|_{H^1(\mathbb{R})} + \|\pi^\perp_{0211}\hat\xi^\nu|_{t=\delta^\nu}\|_{H^1(\mathbb{R})}\big)\\
&\to 0
\end{aligned}$$

with uniform constants C, C' by (16), (24), (25), and a bound on the operator norm of π_{02}. Now combining $\|\hat\xi^\nu|_{t=\delta^\nu}\|_{H^1(\mathbb{R})}\to 0$ with (25) proves $\|\hat\xi^\nu\|_{\mathcal{C}^0([0,\delta^\nu],H^1(\mathbb{R}))}\to 0$ in contradiction to the assumption and the previously established fact that $\|\xi^\nu_{02}\|_{\mathcal{C}^0([0,1],H^1(\mathbb{R}))}\to 0$. □

The solution u of the 0-equation corresponds to $\xi=0$, which is an almost zero of $\mathcal{F}_u$. This and a quadratic estimate for $d\mathcal{F}_u$ near 0 is the content of the next lemma. For later purposes we also compare $d\mathcal{F}_u(\xi)$ with the linearized operator $D_{e_u(\xi)}$ of $\overline\partial_J = (\overline\partial_{J_{02}}, \overline\partial_{\hat J})$ at $e_u(\xi)$. To state the comparison we will need the pointwise linear operator

$$E_u(\xi)\eta := \tfrac{d}{d\tau} e_u(\xi+\tau\eta)|_{\tau=0}.$$

It satisfies $E_u(0)=\mathrm{Id}$, and since e_u maps $\Gamma_{1,\delta}$ to the space of maps satisfying the boundary conditions in (13), the linearization $E_u(\xi)$ maps $\Gamma_{1,\delta}$ to the space of sections $\zeta\in\Gamma(v_{02}^*TM_{02})\times\Gamma(\hat v^*TM_{0211})$ over $v=(v_{02},\hat v) := e_u(\xi)$, that satisfy the linearized boundary conditions

$$(\zeta_{02},\zeta)|_{t=0}\in T_v(\Delta_0\times\Delta_2\times\Delta_1),\quad \hat\zeta|_{t=\delta}\in T_{\hat v}(L_{01}\times L_{12}),\quad \zeta_{02}|_{t=1}\in T_{v_{02}}(L_0\times L_2).$$

The linearized operator D_v acts on this space of sections and is given by

$$D_v\zeta = \tilde\nabla_\tau \overline\partial_J e_v(\tau\zeta)|_{\tau=0},$$

with the connection $\tilde\nabla$ introduced on page 15. In this notation we have $D_{e_u(0)} = d\mathcal{F}_u(0)$.

Lemma 3.1.5. *(Uniform quadratic and error estimates) There are uniform constants* $\epsilon>0$ *and* C_1, C_2, C_3 *such that for all* $\delta\in(0,1]$ *and* $\xi\in\Gamma_{1,\delta}(\epsilon)$, $\eta\in\Gamma_{1,\delta}$

$$\begin{aligned}
\|\mathcal{F}_u(0)\|_{\Omega_{1,\delta}} &\le C_1\sqrt{\delta},\\
\|d\mathcal{F}_u(\xi)\eta - d\mathcal{F}_u(0)\eta\|_{\Omega_{1,\delta}} &\le C_2\|\xi\|_{\Gamma_{1,\delta}}\|\eta\|_{\Gamma_{1,\delta}},\\
\|d\mathcal{F}_u(\xi)\eta - \Phi_u(\xi)^{-1}D_{e_u(\xi)}E_u(\xi)\eta\|_{\Omega_{1,\delta}} &\le C_3\|\xi\|_{\Gamma_{1,\delta}}\|\eta\|_{\Gamma_{1,\delta}}.
\end{aligned}$$

Proof. To estimate $\mathcal{F}_u(0)$ we recall that u_{02} is holomorphic and $\bar u$ is constant in t, so

$$\|\mathcal{F}_u(0)\|^2_{\Omega_{1,\delta}} = \|(0,\partial_s\bar u)\|^2_{H^1_{1,\delta}} = \delta\big(\|\partial_s u_{02}|_{t=0}\|^2_{H^1(\mathbb{R})} + 2\|\partial_s\bar u_1\|^2_{H^1(\mathbb{R})}\big) = C_1^2\delta.$$

Here $\partial_s u_{02}\to 0$ converges exponentially as $s\to\pm\infty$, and so does $\partial_s\bar u_1 = d\ell_1(\partial_s\bar u_{02})$, where ℓ_1 from (12) has bounded differential. This shows that the above constant C_1 is indeed finite. For the third estimate we differentiate as in [8, p.68] the identity $\Phi_u(\xi+\tau\eta)\mathcal{F}_u(\xi+\tau\eta) = \overline\partial_J(e_u(\xi+\tau\eta))$ to obtain

$$\Phi_u(\xi)d\mathcal{F}_u(\xi)\eta - D_{e_u(\xi)}E_u(\xi)\eta = -\Psi_u(\xi,\eta,\mathcal{F}_u(\xi)), \tag{26}$$

where the estimate for the right hand side

$$\Psi_u(\xi,\eta,\zeta) := \tilde\nabla_\tau(\Phi_u(\xi+\tau\eta)\zeta)|_{\tau=0}$$

is part of the estimates below. The first component of $\mathcal{F}_u$ is independent of δ, so the quadratic estimates for it simply follow from the continuous differentiability of $\mathcal{F}_u$. For the second component we follow the argument in [8, Prop.3.5.3.] to obtain a uniform constant for all $\delta \in (0,1]$. We need to consider

$$\mathcal{F}_{\bar{u}}(\hat{\xi}) := \Phi_{\bar{u}}(\hat{\xi})^{-1}(\overline{\partial}_{\hat{J}} e_{\bar{u}}(\hat{\xi})),$$

where $e_{\bar{u}}(\hat{\xi}) = \exp_{\bar{u}}(\hat{\xi} + Q(\hat{\xi}))$ is the exponential map with quadratic correction defined in (20). Note that our parallel transport $\Phi_{\bar{u}}(\hat{\xi})$ is defined with respect to the path $\tau \mapsto e_{\bar{u}}(\tau\hat{\xi})$ and the Hermitian connection $\tilde{\nabla}$ on $T(M_0 \times M_2 \times M_1 \times M_1)$ that leaves $\hat{J}$ invariant. Since $e_{\bar{u}}(0) = \bar{u}$ and $de_{\bar{u}}(0) = \mathrm{Id}$, the same path can be used in the definition of $\nabla_{\hat{\xi}}$ instead of the geodesic. Now let $\xi, \eta \in \Gamma_{1,\delta}$ with $\|\xi\|_{H^2_{1,\delta}} \leq \epsilon$. Then by Lemma 3.1.4

$$\|\hat{\xi}\|_{\mathcal{C}^0} \leq C_S \|\xi\|_{H^2_{1,\delta}} \leq C_S \epsilon =: c_0, \qquad \|\hat{\eta}\|_{\mathcal{C}^0} \leq C_S \|\eta\|_{H^2_{1,\delta}}$$

with a uniform constant C_S thus a uniform constant c_0 that only depends on ϵ. In the following, all constants will be uniform in the sense that they only depend on c_0 and hence ϵ. Next, we consider

$$E_{\bar{u}}(\hat{\xi})\hat{\eta} := \tfrac{d}{d\tau} e_{\bar{u}}(\hat{\xi} + \tau\hat{\eta})|_{\tau=0}, \qquad \Psi_{\bar{u}}(\hat{\xi}, \hat{\eta}, \zeta) := \tilde{\nabla}_\tau (\Phi_{\bar{u}}(\hat{\xi} + \tau\hat{\eta})\zeta)|_{\tau=0}.$$

Note that $E_{\bar{u}}(0) = \mathrm{Id}$ and that $\Psi(0, \hat{\eta}, \zeta) = 0$ since the covariant derivative exactly uses the parallel transport $\Phi_{\bar{u}}(\tau\hat{\eta})$. Moreover, these maps are linear in $\hat{\eta}$ and ζ, and they depend smoothly on $\hat{\xi}$. So given ϵ and thus $|\hat{\xi}| \leq c_0$ we have linear bounds

$$|E_{\bar{u}}(\hat{\xi})| \leq c_1, \qquad |\nabla(E_{\bar{u}}(\hat{\xi}))| \leq c_1\big(|\nabla\hat{\xi}| + |d\bar{u}||\hat{\xi}|\big), \qquad |\Psi_{\bar{u}}(\hat{\xi}, \hat{\eta}, \zeta)| \leq c_1 |\hat{\xi}||\hat{\eta}||\zeta|$$

with a uniform constant c_1. With these preparations we calculate from (26), using the notation of [8, Prop.3.5.3.],

$$\begin{aligned}
&\Phi_{\bar{u}}(\hat{\xi})\big(d\mathcal{F}_{\bar{u}}(\hat{\xi})\hat{\eta} - d\mathcal{F}_{\bar{u}}(0)\hat{\eta}\big) \\
&= -\Psi_{\bar{u}}(\hat{\xi}, \hat{\eta}, \mathcal{F}_{\bar{u}}(\hat{\xi})) + \big(\nabla(E_{\bar{u}}(\hat{\xi}))\hat{\eta}\big)^{0,1} + \big(\big(E_{\bar{u}}(\hat{\xi}) - \Phi_{\bar{u}}(\hat{\xi})\big)\nabla\hat{\eta}\big)^{0,1} \\
&\quad - \tfrac{1}{2}\hat{J}(e_{\bar{u}}(\hat{\xi}))\big(\big((\nabla_{(E_{\bar{u}}(\hat{\xi})\hat{\eta} - \Phi_{\bar{u}}(\hat{\xi})\hat{\eta})}\hat{J})(e_{\bar{u}}(\hat{\xi}))\big)\Phi_{\bar{u}}(\hat{\xi})d\bar{u}\big)^{0,1} \\
&\quad - \tfrac{1}{2}\hat{J}(e_{\bar{u}}(\hat{\xi}))\big(\big((\nabla_{\Phi_{\bar{u}}(\hat{\xi})\hat{\eta}}\hat{J})(e_{\bar{u}}(\hat{\xi})) - \Phi_{\bar{u}}(\hat{\xi})(\nabla_{\hat{\eta}}\hat{J})(\bar{u})\Phi_{\bar{u}}(\hat{\xi})^{-1}\big)\Phi_{\bar{u}}(\hat{\xi})d\bar{u}\big)^{0,1} \\
&\quad - \tfrac{1}{2}\hat{J}(e_{\bar{u}}(\hat{\xi}))\big((\nabla_{E_{\bar{u}}(\hat{\xi})\hat{\eta}}\hat{J})(e_{\bar{u}}(\hat{\xi}))(d(e_{\bar{u}}(\hat{\xi})) - \Phi_{\bar{u}}(\hat{\xi})d\bar{u})\big)^{0,1}.
\end{aligned}$$

We then use the uniform bounds on $\|\nabla\hat{J}\|_\infty$, $|d\bar{u}|$, $|\hat{\xi}|$, and the estimates

$$\begin{aligned}
&|\mathcal{F}_{\bar{u}}(\hat{\xi})| \leq |d(e_{\bar{u}}(\hat{\xi}))| \leq c_2\big(|\nabla\hat{\xi}| + |d\bar{u}|\big), \qquad |d(e_{\bar{u}}(\hat{\xi})) - \Phi_{\bar{u}}(\hat{\xi})d\bar{u}| \leq c_2\big(|\nabla\hat{\xi}| + |d\bar{u}||\hat{\xi}|\big), \\
&\big|E_{\bar{u}}(\hat{\xi}) - \Phi_{\bar{u}}(\hat{\xi})\big| \leq c_2|\hat{\xi}|, \qquad \big|(\nabla_{\Phi_{\bar{u}}(\hat{\xi})\hat{\eta}}\hat{J})(e_{\bar{u}}(\hat{\xi})) - \Phi_{\bar{u}}(\hat{\xi})(\nabla_{\hat{\eta}}\hat{J})(\bar{u})\Phi_{\bar{u}}(\hat{\xi})^{-1}\big| \leq c_2|\hat{\xi}||\hat{\eta}|
\end{aligned}$$

with a uniform constant c_2 to obtain with a further uniform constant c_3

$$\big|d\mathcal{F}_{\bar{u}}(\hat{\xi})\hat{\eta} - d\mathcal{F}_{\bar{u}}(0)\hat{\eta}\big| \leq c_3\big(|\hat{\xi}||\hat{\eta}| + |\hat{\eta}||\nabla\hat{\xi}| + |\hat{\xi}||\nabla\hat{\eta}|\big).$$

So far these pointwise estimates were standard calculations. Now we have to check that they actually lead to uniform bounds in the δ-dependent norms. The zeroth order part of the $\Omega_{1,\delta}$-norm over $\mathbb{R}\times[0,\delta]$ can be estimated with the help of Lemma 3.1.4 by

$$\begin{aligned}\bigl\| d\mathcal{F}_{\bar u}(\hat\xi)\hat\eta - d\mathcal{F}_{\bar u}(0)\hat\eta\bigr\|_{L^2} &\le c_3\bigl(\|\hat\xi\|_{L^4}\|\hat\eta\|_{L^4} + \|\hat\eta\|_{\mathcal{C}^0}\|\nabla\hat\xi\|_{L^2} + \|\hat\xi\|_{\mathcal{C}^0}\|\nabla\hat\eta\|_{L^2}\bigr)\\ &\le c_3(C_S^2+2C_S)\|\xi\|_{H^2_{1,\delta}}\|\eta\|_{H^2_{1,\delta}},\\ \bigl\| d\mathcal{F}_{\bar u}(\hat\xi)\hat\eta - d\mathcal{F}_{\bar u}(0)\hat\eta\bigr\|_{L^4} &\le c_3\bigl(\|\hat\xi\|_{L^8}\|\hat\eta\|_{L^8} + \|\hat\eta\|_{\mathcal{C}^0}\|\nabla\hat\xi\|_{L^4} + \|\hat\xi\|_{\mathcal{C}^0}\|\nabla\hat\eta\|_{L^4}\bigr)\\ &\le c_3(C_S^2+2C_S)\bigl(\|\xi\|_{H^2_{1,\delta}} + \|\xi\|_{L^4_{1,\delta}}\bigr)\bigl(\|\eta\|_{H^2_{1,\delta}} + \|\nabla\eta\|_{L^4_{1,\delta}}\bigr).\end{aligned}$$

For the first order part of the $\Omega_{1,\delta}$-norm one differentiates the above identity and uses further bounds on $\|\nabla^2\hat J\|_\infty$ and $|\nabla d\bar u|$ to find a pointwise bound

$$\begin{aligned}\bigl|\nabla\bigl(d\mathcal{F}_{\bar u}(\hat\xi)\hat\eta - d\mathcal{F}_{\bar u}(0)\hat\eta\bigr)\bigr| &\le c_4\bigl(|\hat\xi| + |\nabla\hat\xi|\bigr)\bigl(|\hat\eta| + |\nabla\hat\eta|\bigr)\\ &\quad + c_4\bigl(|\nabla^2\hat\xi||\hat\eta| + |\nabla\hat\xi|^2|\hat\eta| + |\nabla\hat\xi||\nabla\hat\eta| + |\hat\xi||\nabla^2\hat\eta|\bigr).\end{aligned}$$

Then we again use Lemma 3.1.4 and $\|\nabla\hat\xi\|_{L^2}\le\epsilon$ to obtain with a final uniform constant c_5

$$\begin{aligned}&\bigl\|\nabla\bigl(d\mathcal{F}_{\bar u}(\hat\xi)\hat\eta - d\mathcal{F}_{\bar u}(0)\hat\eta\bigr)\bigr\|_{L^2}\\ &\le c_4\bigl(\|\hat\xi\|_{L^4} + \|\nabla\hat\xi\|_{L^4}\bigr)\bigl(\|\hat\eta\|_{L^4} + \|\nabla\hat\eta\|_{L^4}\bigr)\\ &\quad + c_4\bigl(\|\nabla^2\hat\xi\|_{L^2}\|\hat\eta\|_{\mathcal{C}^0} + \|\nabla\hat\xi\|_{L^2}\|\nabla\hat\xi\|_{L^4}\|\hat\eta\|_{L^4} + \|\nabla\hat\xi\|_{L^4}\|\nabla\hat\eta\|_{L^4} + \|\hat\xi\|_{\mathcal{C}^0}\|\nabla^2\hat\eta\|_{L^2}\bigr)\\ &\le c_5\bigl(\|\xi\|_{H^2_{1,\delta}} + \|\nabla\xi\|_{L^4_{1,\delta}}\bigr)\bigl(\|\eta\|_{H^2_{1,\delta}} + \|\nabla\eta\|_{L^4_{1,\delta}}\bigr).\end{aligned}$$

□

Theorem 3.1.1 now follows from the implicit function theorem [8, A.3.4] if we can establish surjectivity and a uniform bound on the right inverse for the linearized operator

$$\begin{aligned}(27)\qquad D^\delta:\Gamma_{1,\delta}\to\Omega_{1,\delta},\qquad D^\delta\xi &:= d\mathcal{F}_u(0)\xi = \bigl(D_{u_{02}}\xi_{02}\,,\,D_{\bar u}\hat\xi\bigr);\\ D_{u_{02}}\xi_{02} &= \nabla_s\xi_{02} + J(u_{02})\nabla_t\xi_{02} + \nabla_{\xi_{02}}J_{02}(u_{02})\partial_t u_{02},\\ D_{\bar u}\hat\xi &= \nabla_s\hat\xi + \hat J(\bar u)\nabla_t\hat\xi + \tfrac12\nabla_{\hat\xi}\hat J(\bar u)\hat J(\bar u)\partial_s\bar u.\end{aligned}$$

Here $D_{u_{02}}$ and $D_{\bar u}$ are the linearized operators of $\overline{\partial}_{J_{02}}$ at u_{02} (which is holomorphic) and of $\overline{\partial}_{\hat J}$ at $\bar u$ (which satisfies $\partial_t\bar u=0$) respectively. (See [8, Prop.3.1.1.] for an explicit calculation of the linearized operators, and note that we identify $\Omega^{0,1}(\mathbb{R}\times[0,1],u^*TM)$ with sections of u^*TM by $\gamma ds + J\gamma dt\mapsto\gamma$.) We can identify the cokernel of D^δ with $(\operatorname{im}D^\delta)^\perp\subset(H^1_{1,\delta})^*$. By elliptic regularity any element in this cokernel can be represented by the L^2-inner product $\langle\eta,\operatorname{im}D^\delta\rangle=0$ with a smooth section η. Partial integration then shows that $\eta\in\Gamma_{1,\delta}$ satisfies the boundary conditions (16) and lies in the kernel of the formal adjoint operator, $(D^\delta)^*\eta=0$. Note that $(D^\delta)^*$ is given by $\bigl(-\nabla_s + J_{02}(u_{02})\nabla_t, -\nabla_s + \hat J(\bar u)\nabla_t\bigr)$ plus lower order terms. So $(D^\delta)^*$ has the same analytic properties as D^δ, and we will prove the surjectivity of D^δ by establishing injectivity for $(D^\delta)^*$.

By our assumptions on the index and regularity of $(u_0,u_2)\in\widetilde{\mathcal{M}}^1_0(x^-,x^+)$ we know that the operator $D_{u_{02}}\oplus\pi_{02}^\perp$ on the space of sections in $H^2(u_{02}^*T(M_0\times M_2))$ with boundary conditions at $t=1$ in $T(L_0\times L_2)$ (where $\pi_{02}^\perp$ is the projection at $t=0$) is surjective and has a one dimensional kernel $\ker(D_{u_{02}}\oplus\pi_{02}^\perp)$. This is not a subspace of $\Gamma_{1,\delta}$, but we will fix a complement for every $\delta>0$ in the following sense,

$$K_0 := \bigl\{\xi=(\xi_{02},\hat\xi)\in\Gamma_{1,\delta}\,\big|\,\langle\xi_{02},\ker(D_{u_{02}}\oplus\pi_{02}^\perp)\rangle_{L^2}\equiv 0\bigr\}.$$

Here we used the L^2-inner product on $H^2(\mathbb{R}\times[0,1],u_{02}^*T(M_0\times M_2))$.

Combining the uniform linear estimates Lemma 3.2.1 and Lemma 3.2.2 we can choose $\delta_0 := \frac{1}{16}c_1^2c_2^2 > 0$ such that for all $\delta \in (0,\delta_0)$ and $\xi \in \Gamma_{1,\delta}$

$$\begin{aligned}(1+c_2^{-1})\|(D^\delta)^*\xi\|_{\Omega_{1,\delta}} &\geq \tfrac{1}{2}\|(D^\delta)^*\xi\|_{H^1_{1,\delta}} + \tfrac{1}{2}\|(D^\delta)^*\xi\|_{L^4_{1,\delta}} + c_2^{-1}\|D^*_{u_{02}}\xi_{02}\|_{H^1(\mathbb{R}\times[0,1])}\\ &\geq \tfrac{1}{2}c_1\|\xi\|_{\Gamma_{1,\delta}} - c_2^{-1}\sqrt{\delta}\|\nabla_t\hat{\xi}\|_{H^1(\mathbb{R}\times[0,\delta])} \;\geq\; \tfrac{1}{4}c_1\|\xi\|_{\Gamma_{1,\delta}},\end{aligned}$$

and similarly for all $\xi \in \Gamma_{1,\delta}\cap K_0$

$$\|D^\delta\xi\|_{\Omega^1_{1,\delta}} \geq \frac{c_1c_2}{4(c_2+1)}\|\xi\|_{\Gamma_{1,\delta}}. \tag{28}$$

The first estimate shows that $(D^\delta)^*$ is injective and hence D^δ is surjective. The second estimate shows that its right inverse is uniformly bounded. It remains to check that D^δ stays surjective when restricted to K_0. This follows from the fact that both $D_{u_{02}}$ with boundary conditions in $(L_{02}, L_0\times L_2)$ and $D^\delta = (D_{u_{02}}, D_{\bar{u}})$ with boundary conditions (16) are surjective and have the same index 1 by Lemma 2.1.3 and the identification $\widetilde{\mathcal{M}}^1_{\delta}(x^-,x^+) \cong \widehat{\mathcal{M}}^1_{\delta}(x^-,x^+)$. So D^δ has a 1-dimensional kernel, which is transversal to K_0 by the last estimate, and hence $D^\delta|_{K_0}$ must be surjective. This finishes the proof of theorem 3.1.1. Here $\epsilon > 0$ is fixed such that the exponential map e_u is defined on $\Gamma_{1,\delta}(\epsilon)$ and such that Lemma 3.1.5 holds.

Corollary 3.1.6. *There exists $\delta_0 > 0$ such that the map $\mathcal{T}_\delta : \mathcal{M}^1_0(x^-,x^+) \to \mathcal{M}^1_\delta(x^-,x^+)$ given by $\mathcal{T}_\delta([u]) := [v_u]$ is well defined and injective for all $\delta \in (0,\delta_0]$.*

Proof. We choose $\delta_0 \leq \epsilon^2 C_0^{-2}$ such that Theorem 3.1.1 applies. Then let $v_u = e_u(\xi)$ be the solution constructed from $u \in \widetilde{\mathcal{M}}^1_0(x^-,x^+)$ and consider a shifted 0-solution $\tilde{u} = u(\cdot+\sigma) \in [u]$. Then $\tilde{\xi} := \xi(\cdot+\sigma)$ satisfies $\|\tilde{\xi}\| = \|\xi\| \leq C_0\sqrt{\delta} \leq \epsilon$, $\mathcal{F}_u(\tilde{\xi}) = 0$, and the orthogonality condition to $\ker(D_{\tilde{u}_{02}} \oplus \pi^\perp_{02})$. Hence $v_{\tilde{u}} = e_{u(\cdot+\sigma)}(\xi(\cdot+\sigma)) = v_u(\cdot+\sigma) \in [v_u]$, so $\mathcal{T}_\delta([u]) = [v_u]$ is well defined.

The injectivity of $\mathcal{T}_\delta$ follows from the fact that $\mathcal{M}^1_0(x^-,x^+)$ consists of isolated points, so the $\mathcal{C}^0$-distance $d_{\mathcal{C}^0}([u],[u']) > \Delta_0$ is bounded below by some $\Delta_0 > 0$ for all $[u] \neq [u']$. On the other hand, $d_{\mathcal{C}^0}([\bar{u}], \mathcal{T}_\delta([u]) \leq C_0C_S(1+C_Q)\sqrt{\delta}$ by (15), (17), and Lemma 3.1.4. So if we had $\mathcal{T}_\delta([u]) = \mathcal{T}_\delta([u'])$ then $d_{\mathcal{C}^0}([u],[u']) \leq d_{\mathcal{C}^0}([\bar{u}],[\bar{u}']) \leq 2C_0C_S(1+C_Q)\sqrt{\delta}$. This implies $[u] = [u']$ whenever $\delta \leq \delta_0$, where we choose $\delta_0 \leq (2C_0C_S(1+C_Q))^{-2}\Delta_0^2$. ☐

3.2. **Uniform estimates.**

In this section we establish the uniform linear and nonlinear estimates that are used in Sections 3.1 and 3.3. We will work in the setup of section 3.1 and fix a solution $u \in \widetilde{\mathcal{M}}^1_0(x^-,x^+)$. For convenience we denote the target spaces by $M_{02} := M_0\times M_2$ and $M_{0211} := M_0\times M_2\times M_1\times M_1$ and the symplectic structures by $\omega_{02} = (-\omega_0)\oplus\omega_2$ and $\omega_{0211} = \omega_0\oplus(-\omega_2)\oplus(-\omega_1)\oplus\omega_1$ respectively. The nonlinear equation for $v = (v_{02},\hat{v})$, $v_{02} : \mathbb{R}\times[0,1] \to M_{02}$, $\hat{v} : \mathbb{R}\times[0,\delta] \to M_{0211}$ is

$$\overline{\partial}_J v := \partial_s v + J(v)\partial_t v := \big(\partial_s v_{02} + J_{02}(v_{02})\partial_t v_{02}\,,\,\partial_s\hat{v} + \hat{J}(\hat{v})\partial_t\hat{v}\big).$$

We will need uniform estimates for the nonlinear operator $\xi \mapsto \overline{\partial}_J e_u(\xi)$ on $\xi \in \Gamma_{1,\delta}(\epsilon)$ and the linearized operator D^δ. For that purpose we use the Levi-Civita connection on $M = M_{02}$ and $M = M_{0211}$ respectively to identify $T_uM\times T_uM \cong T_\xi T_uM$ for every $\xi \in T_uM$. With this we decompose $Te(u,\xi) : T_\xi T_uM \to T_{e_u\xi}M$ as

$$Te(u,\xi)(X,\eta) = \partial_1 e(u,\xi)X + de_u(\xi)\eta \qquad \forall \xi, X, \eta \in T_uM.$$

We denote the pullback almost complex structure on $H^2_{1,\delta}$ under $de_u(\xi)$ by

$$\begin{aligned} J(\xi) &:= (J_{02}(\xi_{02}), \hat{J}(\hat{\xi})) \\ &:= \big((de_{u_{02}}(\xi_{02}))^{-1} J_{02}(e_{u_{02}}(\xi_{02})) de_{u_{02}}(\xi_{02}), (de_{\bar{u}}(\hat{\xi}))^{-1} \hat{J}(e_{\bar{u}}(\hat{\xi})) de_{\bar{u}}(\hat{\xi})\big) \end{aligned}$$

for $\xi = (\xi_{02}, \hat{\xi}) \in \Gamma_{1,\delta}(\epsilon)$. With this we can express

$$\overline{\partial}_J(e_u(\xi)) = de_u(\xi)\big(\nabla_s \xi + J(\xi)\nabla_t \xi\big) + \partial_1 e(u,\xi)\partial_s u + J(u)\partial_1 e(u,\xi)\partial_t u \tag{29}$$

in terms of the nonlinear operator on $H^2_{1,\delta}$,

$$\nabla_s \xi + J(\xi)\nabla_t \xi := \big(\nabla_s \xi_{02} + J_{02}(\xi_{02})\nabla_t \xi_{02}\, ,\, \nabla_s \hat{\xi} + \hat{J}(\hat{\xi})\nabla_t \hat{\xi}\big).$$

Note that $J(0) = (J_{02}, \hat{J})$ is the usual almost complex structure, so we can express the linearized operator (27) as

$$D^\delta \xi = \nabla_s \xi + J(0)\nabla_t \xi + \big(\nabla_{\xi_{02}} J_{02}(u_{02})\partial_t u_{02}\, ,\, \tfrac{1}{2}\nabla_{\hat{\xi}} \hat{J}(\bar{u})\hat{J}(\bar{u})\partial_s \bar{u}\big).$$

The following lemma provides uniform elliptic estimates.

Lemma 3.2.1.

(a) *There is a constant C_1 such that for all $\delta \in (0,1]$ and $\xi \in \Gamma_{1,\delta}$*

$$\left|\int_{\{1\}\times\mathbb{R}} \omega_{02}(\xi_{02}, \nabla_s \xi_{02})\right| + \left|\int_{\{\delta\}\times\mathbb{R}} \omega_{0211}(\hat{\xi}, \nabla_s \hat{\xi})\right| \le C_1\big(\|\xi_{02}|_{t=1}\|_{H^0(\mathbb{R})} + \|\hat{\xi}|_{t=\delta}\|_{H^0(\mathbb{R})}\big)^2,$$

$$\left|\int_{\{1\}\times\mathbb{R}} \omega_{02}(\nabla_s \xi_{02}, \nabla_s^2 \xi_{02})\right| + \left|\int_{\{\delta\}\times\mathbb{R}} \omega_{0211}(\nabla_s \hat{\xi}, \nabla_s^2 \hat{\xi})\right| \le C_1\big(\|\xi_{02}|_{t=1}\|_{H^1(\mathbb{R})} + \|\hat{\xi}|_{t=\delta}\|_{H^1(\mathbb{R})}\big)^2.$$

(b) *There is a constant $\epsilon > 0$ and for every $c_0 > 0$ there is a constant C_1 such that for all $\delta \in (0,1]$ and $\xi, \zeta \in H^2_{1,\delta}$ with $\|\zeta\|_\infty \le \epsilon$, $\|\nabla\zeta\|_\infty \le c_0$*

$$\begin{aligned} \|\xi\|_{H^1_{1,\delta}} \le C_1\Bigg(&\|\nabla_s \xi + J(\zeta)\nabla_t \xi\|_{H^0_{1,\delta}} + \|\xi\|_{H^0_{1,\delta}} \\ &+ \left|\int_{\{\delta\}\times\mathbb{R}} \omega_{0211}(\hat{\xi}, \nabla_s \hat{\xi})\right|^{1/2} + \left|\int_{\{1\}\times\mathbb{R}} \omega_{02}(\xi_{02}, \nabla_s \xi_{02})\right|^{1/2}\Bigg), \\ \|\xi\|_{H^2_{1,\delta}} \le C_1\Bigg(&\|\nabla_s \xi + J(\zeta)\nabla_t \xi\|_{H^1_{1,\delta}} + \|\xi\|_{H^0_{1,\delta}} \\ &+ \left|\int_{\{\delta\}\times\mathbb{R}} \omega_{0211}(\hat{\xi}, \nabla_s \hat{\xi})\right|^{1/2} + \left|\int_{\{\delta\}\times\mathbb{R}} \omega_{0211}(\nabla_s \hat{\xi}, \nabla_s^2 \hat{\xi})\right|^{1/2} \\ &+ \left|\int_{\{1\}\times\mathbb{R}} \omega_{02}(\xi_{02}, \nabla_s \xi_{02})\right|^{1/2} + \left|\int_{\{1\}\times\mathbb{R}} \omega_{02}(\nabla_s \xi_{02}, \nabla_s^2 \xi_{02})\right|^{1/2}\Bigg), \\ \|\nabla\xi\|_{L^4_{1,\delta}} \le C_1\Big(&\|\xi\|_{H^2_{1,\delta}} + \|\nabla_s \xi + J(\zeta)\nabla_t \xi\|_{L^4_{1,\delta}} + \|\hat{\xi}|_{t=\delta}\|_{H^1(\mathbb{R})}\Big). \end{aligned}$$

(c) *There is a constant $c_1 > 0$ such that for all $\delta \in (0,1]$ and $\xi \in \Gamma_{1,\delta}$*

$$\begin{aligned} c_1\|\xi\|_{H^2_{1,\delta}} &\le \|D^\delta \xi\|_{H^1_{1,\delta}} + \|\xi\|_{H^0_{1,\delta}} + \|\hat{\xi}|_{t=\delta}\|_{H^1(\mathbb{R})} + \|\xi_{02}|_{t=1}\|_{H^1(\mathbb{R})}, \\ c_1\|\nabla\xi\|_{L^4_{1,\delta}} &\le \|D^\delta \xi\|_{H^1_{1,\delta}} + \|D^\delta \xi\|_{L^4_{1,\delta}} + \|\xi\|_{H^0_{1,\delta}} + \|\hat{\xi}|_{t=\delta}\|_{H^1(\mathbb{R})} + \|\xi_{02}|_{t=1}\|_{H^1(\mathbb{R})}, \end{aligned}$$

and the same holds with D^δ replaced by $(D^\delta)^*$.

Proof. We prove (a) in general for $\int_{\mathbb{R}}\omega(\xi,\nabla_s\xi)$ and $\int_{\mathbb{R}}\omega(\nabla_s\xi,\nabla_s^2\xi)$ with a Lagrangian section $\xi:\mathbb{R}\to u^*TL$ over a path $u:\mathbb{R}\to L$. These expressions vanish if L is totally geodesic. To estimate them in general we pick a smooth family of orthonormal frames $(\gamma_i(s))_{i=1,\dots,k}\in u(s)^*TL$, then

$$\xi=\sum\lambda^i\gamma_i,\quad \nabla_s\xi=\sum\Bigl(\partial_s\lambda^i\gamma_i+\lambda^i\nabla_s\gamma_i\Bigr),\quad \nabla_s^2\xi=\sum\Bigl(\partial_s^2\lambda^i\gamma_i+2\partial_s\lambda^i\nabla_s\gamma_i+\lambda^i\nabla_s^2\gamma_i\Bigr)$$

with $\lambda:\mathbb{R}\to\mathbb{R}^k$. By the orthonormality we have $|\lambda(s)|=|\xi(s)|$, and using $(\gamma,J\gamma)$ as a trivialization for the definition of Sobolev norms on u^*TM we obtain $\|\lambda\|_{H^s(\mathbb{R})}=\|\xi\|_{H^s(\mathbb{R})}$. We now use the identities $\omega(\gamma_i,\gamma_j)=0$ to obtain

$$\left|\int_{\mathbb{R}}\omega(\xi,\nabla_s\xi)\right|\le\left|\int_{\mathbb{R}}C|\xi(s)||\lambda(s)|ds\right|=C\|\xi\|^2_{L^2(\mathbb{R})},$$

$$\left|\int_{\mathbb{R}}\omega(\nabla_s\xi,\nabla_s^2\xi)\right|\le\left|\int_{\mathbb{R}}C\bigl(|\nabla_s\xi||\lambda|+|\nabla_s\xi||\partial_s\lambda|+|\partial_s\lambda|^2+|\lambda|^2\bigr)\right|\le 4C\|\xi\|^2_{H^1(\mathbb{R})},$$

where the constant C only depends on γ (that is on $u:\mathbb{R}\to L$) up to third derivatives. Here we used partial integration

$$\int_{\mathbb{R}}\sum_{i,j}\lambda^i\partial_s^2\lambda^j\omega(\nabla_s\gamma_i,\gamma_j)=-\int_{\mathbb{R}}\sum_{i,j}\Bigl(\partial_s\lambda^i\partial_s\lambda^j\omega(\nabla_s\gamma_i,\gamma_j)+\lambda^i\partial_s\lambda^j\partial_s\omega(\nabla_s\gamma_i,\gamma_j)\Bigr).$$

To prove (c) we can replace D^δ by $\nabla_s\xi+J(0)\nabla_t\xi$ since the difference of the operators is bounded in the different components and norms by

$$\begin{aligned}
&\bigl\|\nabla_{\xi_{02}}J_{02}(u_{02})\partial_t u_{02}\bigr\|_{H^0(\mathbb{R}\times[0,1])}+\bigl\|\nabla_{\hat\xi}\hat J(\bar u)J(\bar u)\partial_s\bar u\bigr\|_{H^0(\mathbb{R}\times[0,\delta])}\le C\bigl\|\xi\bigr\|_{H^0_{1,\delta}},\\
&\bigl\|\nabla_{\xi_{02}}J_{02}(u_{02})\partial_t u_{02}\bigr\|_{L^4(\mathbb{R}\times[0,1])}\le C\bigl\|\nabla_{\xi_{02}}J_{02}(u_{02})\partial_t u_{02}\bigr\|_{H^1(\mathbb{R}\times[0,1])}\le C\bigl\|\xi\bigr\|_{H^1_{1,\delta}},\\
&\bigl\|\nabla_{\hat\xi}\hat J(\bar u)J(\bar u)\partial_s\bar u\bigr\|_{H^1(\mathbb{R}\times[0,\delta])}\le C\bigl\|\xi\bigr\|_{H^1_{1,\delta}},\\
&\bigl\|\nabla_{\hat\xi}\hat J(\bar u)J(\bar u)\partial_s\bar u\bigr\|_{L^4(\mathbb{R}\times[0,\delta])}\le C\|\nabla\hat J\|_\infty\|\partial_s\bar u\|_\infty\|\hat\xi\|_{L^4(\mathbb{R}\times[0,\delta])}\le C\bigl\|\xi\bigr\|_{H^2_{1,\delta}},
\end{aligned}\tag{30}$$

where C denotes any uniform constant. The extra terms on the right hand side will fit into the proof and will be recalled for the relevant estimates. The proof for $(D^\delta)^*$ is completely analogous. We will use the notation $\nabla_s\xi+J(\sigma\zeta)\nabla_t\xi$ to make partial integration calculations for the nonlinear ($\sigma=1$) and linear ($\sigma=0$) operator at the same time. In the nonlinear case the almost complex structure $J(\zeta)$ is not skew-adjoint. In order to restore this property we work with the $L^2_{1,\delta}(\sigma\zeta)$-metric, which uses the pullback metric $g_{\sigma\zeta}=\langle\cdot,\cdot\rangle_{\sigma\zeta}$ under $de_{u_{02}}(\sigma\zeta_{02})$ on M_{02} and $de_{\bar u}(\sigma\zeta)$ on M_{0211} respectively. In the linear case $\sigma=0$ nothing has happened; in the nonlinear case we can pick $\epsilon>0$ and hence $\|\zeta\|_\infty$ sufficiently small such that $de_u(\zeta)$ is $\mathcal{C}^0$-close to the identity, and hence the induced $L^2_{1,\delta}(\zeta)$-norm is uniformly equivalent to the standard $L^2_{1,\delta}$-norm. With this in mind we start by calculating for any $\zeta,\eta\in H^2_{1,\delta}$ with $\|\zeta\|_\infty\le\epsilon$ (unless otherwise specified integrals are over two infinite strips of width δ and 1)

$$\begin{aligned}
&\|\nabla_s\eta+J(\sigma\zeta)\nabla_t\eta\|^2_{L^2_{1,\delta}(\sigma\zeta)}\\
&=\int\Bigl(|\nabla_s\eta|^2_{\sigma\zeta}+|\nabla_t\eta|^2_{\sigma\zeta}+\langle\nabla_s\eta,J(\sigma\zeta)\nabla_t\eta\rangle_{\sigma\zeta}-\langle\nabla_t\eta,J(\sigma\zeta)\nabla_s\eta\rangle_{\sigma\zeta}\Bigr)
\end{aligned}$$

$$
\begin{aligned}
&= \|\nabla\eta\|^2_{L^2_{1,\delta}(\sigma\zeta)} - \int \Big(\nabla_s g_{\sigma\zeta}\big(\eta, J(\sigma\zeta)\nabla_t\eta\big) - \nabla_t g_{\sigma\zeta}\big(\eta, J(\sigma\zeta)\nabla_s\eta\big) \Big) \\
&\quad - \int \Big(\langle \eta, \big(\nabla_s\big(J(\sigma\zeta)\nabla_t\eta\big) - \nabla_t\big(J(\sigma\zeta)\nabla_s\eta\big)\big)\rangle_{\sigma\zeta} \Big) \\
&\quad - \lim_{S\to\infty}\int_{\{s=-S\}} \langle \eta, J(\sigma\zeta)\nabla_t\eta\rangle_{\sigma\zeta} + \lim_{S\to\infty}\int_{\{s=S\}} \langle \eta, J(\sigma\zeta)\nabla_t\eta\rangle_{\sigma\zeta} \\
&\quad + \int_{\{0\}\times\mathbb{R}} \langle \eta, J(\sigma\zeta)\nabla_s\eta\rangle_{\sigma\zeta} - \int_{\{1\}\times\mathbb{R}} \langle \eta_{02}, J_{02}(\sigma\zeta_{02})\nabla_s\eta_{02}\rangle_{\sigma\zeta_{02}} - \int_{\{\delta\}\times\mathbb{R}} \langle \hat\eta, \hat J(\sigma\hat\zeta)\nabla_s\hat\eta\rangle_{\sigma\hat\zeta} \\
&\geq \|\nabla\eta\|^2_{L^2_{1,\delta}(\sigma\zeta)} - \int C\big((1+\sigma c_0)|\eta||\nabla\eta| + |\eta|^2\big) - \Omega_{02}(\eta_{02}|_{t=1}) - \Omega_{0211}(\hat\eta|_{t=\delta}),
\end{aligned}
$$

where we abbreviated

$$
\Omega_{02}(\eta_{02}|_{t=1}) := \left| \int_{\{1\}\times\mathbb{R}} \omega_{02}(\eta_{02}, \nabla_s\eta_{02}) \right|, \qquad \Omega_{0211}(\hat\eta|_{t=\delta}) := \left| \int_{\{\delta\}\times\mathbb{R}} \omega_{0211}(\hat\eta, \nabla_s\hat\eta) \right|.
$$

These boundary terms occur on the right hand side of (c) and they will be estimated by (a) to prove (b). The boundary term at $t=0$ vanishes by the diagonal boundary conditions, and the boundary terms at $S\to\pm\infty$ vanish since $\eta|_{\{s\in[S,S+1]\}}\to 0$ in the $H^2_{1,\delta}$-norm. The error term can be estimated by

$$
\int C\big((1+\sigma c_0)|\eta||\nabla\eta| + |\eta|^2\big) \leq C\|\eta\|^2_{L^2_{1,\delta}(\sigma\zeta)} + \frac{1}{2}\|\nabla\eta\|^2_{L^2_{1,\delta}(\sigma\zeta)} + \frac{1}{2}C^2(1+\sigma c_0)^2\|\eta\|^2_{L^2_{1,\delta}(\sigma\zeta)},
$$

where the highest order term $\|\nabla\eta\|$ can be absorbed on the right hand side. From now on C will denote any uniform constant (which is allowed to depend on c_0 in the nonlinear case $\sigma=1$). In summary, the estimates for $\eta=\xi$ and $\eta=\nabla_s\xi$ are

$$
\begin{aligned}
\tfrac{1}{C}\|\nabla\xi\|^2_{L^2_{1,\delta}} &\leq \big\|\nabla_s\xi + J(\sigma\xi)\nabla_t\xi\big\|^2_{L^2_{1,\delta}} + \|\xi\|^2_{L^2_{1,\delta}} + \Omega_{02}(\xi_{02}|_{t=1}) + \Omega_{0211}(\hat\xi|_{t=\delta}), \\
\tfrac{1}{C}\|\nabla\nabla_s\xi\|^2_{L^2_{1,\delta}} &\leq \big\|\nabla_s\big(\nabla_s\xi + J(\sigma\xi)\nabla_t\xi\big)\big\|^2_{L^2_{1,\delta}} + \|\nabla\xi\|^2_{L^2_{1,\delta}} \\
&\qquad + \Omega_{02}(\nabla_s\xi_{02}|_{t=1}) + \Omega_{0211}(\nabla_s\hat\xi|_{t=\delta}).
\end{aligned}
$$

This already proves the first estimate in (b). We can moreover use the identity $\nabla_t\xi = J(\sigma\zeta)\nabla_s\xi - J(\sigma\zeta)(\nabla_s\xi + J(\sigma\zeta)\nabla_t\xi)$ to obtain

$$
\|\nabla\nabla_t\xi\|_{L^2_{1,\delta}} \leq \|\nabla\nabla_s\xi\|_{L^2_{1,\delta}} + \|\nabla(\nabla_s\xi + J(\sigma\zeta)\nabla_t\xi)\|_{L^2_{1,\delta}} + C\|\nabla\xi\|_{L^2_{1,\delta}} + \sigma C c_0\|\nabla\xi\|_{L^2_{1,\delta}}.
$$

In the linear case (c) these estimates combined with (a) and (30) to prove the first estimate:

$$
c_1\|\xi\|_{H^2_{1,\delta}} \leq \big\|D^\delta\xi\big\|_{H^1_{1,\delta}} + \|\xi_{02}|_{t=1}\|_{H^1(\mathbb{R})} + \|\hat\xi|_{t=\delta}\|_{H^1(\mathbb{R})} + \|\xi\|_{L^2_{1,\delta}}
$$

with a uniform constant $c_1>0$. In the nonlinear case (b) we obtain similarly

$$
\begin{aligned}
C_1^{-1}\|\xi\|_{H^2_{1,\delta}} &\leq \big\|\nabla_s\xi + J(\zeta)\nabla_t\xi\big\|_{H^1_{1,\delta}} + \|\xi\|_{L^2_{1,\delta}} + \Omega_{02}(\xi_{02}|_{t=1}) + \Omega_{0211}(\hat\xi|_{t=\delta}) \\
&\qquad + \Omega_{02}(\nabla_s\xi_{02}|_{t=1}) + \Omega_{0211}(\nabla_s\hat\xi|_{t=\delta})
\end{aligned}
$$

with a constant C_1 that depends on $\|\nabla\xi\|_\infty \leq c_0$.

The L^4-estimate for the linear and nonlinear operators will arise by rescaling from the following basic estimate. Here $\hat u:\mathbb{R}\times[0,1]\to M_{0211}$ will be given by $\hat u(s,t)=\bar u(\delta s)$ for any $\delta\in(0,1]$. Then for every $\hat\eta\in H^1(\mathbb{R}\times[0,1],\hat u^*TM_{0211})$

$$
\|\hat\eta\|_{L^4(\mathbb{R}\times[0,1])} \leq C_0\big(\|\hat\eta|_{t=1}\|_{L^2(\mathbb{R})} + \|\nabla\hat\eta\|_{L^2(\mathbb{R}\times[0,1])}\big).
$$

This simply follows from the Sobolev embedding $H^1(\mathbb{R}\times[0,1])\hookrightarrow L^4(\mathbb{R}\times[0,1])$ and

$$\|\hat\eta\|^2_{L^2(\mathbb{R}\times[0,1])}\le\int_0^1\left\|\hat\eta(\cdot,1)-\int_t^1\nabla_t\hat\eta(\cdot,\tau)d\tau\right\|^2_{L^2(\mathbb{R})}dt\le 2\|\hat\eta|_{t=1}\|^2_{L^2(\mathbb{R})}+2\|\nabla_t\hat\eta\|^2_{L^2(\mathbb{R}\times[0,1])}.$$

When applying this to $\hat\eta(s,t):=\nabla_s\hat\xi(\delta s,\delta t)$ we encounter the following terms:

$$\begin{aligned}\|\hat\eta\|^2_{L^4(\mathbb{R}\times[0,1])}&=\left(\int_{\mathbb{R}\times[0,1]}|\nabla_s\hat\xi(\delta s,\delta t)|^4dsdt\right)^{1/2}=\delta^{-1}\|\nabla_s\hat\xi\|^2_{L^4(\mathbb{R}\times[0,\delta])},\\ \|\hat\eta|_{t=1}\|^2_{L^2(\mathbb{R})}&=\int_{\mathbb{R}}|\nabla_s\hat\xi(\delta s,\delta)|^2ds=\delta^{-1}\|\nabla_s\hat\xi|_{t=\delta}\|^2_{L^2(\mathbb{R})},\\ \|\nabla\hat\eta\|^2_{L^2(\mathbb{R}\times[0,1])}&=\int_{\mathbb{R}\times[0,1]}\delta^2|\nabla\nabla_s\hat\xi(\delta s,\delta t)|^2dsdt=\|\nabla\nabla_s\hat\xi\|^2_{L^2(\mathbb{R}\times[0,\delta])}.\end{aligned}$$

Putting this together we find that

$$\|\nabla_s\hat\xi\|_{L^4(\mathbb{R}\times[0,\delta])}\le C_0\big(\|\nabla_s\hat\xi|_{t=\delta}\|_{L^2(\mathbb{R})}+\|\nabla\nabla_s\hat\xi\|_{H^2(\mathbb{R}\times[0,\delta])}\big)\le C_0\big(\|\hat\xi|_{t=\delta}\|_{H^1(\mathbb{R})}+\|\xi\|_{H^2_{1,\delta}}\big),$$

where the estimate for $\|\xi\|_{H^2_{1,\delta}}$ is already established. The L^4-estimate for $\nabla\xi_{02}$ follows from the Sobolev embedding $H^1(\mathbb{R}\times[0,1])\hookrightarrow L^4(\mathbb{R}\times[0,1])$, and for the last component we have

$$\|\nabla_t\hat\xi\|_{L^4(\mathbb{R}\times[0,\delta])}\le\|\nabla_s\hat\xi+\hat J(\sigma\hat\zeta)\nabla_t\hat\xi\|_{L^4(\mathbb{R}\times[0,\delta])}+\|\nabla_s\hat\xi\|_{L^4(\mathbb{R}\times[0,\delta])}.$$

This finishes the proof of the second estimate, where we allow $\|\nabla_s\xi+J(\sigma\zeta)\nabla_t\xi\|_{L^4_{1,\delta}}$ on the right hand side, and the constant in the nonlinear case depends on $\|\nabla\zeta\|_\infty\le c_0$. In the linear case the difference to $\|D^\delta\xi\|_{L^4_{1,\delta}}$ in (30) is bounded by the previous estimate. □

The lemma below gives control of the lower-order terms appearing in Lemma 3.2.1 and in particular will be used to prove surjectivity of the linearized operator.

Lemma 3.2.2. (a) *There is a constant $\epsilon>0$ and for every $c_0>0$ there is a constant C_2 such that for all $\delta\in(0,1]$ and $\xi,\zeta\in H^2_{1,\delta}$ with $\|\zeta\|_\infty\le\epsilon$, $\|\nabla\zeta\|_\infty\le c_0$ we have*

$$\begin{aligned}&\|\hat\xi|_{t=\delta}\|_{H^1(\mathbb{R})}+\|\xi_{02}|_{t=1}\|_{H^1(\mathbb{R})}\\ &\le C_2\big(\|\nabla_s\xi_{02}+J_{02}(\zeta_{02})\nabla_t\xi_{02}\|_{H^1(\mathbb{R}\times[0,1])}+\sqrt{\delta}\|\nabla_t\hat\xi\|_{H^1(\mathbb{R}\times[0,\delta])}+\|\pi^\perp_{0211}\hat\xi|_{t=\delta}\|_{H^1(\mathbb{R})}\\ &\qquad+\|\xi_{02}\|_{L^2(\mathbb{R}\times[0,1])}+\|(\xi_1'-\xi_1)|_{t=0}\|_{H^1(\mathbb{R})}+\|(\xi_{02}'-\xi_{02})|_{t=0}\|_{H^1(\mathbb{R})}\big).\end{aligned}$$

(b) *There is a constant $c_2>0$ such that for all $\delta\in(0,1]$ and $\xi\in\Gamma_{1,\delta}$*

$$c_2\big(\|\hat\xi|_{t=\delta}\|_{H^1(\mathbb{R})}+\|\xi_{02}|_{t=1}\|_{H^1(\mathbb{R})}+\|\xi\|_{H^0_{1,\delta}}\big)\le\|D^*_{u_{02}}\xi_{02}\|_{H^1(\mathbb{R}\times[0,1])}+\sqrt{\delta}\|\nabla_t\hat\xi\|_{H^1(\mathbb{R}\times[0,\delta])},$$

and for all $\xi\in\Gamma_{1,\delta}\cap K_0$

$$c_2\big(\|\hat\xi|_{t=\delta}\|_{H^1(\mathbb{R})}+\|\xi_{02}|_{t=1}\|_{H^1(\mathbb{R})}+\|\xi\|_{H^0_{1,\delta}}\big)\le\|D_{u_{02}}\xi_{02}\|_{H^1(\mathbb{R}\times[0,1])}+\sqrt{\delta}\|\nabla_t\hat\xi\|_{H^1(\mathbb{R}\times[0,\delta])}.$$

Proof. The constant $\epsilon>0$ in case (a) is chosen such that $e_{u_{02}}(\zeta_{02})$ and thus $J_{02}(\zeta_{02})$ is defined. To prove (a) (and similar for (b)) we assume by contradiction that we have sequences $\delta^\nu>0$ and $\xi^\nu,\zeta^\nu\in H^2_{1,\delta^\nu}$ such that $\|\hat\xi^\nu|_{t=\delta^\nu}\|_{H^1(\mathbb{R})}+\|\xi^\nu_{02}|_{t=1}\|_{H^1(\mathbb{R})}=1$ (in case (b) add $\|\xi^\nu\|_{H^0_{1,\delta}}$ here), but the right hand sides converges to zero. For technical reasons we assume in addition $\|\xi_{02}^\nu\|_{H^1(\mathbb{R}\times[0,1])}\leq 1$, which we will also disprove (i.e. we actually prove a stronger estimate with this term on the left hand side). First we integrate for all $t\in[0,\delta^\nu]$

$$\|\hat\xi^\nu|_{t=t_0}-\hat\xi^\nu|_{t=\delta^\nu}\|_{H^1(\mathbb{R})}\leq\int_0^{\delta^\nu}\|\nabla_t\hat\xi^\nu\|_{H^1(\mathbb{R})}\leq\sqrt{\delta^\nu}\|\nabla_t\hat\xi^\nu\|_{H^1(\mathbb{R}\times[0,\delta^\nu])}\to 0. \tag{31}$$

Next, Lemma 3.1.3 implies

$$\begin{aligned}\|\pi_{02}^\perp\xi_{02}^\nu|_{t=0}\|_{L^2(\mathbb{R})} &\leq \|\pi_{02}^\perp\xi_{02}'^\nu|_{t=\delta^\nu}\|_{L^2(\mathbb{R})}+\|\xi_{02}'^\nu|_{t=0}-\xi_{02}'^\nu|_{t=\delta^\nu}\|_{L^2(\mathbb{R})}+\|(\xi_{02}'^\nu-\xi_{02}^\nu)|_{t=0}\|_{L^2(\mathbb{R})}\\ &\leq C\big(\|\pi_{0211}^\perp\hat\xi^\nu|_{t=\delta^\nu}\|_{L^2(\mathbb{R})}+\|\hat\xi^\nu|_{t=0}-\hat\xi^\nu|_{t=\delta^\nu}\|_{L^2(\mathbb{R})}\\ &\qquad+\|(\xi_1'^\nu-\xi_1^\nu)|_{t=0}\|_{L^2(\mathbb{R})}+\|(\xi_{02}'^\nu-\xi_{02}^\nu)|_{t=0}\|_{L^2(\mathbb{R})}\big)\to 0,\end{aligned}$$

(32)
$$\begin{aligned}\|\pi_{02}^\perp\xi_{02}^\nu|_{t=0}\|_{H^1(\mathbb{R})} &\leq \|\pi_{02}^\perp\xi_{02}'^\nu|_{t=\delta^\nu}\|_{H^1(\mathbb{R})}+\|\xi_{02}'^\nu|_{t=0}-\xi_{02}'^\nu|_{t=\delta^\nu}\|_{H^1(\mathbb{R})}+\|(\xi_{02}'^\nu-\xi_{02}^\nu)|_{t=0}\|_{H^1(\mathbb{R})}\\ &\leq C\big(\|\pi_{0211}^\perp\hat\xi^\nu|_{t=\delta^\nu}\|_{H^1(\mathbb{R})}+\|\hat\xi^\nu|_{t=0}-\hat\xi^\nu|_{t=\delta^\nu}\|_{H^1(\mathbb{R})}+\|(\xi_1'^\nu-\xi_1^\nu)|_{t=0}\|_{H^1}\\ &\qquad+\|(\xi_{02}'^\nu-\xi_{02}^\nu)|_{t=0}\|_{H^1(\mathbb{R})}+\big\||\partial_s\bar u|\cdot|\hat\xi^\nu|_{t=\delta^\nu}|\big\|_{L^2(\mathbb{R})}\big).\end{aligned}$$

In the two cases of (b) we use the boundary conditions for $\xi^\nu\in\Gamma_{1,\delta}$ here. In all three cases the hardest step is now to prove that $\big\||\partial_s\bar u|\cdot|\hat\xi^\nu|_{t=\delta^\nu}|\big\|_{L^2(\mathbb{R})}\to 0$. Here we exploit the assumption that $\|\xi_{02}^\nu\|_{H^1(\mathbb{R}\times[0,1])}$ is bounded. This implies a bound on $\|\xi_{02}^\nu|_{t=0}\|_{L^2(\mathbb{R})}$. Now we find a convergent subsequence $\xi_{02}^\nu\to\xi_{02}^\infty\in H^1(\mathbb{R}\times[0,1],u_{02}^*TM_{02})$ in the weak H^1-topology, and at the same time $\xi_{02}^\nu|_{t=0}\to\xi_{02}^\infty|_{t=0}$ in the L^2-norm on every compact set. (The Sobolev embedding $H^1(\Omega)\hookrightarrow L^2(\partial\Omega)$) is compact for compact domains $\Omega\subset\mathbb{R}\times[0,1]$ with smooth boundary $\partial\Omega$, see e.g. [1, Theorem 6.3].) In case (a) the limit has to be $\xi_{02}^\infty=0$ since $\|\xi_{02}^\infty\|_{L^2(\mathbb{R}\times[0,1])}\leq\liminf_{\nu\to\infty}\|\xi_{02}^\nu\|_{L^2(\mathbb{R}\times[0,1])}=0$. This also holds in case (b) since the limit satisfies with $D=D_{u_{02}}$ or $D=D_{u_{02}}^*$

$$\begin{aligned}\|D\xi_{02}^\infty\|_{L^2(\mathbb{R}\times[0,1])}&\leq\liminf_{\nu\to\infty}\|D\xi_{02}^\nu\|_{L^2(\mathbb{R}\times[0,1])}=0,\\ \|\pi_{02}^\perp\xi_{02}^\infty|_{t=0}\|_{L^2(\mathbb{R})}&\leq\liminf_{\nu\to\infty}\|\pi_{02}^\perp\xi_{02}^\nu|_{t=0}\|_{L^2(\mathbb{R})}=0.\end{aligned}$$

Since u_{02} is assumed regular, $D_{u_{02}}^*\oplus\pi_{02}^\perp$ is injective, and in the second part of case (b) we have in addition $\xi_{02}^\infty\in\ker(D_{u_{02}}\oplus\pi_{02}^\perp)^\perp$. So in all three cases we obtain

$$\|\xi_{02}^\nu|_{t=0}\|_{L^2(\mathbb{R})}\leq C\qquad\text{and}\qquad\|\xi_{02}^\nu|_{t=0}\|_{L^2([-T,T])}\to 0\qquad\text{for all } T>0.$$

The same holds for $\hat\xi^\nu|_{t=\delta^\nu}$ since we can apply Lemma 3.1.3 on the interval $(-T,T)$ for any $T\in(0,\infty]$ to obtain

$$\begin{aligned}\|\hat\xi^\nu|_{t=\delta^\nu}\|_{L^2}&\leq C\big(\|\pi_{02}\xi_{02}'^\nu|_{t=\delta^\nu}\|_{L^2}+\|\pi_{0211}^\perp\hat\xi^\nu|_{t=\delta^\nu}\|_{L^2}+\|(\xi_1'^\nu-\xi_1^\nu)|_{t=\delta^\nu}\|_{L^2}\big)\\ &\leq C'\big(\|\xi_{02}^\nu|_{t=0}\|_{L^2}+\|(\xi_{02}'^\nu-\xi_{02}^\nu)|_{t=0}\|_{L^2}+\|\hat\xi^\nu|_{t=0}-\hat\xi^\nu|_{t=\delta^\nu}\|_{L^2}\\ &\qquad+\|\pi_{0211}^\perp\hat\xi^\nu|_{t=\delta^\nu}\|_{L^2}+\|(\xi_1'^\nu-\xi_1^\nu)|_{t=0}\|_{L^2}\big).\end{aligned}$$

This together with the fact that $\sup_{|s|\geq T}|\partial_s\bar u(s)|\to 0$ as $T\to\infty$ implies that $\big\||\partial_s\bar u|\cdot|\hat\xi^\nu|_{t=\delta^\nu}|\big\|_{L^2(\mathbb{R})}\to 0$ and hence $\|\pi_{02}^\perp\xi_{02}^\nu|_{t=0}\|_{H^1(\mathbb{R})}\to 0$ by (32). From this we will move on to prove that

$$\|\xi_{02}^\nu\|_{H^{3/2}(\mathbb{R}\times[0,1])}\to 0. \tag{33}$$

For that purpose we denote by D any of the three operators $\nabla_s+J_{02}(\zeta_{02})\nabla_t$ in case (a) and $D_{u_{02}}^*$ or $D_{u_{02}}$ in case (b). Then we use the fact that in all three cases the operator

$D \oplus \pi_{02}^\perp$ is Fredholm on the space of sections η that satisfy the boundary conditions $\eta|_{t=1} \in T_{u_{02}}(L_0 \times L_2)$, see e.g. [5, Theorem 20.1.2] for compact domains. The corresponding estimates add up to

$$\|\xi_{02}^\nu\|_{H^{3/2}(\mathbb{R}\times[0,1])} \leq C\big(\|D\xi_{02}^\nu\|_{H^1(\mathbb{R}\times[0,1])} + \|\pi_{02}^\perp\xi_{02}^\nu|_{t=0}\|_{H^1(\mathbb{R})} + \|\xi_{02}^\nu\|_{H^0(\mathbb{R}\times[0,1])}\big). \tag{34}$$

In the nonlinear case (a) the constant in this estimate depends continuously on $J_{02}(\zeta_{02})$ in the $\mathcal{C}^1$-topology, see e.g. [8, Appendix B]. In this case the above estimate already implies the claim (33) since we assumed $\|\xi_{02}^\nu\|_{L^2} \to 0$. In the linear cases we need to use the injectivity of the operators to remove the last term from the right hand side of (34). Since $H^{3/2}(\mathbb{R}) \hookrightarrow H^0((-T,T))$ is compact only for $T<\infty$, we first have to achieve a lower order term on a compact domain:

Consider the operator $D_{x^\pm} = \partial_s - A$, where $A := -J(x^\pm)\partial_t$ (or $A := J(x^\pm)\partial_t$ in the case $D = D_{u_{02}}^*$) is self-adjoint and invertible on its constant domain $H^1([0,1], T_{x^\pm}M_{02})$ with boundary conditions $\eta|_{t=0} \in T_{x^\pm}L_{02}$, $\eta|_{t=1} \in T_{x^\pm}(L_0 \times L_2)$. Then abstract theory (e.g. [13, Lemma 3.9, Proposition 3.14]) implies the Fredholm property and bijectivity,

$$\|\eta\|_{H^1(\mathbb{R}\times[0,1])} \leq C\|D_{x^\pm}\eta\|_{H^0(\mathbb{R}\times[0,1])}.$$

In order to apply this estimate to ξ_{02}^ν we first find an extension $\zeta \in H^1(\mathbb{R}\times[0,1])$ of $\zeta|_{t=0} = \pi_{02}^\perp\xi_{02}^\nu|_{t=0}$ such that $\|\zeta\|_{H^1} \leq C\|\pi_{02}^\perp\xi_{02}^\nu|_{t=0}\|_{H^{1/2}}$. We moreover fix a cutoff function $h \in \mathcal{C}_0^\infty(\mathbb{R},[0,1])$ with $h|_{\{|s|\leq T-1\}} \equiv 0$ and $h|_{\{|s|\geq T\}} \equiv 1$, where we fix $T>1$ sufficiently large such that $u_{02}|_{\mathrm{supp}(h)} = e_{x^\pm}(\vartheta_{02})$ for some smooth map $\vartheta_{02} : \{\pm s \geq (T-1)\} \to T_{x^\pm}M_{02}$. Then we can apply the estimate to $\eta := \Phi_{x^\pm}(\vartheta_{02})^{-1}\big(h(\xi_{02}^\nu - \zeta)\big)$, where $\Phi_{x^\pm}(\vartheta_{02})$ denotes parallel transport along the path $[0,1] \ni \tau \mapsto e_{x^\pm}(\tau\vartheta_{02})$. We obtain, denoting all uniform constants by C,

$$\begin{aligned}
&\|h\xi_{02}^\nu\|_{H^1(\mathbb{R}\times[0,1])} \\
&\leq C\|\eta\|_{H^1(\mathbb{R}\times[0,1])} + \|h\zeta\|_{H^1(\mathbb{R}\times[0,1])} \\
&\leq C\big(\|\big(D_{x^\pm} - D\circ\Phi_{x^\pm}(\vartheta_{02})\big)\eta\|_{H^0(\mathbb{R}\times[0,1])} + \|D(h\xi_{02}^\nu)\|_{H^0(\mathbb{R}\times[0,1])} + \|h\zeta\|_{H^1(\mathbb{R}\times[0,1])}\big) \\
&\leq C\big(\|\big(D_{x^\pm} - D\circ\Phi_{x^\pm}(\vartheta_{02})\big)\big|_{\{|s|>T-1\}}\| \cdot \|h(\xi_{02}^\nu - \zeta)\|_{H^1(\mathbb{R}\times[0,1])} + \|D\xi_{02}^\nu\|_{H^0(\mathbb{R}\times[0,1])} \\
&\qquad + \|\xi_{02}^\nu\|_{H^0([-T,T]\times[0,1])} + \|\pi_{02}^\perp\xi_{02}^\nu|_{t=0}\|_{H^{1/2}(\mathbb{R})}\big).
\end{aligned}$$

Here the difference of the operators goes to zero for $T\to\infty$ since $u_{02}|_{\{|s|\geq T-1\}} \to x^\pm$ with all derivatives. Thus for sufficiently large $T>0$ we can absorb the first term into the left hand side and $\|h\zeta\|_{H^1} \leq C\|\pi_{02}^\perp\xi_{02}^\nu|_{t=0}\|_{H^{1/2}}$. After all this we can finally replace the last term in (34) by $\|\xi_{02}^\nu\|_{H^0([-T,T]\times[0,1])}$.

Now in the first case of (b) we can deduce (33) from the fact that $D_{u_{02}} \oplus \pi_{02}^\perp$ is surjective by assumption and hence $D_{u_{02}}^* \oplus \pi_{02}^\perp$ is injective. So the compact embedding $H^{3/2}(\mathbb{R}\times[0,1]) \hookrightarrow H^0([-T,T]\times[0,1])$ allows the removal of the lower order term. Similarly, in the second case of (b) we can employ the injectivity of the operator on $\ker(D_{u_{02}} \oplus \pi_{02}^\perp)^\perp \ni \xi_{02}^\nu$ to deduce (33).

Next, (33) and the Sobolev trace theorem provide $\|\xi_{02}^\nu|_{t=0}\|_{H^1(\mathbb{R})} + \|\xi_{02}^\nu|_{t=1}\|_{H^1(\mathbb{R})} \to 0$, and again using Lemma 3.1.3 we can deduce that

$$\begin{aligned}
&\|\hat{\xi}^\nu|_{t=\delta^\nu}\|_{H^1(\mathbb{R})} \\
&\le C\big(\|\pi_{02}\xi_{02}'^\nu|_{t=\delta^\nu}\|_{H^1(\mathbb{R})} + \|\pi_{0211}^\perp\hat{\xi}^\nu|_{t=\delta^\nu}\|_{H^1(\mathbb{R})} + \|(\xi_1'^\nu - \xi_1^\nu)|_{t=\delta^\nu}\|_{H^1(\mathbb{R})}\big) \\
&\le C\big(\|\xi_{02}^\nu|_{t=0}\|_{H^1(\mathbb{R})} + \|(\xi_{02}'^\nu - \xi_{02}^\nu)|_{t=0}\|_{H^1(\mathbb{R})} + \|\pi_{0211}^\perp\hat{\xi}^\nu|_{t=\delta^\nu}\|_{H^1(\mathbb{R})} \\
&\qquad\qquad + \|\hat{\xi}^\nu|_{t=0} - \hat{\xi}^\nu|_{t=\delta^\nu}\|_{H^1(\mathbb{R})} + \|(\xi_1'^\nu - \xi_1^\nu)|_{t=0}\|_{H^1(\mathbb{R})}\big) \to 0.
\end{aligned}$$

Finally, combining this with (31) in case (b) implies

$$\|\hat{\xi}^\nu\|_{L^2(\mathbb{R}\times[0,\delta^\nu])} \to 0$$

in contradiction to the assumption. □

Finally, we establish uniform exponential decay for the solutions of Floer's equation (13) on the triple strip. For that purpose we introduce the following notation for integration over finite strips,

$$\int_{[0,1]\sqcup[0,\delta]} |\partial_s v(s,t)|^2\,dt := \int_0^1 |\partial_s v_{02}(s,t)|^2\,dt + \int_0^\delta |\partial_s \hat{v}(s,t)|^2\,dt,$$

and similarly for the $\mathcal{C}^0$-norm

$$\begin{aligned}
\|\partial_s v\|_{\mathcal{C}^0_{1,\delta}([s_0,s_1])} &:= \|\partial_s v_{02}\|_{L^\infty([s_0,s_1]\times[0,1])} + \|\partial_s \hat{v}\|_{L^\infty([s_0,s_1]\times[0,\delta])}, \\
d_{\mathcal{C}^0_{1,\delta}([s_0,s_1])}(v,x^\pm) &:= \sup_{(s,t)\in[s_0,s_1]\times[0,1]} d_{M_{02}}(v_{02}(s,t),x^\pm), \\
&\qquad + \sup_{(s,t)\in[s_0,s_1]\times[0,\delta]} d_{M_{0211}}(\hat{v}(s,t),(x^\pm,x_1^\pm,x_1^\pm)).
\end{aligned}$$

Lemma 3.2.3. *There are constants $\hbar,\Delta>0$ and C such that the following holds for every $\delta\in(0,1]$. If $v\in\widehat{\mathcal{M}}_\delta(x^-,x^+)$ is a smooth solution of (13) satisfying*

$$\int_0^\infty \int_{[0,1]\sqcup[0,\delta]} |\partial_s v(s,t)|^2\,dtds \;<\; \hbar, \tag{35}$$

then for every $S\ge 3$

$$d_{\mathcal{C}^0_{1,\delta}([S,\infty))}(v,x^+)^2 + \|\partial_s v\|^2_{\mathcal{C}^0_{1,\delta}([S,\infty))} \le Ce^{-\Delta S}\int_0^2\int_{[0,1]\sqcup[0,\delta]} |\partial_s v(s,t)|^2\,dtds,$$

and the analogous statement holds on $(-\infty,0]$ for the convergence to x^-.

Proof. **Step 1:** *For every $\kappa>0$ there is an $\epsilon_\kappa>0$ such that the following holds for all $\delta\in(0,1]$. If $v\in\widehat{\mathcal{M}}_\delta(x^-,x^+)$ satisfies (35) with $\hbar=\epsilon_\kappa$, then*

$$\|\partial_s v\|_{\mathcal{C}^0_{1,\delta}([\frac12,\infty))} \le \kappa. \tag{36}$$

Assume by contradiction that this is wrong. Then there exist $\kappa>0$ and sequences $\delta^\nu\in(0,1]$ and $v^\nu\in\widehat{\mathcal{M}}_{\delta^\nu}(x^-,x^+)$ such that

$$\lim_{\nu\to\infty}\int_0^\infty\int_{[0,1]\sqcup[0,\delta^\nu]} |\partial_s v^\nu(s,t)|^2\,dtds = 0, \tag{37}$$

but the assertion fails. So after a time-shift we can assume that

$$\|\partial_s v^\nu\|_{\mathcal{C}^0_{1,\delta^\nu}([\frac12,1])} > \tfrac12\kappa.$$

The equation $\overline{\partial}_J v^\nu = 0$ together with (37) implies that $dv^\nu|_{s\geq 0} \to 0$ in the L^2-norm. If δ^ν is bounded away from zero, then the standard compactness for holomorphic curves with Lagrangian boundary conditions implies that $dv^\nu|_{s>0} \to 0$ in $\mathcal{C}^\infty$ on every compact set (for a subsequence), in contradiction to the assumption. In the case $\delta^\nu \to 0$ the standard compactness theory still implies $dv_{02}^\nu|_{(0,1]\times(0,\infty)} \to 0$ in $\mathcal{C}^\infty$ on every compact set. For $\hat{v}$ and v_{02} near the boundary $t = 0$ we obtain a $\mathcal{C}^1$-bound from Lemma 3.3.2. So we obtain $\mathcal{C}^0$-convergence of a subsequence $v_{02}^\nu \to x_{02}$, $\hat{v}^\nu \to (x_{02}, x_1, x_1)$ to constants $x_{02} \in L_0 \times L_2$, $x_1 \in M_1$ such that $(x_{02}, x_1, x_1) \in L_{01} \times L_{12}$. Now we can use the same compactness arguments as in the proof of Lemma 3.3.2 (step 2, using a cutoff function only in s) to deduce that $dv^\nu|_{s\in[\frac{1}{2},1]} \to 0$ in the $\mathcal{C}^0$-norm. This again is a contradiction.

Step 2: *There are constants $\epsilon_1 > 0$ and C_1 such that the following holds for all $\delta \in (0,1]$. If $v \in \widehat{\mathcal{M}}_\delta(x^-, x^+)$ satisfies (35) with $\hbar = \epsilon_1$, then*

$$\|\partial_s v(1,\cdot)\|^2_{\mathcal{C}^0([0,1]\sqcup[0,\delta])} \leq C_1 \int_{[0,1]\sqcup[0,\delta]} |\nabla_t \partial_s v(1,t)|^2 \, dt.$$

By contradiction we find sequences $\delta^\nu \in (0,1]$ and $v^\nu \in \widehat{\mathcal{M}}_{\delta^\nu}(x^-, x^+)$ that satisfy (37), but there is no uniform constant C_1 with which the estimate holds. Then as in Step 1 we obtain (for a subsequence) $\mathcal{C}^1$-convergence $v^\nu \to x = (x_{02}, \hat{x})$ on $[\frac{1}{2}, 2] \times ([0,1] \sqcup [0,\delta^\nu])$ to constants $x_{02} \in L_0 \times L_2$, $x_1 \in M_1$ with $\hat{x} = (x_{02}, x_1, x_1) \in L_{01} \times L_{12}$. By assumption L_{02} and $(L_0 \times L_2)$ intersect transversely in x_{02}, and hence we have for all $\xi_{02} : [0,1] \to T_{x_{02}} M_{02}$ with $\xi_{02}(1) \in T_{x_{02}}(L_0 \times L_2)$

$$\|\xi_{02}\|_{\mathcal{C}^0([0,1])} \leq C\big(\|\nabla_t \xi_{02}\|_{L^2([0,1])} + \big|\pi_{02}^\perp \xi_{02}(0)\big|\big).$$

Now consider in addition $\hat{\xi} : [0,\delta] \to T_{\hat{x}} M_{0211}$ such that $\hat{\xi}(\delta) \in T_{\hat{x}}(L_{01} \times L_{12})$ and $\xi|_{t=0} = (\xi_{02}, \hat{\xi})|_{t=0} \in T_x(\Delta_{M_0\times M_2} \times \Delta_1)$. We integrate for all $t \in [0,\delta]$

$$\big|\hat{\xi}(t) - \hat{\xi}(\delta)\big| \leq \int_0^\delta |\nabla_t \hat{\xi}(t)| dt \leq \sqrt{\delta} \bigg(\int_0^\delta |\nabla_t \hat{\xi}(t)|^2 dt \bigg)^{1/2}. \tag{38}$$

Combining this with Lemma 3.1.3 and using the boundary conditions we obtain

$$\big|\pi_{02}^\perp \xi_{02}(0)\big| \leq \big|\pi_{0211}^\perp \hat{\xi}(\delta)\big| + \big|\pi_{0211}^\perp\big(\hat{\xi}(0) - \hat{\xi}(\delta)\big)\big| + \big|\xi_1'(0) - \xi_1(0)\big| \leq C\sqrt{\delta} \bigg(\int_0^\delta |\nabla_t \hat{\xi}(t)|^2 dt \bigg)^{1/2},$$

and thus

$$\|\xi_{02}\|^2_{\mathcal{C}^0([0,1])} \leq C^2 \int_{[0,1]\sqcup[0,\delta]} |\nabla_t \xi|^2 \, dt.$$

We moreover obtain from Lemma 3.1.3 with uniform constants C, C', C''

$$\begin{aligned} \big|\hat{\xi}(\delta)\big| &\leq C\big(\big|\pi_{02}\xi_{02}'(\delta)\big| + \big|(\xi_1'(\delta) - \xi_1(\delta))\big|\big) \\ &\leq C'\big(\big|\xi_{02}(0)\big| + \big|\hat{\xi}(0) - \hat{\xi}(\delta)\big|\big) \;\leq\; C'' \bigg(\int_{[0,1]\sqcup[0,\delta]} \big|\nabla_t(\xi_{02}, \hat{\xi})\big|^2 dt \bigg)^{1/2}. \end{aligned}$$

Together with (38) this implies

$$\|\xi\|^2_{\mathcal{C}^0([0,1]\sqcup[0,\delta])} \leq C_1 \int_{[0,1]\sqcup[0,\delta]} |\nabla_t \xi|^2 \, dt$$

with some uniform constant C_1 for all $\delta \in (0,1]$ and all sections ξ over x satisfying the boundary conditions. Due to the $\mathcal{C}^1$-convergence $v^\nu \to x$ this estimate continues to hold with a uniform constant for sufficiently large ν for sections $\xi_{02} \in \mathcal{C}^1([0,1], v_{02}^\nu|_{s=1}{}^*TM_{02})$, $\hat\xi \in \mathcal{C}^1([0,\delta^\nu], \hat v|_{s=1}^* TM_{0211})$ that satisfy the analogous boundary conditions. (We can write $v^\nu|_{s=1} = e_x(\zeta^\nu)$ with $\|\zeta^\nu\|_{\mathcal{C}^1} \to 0$ and use $de_x(\zeta^\nu)^{-1}$ to map $(\xi_{02}, \hat\xi)$ to a section over x. This preserves the boundary conditions by construction of e.) In particular, we can apply this new estimate to $\xi = \partial_s v^\nu|_{s=1}$, which provides a uniform estimate and thus finishes the proof by contradiction.

Step 3: *There are uniform constants $\epsilon_2, \Delta > 0$ and C_2 such that the following holds for all for all $\delta \in (0,1]$. If $v \in \widehat{\mathcal{M}}_\delta(x^-, x^+)$ satisfies (35) with $\hbar = \epsilon_2$, then for all $s_0 \geq 2$*

$$\int_{[0,1]\sqcup[0,\delta]} |\partial_s v(s_0,t)|^2 dt \leq C_2 e^{-\Delta s_0} \int_1^2 \int_{[0,1]\sqcup[0,\delta]} |\partial_s v(s,t)|^2 dt ds.$$

Consider the function $f : [1,\infty) \to [0,\infty)$ defined by

$$f(s) := \tfrac{1}{2} \int_{[0,1]\sqcup[0,\delta]} |\partial_s v(s,t)|^2 dt.$$

We can use the equation $\overline{\partial}_J v = \bigl(\partial_s v_{02} + J_{02}(v_{02})\partial_t v_{02}\,,\, \partial_s \hat v + \hat J(\hat v)\partial_t \hat v\bigr) = 0$ and the bound $\|\partial_s v\|_\infty \leq \kappa$ from Step 1 to calculate for $s \geq 1$

$$\begin{aligned}
f''(s) &= \int_{[0,1]\sqcup[0,\delta]} \Bigl(|\nabla_s \partial_s v|^2 + \langle \partial_s v\,,\, \nabla_s^2 \partial_s v\rangle \Bigr) \\
&= \int_{[0,1]\sqcup[0,\delta]} \Bigl(|J\nabla_t \partial_s v + (\nabla_{\partial_s v} J)\partial_t v|^2 - \langle \partial_s v\,,\, J\nabla_t \nabla_s \partial_s v\rangle \Bigr) \\
&\quad - \int_{[0,1]\sqcup[0,\delta]} \Bigl(\langle \partial_s v\,,\, JR(\partial_s v, \partial_t v)\partial_s v + 2(\nabla_{\partial_s v} J)\nabla_s \partial_t v + \nabla_s(\nabla_{\partial_s v} J)\partial_t v\rangle \Bigr) \\
&\geq \int_{[0,1]\sqcup[0,\delta]} \Bigl(2|J\nabla_t \partial_s v|^2 + \partial_t\bigl(\omega(\partial_s v, \nabla_s \partial_s v)\bigr) - C|\partial_s v|^2 \bigl(|\partial_s v|^2 + |\nabla_t \partial_s v|\bigr) \Bigr) \\
&\geq \bigl(2 - C\kappa\bigr) \int_{[0,1]\sqcup[0,\delta]} |J\nabla_t \partial_s v(s,t)|^2 dt \; - C'\bigl(\kappa + \kappa^2\bigr) \|\partial_s v(s,\cdot)\|^2_{\mathcal{C}^0([0,1]\sqcup[0,\delta])}.
\end{aligned}$$

The last step uses $2|\partial_s v|^2 |\nabla_t \partial_s v| \leq \kappa |\partial_s v|^2 + \kappa |\nabla_t \partial_s v|^2$ and the claim

$$\left| \int_{[0,1]\sqcup[0,\delta]} \partial_t\bigl(\omega(\partial_s v, \nabla_s \partial_s v)\bigr) \right| \leq C\bigl(|\partial_s v_{02}(1)|^3 + |\partial_s \hat v(\delta)|^3\bigr).$$

To prove the claim we first use the diagonal boundary conditions to obtain

$$\left| \int_{[0,1]\sqcup[0,\delta]} \partial_t\bigl(\omega(\partial_s v, \nabla_s \partial_s v)\bigr) \right| = \bigl| \omega_{02}(\partial_s v_{02}, \nabla_s \partial_s v_{02})|_{t=1} + \omega_{02}(\partial_s \hat v, \nabla_s \partial_s \hat v)|_{t=\delta} \bigr|.$$

Then we use a smooth family of orthonormal frames $(\gamma_i)_{i=1,\ldots,k} \in \Gamma(T(L_0 \times L_2))$ near $w(s) := v_{02}(s,1)$ (and similarly for $\hat v$),

$$\partial_s w(s) = \sum \lambda^i(s)\gamma_i(w(s)), \quad \nabla_s \partial_s w(s) = \sum \Bigl(\partial_s \lambda^i(s) \gamma_i(w(s)) + \lambda^i(s) \nabla_{\partial_s w(s)} \gamma_i \Bigr)$$

with $\lambda : \mathbb{R} \to \mathbb{R}^k$. By the orthonormality we have $|\lambda(s)| = |\partial_s w(s)|$, and using the identities $\omega(\gamma_i, \gamma_j) = 0$ one obtains $\bigl|\omega(\partial_s w, \nabla_s \partial_s w)\bigr| \leq C|\partial_s w|^3$, where the constant C only depends on $\nabla \gamma_i$. Since L is compact this holds with a uniform constant.

We can now choose $\kappa > 0$ sufficiently small and then fix $\hbar \leq \min\{\epsilon_1, \epsilon_\kappa\}$ such that Step 1 and Step 2 (applied to time-shifts of v) together with the above calculation yield for all $s \geq 1$

$$f''(s) \geq \int_{[0,1]\sqcup[0,\delta]} |J\nabla_t \partial_s v(s,t)|^2 dt \geq ((1+\delta)C_1)^{-1} \int_{[0,1]\sqcup[0,\delta]} |\partial_s v(s,t)|^2 dt \geq \Delta^2 f(s)$$

with $\Delta > 0$. Any such nonnegative convex function satisfies for all $s \geq 2$ and $T \geq s$

$$f(s) \leq Ce^{-\Delta s}\left(\int_{[1,2]} f(t)dt + \int_{[2T,2T+1]} f(t)dt\right)$$

with a constant C that only depends on Δ. A detailed proof can be found in e.g. [14, Lemma 3.7] (use the estimate for $\hat{f}(s-T-1)$, where the function $\hat{f}$ is shifted by $T+1$). If we let $T \to \infty$ then $\int_{[2T,2T+1]} f(t)dt \to 0$ by the finite energy condition $\int_0^\infty f(s)ds < \hbar$, and this proves the claim.

Step 4: *There are constants $\epsilon_3 > 0$ and C_3 such that the following holds for all $\delta \in (0,1]$. If $v \in \widehat{\mathcal{M}}_\delta(x^-, x^+)$ satisfies (35) with $\hbar = \epsilon_3$, then*

$$\|\partial_s v\|_{\mathcal{C}^0_{1,\delta}([1,2])} \leq C_3 \|\partial_s v\|_{L^2_{1,\delta}([\frac{1}{2},\frac{5}{2}])}.$$

By contradiction we find sequences $\delta^\nu \in (0,1]$ and $v^\nu \in \widehat{\mathcal{M}}_{\delta^\nu}(x^-, x^+)$ that satisfy (37), but the assertion fails, i.e. we cannot find a constant C_3 for which the estimate is satisfied. Then as in Step 1 we obtain (for a subsequence) $\mathcal{C}^1$-convergence $v^\nu \to x = (x_{02}, \hat{x})$ on $[\frac{1}{2},\frac{5}{2}]\times([0,1]\sqcup[0,\delta^\nu])$ to constants $x_{02} \in L_0 \times L_2$, $x_1 \in M_1$ with $\hat{x} = (x_{02}, x_1, x_1) \in L_{01} \times L_{12}$. So we can find sections $\xi^\nu \in \Gamma_{1,\delta^\nu}$ over $u = x$ such that $v^\nu|_{s\in[\frac{1}{2},\frac{5}{2}]} = e_x(\xi^\nu)$. The equation $\overline{\partial}_J v^\nu$ then becomes

$$\nabla_s \xi^\nu + J(\xi^\nu)\nabla_t \xi^\nu = 0$$

and we have the boundary conditions $\nabla_s \xi^\nu_{02}|_{t=1} \in T_{x_{02}}(L_0 \times L_2)$ and $\nabla_s \hat{\xi}^\nu|_{t=\delta^\nu} \in T_{\hat{x}}(L_{01} \times L_{12})$. We fix two cutoff functions $h, \tilde{h} \in \mathcal{C}^\infty(\mathbb{R},[0,1])$ with $h|_{[1,2]} \equiv 1$, $\tilde{h}|_{\operatorname{supp} h} \equiv 1$ and $\operatorname{supp}(h), \operatorname{supp}(\tilde{h}) \subset (\frac{1}{2},\frac{5}{2})$ and consider the sections $h\xi^\nu, \tilde{h}\xi^\nu \in \Gamma_{1,\delta^\nu}$. Note that $\partial_s v^\nu = de_x(\xi^\nu)\nabla_s \xi^\nu$ with $de_x(\xi^\nu) \approx \mathrm{Id}$. So for sufficiently large ν we have

$$\|\partial_s v^\nu\|_{\mathcal{C}^0_{1,\delta^\nu}([1,2])} \leq 2\|h\nabla_s \xi^\nu\|_{\mathcal{C}^0_{1,\delta^\nu}} \leq 2C_S \|h\nabla_s \xi^\nu\|_{H^2_{1,\delta^\nu}},$$
$$\|\nabla_s \xi^\nu\|_{L^2_{1,\delta^\nu}([\frac{1}{2},\frac{5}{2}])} \leq 2\|\partial_s v^\nu\|_{L^2_{1,\delta^\nu}([\frac{1}{2},\frac{5}{2}])},$$

where we used Lemma 3.1.4. Now we apply Lemma 3.2.1 (b) to the sections $\xi = h\nabla_s \xi^\nu$ and $\xi = \tilde{h}\nabla_s \xi^\nu$ (for which the boundary terms vanish since $\nabla_s \xi^\nu, \nabla_s^2 \xi^\nu, \nabla_s^3 \xi^\nu$ satisfy the boundary conditions) and $\zeta = \xi^\nu$ (which satisfy $\|\xi^\nu\|_\infty \to 0$ and $\|\nabla \xi^\nu\|_\infty \to 0$) to obtain with uniform constants C, C'

$$\begin{aligned}
\|h\nabla_s \xi^\nu\|_{H^2_{1,\delta^\nu}} &\leq C_1\big(\|\big(\nabla_s + J(\xi^\nu)\nabla_t\big)h\nabla_s \xi^\nu\|_{H^1_{1,\delta^\nu}} + \|h\nabla_s \xi^\nu\|_{H^0_{1,\delta^\nu}}\big) \\
&= C_1\big(\|h'\nabla_s \xi^\nu\|_{H^1_{1,\delta^\nu}} + \|h\nabla_s \xi^\nu\|_{H^0_{1,\delta^\nu}}\big) \\
&\leq C\|\nabla_s \xi^\nu\|_{H^1_{1,\delta^\nu}(\operatorname{supp} h)} \;\leq\; C\|\tilde{h}\nabla_s \xi^\nu\|_{H^1_{1,\delta^\nu}} \\
&\leq CC_1\big(\|\big(\nabla_s + J(\xi^\nu)\nabla_t\big)\tilde{h}\nabla_s \xi^\nu\|_{H^0_{1,\delta^\nu}} + \|\tilde{h}\nabla_s \xi^\nu\|_{H^0_{1,\delta^\nu}}\big) \\
&\leq C'\|\nabla_s \xi^\nu\|_{H^0_{1,\delta^\nu}([\frac{1}{2},\frac{5}{2}]}.
\end{aligned}$$

Now the contradiction follows,

$$\|\partial_s v^\nu\|_{\mathcal{C}^0_{1,\delta^\nu}([1,2])} \leq 2\|h\nabla_s\xi^\nu\|_{H^2_{1,\delta^\nu}} \leq 2C'\|\nabla_s\xi^\nu\|_{H^0_{1,\delta^\nu}([\frac{1}{2},\frac{5}{2}])} \leq 4C'\|\partial_s v^\nu\|_{L^2_{1,\delta^\nu}([\frac{1}{2},\frac{5}{2}])}.$$

Step 5: *We prove the claim, that is for every $s \geq 3$*

$$d_{\mathcal{C}^0([0,1]\sqcup[0,\delta])}(v(s,\cdot),x^+)^2 + \|\partial_s v(s,\cdot)\|^2_{\mathcal{C}^0([0,1]\sqcup[0,\delta])} \leq Ce^{-\Delta s}E'(v)$$

with

$$E'(v) := \int_0^2 \int_{[0,1]\sqcup[0,\delta]} |\partial_s v(s,t)|^2 dtds.$$

We choose $\hbar = \min\{\epsilon_2, \epsilon_3\}$, then Step 3 and Step 4 (applied to appropriately shifted solutions) combine as follows for all $s \geq 3$

$$\begin{aligned}\|\partial_s v\|^2_{\mathcal{C}^0_{1,\delta}([s-\frac{1}{2},s+\frac{1}{2}])} &\leq C_3^2 \int_{s-1}^{s+1} \int_{[0,1]\sqcup[0,\delta]} |\partial_s v(s,t)|^2 dt \\ &\leq C_3^2 C_2 \int_{s-1}^{s+1} e^{-\Delta s}E'(v)ds \;\leq\; C_3^2 C_2 \Delta^{-1} e^{\Delta} e^{-\Delta s} E'(v).\end{aligned}$$

This proves the second part of the claim. The estimate on $d_{\mathcal{C}^0([0,1]\sqcup[0,\delta])}(v(S,\cdot),x^+)$ now simply follows by integration: For all $S \geq 3$ and $t \in [0,1]$

$$\begin{aligned} d_{M_{02}}(v_{02}(S,t),x^+) &\leq \int_S^\infty |\partial_s v_{02}(s,t)|ds \\ &\leq C\int_S^\infty e^{-\Delta s/2}\sqrt{E'(v)}ds \\ &= 2C\Delta^{-1}e^{-\Delta S/2}\sqrt{E'(v)}, \end{aligned}$$

and similarly for $\hat{v}$. □

3.3. **Compactness.** The surjectivity of the map $\mathcal{T}_\delta : \mathcal{M}^1_0(x^-,x^+) \to \mathcal{M}^1_\delta(x^-,x^+)$, as introduced in the previous section, will be a direct consequence of the following compactness result. Here we choose $\epsilon_0 \in (0,\epsilon]$ with $\epsilon > 0$ from in Theorem 3.1.1. Then $v = e_u(\xi)$ with $\xi \in \Gamma_{1,\delta}(\epsilon_0) \cap K_0$ implies that $[v_u] = \mathcal{T}_\delta([u])$ by the definition of $\mathcal{T}_\delta$ via theorem 3.1.1. We will denote the time-shift by $\tau^\sigma v(s,t) := v(\sigma+s,t)$.

Theorem 3.3.1. *Given $\epsilon_0 > 0$ there exists $\delta_0 > 0$ such that for every $\delta \in (0,\delta_0]$ and $v \in \widehat{\mathcal{M}}^1_\delta(x^-,x^+)$ there exist $u \in \widetilde{\mathcal{M}}^1_0(x^-,x^+)$ and $\sigma \in \mathbb{R}$ such that $\tau^\sigma v = e_u(\xi)$ with $\xi \in \Gamma_{1,\delta} \cap K_0$ and $\|\xi\|_{\Gamma_{1,\delta}} \leq \epsilon_0$. Moreover, the moduli space $\widehat{\mathcal{M}}^1_\delta(x^-,x^+)$ is regular for all $\delta \in (0,\delta_0]$ in the sense that the linearized operator D_v is surjective for every $v \in \widehat{\mathcal{M}}^1_\delta(x^-,x^+)$.*

Proof. We assume by contradiction that there is an $\epsilon_0 > 0$, a sequence $\delta^\nu \to 0$, and solutions $v^\nu = (v^\nu_{02}, \hat{v}^\nu) \in \widehat{\mathcal{M}}^1_{\delta^\nu}(x^-,x^+)$ for which the assertion of the theorem fails. Their energy is fixed, $E(v^\nu) = \frac{1}{2}\tau + \frac{1}{2}c(x_-,x_+)$, by the analogue of Proposition 2.1.1 for strips of different widths: For any pair of maps $(v_{02},\hat{v})$ that are not necessarily holomorphic but satisfy the limits and seam conditions of $\widehat{\mathcal{M}}^1_\delta(x^-,x^+)$ we have

$$\begin{aligned} E(v_{02},\hat{v}) &= \int v_{02}^*\big((-\omega_0)\oplus\omega_2\big) + \int \hat{v}^*\big(\omega_0 \oplus (-\omega_2) \oplus (-\omega_1) \oplus \omega_1\big) \\ &= \tfrac{1}{2}\tau \mathrm{Ind}(D_{(v_{02},\hat{v})}) + \tfrac{1}{2}c_\delta(x_-,x_+). \end{aligned} \tag{39}$$

Here $c_\delta(x_-,x_+)$ is independent of δ since the equations for different δ apply to the same map, rescaled to different widths, which has the same energy and index. Next, we can exclude bubbling by the following argument based on Lemma 3.3.2 below:

If $|dv^\nu_{02}|$ is unbounded near a point $z\in\mathbb{R}\times(0,1]$, then the standard rescaling method gives rise to a nontrivial holomorphic sphere or disk in (M_0,L_0), or in (M_2,L_2), or in both. Thus some fixed amount of energy $\hbar>0$ would have to concentrate near z. The same energy quantization holds for blowup of $d\hat{v}$ or $dv_{02}|_{t=0}$ by Lemma 3.3.2. So the energy densities $|dv^\nu|$ can only blow up at finitely many points. On the complement the same compactness proof as in the next paragraph provides a $\mathcal{C}^0_{\rm loc}$ convergent subsequence $v^\nu_{02}\to u_{02}$, where the limit corresponds to a solution $u\in\widetilde{\mathcal{M}}_0(y^-,y^+)$ with finitely many singularities and energy $E(u)<E(v^\nu)$. The singularities can be removed by the standard proofs for pseudoholomorphic curves with Lagrangian boundary condition [8, Theorem 4.1.2], so we would obtain a solution $\tilde{u}\in\widetilde{\mathcal{M}}_0(y^-,y^+)$ of energy $E(\tilde{u})<E(v^\nu)$. Its limits $y_\pm$ may not be the same as those of v^ν, in which case we find a sequence of trajectories $\underline{\tilde{u}}=(\tilde{u}_1,\ldots,\tilde{u}_N)\subset\widetilde{\mathcal{M}}_0(\cdot,\cdot)$ connecting x_- to x_+, with total energy $E(\underline{\tilde{u}})=\sum E(\tilde{u}_j)<E(v^\nu)$. We claim that monotonicity forces $\underline{\tilde{u}}$ to have total index $\sum \mathrm{Ind}(D_{\tilde{u}_j})<\mathrm{Ind}(D_{v^\nu})=1$, and hence by regularity of the moduli spaces $\widetilde{\mathcal{M}}_0(\cdot,\cdot)$ consists of a single constant trajectory. This however would mean that v^ν were self-connecting trajectories of $x_-\neq x_+$, i.e. we have annuli with $\mathrm{Ind}(D_{v^\nu})=1$ – in contradiction to assumption (d).

To control the index of $\underline{\tilde{u}}$ we glue the trajectories to a single map $\tilde{w}:\mathbb{R}\times[0,1]\to M_0^-\times M_2$ satisfying all limit and boundary conditions of $\widetilde{\mathcal{M}}_0(x^-,x^+)$ except for holomorphicity. Its index and energy coincide with the total energy and index of $\underline{\tilde{u}}$. With that we obtain

$$\tau\mathrm{Ind}(D_{\tilde{w}})+c(x_-,x_+)=2E(\tilde{w})<2E(v^\nu)=\tau+c(x_-,x_+)$$

from the monotonicity formula (39) together with the index and energy identities in Lemma 2.1.3 applied to $(\tilde{w},\hat{w})$, where $\hat{w}$ is the t-independent map given by the lift of $\tilde{w}|_{t=0}\subset L_{02}$ to $(L_{01}\times_{\Delta_1}L_{12})^T$. This proves $\sum\mathrm{Ind}(D_{\tilde{u}_j})\le 0$ as claimed and hence excludes bubbling.

So from now on we assume that $|dv^\nu|\le C_0$ is uniformly bounded. Then we have $d_{\mathcal{C}^0}\big(v^\nu_{02}|_{t=\delta^\nu},L_{02}\big)\to 0$ since as in Lemma 3.1.3 it is bounded by $d_{\mathcal{C}^0}\big(v'^\nu_1|_{t=\delta^\nu},v^\nu_1|_{t=\delta^\nu}\big)\le d_{\mathcal{C}^0}\big(\hat{v}^\nu|_{t=\delta^\nu},\hat{v}^\nu|_{t=0}\big)\le C_0\delta^\nu$. So we can fix $p>2$ and find a subsequence and map $u_{02}\in\mathcal{C}^0\cap W^{1,p}_{\rm loc}(\mathbb{R}\times[0,1],M_0\times M_2)$ such that $v^\nu_{02}\to u_{02}$ in the $\mathcal{C}^0$-topology and the weak $W^{1,p}$-topology on every compact subset of $\mathbb{R}\times[0,1]$. The limit u_{02} corresponds to a solution $(u_0,u_2)\in\widetilde{\mathcal{M}}^1_0(x^-,x^+)$. We also conclude that $\hat{v}\to\bar{u}=(u_{02}|_{t=0},\bar{u}_1,\bar{u}_1)$ in $\mathcal{C}^0([-T,T]\times[0,\delta^\nu])$ for all $T>0$, where $\bar{u}_1$ is determined uniquely by $(u_{02}|_{t=0},\bar{u}_1,\bar{u}_1)\in L_{01}\times L_{12}$. Indeed, $\hat{v}^\nu|_{t=0}=(v^\nu_{02},v^\nu_1,v^\nu_1)|_{t=0}$ satisfies $d_{\mathcal{C}^0}(\hat{v}^\nu|_{t=0},u_{02}\times\Delta_1)\to 0$ as well as $d_{\mathcal{C}^0}(\hat{v}^\nu|_{t=0},L_{01}\times L_{12})\le d_{\mathcal{C}^0}(\hat{v}^\nu|_{t=0},\hat{v}^\nu|_{t=\delta^\nu})\to 0$, so $v_1|_{t=0}$ must converge to $\bar{u}_1$ on compact sets, and the convergence for $t_0\in[0,\delta^\nu]$ follows from $d_{\mathcal{C}^0}(\hat{v}^\nu|_{t=0},\hat{v}^\nu|_{t=t_0})\le C_0\delta^\nu\to 0$.

In summary we have $v^\nu\to u:=(u_{02},\bar{u})$ in the $\mathcal{C}^0$-topology on every set $\{|s|\le T\}$ for fixed T. In the following, we will strengthen this convergence using uniform nonlinear estimates and exponential decay, to find sections $\xi^\nu\in\Gamma_{1,\delta^\nu}(\epsilon_0)$ such that $v^\nu=e_u(\xi^\nu)$ and D_{v^ν} is surjective in contradiction to the assumption. Let us first note that, by the same monotonicity arguments as above, the limit must be a nonbroken trajectory $u\in\widetilde{\mathcal{M}}^1_0(x^-,x^+)$ of the same index and energy $E(u)=E(v^\nu)$. In the next step we strengthen the local convergence.

For fixed $T>0$ and sufficiently large $\nu\ge\nu_0$ we can write $v^\nu|_{\{|s|\le T\}}=e_u(\xi^\nu)$ with a section $\xi^\nu\in\Gamma_{1,\delta^\nu}$ (extended smoothly to $\{|s|>T\}$). The extension of ξ^ν can be chosen such that $\|\xi^\nu\|_\infty \to 0$ and $\sup_\nu \|\nabla\xi^\nu\|_\infty < \infty$ follows from the $\mathcal{C}^0$-convergence and $\mathcal{C}^1$-boundedness of $v^\nu|_{\{|s|\leq T\}}$. For the latter note that $\nabla\xi^\nu = de_u(\xi^\nu)^{-1}\nabla v^\nu - \partial_1 e(u,\xi^\nu)\nabla u$, where ∇v^ν is uniformly bounded, and $de_u(\xi^\nu) \to \mathrm{Id}$ as $|\xi^\nu| \to 0$. This puts us into the position where Lemma 3.2.1 applies with $\zeta = \xi^\nu$. We fix a cutoff function $h \in \mathcal{C}_0^\infty([-T,T],[0,1])$ with $h|_{[-T+1,T-1]} \equiv 1$, then

$$\|h\xi^\nu\|_{H^1_{1,\delta}} \leq C_1\Big(\|\big(\nabla_s + J(\xi^\nu)\nabla_t\big)h\xi^\nu\|_{H^0_{1,\delta^\nu}} + \|h\xi^\nu\|_{H^0_{1,\delta^\nu}} + \|h\hat{\xi}^\nu|_{t=\delta^\nu}\|_{H^0(\mathbb{R})} + \|h\xi^\nu_{02}|_{t=1}\|_{H^0(\mathbb{R})}\Big).$$

Now we can use (29), $\overline{\partial}_J v^\nu = 0$, $\overline{\partial}_{J_{02}} u_{02} = 0$, and $\partial_t \bar{u} = 0$ to obtain

$$\begin{aligned}\big\|h\big(\nabla_s + \hat{J}(\hat{\xi}^\nu)\nabla_t\big)\hat{\xi}^\nu\big\|_{L^2(\mathbb{R}\times[0,\delta^\nu])} &= \big\|h\cdot de_{\bar{u}}(\hat{\xi}^\nu)^{-1}\big(\partial_1 e(\bar{u},\hat{\xi}^\nu)\partial_s\bar{u}\big)\big\|_{L^2([-T,T]\times[0,\delta^\nu])} \\ &\leq C\|\partial_s\bar{u}\|_{L^2([-T,T]\times[0,\delta^\nu])} \leq C\sqrt{\delta^\nu}\|\partial_s\bar{u}\|_{L^2([-T,T])},\end{aligned}$$

and furthermore, using the fact that $\partial_1 e(u_{02},0) = \mathrm{Id}$ commutes with $J(u_{02})$,

$$\begin{aligned}&\big\|h\big(\nabla_s + J_{02}(\xi^\nu_{02})\nabla_t\big)\xi^\nu_{02}\big\|_{L^2(\mathbb{R}\times[0,1])} \\ &= \big\|h\cdot de_{u_{02}}(\xi^\nu_{02})^{-1}\big(\partial_1 e(u_{02},\xi^\nu_{02})J(u_{02})\partial_t u_{02} - J_{02}(u_{02})\partial_1 e(u_{02},\xi^\nu_{02})\partial_t u_{02}\big)\big\|_{L^2(\mathbb{R}\times[0,1])} \\ &\leq C\|\xi^\nu_{02}\|_{L^2([-T,T]\times[0,1])}.\end{aligned}$$

Hence we have

$$\|\xi^\nu\|_{H^1_{1,\delta^\nu}(\{|s|\leq T-1\})} \leq C\Big(\sqrt{\delta^\nu} + \|\xi^\nu\|_{H^0_{1,\delta^\nu}(\{|s|\leq T\})} + \|h\hat{\xi}^\nu|_{t=\delta^\nu}\|_{H^0(\mathbb{R})} + \|h\xi^\nu_{02}|_{t=1}\|_{H^0(\mathbb{R})}\Big),$$

which converges to zero, and thus $v^\nu_{02} \to u_{02}$ in the H^1-norm on every compact set. Now we can verify the assumptions of Lemma 3.2.3 (with the constant $\hbar > 0$) and achieve uniform exponential decay: Pick $T > 0$ such that $\int_{[-T,T]\times[0,1]} |\partial_s u_{02}|^2 \geq E(u) - \frac{1}{2}\hbar$ and pick ν_0 such that for all $\nu \geq \nu_0$ we have $\|\partial_s u_{02}\|^2_{L^2([-T,T]\times[0,1])} - \|\partial_s v^\nu_{02}\|^2_{L^2([-T,T]\times[0,1])} \leq \frac{1}{2}\hbar$ and thus

$$\int_{\{|s|>T\}}\left(\int_{[0,1]} |\partial_s v^\nu_{02}|^2 + \int_{[0,\delta^\nu]} |\partial_s \hat{v}|^2\right) \;\leq\; E(v^\nu) + \tfrac{1}{2}\hbar - E(u) + \tfrac{1}{2}\hbar \;=\; \hbar.$$

Now the exponential decay Lemma 3.2.3 combined with the local $\mathcal{C}^0$-convergence implies that

$$d_{\mathcal{C}^0}(v^\nu_{02}, u_{02}) + d_{\mathcal{C}^0}(\hat{v}^\nu, \bar{u}) \to 0$$

uniformly for all s,t. Thus for sufficiently large ν we can write $v^\nu = e_u(\xi^\nu)$ with $\xi^\nu \in H^2_{1,\delta^\nu}$ and $\|\xi^\nu\|_\infty \to 0$. In fact, the uniform exponential decay implies global convergence,

$$\|\xi^\nu\|_\infty \to 0, \qquad \|\xi^\nu\|_{L^p_{1,\delta}} \to 0 \quad \forall p \geq 1, \qquad \|\nabla\xi^\nu\|_\infty \leq c_0 < \infty.$$

This puts us into the position where Lemma 3.2.1 and 3.2.2 apply with $\zeta = \xi^\nu$,

$$\begin{aligned}&\|\xi^\nu\|_{H^2_{1,\delta^\nu}} + \|\nabla\xi^\nu\|_{L^4_{1,\delta^\nu}} \\ &\leq C_1\Big(\|\nabla_s\xi^\nu + J(\xi^\nu)\nabla_t\xi^\nu\|_{H^1_{1,\delta^\nu}} + \|\nabla_s\xi^\nu + J(\xi^\nu)\nabla_t\xi^\nu\|_{L^4_{1,\delta^\nu}} \\ &\qquad + \|\xi^\nu\|_{H^0_{1,\delta^\nu}} + \|\hat{\xi}^\nu|_{t=\delta^\nu}\|_{H^1(\mathbb{R})} + \|\xi^\nu_{02}|_{t=1}\|_{H^1(\mathbb{R})}\Big) \\ &\leq C_1(1+C_2)\Big(\|\nabla_s\xi^\nu + J(\xi^\nu)\nabla_t\xi^\nu\|_{H^1_{1,\delta^\nu}} + \|\nabla_s\xi^\nu + J(\xi^\nu)\nabla_t\xi^\nu\|_{L^4_{1,\delta^\nu}} \\ &\qquad + \|\xi^\nu\|_{H^0_{1,\delta^\nu}} + \sqrt{\delta^\nu}\|\nabla_t\hat{\xi}^\nu\|_{H^1(\mathbb{R}\times[0,\delta^\nu])}\Big).\end{aligned}$$

The terms in the last line converge to zero or can be absorbed into the left hand side for δ^ν sufficiently small. We claim that the penultimate line also converges to zero and we thus obtain the convergence $\|\xi^\nu\|_{\Gamma_{1,\delta}} \to 0$. To check this we recall from (29) that $\overline{\partial}_J v^\nu = 0$ implies

$$\nabla_s \xi^\nu + J(\xi^\nu)\nabla_t \xi^\nu = -de_u(\xi^\nu)^{-1}\big(\partial_1 e(u,\xi^\nu)\partial_s u + J(u)\partial_1 e(u,\xi^\nu)\partial_t u\big). \tag{40}$$

Recall that

$$\partial_1 e(u,0) = \mathrm{Id}_{T_uM}, \qquad \partial_2 e(u,0) = de_u(0) = \mathrm{Id}_{T_uM}. \tag{41}$$

So in zeroth order we have, using the equations $\partial_t \bar{u} = 0$ and $\partial_s u_{02} = -J_{02}(u_{02})\partial_t u_{02}$,

$$\begin{aligned}
\big|\nabla_s \hat{\xi}^\nu + \hat{J}(\hat{\xi}^\nu)\nabla_t \hat{\xi}^\nu\big| &\le \big|de_{\bar{u}}(\hat{\xi}^\nu)^{-1}\big(\partial_1 e(\bar{u},\hat{\xi}^\nu)\partial_s \bar{u}\big)\big| \le C|\partial_s \bar{u}|, \\
\big|\nabla_s \xi_{02}^\nu + J_{02}(\xi_{02}^\nu)\nabla_t \xi_{02}^\nu\big| &\le \big|de_{u_{02}}(\xi_{02}^\nu)^{-1}\big(\partial_1 e(u_{02},\xi_{02}^\nu)J_{02}(u_{02}) \\
&\qquad - J_{02}(u_{02})\partial_1 e(u_{02},\xi_{02}^\nu)\big)\partial_t u_{02}\big| \;\le\; C|\xi_{02}^\nu|,
\end{aligned}$$

and thus

$$\begin{aligned}
&\|\nabla_s \xi^\nu + J(\xi^\nu)\nabla_t \xi^\nu\|_{L^2_{1,\delta^\nu}} + \|\nabla_s \xi^\nu + J(\xi^\nu)\nabla_t \xi^\nu\|_{L^4_{1,\delta^\nu}} \\
&\le C\big(\|\xi_{02}^\nu\|_{L^2(\mathbb{R}\times[0,1])} + \|\xi_{02}^\nu\|_{L^4(\mathbb{R}\times[0,1])} + (\delta^\nu)^{1/2}\|\partial_s \bar{u}\|_{L^2(\mathbb{R})} + (\delta^\nu)^{1/4}\|\partial_s \bar{u}\|_{L^4(\mathbb{R})}\big) \to 0.
\end{aligned}$$

For the first derivative we calculate from (40), denoting all uniform constants by C,

$$\begin{aligned}
\big|\nabla\big(\nabla_s \hat{\xi}^\nu + \hat{J}(\hat{\xi}^\nu)\nabla_t \hat{\xi}^\nu\big)\big| &\le C(1+|\nabla\hat{\xi}^\nu|)\big|\partial_1 e(\bar{u},\hat{\xi}^\nu)\partial_s \bar{u}\big| + C\big|\nabla\big(\partial_1 e(\bar{u},\hat{\xi}^\nu)\partial_s \bar{u}\big)\big| \\
&\le C\big(1+|\nabla\hat{\xi}^\nu|\big)\big(|\partial_s \bar{u}| + |\nabla_s \partial_s \bar{u}|\big),
\end{aligned}$$

and (in between dropping the subscript from ξ_{02}^ν)

$$\begin{aligned}
\big|\nabla\big(\nabla_s \xi_{02}^\nu + J_{02}(\xi_{02}^\nu)\nabla_t \xi_{02}^\nu\big)\big| &\le C(1+|\nabla\xi^\nu|)\big|\partial_1 e(u,\xi^\nu)J(u)\partial_t u - J(u)\partial_1 e(u,\xi^\nu)\partial_t u\big| \\
&\quad + C\big|\nabla\big(\partial_1 e(u,\xi^\nu)J(u) - J(u)\partial_1 e(u,\xi^\nu)\big)\big| \cdot |\partial_t u| \\
&\quad + C\big|\partial_1 e(u,\xi^\nu)J(u) - J(u)\partial_1 e(u,\xi^\nu)\big| \cdot |\nabla\partial_t u| \\
&\le C|\xi_{02}^\nu|\big(1+|\nabla\xi_{02}^\nu|\big).
\end{aligned}$$

Here the estimate for the second summand follows from (41) and the identity

$$\nabla_s(\partial_1 e(u,\xi)X) = \partial_1 e(u,\xi)\nabla_s X + (\nabla_{(\partial_s u,\nabla_s \xi)}\partial_1 e)(u,\xi)X$$

(and similarly for $\nabla_t(\partial_1 e(u,\xi)X)$), where we have $\big(\nabla_{(\partial_s u,\nabla_s\xi)}\partial_1 e\big)(u,0) = 0$ since

$$\big(\nabla_{(Y,0)}\partial_1 e\big)(u,0) = \nabla_Y \mathrm{Id}_{T_uM} = 0$$

and, calculating in local normal coordinates with an extension $\tilde{Y} \in \Gamma(TM)$ of $Y \in T_uM$ that is covariantly constant along $\tau \mapsto \exp_u(\tau X)$,

$$\big(\nabla_{(0,Y)}\partial_1 e\big)(u,0)X = \partial_\sigma|_{\sigma=0}\partial_\tau|_{\tau=0} e(\exp_u(\tau X),\sigma Y) = \partial_\tau|_{\tau=0}\tilde{Y}(\exp_u(\tau X)) = 0.$$

Now the uniform estimate $\|\nabla\xi^\nu\|_\infty \le c_0$ and the exponential decay of $\bar{u} = \bar{u}(s)$ imply

$$\big\|\nabla\big(\nabla_s \xi^\nu + J(\xi^\nu)\nabla_t \xi^\nu\big)\big\|_{L^2_{1,\delta^\nu}} \le C(1+c_0)\big(\|\xi_{02}^\nu\|_{L^2(\mathbb{R}\times[0,1])} + (\delta^\nu)^{1/2}\|\partial_s \bar{u}\|_{H^1(\mathbb{R})}\big) \to 0.$$

This proves

$$\|\xi^\nu\|_{\Gamma_{1,\delta^\nu}} \to 0.$$

It remains to find a time-shift such that $\tau^\sigma v^\nu = e_u(\xi^\nu(\sigma))$ with some $\xi^\nu(\sigma) \in K_0$ but still $\|\xi^\nu(\sigma)\|_{\Gamma_{1,\delta^\nu}} \leq \epsilon_0$. In order to find this shift we write $\tau^\sigma v^\nu = e_u(\xi^\nu(\sigma))$ with

$$\xi^\nu(\sigma) := \bigl(e_u^{-1} \circ \tau^\sigma \circ e_u\bigr)(\xi^\nu) \in \Gamma_{1,\delta^\nu}. \tag{42}$$

This will satisfy

$$\|\xi^\nu(\sigma)\|_{\Gamma_{1,\delta^\nu}} \leq C\bigl(\|\xi^\nu\|_{\Gamma_{1,\delta^\nu}} + |\sigma|\|du\|_{\Gamma_{1,\delta^\nu}}\bigr),$$

so it is well defined whenever $|\sigma| \leq \sigma_0$, where we fixed $\sigma_0 = \frac{1}{2}\epsilon_0 C^{-1}\|du\|^{-1}_{\Gamma_{1,\delta^\nu}}$ such that $\|\xi^\nu(\sigma)\|_{\Gamma_{1,\delta^\nu}} \leq \epsilon_0$ is ensured for sufficiently large $\nu \geq \nu_0$. The L^2-estimate on $\xi^\nu(\sigma)$ can be seen from the pointwise estimate

$$\begin{aligned}
\bigl|e_u^{-1}\tau^\sigma e_u(\xi)\bigr| &\leq \bigl|e_u^{-1}\tau^\sigma e_u(\xi) - e_u^{-1}\tau^\sigma e_u(0)\bigr| + \bigl|e_u^{-1}\tau^\sigma u - e_u^{-1}u\bigr| \\
&\leq C\bigl(d\bigl(\tau^\sigma e_u(\xi), \tau^\sigma e_u(0)\bigr) + d\bigl(\tau^\sigma u, u\bigr)\bigr) \\
&\leq C\bigl(\bigl|\tau^\sigma \xi\bigr| + \sigma|\partial_s u|\bigr).
\end{aligned}$$

Here C is a continuity constant for e_u^{-1}. The higher derivatives of $\xi(\sigma) = e_u^{-1}\tau^\sigma e_u(\xi)$ are estimated similarly. Now consider the function

$$\Theta^\nu(\sigma) := \langle \xi^\nu_{02}(\sigma), \partial_s u_{02}\rangle_{L^2}.$$

It satisfies

$$|\Theta^\nu(0)| \leq \|\partial_s u\|_{L^2_{1,\delta^\nu}} \|\xi^\nu\|_{L^2_{1,\delta^\nu}} \to 0$$

and (dropping the 02-subscript) we obtain from (42)

$$\begin{aligned}
&\Bigl|\tfrac{\partial}{\partial\sigma}\Theta^\nu(\sigma) - \|\partial_s u\|^2_{L^2}\Bigr| \\
&= \Bigl|\langle \bigl(de_u(\xi^\nu(\sigma))^{-1}\tau^\sigma\bigl(\partial_1 e(u,\xi^\nu)\partial_s u + de_u(\xi^\nu)\partial_s\xi^\nu\bigr) - \tau^\sigma\partial_s u\bigr), \partial_s u\rangle_{L^2} \\
&\qquad\qquad + \langle\bigl(\tau^\sigma\partial_s u - \partial_s u\bigr), \partial_s u\rangle_{L^2}\Bigr| \\
&\leq C\bigl(\|\xi^\nu\|_{H^1}\|\partial_s u\|_{L^2} + \|\xi^\nu\|_\infty\|\partial_s u\|^2_{L^2} + |\sigma|\|\nabla_s\partial_s u\|_{L^2}\|\partial_s u\|_{L^2}\bigr).
\end{aligned}$$

The latter is an arbitrarily small error for large ν and small σ. Hence we will find solutions $\sigma^\nu \sim -\Theta^\nu(0)/\|\partial_s u_{02}\|^2_{L^2} \in [-\sigma_0, \sigma_0]$ of $\Theta^\nu(\sigma^\nu) = 0$. With these we have $\tau^{\sigma^\nu}v^\nu = e_u(\xi^\nu(\sigma))$, where $\xi^\nu \in K_0 = \bigl\{\xi \in \Gamma_{1,\delta}\bigl|\langle\xi_{02}, \partial_s u_{02}\rangle_{L^2} = 0\bigr\}$ and $\|\xi^\nu(\sigma)\|_{\Gamma_{1,\delta^\nu}} \leq \epsilon_0$. So with this small time-shift on v^ν we obtain a contradiction to the assumption that $\mathcal{T}^\nu_\delta$ is not surjective.

Finally, to prove the transversality we need to check that $D_{v^\nu} = D_{e_u(\xi^\nu)}$ is surjective. (The same then holds for the time shifts $\tau^{\sigma^\nu}v^\nu$.) This follows from the quadratic estimate in Lemma 3.1.5 : Let $Q : \Omega_{1,\delta^\nu} \to \Gamma_{1,\delta^\nu}$ be the right inverse of $D^\delta = d\mathcal{F}_u(0)$, then

$$\begin{aligned}
\|\Phi_u(\xi^\nu)^{-1}D_{e_u(\xi^\nu)}E_u(\xi^\nu)Q - \mathrm{Id}\| &\leq \|\Phi_u(\xi^\nu)^{-1}D_{e_u(\xi^\nu)}E_u(\xi^\nu) - d\mathcal{F}_u(0)\|\cdot\|Q\| \\
&\leq 2C_2\|Q\|\|\xi^\nu\|_{\Gamma_{1,\delta^\nu}},
\end{aligned}$$

where $\|Q\| < \infty$ by (28) and $\|\xi^\nu\|_{\Gamma_{1,\delta^\nu}} \to 0$. This shows that $\Phi_u(\xi^\nu)^{-1}D_{e_u(\xi^\nu)}E_u(\xi^\nu)Q$ and hence also the operator $\Phi_u(\xi^\nu)^{-1}D_{e_u(\xi^\nu)}E_u(\xi^\nu)$ has a right inverse for all sufficiently large $\nu \geq \nu_0$. Here the parallel transport $\Phi_u(\xi^\nu)$ is an isomorphism on the target and $E_u(\xi^\nu)$ identifies $\Gamma_{1,\delta}$ with the domain of $D_{e_u(\xi^\nu)}$. For the latter see the discussion before Lemma 3.1.5 and recall that $E_u(0) = \mathrm{Id}$. So we have established that D_{v^ν} is surjective, and this finishes the proof. □

Lemma 3.3.2. *There exists a universal constant $\hbar > 0$ such that the following holds for any sequence of Floer trajectories $v^\nu \in \widehat{\mathcal{M}}_{\delta^\nu}(x^+,x^-)$ with $\delta^\nu \to 0$. If for some $s \in \mathbb{R}$*

$$\liminf_{\nu\to\infty}\bigl(\|dv_{02}^\nu\|_{L^\infty(B_\epsilon(s,0))} + \|d\hat{v}^\nu\|_{L^\infty(B_\epsilon(s,0))}\bigr) = \infty \qquad \forall \epsilon > 0,$$

then there exists a sequence $\epsilon^\nu \to 0$ such that

$$\liminf_{\nu\to\infty}\left(\int_{B_{\epsilon^\nu}(s,0)} |dv_{02}^\nu|^2 + \int_{B_{\epsilon^\nu}(s,0)} |d\hat{v}^\nu|^2\right) \geq \hbar.$$

Here $B_\epsilon(s,0)$ is the ϵ-ball in $\mathbb{R}\times[0,1]$ or $\mathbb{R}\times[0,\delta^\nu]$ respectively.

In the usual analysis of bubbling effects, one would prove this lemma by rescaling around points where the differentials blow up, identifying the limits with holomorphic spheres or disks, and hence obtaining an energy quantization constant $\hbar$ that is geometrically determined by the minimal nonzero energy of spheres or disks. In the present case however, depending on the relative speed of blow-up and strip-shrinking $\delta^\nu \to 0$, the rescaling may lead to sphere bubbles in M_0, M_1, or M_2, disk bubbles in $(M_0\times M_1, L_{01})$, $(M_1 \times M_2, L_{12})$, or $(M_0 \times M_2, L_{01}\circ L_{12})$, or the novel *figure eight bubble* described in the introduction. Since we do not have a geometric bound on the minimal energy of figure eight bubbles, we use a mean value inequality to obtain $\hbar$ by purely analytic methods.

Proof of Lemma 3.3.2. For notational convenience we introduce the noncontinuous function $|dv| : \mathbb{R}\times[0,1] \to [0,\infty)$ given by $|dv(s,t)|^2 = |dv_{02}(s,t)|^2 + |d\hat{v}(s,t)|^2$ for $t \in [0,\delta]$ and $|dv(s,t)| = |dv_{02}(s,t)|$ for $t \in (\delta,1]$.

Suppose the lemma is false, that is, for every $k \in \mathbb{N}$ there exists a sequence $v^{k,\nu} \in \widehat{\mathcal{M}}_{\delta^{k,\nu}}(x^+,x^-)$ with $\delta^{k,\nu} \to 0$ such that (after time shift to $s=0$) $R_k^\nu := |dv^{k,\nu}(s_k^\nu,t_k^\nu)| \to \infty$ for some $(s_k^\nu,t_k^\nu) \to (0,0)$, but

$$\liminf_{\nu\to\infty}\int_{B_{\epsilon^\nu}(0)} |dv^{\nu,k}|^2 \leq \frac{1}{k}.$$

for every sequence $\epsilon^\nu \to 0$. In particular, this will hold for a fixed sequence $\epsilon_k^\nu \to 0$ that satisfies in addition $\epsilon_k^\nu \geq \delta_k^\nu$, $(s_k^\nu,t_k^\nu) \in B_{\frac14\epsilon_k^\nu}(0)$ and $\epsilon_k^\nu R_k^\nu \to \infty$. We can then find diagonal sequences $v^k \in \widehat{\mathcal{M}}_{\delta_k}(x^+,x^-)$ with $\delta_k \to 0$, and $\epsilon_k \to 0$, $(s_k,t_k) \in B_{\frac14\epsilon_k}(0)$ such that $\epsilon_k R_k := \epsilon_k|dv^k(s_k,t_k)| \to \infty$ and

$$\int_{B_{\epsilon_k}(0)} |dv^k|^2 \to 0. \tag{43}$$

Next, we use Lemma 3.3.3 to refine the choice of the blowup points (s_k,t_k). For that purpose we consider the spaces $X_{02} = \mathbb{R}\times[0,1]$, $\hat{X} = \mathbb{R}\times[0,\delta_k]$, and $X = \mathbb{R}\times[0,1]$, with the obvious inclusion $\pi : X_{02}\cup\hat{X} \to X$. Using the function $f = |dv_{02}^k|$ on X_{02} and $f = |d\hat{v}^k|$ on $\hat{X}$ one can then vary the point $\pi(x) = (s_k,t_k) \in \mathbb{R}\times[0,1]$ by $2\rho = \frac14\epsilon_k$ to find $(s_k,t_k) \in B_{\frac12\epsilon_k}(0)$ and $\epsilon_k' \leq \frac18\epsilon_k$, such that $\epsilon_k' R_k := \epsilon_k'|dv^k(s_k,t_k)| \to \infty$ and $|dv^k| \leq 4R_k$ on $B_{\epsilon_k'}(s_k,t_k)$. Here (43) continues to hold on $B_{\epsilon_k(0)} \supset B_{\epsilon_k'(s_k,t_k)}$.

Now in a first step we will prove that figure eight bubbles (arising from rescaling in the case $\delta_k R_k \to \Delta \in (0,\infty)$) have a minimal energy (possibly depending on $\Delta > 0$.) More precisely, we claim that (43) implies

$$t_k R_k \to 0, \qquad \text{and} \qquad \delta_k R_k \to 0. \tag{44}$$

In a second step we will then see that this gives rise to a disk bubble in $(M_0\times M_2, L_{01}\circ L_{12})$.

Step 1: *We prove (44).*

First consider the case $|dv_{02}^k(s_k,t_k)|\geq\frac12|dv^k(s_k,t_k)|$ and $t_k\geq\frac12\delta_k$. Then for all sufficiently large k we can apply the mean value inequality [8, Lemma 4.3.1] to $|dv_{02}^k|$ on the ball $B_{r_k}(s_k,t_k)\subset\mathbb{R}\times(0,1)\cap B_{\epsilon_k}(0)$ with $r_k:=\min\{t_k,\epsilon_k'\}$,

$$\tfrac14(r_kR_k)^2\leq r_k^2|dv_{02}^k(s_k,t_k)|^2\leq c\int_{B_{r_k}(s_k,t_k)}|dv_{02}^k|^2\to 0.$$

Here we cannot have $r_k=\epsilon_k'$ since $\epsilon_k'R_k\to\infty$, so we have $r_k=t_k$ and thus $\frac12\delta_kR_k\leq t_kR_k\to 0$ as claimed.

In the case $|d\hat v^k(s_k,t_k)|\geq\frac12|dv^k(s_k,t_k)|$ and $\delta_k\geq t_k\geq\frac12\delta_k$ we can apply the mean value inequality [19, Theorem 1.3, Lemma A.1] to $|d\hat v^k|$ with boundary condition $\hat v^k|_{t=\delta_k}\in L_{01}\times L_{12}$ on the partial ball $B_{r_k(s_k,t_k)}\subset\mathbb{R}\times(0,\delta_k]\cap B_{\epsilon_k}(0)$ for $r_k:=\min\{\frac12\delta_k,\epsilon_k'\}$,

$$\tfrac14(r_kR_k)^2\leq r_k^2|d\hat v^k(s_k,t_k)|^2\leq c\int_{B_{r_k}(s_k,t_k)}|d\hat v^k|^2\to 0.$$

As before we cannot have $r_k=\epsilon_k'$ since $\epsilon_k'R_k\to\infty$, so we have $r_k=\frac12\delta_k$ and thus $t_kR_k\leq\delta_kR_k\to 0$ as claimed.

In the remaining case $t_k\leq\frac12\delta_k$ we consider the holomorphic curve

$$w^k:=(v_{02}^k,\hat v^k):\mathbb{R}\times[0,\delta_k]\to M_0\times M_2\times M_0\times M_2\times M_1\times M_1,$$

which satisfies the Lagrangian boundary condition $w^k|_{t=0}\in\Delta_0\times\Delta_2\times\Delta_1$. By the above we have $|dw^k(s_k,t_k)|\geq R_k\to\infty$ and $\int_{B_{\epsilon_k}(0)}|dw^k|^2\to 0$. So for all sufficiently large k we can apply the mean value inequality [19, Theorem 1.3, Lemma A.1] on the partial ball $B_{r_k(s_k,t_k)}\subset\mathbb{R}\times[0,\delta_k)\cap B_{\epsilon_k}(0)$ for $r_k:=\min\{\frac12\delta_k,\epsilon_k'\}$,

$$(r_kR_k)^2\leq r_k^2|dw^k(s_k,t_k)|^2\leq c\int_{B_{r_k}(s_k,t_k)}|dw^k|^2\to 0.$$

Again we cannot have $r_k=\epsilon_k'$ since $\epsilon_k'R_k\to\infty$, so we have $r_k=\frac12\delta_k$ and thus $2t_kR_k\leq\delta_kR_k\to 0$ as claimed.

Step 2: *We prove the lemma.*

We consider the rescaled maps $w^k=(w_{02}^k,\hat w^k)$, where $w_{02}^k:B_{\epsilon_kR_k}(0)\cap\mathbb{H}^2\to M_0\times M_2$ is defined on half balls of radius $\epsilon_kR_k\to\infty$ in the half space $\mathbb{H}^2:=\mathbb{R}\times[0,\infty)$ by $w_{02}^k(s,t):=v_{02}^k(s_k+s/R^k,t/R^k)$, and $\hat w^k:B_{\epsilon_kR_k}(0)\cap(\mathbb{R}\times[0,\delta_kR_k])\to M_0\times M_2\times M_1\times M_1$ is defined by $\hat w^k(s,t):=\hat v^k(s_k+s/R^k,t/R^k)$ on balls of radius ϵ_kR_k intersected with the strip of width $\delta_kR_k\to 0$.

This rescaling preserves the nontriviality $|dw^k(0,t_kR_k)|\geq 1$, but on both domains $|dw^k|$ is uniformly bounded. Hence we can find a subsequence of the w_{02}^k that converges in the $\mathcal{C}^0$-topology on the unit half ball $D_1:=B_1(0)\cap\mathbb{H}^2$. The (scaling invariant) energy $\int_{B_{\epsilon_kR_k}(0)}|dw_{02}^k|^2$ converges to zero by (43), so the limit has to be constant. In fact, we have $w_{02}^k\to x_{02}\in L_{02}$ since the boundary values $w_{02}^k|_{t=0}$ converge to $L_{01}\circ L_{12}=L_{02}$ in $\mathcal{C}^0([-1,1])$. To see the latter use the transversality of the Lagrangians as in Lemma 3.1.3 and integrate the bound on $|\partial_t\hat w^k|$ to obtain

$$d\bigl(\hat w^k(s,0),\hat w^k(s,\delta_k)\bigr)\leq\int_0^{\delta_k}|\partial_t\hat w^k(s,t)|dt\leq\delta_k2R_k\to 0.$$

This also proves that $\hat{w}^k \to x_1$ in $\mathcal{C}^0([-1,1]\times[0,\delta_k R_k])$, where $x_1 \in M_1$ is uniquely determined by $\bar{x} := (x_{02}, x_1, x_1) \in L_{01}\times L_{12}$. The maps w_{02}^k are $\bar{J}_{02}$-holomorphic, so by elliptic regularity the convergence $w_{02}^k \to x_{02}$ is in the $\mathcal{C}^\infty$-topology on every compact subset of $\mathbb{H}^2 \setminus \partial\mathbb{H}^2$. However, in order to obtain a contradiction to the fact that $|dw^k(0,t_kR_k)| \geq 1$ with $t_kR_k \to 0$ we need to establish $\mathcal{C}^1$-convergence on D_1 up to the boundary.

We begin by noting that due to the $\mathcal{C}^0$-convergence we can express $w^k = e_x(\xi^k)$ in terms of sections $\xi^k = (\xi_{02}^k, \hat{\xi}^k) \in H^2(D_1, x_{02}^* T(M_0\times M_2)) \times H^2([0,1]\times[0,\delta_kR_k], \bar{x}^*T(M_0\times M_2\times M_1\times M_1))$ using the exponential map centered at $x = (x_{02},\bar{x})$. These sections satisfy the diagonal and Lagrangian boundary conditions $\xi^k|_{t=0} \in T_x(\Delta_0\times\Delta_2\times\Delta_1)$ and $\hat{\xi}^k|_{t=\delta_kR_k} \in T_{\bar{x}}(L_{01}\times L_{12})$, the $\mathcal{C}^0$-convergence $\|\xi^k\|_\infty \to 0$, and a uniform bound $\|\nabla\xi^k\|_\infty \leq c_0$. Since $\overline{\partial}_J w^k = 0$ and $\nabla x = 0$ we obtain from (29)

$$\nabla_s\xi^k + J(\xi^k)\nabla_t\xi^k = 0.$$

Now $dw^k = de_x(\xi^k)\nabla_s\xi^k ds + de_x(\xi^k)J(\xi^k)\nabla_s\xi^k dt$, so it suffices to prove the $\mathcal{C}^0$-convergence of $\nabla_s\xi^k$ near 0. For that purpose we multiply the sections by cutoff functions $h = (h_{02}, \hat{h})$ with $h_{02} : \mathbb{R}\times[0,1]\to[0,1]$ supported in D_1, $\hat{h}:\mathbb{R}\to[0,1]$ supported in $[-1,1]$, and both equal to 1 near 0. Then we obtain sections on the multistrip $h\xi^k := (h_{02}\xi_{02}^k, \hat{h}\hat{\xi}^k) \in \Gamma_{1,\delta_kR_k}$ that also satisfy the boundary condition $h_{02}\xi_{02}^k|_{t=1} = 0$. These satisfy a uniform bound

$$\sup_k\Big(\|\nabla_s(h\xi^k) + J(\xi^k)\nabla_t(h\xi^k)\|_{H^1_{1,\delta_kR_k}} + \|h\xi^k\|_{H^0_{1,\delta_kR_k}}\Big) \leq \sup_k C\|\xi^k\|_{H^1_{1,\delta_kR_k}(\operatorname{supp}(h))} < \infty$$

due to the bounds on $\|\xi^k\|_\infty$ and $\|\nabla\xi^k\|_\infty$ and the compact support of h. From this Lemma 3.2.1 (b) provides a uniform bound

$$\sup_k \|h\xi^k\|_{H^2_{1,\delta_kR_k}} \leq C_\Gamma < \infty.$$

Indeed, the boundary terms vanish since the constant boundary conditions directly transfer to the derivatives, $\nabla_s\xi_{02}^k|_{t=1}, \nabla_s^2\xi_{02}^k|_{t=1} \in T_{x_{02}}(L_0\times L_2)$ and $\nabla_s\hat{\xi}^k|_{t=\delta_kR_k}, \nabla_s^2\hat{\xi}^k|_{t=\delta_kR_k} \in T_{\hat{x}}(L_{01}\times L_{12})$.

We now fix a pair of cutoff functions h' with support in $h^{-1}(1)$ and still equal to 1 near 0. Then we apply Lemma 3.2.1 (b) to $h'\nabla_s\xi^k$, again with vanishing boundary terms, to obtain

$$\begin{aligned}\sup_k \|h'\nabla_s\xi^k\|_{H^2_{1,\delta_kR_k}} &\leq \sup_k C_1\Big(\big\|\big(\nabla_s + J(\xi^k)\nabla_t\big)h'\nabla_s\xi^k\big\|_{H^1_{1,\delta_kR_k}} + \|h'\nabla_s\xi^k\|_{H^0_{1,\delta_kR_k}}\Big)\\ &\leq \sup_k C(1+c_0)\|h\xi^k\|_{H^2_{1,\delta_kR_k}} < \infty.\end{aligned}$$

We can pick the cutoff functions such that $h'_{02}|_{D_{1/2}} \equiv 1$ on the half ball $D_{1/2}\subset\mathbb{H}^2$ and $\hat{h}|_{[-\frac12,\frac12]} \equiv 1$. Then the compact Sobolev embedding $H^2(D_{1/2}) \hookrightarrow \mathcal{C}^0(D_{1/2})$ provides $\mathcal{C}^0$-convergence of a subsequence $\nabla_s\xi_{02}^k$. We already know that the limit is 0, so we obtain $\nabla_s\xi_{02}^k \to 0$ and $\partial_s w_{02}^k \to 0$ in $\mathcal{C}^0(D_{1/2})$. It remains to establish $\|\nabla_s\hat{\xi}^k\|_{\mathcal{C}^0([-\frac12,\frac12]\times[0,\delta_kR_k])} \to 0$ and thus $\|\partial_s\hat{w}^k\|_{\mathcal{C}^0([-\frac12,\frac12]\times[0,\delta_kR_k])} \to 0$ in contradiction to $|dw^k(0,t_kR_k)| \geq 1$ with $t_kR_k \to 0$. To see this we follow the argument in Lemma 3.1.4. Using the standard Sobolev embedding

$H^1([-\frac12,\frac12]) \hookrightarrow \mathcal{C}^0([-\frac12,\frac12])$ we obtain for all $t_0 \in [0,\delta_k R_k]$

$$\begin{aligned} \tfrac{1}{C}\|\nabla_s\hat\xi^k|_{t=t_0} - \nabla_s\hat\xi^k|_{t=\delta_k R_k}\|^2_{\mathcal{C}^0([-\frac12,\frac12])} &\le \|\nabla_s\hat\xi^k|_{t=t_0} - \nabla_s\hat\xi^k|_{t=\delta_k R_k}\|^2_{H^1([-\frac12,\frac12])} \\ &\le \delta_k R_k \int_0^{\delta_k R_k} \|\nabla_t\nabla_s\hat\xi^k\|^2_{H^1([-\frac12,\frac12])} \\ &\le \delta_k R_k \|\nabla_s\hat\xi^k\|^2_{H^2([-\frac12,\frac12]\times[0,\delta_k R_k])} \to 0. \end{aligned} \tag{45}$$

From the above we moreover have $\|\nabla_s\xi'^k_{02}|_{t=0}\|_{\mathcal{C}^0([-\frac12,\frac12])} = \|\nabla_s\xi^k_{02}|_{t=0}\|_{\mathcal{C}^0([-\frac12,\frac12])} \to 0$. Now using Lemma 3.1.3 and the boundary conditions, in particular $(\xi^k_1 - \xi'^k_1)|_{t=0} = 0$, we obtain

$$\begin{aligned} &\|\nabla_s\hat\xi^k|_{t=\delta_k R_k}\|_{\mathcal{C}^0([-\frac12,\frac12])} \\ &\le C\big(\|\pi_{02}(\nabla_s\hat\xi^k)|_{t=\delta_k R_k}\|_{\mathcal{C}^0([-\frac12,\frac12])} + \|\nabla_s(\xi^k_1 - \xi'^k_1)|_{t=\delta_k R_k}\|_{\mathcal{C}^0([-\frac12,\frac12])}\big) \\ &\le C\big(\|\nabla_s\xi'^k_{02}|_{t=0}\|_{\mathcal{C}^0([-\frac12,\frac12])} + 3\|\nabla_s\hat\xi^k|_{t=\delta_k R_k} - \nabla_s\hat\xi^k|_{t=0}\|_{\mathcal{C}^0([-\frac12,\frac12])}\big) \to 0. \end{aligned}$$

Combining $\|\nabla_s\hat\xi^k|_{t=\delta_k R_k}\|_{\mathcal{C}^0([-\frac12,\frac12])} \to 0$ with (45) then proves $\|\nabla_s\hat\xi^k\|_{\mathcal{C}^0([-\frac12,\frac12]\times[0,\delta_k R_k])} \to 0$ and thus $|dw^k(0,t_kR_k)| \to 0$ in contradiction to the assumption. □

Lemma 3.3.3. *Let (X,d) be a metric space, $X_1,\ldots,X_n$ topological spaces, $\pi : X_1\cup\ldots\cup X_n \to X$ a continuous map, and $f : X_1\cup\ldots X_n \to \mathbb{R}$ a non-negative continuous function. Fix $x\in X_i$ for some $i=1,\ldots,n$ and $\rho>0$. Suppose that $\pi^{-1}(B_{2\rho}(\pi(x)))\cap X_i$ is complete for each $i=1,\ldots,n$. Then there exists an $x'\in X_1\cup\ldots X_n$ and a positive number $\rho'\le\rho$ such that*

$$d(\pi(x'),\pi(x)) < 2\rho, \qquad \sup_{\pi^{-1}B_{\rho'}(\pi(x'))} f \le 2f(x'), \qquad \rho' f(x') \ge \rho f(x).$$

Proof. Otherwise, the same argument as in the proof of Hofer's lemma [8, p.93] shows that there exists a sequence $x_\alpha \in X_1\cup\ldots\cup X_n$ such that

$$x_0 = x, \quad d(\pi(x_\alpha),\pi(x_{\alpha+1})) \le \rho/2^\alpha, \quad f(x_{\alpha+1}) > 2f(x_\alpha).$$

After passing to a subsequence, we obtain a Cauchy sequence x_α in some X_i with $f(x_\alpha)\to\infty$, which contradicts completeness and continuity of f. □

References

[1] Robert A. Adams. *Sobolev spaces*. Academic Press [A subsidiary of Harcourt Brace Jovanovich, Publishers], New York-London, 1975. Pure and Applied Mathematics, Vol. 65.
[2] A. Floer. A relative Morse index for the symplectic action. *Comm. Pure Appl. Math.*, 41(4):393–407, 1988.
[3] A. Floer, Helmut Hofer, and Dietmar Salamon. Transversality in elliptic Morse theory for the symplectic action. *Duke Math. J.*, 80(1):251–292, 1995.
[4] Victor Guillemin and Shlomo Sternberg. The moment map revisited. *J. Differential Geom.*, 69(1):137–162, 2005.
[5] L. Hörmander. *The analysis of linear partial differential operators. III*. Springer-Verlag, Berlin, 1994. Pseudo-differential operators, Corrected reprint of the 1985 original.
[6] Mikhail Khovanov and Richard Thomas. Braid cobordisms, triangulated categories, and flag varieties. Homology, Homotopy and Applications 9, 19–94, 2007.
[7] Yanki Lekili. Heegaard Floer homology of broken fibrations over the circle, 2009. arXiv.org:0903.1773.
[8] Dusa McDuff and Dietmar Salamon. *J-holomorphic curves and symplectic topology*, volume 52 of *American Mathematical Society Colloquium Publications*. American Mathematical Society, Providence, RI, 2004.

[9] Y.-G. Oh. Floer cohomology of Lagrangian intersections and pseudo-holomorphic disks. I. *Comm. Pure Appl. Math.*, 46(7):949–993, 1993.
[10] Tim Perutz. Lagrangian matching invariants for fibred four-manifolds. I. *Geom. Topol.*, 11:759–828, 2007.
[11] Tim Perutz. Lagrangian matching invariants for fibred four-manifolds. II. *Geom. Topol.*, 12(3):1461–1542, 2008.
[12] Reza Rezazadegan. Seidel-Smith cohomology for tangles, 2008. arXiv.org:0808.3381.
[13] Joel Robbin and Dietmar Salamon. The spectral flow and the Maslov index. *Bull. London Math. Soc.*, 27(1):1–33, 1995.
[14] D. Salamon and K. Wehrheim. Instanton Floer homology with Lagrangian boundary conditions. in preparation.
[15] Paul Seidel. A long exact sequence for symplectic Floer cohomology. *Topology*, 42(5):1003–1063, 2003.
[16] Paul Seidel. *Fukaya categories and Picard-Lefschetz theory*. Zurich Lectures in Advanced Mathematics. European Mathematical Society (EMS), Zürich, 2008.
[17] Paul Seidel and Ivan Smith. A link invariant from the symplectic geometry of nilpotent slices. *Duke Math. J.*, 134(3):453–514, 2006.
[18] Claude Viterbo. Intersection de sous-variétés lagrangiennes, fonctionnelles d'action et indice des systèmes hamiltoniens. *Bull. Soc. Math. France*, 115(3):361–390, 1987.
[19] Katrin Wehrheim. Energy quantization and mean value inequalities for nonlinear boundary value problems. *J. Eur. Math. Soc. (JEMS)*, 7(3):305–318, 2005.
[20] K. Wehrheim and C.T. Woodward. Functoriality for Lagrangian correspondences in Floer theory. arXiv:0708.2851.
[21] K. Wehrheim and C.T. Woodward. Quilted Floer cohomology. 2009 preprint.
[22] K. Wehrheim and C.T. Woodward. Pseudoholomorphic quilts. 2009 preprint.
[23] K. Wehrheim and C.T. Woodward. Orientations for pseudoholomorphic quilts. in preparation.
[24] K. Wehrheim and C.T. Woodward. Floer field theory. 2008 preprint.
[25] K. Wehrheim and C.T. Woodward. Floer field theory for tangles. 2008 preprint.
[26] A. Weinstein. The symplectic "category". In *Differential geometric methods in mathematical physics (Clausthal, 1980)*, volume 905 of *Lecture Notes in Math.*, pages 45–51. Springer, Berlin, 1982.
[27] A. Weinstein. *Lectures on Symplectic Manifolds*, volume 29 of *CBMS Regional Conf. Series in Math.* Amer. Math. Soc., 1983. third printing.

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