The index formula for the moduli of G-bundles on a curve

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Dedicated to the memory of Raoul Bott

Abstract

We prove the formulae conjectured by the first author for the index of K-theory classes over the moduli stack of algebraic G-bundles on a smooth projective curve. The formulae generalise E. Verlinde's for line bundles and have Witten's integrals over the moduli space of stable bundles as their large level limits. As an application, we prove the Newstead-Ramanan conjecture on the vanishing of high Chern classes of certain moduli spaces of semi-stable G-bundles.

Introduction

Let G be a reductive, connected complex Lie group and $\mathfrak{M}$ the moduli stack of algebraic G-bundles over a smooth projective curve Σ of genus g. In this paper, we determine the *analytic index* on a dense subring of the topological K-theory of $\mathfrak{M}$. For line bundles, we recover the famous formula due to E. Verlinde [V], which we extend to include the *Atiyah-Bott classes*, described in §1. From this angle, our index is analogous to Witten's cohomological integration formula [W] over the moduli space of semi-stable bundles, which appears for us in the large level limit of the index. Like the Verlinde formula, but unlike Witten's, our index is expressed as a finite sum; this removes the convergence problems and consequent regularisation in [W]. While other regularisations have been considered in the literature [JK], ours is intrinsically meaningful in topological K-theory, and expresses the fact that indexes of vector bundles over $\mathfrak{M}$, and not just those of line bundles, are controlled by finite-dimensional Frobenius algebras [T3].

For a smooth projective variety X, the analytic index of a holomorphic vector bundle V can be defined as the alternating sum $\chi(X;V)$ of its sheaf cohomologies. This agrees with the *topological index* of V, defined by the Gysin map to a point in topological K-theory. The construction extends to certain well-behaved Artin stacks. Thus, when G acts on X, a vector bundle over the quotient stack X/G corresponds to an equivariant bundle V over X, and the holomorphic Euler characteristic $\chi(X/G;V)$ (defined, say, using the simplicial bar construction of X/G) agrees with the invariant part of the virtual G-representation $\chi(X;V)$. Now, $\chi(X;V)$ agrees with the *equivariant topological index*, the image of V under the Gysin map from $K^0_{G_k}(X)$ to R_G, the representation ring of the maximal compact subgroup $G_k \subset G$. Regarding the map $R_G \to \mathbb{Z}$ which extracts the invariant part of a representation as the Gysin projection from the classifying stack BG_k to a point gives us an "analytic = topological" index theorem for X/G.

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Our stack $\mathfrak{M}$ fails a basic test for good behaviour: it has infinite type. Thus, when G is a torus, $\mathfrak{M}$ has infinitely many connected components, labelled by $H^2(\Sigma, \pi_1 G)$. Nonetheless, $\mathfrak{M}$ has a distinguished *Shatz stratification.* For a torus, the strata are the connected components. Finite, open unions of strata can be presented as quotients of smooth quasi-projective varieties by reductive groups; this allows us to use familiar techniques of sheaf cohomology. In addition, special geometric features of the stratification — reflected in the properties of canonical parabolic reductions of G-bundles — ensure the finiteness of sheaf cohomology, and allows us to define the index, for a sub-ring of *admissible* K-theory classes. When G is simply connected, these classes are dense in the rational K-theory of $\mathfrak{M}$, in the topology induced by the stratification. The index is not continuous in this topology and does not extend to all of K^0; nevertheless, interesting limits do exist, such as in our application to the Newstead conjecture in §7.

This extension of Verlinde's formula, capturing the index of vector bundles, emerged from the discovery that a certain twisted K-theory was the topological home for the index of line bundles over $\mathfrak{M}$ [FHT, §8]. Thus motivated, formulae for the index of admissible classes were proposed in [T3], equating the analytic index over $\mathfrak{M}$, defined from coherent sheaf cohomology, with a topological index defined in twisted K-theory. (See also the informal notes [T4].) The topological index can be calculated by the Atiyah-Bott fixed-point method, and when $\pi_1 G$ is free, one obtains a formula in terms of the maximal torus T and the Weyl denominator. In this situation, [T3] offers two conjectural formulae for the analytic index over $\mathfrak{M}$ which do not involve twisted K-theory: a localisation formula, which reduces the index to the stack of principal T-bundles, and a Verlinde-like formula involving a finite sum over conjugacy classes.

We prove these formulae here. It is clear that our method leads to the equality of analytic and topological indexes for all compact groups, but for simplicity we confine ourselves to *connected groups with free* π_1; explicit formulae for more general groups require additional calculations (Remark 4.14).

In principle, we also solve the index problem over the more traditional moduli *space* M of semi-stable bundles. For large levels (twists by large line bundles), the contribution of unstable strata of $\mathfrak{M}$ vanishes, and the index over $\mathfrak{M}$ is equal to that over the open sub-stack $\mathfrak{M}^{\mathrm{ss}}$ of semi-stable bundles (1.6). The cohomology of a coherent sheaf over $\mathfrak{M}^{\mathrm{ss}}$ agrees with that of its direct image to M. The index over M, which is a projective variety, depends (quasi-)polynomially on the level, so we can give a formula (unpleasant, but explicit) for the index of (the direct image to M of) admissible classes, at any level. When all semi-stable G-bundles are stable,[1] $\mathfrak{M}^{\mathrm{ss}}$ is an orbifold with coarse quotient M, and the rational cohomology calculation of [AB] shows that we generate all of $K^0(M; \mathbb{Q})$ in this way.

Our proof relies on a remarkable symmetry of the index over $\mathfrak{M}$ which is *absent* on M, or on any finite-type approximation. The symmetry arises from a loop group version of Bott's reflection argument [B], a *Hecke correspondence.* (This device was already used in [BS] in relation to the Verlinde formula.) The reader should refer to §1.1 below for the definitions in what follows. Choose an *admissible line bundle* $\mathfrak{L}$ and an *index bundle* $E^*_\Sigma V$. For a weight λ of the maximal torus T, denote by V_λ the holomorphically induced virtual G-representation. Regard the index of $\mathfrak{L} \otimes \exp[tE^*_\Sigma V] \otimes E^*_x V_\lambda$, a formal series in t, as the e^λ-coefficient for a Fourier series on T, with values in $\mathbb{Q}[[t]]$. This series turns out to be anti-invariant for a certain action of the affine Weyl group, and is thereby constrained to represent a sum of δ-functions at prescribed, *regular* points of T. Regularity of its support, combined with Atiyah's localisation theorem for the index of transversally elliptic operators, implies that the index distribution only sees the contribution of principal bundles whose structure group reduces to T. That can be calculated by Riemann-Roch, leading to our explicit index formula.

[1]When $G \neq \mathrm{GL}(n)$, this condition can only hold if we include parabolic structures.

The paper is organised as follows. In §1 we describe the admissible K-classes and define their analytic index. We include a brief review of the stratification of $\mathfrak{M}$ and the local cohomology vanishing results of [T2]. Section 2 contains the precise statements of our formulae. The proof is split into §3, where we check the anti-symmetry of the index distribution, and §4 where we eliminate the contributions of non-toric principal bundles.

The last sections contain two applications. In §5, we show how Witten's integration formulae over M arise from our index formula in the large level limit; we only give full details for SL(2). (The formulae were proven for SL(r) by Jeffrey-Kirwan [JK] and, independently of our work but simultaneously, by Meinrenken [Me] for compact, 1-connected G.) Section 6 enhances our index formulae by incorporating Kähler differentials, needed in our next application in §7 to a conjecture of Newstead and Ramanan. The original version, proved by Gieseker [G], asserted the vanishing of the top $2g-1$ Chern classes of the moduli space of stable, odd degree vector bundles of rank 2 on Σ. An analogue in rank 3 was settled by Kiem and Li [KL]. We generalise this to the vanishing of the top $(g-1)\ell$ rational Chern classes of the moduli space M of stable principal bundles with semi-simple structure group of rank ℓ, whenever M (or a variant decorated with parabolic structures) is a compact orbifold.

The appendix reviews some background on the topological K-theory of $\mathfrak{M}$ and on its variants decorated with parabolic structures; the exotic parabolic structure associated to the simple affine root of $\mathfrak{g}$ plays a special rôle in the proof. We do not review general properties of stacks and their cohomology, these matters having had increasing coverage in the literature since the detailed treatments [BL, LS]; a review suited to our needs is found in [T1, T2].

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Notation. G is a reductive group, T a maximal torus and $B \supset T$ a Borel subgroup. G_k, T_k will be the compact forms and $\mathfrak{g}_k, \mathfrak{t}_k$ their Lie algebras. The co-weight lattice of T, $\log(1)/2\pi\mathrm{i}$, lies in $\mathrm{i}\mathfrak{t}_k$; its $\mathbb{Z}$-dual is the weight lattice in $\mathrm{i}\mathfrak{t}_k^\vee$. W is the Weyl group and $\Delta := \prod_{\alpha>0} 2\sin(\mathrm{i}\alpha/2)$ the Weyl denominator. The Weyl vector ρ is the half-sum of the positive roots. The simple roots are $\alpha_1, \ldots, \alpha_\ell$; when $\mathfrak{g}$ is simple, the simple affine root α_0 sends $\xi \in \mathfrak{t}$ to $1 - \vartheta(\xi)$, with the highest root ϑ. (The affine root vector of α_0 is $z^{-1}e_\vartheta$.) The representation ring of G_k is denoted by R_G, and $\mathbb{C}R_G := \mathbb{C} \otimes R_G$.

1. Atiyah-Bott classes

In this section, we introduce the Atiyah-Bott classes and admissible classes. We then define their analytic index and derive its finiteness from the local cohomology vanishing results of [T2]. This requires a brief review of the Shatz stratification.

1.1 Admissible classes. Given a representation V of G, call E^*V the vector bundle over $\Sigma \times \mathfrak{M}$ associated to the universal G-bundle. Call π the projection along Σ, $\sqrt{K}$ a square root of the relative canonical bundle, and $[C]$ the topological K_1-homology class of a 1-cycle C on Σ. Consider the following classes in the topological K-theory of $\mathfrak{M}$:

(i) The restriction $E_x^*V \in K^0(\mathfrak{M})$ of E^*V to a point $x \in \Sigma$;

(ii) The slant product $E_C^*V := E^*V/[C] \in K^{-1}(\mathfrak{M})$ of E^*V with $[C]$;

(iii) The Dirac index bundle $E_\Sigma^*V := R\pi_*(E^*V \otimes \sqrt{K}) \in K^0(\mathfrak{M})$ of E^*V along Σ;

(iv) The inverse *determinant of cohomology*, $D_\Sigma V := \det^{-1} E_\Sigma^* V$.

We call the classes (i)–(iii) the *Atiyah-Bott generators*; they are introduced in [AB, §2], along with their counterparts in cohomology, and can also be described from the Künneth decomposition of E^*V in

$$K^0(\Sigma \times \mathfrak{M}) \cong K^0(\Sigma) \otimes K^0(\mathfrak{M}) \oplus K^1(\Sigma) \otimes K^1(\mathfrak{M}),$$

by contraction with the various classes in Σ. Classes (i) and (iv) are represented by algebraic vector bundles, while (iii) can be realised as a perfect complex of $\mathcal{O}$-modules. The class $E^*_C V$ in (ii) is not algebraic. Note that $\det E^*_\Sigma V = \det R\pi_*(E^*V)$ when $\det V$ is trivial; an important example is the canonical bundle $\mathcal{K} = \det E^*_\Sigma \mathfrak{g}$ of $\mathfrak{M}$, defined from the adjoint representation $\mathfrak{g}$.

For simply connected groups, we could stay with determinant line bundles, but those are too restrictive in general. Call a line bundle $\mathcal{L}$ *positive* if $\mathcal{L}^{\otimes n}$ is topologically isomorphic to $D_\Sigma V$ for some $n > 0$ and an infinitesimally faithful representation V. Call $\mathcal{L}$ *admissible* if $\mathcal{L} > \mathcal{K}^{1/2}$. ($\mathcal{K}$ has a distinguished *Pfaffian* square root [LS].) Products of an admissible line bundle and any number of Atiyah-Bott generators span the ring of *admissible classes.*

1.2 Remark. (i) Positivity of $\mathcal{L}$ depends only on its rational Chern class in $H^2(\mathfrak{M};\mathbb{Q})$: namely, $c_1(\mathcal{L})$ must lie in the $\mathbb{Q}_+$-span of the $c_1(D_\Sigma V)$'s with $\mathfrak{g}$-faithful V.

(ii) Taking $V = \mathfrak{g}$ for semi-simple G shows that $\mathcal{K}$ is negative, and so $\mathcal{O}$ is admissible. This fails for a torus, but positive line bundles are admissible (that is, $\mathcal{K} \le 0$) for any G; cf. §1.3 below.

(iii) For $g > 1$ and simply connected G, our positivity agrees with ampleness on the moduli space. (Suffices to check this for simple G: recall then that $\mathrm{Pic}(\mathfrak{M}) = \mathbb{Z}$ and that $\mathcal{K}^{-1}$ is ample.) When $\pi_1 G \neq 0$, positivity as defined is much more restrictive, since it relates the Chern classes over the different components of $\mathfrak{M}$; cf. Remark 1.4 below.

1.3 Level of a line bundle. We associate to an admissible line bundle on $\mathfrak{M}$ a quadratic form on the Lie algebra, which we call its *level.* For any V, Riemann-Roch along Σ expresses $c_1(E^*_\Sigma V)$ as the image of $ch_2(V) = \frac{1}{2}c_1^2(V) - c_2(V)$ under the *transgression* $\tau : H^4(BG;\mathbb{Q}) \to H^2(\mathfrak{M};\mathbb{Q})$ (the construction (1.1.iii) in cohomology). It is important that τ is injective (Remark 4.11). We now identify $H^4(BG;\mathbb{R})$ with the space of invariant symmetric bilinear forms on $\mathfrak{g}_k$ so that the trace $\xi, \eta \mapsto \mathrm{Tr}_V(\xi\eta)$ corresponds[2] to $ch_2(V)$. The traces span the negative definite cone in $H^4(BG;\mathbb{R})$, and we conclude that the Chern class of a positive line bundle is transgressed from a unique positive definite quadratic form on $\mathfrak{g}_k$, which we call the *level.* Thus, for SL_n, the level of the positive generator of Pic is $-\mathrm{Tr}_{\mathbb{C}^n}$, in the standard representation. For another example, $c := -\frac{1}{2}\mathrm{Tr}_\mathfrak{g}$ is the level of $\mathcal{K}^{-1/2}$. Similarly, $\mathcal{L}$ is admissible iff $c_1(\mathcal{L})$ is transgressed from a form $h > -c$.

1.4 Remark. When G is simply connected, $\tau : H^4(BG;\mathbb{Z}) \to H^2(\mathfrak{M};\mathbb{Z})$ is an isomorphism, but this fails (even rationally) as soon as $\pi_1 G \neq 0$. Admissible line bundles are then very special.

1.5 The index of an admissible class. We first recall the finiteness result which enables us to define the index of admissible classes by means of sheaf cohomology. It is a consequence of [T2], combining the relative case of the main theorem in *loc. cit.*, §5 with the discussion of $\mathfrak{M}$ in §8 and §9. For the reader's convenience we will also outline the proof in §1.12, after we review the stratification of $\mathfrak{M}$.

Let $\mathcal{E}$ be the twist of an external tensor product $\boxtimes E^*V_k$ of universal bundles over $\Sigma^n \times \mathfrak{M}$ by an admissible line bundle $\mathcal{L}$. Call $\underline{\mathcal{E}}$ the direct image to $\Sigma^n \times M$, the moduli space, of the restriction of $\mathcal{E}$ to the semi-stable part $\mathfrak{M}^{\mathrm{ss}}$. Consider the projections ϕ and $\underline{\phi}$ from $\Sigma^n \times \mathfrak{M}$, resp. $\Sigma^n \times M$, to Σ^n.

[2] It is more standard to identify Tr with $2\,ch_2$; our choice here avoids factors of 2 elsewhere.

1.6 Lemma. *The total direct image* $\bigoplus_i R^i\phi_*\mathcal{E}$ *is coherent on* Σ^n*. For large enough* $\mathcal{L}$*, depending on the* V_k*, it agrees with* $\bigoplus_i R^i\underline{\phi}_*\underline{\mathcal{E}}$*.* $\square$

A lower bound for the level of $\mathcal{L}$ can be given, linear in the highest weights of the V_k, see §1.12.

Choose now cycles C_k on Σ of various dimensions, but with even total degree. We wish to define the index of $\mathcal{L}\otimes\bigotimes_k E^*_{C_k}V_k$ over $\mathfrak{M}$ as the Euler characteristic of its coherent sheaf cohomology, but the non-algebraic classes (1.1.ii) impose the indirect

1.7 Definition. The *index* over $\mathfrak{M}$ of $\mathcal{L}\otimes\bigotimes_k E^*_{C_k}V_k$ is the pairing of $\times_k[C_k]\in K_0(\Sigma^n)$ with the topological K-theory class of $\sum_i(-1)^iR^i\phi_*\mathcal{E}$.

When all the C_k are even, we can switch and push down along Σ^n first, recovering thus the Euler characteristic. When G is abelian, the index theorem applied to the components of $\mathfrak{M}$ shows that our index only depends on the underlying topological K-class of the bundle. That is less obvious in general, but will follow for instance from our abelianisation formula (2.20).

1.8 Shatz stratification. Recall that any G-bundle over Σ admits a canonical reduction of structure group to a standard parabolic subgroup P of G, for which the associated bundle with Levi structure group is semi-stable. Topologically, this reduction is classified by a co-weight of $P/[P,P]$; we identify this with a (possibly fractional) dominant co-weight ξ of $\mathfrak{g}$, called the *instability type* of the original bundle. Then, P is the standard parabolic subgroup defined by ξ; we will denote it by P_ξ and its Levi subgroup by G_ξ. If $\mathfrak{M}_\xi$ denotes the stack of G-bundles of type ξ, we have an algebraic stratification [Sh, AB]

$$\mathfrak{M}=\bigcup\nolimits_\xi\mathfrak{M}_\xi.$$

Sending a P_ξ-bundle to its associated Levi bundle gives a morphism from $\mathfrak{M}_\xi$ to the stack $\mathfrak{M}^{\mathrm{ss}}_{G_\xi,\xi}$ of semi-stable principal G_ξ-bundles of type ξ; the fibres are quotient stacks of affine spaces by nilpotent groups. The virtual normal bundle for the morphism $\mathfrak{M}^{\mathrm{ss}}_{G_\xi,\xi}\to\mathfrak{M}$ is the complex

$$\nu_\xi=R\pi_*E^*(\mathfrak{g}/\mathfrak{g}_\xi)[1].$$

Its K-theory Euler class should be the alternating sum of exterior powers[3]

$$\lambda_{-1}(\nu_\xi^\vee):=\sum(-1)^p\lambda^p(\nu_\xi^\vee),$$

but for now this infinite sum is only a formal expression, whose meaning is to be spelt out.

1.9 Local cohomology. Finite, open unions of Shatz strata

$$\mathfrak{M}_{\le\xi}=\bigcup\nolimits_{\mu\le\xi}\mathfrak{M}_\mu$$

can be presented as quotient stacks of smooth quasi-projective varieties by reductive groups. The cohomology with supports over $\mathfrak{M}_\xi$ of a vector bundle $\mathcal{E}$ is

$$H^\bullet_{\mathfrak{M}_\xi}(\mathfrak{M}_{\le\xi},\mathcal{E}_{\le\xi})=H^{\bullet+d_\xi}(\mathfrak{M}_\xi,\mathcal{R}_\xi\mathcal{E})\tag{1.10}$$

where d_ξ is the co-dimension of $\mathfrak{M}_\xi$ and $\mathcal{R}_\xi\mathcal{E}\to\mathfrak{M}_\xi$ the sheaf of "$\mathcal{E}$-valued residues along $\mathfrak{M}_\xi$", the cohomology sheaves relative to the complement of $\mathfrak{M}_\xi$. Pushing down to $\mathfrak{M}^{\mathrm{ss}}_{G_\xi,\xi}$ and passing to the associated graded sheaf for the filtration by order of the pole leads to

$$\cdot H^\bullet(\mathfrak{M}^{\mathrm{ss}}_{\xi,G_\xi},\mathcal{E}_\xi\otimes\mathrm{Eul}(\nu_\xi)^{-1}_+)\tag{1.11}$$

[3]Recall that the pth exterior power λ^p of a complex $V^0\xrightarrow{\delta}V^1$ is the complex with qth space $\Lambda^{p-q}V^0\otimes\mathrm{Sym}^qV^1$ and obvious differential induced by δ. A similar definition applies to symmetric powers.

where $\mathcal{E}_\xi$ is the restriction to $\mathfrak{M}^{\mathrm{ss}}_{G_\xi,\xi}$, while the complex of sheaves

$$\mathrm{Eul}(\nu_\xi)_+^{-1} := \mathrm{Sym} R\pi_* \left(E^*(\mathfrak{p}_\xi/\mathfrak{g}_\xi)[1]^\vee \oplus R\pi_* E^*(\mathfrak{g}/\mathfrak{p}_\xi)[1]\right) \otimes \det(R\pi_* E^*(\mathfrak{g}/\mathfrak{p}_\xi)[1])[d_\xi]$$

is formally an inverse to the Euler class $\lambda_{-1}(\nu_\xi^\vee)$ which "prefers" the ξ-negative eigenvalues in the geometric expansion.

1.12 Finiteness and vanishing. All eigenvalues of ξ appearing in $\mathrm{Eul}(\nu_\xi)_+^{-1}$ are negative, with finite multiplicity. The determinant factor has weight $c(\xi,\xi)$ (negative, as $\xi \in \mathrm{i}\mathfrak{t}_k$). An admissible line bundle factor $\mathcal{L}$ in $\mathcal{E}$ changes this behaviour to $(h+c)(\xi,\xi)$. Atiyah-Bott bundles $E^*_?V$ alter this behaviour linearly in ξ. Overall, for any admissible $\mathcal{E}$, the ξ-invariant part of $\mathcal{E} \otimes \mathrm{Eul}(\nu_\xi)_+^{-1}$ is finite-dimensional, and vanishes for all but finitely many ξ.

It follows that almost all cohomologies (1.10) vanish, and the index of $\mathcal{E}$ over $\mathfrak{M}$ is the sum of finitely many local contributions over the $\mathfrak{M}_\xi$. Passage to the Gr does not change the index and we obtain

$$\mathrm{Ind}(\mathfrak{M},\mathcal{E}) = \sum\nolimits_\xi \mathrm{Ind}(\mathfrak{M}^{\mathrm{ss}}_{G_\xi,\xi}, \mathcal{E}_\xi \otimes \mathrm{Eul}(\nu_\xi)_+^{-1}). \tag{1.13}$$

Lemma 1.6 is the relative version of this story for the projection ϕ to Σ^n, with $R^i\phi_*$ replacing cohomology and $\sum(-1)^i R^i\phi_* \in K^0(\Sigma^n)$ replacing the index.

1.14 Remark. Formula (1.13) is related to the *non-abelian localisation* principle of Witten [W]. When presenting $\mathfrak{M}_{\le\xi}$ as a quotient of a manifold by a reductive group, the $\bar\partial$ operator can be deformed so that the invariant part of its kernel localises at the critical points of the norm-square of the moment map, leading to the individual contributions in (1.13), see Paradan [P].

1.15 Remark. Inadmissible $\mathcal{L}$'s can have infinitely many contributions to (1.13). However, when G is semi-simple, the theorems of Kumar [Ku] and Mathieu [Ma] imply the vanishing of all the direct images for negative $\mathcal{L}$: $\mathfrak{M}$ is then isomorphic to a quotient of the generalised flag variety $X := G((z))/G[[z]]$ for the loop group by the group $G[\Sigma\setminus\{x\}]$ of algebraic maps on the punctured curve, and the cohomology of $\mathcal{L}$ vanishes over X.

2. The index formulae

The index formulae involve a sum over deformations of certain *Verlinde conjugacy classes* in G, which appear in the formula for line bundles. We start by recalling that story.

2.1 Isogenies from admissible levels. Contraction $\xi \mapsto \iota(\xi)h$ with the level h of an admissible line bundle $\mathcal{L}$ maps the co-weight lattice to its dual, the weight lattice. This map descends to a homomorphism $\chi : T \to T^\vee$, the dual torus. The homomorphism χ' defined from $h' := h + c$ is an isogeny, with kernel $F \subset T$. Let F_ρ be the translate of F lying over $e^{2\pi \mathrm{i}\rho} \in T^\vee$. This last point does not depend on the Weyl chamber used to define ρ, and gives the Spin covering of T in the adjoint representation $\mathfrak{g}$.

2.2 Example. If $G = \mathrm{SL}(n)$, $\mathrm{Pic}(\mathfrak{M}) \cong \mathbb{Z}$, with positive generator $\mathcal{O}(1) = D_\Sigma\mathbb{C}^n$ and $\mathcal{K} = \mathcal{O}(-2n)$. $T^\vee$ is the maximal torus of $\mathbb{P}\mathrm{SL}(n)$ and χ, for $\mathcal{O}(1)$, is the natural projection. Hence, for $\mathcal{L} = \mathcal{O}(l)$, $F = F_\rho$ comprises the $(l+n)^{th}$ roots of the centre of $\mathrm{SL}(n)$. The analogue holds for simply connected, simply laced groups, if n is replaced by the dual Coxeter number.

A formula of E. Verlinde (first given in the context of conformal field theory) describes the index of a determinant line bundle over $\mathfrak{M}$. Let Θ be the sum of delta-functions on the *regular* G_k-conjugacy classes through F_ρ, divided by the order $|F|$ of F. Define a linear map $R_G \to \mathbb{Z}$

on representations by

$$U \mapsto \Theta(U) = \int_{G_k} \mathrm{Tr}_U(g) \cdot \Theta(g) dg = \sum_{f \in F_\rho^{\mathrm{reg}}/W} \mathrm{Tr}_U(f) \cdot \frac{\Delta(f)^2}{|F|}. \tag{2.3}$$

(Recall that we normalised the Weyl denominator so that $\Delta(f)^2$ is the volume of the conjugacy class.) Let $\theta(f) = \Delta(f)^2/|F|$; Verlinde's formula[4] gives the index of $\mathfrak{L}$ as

$$\mathrm{Ind}(\mathfrak{M}; \mathfrak{L}) = \sum_{f \in F_\rho^{\mathrm{reg}}/W} \theta(f)^{1-g}. \tag{2.4}$$

2.5 Remark. There is a version of formula (2.4) with F_ρ replaced by F. The components $\mathfrak{M}^{(\gamma)}$ of $\mathfrak{M}$ are labeled by $\gamma \in \pi_1 G$, and the Spin covering of the adjoint representation of G defines a character $\sigma : \pi_1 G \to \{\pm 1\}$. The calculations of §4 give a *graded* index formula

$$\sum_{\gamma \in \pi_1 G} \sigma(\gamma) \cdot \mathrm{Ind}(\mathfrak{M}^{(\gamma)}; \mathfrak{L}) = \sum_{f \in F^{\mathrm{reg}}/W} \theta(f)^{1-g}. \tag{2.6}$$

The same applies to our generalised index formulae below.

2.7 Remark. The kernel of the pairing $(U, U') \mapsto \Theta(U \otimes U')$ is the ideal $I_h \subset R_G$ of virtual characters which vanish on F_ρ^{reg}. We obtain a non-degenerate pairing on the quotient R_G/I_h, which becomes an integral Frobenius algebra, the *Verlinde ring at level h*. Its complex spectrum is F_ρ^{reg}/W. A folk result asserts that a Frobenius algebra is the same as a 2-dimensional topological field theory, and formula (2.4) is the "partition function" for a genus g surface in the Verlinde ring.

2.8 Deformations. Given a representation V of G, consider the following formal one-parameter family of transformations on G:

$$g \mapsto m_t(g) := g \cdot \exp\left[t\, \nabla \mathrm{Tr}_V(g)\right], \tag{2.9}$$

with the gradient in the bilinear form h'. This descends to the space $G/\mathrm{Ad}G$ of conjugacy classes; note from the Ad-invariance of Tr_V that singular classes remain singular. Restricting to conjugacy classes in G_k and composing with Θ gives a formal t-family $\Theta_t := \Theta \circ m_t$ of distributions, even though the points f_t of its support, the solutions to $m_t(f_t) = f$, can move in complex directions (when Tr_V is not real):

$$\Theta_t(U) := \int_{G_k} \mathrm{Tr}_U(g) \cdot \Theta(m_t(g))\, dg = \sum_{f \in F_\rho^{\mathrm{reg}}/W} \mathrm{Tr}_U(f_t) \cdot \theta_t(f_t) \in \mathbb{C}[[t]].$$

The θ_t are described as follows. Call H_V the Hessian of Tr_V: $H_V(u)(\xi, \eta) = \mathrm{Tr}_V(u\xi\eta)$, for $u \in T$ and $\xi, \eta \in \mathfrak{t}$, and denote by $H_V(u)^\dagger$ its conversion via h' to an endomorphism of $\mathfrak{t}$. In view of the volume scaling under (2.9), we have

$$\theta_t(f_t) = \det{}^{-1}\left[1 + tH_V(f_t)^\dagger\right] \cdot \frac{\Delta(f_t)^2}{|F|}. \tag{2.10}$$

Here is an alternate description of θ_t. Strictly speaking, it applies only when Tr_V is real; see the closely related Fourier expansion of Θ_t in §4.12, which is free of this flaw. The push-down of Θ to the space $G_k/\mathrm{Ad}G_k = T_k/W$ of unitary conjugacy classes is

$$\sum\nolimits_{F_\rho/W} \theta(f) \cdot \delta_f = \Delta^2 \cdot \delta_\rho \circ \chi', \tag{2.11}$$

[4] See, e.g. [AMW] for semi-simple G; we shall reprove it below when $\pi_1 G$ is free.

with the delta-function δ_ρ at $e^{2\pi i\rho} \in T^\vee$ and the isogeny $\chi' : T \to T^\vee$ of §2.1. Viewing $d\,\mathrm{Tr}_V$ as a map $T \to \mathfrak{t}^\vee$, χ' has a formal deformation to $\chi'_t := \chi' \cdot \exp(td\,\mathrm{Tr}_V)$, under which (2.11) deforms to

$$\sum \theta_t(f_t) \cdot \delta_{f_t} = \Delta^2 \cdot \delta_\rho \circ \chi'_t. \tag{2.12}$$

2.13 Remark. Pulling back Fourier modes on $T^\vee$ by χ'_t defines a group homomorphism from $\pi_1 T$ to the units in $\mathbb{C}R_T[[t]]$. This defines a *(higher) twisting* for the equivariant K-theory $K_T\,(T;\mathbb{C}[[t]])$. This extends to a twisting for the conjugation action of G on itself, and the twisted K-group $K_G(G)$ turns out to be the quotient of $\mathbb{C}R_G[[t]]$ by the kernel of the pairing $\Theta_t(U \otimes U')$. It is a Frobenius algebra over $\mathbb{C}[[t]]$, deforming the complex Verlinde ring at $t = 0$. See [T3] for more details.

2.14 Even Atiyah-Bott generators. We incorporate the index bundles (1.1.iii) into Verlinde's formula by means of a generating function

$$\mathcal{L} \otimes \exp[t_1 E^*_\Sigma V_1 + \ldots + t_n E^*_\Sigma V_n] \otimes E^*_x U \in K^\bullet(\mathfrak{M})[[t_1, \ldots, t_n]].$$

Let $\theta_{\mathbf{t}}(f)$ denote the multi-parameter version of $\theta_t(f)$, for $\mathbf{t} = (t_1, \ldots, t_n)$.

2.15 Theorem (Index formula for even classes)**.**

$$\mathrm{Ind}\,(\mathfrak{M}; \mathcal{L} \otimes \exp[t_1 E^*_\Sigma V_1 + \ldots + t_n E^*_\Sigma V_n] \otimes E^*_x U) = \sum_{f \in F^{\mathrm{reg}}_\rho / W} \theta_{\mathbf{t}}(f)^{1-g} \cdot \mathrm{Tr}_U(f_{\mathbf{t}}).$$

With $\mathbf{t} = 0$ and the trivial representation U, this recovers (2.4).

2.16 Example. When $G = \mathrm{SL}(2)$, $\mathcal{L} = \mathcal{O}(l)$ and $\mathrm{Tr}_V = \sum \varphi_n u^n$ on matrices with eigenvalues $\{u, u^{-1}\}$, we have, as conjectured in [T3],

$$\mathrm{Ind}\,(\mathfrak{M}; \mathcal{L} \otimes \exp[tE^*_\Sigma V]) = \sum_{\zeta_t} \left[\frac{2l + 4 + t\ddot{\varphi}(\zeta_t)}{\left|\zeta_t - \zeta_t^{-1}\right|^2} \right]^{g-1}$$

where the ζ_t range over the solutions of $\zeta_t^{2l+4} \cdot \exp\left(t\dot{\varphi}(\zeta_t)\right) = 1$ with positive imaginary part, $\dot{\varphi}(u) = \sum n\varphi_n u^n$ and $\ddot{\varphi}(u) = \sum n^2 \varphi_n u^n$.

2.17 Odd generators. The bilinear form $h' + tH_V(u)$ on $\mathfrak{t}$ is non-degenerate; denote by $\langle\,|\,\rangle(u)$ the inverse form on $\mathfrak{t}^\vee$. To an even product ψ of odd Atiyah-Bott generators (1.1.ii), we assign a function $[\psi](u)$ on T as follows: split ψ into quadratic factors $E^*_C U \wedge E^*_{C'} U'$, replace each factor by the number

$$-\#(C \cap C') \cdot \langle d\,\mathrm{Tr}_U(u) | d\,\mathrm{Tr}_{U'}(u) \rangle(u),$$

where $\#(C \cap C')$ is the intersection pairing, and sum over all possible quadratic splittings, with signs as required by re-ordering. Set $[\psi](u) = 0$ if ψ is odd. We shall see in §4 that $[\psi](u)$ is expressible in terms of the integral of the Chern character $\mathrm{Ch}(\psi)$ against a Gaussian form $\exp\{[h' + tH_V(u)] \otimes \eta\}$ on the Jacobian of T-bundles on Σ; in particular, it only depends on the K-theory class of ψ. The following gives the index for odd and even classes; for simplicity we use a single V.

2.18 Theorem (Index formula for general classes)**.**

$$\mathrm{Ind}\,(\mathfrak{M}; \mathcal{L} \otimes \exp[tE^*_\Sigma V] \otimes E^*_x U \otimes \psi) = \sum_{f \in F^{\mathrm{reg}}_\rho / W} \mathrm{Tr}_U(f_t) \theta_t(f_t)^{1-g} \cdot [\psi](f_t).$$

2.19 Abelianisation. We will derive our index formulae from a more conceptual "virtual localisation" to the stack $\mathfrak{M}_T$ of T-bundles. Let $\nu := R\pi_* E^*(\mathfrak{g}/\mathfrak{t})[1]$ be the virtual normal bundle for the morphism $j : \mathfrak{M}_T \to \mathfrak{M}$. In §4.2, we will see that the K-theoretic Euler class $\lambda_{-1}(\nu^\vee)$ is well-defined after inverting the Weyl denominator and equals

$$\lambda_{-1}(\nu^\vee) = (-1)^{2\rho(\gamma)} \Delta^{2g-2} \mathcal{K}^{1/2}$$

on the component of $\mathfrak{M}_T$ of topological type γ.

2.20 Theorem. *For admissible* $\mathcal{E}$, $\operatorname{Ind}(\mathfrak{M}; \mathcal{E}) = |W|^{-1}$"Ind" $\left(\mathfrak{M}_T; j^*\mathcal{E} \otimes \lambda_{-1}(\nu^\vee)^{-1}\right)$.

The right-hand side needs clarification. Each component of $\mathfrak{M}_T$ is the product of the classifying stack BT with a Jacobian of T-bundles, and the index over $\mathfrak{M}_T$ should be the sum of the T-invariant parts of indexes over these Jacobians. Because of the Weyl denominator, $\lambda_{-1}(\nu^\vee)$ is not invertible in $K^\bullet(\mathfrak{M}_T)$; the index of $j^*\mathcal{E} \otimes \lambda_{-1}(\nu^\vee)^{-1}$ over each Jacobian lands in $R_T[\Delta^{-1}]$, and its T-invariant part is not *a priori* well-defined. However, we will see in §4.12 that summing over all Jacobians leads to a well-defined distribution on the regular part of T_k, supported on F_ρ^{reg}. We *declare* the index over $\mathfrak{M}_T$ to be the invariant part (= integral over T_k) of this index distribution, *after extension by zero to the singular locus.*

A formula which does not require inverting $|W|$ will be given in Proposition 4.1. In a family of curves, the index is replaced with a K-theory class on the base, and the alternate formula loses slightly less torsion. We hope to return to this in future work.

3. Affine Weyl symmetry

In this section, we establish the anti-symmetry of the index under an action of the affine Weyl group (Proposition 3.3). This constrains the general form of the answer (Corollary 3.8).

3.1 Affine Weyl action. Define a group homomorphism from the co-root lattice Π to the units in $\mathbb{C}R_T[[t]]$ by

$$\Pi \ni \gamma \mapsto \exp\left[\iota(\gamma)h' + t\frac{\partial \mathrm{Tr}_V}{\partial \gamma}\right].$$

(This is the homomorphism mentioned in Remark 2.13.) Multiplication by these units combines with the Weyl transformations into an action of the *affine Weyl group* $W_{\mathrm{aff}} := W \ltimes \Pi$ on $\mathbb{C}R_T[[t]]$. This action extends to the space of formal (unrestricted) Fourier series on T with coefficients in $\mathbb{C}[[t]]$.

For a weight μ of B, call V_μ the *holomorphically induced* virtual representation of G, that is, the G-equivariant index of the *weight line bundle* $\mathcal{O}(\mu)$ over the flag variety G/B. Define the following *index series* on T, a formal Fourier series with coefficients in $\mathbb{C}[[t]]$:

$$\mathcal{I} := \sum\nolimits_\mu \operatorname{Ind}(\mathfrak{M}; \mathcal{L} \otimes \exp[tE_\Sigma^* V] \otimes E_x^* V_{\mu+\rho}) e^{-\mu}. \tag{3.2}$$

We use a single t and only even classes to keep the notation manageable, but this restriction is not necessary (cf. §3.9 below).

3.3 Proposition. *The index series* $\mathcal{I}$ *is anti-invariant under the affine Weyl action.*

Proof. Weyl anti-invariance being clear from the holomorphic induction step, it suffices to confirm, for each simple factor of $\mathfrak{g}$, the sign change under the following affine reflection S: the highest Weyl reflection s_0, followed by subtraction of the co-root H of the highest root ϑ. This is the affine analogue of the famous Bott reflection [B].

The stack $\mathfrak{M}(x,\mathcal{B})$ of G-bundles over Σ with B-reduction at x is a G/B-fibre bundle over $\mathfrak{M}$ (Example A.4.i) and carries natural extensions of the weight line bundles $\mathcal{O}(\mu)$ on the fibre. From the Borel-Weil-Bott and Leray theorems, the e^μ Fourier coefficient is

$$\mathcal{I}(e^\mu) = \mathrm{Ind}\left(\mathfrak{M}(x,\mathcal{B}); \mathcal{L}(\mu+\rho)\otimes\exp[tE_\Sigma^*V]\right),$$

where () is the twist by $\mathcal{O}(\mu)$ and, abusively, $\mathcal{L}$ stands for its own lift to $\mathfrak{M}(x,\mathcal{B})$.

Let $\mathfrak{M}'$ be the stack of G-bundles with parabolic structure at x defined by the simple affine root α_0 (Example A.4.iii). We have $\mathfrak{M}(x,\mathcal{B}) \cong \mathfrak{P}\times_{\mathbb{P}\mathrm{SL}(2)}\mathbb{P}^1$, for the principal $\mathbb{P}\mathrm{SL}(2)$-bundle $\mathfrak{P}\to\mathfrak{M}'$ determined by α_0. Call p the projection to $\mathfrak{M}'$. For a vector bundle $\mathcal{E}$ over $\mathfrak{M}(x,\mathcal{B})$ which tensors into SL(2)-equivariant bundles over the two factors,[5] define a new bundle $\mathbb{D}\mathcal{E}$ by dualising the $\mathbb{P}^1$-factor and then twisting by the relative canonical bundle $\mathcal{O}(-\vartheta)$. Relative Serre duality along $\mathbb{P}^1$ gives $R^ip_*\mathbb{D}\mathcal{E} = R^{1-i}p_*\mathcal{E}$ (as can be seen from SL(2)-equivariant Serre duality on $\mathbb{P}^1$ and self-duality of SL(2)-representations). Integrating over $\mathfrak{M}'$ shows that the indexes of $\mathcal{E}$ and $\mathbb{D}\mathcal{E}$ over $\mathfrak{M}(x,\mathcal{B})$ differ by a sign, and we will prove our proposition by relating S to $\mathbb{D}$.

We claim that $\mathcal{L}$ factors for the fibre product presentation of $\mathfrak{M}(x,\mathcal{B})$ as

$$\mathcal{L}\cong\mathcal{L}(-\tfrac{1}{2}\iota(H)h)\boxtimes\mathcal{O}(\tfrac{1}{2}\iota(H)h). \tag{3.4}$$

Then, $\mathbb{D}\mathcal{L} = \mathcal{L}(-\iota(H)h-\vartheta)$. Further,[6] $\iota(H)c = \rho - s_0\rho+\vartheta$, and we get

$$\mathbb{D}\left[\mathcal{L}(\mu+\rho)\right] = \mathcal{L}(s_0\mu-\iota(H)h'+\rho) = \mathcal{L}(S\mu+\rho), \tag{3.5}$$

confirming the proposition for $t=0$.

To verify (3.4), note that *some* such formula must hold, with h replaced by a fixed multiple of itself; namely, the one which renders the first factor trivial along the fibres $\mathbb{P}^1$. From its definition, it follows that $\mathbb{D}$ preserves any square root $\mathcal{K}^{1/2}(\rho)$ of the canonical bundle of $\mathfrak{M}(x,\mathcal{B})$. Setting $h=-c, \mu=0$ gives a fixed-point for (3.5) and shows the Ansatz (3.4) to be correct.

For more general admissible classes, S and $\mathbb{D}$ are only related after splitting some filtrations. Denote by $\partial E_x^*V/\partial H$ the sum of weight line bundles on $\mathfrak{M}(x,\mathcal{B})$ defined by the virtual character $\partial\mathrm{Tr}_V/\partial H$ of T. We claim that

$$\begin{aligned}\mathbb{D}\,\mathrm{gr}\left\{\mathcal{L}(\mu+\rho)\otimes\exp[tE_\Sigma^*V]\right\} &= \\ = \mathcal{L}(S\mu+\rho)\otimes\mathrm{gr}\left\{\exp[-t\cdot\partial E_x^*V/\partial H)]\otimes\exp[tE_\Sigma^*V]\right\},\end{aligned} \tag{3.6}$$

for certain finite filtrations (term by term in t) on the two sides. To see this, let ν be the highest weight of V and let $\mathcal{E}'$ be the sheaf of sections of E^*V whose λ-weight component vanishes at x to order $\frac{1}{2}(\nu-\lambda)(H)$ or higher. This condition is stable under the α_0-root $\mathfrak{sl}(2)$, so $\mathcal{E}'$ descends to $\Sigma\times\mathfrak{M}'$. The quotient $\mathcal{Q}=E^*V/\mathcal{E}'$ is supported on $\{x\}\times\mathfrak{M}(x,\mathcal{B})$. It has a finite filtration whose associated graded sheaf is a sum of weight line bundles $\mathcal{O}(\lambda)$ on $\mathfrak{M}(x,\mathcal{B})$, for the weights λ of V and various multiplicities. By construction,

$$s_0(\mathrm{gr}\mathcal{Q}) - \mathrm{gr}\mathcal{Q} = \partial E_x^*V/\partial H.$$

Dualising $\mathrm{gr}\,E_\Sigma^*V$ along the fibres of p then results in $\mathrm{gr}\,E_\Sigma^*V - \partial E_x^*V/\partial H$, proving (3.6).

The sign change of $\mathcal{I}$ under the action (3.1) of S follows now by factoring the index map via $\mathfrak{M}'$, since splitting the filtrations does not change the index. $\square$

[5] But such that the diagonal action factors through $\mathbb{P}\mathrm{SL}(2)$.

[6] Both sides are parallel to ϑ, so the equality only needs testing against $H/2$, when the two sides become $c(H,H)/2$ and $\rho(H)+1$, which are equal to the dual Coxeter number.

3.7 Example. When $G = \mathrm{SL}(2)$, $\mathfrak{M}(x, \mathcal{B})$ is the moduli stack of pairs $(\mathcal{E}, L)$, where $\mathcal{E}$ is a rank 2 bundle with trivial determinant and L a line in the fibre at x. $\mathfrak{M}'$ is naturally equivalent to the stack of rank 2 bundles with determinant identified with the line bundle $\mathcal{O}_\Sigma(-x)$. The morphism p takes $(\mathcal{E}, L)$ to its sub-sheaf $\mathcal{E}'$ of sections whose value at x lies in L. The lines L assemble to the weight line bundle $\mathcal{O}(-1)$ over $\mathfrak{M}(x, B)$. The vector bundles associated to irreducible representations of SL(2) are the symmetric powers of $\mathcal{E}$, and the maximal sub-sheaves in the proof of (3.3) are the symmetric powers of $\mathcal{E}'$. The quotient $S^n\mathcal{E}/S^n\mathcal{E}'$ is supported at x, its associated graded sheaf over $\mathfrak{M}(x, B)$ is $\bigoplus_{0 \le k \le n} \mathcal{O}(n-2k)^{\oplus k}$ and the anti-symmetrisation is $\bigoplus \mathcal{O}(k)^{\oplus k}$ ($k = n \mod 2$, $|k| \le n$).

3.8 Corollary. *The series $\mathcal{I}$ represents a Weyl anti-symmetric linear combination of δ-functions on T_k. In particular, it is supported at regular points only.*

Proof. Weyl anti-symmetry is clear. Assume first that G is simply connected, so that $\Pi = \pi_1 T$. At $t = 0$, invariant functionals under the lattice Π are spanned by δ-functions supported on F. However, the t-deformed action is obtained from the one at $t = 0$ by the change of coordinates (2.9); so the Π-invariant Fourier series are spanned by the δ-functions at the regular f_t.

In general, the W_{aff}-symmetry of §3.1 can be enhanced by the action of the co-weights of the centre $Z(G) \subset G$: geometrically, these central co-weights define elementary transformations on bundles which translate the components of $\mathfrak{M}$, and the multiplicative factor in the W_{aff}-action corrects for the change in $\mathcal{L} \otimes \exp[tE^*_\Sigma V]$. The extended lattice is co-compact in $\mathfrak{t}_k$, so our index functional is a span of δ-functions, as before. ☐

3.9 Odd classes. The arguments of this section also apply to more general bundles $\bigotimes_k E^*_{C_k} V_k \otimes \mathcal{L} \otimes \exp[tE^*_\Sigma V]$ which include odd factors $E^*_{C_k} V_k$ from (1.1.ii): each C_k can be moved to avoid the Hecke point x, and $E^*_{C_k} V_k$ remains unchanged in the Serre duality step. Let us rephrase this observation to match our indirect definition 1.7 of the index.

The index is obtained by first pushing down $\boxtimes_k E^* V_k \otimes \mathcal{L} \otimes \exp[tE^*\Sigma V] \otimes E^*_x V_\mu$ to Σ^n, and then taking the index over $\times C_k$. Now, $\times C_k$ lies within $(\Sigma^\circ)^n$, with $\Sigma^\circ := \Sigma \setminus \{x\}$, so we can restrict our bundle to $\mathfrak{M}(x, \mathcal{B}) \times (\Sigma^\circ)^n$. In repeating the arguments, we note that each bundle $E^* V_k$ is in fact pulled back from $\mathfrak{M}' \times (\Sigma^\circ)^n$: this is because the corresponding $\mathcal{E}'$ used the proof of (3.3) is part of an exhaustive filtration, once we allow arbitrary poles at x. Therefore, each factor survives Serre duality unchanged, leading to the same symmetry.

4. Abelianisation

We now prove Theorems 2.18 and 2.20. For technical reasons, we must use the stack $\mathfrak{M}(x, \mathcal{B})$ of bundles decorated with a Borel structure at $x \in \Sigma$. Fix an admissible class $\mathcal{E}$ and let

$$\mathcal{I}_\mathcal{E} := \sum\nolimits_\mu \mathrm{Ind}\left(\mathfrak{M}(x, \mathcal{B}); \mathcal{E}(\mu + \rho)\right) \cdot e^{-\mu}.$$

Taylor expansion in Corollary 3.8 (and §3.9, if $\mathcal{E}$ contains odd classes) shows that $\mathcal{I}_\mathcal{E}$ is a distribution supported on a finite set of regular points in T_k. Let $\widetilde{\mathfrak{M}}_T$ denote the moduli of T-bundles trivialised at x (a disjoint union of Jacobians), and Ind_T the T-equivariant index of a vector bundle lifted from $\mathfrak{M}_T$. Call ν_B the virtual normal bundle of the morphism $\mathfrak{M}_T \to \mathfrak{M}(x, \mathcal{B})$.

4.1 Proposition. *Over the regular part of T_k, $\mathcal{I}_\mathcal{E} = \mathrm{Ind}_T\left(\widetilde{\mathfrak{M}}_T; \mathcal{E}(\rho)/\lambda_{-1}(\nu_B^\vee)\right)$ as distributions.*

For the proof, we will show that the decomposition (1.13) for $\mathcal{I}_\mathcal{E}$ converges as a distribution, that the terms with $G_\xi \ne T$ do not contribute over the regular part of T_k, and finally that the terms with $G_\xi = T$ contribute the Ind_T of (4.1). We need some preliminary calculations.

4.2 The Euler complex. For any ξ labelling a Shatz stratum, recall the complex

$$\mathrm{Eul}(\nu_\xi)_+^{-1} = \mathrm{Sym}\left(R\pi_*E^*(\mathfrak{p}_\xi/\mathfrak{g}_\xi)[1]^\vee \oplus R\pi_*E^*(\mathfrak{g}/\mathfrak{p}_\xi)[1]\right) \otimes \det(R\pi_*E^*(\mathfrak{g}/\mathfrak{p}_\xi)[1])[d_\xi].$$

It splits by ξ-eigenvalue into bounded complexes with coherent cohomologies, and for index purposes we may perform K-theoretic cancellations. One such arises from Serre duality

$$R\pi_*E^*\mathfrak{p}_\xi/\mathfrak{g}_\xi[1]^\vee = R\pi_*(E^*\mathfrak{g}/\mathfrak{p}_\xi \otimes K), \tag{4.3}$$

using the G_ξ-isomorphism $(\mathfrak{g}/\mathfrak{p}_\xi)^\vee = \mathfrak{p}_\xi/\mathfrak{g}_\xi$. Replacing the second complex, in K-theory, by $R\pi_*E^*\mathfrak{g}/\mathfrak{p}_\xi \oplus (2g-2)E_x^*\mathfrak{g}/\mathfrak{p}_\xi$ simplifies the symmetric factor to

$$\left(\mathrm{Sym}\, E_x^*\mathfrak{g}/\mathfrak{p}_\xi\right)^{\otimes(2g-2)}.$$

Similarly, the determinant can be rewritten as $\det^{g-1} E_x^*\mathfrak{g}/\mathfrak{p}_\xi \otimes D_\Sigma(\mathfrak{g}/\mathfrak{p}_\xi)$. Using Serre duality as in (4.3) again, we get $D_\Sigma(\mathfrak{g}/\mathfrak{p}_\xi) \cong D_\Sigma(\mathfrak{p}_\xi/\mathfrak{g}_\xi)$. With Δ_ξ, ρ_ξ and $\mathcal{K}_\xi$ denoting the G_ξ-counterparts of $\Delta, \rho, \mathcal{K}$, and using the codimension formula $d_\xi = (g-1)\dim(\mathfrak{g}/\mathfrak{p}_\xi) + 2(\rho-\rho_\xi)(\xi)$, we obtain the following K-theoretic replacement of the inverse Euler complex:

$$\mathbb{E}(\nu_\xi)_+^{-1} := (-1)^{2(\rho-\rho_\xi)(\xi)} E_x^*\,(\Delta_\xi/\Delta)_+^{2g-2} \otimes (\mathcal{K}_\xi/\mathcal{K})^{1/2}, \tag{4.4}$$

remarkably, a line bundle twist of a geometric series of weight line bundles. The subscript denotes that ξ-negative modes are to be chosen for the Fourier expansion of the Weyl denominator.

For simplicity, we have avoided parabolic structures; in the special case of Borel structure at a single point x, to be used in the proof below, (4.4) carries an additional factor of $e^{\rho_\xi-\rho}(\Delta_\xi/\Delta)_+$, from the flag varieties of G and G_ξ.

Proof of Proposition 4.1. Stratify $\mathfrak{M}(x,\mathcal{B})$ using a generic polarisation (§A.5) and express the index distribution $\mathcal{I}_\mathcal{E}$ as a sum of contributions from Shatz strata $\mathfrak{M}_\xi$ as in (1.13). Each $\mathcal{I}_{\mathcal{E},\xi}$ is a formal Fourier series whose coefficients are the indices of $\mathcal{E}_\xi \otimes \mathbb{E}(\nu_\xi)_+^{-1}$ over the moduli stack $\mathfrak{M}^{\mathrm{ss}}_{G_\xi,\xi}$ of semi-stable G_ξ-bundles of topological type ξ, with Borel reduction at x. We claim that

(i) Each $\mathcal{I}_{\mathcal{E},\xi}$ is a distribution on T_k.
(ii) Unless $G_\xi = T$, $\mathcal{I}_{\mathcal{E},\xi}$ is supported in the $\mathfrak{g}$-singular locus of T_k.
(iii) As distributions, $\sum \mathcal{I}_{\mathcal{E},\xi} = \mathcal{I}_\mathcal{E}$.

When $\mathfrak{g}_\xi = \mathfrak{t}$, $\mathbb{E}(\nu_B)_+^{-1} = \lambda_{-1}(\nu_B^\vee)^{-1}$ over the regular locus of T_k, proving our proposition subject to the three claims.

We now prove the claims. Since the polarisation is generic, the stack $\mathfrak{M}^{\mathrm{ss}}_{G_\xi,\xi}$ is the quotient by T of a smooth, quasi-projective variety on which the centre $\mathfrak{z}_\xi$ of $\mathfrak{g}_\xi$ acts trivially and $\mathfrak{t}/\mathfrak{z}_\xi$ acts freely. If we ignore the automorphisms coming from the trivial action of Z_ξ, then $\mathfrak{M}^{\mathrm{ss}}_{G_\xi,\xi}$ is a smooth, proper Deligne-Mumford stack. The Narasimhan-Mehta-Seshadri construction [MS] presents the underlying orbifold as a (locally free) quotient of a compact manifold by a compact group: namely the quotient by $(T/Z_\xi)_k$-conjugation of the manifold M_ξ^* of flat unitary G_ξ-connections on $\Sigma \setminus \{x\}$ with a prescribed, regular value in T_k of the monodromy at x.

The index of a vector bundle over $\mathfrak{M}^{\mathrm{ss}}_{G_\xi,\xi}$ is the Z_ξ-invariant part of the index of its direct image to the orbifold. GAGA, applied to the coarse moduli space, allows us to use the holomorphic Euler characteristic instead. As in the manifold case, that can be identified with the index of a twisted Dolbeault operator (see e.g. Duistermaat [D]). By Kawasaki [K, Example II], the latter is the invariant part of the distributional index of a twisted Dolbeault operator on

M_ξ^*, which is transversally elliptic for the T_k-action.[7] By (4.4), expansion into Fourier modes equates the index series $\mathcal{I}_{\mathcal{E},\xi}$ with the distributional Dolbeault index of $\mathcal{E}_\xi \otimes (\mathcal{K}_\xi/\mathcal{K})^{1/2}$ on M_ξ^*, multiplied by $(\Delta_\xi/\Delta)_+^{2g-2}$. Since Z_ξ acts trivially, the distributional index is in fact a Fourier polynomial along Z_ξ, and the ξ-negative choice of the Fourier expansion ensures convergence of the sum to a distribution. This proves claim (i).

We derive claim (ii) from Atiyah's localisation theorem [A, Thm. 4.6], which asserts that the distributional index of a transversally elliptic operator is supported over the union of all stabiliser subgroups. Now, the freedom of $\pi_1 G$ implies that all stabilisers of the T_k-action on $M_{G_\xi,\xi}^*$ lie in the $\mathfrak{g}$-singular locus. Indeed, a result of Borel's ensures that the G_k-centraliser of any $\mathfrak{g}$-regular torus element is T_k itself; but if all monodromies were in T_k, then the monodromy around x would be trivial.

Finally, for (iii), it suffices to fix ξ and G_ξ and show convergence of the sum of $\mathcal{I}_{\mathcal{E},\xi+\gamma}$ over the co-weights γ of Z_ξ. We'll also need to divide into co-sets of W/W_ξ, for the different expansions of the inverse Euler class. Compared with $\mathcal{I}_{\mathcal{E},\xi}$, the Atiyah-Bott factors (i) and (ii) in $\mathcal{I}_{\mathcal{E},\xi+\gamma}$ are unchanged, while each index bundle $E_\Sigma^* V$ acquires a summand $E_x^*(\partial \mathrm{Tr}_V/\partial\gamma)$. This is a sum of weight spaces of V, with multiplicities linear in γ, and factors out of the index. Finally, a line bundle $\mathcal{L}$ of $\mathcal{E}$ gets shifted by the weight $\iota(\gamma)h$ of T (see for instance (4.8) below), while $\mathbb{E}(\nu_\xi)_+^{-1}$ acquires a factor of $e^{\iota(\gamma)c}$ from the canonical bundles. This gives a sum of the form

$$\sum_\xi \mathcal{I}_{\mathcal{E},\gamma} = \sum_{\gamma;j;\mu} p_{j,\mu}(\gamma)\mathcal{I}'_{\mathcal{E},j,\mu}\cdot e^\mu \cdot e^{\iota(\gamma)h'} \prod_{\alpha(\xi+\gamma)<0} (1-e^\alpha)^{2-2g},$$

over finitely many values of j,μ, with distributional Dolbeault indices $\mathcal{I}'_{\mathcal{E},j,\mu}$ of vector bundles over the M_ξ^* and polynomials $p_{j,\mu}$ in γ. The distributions $\mathcal{I}'_{\mathcal{E},j,\mu}$ are in fact Fourier polynomials along Z_ξ, so the negativity constraint on the roots α and negative-definiteness of h' on the co-weight lattice assures distributional convergence after expanding out into a Fourier series. $\square$

4.5 The Jacobian contributions. In preparation for the proof of Theorem 2.18, we now spell out the Riemann-Roch formula for the T-Jacobians. The components of $\mathfrak{M}_T$ are labelled by the first Chern classes of T-bundles, valued in the co-weight lattice. Each component $\mathfrak{M}_T^{(\gamma)}$ factors as $J_\gamma \times BT$, so that the projection to BT lifts the T-representation $\mathbb{C}_\mu$ with weight μ to the line bundle $E_x^*\mathbb{C}_\mu$, and each J_γ is identified with the T-Jacobian $J := J_0$ by an elementary transformation at x. Call ω the positive integral generator of $H^2(\Sigma)$ and Ψ the duality tensor in $H_1(\Sigma)\otimes H^1(\Sigma)$. After the natural identifications

$$H^1(J) \cong T^\vee J \cong H^1(\Sigma;\mathfrak{t})^\vee \cong \mathfrak{t}^\vee \otimes H_1(\Sigma),$$

we have on $\mathfrak{M}_T^{(\gamma)} \times \Sigma$

$$c_1(E^*\mathbb{C}_\mu) = \pi^* c_1(E_x^*\mathbb{C}_\mu) + \mu(\gamma)\cdot\omega + \mathrm{i}\mu\otimes\Psi. \tag{4.6}$$

With the cup-product form $\eta \in \Lambda^2 H_1(\Sigma)$, we note the relation

$$(\mu\otimes\Psi)^2 = -2\mu^2 \otimes \eta\wedge\omega \in H^4(J\times\Sigma),$$

where μ^2 is the square in $\mathrm{Sym}^2\mathfrak{t}^\vee$. We now convert K-classes over $\mathfrak{M}_T = J\times BT$ into cohomology classes on $J\times\Sigma$ with coefficients in R_T: thus, the Chern character $\mathrm{Ch}(E_x^*\mathbb{C}_\mu)$ becomes the group

[7] The reader may refer to [Ve, P] for further discussion of these methods.

character e^{μ}. Formula (4.6) gives

$$\begin{aligned}
\mathrm{Ch}(E^*\mathbb{C}_\mu) &= e^{\mu}\,(1+\mu(\gamma)\cdot\omega)\,(1+\mathrm{i}\mu\otimes\Psi+\mu^2\otimes\eta\wedge\omega),\\
\mathrm{Ch}\,(E^*_\Sigma\mathbb{C}_\mu) &= e^{\mu}\,\big(\mu(\gamma)+\mu^2\otimes\eta\big)\\
\mathrm{Ch}\,(E^*_C\mathbb{C}_\mu) &= e^{\mu}\cdot C\otimes\mathrm{i}\mu\in H^1(J),\\
\mathrm{Ch}(D_\Sigma\mathbb{C}_\mu) &= e^{-\mu\cdot\mu(\gamma)}\exp(-\mu^2\otimes\eta),\\
\mathrm{Ch}\,(\exp\,[tE^*_\Sigma\mathbb{C}_\mu)]) &= \exp\left\{te^{\mu}\left[\mu(\gamma)+\mu^2\otimes\eta\right]\right\},
\end{aligned}\tag{4.7}$$

whence we get on $BT\times J_\gamma$, for any T-representations U,V, the two formulae

$$\mathrm{Ch}(D_\Sigma U) \;=\; e^{\iota(\gamma)h}\cdot\exp(h\otimes\eta),\tag{4.8}$$
$$\mathrm{Ch}\,(\exp[tE^*_\Sigma V])\,(u) \;=\; \exp\left\{t\left[\partial\mathrm{Tr}_V(u)/\partial\gamma+H_V(u)\otimes\eta\right]\right\},\tag{4.9}$$

with the metric $h=-\mathrm{Tr}_U$ on $\mathfrak{t}$ and the Hessian 2-form $H_V(u)$ of Tr_V at $u\in T$.

Finally, to find the Riemann-Roch expression for $\lambda_{-1}(\nu^\vee)$, we apply the argument of §4.2 over a γ-component of $\mathfrak{M}_T$, restricting to the regular points of T_k (where the choice of expansion of the series is immaterial). We get from (4.8)

$$\mathrm{Ch}(\lambda_{-1}\nu^\vee)^{-1}=(-1)^{2\rho(\gamma)}\Delta^{2-2g}e^{c(\gamma)}\exp[c\otimes\eta].\tag{4.10}$$

4.11 Remark. Note from (4.8) that h can be recovered from $c_1(D_\Sigma U)$ when G is a torus, and then for any G by passing to the maximal torus.

Proof of Theorem 2.18. Summing over γ the products of contributions in (4.8), (4.9) and (4.10) gives the following answer on $T\times J$:

$$\begin{aligned}
&\sum_{\gamma\in\pi_1T}\mathrm{Ch}\,(\mathcal{L}\otimes\exp[tE^*_\Sigma V])\wedge\mathrm{Ch}(\lambda_{-1}\nu^\vee)^{-1}=\\
&\quad=\sum_{\gamma}\frac{(-1)^{2\rho(\gamma)}}{\Delta(u)^{2g-2}}\left[u^{h'}\exp\left[td\,\mathrm{Tr}_V(u)\right]\right]^{\gamma}\cdot\exp\left\{[h'+tH_V(u)]\otimes\eta\right\}=\\
&\quad=\delta_\rho\circ\chi'_t(u)\cdot\exp\left\{[h'+tH_V(u)]\otimes\eta\right\}/\,\Delta(u)^{2g-2}.
\end{aligned}\tag{4.12}$$

Observe now that

$$\int_J\exp\left\{[h'+tH_V(u)]\otimes\eta\right\}=|F|^g\det{}^g\left[1+tH_V(u)^\dagger\right].\tag{4.13}$$

At $t=0$, this follows because $|F|$ is the determinant of $h':\mathfrak{t}\to\mathfrak{t}^\vee$ (with volume form normalised by the respective lattices), and the polarisation η on the GL(1) Jacobian is principal; while from $t=0$ the formula is clear in general. Theorem (2.15) now follows from (2.10) and (2.12).

To prove Theorem 2.18, recall from (4.7) the Chern characters $\mathrm{i}\,d\,\mathrm{Tr}_V(u)\otimes[C]\in H^1(J)$ of odd classes E^*_CV. Including a monomial ψ in these classes in the integral (4.13) multiplies it precisely by the $[\psi](u)$ defined by the contraction procedure in §2.17. □

4.14 Remark. Summing over the relevant part of $\mathfrak{M}_T$ gives the correct answer for each component of $\mathfrak{M}_G$ separately. Similarly, we can produce a formula for the index over the moduli of vector bundles with fixed but non-trivial determinant from the sum over appropriate Jacobians. However, torsion in π_1 brings in additional contributions from principal bundles under the normaliser of T in G; see the closely related calculation in [AMW] for line bundles.

Proof of Theorem 2.20. In $K^0(\mathfrak{M}_T)$, $\nu_B=\nu+E^*_x(\mathfrak{g}/\mathfrak{b})$, cf. §2.19, so $\lambda_{-1}(\nu_B)=\lambda_{-1}(\nu)\otimes\lambda_{-1}(E^*_x(\mathfrak{g}/\mathfrak{b}))$. Anti-symmetry (3.3) allows us to sign-average over W: Weyl's character formula converts $e^{\mu}/\lambda_{-1}(E^*_x(\mathfrak{g}/\mathfrak{b})^\vee)$ into $E^*_xV_\mu/|W|$ while leaving the factor $\mathcal{E}/\lambda_{-1}(\nu^\vee)$ unchanged. □

5. Witten's formulae from the large level limit

Assume now that the genus g is 2 or more. If $\mathfrak{M}$ were a compact manifold of complex dimension $d = (g-1)\dim G$, Riemann-Roch would enforce the behaviour

$$\mathrm{Ind}\,(\mathfrak{M};\psi^n\mathcal{E}) = n^d \int_{\mathfrak{M}} \mathrm{Ch}(\mathcal{E}) + O(n^{d-1}) \tag{5.1}$$

for any K-class $\mathcal{E}$ and its nth Adams power $\psi^n\mathcal{E}$. (Recall that $\psi^n L = L^n$ for a line bundle L, and ψ^n extends to K-theory additively using the splitting principle.)

In general, even the meaning of the integral on the right is unclear. Suppose, however, that $\mathcal{E}$ is a product of a polynomial in the Atiyah-Bott classes with a sufficiently large admissible line bundle. Then, for all n, $\mathrm{Ind}\,(\mathfrak{M};\psi^n\mathcal{E})$ has vanishing contribution from the unstable strata (Lemma 1.6 and §1.12), so the leading n asymptotic term in the index comes from the semi-stable stratum in (1.13). This contribution is slightly complicated by the singularities of the moduli space M. More precisely, the index of $\psi^n\mathcal{E}$ over the semi-stable stratum is that of its direct image from $\mathfrak{M}^{\mathrm{ss}}$ to M. Consider instead the direct image of the pull-back bundle $\tilde{\mathcal{E}}$ to the orbifold desingularisation $\tilde{M}$ of M, obtained by Kirwan's method [Ki]. Because M has rational singularities, the two indices agree, and when $\psi^n\mathcal{E}$ descends to $\tilde{M}$, the leading term in the M-index is $n^d\int_{\tilde{M}}\mathrm{Ch}(\tilde{\mathcal{E}})$. Descent holds when all stabilisers on $\tilde{M}$ act trivially on the fibres, in particular $\psi^n\mathcal{E}$ descends when all stabiliser orders in $\tilde{M}$ divide n.

It is more convenient to find the leading term in the twisted limit $\mathcal{K}^{1/2}\otimes\psi^n(\mathcal{K}^{-1/2}\mathcal{E})$. Let $\mathcal{E} = \mathcal{L}\otimes\exp[tE^*_\Sigma V]$, specialising to even generators for simplicity. Riemann-Roch implies

$$\psi^n E^*_\Sigma V = \frac{1}{n} E^*_\Sigma(\psi^n V),$$

and the properties of ψ give

$$\psi^n \exp\left[tE^*_\Sigma V\right] = \exp\left[tE^*_\Sigma(\psi^n V)/n\right]. \tag{5.2}$$

Since h' scales by n and $d\,\mathrm{Tr}_{\psi^n V}(u) = n\cdot d\,\mathrm{Tr}_V(u^n)$, the transformation (2.9) is unchanged, and the effect of the twisted ψ^n operation is to pre-compose the map $\chi'_t : T \to T^\vee$ in (2.12) with the nth power map on T. The key observation now is that the n^d contribution to the sum in Theorem 2.15, as $n\to\infty$, come from those points f_t located near the centre of G. Now, the descent condition on $\mathcal{E}$ requires the centre of G to act trivially on the fibres, with the result that the contributions near the various central elements agree, and summation over the centre can be concealed in the answer. Rescaling the $\log f_t$'s in the Lie algebra by n recovers Witten's sum over integral weights in [W, §5], with potential $Q = h'(\phi,\phi) + t\cdot\mathrm{Tr}_V(e^\phi)$ ($\phi\in\mathfrak{g}$). For example, the rescaled Weyl denominator in the θ_t converges to the dimension formula for the representations. We only spell out the complete details for $G = \mathrm{SL}(2)$, but the method works in general (Remark 5.8).

Let $\mathcal{E}$ be as above, with $c_1(\mathcal{L}) = l \in H^2(M,\mathbb{Z})$ and V of even spin $2j$. In the notation of §2.16, a solution ζ_t of

$$\zeta_t^{(2l+4)n}\cdot\exp\left[t\dot{\varphi}(\zeta_t^n)\right] = 1$$

can be written

$$\zeta_t = \exp\frac{\pi\mathrm{i}k_t}{(l+2)n}, \tag{5.3}$$

where for each $k\in\mathbb{Z}^+$, $k_t = k + k_1 t + k_2 t^2 + \cdots$ formally solves the equation

$$k_t + t\dot{\varphi}\left(\exp\frac{\pi\mathrm{i}k_t}{l+2}\right) = k. \tag{5.4}$$

5.5 Proposition. *With $\tilde{\mathcal{E}}$ as above, we have*

$$\int_{\tilde{M}} \mathrm{Ch}(\mathcal{K}^{-1/2} \otimes \tilde{\mathcal{E}}) = 2(l+2)^d \cdot \sum_{k=1}^{\infty} \left[1 + \frac{t\,\ddot{\varphi}(\exp \frac{\pi i k_t}{l+2})}{2l+4} \right]^{g-1} \cdot (\sqrt{2}\pi k_t)^{2-2g} \pmod{t^{(l+2)/j}}.$$

5.6 Remark. (i) Note that $l+2 = c_1(\mathcal{K}^{-1/2}\mathcal{E})$.

(ii) To finite order in t, our formula involves integrals of polynomials in $\exp(c_1)$ and Atiyah-Bott cohomology classes; so our ingredients are equivalent to Witten's, the exponential term $\mathrm{Tr}(e^{\phi})$ notwithstanding. But our truncation is needed precisely because the presence of exponentials; for no l does the formula hold to all orders in t.

(iii) Our answer seems at first to differ from [W]: the dimensions k of the irreducible representations of SU(2) have been deformed to k_t. To reconcile the formulae, note that our first factor in the sum (5.5) is the Jacobian determinant of the map $\xi \mapsto \xi + t\nabla\mathrm{Tr}_V(e^{\xi})$ on $\mathfrak{t}$, whereas its counterpart in [W] is the corresponding Jacobian on $\mathfrak{g}$. The ratio of the two is the volume ratio k_t^2/k^2 of the two co-adjoint orbits.

Proof. We have t-truncated the formula to the place where unstable strata begin contributing to the index. We must then only check that (5.5) gives the limiting n^d-coefficient in the index over $\mathfrak{M}$. To do so, we subdivide the summation range $1 \le k < n(l+2)$ into an interior region and two ends. We then check that the interior sum is bounded by $o(n^d)$, while the ends gives the wanted n^d-contribution.

The periodicity $k_t \mapsto k_t + 2l + 4$ shows that (5.4) involves only finitely many equations. All have analytic solutions for small t, so we can find a bound, independent of n and k, for the variation of k_t with t. This will allow us to replace k_t by k in some estimates.

We now cut off at $k_- = \sqrt{n}$ and $k_+ = (l+2)n - \sqrt{n}$. In-between, $|\zeta - \zeta^{-1}| > \pi/\sqrt{n}(l+2)$, so $|\theta^{1-g}| = O(n^{2g-2})$ and the sum over k is bounded by $O(n^{2g-1})$, less than $o(n^{3g-3})$ when $g > 2$.

On the other hand, for $k < k_-$, Taylor expansion of the Weyl denominator gives

$$\zeta_t - \zeta_t^{-1} = \frac{2\pi i k_t}{n(l+2)} \left(1 + O(n^{-1})\right),$$

with k-independent error bound, so the kth term in the index sum is

$$(l+2)^{2(g-1)} n^{3(g-1)} \cdot \left[\frac{2l+4+t\,\ddot{\varphi}(\exp \frac{\pi i k_t}{l+2})}{(2\pi k_t)^2} \right]^{g-1} \left(1 + O(n^{-1})\right)$$

and convergence of the series allows us to ignore the error. This and (5.1) give half of (5.5), the other half coming from the neighbourhood of $\zeta = -1$, by the central symmetry. □

5.7 Remark. The central symmetry relies on our choice of V with even spin. Its absence for odd spin reflects the fact that the central automorphism of SL(2)-bundles obstructs the descent of $E_{\Sigma}^{*}V$ to $\tilde{M}$. Similarly, there is an integration formula for the moduli of bundles with fixed determinant of degree 1, which introduces a sign $(-1)^k$ in the sum (cf. Remark 4.14). The level l must now be even; else, the contributions near $\zeta = 1$ and $\zeta = -1$ cancel instead of agreeing, even for line bundles. This reflects the fact that odd-level line bundles do not descend to the moduli space (again, the central automorphism acts by a sign).

5.8 Remark. This argument works for any simple G. Subdivide the simplex T_k/W of conjugacy classes into thickenings of width $1/\sqrt{n}$ of the faces. (First thicken the vertices, then the remainder of the edges, etc.) For each face Φ, the factors in the Weyl denominator are bounded as above: $|\sin\alpha| > 1/n$ if the root α vanishes on Φ, otherwise $|\sin\alpha| > 1/\sqrt{n}$. With Z_{Φ} denoting the centraliser of Φ, there are $\frac{1}{2}(\dim Z_\Phi - \dim T)$ of the former and $\frac{1}{2}(\dim G - \dim Z_\Phi)$ of the latter. As $|F| = O(n^{\dim T})$, the contribution of each point near Φ to the index formula can be overestimated by $O(n^{(g-1)p})$, with

$$p = \dim T + (\dim Z_\Phi - \dim T) + \frac{\dim G - \dim Z_\Phi}{2} = \frac{\dim Z_\Phi + \dim G}{2}.$$

Even after adding $\dim \Phi$ to account for the number of terms, $(g-1)p$ is less than the dimension $(g-1)\dim G$ of M, unless Φ is a central vertex of T_k/W, so that $Z_\Phi = G$. Thus, only the f_t near the centre contribute. The error estimate in their contribution proceeds as before.

6. Kähler differentials

In this section, we include the Kähler differentials $\Omega^\bullet$ over $\mathfrak{M}$ in our index. Recall that $\Omega^p = \lambda^p\left(R\pi_*(E^*\mathfrak{g} \otimes K)\right)$, where $R\pi_*(\dots)$ is the (perfect) cotangent complex of $\mathfrak{M}$. Thus, Ω^p does not quite land in the admissible K-theory ring, but rather in its enlargement by the λ-operations. While the Abelian reduction formula (2.20) and its proof carry over to this more general setting, our explicit index formula (2.18) does not immediately provide an answer. In this section, we show how to extend the formula to these more general K-classes.

As we will transfer the result to the moduli space M, we note the following improvement of Lemma 1.6: the indexes of $\Omega^\bullet \otimes \mathcal{L} \otimes \mathcal{E}$ over the stack $\mathfrak{M}$ and over its semi-stable part $\mathfrak{M}^{\mathrm{ss}}$ agree for large enough $\mathcal{L}$, depending on the Atiyah-Bott monomial $\mathcal{E}$ but *not on the degree of the differentials*. The proof requires the finer calculation in [T2, §7]. Note also that, for semi-simple G, the differentials on the stack of *stable* bundles are the orbifold differentials over the moduli space of the same; in the reductive case, infinitesimal automorphisms cause a discrepancy which we leave in the care of the reader.

Recall the notations of §2, in particular fix a representation V of G. As $1 + te^\alpha$ is a function on T, $(1 + te^\alpha)^\alpha$ is a $T^\vee$-valued map. Set

$$\chi'_{s,t} = \chi' \cdot e^{s \cdot d\,\mathrm{Tr}_V(.)} \cdot \prod_{\alpha>0} \left[\frac{1 + te^\alpha}{1 + te^{-\alpha}}\right]^\alpha : T \to T^\vee. \tag{6.1}$$

Denote by $F_{s,t}$ the set of solutions of the equation

$$\chi'_{s,t}(f) = (-1)^{2\rho} \in T^\vee \tag{6.2}$$

and by $F_{s,t}^{\mathrm{reg}}$ the subset of those which are *regular* as G-conjugacy classes at $s = t = 0$. Call $H(f)$ the differential of $\chi'_{s,t}$ at $f \in T$; the notation H stems from its agreement with the Hessian of the function on $\mathfrak{t}$

$$\xi \mapsto \frac{h+c}{2}(\xi, \xi) + s\mathrm{Tr}_V(e^\xi) - \mathrm{Tr}_{\mathfrak{g}}\left(\mathrm{Li}_2(te^\xi)\right),$$

with Euler's dilogarithm Li_2. Using the metric $(h+c)$, we convert H to an endomorphism $H^\dagger$ of $\mathfrak{t}$ and define

$$\theta_{s,t}(f)^{-1} = |F| \cdot \prod_\alpha \frac{1 + te^\alpha}{1 - e^\alpha} \cdot \det H^\dagger(f), \tag{6.3}$$

the product ranging over all roots. Note that $\det H^\dagger = 1$ at $s = t = 0$.

6.4 Theorem. *With $\Omega_t := \bigoplus_p t^p \cdot \Omega^p$, we have the index formula*

$$\mathrm{Ind}\,(\mathfrak{M}; \Omega_t \otimes \mathcal{L} \otimes \exp[sE^*_\Sigma V] \otimes E^*_x U) = (1+t)^{(g-1)\ell} \sum\nolimits_f \theta_{s,t}(f)^{1-g} \cdot \mathrm{Tr}_U(f),$$

with $f \in F^{\mathrm{reg}}_{s,t}$ ranging over a complete set of Weyl orbit representatives.

Proof. In topological K-theory, $R\pi_*(E^*\mathfrak{g}\otimes K) = E^*_\Sigma\mathfrak{g}\oplus E^*_x\mathfrak{g}$, so $\Omega_t = \lambda_t\left(E^*_\Sigma\mathfrak{g}\right)\otimes\lambda_t\left(E^*_x\mathfrak{g}\right)^{\otimes(g-1)}$. In terms of the Adams operations ψ^p,

$$\lambda_t = \textstyle\sum_p t^p\lambda^p = \exp\left[-\textstyle\sum_{p>0}(-t)^p\psi^p/p\right].$$

Using (5.2), we see that Theorem 6.4 refers to the index over $\mathfrak{M}$ of

$$\mathcal{L}\otimes\exp\left[sE^*_\Sigma V - \sum\nolimits_{p>0}\frac{(-t)^p}{p^2}E^*_\Sigma\,\psi^p(\mathfrak{g})\right]\otimes E^*_x\lambda_t(\mathfrak{g})^{\otimes(g-1)}\otimes E^*_x U,$$

which now has the form covered in Theorem 2.15. The associated equation

$$\exp\left[(h+c)+s\cdot d\mathrm{Tr}_V - \sum\nolimits_{p>0;\alpha}\frac{(-t)^p}{p}e^{p\alpha}\cdot\alpha\right] = (-1)^{2\rho}$$

is precisely (6.2). To reduce formula (6.4) to Theorem 2.15, observe that the pre-factor $(1+t)^\ell$ and the factors $1+te^\alpha$ in (6.3) come from the character of $\lambda_t(\mathfrak{g})$, which factors as $(1+t)^\ell\cdot\prod_\alpha(1+te^\alpha)$. □

6.5 Remark. Odd generators are included as in Theorem 2.18, using the contraction procedure with the inverse of the bilinear forms $H(f)$.

6.6 Full-flag parabolic structures. A formula for the stack $\mathfrak{M}(x,\mathcal{B})$ follows by considering the projection $\mathfrak{M}(x,\mathcal{B})\to\mathfrak{M}$, with fibres G/B: replace Tr_U in (6.4) by

$$\mathrm{Tr}_U\cdot\prod_{\alpha>0}\frac{(1+te^\alpha)}{(1-e^\alpha)},$$

and sum over all points of F^{reg} instead of Weyl orbits. The numerator accounts for the differentials on the fibres G/B, while the denominator and summation over W together constitute the Weyl character formula.

7. The Newstead-Ramanan conjecture

In important special cases, all semi-stable bundles over Σ are stable and then M is a compact orbifold. This happens when $G = \mathrm{GL}(n)$, for the components of degree prime to n, or else if we enrich the bundle with a sufficiently generic parabolic structure.[8] Henceforth, we place ourselves in one of these favourable situations. Let ℓ^{ss} and ℓ^{c} be the semi-simple and central ranks of G. The following result generalises an old conjecture of Newstead and Ramanan [N, R].

7.1 Theorem. *The top $(g-1)\ell^{\mathrm{ss}}+g\ell^{\mathrm{c}}$ rational Chern classes of M vanish.*

In other words, they vanish above degree $\dim(G/T)(g-1)$. For rational cohomology, we can pass to finite covers with impunity [AB, §7] and split G as a product of a torus and simple groups; so the only content of the theorem concerns ℓ^{ss}. We will prove an equivalent result in topological K-theory. Let G be semi-simple of rank ℓ.

7.2 Theorem. *The top $(g-1)\ell$ rational Grothendieck γ-classes of M vanish.*

[8] Note that the moduli of stable vector bundles of degree d is also that of stable vector bundles of degree 0 but with parabolic structure defined by the vertex $\mathrm{diag}[2\pi i d/n]$ of the Weyl alcove of $\mathfrak{gl}(n)$; cf. Example A.4.

The γ-classes are recalled below, along with the equivalence of the two theorems above. In some cases, such as $G = \mathrm{GL}(n)$, $\mathrm{SL}(n)$ or $\mathrm{Sp}(n)$, M is known to be free of homology torsion [AB], and we get an integral result. It seems to be unknown whether $K(M)$ is torsion-free for other (e.g. simply connected) groups.

To prove Theorem 7.2, we pair the total γ-class $\sum t^k\gamma^k$ of TM against any test class $\mathcal{E}$ in $K^0(M)$ and show that we obtain a polynomial in t of degree no more than $\dim M - (g-1)\ell$. Since the index over the orbifold M varies quasi-polynomially in the Chern classes of $\mathcal{E}$, it suffices to check this behaviour when $\mathcal{E}$ contains a large line bundle factor, which we will do using using the index formula (6.4).

This strategy is not new, cf. Zagier [Z] for SL(2), but the integration formulae over M turned out to be unwieldy. Our index formula seems to be a better fit; the reason is the abelian localisation (2.20). Indeed, over $\mathfrak{M}_T$, the tangent complex to $\mathfrak{M}$ has a trivial summand of rank predicted by the vanishing. The proof then consists in checking that nothing in the index formula spoils the vanishing that is already apparent. Still, the method has limits: thus, we were unable to decide whether the γ-classes vanish in *algebraic* K-theory.

7.3 The γ-classes. For a complex vector bundle V of rank r over a compact space X, define the classes $\gamma^p(V) \in K^0(X)$ as the coefficients of the following polynomial of degree r:

$$\gamma_t(V) = \sum\nolimits_p t^p\gamma^p(V) := (1-t)^r\lambda_{t/(1-t)}(V),$$

with the total λ-class $\lambda_s(V) = \sum s^p\lambda^p(V)$, as before. Note that

$$\gamma_t(V\oplus W) = \gamma_t(V)\cdot\gamma_t(W)$$

for vector bundles V and W, while

$$\gamma_t(L) = (1-t) + tL$$

for a line bundle L; these conditions determine γ_t from the splitting principle. Also,

$$\gamma^1(L) = L - 1,$$

the K-theory Euler class of the line bundle, and in this sense γ_t is the total K-theory Chern class. The next exercise is included for the reader's convenience.

7.4 Proposition. *The following assertions are equivalent.*

(i) The top d rational Chern classes of V vanish.

(ii) The top d rational γ-classes of V vanish.

(iii) The polynomial $\lambda_t(V) \in K^0(X;\mathbb{Q})[t]$ is divisible by $(1+t)^d$.

When $K^0(X;\mathbb{Q})$ satisfies Poincaré duality with respect to a map $\mathrm{Ind} : K^0(X) \to \mathbb{Q}$, these conditions are equivalent to

(iv) For every $W \in K^0(X)$, $\mathrm{Ind}(\lambda_t(V)\cdot W) \in \mathbb{Q}[t]$ vanishes to order d or more at $t=-1$.

Proof. Equivalence of (ii) and (iii) is clear from the inversion formula $\lambda_t = (1+t)^r\gamma_{t/(1+t)}$. Observe next that in the ring R of symmetric power series in variables $x_1,\dots,x_r$ the ideal $(e_{r-d+1},\dots,e_r)$ generated by the top d elementary symmetric functions is the intersection of R with the ideal $(x_{r-d+1},\dots,x_r) \in \mathbb{Q}[[x_1,\dots,x_r]]$. The transformation

$$x_i \mapsto y_i := e^{x_i} - 1$$

defines an automorphism of $\mathbb{Q}[[x_1,\dots,x_r]]$ which preserves $(x_{r-d+1},\dots,x_r)$. It follows that $(e_{r-d+1},\dots,e_r)$ agrees with the ideal of the top d elementary symmetric functions in the y_k. Let x_k be the Chern roots of V; then,

$$\mathrm{Ch}\ \gamma_t(E) = \prod(1+ty_i),$$

so the γ-classes are the elementary symmetric functions in y, and we conclude that (i) $\Leftrightarrow$ (ii). $\square$

7.5 Reduction to Borel structures. We now show that if Theorem 7.2 holds for moduli of bundles with Borel structures, then it holds for all parabolic structures. Let $\mathfrak{M}(x,\mathcal{P})$ denote the stack of bundles with a $\mathcal{P}$-parabolic structure at x and call $\pi:\mathfrak{M}(x,\mathcal{B})\to\mathfrak{M}(x,\mathcal{P})$ the projection (A.3). Over $\mathfrak{M}(x,\mathcal{B})$, we have a distinguished triangle of tangent complexes

$$T_\pi\mathfrak{M}(x,\mathcal{B})\to T\mathfrak{M}(x,\mathcal{B})\to\pi^*T\mathfrak{M}(x,\mathcal{P})\to T_\pi\mathfrak{M}(x,\mathcal{B})[1],$$

leading to an equality in K-theory,

$$\lambda_t\left(T^\vee\mathfrak{M}(x,\mathcal{B})\right) = \lambda_t\left(T_\pi^\vee\mathfrak{M}(x,\mathcal{B})\right)\otimes\pi^*\lambda_t(T^\vee\mathfrak{M}(x,\mathcal{P})).$$

The fibres $\mathcal{P}/\mathcal{B}$ of π are flag varieties; they are smooth and proper, with cohomology of type (p,p). Hodge decomposition gives

$$R\pi_*\left[\lambda_t\left(T^\vee\mathfrak{M}(x,\mathcal{B})\right)\otimes\pi^*\mathcal{E}\right] = \lambda_t(T^\vee\mathfrak{M}(x,\mathcal{P}))\otimes\mathcal{E}\cdot\sum(-t)^p b_{2p}(\mathcal{P}/\mathcal{B})$$

where the b_{2p} are the Betti numbers. For $t=-1$, the last factor is positive and so it does not affect the vanishing order of the index.

7.6 Limit of the index as $t\to-1$. In Theorem 6.4, the desired factor $(1+t)^{(g-1)\ell}$ appears explicitly in the index formula, so to prove Theorem 7.2 we must check that no singularities in $\theta_{s,t}(f)^{1-g}$ or in $H(f_t)$ (cf. Remark 6.5) reduce the order of vanishing at $t=-1$. To do so, we study the roots of (6.2). When $h>0$ and $t=s=0$, χ' is an isogeny and all roots are simple. The following lemma will ensure that they remain simple for all $t\in(-1,0]$ and small s.

7.7 Lemma. *If $h>c$, s is small and $t\in(-1,0]$, the differential $H=d\chi'_{s,t}$ is non-degenerate on T_k.*

Proof. With $H_V(f)$ denoting the Hessian of Tr_V at f, we have

$$H = (h+c)+sH_V(f)+t\sum\nolimits_\alpha\frac{e^\alpha}{1+te^\alpha}(f)\cdot\alpha^{\otimes 2}. \tag{7.8}$$

Note that $\alpha^{\otimes2}$ is negative semi-definite, $t\le0$ and $\Re\frac{e^\alpha}{1+te^\alpha}\ge-1$ for $|e^\alpha|=1$. As $\sum_\alpha\alpha^{\otimes2}=-2c$, H is bounded below by $(h-c)+sH_V$. $\square$

Skew-adjointness of χ' for $s=0$ then keeps the solutions in the *compact* torus T_k for small variations in the real time t, and thus for all times $t\in[-1,0]$. Non-degeneracy of H also shows that the s-dependence in (6.2) can be solved order-by order, when $t\in(-1,0]$, and keeps $F_{s,t}$ in a formal neighbourhood of T_k. We will now show that H remains regular at $t=-1$, so the solution can be perturbed analytically in s even there. As certain regular solutions *do* wander into the singular locus of T as $t\to-1$, we need to control this behaviour. Let $f_t=f_{0,t}$.

7.9 Lemma. *Let $f_t\in F$ be regular at $t=0$ but singular at $t=-1$. For small $x=\sqrt{t+1}$, f_t has a convergent expansion*

$$f_t = f_{-1}\cdot\exp\left[\sum\nolimits_{k>0}x^k\xi_k\right].$$

Moreover, $\beta(\xi_1)\neq0$ for any root β such that $e^\beta(f_{-1})=1$.

Thus, the tangent line to f_t at f_{-1} is regular in the Lie algebra centraliser $\mathfrak{z}$ of f_{-1}. We obtain

$$\begin{aligned} \lim_{t\to-1} \frac{1+te^{\alpha}}{1-e^{\alpha}}(f_t) &= 1 \qquad \text{for all roots } \alpha \text{ of } \mathfrak{g}, \\ \lim_{t\to-1}\left(\frac{e^{\beta}}{1+te^{\beta}}(f_t)+\frac{e^{-\beta}}{1+te^{-\beta}}(f_t)\right) &= -1-\frac{2}{\beta(\xi_1)^2} \quad \text{for roots } \beta \text{ of } \mathfrak{z}. \end{aligned} \tag{7.10}$$

The limiting value $H(f_{-1})$ in (7.8) is then the positive definite form $h + sH_V + \sum_\beta \frac{\beta^{\otimes 2}}{\beta(\xi_1)^2}$, summing over roots of $\mathfrak{z}$. This excludes unexpected singularities in the index formula.

Proof of (7.9). At $t=-1$, equation 6.2 simplifies to

$$\exp[h + s\cdot d\mathrm{Tr}_V(.)] = 1; \tag{7.11}$$

however, the cancellation involved conceals multiple solutions on the singular locus in T. The latter partitions T_k into alcoves that are simply permuted by the Weyl group. We claim that each *singular* solution of (7.11) is a limit of at least one solution behaving as in Lemma 7.9. If so, then by Weyl symmetry there must be such a solution from each adjacent alcove. Now, every *regular* solution of (7.11) is also the limit of a regular solution of (6.2): this is because it is the limit of *some* solution, and Weyl symmetry plus Lemma 7.7 ensures that the points of F that are singular at $t=0$ stay so until $t=-1$. Finally, recall that the solutions of (7.11) in a *closed* Weyl alcove are in bijection with the regular solutions of (6.2) in that alcove. Our claim then accounts for the $t=-1$ limits of *all* regular points of F and proves Lemma 7.9.

To prove the claim, it suffices to find a *formal* solution f_t as in the Lemma. As t converges faster than f_t becomes singular, the function $\chi'_{s,t}$ converges to (7.11), so equation (6.2) is verified to zeroth order precisely when f_{-1} solves (7.11). To obtain the constraint on ξ_1, we differentiate in x:

$$\frac{d}{dx}\chi'_{s,t}(f_t) = \iota(\xi')\left[(h+c)+sH_V(f_t)\right] + \sum_\alpha \frac{[2x+t\alpha(\xi')]e^{\alpha}}{1+te^{\alpha}}(f_t)\cdot\alpha,$$

with $\xi = \sum_k \xi_k x^k$. The limit at $x=0$ is found from (7.10) and leads to

$$\iota(\xi_1)\left[h+sH_V(f_{-1})\right] = \sum\nolimits_\beta \frac{\beta}{\beta(\xi_1)}, \tag{7.12}$$

summed over the roots β of $\mathfrak{z}$. Its solutions are the critical points of the function

$$\mathfrak{t}\ni\zeta\mapsto\frac{1}{2}\left[h(\zeta,\zeta)+sH_V(\zeta,\zeta)\right]-\sum\nolimits_\beta \log|\beta(\zeta)|.$$

This function is real-valued for $s=0$, blows up on each walls of the Weyl chamber of $\mathfrak{z}$ and is dominated by the quadratic term at large ζ, so a minimum must exist *inside the chamber.* Further, the Hessian

$$h+\sum\nolimits_\beta \frac{\beta^{\otimes 2}}{\beta(\zeta)^2}$$

is positive-definite, so the minimum is non-degenerate and the s-perturbed equation can also be solved for small s. Continuing to higher order in x, we get a recursive family of equations for $k>1$

$$\iota(\xi_k)\left(h+sH_V+\sum\nolimits_\beta \frac{\beta^{\otimes 2}}{\beta(\xi_1)^2}\right) = (\text{expression in } \xi_j,\ j<k), \tag{7.13}$$

solvable because of the same non-degeneracy. This proves our claim and thus the lemma. □

A. Background on $\mathfrak{M}$

For the more analytically minded, the stack $\mathfrak{M}$ admits the *Atiyah-Bott presentation* as a quotient of the space of $(0,1)$-connections by the group of complex gauge transformations; but its underlying *algebraic* structure is essential for us. The algebraic geometry of the stack was discussed in [BL, LS] and further properties were developed in [T1, T2]. In particular, $\mathfrak{M}$ is covered by quotients of smooth varieties by reductive groups; many general properties of sheaf cohomology follow, without the need of simplicial topos theory as in [T1]. In this appendix, we quickly review the variants of $\mathfrak{M}$ with parabolic structures and discuss the topological K-theory of $\mathfrak{M}$.

A.1 Parabolic structures. Call $\mathcal{B}$ the *Iwahori subgroup* of the loop group $G(\!(z)\!)$, consisting of those formal Taylor loops whose value at $z=0$ lies in a fixed Borel subgroup B. For any subset Φ of simple affine roots, let $\mathcal{P}_\Phi$ denote the *standard parabolic subgroup* of $G(\!(z)\!)$ generated by $\mathcal{B}$ and by the root SL_2 subgroups from Φ. If $\Psi \subset \Phi$, then the quotient $\mathcal{P}_\Phi/\mathcal{P}_\Psi$ is isomorphic (possibly non-canonically) to a homogeneous space for a subgroup of G.

A.2 Example. (i) $\mathcal{P}_\emptyset = \mathcal{B}$. More generally, if Φ consists of (linear) roots of $\mathfrak{g}$, then $\mathcal{P}_\Phi$ is the subgroup of formal Taylor loops whose value at $z=0$ lies in the parabolic subgroup $\mathcal{P}_\Phi \cap G$ of G.

(ii) If $\Phi = \{\alpha_0\}$ the non-linear simple root, $\mathcal{P}_\Phi$ has Lie algebra $\mathrm{Lie}(\mathcal{B}) \oplus z^{-1}\mathfrak{g}_\vartheta$. We have $\mathcal{P}_\Phi/\mathcal{B} \cong \mathbb{P}^1$. This parabolic subgroup appears in the proof of Proposition 3.3.

For distinct $x_1,\dots,x_n \in \Sigma$ and $\mathcal{P}_1,\dots,\mathcal{P}_n$ standard parabolics, let $\mathfrak{M}(\mathbf{x};\mathcal{P})$ denote the moduli stack of G-bundles with quasi-parabolic structures at $x_1,\dots,x_n$. These are G-bundles over Σ with singularites at the x_i, but with a reduction of the gauge group to $\mathcal{P}_i$ near x_i. When G is semi-simple, the uniformisation theorem [LS, Theorem 9.5] shows that this is the quotient of the product of *generalised flag varieties* $G(\!(z)\!)/\mathcal{P}_i$ by the gauge group $G[\Sigma \setminus \{x_1,\dots,x_n\}]$ of the punctured curve (cf. also [T2, §9]). Let $\mathcal{P}'_1,\dots,\mathcal{P}'_n$ be standard parabolics contained in $\mathcal{P}_1,\dots,\mathcal{P}_n$ respectively. The projections $G(\!(z)\!)/\mathcal{P}_i \to G(\!(z)\!)/\mathcal{P}'_i$ induce a fibration

$$\mathfrak{M}(\mathbf{x};\mathcal{P}') \to \mathfrak{M}(\mathbf{x};\mathcal{P}) \tag{A.3}$$

with fibres $\mathcal{P}_1/\mathcal{P}'_1 \times \cdots \times \mathcal{P}_n/\mathcal{P}'_n$.

A.4 Example. (i) When each of the parabolics is defined by a subset of the linear roots of $\mathfrak{g}$, $\mathfrak{M}(\mathbf{x};\mathcal{P})$ is the stack of G bundles on Σ with reductions to the parabolic subgroups $P_1,\dots,P_n$ over $x_1,\dots,x_n$. In this case, the G-bundles have no singularities and $\mathfrak{M}(\mathbf{x};\mathcal{P})$ admits a forgetful morphism to the moduli stack $\mathfrak{M}$ with fibre $G/P_1 \times \cdots \times G/P_n$.

(ii) For $G = \mathrm{GL}(n,\mathbb{C})$, every parabolic subgroup of $G(\!(z)\!)$ is conjugate to one defined by linear roots of $\mathfrak{g}$, so all parabolic bundles can be described as vector bundles with a choice of flags at the marked points.

(iii) If $\Phi = \{\alpha_0\}$, then $\mathfrak{M}(x,\mathcal{P}_\Phi)$ fibres over $\mathfrak{M}(x,\mathcal{B})$ with fibre $\mathbb{P}^1$.

A.5 Shatz stratification. Each stack $\mathfrak{M}(\mathbf{x};\mathcal{P})$ is equivalent to a stack $\mathfrak{M}_\Gamma$ of equivariant bundles on a suitable Galois cover $\tilde{\Sigma} \to \Sigma$ [TW, §2.2]. The Shatz stratification of $\mathfrak{M}_\Gamma$ induces a stratification on $\mathfrak{M}(\mathbf{x};\mathcal{P})$. This depends on the choice of the cover, but the dependence can be reduced to a choice of of *polarisation* on $\mathfrak{M}(\mathbf{x};\mathcal{P})$. For a Borel structure at a single point x, this is equivalent to a choice of finite-order, regular conjugacy class G_k; to label the strata, we choose a lifting $u \in T_k$.

The co-weights ξ labelling the Shatz strata of $\mathfrak{M}$ in §1.8 have a geometric meaning: every stable bundle in $\mathfrak{M}^{\mathrm{ss}}_{G_\xi,\xi}$ has a unique Hermitian connection with constant, G_ξ-central curvature

$2\pi\mathrm{i}\xi$. The construction above shows that every stable bundle in $\mathfrak{M}_{G_\xi,\xi}(x,\mathcal{B})^{\mathrm{ss}}$ has a Hermitian connection over $\Sigma\setminus\{x\}$ with constant central curvature $2\pi\mathrm{i}\xi$ and holonomy u at x. The central part of u stems from the curvature, while the projection to $\mathrm{Ad}G_\xi$ comes from the global monodromy.

The index of an admissible class $\mathcal{E}$ over $\mathfrak{M}(\mathbf{x},\mathcal{P})$ breaks up as before into a sum over strata. There is also an extra factor in the Euler complex, relating the flag varieties of G and G_ξ. The key finiteness result (1.6) applies to this more general setting, but the vanishing of unstable local cohomologies requires the line bundle $\mathcal{L}$ to match the choice of stratification; see [T2, §9].

A.6 K-theory of $\mathfrak{M}$. The homotopy type of the stack $\mathfrak{M}$ (which, by definition, is that of the geometric realisation of an underlying simplicial scheme) is that of the space of continuous maps from Σ to BG. (This is GAGA plus the Atiyah-Bott construction of holomorphic bundles.) But it is more natural to assign to $\mathfrak{M}$ the equivariant homotopy type given by the conjugation action of $G_k\subset G$ on the space $C_*(\Sigma,BG)$ of continuous maps based at $x\in\Sigma$ to BG. This space is a principal fibration over a product of copies of G, with fibre the group ΩG of based loops in G [AB]. Then, $K^\bullet(\mathfrak{M})$ is defined to be the G_k-equivariant K-theory of $C_*(\Sigma,BG)$. It is an inverse limit of finite modules over the representation ring R_G, taken over the finite parts of a G_k-cellular model of $C_*(\Sigma,BG)$. Similarly, $K^\bullet(\mathfrak{M}(x,\mathcal{B}))=K^\bullet_{T_k}(C_*(\Sigma,BG))$; it is a module over $K^\bullet(\mathfrak{M})$ via the natural projection, and $K^\bullet(\mathfrak{M})$ is a split summand.

Another description of $K^\bullet(\mathfrak{M})$ arises by exhausting $\mathfrak{M}$ with open sub-stacks of finite type. Such sub-stacks are presentable as quotients of quasi-projective manifolds by linear algebraic groups, and their topological K-theory can be defined from continuous vector bundles that are equivariant under the maximal compact part of the acting group. (This can be shown to be independent of the quotient presentation.) If we use the finite, open unions $\mathfrak{M}_{\le\xi}$ of Shatz strata to exhaust $\mathfrak{M}$, the argument of Atiyah and Bott (see [HL] for the K-theory version) shows the surjectivity of the restriction maps between the $K^\bullet(\mathfrak{M}_{\le\xi})$ and leads to the description

$$K^\bullet(\mathfrak{M})=\lim\nolimits_\xi K^\bullet(\mathfrak{M}_{\le\xi}),\qquad \mathrm{gr}\,K^\bullet(\mathfrak{M})=\bigoplus\nolimits_\xi K^\bullet(\mathfrak{M}_\xi).\tag{A.7}$$

The two constructions of $K(\mathfrak{M})$ just described can be related by presenting $\mathfrak{M}$ as a quotient $\mathfrak{M}_*/G$ of the stack of G-bundles with a framing over x modulo the action of G on the fibre: $\mathfrak{M}_*$ can be presented as a quotient of a pro-variety with the homotopy type of $C_*(\Sigma,BG)$ by a pro-unipotent group.

Comparison with the stack $\mathfrak{M}_T$ of T-bundles gives more information. Consider for simplicity $\mathfrak{M}(x,\mathcal{B})$. When π_1G is free, the stabilisers of the T_k-action on the complement of $C_*(\Sigma,BT)$ in $C_*(\Sigma,BG)$ are contained in the singular locus. Consequently, after inverting the Weyl denominator Δ in the coefficients of K-theory, the restriction $j^*:K(\mathfrak{M}(x,\mathcal{B}))\to K(\mathfrak{M}_T)$ becomes an isomorphism, compatible with the inverse limit (A.7). Poincare duality on $\mathfrak{M}_T$ and our index formula show that

$$(j^*)^{-1}=(-1)^{2\rho}\cdot\mathcal{K}^{1/2}\Delta^{2g-2}\cdot j_*,$$

with j_* defined using the finite-dimensional stack structure. However, our index formula carries the additional information that inverting Δ does not damage the index.

A.8 Remark. Rationally, $C_*(\Sigma,BG)$ is a product $\Omega G\times G^{2g}$. The rational cohomology factors [AB] as

$$H^\bullet(\mathfrak{M})\cong H^\bullet_{G_k}(\Omega G)\otimes_R H^\bullet_{G_k}(G)^{\otimes_R 2g},\tag{A.9}$$

with $R=H^\bullet(BG;\mathbb{Q})$. A similar factorisation follows for rational K-theory, with $R=\mathbb{Q}\otimes R_G$, by using Chern characters and fixed-point formulae. It is tempting to suggest that the analogue of (A.9) holds for integral K-theory when π_1G is free, but we only know how to prove this for the groups GL,SL and Sp.

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