SPHERICAL VARIETIES AND EXISTENCE OF INVARIANT KÄHLER STRUCTURE

CHRIS WOODWARD

Contents

1. Introduction

Let K be a compact connected Lie group acting on projective n-space $\mathbb{P}^n$ via a unitary representation $K \to U(n+1)$. If one gives $\mathbb{P}^n$ a symplectic structure via the Fubini-Study form, then the action of K on $\mathbb{P}^n$ is Hamiltonian, and any smooth K-invariant sub-variety $M \subset \mathbb{P}^n$ inherits the structure of a Hamiltonian K-manifold. Naturally one wants to know what class of Hamiltonian actions arise in this way. To phrase a related question, what class of Hamiltonian actions admit an invariant, compatible Kähler structure?

One expected the answers to these questions to depend on the "degree of symmetry" of the symplectic manifold in question. There are many examples of symplectic manifolds without group actions that admit no compatible Kähler structure (see e.g. [14],[15].) On the other hand, as observed by Kostant and Souriau, transitive Hamiltonian actions of compact groups are coadjoint orbits and therefore Kähler. Coadjoint orbits are examples of **multiplicity-free Hamiltonian actions**, which are a class of symplectic manifolds with a very high degree of symmetry. Multiplicity-free torus actions were studied by Delzant [11] (under the name **completely integrable torus actions**) who showed that, under certain assumptions, each of these actions admits an invariant compatible Kähler structure.

Counterexamples in the non-abelian case were found independently by Knop [29] and the author [43]. In this paper we consider the question in more detail. Recall that if M is a compact connected Hamiltonian K-manifold with moment map $\Phi : M \to \mathfrak{k}^*$, then the intersection $\Phi(M) \cap \mathfrak{t}^*_+$ of the image with the positive Weyl chamber is a convex polytope, called the **moment or Kirwan polytope**. The main result (Theorem 6.2 below) is that in good cases one can read off whether a multiplicity-free action might admit an invariant Kähler structure from the moment polytope. The proof of the criterion is an application of the theory of spherical varieties. One shows that after perturbing the symplectic form, and replacing the complex structure, we can assume that $M \subset \mathbb{P}(V)$ is embedded in the projectivization of some representation V of K. By an observation of Brion [5], such a multiplicity-free projective variety is **spherical**; that is, if G denotes the complexification of K, then a Borel subgroup $B \subset G$ has a dense orbit. The variety M is then an equivariant embedding of some homogeneous space G/H. These have been classified by Luna and Vust [33]. A result of Brion [6] shows that each facet F of the moment polytope Δ of a spherical variety corresponds to a (not necessarily unique) B-stable prime divisor in M. The B-stable prime divisors that are not G-stable are called **colors**. An interesting game (which does not exist in the toric case) is to try to determine from the polytope which facets are defined by colors. The criterion is derived from an answer to this problem in a special case.

In the second half of the paper we prove a sufficient criterion for the existence of an invariant Kähler structure. For certain actions of $SO(5)$, we show that our existence and non-existence results combine to give a complete answer. Finally we show that in the $SO(5)$ case our criterion is equivalent to a criterion for the Kählerizability of Hamiltonian torus actions due to S. Tolman [40].

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2. The moment polytope of a spherical variety

Let K be a compact, connected Lie group with complexification G and maximal torus $T \subset K$. Let $\mathfrak{t} \subset \mathfrak{k}$ be the corresponding Cartan subalgebra. Choose a positive chamber $\mathfrak{t}^*_+ \subset \mathfrak{t}^*$ and let $B \subset G$ denote the Borel subgroup containing T whose Lie algebra contains the positive root spaces. Recall that to any G-variety X and G-line bundle L we can associate a convex set $\Delta(L) \subset \mathfrak{t}^*_+$ as follows. For any B-module V let $V^{(B)}$ be the set of non-zero B-eigenvectors in V. For any element $v \in V^{(B)}$, we denote by $\chi(v) : B \to \mathbb{C}^*$ the associated character of B, which we identify with a weight in $\mathfrak{t}^*$. Following Brion [5] we define

$$\Delta(L) = \overline{\bigcup_{n \in \mathbb{N}} \chi(H^0(L^n))^{(B)}/n}. \tag{1}$$

One can show, by tensoring highest-weight sections, that the set $\Delta(L)$ is convex. Now let $K \subset G$ be a maximal compact subgroup with maximal torus T as before. If X is smooth and compact and L has an invariant metric connection with positive curvature $\omega \in \Omega^{1,1}(X)$ and moment map Φ, then $\Delta(L)$ equals the moment polytope $\Phi(X) \cap \mathfrak{t}^*_+$. This is an easy consequence of "quantization commutes with reduction" (Guillemin-Sternberg multiplicity formula) [38, 36].

A G-variety X is called **spherical** if X is normal and B has a dense orbit. If X is spherical, then by work of Brion [6] the facets of $\Delta(L)$ are defined by B-stable prime divisors in X, in a sense that we now explain. Let $\mathbb{C}(X)$ denote the field of rational functions on X. The set of B-eigenfunctions $\mathbb{C}(X)^{(B)}$ has the structure of an abelian group under multiplication, and since X is spherical, we have an exact sequence

$$\mathbb{C}^* \to \mathbb{C}(X)^{(B)} \xrightarrow{\chi} \mathfrak{t}^*.$$

The image $\Lambda = \chi(\mathbb{C}(X)^{(B)})$ is a finitely generated free abelian subgroup of $\mathfrak{t}^*$. Its rank is called the **rank** of X.

Let $\mathcal{D}(X)$ denote the set of B-stable prime divisors in X. Each element $D \in \mathcal{D}(X)$ defines a valuation

$$v_D : \mathbb{C}(X) \to \mathbb{Z}$$

measuring the order of vanishing of any rational function at D. By restriction to $\mathbb{C}(X)^{(B)}$, the divisor D defines an element of $\mathrm{Hom}_{\mathbb{Z}}(\Lambda, \mathbb{Z})$ which we also denote by v_D.

Let $\mathbb{C}(X, L)$ denote the space of rational sections of L, and assume that $H^0(L) \subset \mathbb{C}(X, L)$ is non-trivial. Since $H^0(L)$ is locally finite [30, p. 67], there exists a B-eigensection $\sigma \in H^0(L)^{(B)}$ which defines an isomorphism

$$\mathbb{C}(X)^{(B)} \cong \mathbb{C}(X, L)^{(B)}, \quad f \mapsto f\sigma.$$

For any element $D \in \mathcal{D}(X)$ we denote by $v_D(\sigma)$ the order of vanishing of σ at D. An element $f\sigma \in \mathbb{C}(X, L)^{(B)}$ is a global section if and only if

$$v_D(f) + v_D(\sigma) \geq 0 \text{ for all } D \in \mathcal{D}(X).$$

Let $\Lambda_{\mathbb{R}} = \Lambda \otimes_{\mathbb{Z}} \mathbb{R}$. One sees that

$$\Delta(L) = \chi(\sigma) + \{x \in \Lambda_{\mathbb{R}} \mid v_D(x) \geq -v_D(\sigma), \text{for all } D \in \mathcal{D}(X)\} \tag{2}$$

$$= \{x \in \Lambda_{\mathbb{R}}^L \mid v_D(x) \geq -v_D(\sigma) + v_D(\chi(\sigma)) \text{ for all } D \in \mathcal{D}(X)\} \tag{3}$$

where $\Lambda_{\mathbb{R}}^L = \Lambda_{\mathbb{R}} + \chi(\sigma)$. Since X is spherical, the set $\mathcal{D}(X)$ is finite so that (3) expresses $\Delta(L)$ as a finite intersection of half-spaces. It follows that if F is a facet of $\Delta(L)$ then there exists a divisor $D \in \mathcal{D}(X)$ such that $F = \Delta \cap H_D$ where

$$H_D = \{x \in \Lambda_{\mathbb{R}}^L \mid v_D(x) = -v_D(\sigma) + v_D(\chi(\sigma))\}. \tag{4}$$

We call any D with $F = \Delta \cap H_D$ a **divisor corresponding to F**.

Remark 2.1. There are two important differences from the toric case:

1. Several divisors may correspond to a single facet;
2. Not every set of the form (4) is a facet, even if the line bundle L defines a projective embedding of X. See the example in Section 7, and also Lemma 2.4.

2.1. **Relation of the moment polytope to the colored fan.** Let X be a spherical G-variety, $x \in X$ be any point in the open B-orbit, and $H = G_x$ its isotropy subgroup. Then X is an equivariant embedding of the homogeneous space G/H. The Luna-Vust theory classifies such embeddings by a combinatorial invariant of X called the colored fan, which we will now describe. Let $\mathcal{D}(X)$ denote the set of B-stable prime divisors in X. For each G-orbit $Y \subset X$, let $C_Y \subset \mathrm{Hom}_{\mathbb{Z}}(\Lambda, \mathbb{Q})$ be the cone on the vectors v_D, for $D \in \mathcal{D}(X)$ containing Y. Let E_Y denote the set

$$E_Y = \{D \in \mathcal{D}(G/H) \mid \overline{D} \supset Y\}.$$

The pair

$$C_Y^c = (C_Y, E_Y)$$

is the **colored cone** associated to Y. The set

$$\mathcal{F}(X) = \{C^c_Y \mid Y \subset X\}$$

is the **colored fan** of X. Note that if $D \subset X$ is a G-stable divisor, then the corresponding valuation v_D is G-invariant. Let $\mathcal{V}^G \subseteq \mathrm{Hom}_{\mathbb{Z}}(\Lambda, \mathbb{Q})$ denote the image of the set of G-invariant discrete valuations (with rational values) on $\mathbb{C}(G/H) \cong \mathbb{C}(X)$. It is a convex cone containing $\mathrm{Hom}_{\mathbb{Z}}(\Lambda, \mathbb{Q}) \cap -\mathfrak{t}_+$ (see [25, Corollary 5.3].) A pair (C, E) with $C \subset \mathrm{Hom}_{\mathbb{Z}}(\Lambda, \mathbb{Q})$ and $E \subset \mathcal{D}(G/H)$ is called a **colored cone** if

1. C is generated by vectors v_D with $D \in E$, together with finitely many elements of $\mathcal{V}^G$.
2. The relative interior C° of C intersects $\mathcal{V}^G$.

A colored cone (C, E) is called **strictly convex** if C is strictly convex, and $v_D \neq 0$ for $D \in E$. A colored cone (C', E') is called a **colored face** of (C, E) if C' is a face of C, and $E' = \{D \in E \mid v_D \in C'\}$. A **colored fan** is a non-empty finite set $\mathcal{F}$ of colored cones such that

1. If $C^c \in \mathcal{F}$, then every colored face of C^c lies in $\mathcal{F}$.
2. For every $v \in \mathcal{V}^G$ there is at most one $(C, E) \in \mathcal{F}$ such that $v \in C^\circ$.

A colored fan $\mathcal{F}$ is called **strictly convex** if each colored cone in $\mathcal{F}$ is strictly convex. For the following see also [25, Theorem 3.3].

Theorem 2.2 (Luna-Vust [33]). *The map $X \mapsto \mathcal{F}(X)$ induces a bijection between isomorphism classes of embeddings and strictly convex colored fans.*

Define

$$C(X) = \bigcup_{(C,E) \in \mathcal{F}} C.$$

The variety X is complete if and only if $C(X)$ contains $\mathcal{V}^G$ [25, p.12].

The colored fan of a spherical variety is related to the fan of the polytope of an ample line bundle, if it exists. Recall that if Δ is a convex polytope, its associated fan $\mathcal{F}(\Delta)$ is the set of dual cones to faces of Δ. Here the **dual cone** to a face F' of Δ is the cone generated by normal vectors v_F to facets F of Δ containing F'. If X is a spherical variety, let $\mathcal{C}(X)$ denote the set of cones C such that for some set of colors E, the pair (C, E) lies in the colored fan of X. The following results are due, in a somewhat different form, to Brion [5].

Theorem 2.3 (Brion). *Let X be a complete spherical G-variety of maximal rank with generic stabilizer H and colored fan $\mathcal{F}$, and let $L \to X$ be an ample G-line bundle with polytope $\Delta = \Delta(L)$. Then $\mathcal{C}(X)$ equals the subset of $\mathcal{F}(\Delta)$ consisting of cones $C \in \mathcal{F}(\Delta)$ such that $C^\circ \cap \mathcal{V}^G$ is non-empty.*

The proof relies on the

Lemma 2.4. *Let X, L and Δ be as above, $F \subset \Delta$ a facet, $\mathcal{D}(X)$ a divisor corresponding to F, $H_D \subset \Lambda^L_{\mathbb{R}}$ the hyperplane defined by D as in Equation (4), and $Y \subset X$ a G-orbit. Let $\Delta_{\overline{Y}}$ denote the polytope of the restriction of L to Y as in Equation (1). Then D contains Y if and only if H_D contains $\Delta_{\overline{Y}}$.*

Proof. This follows from Brion's [7, Theorem p.409]. Alternatively, by Equation (4) H_D contains $\Delta_{\overline{Y}}$ if and only if any element s of $H^0(L^n)^{(B)}$ zero on D also is zero on Y, for any $n \in \mathbb{N}$.

By [25, Corollary 1.7] (where we let v_0 be a valuation with center $\overline{Y}$), this holds if and only if any global section s of L^n vanishing on D vanishes on Y, for any $n \in \mathbb{N}$; that is, Y is contained in D. □

Proof of Theorem 2.3 - First, note that $\Delta_{\overline{Y}}$ is a face of Δ. Indeed, the locus of vanishing of a section $s \in H^0(L^n)^{(B)}$ is the union of $D \in \mathcal{D}(X)$ such that $v_D(s) > 0$. Since $\mathcal{D}(X)$ is finite, s does not vanish identically on Y if and only if $v_D(s) = 0$ for every $D \in \mathcal{D}(X)$ containing Y. Therefore,

$$\Delta_{\overline{Y}} = \Delta \cap \bigcap_{D \supset Y} H_D.$$

It follows from Lemma 2.4 and Equation (3) that the dual cone to $\Delta_{\overline{Y}}$ equals C_Y. On the other hand, since X is complete, any cone $C \in \mathcal{F}(\Delta)$ of maximal dimension such that $C^\circ \cap \mathcal{V}^G$ is non-empty must be contained in $\mathcal{C}(X)$. Any other cone C with $C^\circ \cap \mathcal{V}^G$ non-empty is a face of a cone of maximal dimension with this property and is therefore contained in $\mathcal{C}(X)$.

2.2. **Facets defined by colors.** We are particularly interested in facets of $\Delta(L)$ defined by colors, since these do not exist in the toric case. It follows from work of Knop [27] (see also Foschi [13]) that these facets of $\Delta(L)$ can be three types with respect to a given minimal parabolic:

Theorem 2.5. *Let X be a spherical G-variety of maximal rank and $L \to X$ a G-line bundle with polytope $\Delta(L)$. Let α be a simple root and P the corresponding minimal parabolic. Then there are either $n = 0, 1,$ or 2 colors which are not stable under P. If case $n = 1$, let D be the color such that $PD \neq D$. Then $H_D = H_\alpha$. If $n = 2$, let $D_\pm$ be the colors such that $PD_\pm \neq D_\pm$. Then $H_{D_+} = s_\alpha H_{D_-}$.*

Proof. Let $D \subset X$ be a color such that $PD \neq D$. According to a result of Brion [5] and Vinberg [41], B has a finite number of orbits in X, and so D contains a dense B-orbit Bx. According to Knop [27], B-orbits in Px correspond to P_x-orbits in $P/B \cong \mathbb{P}^1$, and there can be either two or three in number. The closed orbits correspond exactly to colors that are not P-stable.

Two orbits: Suppose $s \in H^0(nL)^{(B)}$ is non-vanishing on D. Since all other colors are P-stable, the divisor $s = 0$ is P-stable, so $\chi(s)/n \in H_\alpha$. Characters of this form are dense in H_D, which proves the forward inclusion. The reverse inclusion follows from the fact that H_D, H_α are both codimension 1.

Three orbits: According to Knop [27] this corresponds to the case that the image of P_x in Aut $(\mathbb{P}^1)$ is a copy of $\mathbb{C}^*$. By [27, 3.2 Lemma] the character groups $\chi(D_\pm) := \chi(\mathbb{C}(D_\pm)^{(B)})$ satisfy codim $(\chi(D_\pm) \otimes_{\mathbb{Z}} \mathbb{R}) = 1$ which implies that $\ker v_{D_\pm} = \chi(D_\pm) \otimes_{\mathbb{Z}} \mathbb{R}$. Furthermore $s_a \chi(D_+) = \chi(D_-)$ which proves the claim for L trivial. For L non-trivial, we can identify rational sections with rational functions on the geometric realization Y of L with weight one with respect to the $\mathbb{C}^*$ action on the fibers. Considering Y as a $G \times \mathbb{C}^*$ variety, the same argument proves the claim. □

2.3. **Stability of the moment polytope.** In this section we describe a situation in which one can identify from the polytope which facets are defined by colors. Let $L \to X$ be a G-line bundle. For each simple root $\alpha \in \mathfrak{t}^*$, let $H_\alpha \subset \mathfrak{t}^*$ denote the corresponding reflection hyperplane. We will call $\Delta(L)$ **stable** if, for any other G-line bundle $L_1 \to X$ and reflection hyperplane H_α meeting $\Delta(L)$, the polytope $\Delta(pL + qL_1)$ meets H_α for $p \gg q$. It is easy to see that "generic" G-line bundles over spherical varieties have stable polytopes. See for example Theorem 7.1.

Theorem 2.6. *Let G be a connected complex reductive group, X a spherical G-variety, and L a G-line bundle such that $\Delta(L)$ is stable and $\Delta(L) \cap H_\alpha$ is non-empty for all simple roots α. Let $F \subset \Delta(L)$ be a facet and $D \in \mathcal{D}(X)$ a corresponding B-stable divisor. For any simple root α, let P_α denote the corresponding minimal parabolic. Then $P_\alpha D \neq D$ if and only if*

$$F \supset \Delta \cap H_\alpha. \tag{5}$$

In particular D is G-stable if and only if (5) holds for no α.

Proof. We can assume that the action of G lifts to the line bundle $[D]$ [30]. Let $\sigma_D \in H^0([D])^{(B)}$ define D, and let $\chi(D) = \chi(\sigma_D)$ be the corresponding character of B (unique up to a character of G). Let $L_{p,q}$ be the line bundle $pL + q[D]$. If σ is a B-eigensection of L, then $\sigma^p \sigma_D^q$ is a B-eigensection of $L_{p,q}$. By (3)

$$\Delta(L_{p,q}) = \chi(\sigma^p \sigma_D^q) + \{x \in \Lambda_\mathbb{R} \mid v_{D'}(x) \geq -v_{D'}(\sigma^p \sigma_D^q), \text{ for all } D' \in \mathcal{D}(X)\}.$$

Since $v_{D'}(\sigma_D) = 1$ if $D' = D$, and $v_{D'}(\sigma_D) = 0$ otherwise, we have that

$$\Delta(L_{p,q}) = q\chi(D) + \widetilde{\Delta}_{p,q}$$

where

$$\widetilde{\Delta}_{p,q} = \{x \in p\Delta \mid v_D(x) \geq -pv_D(\sigma) + pv_D(\chi(\sigma)) + q\}.$$

Since $\Delta(L)$ is stable,

$$\min_{x \in \widetilde{\Delta}_{p,q}} (x, \check{\alpha}) = -q(\chi(D), \check{\alpha})$$

where $\check{a}$ is the coroot of α. Therefore $(\chi(D), \check{\alpha})$ vanishes if and only if $\widetilde{\Delta}_{p,q}$ meets H_α. For $p \gg q$, this happens exactly when (5) holds. ☐

3. Little Weyl groups and collective functions

In this section we will make an application of Brion and Knop's theory of the little Weyl group to the smoothness of invariant collective functions. For the following, see Brion [8] or Knop [26, Theorem 1.3].

Theorem 3.1 (Brion). *Let X be a spherical G-variety and $\mathcal{V}^G(X) \subseteq \mathrm{Hom}_\mathbb{Z}(\Lambda, \mathbb{Q})$ the cone of invariant valuations. There is a finite reflection subgroup $W_X \subseteq W$ such that $\mathcal{V}^G(X)$ is a fundamental domain of W_X acting on $\mathrm{Hom}_\mathbb{Z}(\Lambda, \mathbb{Q})$.*

Knop has given a geometric definition of the little Weyl group [26] and generalized Brion's result to the non-spherical case. We present here Knop's definition in the case that X is smooth and maximal rank. Let $T_h^* X$ denote the bundle of holomorphic cotangent vectors. The action of G on X induces a holomorphic moment map $\Phi_h : T_h^* \to \mathfrak{g}^*$. Let $\mathfrak{l} \subset \mathfrak{g}$ be the complexification of the real Cartan subalgebra $\mathfrak{t} \subset \mathfrak{k}$. Composing with the quotient map $q_h : \mathfrak{g}^* \to \mathfrak{g}^*//G \cong \mathfrak{l}^*/W$

one obtains a morphism $\tilde\Phi_h^*: T_h^* \to \mathfrak{l}^*/W$. Consider the fiber product

$$\begin{tikzcd}[column sep=small] & T_h^*(X) \times_{\mathfrak{l}^*/W} \mathfrak{l}^* \arrow[dl, "\pi_1"'] \arrow[dr, "\pi_2"] & \\ T_h^*(X) \arrow[dr, "\tilde\Phi_h^*"'] & & \mathfrak{l}^* \arrow[dl, "q_h"] \\ & \mathfrak{l}^*/W & \end{tikzcd}$$

Because X is maximal rank, the inverse image $\pi_2^{-1}(\mathfrak{l}^*_{\mathrm{reg}})$ is non-empty, and π_1 is a ramified cover with generic fiber W. The fiber product $T_h^*X \times_{\mathfrak{l}^*/W} \mathfrak{l}^*$ may have several irreducible components, which are closures of the components of $\pi_2^{-1}(\mathfrak{l}^*_{\mathrm{reg}})$ and are permuted transitively by W. By [26, p. 317] there is a distinguished component $\hat{T}_h^*$ of $T_h^*X \times_{\mathfrak{l}^*/W} \mathfrak{l}^*$, called the **polarized cotangent bundle**. The **little Weyl group** $W_X \subseteq W$ is the set of elements $w \in W$ such that $w\hat{T}_h^* = \hat{T}_h^*$.

Now let M be a Hamiltonian K-manifold with moment map $\Phi : M \to \mathfrak{k}^*$, and let $\tilde\Phi : M \to \mathfrak{t}_+^*$ be the composition of Φ with the quotient map $q : \mathfrak{k}^* \to \mathfrak{t}_+^*$ which assigns to any $x \in \mathfrak{k}^*$ the unique point of intersection $Kx \cap \mathfrak{t}_+^*$. A **collective function** on M is a function of the form $\Phi^* f$, for some continuous function f on $\mathfrak{k}^*$. A K-invariant collective function can be written $\tilde\Phi^* f$. Our application of the little Weyl group is the following theorem which was suggested by Knop.[1]

Theorem 3.2. *Let V be a G-representation and $M \subset \mathbb{P}(V)$ a smooth invariant subvariety of maximal rank, i.e., with moment polytope of maximal dimension. If $f \in C^\infty(\mathfrak{t}^*)^{W_M}$ is a W_M-invariant smooth function, then the function $\tilde\Phi^* f$ is smooth.*

Proof. First, consider the case that $f \in \mathbb{R}[\mathfrak{t}^*]^{W_M}$ is polynomial. Let $L\backslash 0$ denote the geometric realization of the pull-back of the hyperplane bundle, minus its zero section. The product $G \times \mathbb{C}^*$ acts on $L\backslash 0$ with little Weyl group $W_{L\backslash 0}$ which is isomorphic to W_M [28, p.11]. Let $\pi : L\backslash 0 \to M$ denote projection onto the base.

The restriction $\pi_1|_{\hat{T}_h^*(L\backslash 0)}$ is a ramified cover with generic fiber W_M, and the quotient $\hat{T}_h^*(L\backslash 0)/W_M$ equals $T_h^*(L\backslash 0)$. Indeed, since π_1 is W_M-invariant, its restriction to $\hat{T}_h^*(L\backslash 0)$ induces an affine birational morphism $\hat{T}_h^*(L\backslash 0)/W_M \to T_h^*(L\backslash 0)$. This map has finite fibers and normal target space, and is therefore an isomorphism by Zariski's Main Theorem. Hence $\pi_2|_{\hat{T}_h^*}$ induces a morphism

$$\pi_2/W_M : T_h^*(L\backslash 0) \to \mathfrak{l}^*/W_M \times \mathbb{C}^* \to \mathfrak{l}^*/W_M.$$

The last map is just projection onto the first factor. Let $f_h \in \mathbb{C}[\mathfrak{l}^*]^{W_M}$ be the analytic continuation of f, so that $f = f_h|_{\mathfrak{t}^*}$. We write f_h as a polynomial $\overline{f_h} \in \mathbb{C}[\mathfrak{l}^*/W_M]$ in the generators of $\mathbb{C}[\mathfrak{l}^*]^{W_M}$, so that f_h is the pullback of $\overline{f_h}$ by the quotient map map $\mathfrak{l}^* \to \mathfrak{l}^*/W_M$. The function

$$\alpha^*(\pi_2/W_M)^*\overline{f_h}$$

[1] The following theorem is related to Knop's forthcoming work on the "symplectic little Weyl group".

(where $\alpha = \sqrt{-1}\partial \ln |z|^2/2\pi$ is the Fubini-Study 1-form) is a smooth function on $L\backslash 0$ and we claim it equals $\pi^*\tilde{\Phi}^* f$.

$$\begin{tikzcd}
T_h^*(L\backslash 0) \arrow[rr, "\pi_2/W_M"] \arrow[dr, "\Phi_h"'] & & \mathfrak{l}^*/W_M \arrow[dr, "\gamma_h"] & \\
& \mathfrak{g}^* \arrow[rr, "q_h"] & & \mathfrak{l}^*/W \\
L\backslash 0 \arrow[uu, "\alpha"] \arrow[rr, "\rho"] \arrow[dr, "\pi^*\Phi"'] & & -(\mathcal{V}_{\mathbb{R}}^G)^* \arrow[uu, "r"'] \arrow[dr, "\gamma"] & \\
& \mathfrak{k}^* \arrow[uu] \arrow[rr, "q"] & & \mathfrak{t}^*_+ \arrow[uu]
\end{tikzcd}$$

FIGURE 1. Relation of the unitary and holomorphic moment maps

Consider the commutative diagram in Figure 1. In the top square we have left off extra factors of $\mathbb{C}$; for example, Φ_h denotes the holomorphic moment map for the action of $G \times \mathbb{C}^*$ on $L\backslash 0$, composed with projection onto $\mathfrak{g}^*$. The notation $(\mathcal{V}_{\mathbb{R}}^G)^*$ denotes the fundamental domain of W_M acting on $\mathfrak{t}^*$, containing $-\mathfrak{t}^*_+$. Note that $\mathfrak{k}^*$ denotes $\mathrm{Hom}_{\mathbb{R}}(\mathfrak{k}, \mathbb{R})$, while $\mathfrak{g}^*$ denotes $\mathrm{Hom}_{\mathbb{C}}(\mathfrak{g}, \mathbb{C})$. Since $\mathfrak{g}$ is isomorphic to $\mathfrak{k} \oplus i\mathfrak{k}$, analytic continuation defines an inclusion $\mathfrak{k}^* \to \mathfrak{g}^*$. The map r is the restriction of $\mathfrak{l}^* \to \mathfrak{l}^*/W_M$ to $-(\mathcal{V}_{\mathbb{R}}^G)^*$. The map ρ is defined as follows. Let $\hat{\pi_1}$ denote the restriction of π_1 to $\hat{T_h^*}(L\backslash 0)$. Since $\Phi_h(\alpha(L\backslash 0))$ is contained in $\mathfrak{k}^*$, we have that

$$\pi_2(\hat{\pi_1}^{-1}(\alpha(L\backslash 0))) \subset \mathfrak{t}^*.$$

Since π_2 is W_M-equivariant, we can quotient by the action of W_M to obtain a continuous map

$$\alpha(L\backslash 0) \to -(\mathcal{V}_{\mathbb{R}}^G)^*$$

whose composition with α we define to be ρ. By definition $\rho \circ r$ equals $\alpha \circ \pi_2/W_M$. The map γ_h is the polynomial map obtained by expressing a set of generators of $\mathbb{C}[\mathfrak{l}^*]^W$ in terms of a set of generators of $\mathbb{C}[\mathfrak{l}^*]^{W_M}$. The map γ is the restriction of q to $-(\mathcal{V}_{\mathbb{R}}^G)^*$. The commutativity of the diagram is clear from the definition of ρ and the fact that $\Phi_h \circ \alpha = \pi^*\Phi$.

Lemma 3.3. *The restriction of γ to $\rho(L\backslash 0)$ is the identity.*

Proof. We first show that there exists an element $w \in W$ such that $\gamma(x) = wx$ for all $x \in \rho(L\backslash 0)$. Recall from e.g. [32] that if M is a compact connected Hamiltonian K-manifold with moment map $\Phi : M \to \mathfrak{k}^*$ such that $\Phi^{-1}(\mathfrak{k}^*_{\mathrm{reg}})$ is non-empty, then $\Phi^{-1}(\mathfrak{k}^*_{\mathrm{reg}})$ is connected and dense. Therefore, $\pi^{-1}\Phi^{-1}(\mathfrak{k}^*_{\mathrm{reg}})$ is connected and dense. The image $\rho(\pi^{-1}\Phi^{-1}(\mathfrak{k}^*_{\mathrm{reg}}))$ is connected and therefore contained in some chamber $w\ (\mathfrak{t}^*_+)^\circ$ in $\mathfrak{t}^*_{\mathrm{reg}}$. By continuity, $\gamma(x) = wx$ for all $x \in \rho(L\backslash 0)$.

That $w = \mathrm{Id}$ follows from Knop's definition of $\hat{T}^*_h$. We will use freely the notation developed in his paper [26, p.5]. Let $\eta \in (\mathfrak{t}^*_+)^\circ$ be a generic point in Δ and let $l \in \pi^{-1}\Phi^{-1}(\eta)$ be a point in its inverse image. It suffices to show that $\rho(l) = \eta$. By [26, Corollary 3.3], since M is maximal rank there exists an element $\overline{b} \in A^s_r$ such that $\chi_{D(\overline{b})} = \eta$. The image $\hat{\psi}(l,\overline{b}) \in \hat{T}^*_h(L\backslash 0)$ is therefore

$$\hat{\psi}(l,\overline{b}) = (\psi^*(l,\overline{b}), \eta)$$

and as in [26, p.5],

$$\Phi_h(\psi^*(l,\overline{b})) \in \eta + \mathfrak{b}^\circ$$

where $\mathfrak{b}^\circ \subset \mathfrak{g}^*$ is the annihilator of the Lie algebra of B. Now let

$$\zeta_t = t\alpha_l + (1-t)\psi^*(l,\overline{b})$$

for $t \in [0,1]$. The image $\Phi_h(\zeta_t)$ lies in $\eta + \mathfrak{b}^\circ$, and in particular the coadjoint orbit of η. Therefore the path

$$(\zeta_t, \eta), \qquad t \in [0,1]$$

lies in the fiber product $T^*_h(L\backslash 0) \times_{\mathfrak{t}^*/W} \mathfrak{t}^*$. Therefore (α_l, η) is contained in $\hat{T}^*_h(L\backslash 0)$. It follows that $\rho(\alpha_l) = \eta$, as required. □

By Lemma 3.3 the pullback $\pi^*\tilde{\Phi}^* f$ equals $\rho^* f$, which equals $\alpha^*(\pi_2/W_M)^*\overline{f_h}$ as claimed.

Now let $f \in C^\infty(\mathfrak{t}^*)^{W_M}$ be any smooth W_M-invariant function. By a theorem of G. Schwarz [37], f can be written as a smooth function of the generators of the W_M-invariant polynomials on $\mathfrak{t}^*$, so the result follows from the previous case. □

Theorem 3.4. *Let $M \subset \mathbb{P}(V)$ be a smooth K-variety of maximal rank with moment map $\Phi : M \to \mathfrak{k}^*$. Suppose that Φ is transversal to a face $\sigma \subset \mathfrak{t}^*_+$, and let W_σ be the Weyl group of T in K_σ. Then $W_\sigma \subset W_M$.*

Proof. Let $W' \subseteq W$ be the subgroup generated by reflections contained in both W_σ and W_M. We want to show that W' equals W_σ. Let $\sigma^\perp \subset \mathfrak{t}^*$ be the subspace perpendicular to σ. It suffices to prove that $\mathbb{R}[\sigma^\perp]^{W'}$ is contained in $\mathbb{R}[\sigma^\perp]^{W_\sigma}$ (see e.g. [23, Theorem 3.9]). Let $\pi_\sigma : \mathfrak{t}^* \to \sigma^\perp$ denote orthogonal projection onto $\sigma^\perp$, and let f be any element of $\mathbb{R}[\sigma^\perp]^{W'}$.

Step 1: $\tilde{\Phi}^*\pi^*_\sigma f$ is smooth at any point $m \in \Phi^{-1}(\sigma)$. We construct a W-invariant smooth function $h \in C^\infty(\mathfrak{t}^*)$ equal to $\pi^*_\sigma f$ near $\Phi(m)$ as follows. Let $\rho \in C^\infty(\mathfrak{t}^*)$ be a cutoff function supported near $\Phi(m)$, with $\rho = 1$ in a neighborhood of $\Phi(m)$, and

$$h = \sum_{w \in W} w^*(\rho \cdot \pi^*_\sigma f)$$

By Theorem 3.2 $\tilde{\Phi}^* h$ is smooth at $\Phi(m)$, which shows that $\tilde{\Phi}^*\pi^*_\sigma f$ is smooth, too.

Step 2: $q^*\pi^*_\sigma f$ is smooth at $\Phi(m)$. Since Φ is transversal to σ, we can choose a submanifold $U \subset \Phi^{-1}(\mathfrak{k}^*_\sigma)$ such that Φ is a diffeomorphism on U and $\Phi(U)$ meets σ transversally at $\Phi(m)$. The function $q^*\pi^*_\sigma f$ is therefore smooth on $\Phi(U)$, and since the restriction of $q^*\pi^*_\sigma f$ to $\mathfrak{k}^*_\sigma$ is locally constant on the fibers of π_σ^* near $\Phi(m)$, this restriction is smooth at $\Phi(m)$. Let V be a small K_σ-invariant neighborhood of $\Phi(m)$ in $\mathfrak{k}_\sigma^*$. Since $KV \subset \mathfrak{k}^*$ is isomorphic to $K \times_{K_\sigma} V$ and $q^*\pi_\sigma^* f$ is K-invariant, the claim follows.

Step 3: f is W_σ-invariant. Let $R(\sigma)$ be the set of simple roots perpendicular to σ. For any $\alpha \in R(\sigma)$, let $r_\alpha \in W_\sigma$ denote the corresponding reflection. The function $q^*\pi_\sigma^* f$ is r_α-invariant, and therefore

$$(D_\alpha^n q^*\pi_\sigma^* f)(\Phi(m)) = (D_\alpha^n \pi_\sigma^* f)(\Phi(m)) = 0$$

for $n \in \mathbb{N}$ odd, where D_α is partial differentiation with respect to α. Since f is polynomial, this shows that f is itself r_α-invariant, for each $\alpha \in R(\sigma)$, and therefore f is W_σ-invariant. □

Corollary 3.5. *Let $M \subset \mathbb{P}(V)$ be a smooth K-invariant variety whose moment map is transversal to every face $\sigma \subset \mathfrak{t}_+^*$ and whose polytope Δ is of maximal dimension and meets the hyperplane H_α for each simple root α. Then $W_M = W$.*

4. Multiplicity-free Hamiltonian actions

A complex representation V of a compact connected Lie group K is called **multiplicity-free** if each irreducible representation appears at most once in V, or equivalently, if the algebra $\mathrm{End}_K(V)$ of K-equivariant endomorphisms is abelian. A Hamiltonian K-manifold M is called **multiplicity-free** if the set $C_K^\infty(M)$ of K-invariant smooth functions forms an abelian Poisson algebra [18]. That is, for any $f_1, f_2 \in C_K^\infty(M)$ we have $\{f_1, f_2\} = 0$.

There is an important, alternative definition of multiplicity-free in terms of **symplectic reduction.** Recall that if M is a Hamiltonian K-manifold with moment map $\Phi : M \to \mathfrak{k}^*$, then for each coadjoint orbit $Kx \subset \mathfrak{k}^*, x \in \mathfrak{k}^*$ there is an associated **Marsden-Weinstein symplectic reduced space** M_x defined by

$$M_x = \Phi^{-1}(Kx)/K.$$

If x is a regular value of Φ then M_x is a symplectic orbifold. Sjamaar (based on a result of Arms-Cushman-Gotay [1]) has observed that

Proposition 4.1 (Sjamaar). *A compact connected Hamiltonian K-manifold is multiplicity-free if and only if M_x is a point for any $x \in \mathfrak{k}^*$. If the principal isotropy subgroup is discrete, then M is multiplicity-free if and only if $\dim M = \dim K + \mathrm{rank}\ K$.*

This follows from the main result in [39]; see [42, Proposition A.1]. By Proposition 4.1, a compact connected Hamiltonian K-manifold M is multiplicity-free if and only if Φ induces a bijection $M/K \cong \Delta$. (See [5] for the proof when M is a projective K-variety.) The decomposition of M/K into orbit-types is related to the face decomposition of Δ:

Lemma 4.2 (Delzant). *Let M be a compact, connected multiplicity-free K-manifold with discrete principal isotropy subgroup, and $F \subset \Delta$ an open face contained in the interior $(\mathfrak{t}_+^*)^\circ$ of $\mathfrak{t}_+^*$. Then the Lie algebra $\mathfrak{k}_m$ of the isotropy subgroup K_m of any point $m \in \Phi^{-1}(F)$ equals the annihilator $F^\circ \subset \mathfrak{t}$ of F. Furthermore, if the principal isotropy subgroup is trivial then K_m is connected.*

For a proof see [12] or [42, Lemma 3.2]. In particular, a face $F \subset \Delta \cap (\mathfrak{t}_+^*)^\circ$ is a vertex of Δ if and only if $\Phi^{-1}(F)$ is a T-fixed point. Describing the orbit types of points in $\Phi^{-1}(F) \subset \partial\mathfrak{t}_+^*$ is in general an open problem. For the transversal case, see [42, Theorem 7.2].

Note that if M is a connected Hamiltonian K-manifold, then the principal isotropy subgroup (which in general is only defined up to conjugacy) is fixed by the choice of maximal torus and positive chamber $\mathfrak{t}_+^*$. Indeed, there exists a face σ of $\mathfrak{t}_+^*$ such that $\Phi^{-1}(K\sigma)$ is connected and dense (see e.g. [32, Theorem 3.7]), and we define the **principal isotropy subgroup** of M to be the isotropy subgroup K_x of any point x in the principal orbit-type stratum for the action of K_σ on $\Phi^{-1}(\sigma)$. Since K_x contains the semisimple part of K_σ (see e.g. [32, Remark 3.10]) K_x is independent of the choice of x. The following conjecture has been proved in special cases [11, 24, 12, 42]:

Conjecture 4.3 (The Multiplicity-free or Delzant Conjecture). *Let M_1 and M_2 be compact connected multiplicity-free Hamiltonian K-manifolds with the same moment polytope and the same principal isotropy subgroup. Then M_1 and M_2 are equivariantly symplectomorphic.*

Remark 4.4. A multiplicity-free Hamiltonian K-manifold may admit invariant complex structures that are not equivariantly isomorphic (F. Knop). The simplest examples are the $SL(2,\mathbb{C})$-spherical varieties $\mathbb{P}^1 \times \mathbb{P}^1$ with diagonal $SL(2,\mathbb{C})$-action, and $SL(2,\mathbb{C}) \times_B \mathbb{P}^1$, with action of B on $\mathbb{P}^1$ given by the trivial action of the unipotent subgroup, and action of $\mathbb{C}^* \subset B$ by $z \cdot [w_0, w_1] = [zw_0, w_1]$. These varieties are $SU(2)$-equivariantly symplectomorphic for a suitable choice of symplectic forms.

If the conjecture holds, it suggests that geometric properties of multiplicity-free actions should be "translatable" into the language of convex polytopes. We call a Hamiltonian K-manifold **transversal** if the moment map is transversal to a Cartan subalgebra. There is a characterization of transversality in terms of isotropy subgroups [42, Lemma 2.2]:

Lemma 4.5 (Guillemin-Souza). *Let M be a Hamiltonian K-manifold, $\sigma \subset \mathfrak{t}_+^*$ any face of the positive chamber, and K_σ the isotropy subgroup K_σ of any point in σ. Then Φ is transversal to $\mathfrak{t}^*$ at $\Phi^{-1}(\sigma)$ if and only if for any $x \in \sigma$ the semisimple part (K_σ, K_σ) of K_σ acts locally freely on $\Phi^{-1}(x)$, or equivalently, if Φ is transversal to σ.*

The main result of [42] is that transversality has the following description in terms of convex polytopes: For each $x \in \Delta$ we denote by $H(x)$ (resp. $V(x)$) the set of hyperplanes in $\mathfrak{t}^*$ intersecting Δ in facets meeting x (resp. inward pointing normal vectors to these facets.) The polytope Δ is called **simple** if $V(x)$ is linearly independent, for all $x \in \Delta$. We call Δ **reflective** if

1. the set $H(x)$ is W_x-invariant, for all $x \in \Delta$, and
2. the intersection $\Delta \cap \partial\mathfrak{t}_+^*$ is a union of faces of codimension at least 2.

Here W denotes the Weyl group of $T \subset K$, and W_x the isotropy subgroup of x.

Theorem 4.6. [42] *Let M be a compact, connected multiplicity-free K-manifold with moment polytope Δ and discrete principal isotropy subgroup. If M is transversal then Δ is simple and reflective.*

In fact the converse is also true.

We will need the following combinatorial result on reflective simple polytopes. For any convex polytope Δ, let $H(\Delta)$ denote the set of hyperplanes intersecting Δ in facets.

Proposition 4.7. [42, Proposition 5.1] *Let $\Delta \subset \mathfrak{t}_+^*$ be a simple reflective convex polytope and α a simple root such that Δ meets the hyperplane H_α. Then there are exactly two elements*

$H_\pm \in H(\Delta)$ such that $H_\pm$ contains $\Delta \cap H_\alpha$. The corresponding normal vectors $v_\pm \in \mathfrak{t}$ satisfy $(v_\pm, \alpha) > 0$, and the intersection $\Delta \cap H_+ \cap H_-$ equals $\Delta \cap H_\alpha$. Any other element H of $H(\Delta)$ meeting $H_\alpha \cap \Delta$ intersects H_α orthogonally.

Finally in Section 7 we will need the following definitions. We call a compact, connected, transversal Hamiltonian K-manifold **torsion-free** if (K_x, K_x) acts freely on $\Phi^{-1}(x)$ for all $x \in \Delta$. The elements of the set $V(x)$ can be chosen to be minimal in the lattice $\exp^{-1}(\mathrm{Id})$. If (K_x, K_x) is simply-connected, we call Δ **torsion-free** at x if $V(x)$ forms part of a lattice basis. Otherwise, we say that Δ is torsion-free at x if the conditions in [42, Remark 10.2] hold. We call Δ torsion-free if Δ is torsion-free at all $x \in \Delta$. A compact connected transversal multiplicity-free K-manifold with trivial principal isotropy is torsion-free if and only if its moment polytope is torsion-free . This is proved in the case $\pi_1((K_x, K_x)) = \{1\}$ in [42, Theorem 6.2].

Theorem 4.8. [42] *The map $M \to \Delta(M)$ induces a bijection between compact connected transversal torsion-free multiplicity-free K-manifolds with trivial principal isotropy and reflective, torsion-free polytopes.*

In particular, Delzant's conjecture 4.3 holds for these actions.

5. Algebraization

As above let K be a compact connected Lie group with complexification G. Brion [5] has noted that

Proposition 5.1 (Brion). *Let V be a complex G-representation, and $M \subset \mathbb{P}(V)$ a smooth invariant sub-variety. Then M is a multiplicity-free K-manifold if and only if M is a spherical G-variety.*

This is because a generic symplectic quotient of M is homeomorphic to a geometric invariant theory quotient of $M \times G/B$, which is zero dimensional if and only if B has a dense orbit. In order to apply Brion's result, we will need the following application of the Kodaira embedding theorem:

Proposition 5.2. *Let (M, ω) be a compact Hamiltonian K-manifold and J an invariant compatible Kähler structure. Suppose that the fixed points of a maximal torus T are isolated. Then there exists a perturbation $\overline{\omega}$ of ω, an integer $n \in \mathbb{N}$, and an invariant compatible Kähler structure $\overline{J}$ such that $(M, n\overline{\omega}, \overline{J})$ embeds in projective space.*

Proof. Since M admits a $\mathbb{C}^*$-action with isolated fixed points, then by the results of Carrell-Liebermann [10] or Carrell-Sommese the cohomology $H^{i,j}(M)$ vanishes unless $i = j$. In particular, $H^{2,0}(M) \cong H^{0,2}(M)$ vanishes, so there exists an invariant perturbation of $\overline{\omega}$ of ω such that $\overline{\omega} \in \Omega^{1,1}(M)$ and $[\overline{\omega}] \in H^2(M, \mathbb{Q})$. Since $\overline{\omega} \in \Omega^{1,1}$, the pair $(\overline{\omega}, J)$ defines an invariant Kähler structure on M. Let $n_1 \in \mathbb{Z}$ be an integer such that $[n_1\overline{\omega}] \in H^2(M, \mathbb{Z})$. Let L be a holomorphic metric line bundle with invariant connection and curvature $\overline{\omega}$ [16, p. 149]. By the Kodaira embedding theorem, there exists an integer $n_2 \in \mathbb{N}$ such that the sections of L^{n_2} give an equivariant embedding $i : M \to \mathbb{P}^N$ of (M, J) in projective N-space. Let $\omega_{FS} \in \Omega^2(\mathbb{P}^N)$ denote the Fubini-Study 2-form. Unfortunately, $i^*\omega_{FS}$ will not usually equal $n_1 n_2 \overline{\omega}$. However, the metrics $i^*\omega_{FS}(\cdot, J\cdot)$ and $n_1 n_2 \overline{\omega}(\cdot, J\cdot)$ are positive definite. If ω_t is the invariant closed 2-form defined by

$$\omega_t = t i^*\omega_{FS} + (1 - t) n_1 n_2 \overline{\omega}$$

then for $t \in [0,1]$ the metric $\omega_t(\cdot, J\cdot)$ is also positive definite, and so ω_t is symplectic for $t \in [0,1]$. Furthermore, ω_0 is cohomologous to ω_1, so by Moser isotopy (see e.g. [35, p. 91]) there exists a K-equivariant symplectomorphism

$$\varphi : (M, i^*\omega_{FS}) \cong (M, n_1 n_2 \overline{\omega}).$$

Defining $\overline{J} = \varphi^* J$ and $n = n_1 n_2$ completes the proof. □

To apply this result to Kähler multiplicity-free actions, we need to note that if M is a compact multiplicity-free K-manifold, then any maximal torus $T \subset K$ acts with isolated fixed points. Indeed, let M_T denote the T-fixed point set. Let $m \in M_T$ and let $N \subseteq M_T$ be the connected component of M_T containing m. The image $\Phi(N)$ lies in $\mathfrak{t}^*$, by equivariance of Φ, and because N is a smooth connected symplectic submanifold on which T acts trivially, in fact $\Phi(N)$ equals $\Phi(m)$. Since M is multiplicity-free,

$$\Phi^{-1}(\Phi(m)) \cong K_{\Phi(m)}/K_m$$

and the fixed point set of the action of T on any K-homogeneous space is finite.[2] In case M is transversal, one can argue alternatively that $\Phi(M_T) \subset \mathfrak{t}^*_{\mathrm{reg}}$ and so M_T is discrete by Lemma 4.2.

Combining with 5.2, 5.1 and Chow's theorem we have proved that

Corollary 5.3. *Any compact connected multiplicity-free K-manifold which admits an invariant Kähler structure admits (after a perturbation) the structure of a projective spherical variety.*

Remark 5.4. These arguments work only if M is compact. For results in the general case, see [22].

We want to check that certain properties and invariants of the Hamiltonian K-manifold M are invariant under perturbation. Let $V(\Delta)$ denote the set of inward pointing normal vectors to facets of Δ. (The elements of this set are unique up to multiplication by positive scalars.)

Proposition 5.5. *Let (M, ω) be a compact connected Hamiltonian K-manifold, and $t \mapsto \omega_t$ an (affine) linear map of $\mathbb{R}^k$ into the space of closed, K-invariant 2-forms on M, with $\omega_0 = \omega$. There exists a linear map $t \mapsto \Phi_t$ such that each $\Phi_t : M \to \mathfrak{k}^*$ is a moment map for the action of K on (M, ω_t). There also exists a neighborhood, U, of $0 \in \mathbb{R}^k$ such that for $t \in U$*

1. *the form ω_t is symplectic.*
2. *If Φ_0 is transversal to $\mathfrak{t}^*$, then*
   (a) *Φ_t is transversal to $\mathfrak{t}^*$;*
   (b) *if $\Phi_0(M)$ meets a face $\sigma \subset \mathfrak{t}^*_+$ then $\Phi_t(M)$ does; and*
   (c) *if M is multiplicity-free and has discrete principal isotropy then $V(\Delta_t) = V(\Delta_0)$.*

Proof. The existence of Φ_t follows from the discussion on [3, p.23], by which the contraction $\imath(X_M)\omega_t$ is exact for any $X \in \mathfrak{k}$ and $t \in \mathbb{R}^k$. It follows that for any t there exists a map $\Phi_t : M \to \mathfrak{k}^*$ such that $\imath(X_M)\omega_t = d\langle \Phi, X \rangle$. The map Φ_t may be made equivariant by [21, Section 26]. To construct a linear map $t \mapsto \Phi_t$, choose a basis $t_1, \ldots, t_n$ for $\mathbb{R}^k$, construct Φ_{t_i} as above, and define Φ_t for arbitrary t by linearity.

Statement (1) follows from the compactness of M, and the fact that the set of non-degenerate 2-forms on $T_m M$ is open, for any $m \in M$. Statement 2(a) follows from a similar argument. By

[2]In fact the fixed point set has a transitive action of W.

Lemma 4.5 Φ_0 is transversal to σ, and so any perturbation of Φ_0 also meets σ, which shows 2(b).

To prove 2(c) note that in the transversal case every facet F of Δ meets $(\mathfrak{t}^*_+)^\circ$ by Theorem 4.6 and so by Lemma 4.2 any point $m \in \Phi^{-1}(F)$ has $\mathfrak{t}_m = F^\circ$. In the transversal case the set of such isotropy algebras is stable under perturbation. The argument is technical but straight-forward; the proof is left to the reader. $\square$

We will need later the following Remark.

Remark 5.6. Consider the situation in Lemma 2.4, with the additional assumptions that X is smooth, transversal, and maximal rank. Then the face $\Delta_{\overline{Y}}$ intersects the interior $(\mathfrak{t}^*_+)^\circ$ of the positive chamber. Indeed, let $\sigma \subset \mathfrak{t}^*_+$ be a face of maximal dimension intersecting $\Phi(Y)$, so that for any point $y \in Y \cap \Phi^{-1}(\sigma)$, a neighborhood of y in Y is contained in $\Phi^{-1}(\sigma)$. The tangent space to the orbit $(K_\sigma, K_\sigma)y$ lies in T_yY, since Y is K-invariant, but also lies in the symplectic orthogonal to T_yY, by definition of the moment map. By Lemma 4.5, if the dimension of (K_σ, K_σ) is positive then $T_y((K_\sigma, K_\sigma)y)$ is of positive dimension, which contradicts that Y is a complex and therefore symplectic submanifold of X. Therefore, $\sigma = (\mathfrak{t}^*_+)^\circ$.

6. Characterization of colored facets

We say that a facet $F \subset \Delta$ with inward-pointing normal vector v_F is **negative** if $v_F \in -\mathfrak{t}_+$.

Theorem 6.1. *Let $M \subset \mathbb{P}(V)$ be a smooth K-invariant variety with discrete principal isotropy subgroup and moment map transversal to $\mathfrak{t}^*$. Let $\Delta \subset \mathfrak{t}^*_+$ denote the moment polytope of M, and assume that $\Delta \cap H_\alpha$ is non-empty for all simple roots α. such that the induced Hamiltonian action of K on M satisfies the assumptions of Theorem 6.2. Then the following are equivalent:*

1. *a facet $F \subset \Delta$ corresponds to a color;*
2. *F contains $\Delta \cap H_\alpha$ for some simple root α;*
3. *F is not negative.*

Proof. Let $F \subset \Delta$ be a facet. By Proposition 5.5 2(b), Δ is stable. Theorem 2.6 shows that the first two conditions are equivalent. It therefore suffices to show that F corresponds to a G-stable divisor D if and only if $v_F \in -\mathfrak{t}_+$. By Corollary 3.5, $W_M = W$, so $\mathcal{V}^G(M) = -\mathfrak{t}_+$ and if v_F does not lie in $-\mathfrak{t}_+$ then D cannot be G-stable. On the other hand, if D is not G-stable then by Theorem 2.6 F contains $\Delta \cap H_\alpha$ for some simple root α. By Proposition 4.7, $(v, \alpha) > 0$, so $v \notin -\mathfrak{t}_+$ which completes the proof. $\square$

Although Theorem 6.1 does not apply to many spherical varieties, it does apply to several well-known examples that arise in representation theory, such as the flag variety $GL(n+1, \mathbb{C})/B$ under the action of $GL(n, \mathbb{C})$ and similarly the generalized flag variety of $SO(n+1, \mathbb{C})$ under the action of $SO(n, \mathbb{C})$ (at least for a generic projective embedding.) Other examples will be given later.

The main result of the paper is the following necessary criterion for the existence of an invariant Kähler structure.

Theorem 6.2. *Let K be a compact connected Lie group and let M be a compact, connected, multiplicity-free Hamiltonian K-manifold, with discrete principal isotropy subgroup and moment map transversal to $\mathfrak{t}^*$. Let $\Delta \subset \mathfrak{t}^*_+$ denote the moment polytope of M, and assume that*

$\Delta \cap H_\alpha$ is non-empty for all simple roots α. If M admits a compatible invariant Kähler structure, then a facet F is not negative if and only if F contains $\Delta \cap H_\alpha$ for some simple root α.

Proof. By Corollary 5.3 and Proposition 5.5, we can assume that $M \subset \mathbb{P}(V)$ is a smooth projective K-variety, and the result follows from Theorem 6.1. □

Remark 6.3. Suppose that M satisfies the assumptions of Theorem 6.2, and M admits an invariant compatible Kähler structure. By Theorem 6.2 and Proposition 4.7 its moment polytope can have at most $2r$ non-negative facets, where r is the rank of the semi-simple part of K.

7. Example: Blow-ups of a product of coadjoint orbits of $SO(5)$

In this section we describe an example: symplectic blow-ups of a product M of two coadjoint orbits of $SO(5)$. The Hamiltonian K-manifold M contains exactly two symplectic K-orbits, and we can symplectically blow-up either one. Depending on which orbit we choose, the blow-up admits (resp. does not admit) an invariant, compatible Kähler structure.

Let $K \subset SO(5)$ and $T \subset K$ the standard choice of maximal torus. The usual choice for a basis for $\mathfrak{t}$ gives isomorphisms $\mathfrak{t} \cong \mathbb{R}^2$ and

$$\mathfrak{t}^*_+ = \{(x, y) \in \mathbb{R}^2 \mid 0 \le y \le x\}.$$

Let λ, μ be positive real numbers, and define

$$\overline{\lambda} = (\lambda, \lambda) \text{ and } \overline{\mu} = (\mu, 0)$$

so that $\overline{\lambda}, \overline{\mu}$ lie in the boundary $\partial \mathfrak{t}^*_+$. Let $\Theta_\lambda, \Theta_\mu \subset \mathfrak{so}(5)^*$ be the coadjoint orbits through $\overline{\lambda}$ (resp. $\overline{\mu}$.) Let

$$M = \Theta_\lambda \times \Theta_\mu$$

denote the product, with the diagonal action of K, which has moment map $\Phi : M \to \mathfrak{k}^*$ given by

$$\Phi(v, w) = v + w.$$

Theorem 7.1. *The Hamiltonian K-manifold M is multiplicity-free with trivial principal isotropy subgroup, and its polytope Δ equals*

$$\Delta = \{(x, y) \in \mathbb{R}^2 \mid y \le \lambda \le x \text{ and } x - y \le \mu \le x + y\}.$$

If μ does not equal λ or 2λ then Φ is transversal to $\mathfrak{t}^*$.

Proof. The fixed point set M_T equals

$$\begin{aligned} M_T &= W\overline{\lambda} \times W\overline{\mu} \\ &= \{((\pm\lambda, \pm\lambda), (\pm\mu, 0)), ((\pm\lambda, \pm\lambda), (0, \pm\mu))\} \end{aligned}$$

so that

$$\Phi(M_T) = \{(\pm\lambda \pm \mu, \pm\lambda), (\pm\lambda, \pm\lambda \pm \mu)\}$$

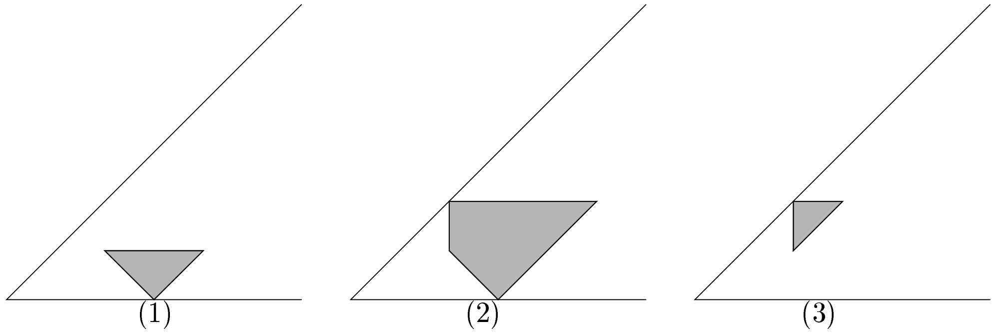

FIGURE 2. The polytopes Δ for (1) $\mu \geq 2\lambda$, (2)$\lambda \leq \mu \leq 2\lambda$, and (3) $\mu \leq \lambda$.

and

$$\begin{aligned} \Phi(M_T) \cap \mathfrak{t}^*_+ &= \{(\mu \pm \lambda, \lambda)\} && \text{if } \mu \geq 2\lambda \\ &= \{(\lambda, \mu - \lambda), (\lambda + \mu, \lambda)\} && \text{if } \lambda \leq \mu \leq 2\lambda \\ &= \{(\lambda, \lambda - \mu), (\lambda + \mu, \lambda)\} && \text{if } \mu \leq \lambda \end{aligned}$$

For simplicity we will consider only the case $\lambda < \mu < 2\lambda$. The weights of T on $T_{(\overline{\lambda}, \overline{\mu})} M$ are the negative roots of $\mathfrak{k}$, and the negative roots $(-1,0), (-1,-1)$ which annihilate neither $\overline{\lambda}$ nor $\overline{\mu}$ appear with multiplicity two. Since for any $m \in Y_+ := \Phi^{-1}((\mathfrak{t}^*_+)^\circ)$

$$T_m M \cong T_m Y_+ \oplus (\mathfrak{k}/\mathfrak{t})^*$$

the weights of T on $T_{(\lambda,\mu)} Y_+$ are $(-1,0)$ and $(-1,-1)$. It follows that $\Phi(Y_+) = \Delta \cap (\mathfrak{t}^*_+)^\circ$ is locally the cone on the vectors $(-1,0)$ and $(-1,-1)$. By similar arguments, near $(\lambda, \mu - \lambda)$ the polytope Δ equals the cone on $(0,1)$ and $(1,-1)$. If M is any Hamiltonian K-manifold, and $x \in (\mathfrak{t}^*_+)^\circ$ is a vertex of Δ, then $\Phi(x) \subset M_T$. Therefore, $(\lambda + \mu, \lambda)$ and $(\lambda, \mu - \lambda)$ are the only vertices of Δ lying in $(\mathfrak{t}^*_+)^\circ$. By the description of the local cones, the only possible additional vertices are $\overline{\lambda}$ and $\overline{\mu}$.

Since the weights $(-1,0)$ and $(-1,-1)$ of T on $T_{(\overline{\lambda},\overline{\mu})} Y_+$ are a lattice basis, the map

$$T \to \text{Aut } (\mathrm{T}_{(\overline{\lambda},\overline{\mu})}\mathrm{Y}_+)$$

is injective, so the principal isotropy subgroup of T acting on Y_+, which equals the principal isotropy subgroup of K acting on M, is trivial.

The assertion on transversality follows from Delzant's list of local models [12], and can also be verified directly. □

7.1. **Symplectic blow-ups of M.** We will define symplectic blow-ups as a special case of Lerman's symplectic cuts [31]. Let M be a Hamiltonian K-manifold, $\mu : M \to \mathbb{R}$ a K-invariant continuous function, and $a \in \mathbb{R}$ a real number such that in a neighborhood U of $\mu^{-1}(a)$, the function μ is a moment map for a Hamiltonian circle action. Let $M_a = \mu^{-1}(a)/S^1$ be the Marsden-Weinstein reduced space at a, and let $M_{>a} = \mu^{-1}(a, \infty)$. Then the disjoint union

$$M_{\geq a} := M_a \cup M_{>a}$$

is called the **symplectic cut** of M at a. If S^1 acts freely on $\mu^{-1}(a)$, then $M_{\geq a}$ has the structure of a smooth Hamiltonian K-manifold as follows: Define $\nu : M \times \mathbb{C} \to \mathbb{R}$ by

$$\nu(m, z) = \mu(m) - |z|^2/2$$

so that ν is a moment map for the diagonal action of S^1 on $U \times \mathbb{C}$ (where $\mathbb{C}$ has the opposite symplectic form). Let

$$U_{\geq a} = \nu^{-1}(a)/S^1$$

be the symplectic reduction of $U \times \mathbb{C}$ at a. Then

$$U_{\geq a} \cong U_a \cup U_{>a}$$

and the map $\varphi : U_{>a} \to M_{>a}$ given by inclusion defines an equivariant symplectomorphism $U_{>a} \cong \varphi(U_{>a})$. Let $M_{\geq a}$ be the union of $U_{\geq a}$ and $M_{>a}$ modulo the identification of $U_{>a}$ with $\varphi(U_{>a})$.

In case X is the minimum of μ, and S^1 acts on the normal bundle of X with weight one, then for $\epsilon > 0$ small $M_{\geq a+\epsilon}$ is a symplectic blow-up of M along the symplectic submanifold X ([31],[34]). We will need one further fact:

Proposition 7.2 (Guillemin-Sternberg). *(see* [19, 42]*) Let M be a Hamiltonian K-manifold with moment map $\Phi : M \to \mathfrak{k}^*$. The composition $\tilde{\Phi} : M \to \mathfrak{t}^*_+$ of Φ with the quotient map is a moment map for a K-equivariant action of T on KY_+, which equals the usual action of T on Y_+.*

This is a consequence of the functoriality of symplectic induction, in the sense that the Hamiltonian action of T on Y_+ induces a K-equivariant action of T on the **symplectic induction** $K \times_T Y_+$. We call this densely-defined, K-equivariant action of T the **induced T-action**. We now come to the main result of this section:

Theorem 7.3. *The Hamiltonian K-manifold $M = \Theta_\lambda \times \Theta_\mu$ contains two symplectic K-orbits: Km_1 and Km_2 where $m_1 = (\overline{\lambda}, \overline{\mu})$ and $m_2 = ((\lambda, -\lambda), (0, \mu))$. Only a symplectic blow-up of Km_1 admits an invariant compatible Kähler structure.*

Proof. For any Hamiltonian K-manifold M, an orbit $Km \subset M$ is symplectic if and only if $K_m = K_{\Phi(m)}$. If M is transversal and multiplicity-free, then $\tilde{\Phi}(x)$ can be a symplectic orbit only if $x \in (\mathfrak{t}^*_+)^\circ$ by Lemma 4.5, and then x must be a vertex of Δ, by Lemma 4.2.

Now let $v_1 = (-1, 0)$ and $v_2 = (2, 1)$, and for $i = 1, 2$ let $S^1_i = \exp(\mathbb{R}v_i)$ be the corresponding one-parameter subgroups, and let $\mu_i = \langle \tilde{\Phi}, v_i \rangle$. Since

$$(v_1, (-1, 0)) = (v_1, (-1, 1)) = 1 \quad \text{and} \quad (v_2, (0, 1)) = (v_2, (1, -1)) = 1$$

the induced action of S^1_i on KY_+ has weight one on the tangent space $T_{m_i}Y_+$, and therefore on the normal bundle to Km_i. Let

$$Bl^i_\epsilon(M) = M_{\geq (\Phi(m_i), v_i) + \epsilon}$$

be the corresponding symplectic blow-ups, which have polytopes (see Figure 3)

$$\Delta^i_\epsilon = \{x \in \Delta \mid (v_i, x) \geq (\Phi(m_i), v_i) + \epsilon\}.$$

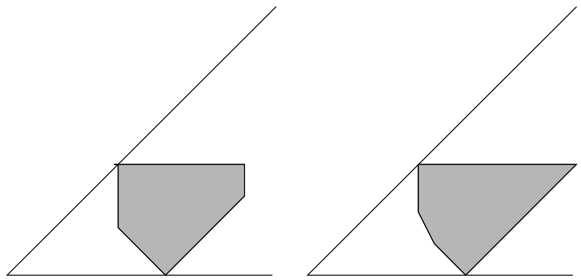

FIGURE 3. The polytopes Δ^1_ϵ and Δ^2_ϵ

The polytope Δ^2_ϵ does not satisfy the condition in Theorem 6.2, so $Bl^2_\epsilon(M)$ admits no invariant compatible Kähler structure. On the other hand, Km_1 is a subvariety, since, if $P_\lambda, P_\mu \subset G$ are parabolics such that

$$\Theta_\lambda \cong G/P_\lambda \qquad \Theta_\mu \cong G/P_\mu$$

then the isotropy group G_{m_1} equals $P_\lambda \cap P_\mu = B$, so $Gm_1 = Km_1$. By the equivalence of Kähler and symplectic blow-ups of subvarieties, $Bl^1_\epsilon(M)$ admits an invariant compatible Kähler structure. (See [34] and, for another argument, the next section.) $\square$

8. Existence results

The main result of this section is a sufficient criterion for a multiplicity-free action to admit a compatible invariant Kähler structure (Theorem 8.7), assuming Delzant's Conjecture 4.3. First, we review a few more topics from the theory of spherical varieties.

8.1. **Local structure theory.** We recall Knop's version [26, Theorem 2.3] of the local structure theorem of Brion-Luna-Vust. Let G be a connected complex reductive group, X be a normal G-variety, and D a B-stable Cartier divisor, which we assume for simplicity is effective. The divisor D induces a line bundle $[D]$, and we assume that the action of G lifts to $[D]$. (This is always possible after taking a finite cover of G.) Let σ be a section of $[D]$ defining D. The parabolic subgroup $P[D]$ of D is the normalizer of the line $\mathbb{C}\sigma$, and the character $\chi(D)$ of D is the character of the action on $\mathbb{C}\sigma$, which is well-defined up to a character of G. One has a morphism

$$\psi_D : X\backslash D \to \mathfrak{g}^* \ , \ x \mapsto l_x \text{ where } \ l_x(\xi) = \frac{\xi \ \sigma}{\sigma} \ (x).$$

Theorem 8.1 (Knop). *Let X be a normal G-variety with effective B-stable divisor D. Then the image of ψ_D is the $P[D]$-orbit through χ_D, and if we set $\Sigma = \psi_D^{-1}(\chi_D)$ and $L = G_{\chi_D}$ then there is an isomorphism*

$$X\backslash D \cong P[D] \times_L \Sigma.$$

Typically one uses the local structure theorem to obtain information about X near a G-orbit Y, and so one wants to choose a D not containing Y, but containing enough B-stable prime divisors so that L is as small as possible. In the case that $X \subset \mathbb{P}(V)$ is a smooth transversal spherical variety of maximal rank there is a particularly good choice of D. By Proposition 4.7 and Theorem 6.1, for any simple root α there are two divisors $D_\pm$ such that $H_{D_\pm}$ contains

$\Delta \cap H_\alpha$. Not both $D_\pm$ contain Y, since $H_{D_+} \cap H_{D_-} \cap \Delta = H_\alpha \cap \Delta$ and by Remark 5.6 $\Delta_{\overline{Y}}$ is not contained in H_α. Therefore, for any α there is exists a divisor $D_{\alpha,Y} \in \mathcal{D}(G/H)$ such that $\overline{D_{\alpha,Y}}$ does not contain Y, and $(v_{D_{\alpha,Y}}, \alpha) > 0$. Define an effective B-divisor by

$$D_Y = \sum_\alpha n_\alpha \overline{D_{\alpha,Y}}. \tag{6}$$

For some choice of $n_\alpha \in \mathbb{N}$, the divisor D_Y has character $\chi(D_Y) \in \mathfrak{t}^*_{\mathrm{reg}}$, so that $P[D_Y] = B$. By Theorem 8.1 there is an isomorphism $X \backslash D_Y \cong B \times_{T_\mathbb{C}} \Sigma$. Since X is spherical the variety Σ is a toric variety, and $Y \cap \Sigma$ is a $T_\mathbb{C}$-orbit in Σ. Furthermore, there is a one-to-one correspondence between B-stable divisors in $X \backslash D$ and $T_\mathbb{C}$-stable divisors in Σ, and the cones $C_{Y \cap \Sigma}$ and C_Y are equal. By the smoothness criterion for toric varieties, we have the following result, which is a special case of Brion's criterion for smoothness in [9].

Corollary 8.2. *Let $X \subset \mathbb{P}(V)$ be a smooth transversal spherical variety with polytope Δ_X of maximal dimension such that $\Delta_X \cap H_\alpha \neq \emptyset$ for all simple roots α. Then for any G-orbit $Y \subset X$, the set of valuations $v_D \in C_Y$ such that $D \supset Y$ form part of a lattice basis.*

8.2. **Line bundles over spherical varieties.** We now review several results of Brion [7] on line bundles over spherical varieties. Let $d = \sum_{D \in \mathcal{D}(X)} n_D D$ be a B-stable divisor and $Y \subset X$ a G-orbit. For each divisor $D \in D(X)$ containing Y, set $l_{d,Y}(v_D) = n_D$. If d is Cartier, then $l_{d,Y}$ extends to a linear map $l_{d,Y} : C_Y \to \mathbb{Q}$ and these maps patch together to form a piecewise-linear map $l_d : C(X) \to \mathbb{Q}$. If d is ample, the function l_d has a simple expression in terms of $\Delta([d])$:

Lemma 8.3. *Let $X \subset \mathbb{P}(V)$ be a spherical G-variety with polytope Δ and d any B-stable hyperplane section. Then for any $v \in C(X)$ we have $l_d(v) = -\min_{x \in \Delta} v(x) + v(\chi(d))$.*

Proof. Suppose that $v \in C_Y$ for some G-orbit $Y \subset X$ and let $D \in \mathcal{D}(X)$ be a divisor containing Y. Note that if σ is the canonical B-eigensection of $[d]$ then $l_d(v_D) = v_D(\sigma)$. By Lemma 2.4 H_D meets Δ so by (3)

$$l_d(v_D) = -\min_{x \in \Delta}(v_D(x)) \ + v_D(\chi(d)).$$

Since l_d is linear on C_Y, the same equation holds with v_D replaced by v. $\square$

As for toric varieties, the association $d \mapsto l_d$ is functorial in the sense that

Lemma 8.4 (Brion)**.** *Let G/H be a spherical homogeneous variety, and $\varphi : X_1 \to X_2$ be a morphism of embeddings of G/H. Then $l_{\varphi^* d}$ is the restriction of l_d to $C(X_1) \subset C(X_2)$.*

Proof. By [7, Section 2] we can assume that X_2 is **simple** (i.e. contains a unique closed G-orbit) and that

$$d = [\phi] \ + \sum_{D \in \mathcal{D}(G/H)} n_D \overline{D}$$

where $\phi \in \mathbb{C}(X_2)^{(B)}$ is a rational function, and $n_D = 0$ if D contains a G-orbit in X. If D does not contain a G-orbit, then $\varphi^* D = D$, since D does not contain the exceptional locus of φ. Therefore,

$$\varphi^* d \ - \sum_{D \in \mathcal{D}(G/H)} n_D \ \overline{D} \ = \ [\varphi^* \phi] \ = \sum_{D' \in \mathcal{D}(X_1)} v_{D'}(\varphi^* \phi) D'$$

so if $D' \in \mathcal{D}(X_1)$ contains a G-orbit then the coefficient of D' in $\varphi^* d$ is $v_{D'}(\varphi^*\phi) = l_d(v_{D'})$ as required. □

8.3. **Existence theorems.** Recall that a fan $\mathcal{F}_2$ is a **subdivision** of a fan $\mathcal{F}_1$ if any cone in $\mathcal{F}_2$ is contained in a cone in $\mathcal{F}_1$. We say that a fan $\mathcal{F}$ is **rational** if any cone $C \in \mathcal{F}$ is spanned by vectors that are rational with respect to the lattice $\exp^{-1}(\mathrm{Id}) \subset \mathfrak{t}$. If a convex polytope Δ has rational fan, then there is a canonical choice of $V(\Delta)$: we can require that each $v \in V(\Delta)$ is a primitive lattice vector.

Theorem 8.5. *Let X_1 be a projective spherical G-variety with polytope Δ_1, and $\Delta_2 \subset \Delta_1$ a convex polytope with rational fan $\mathcal{F}(\Delta_2)$ such that*

1. *$H(\Delta_1)$ is contained in $H(\Delta_2)$,*
2. *$V(\Delta_2)$ is contained in $V(\Delta_1) \cup \mathcal{V}^G(X_1)$, and*
3. *$\mathcal{F}(\Delta_2)$ is a subdivision of $\mathcal{F}(\Delta_1)$.*

Then there exists a spherical variety X_2 such that $\mathcal{C}(X_2) = \mathcal{F}(\Delta_2, \mathcal{V}^G)$ and a morphism $\varphi : X_2 \to X_1$. Furthermore, if X_2 is smooth, and for each $v \in V(\Delta_2) - V(\Delta_1)$ the difference

$$c(v) = \min_{x \in \Delta_1} v(x) - \min_{x \in \Delta_2} v(x)$$

is an integer, then there exists an ample line bundle L_2 over X_2 with polytope $\Delta(L_2) = \Delta_2$.

Remark 8.6. The polytope $\Delta(L)$ of a G-line bundle over a G-variety has rational, but not necessarily integral vertices [5].

Proof of Theorem 8.5 - For any cone $C_2 \in \mathcal{F}(\Delta_2, \mathcal{V}^G)$ let $C_1 \in \mathcal{C}(\Delta_1, \mathcal{V}^G)$ be the cone in $\mathcal{F}(\Delta_1)$ whose interior contains the interior of C_2. By Theorem 2.3, there exists a subset $E_1 \subset \mathcal{D}(G/H)$ such that $(C_1, E_1) \in \mathcal{F}(X_1)$. Let E_2 denote the set of divisors $D \in E_1$ such that $v_D \in C_2$. We claim that (C_2, E_2) is a colored cone. Indeed, suppose that $v \in V(\Delta_2)$ is an extremal vector of C_2 that does not lie in $\mathcal{V}^G$. Then v lies in $V(\Delta_1)$ and C_1. By Theorem 2.3, C_1 is the dual cone to some face F_1 of Δ_1. The vector v is normal to some facet of Δ_1 containing F_1, so that v is extremal in C_1. Since (C_1, E_1) is a colored cone, there exists a divisor $D \in E_1$ such that v_D equals v, and by definition E_2 contains D as required. If we let $\mathcal{F}_2$ be the set of all such pairs (C_2, E_2), then it is straightforward to check that $\mathcal{F}_2$ is a colored fan for G/H. By the Luna-Vust Theorem 2.2 there exists an embedding X_2 of G/H with colored fan $\mathcal{F}_2$ and (see [25, Section 4]) a morphism $\varphi : X_2 \to X_1$.

Now assume that X_2 is smooth. For each $v \in V(\Delta_2) - V(\Delta_1)$ let $D_v \in \mathcal{D}(X_2)$ denote the corresponding G-stable divisor. Let d_1 be a B-stable hyperplane section of X_1, and define

$$d_2 = \varphi^* d_1 + \sum_{v \in V(\Delta_2) - V(\Delta_1)} c(v) D_v = \sum_{D \in \mathcal{D}(X_2)} n_2(D) D.$$

Since X_2 is smooth, any Weil divisor is Cartier and so d_2 defines a line bundle $[d_2]$ over X_2. We claim that $\Delta([d_2]) = \Delta_2$.

Suppose that $d_1 = \sum_{D \in \mathcal{D}(X_1)} n_1(D) D$. By Proposition 8.4, for any $v \in V(\Delta_2) - V(\Delta_1)$ we have that

$$n_2(D_v) = l_{d_1}(v) + c(v).$$

By Corollary 8.3

$$l_{d_1}(v) = -\min_{x \in \Delta_1}(v(x)) \ + v(\chi(d_1))$$

and since $\chi(d_1)$ equals $\chi(d_2)$

$$n_2(D_v) = -\min_{x\in\Delta_2}(v(x)) \ + v(\chi(d_2)).$$

It follows that

$$\Delta(L_2) \ = \ \{y \in \Delta_1 \mid v(y) \geq \min_{x\in\Delta_2} v(x) \text{ for all } v \in V(\Delta_2) - V(\Delta_1)\}.$$

Since $H(\Delta_1) \subset H(\Delta_2)$, $\Delta(L_2)$ equals Δ_2 as required.

We now show that L_2 is ample. Let $Y_2 \subset X_2$ be a G-orbit. There exists a section $s \in H^0(L_2)^{(B)}$ non-vanishing on Y_2 if and only if the intersection $\Delta_{\overline{Y}} \cap (\Lambda \cap \chi(d_2))$ is non-empty. Since the vertices of Δ_2 are rational we can choose an integer $n \in \mathbb{N}$ such that $F \cap (\Lambda/n + \chi(d_2))$ is non-empty for any open face $F \subset \Delta_2$. Let $s \in H^0(L_2^n)^{(B)}$ be a section with $\chi(s)$ in the interior of $\Delta_{\overline{Y}}$. By work of Brion [7, Section 2], it suffices to show that $v_D(s) > 0$, for any $D \in \mathcal{D}(X_2)$ not containing Y_2.

Suppose that $v_D(s) = 0$ for some divisor $D \in \mathcal{D}(X_2)$. The vector v_D must lie in C_{Y_2}, since H_D is a supporting hyperplane containing $\Delta_{\overline{Y}}$. If D is G-stable, then D contains Y_2 by [25, Lemma 2.4].

If D is not G-stable, let $D_1 \in \mathcal{D}(X_1)$ be the closure of $D \cap G/H$ in X_1, and H_{D_1} the hyperplane defined by D_1 as in Equation (4). By definition of d_2, $n_1(D_1)$ equals $n_2(D)$ and since $\chi(d_1) = \chi(d_2)$ the hyperplanes H_D and H_{D_1} are the same. Let $Y_1 \subset X_1$ be the G-orbit such that $C^\circ_{Y_1}$ contains $C^\circ_{Y_2}$. It suffices to show that

$$H_{D_1} \supset \Delta_{\overline{Y_1}}, \tag{7}$$

since in this case D_1 contains Y_1 by Lemma 2.4 and so $D \in E_{Y_2}$ by definition. Equation (7) holds if and only if $n_1(D_1) = l_{d_1}(v_{D_1})$. Since $l_{d_1} \geq l_{d_2}$ on $C(X_2)$ and d_1 is ample we have that

$$n_1(D_1) \geq l_{d_1}(v_D) \geq l_{d_2}(v_D) = n_2(D)$$

which, since $n_1(D_1) = n_2(D)$ implies the claim.

Consider a compact, connected multiplicity-free K-manifold (M, ω_M) with polytope Δ for which Delzant's Conjecture 4.3 applies. To construct a compatible invariant Kähler structure on M, it suffices to construct a compact, connected Kähler multiplicity-free K-manifold M' with the same polytope and principal isotropy.

Theorem 8.7. *Let (M, ω_M) be a transversal, multiplicity-free, compact, connected Hamiltonian K-manifold with trivial principal isotropy and polytope Δ_M. Let (X, ω_X) be a Kähler, transversal, multiplicity-free, compact, connected Hamiltonian K-manifold with trivial principal isotropy and polytope $\Delta_X = \Phi(X) \cap \mathfrak{t}^*_+$ with $\Delta_X \cap H_\alpha$ non-empty for all simple roots α. Suppose that either*

1. *$[\omega_M]$ and $[\omega_X]$ are rational and that $\Delta_M \subset \Delta_X$ satisfies (1)–(3) in Theorem 8.5; or,*
2. *for every rational invariant symplectic perturbation ω'_M of ω_M there exists an invariant compatible symplectic form ω'_X on X such that the corresponding polytopes Δ'_M and Δ'_X satisfy (1)–(3) in Theorem 8.5.*

Then there exists a multiplicity-free, compact, connected, Kähler Hamiltonian K-manifold M_2 with trivial principal isotropy and moment polytope Δ_M.

Proof of (1) - By taking a sufficiently high multiple of $[\omega_M]$ and $[\omega_X]$, we can assume that X is a projective spherical variety and the $c(v)$'s are integral. Let M_2 be the variety given by Theorem 8.5. To prove that M_2 is smooth, let $Y \subset M_2$ be any G-orbit. The image $\varphi(Y)$ is a G-orbit in X and by Remark 5.6 the face $\Delta_{\overline{\varphi(Y)}}$ intersects the interior $(\mathfrak{t}_+^*)^\circ$. Let $D_{\overline{\varphi(Y)}}$ be the B-stable divisor in (6), and D_Y the B-stable divisor in M_2 defined by taking the closure of each $D_{\alpha,\varphi(Y)}$ in M_2. Since the support of $D_{\overline{\varphi(Y)}}$ does not contain $\varphi(Y)$, the support of D_Y does not contain Y, and D_Y is an effective B-stable divisor with $P[D_Y] = B$. By Theorem 8.1 we have an isomorphism $M_2 \backslash D_Y \cong B \times_{T_\mathbb{C}} \Sigma$. The cone $C_{Y \cap \Sigma}$ equals the cone C_Y of Y, which is the dual cone to some face F of Δ_M such that $F \cap (\mathfrak{t}_+^*)^\circ$ is non-empty (since if $F \subset H_\alpha$ then $F \subset H_\pm$ and so $C_Y^\circ \cap -\mathfrak{t}_+ = \emptyset$ which is a contradiction). Since M has trivial principal isotropy, the polytope Δ_M is torsion-free at F (see [12]). Hence, the extremal vectors of C_Y form part of a lattice basis, which implies that $Y \cap \Sigma$ consists of smooth points. This shows that M_2 is smooth, so by Theorem 8.5 there exists a Kähler structure on M_2 with polytope Δ_M.

Proof of (2) - Choose a linear K-invariant family of 2-forms $\omega_t \in \Omega^2(M)$, $t \in \mathbb{R}^n$, with $\omega_0 = \omega_M$, such that the cohomology classes

$$\{\partial_i[\omega_t]\}_{i=1}^n$$

span $H^2(M)$. Since $H^2(M, \mathbb{Q})$ is dense in $H^2(M)$, there exist K-invariant symplectic forms

$$\omega_1, \dots, \omega_n \in \Omega^2_K(M)$$

such that

1. $[\omega_1], \dots, [\omega_n] \in H^2(M, \mathbb{Q})$,
2. each ω_i lies in the neighborhood U of ω_M in Proposition 5.5, and
3. ω_M is contained in the convex hull of the ω_i's.

Let $\Delta_i, i = 1, \dots, n$ denote the polytopes of (M, ω_i). Let $c_1, \dots, c_n$ be such that $\sum c_i = 1$ and

$$\sum_{i=1}^n c_i \omega_i = \omega_M.$$

By assumption, the sets $V(\Delta_i)$ are the same, so by the preceding case there exists a single smooth spherical embedding M_2 of G/H and invariant symplectic forms $\omega_{M_2,i}$ such that the Hamiltonian K-manifold $(M_2, \omega_{M_2,i})$ has polytope Δ_i. The form

$$\omega_{M_2} = \sum_{i=1}^n c_i \omega_{M_2,i}$$

is symplectic and compatible with J, since the set of such forms is convex. Let Φ_i (resp. $\Phi_{M_2,i}$) denote the moment map for the action of K on (M, ω_i) (resp. $(M_2, \omega_{M_2,i})$) so that

$$\Phi_{M_2} = \sum_{i=1}^n c_i \Phi_{M_2,i}$$

is a moment map for the K-action on (M_2, ω_{M_2}). We claim that the polytope

$$\Delta_{M_2} = \Phi_{M_2}(M_2) \cap \mathfrak{t}_+^*$$

equals Δ. The claim follows from localization. Since (M, ω_i) is a transversal multiplicity-free Hamiltonian K-manifold, for any T-fixed point $m \in M_T$ and any $i \in \{1, \dots, n\}$, the image

$\Phi_i(m)$ lies in the regular part $\mathfrak{t}^*_{\mathrm{reg}}$, and by Lemma 4.2 $\Phi_i(m)$ must be a vertex of $w\Delta_i$, for some $w \in W$.

Similarly, the image $\Phi_i(x)$ for any $x \in (M_2)_T$ is a vertex of $w\Delta_i$ for some $w \in \Delta$ and contained in the regular part $\mathfrak{t}^*_{\mathrm{reg}}$. Indeed, the orbit $Gx \subset M_2$ is closed and the polytope Δ_{Gx} is a vertex of Δ. If Δ_{Gx} is contained in a hyperplane H_α then the dual cone, which is generated by normal vectors to facets containing Δ_{Gx}, has interior which meets $-\mathfrak{t}_+$ trivially by Proposition 4.7 which is a contradiction.

Since the classes $[\omega_{M_2,i}]$ are close in $H^2(M_2)$ and $\Phi_{M_2,i}|_{(M_2)_T}$ depends only on the cohomology class of the symplectic form, we can assume that for any $x \in (M_2)_T$, the images $\Phi_{M_2,i}(x)$ are arbitrarily close. Therefore, for any $m \in M_T$, there must exist an element $x \in (M_2)_T$ such that $\Phi_{M_2,i}(x) = \Phi_i(m)$, for each $i = 1, \dots, n$, so that

$$\begin{aligned} \Phi_{M_2}((M_2)_T) &= \{\sum c_i \Phi_{M_2,i}(x) \mid x \in (M_2)_T\} \\ &= \Phi(M_T). \end{aligned}$$

On the other hand, it is clear that the weights of T at m are the same as the weights of T at x, since, for $\Phi(m) \in (\mathfrak{t}^*_+)^\circ$ these are the edge vectors of the polytope $w\Delta_i$ at the vertex $\Phi_i(m) = \Phi_{M_2,i}(x)$, plus some subset of the roots determined by w. Therefore, by localization to the T-fixed point set (see e.g. [17]) the push-forward measures

$$\Phi_* \omega_M = (\Phi_{M_2})_* \omega_X$$

are equal, which implies that (M_2, ω_{M_2}) and (M, ω_M) have the same moment polytope.

We now apply our existence theorems to the case $K = SO(5)$.

Theorem 8.8. *Let $K = SO(5)$ and M a compact, connected, torsion-free, transversal, multiplicity-free K-manifold with polytope Δ that meets both codimension 1 faces of $\mathfrak{t}^*_+$. Then M admits an invariant compatible Kähler structure if and only if every non-negative facet contains $\Delta \cap H_\alpha$ for some α.*

Remark 8.9. Similar results hold for other rank 2 groups. For rank greater than 2 the question of sufficiency is open, even for $K = U(3)$.

Proof. By Theorem 8.7 and the Delzant Conjecture 4.3 in the case rank $(G) = 2$ [12] it suffices to show that the spherical variety $X = \Theta_\lambda \times \Theta_\mu$ has a symplectic structure ω_X such that $\Delta \subset \Delta_X$ satisfies (1)–(3) in Theorem 8.7. First we note that Theorem 2.3, Corollary 3.5 and Theorem 7.1 imply that the colored fan of X consists of a single (non-trivial) colored cone (C, E) where C is the cone on the vectors $(-1, 1), (0, -1) \in \mathfrak{t}$. (That is, X is a two-orbit variety.) Let

$$\alpha_1 = (1, -1) \quad \text{and} \quad \alpha_2 = (0, 1)$$

be the simple roots.

Let λ, μ be real numbers such that

$$\overline{\lambda} = (\lambda, \lambda) = \Delta \cap H_{\alpha_1} \quad \text{and} \quad \overline{\mu} = (\mu, 0) = \Delta \cap H_{\alpha_2}$$

First, we show that $H(\Delta_{\lambda,\mu}) \subset H(\Delta)$ where $\Delta_{\lambda,\mu}$ denotes the polytope of X. Let $v_\pm \in V(\lambda)$ be the normal vectors to facets of Δ meeting $\overline{\lambda}$. By Proposition 4.7, we have $(v_\pm, \alpha) < 0$, and since Δ is reflective, we must have

$$v_\pm = n\alpha_1 \pm m\beta_1$$

for some $\beta_1 \in H_{\alpha_1}$ and $n, m \in \mathbb{Z}/2$ with $n + m \in \mathbb{Z}$. Since Δ is torsion-free ,

$$(1,1),(1,-1) \in \operatorname{span}_{\mathbb{Z}}\{v_\pm\}$$

and so $n = m = 1/2$, in which case $V(\overline{\lambda}) = \{(1,0),(0,-1)\}$. A similar argument (using the complicated definition of torsion-free [42, Remark 10.2] in the case (K_x, K_x) is not simply connected) shows that $V(\overline{\mu}) = \{(\pm 1, 1)\}$.

Since Δ satisfies the criterion in Theorem 6.2, $V(\Delta) \subset V(\Delta_{\lambda,\mu})$. Clearly $F(\Delta)$ is a subdivision of $F(\Delta_{\lambda,\mu})$, which completes the proof. □

9. Equivalence to Tolman's criterion in the $SO(5)$ case

In this section we will use a criterion of Tolman [40] to show

Theorem 9.1. *Let M be a compact connected torsion-free transversal multiplicity-free $SO(5)$-space with moment polytope Δ that meets both codimension 1 faces of $\mathfrak{t}^*_+$. If Δ fails the criterion in Theorem 6.2, then M admits no T-invariant Kähler structure.*

Together with Theorem 8.8 this shows that the following conditions are equivalent:

1. M admits a compatible K-invariant Kähler structure.
2. M admits a compatible T-invariant Kähler structure.
3. The moment polytope Δ of M satisfies the necessary criterion in Theorem 6.2.

9.1. **Tolman's criterion.** Let T be a real torus and Y a compact connected Hamiltonian T-space with moment map $\Phi : Y \to \mathfrak{t}^*$. Let Y_T denote the fixed point set. For simplicity, we will assume that the restriction $\Phi|_{Y_T}$ of Φ to Y_T is injective. For any subgroup $H \subset T$ let

$$Y_{(H)} = \{y \in Y \mid T_y = H\}$$

be the corresponding orbit-type stratum. Let χ denote the set of connected components of the $Y'_{(H)}s$. Tolman defines:

$$\text{X-ray}(Y) = \{\Phi(\overline{X}) \mid X \in \chi\}.$$

X-ray(Y) is a finite set of convex polytopes whose vertices lie in $\Phi(Y_T)$.

Theorem 9.2 (Tolman Extendibility Theorem)**.** *Let Y be a compact, connected Hamiltonian T-space with a compatible, invariant Kähler structure. Let y be a point in Y_T and $\alpha_1, \dots, \alpha_k$ a subset of the weights of T on T_yY such that the cone C on $\alpha_1, \dots, \alpha_k$ is strictly convex. Then there exists a convex polytope $P \in$ X-ray(Y) such that*

1. *there is a neighborhood U of $\Phi(y)$ such that $P \cap U = C \cap U$, and*
2. *for each face F of P there exists a polytope $P_F \in$ X-ray(Y) of the same dimension as F, containing F.*

Tolman proves the above Theorem by constructing an orbit O of the complex torus $T_{\mathbb{C}}$ such that $P = \Phi(\overline{O})$ has property (1). The other properties follow from a theorem of Atiyah [2, Theorem 2].

Proof of Theorem 9.1 - As in the proof of Theorem 8.7 there exist $\lambda, \mu \in \mathbb{R}$ such that Δ is contained in the polytope $\Delta_{\lambda,\mu}$. Let $V_+(\Delta)$ (resp. $V_-(\Delta)$) denote the normal vectors to facets of Δ appearing in clockwise (resp. counterclockwise) order between $\overline{\lambda}$ and $\overline{\mu}$, so that $V(\Delta) = V_+(\Delta) \cup V_-(\Delta)$.

Case 1: $V_-(\Delta) = V_-(\Delta_{\lambda,\mu}) = \{(1,0),(1,1)\}$. Let x equal $(\lambda, \mu - \lambda)$ and let $y_1, \dots, y_m$ be the vertices of Δ appearing between $\overline{\lambda}$ and $\overline{\mu}$, moving clockwise. (See Figure 4.)

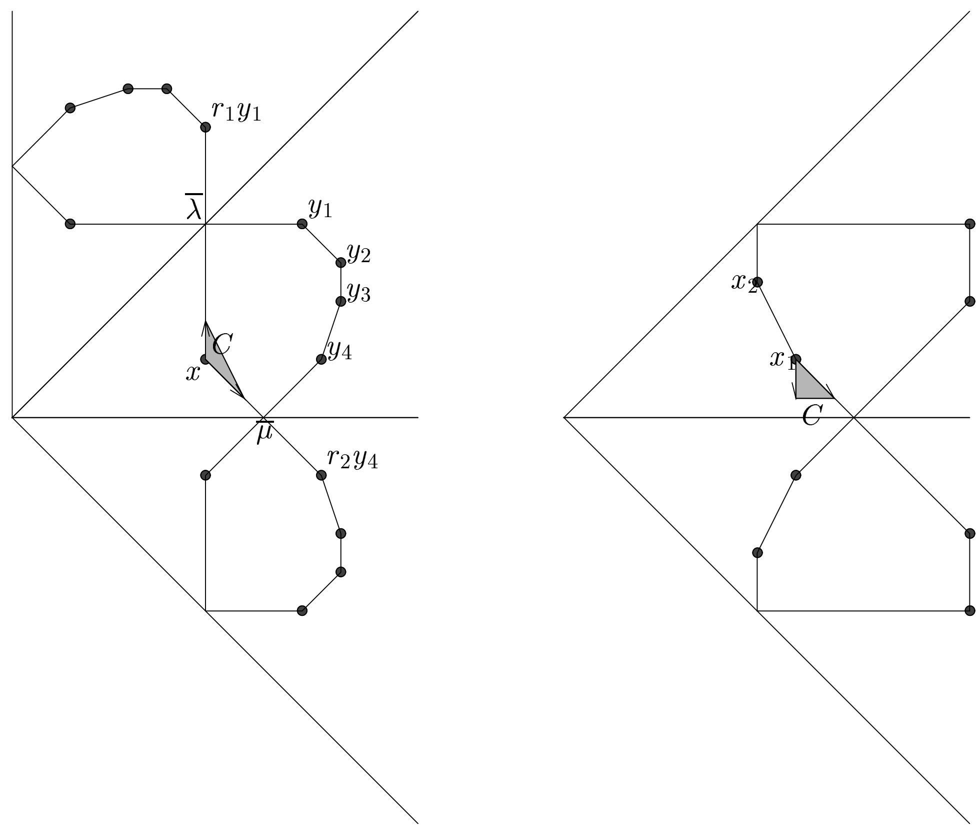

FIGURE 4. An example with (1) $V_-(\Delta) = V_-(\Delta_{\lambda,\mu})$ (2) $V_-(\Delta) \neq V_-(\Delta_{\lambda,\mu})$

Let $\alpha_1 = (1,-1)$ and $\alpha_2 = (0,1)$ be the simple roots and $r_1, r_2 \in W$ the corresponding reflections. Since the maximal torus T acts on Y_+ with discrete principal isotropy subgroup, the weights of T on $T_{\Phi^{-1}(x)}Y_+ \subset T_{\Phi^{-1}(x)}M$ are α_1, α_2. Let C be the cone at x on α_1, α_2 and $P \subset \mathfrak{t}^*$ be the polytope guaranteed by Theorem 9.2, and $p_1, \dots, p_l \in \mathfrak{t}^*$ the vertices of P, starting with $p_1 = x$ and moving clockwise. Since r_1y_1 is the only element of $\Phi(M_T)$ lying in $x + \mathbb{R}_+(0,1)$, we must have $p_2 = r_1y_1$ and by the same argument $p_l = r_2y_m$. The set of weights of T on $T_{\Phi^{-1}(r_1y_1)}M$ is $r_1N \cup \{\beta_1, \beta_2\}$ where N is the set of negative roots and β_1, β_2 are the weights of T on r_1Y_+. By convexity of P, we must have $p_3 \in p_2 + \mathbb{R}_+(1,-1)$. Similarly, $p_{m-1} \in p_m + \mathbb{R}_+(0,1)$. Since Δ does not satisfy the condition in Theorem 6.2, either $v_1 \notin -\mathfrak{t}_+$ or $v_{m+1} \notin -\mathfrak{t}_+$.

Assume the latter. If a vertex y_k with $k \neq m-1$ lies in $r_2y_m + \mathbb{R}_+(0,1)$ then y_k is contained in a facet $F \subset \Delta$ which lies in the interior $(\mathfrak{t}^*_+)^\circ$ of $\mathfrak{t}^*_+$. We can assume that such a vertex does not exist. Indeed, $\tilde{\Phi}^{-1}(F)$ is a codimension 2 submanifold of M, invariant under T. Since the action of T on (M,J) has isolated fixed points, by e.g. Carrell-Liebermann [10] the dual class of $\tilde{\Phi}^{-1}(F)$ is representable by a two-form $\omega_F \in H^{1,1}(M)$. For ϵ small, the triple $(M, \omega + \epsilon\omega_F, J)$

is a T-equivariant Kähler manifold. Let $\Phi_\epsilon : M \to \mathfrak{t}^*$ be the corresponding moment map, and $m \in M$ any T-fixed point. The image $\Phi_\epsilon(m)$ depends only on the cohomology class of ω_F, and therefore we can assume that ω_F is supported in neighborhood of $\tilde{\Phi}^{-1}(F)$. If $m \notin \tilde{\Phi}^{-1}(F)$, then $\Phi_\epsilon(m) = \Phi(m)$, and if $m \in \Phi^{-1}(y_k)$ then

$$\Phi_\epsilon(m) \in y_k + \mathbb{R}_+ \beta,$$

where β is the weight of T on $T_m Y_+ / T_m \Phi^{-1}(F)$. We proceed with (M, ω, Φ) replaced by $(M, \omega + \epsilon\omega_F, J)$.

By a similar argument, we can assume that no vertex of Δ lies in $p_{l-1} + \mathbb{R}_+(-1,1)$ which implies that $p_{l-2} = y_{m-1}$. But then v_{m+1} is a normal vector to P lying clockwise between $(-1,-1)$ and $(0,-1)$ - that is, $v \in -\mathfrak{t}_+$ which is a contradiction. The other case is similar.

Case 2: $V_-(\Delta) \neq V_-(\Delta_{\lambda,\mu})$**.** Let $x_1, \ldots, x_k \in \Delta$ be the vertices of Δ appearing clockwise between $\overline{\mu}$ and $\overline{\lambda}$. (See Figure 4.)

Let C be the cone at x_1 on $(0,-1)$ and $(1,-1)$. The proof that there is no polytope P satisfying the requirements of Theorem 9.2 is similar to the proof for Case 1, and left to the reader.

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Harvard University, Dept. of Mathematics, 1 Oxford Street, Cambridge, Massachusetts 02138
E-mail address: `woodward@math.harvard.edu`