THE YANG-MILLS HEAT FLOW ON THE MODULI SPACE OF FRAMED BUNDLES ON A SURFACE

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Abstract. We study the analog of the Yang-Mills heat flow on the moduli space of framed bundles on a cut surface. Existence and convergence of the heat flow give a stratification of Morse type invariant under the action of the loop group. We use the stratification to prove versions of Kähler quantization commutes with reduction and Kirwan surjectivity.

1. Introduction

Let K be a compact, 1-connected Lie group with complexification G and Lie algebra $\mathfrak{k}$, and $\overline{X}$ a compact, connected Riemann surface. The moduli space $\mathcal{M}(\overline{X})$ of isomorphism classes of flat K-bundles on $\overline{X}$ is homeomorphic to the moduli space of grade-equivalence classes of semistable G-bundles, by the theorems of Narasimhan-Seshadri [25] and Ramanathan [33, 34].

$\mathcal{M}(\overline{X})$ has two presentations as an infinite dimensional quotient which can be used to study its cohomology. The first presentation was introduced by Atiyah and Bott [3] and is rather well understood. Let $\mathcal{A}(\overline{X})$ denote the affine space of connections on the trivial K-bundle over X, with symplectic structure induced by a choice of metric on $\mathfrak{k}$. The group $K(\overline{X})$ of gauge transformations acts symplectically on $\mathcal{A}(\overline{X})$ with moment map given by the curvature, and the symplectic quotient is $\mathcal{M}(\overline{X})$. In the holomorphic description

$$\mathcal{M}(\overline{X}) \cong G(\overline{X}) \backslash\!\backslash \mathcal{A}(\overline{X})$$

where the symbol $\backslash\!\backslash$ means the quotient of the semistable locus. Atiyah and Bott used the stratification of $\mathcal{A}(\overline{X})$ into Harder-Narasimhan types to compute the Betti numbers of $\mathcal{M}(\overline{X})$; they conjectured that the stratification is identical to the stratification into stable manifolds for the gradient flow of minus the Yang-Mills functional. This was proved by Donaldson [8, 9] and Daskalopolous [7], who also proved convergence of the gradient flow up to gauge transformation. Råde [31] proved that

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the gradient flow itself converges, and gave estimates for the rate of convergence.

The second presentation has origins in Weil's double coset construction [41],[3, p.595]; here the related analysis has been less studied. Let $S \subset \overline{X}$ be an embedded circle, and X the Riemann surface with boundary obtained by cutting $\overline{X}$ along S. The Yang-Mills heat flow on the space $\mathcal{A}(X)$ was studied by Donaldson [10], who obtained an analog of the Narasimhan-Seshadri theorem: The moduli space $\mathcal{M}(X)$ of flat K-bundles with framings on the boundary is diffeomorphic to $G(\partial X)/G_{\mathrm{hol}}(X)$, where $G(\partial X) = \mathrm{Map}(\partial X, G)$ and $G_{\mathrm{hol}}(X)$ denotes the subgroup of $G(\partial X)$ consisting of loops that extend holomorphically over the interior. The loop group $K(S)$ acts symplectically on $\mathcal{M}(X)$ with moment map given by the difference of the restriction to the two boundary components, and $\mathcal{M}(\overline{X})$ is homeomorphic to the symplectic quotient. In the holomorphic description

$$\mathcal{M}(\overline{X}) \cong G(S)\backslash\!\backslash(G(\partial X)/G_{\mathrm{hol}}(X)).$$

In recent years, this presentation has become more popular because of its connection with conformal field theory and the Verlinde formulas [4], [12], [19], [40]. Here the circle S is assumed to bound a disk, so that X (in its algebraic manifestation) becomes a punctured curve union a formal disk. Surfaces with boundary do not fit into the algebraic framework.

In this paper we consider the analog of the Yang-Mills heat flow in the second presentation, namely the gradient flow of minus the square of the moment map for the loop group for an arbitrary embedded circle S in $\overline{X}$. We show that the analog of Råde's result holds: The gradient flow exists for all times and converges to a critical point. Although the evolution equation itself is not pseudo-differential, its restriction to the boundary is a (non-linear) heat equation involving the Dirichlet-to-Neumann operator associated to the connection. Calderón observed that this is an elliptic pseudodifferential operator. Because pull-back to the boundary is Fredholm on the space of harmonic forms, we are "up to finite dimensions" in the same situation as for the first presentation, except that the moduli space of framed bundles is not affine.

This analysis implies that $\mathcal{M}(X)$ admits a stratification into stable manifolds for minus the gradient flow, which can be viewed as generalization of the Birkhoff decomposition to arbitrary genus surfaces. By definition, the stable manifold for the zero locus of the moment map is the semistable locus. The other strata are complex submanifolds of finite codimension, and the number of strata of each codimension is finite. Using the stratification, we obtain several cohomological applications which extend known results beyond the case that S bounds a disk. The first, which was the motivation for the paper, is a Kähler "quantization commutes with reduction" theorem, similar to that of Guillemin-Sternberg [13] in the finite dimensional case. This is an instance of Segal's composition axiom for the Wess-Zumino-Witten conformal field theory [38, 28]. In the case S bounds a disk, the algebraic version is due to Beauville-Laszlo [4], Kumar-Narasimhan-Ramanathan [19], and Laszlo-Sorger [20]; see also Teleman [40]. The second application is a surjectivity result for the equivariant cohomology with rational coefficients, similar to that of Kirwan [18]. In the case that S bounds a disk an essentially equivalent result was proved by Bott, Tolman, and Weitsman [6]. An appendix contains a review of the relevant Sobolev spaces.

2. Background on connections on a circle

The following is contained in Pressley-Segal [29] in the context of smooth maps. Let S be a circle, that is, a connected one-manifold. For any $s > 0$, the group

$$K(S)_{s+\frac{1}{2}} := \mathrm{Map}(S, K)_{s+\frac{1}{2}}$$

of free loops of Sobolev class $s + \frac{1}{2}$ acts on the space $\mathcal{A}(S)_{s-\frac{1}{2}}$ of connections on the trivial bundle $S \times K$. Any connection differs from the trivial connection by a $\mathfrak{k}$-valued one-form; using the trivial connection as a base point we identify

$$\mathcal{A}(S)_{s-\frac{1}{2}} \to \Omega^1(S; \mathfrak{k})_{s-\frac{1}{2}}.$$

For any $s > r > 0$ inclusion defines a bijection

$$K(S)_{r-\frac{1}{2}} \backslash \mathcal{A}(S)_{r+\frac{1}{2}} \to K(S)_{s-\frac{1}{2}} \backslash \mathcal{A}(S)_{s+\frac{1}{2}}.$$

For $s > 2$, there is a smooth holonomy map

$$\mathrm{Hol} : \ \mathcal{A}(S)_{s-\frac{1}{2}} \to K$$

depending on the choice of base point x_0 in S; the assumption $s > 2$ implies that A is C^1 which guarantees existence of a solution to the parallel transport equation. For $s > 0$ and $A \in \mathcal{A}(S)_{s-\frac{1}{2}}$ the stabilizer $K(S)_{s+\frac{1}{2},A}$ is a compact, connected Lie group. For $s > 2$, $K(S)_{s+\frac{1}{2},A}$ is isomorphic to the centralizer of the holonomy $\mathrm{Hol}(A)$ via the map

$$K(S)_{s+\frac{1}{2}} \to K, \quad k \mapsto k(x_0).$$

Let $K_{x_0}(S)_{s+\frac{1}{2}}$ denote the space of $k \in K(S)_{s+\frac{1}{2}}$ such that $k(x_0)$ is the identity. For $s > 0$ there are bijections

$$K_{x_0}(S)_{s+\frac{1}{2}} \backslash \mathcal{A}(S)_{s-\frac{1}{2}} \to K, \quad K(S)_{s+\frac{1}{2}} \backslash \mathcal{A}(S)_{s-\frac{1}{2}} \to \mathrm{Ad}(K)\backslash K,$$

which for $s > 2$ are given by taking the holonomy, resp. conjugacy class of the holonomy of the connection.

The orbits of $K(S)_{s+\frac{1}{2}}$ on $\mathcal{A}(S)_{s-\frac{1}{2}}$ can be parametrized by the Weyl alcove as follows. Let Λ denote the coweight lattice of T and

$$W_{\mathrm{aff}} := W \rtimes \Lambda$$

the affine Weyl group. The action of W_{aff} on the Cartan subalgebra $\mathfrak{t}$ has fundamental domain

$$\mathfrak{A} := \{\xi \in \mathfrak{t}_+, \ \alpha_0(\xi) \le 1\}$$

where $\mathfrak{t}_+$ denotes the positive chamber and α_0 the highest root. Inclusion and exponentiation define bijections

$$\mathfrak{A} \to W_{\mathrm{aff}}\backslash \mathfrak{t} \to W\backslash T \to \mathrm{Ad}(K)\backslash K$$

and so for $s > 0$ we have a bijection

$$\mathfrak{A} \to K(S)_{s+\frac{1}{2}} \backslash \mathcal{A}(S)_{s-\frac{1}{2}}. \tag{1}$$

3. Background on connections on a surface

Let X be a compact, connected, oriented surface. Since K is simply-connected, any principal K-bundle is isomorphic to the trivial bundle $X \times K$. Let $\mathcal{A}(X)_s$ denote the affine space of connections on $X \times K$ of Sobolev class $s > 0$. Using the trivial connection as base point we may identify

$$\mathcal{A}(X)_s \to \Omega^1(X; \mathfrak{k})_s.$$

For any $A \in \mathcal{A}(X)_s$ and K-representation V, we have by the Sobolev multiplication theorem a covariant derivative

$$\mathrm{d}_A(V): \ \Omega^0(X;V)_{s+1} \to \Omega^1(X;V)_s \to \Omega^2(X;V)_{s-1}.$$

Let $\mathrm{d}_A := \mathrm{d}_A(\mathfrak{k})$ denote the covariant derivative for the adjoint representation. A is flat if and only if $\mathrm{d}_A^2 = 0$. Choose a Riemannian metric on X and invariant metric $(\ , \ )$ on $\mathfrak{k}$ and let

$$*_X : \Omega^\bullet(X;\mathfrak{k}) \to \Omega^{2-\bullet}(X;\mathfrak{k})$$

denote the resulting Hodge star operator. The operator $\mathrm{d}_A | \Omega^0(X, \partial X; \mathfrak{k})_{s+1}$ has L^2 adjoint

$$\mathrm{d}_A^* : \Omega^1(X;\mathfrak{k})_{-s} \to \Omega^0(X;\mathfrak{k})_{-s-1}, \quad \alpha \mapsto *_X \, \mathrm{d}_A *_X \alpha$$

which restricts to a map $\Omega^1(X;\mathfrak{k})_s \to \Omega^0(X;\mathfrak{k})_{s-1}$.

Lemma 3.0.1. *Suppose ∂X is non-empty. For $A \in \mathcal{A}(X)_s, s > 0$,*

(a) *The generalized Laplacian*

$$\mathrm{d}_A^* \mathrm{d}_A : \Omega^0(X, \partial X; \mathfrak{k})_{s+1} \to \Omega^0(X; \mathfrak{k})_{s-1}$$

is an isomorphism.

(b) *$\Omega^1(X; \mathfrak{k})_s$ has L^2-orthogonal splittings*

$$\Omega^1(X; \mathfrak{k})_s = \mathrm{Im}(\mathrm{d}_A | \Omega^0(X, \partial X; \mathfrak{k})_{s+1}) \oplus \mathrm{Ker}(\mathrm{d}_A^*).$$

$$\Omega^1(X; \mathfrak{k})_s = \mathrm{Im}(*_X \mathrm{d}_A | \Omega^0(X, \partial X; \mathfrak{k})_{s+1}) \oplus \mathrm{Ker}(\mathrm{d}_A).$$

(c) *If A is flat, then the decompositions in* (b) *are compatible, i.e.,*

$$\Omega^1(X; \mathfrak{k})_s = \mathrm{Im}(\mathrm{d}_A \oplus *_X \mathrm{d}_A) \oplus \mathrm{Ker}(\mathrm{d}_A \oplus \mathrm{d}_A^*).$$

Proof. (a) Let $A \in \mathcal{A}(X)$ be smooth. By elliptic regularity $\mathrm{Ker}(\mathrm{d}_A^* \mathrm{d}_A)$ consists of smooth solutions, see [16, Chapter 20], and we have a Hodge decomposition $\Omega^0(X; \mathfrak{k})_{s-1} = \mathrm{Im}(\mathrm{d}_A^* \mathrm{d}_A) \oplus \mathrm{Ker}(\mathrm{d}_A^* \mathrm{d}_A)$. By the Aronszajn-Cordes uniqueness theorem [1], $\mathrm{Ker}(\mathrm{d}_A^* \mathrm{d}_A) = 0$ and so $\mathrm{d}_A^* \mathrm{d}_A$ is an isomorphism. Since $\mathcal{A}(X)$ is dense in $\mathcal{A}(X)_s$ and d_A depends continuously on $A \in \mathcal{A}(X)_s$, $\mathrm{d}_A^* \mathrm{d}_A$ is an isomorphism for any $A \in \mathcal{A}(X)_s$. (b) By part (a), the subspaces are disjoint. For $a \in \Omega^0(X; \mathfrak{k})_s$, we may find $\xi \in \Omega^0(X, \partial X; \mathfrak{k})_{s+1}$ such that $\mathrm{d}_A^* a = \mathrm{d}_A^* \mathrm{d}_A \xi$. Then $a - \mathrm{d}_A \xi \in \ker \mathrm{d}_A^*$ which shows the first splitting; the second is similar. (c) follows immediately from $\mathrm{d}_A^2 = (\mathrm{d}_A^*)^2 = 0$. See also [36, 2.4]. □

For $s > 0$ the gauge group

$$K(X)_{s+1} := \mathrm{Map}(X, K)_{s+1}$$

is a Banach Lie group and acts on $\mathcal{A}(X)_s$ by the formula

$$k \cdot A = \mathrm{Ad}(k) A + k \mathrm{d}(k^{-1}) = \mathrm{Ad}(k) A - \mathrm{d}k \, k^{-1}$$

in any faithful matrix representation of K. It has Lie algebra

$$\mathfrak{k}(X)_{s+1} := \Omega^0(X; \mathfrak{k})_{s+1}.$$

The generating vector fields for the action of $K(X)_{s+1}$ on $\mathcal{A}(X)_s$ are

$$\xi_{\mathcal{A}(X)}(A) := \frac{\mathrm{d}}{\mathrm{d}t} (\exp(-t\xi) \cdot A)|_{t=0} = \mathrm{d}_A \xi, \quad \xi \in \mathfrak{k}(X)_{s+1}.$$

In particular, the Lie algebra $\mathfrak{k}(X)_A$ of the stabilizer $K(X)_A$ of A is

$$\mathfrak{k}(X)_A = \ker(\mathrm{d}_A | \Omega^0(X; \mathfrak{k})).$$

Suppose X is equipped with a complex structure. The map

$$d + \mathrm{ad}(A) \mapsto \overline{\partial}_\alpha := \overline{\partial} + \mathrm{ad}(\alpha)$$

where α is the $(0,1)$-form corresponding to a, defines a one-to-one correspondence between covariant derivatives and holomorphic covariant derivatives

$$\overline{\partial}_\alpha : \ \Omega^0(X;\mathfrak{g}) \mapsto \Omega^{0,1}(X;\mathfrak{g})$$

satisfying the holomorphic Leibniz rule $\overline{\partial}_\alpha(fs) = (\overline{\partial} f)s + f\overline{\partial}_\alpha s$. $G(X)$ acts on the space of holomorphic covariant derivatives by conjugation, and therefore on the space of $\mathfrak{g}$-valued $(0,1)$-forms by

$$g \cdot \alpha = \mathrm{Ad}(g)\alpha - (\overline{\partial} g)g^{-1}.$$

This formula extends to a holomorphic action of $G(X)_{s+1}$ on $\mathcal{A}(X)_s$. The invariant metric on $\mathfrak{k}$ defines a weakly symplectic form (that is, a closed 2-form that defines an injection $T\mathcal{A}(X)_s \to T^*\mathcal{A}(X)_s$) on $\mathcal{A}(X)_s$ for $s > 0$ by

$$\omega_{\mathcal{A}(X)} : \ (a_1, a_2) \mapsto \int_X (a_1 \wedge a_2)$$

where $(a_1 \wedge a_2)$ is the real-valued L^1 two-form on X defined by the wedge and inner products. In the case that the boundary of X is empty, the action of $K(X)_{s+1}$ is Hamiltonian with moment map given by the curvature [3]

$$\mathcal{M}(X)_s \to \Omega^2(X;\mathfrak{k})_{s-1}, \quad A \mapsto F_A.$$

Let $\mathcal{A}_\flat(X)_s$ denote the subspace of flat connections,

$$\mathcal{A}_\flat(X)_s := \{A \in \mathcal{A}(X)_s, \quad F_A = 0\}.$$

The symplectic quotient

$$\mathcal{M}(X)_s = K(X)_{s+1} \backslash\!\backslash \mathcal{A}(X)_s := K(X)_{s+1} \backslash \mathcal{A}_\flat(X)_s$$

is the *moduli space of flat bundles* on X. For $s > 2$, we have a holonomy map

$$\mathrm{Hol} : \ \mathcal{A}_\flat(X)_s \to \mathrm{Hom}(\pi_1(X, x_0), K). \tag{2}$$

Evaluation at the base point x_0 defines a homomorphism $K(X)_{s+1} \to K$ such that

$$k \cdot A = k(x_0) \cdot \mathrm{Hol}(A).$$

It follows that the stabilizer subgroup $K(X)_{s,A}$ is isomorphic to $K_{\mathrm{Hol}(A)}$ and so $K(X)_{s,A}$ is compact. The holonomy map induces a homeomorphism

$$\mathcal{M}(X)_s \to \mathrm{Hom}(\pi_1(X, x_0), K)/K.$$

In the case X has non-empty boundary, the moment map picks up an additional term [2], [10], [23]

$$\mathcal{A}(X)_s \to \Omega^2(X;\mathfrak{k})_{s-1} \oplus \Omega^1(\partial X;\mathfrak{k})_{s-1/2}, \quad A \mapsto (F_A, -r_{\partial X} A) \tag{3}$$

where $r_{\partial X}$ is restriction to the boundary. That is, for all $\xi \in \mathfrak{k}(X)_{s+1}$

$$\iota(\xi_{\mathcal{A}(X)})\omega_{\mathcal{A}(X)} = -\mathrm{d}\int_X (F_A \wedge \xi) + \mathrm{d}\int_{\partial X} (r_{\partial X} A \wedge \xi).$$

Let $K_\partial(X)_{s+1}$ be the subgroup fixing a framing on the boundary,

$$K_\partial(X)_{s+1} = \{k \in K(X)_{s+1},\ k|_{\partial X} = 1\}.$$

For $s > 0$ there is an exact sequence of Banach Lie groups

$$1 \to K_\partial(X)_{s+1} \to K(X)_{s+1} \to K(\partial X)_{s+\frac{1}{2}} \to 1.$$

Surjectivity of the third map follows from triviality of $\pi_1(K)$ and Lemma A.0.2 (e). The moment map for $K_\partial(X)_{s+1}$ is the curvature and the symplectic quotient

$$\mathcal{M}(X)_s = K_\partial(X)_{s+1}\backslash\!\backslash\mathcal{A}(X)_s := K_\partial(X)_{s+1}\backslash\mathcal{A}_\flat(X)_s$$

is the *moduli space of framed flat bundles* on X. Note that the stabilizer $K_\partial(X)_{s,A}$ is trivial, since we can choose the base point to lie on the boundary. For $s > 2$ this gives another proof that the operator $\mathrm{d}_A|\Omega^0(X, \partial X; \mathfrak{k})_{s+1}$ is injective.

Charts for $\mathcal{M}(X)_s, s > 0$ are constructed from local slices for the gauge action as follows. Using Lemma 3.0.1 and the implicit function theorem one sees that for $a \in \Omega^1(X; \mathfrak{k})_s$ sufficiently small there exists a unique gauge transformation $k \in K_\partial(X)_{s+1}$ in a neighborhood of the identity such that $k \cdot (A + a)$ is in Coulomb gauge with respect to A:

$$\mathrm{d}_A^*(k \cdot (A + a) - A) = 0.$$

By the Lemma, the operator $\mathrm{d}_A : \Omega^1(X; \mathfrak{k})_s \to \Omega^2(X; \mathfrak{k})_{s-1}$ has a right inverse $\mathrm{d}_A^{-1} : \Omega^2(X; \mathfrak{k})_{s-1} \to \Omega^1(X; \mathfrak{k})_s$ depending continuously on A. Suppose that A is flat. By the implicit function theorem again, there exists a constant ϵ depending only on $\|\mathrm{d}_A^{-1}\|$, open neighborhoods of A, resp. 0

$$U_A \subset \{A + a \in \mathcal{A}(X)_s,\quad F_{A+a} = 0,\quad \mathrm{d}_A^* a = 0\} \tag{4}$$

$$V_A \subset \{a \in \Omega^1(X; \mathfrak{k})_s,\quad \mathrm{d}_A a = 0,\quad \mathrm{d}_A^* a = 0\} \tag{5}$$

such that V_A is an ϵ-ball around 0, and a smooth map

$$S :\ V_A \to \Omega^2(X; \mathfrak{k})_{s-1}$$

such that

$$F_{A+(I+\mathrm{d}_A^{-1}S)a} = Sa + \frac{1}{2}[a + \mathrm{d}_A^{-1}Sa, a + \mathrm{d}_A^{-1}Sa] = 0. \tag{6}$$

Define

$$\varphi_A :\ V_A \to U_A,\quad a \mapsto A + (I + \mathrm{d}_A^{-1}S)a.$$

The following lemma summarizes the basic properties of $\mathcal{M}(X)_s$:

Lemma 3.0.2. Let X be a compact, connected, oriented surface with $b > 0$ boundary components and genus g.[1]

(a) For any $s > 0$, $\mathcal{M}(X)_s$ is a smooth Banach manifold.
(b) The size of the slice V_A depends only on $\|\mathrm{d}_A^{-1}\|$.
(c) For any $r > s > 0$, the inclusion $\mathcal{A}_\flat(X)_r \to \mathcal{A}_\flat(X)_s$ induces a bijection

$$K(\partial X)_{r+\frac{1}{2}} \backslash (K(\partial X)_{s+\frac{1}{2}} \times \mathcal{M}(X)_r) \to \mathcal{M}(X)_s.$$

(d) If $s > \frac{1}{4}$ restriction to the boundary

$$\mathcal{A}_\flat(X)_s \to \Omega^1(\partial X, \mathfrak{k})_{s-\frac{1}{2}}, \quad A \mapsto r_{\partial X} A$$

is continuous and induces a proper moment map

$$\mathcal{M}(X)_s \to \Omega^1(\partial X, \mathfrak{k})_{s-\frac{1}{2}}, \quad [A] \mapsto r_{\partial X} A$$

for the action of $K(\partial X)_{s+\frac{1}{2}}$.
(e) For any $s > 2$, $\mathcal{M}(X)_s$ is diffeomorphic to a fiber product

$$K^{2(g+b-1)} \times_{K^b} \Omega^1(\partial X; \mathfrak{k})_{s-\frac{1}{2}}.$$

(f) For any $s > \frac{1}{4}$, the quotient $K(\partial X)_{s+\frac{1}{2}} \backslash \mathcal{M}(X)_s$ is compact.
(g) For any $s > 0$, the Hodge star

$$*_X : \ker \mathrm{d}_A \oplus \mathrm{d}_A^* \to \ker \mathrm{d}_A \oplus \mathrm{d}_A^*$$

defines an almost complex structure on $\mathcal{M}(X)_s$.
(h) For any $s \geq 1$, $\mathcal{M}(X)_s$ is diffeomorphic to $G(\partial X)_{s+\frac{1}{2}} / G_{\mathrm{hol}}(X)_{s+\frac{1}{2}}$.
(i) For any $s \geq 1$, the almost complex structure $*_X$ is integrable, that is, there exist charts for which the transition maps are holomorphic.

Proof. The proofs of (a) and (c) are somewhat standard, as in [11], and left to the reader. (b) is a consequence of the implicit function theorem. (d) If $s > \frac{1}{2}$ holds then the trace is a continuous linear map $r_{\partial X} : \Omega^1(X; \mathfrak{k})_s \to \Omega^1(\partial X; \mathfrak{k})_{s-\frac{1}{2}}$. Therefore, the problem is to establish the lemma in the case $\frac{1}{4} < s < \frac{1}{2}$. Let $A \in \mathcal{A}_\flat(X)$ be smooth. We claim that $r_{\partial X} \circ \varphi_A : V_{A,s}^\epsilon \to \Omega^1(\partial X; \mathfrak{k})_{s-\frac{1}{2}}$ is smooth. The element a satisfies $(\mathrm{d}_A \oplus \mathrm{d}_A^*)a = 0$, hence the trace of a is well-defined [5, Theorem 13.8]. The non-linear term $\frac{1}{2}[a + \mathrm{d}_A^{-1} Sa, a + \mathrm{d}_A^{-1} Sa]$, and therefore also Sa by (6), has class $\min(s, 2s-1)$. Therefore, $\mathrm{d}_A^{-1} Sa$ is class $\min(s+1, 2s) > \frac{1}{2}$ which implies that its trace is also defined.

[1] That is, X is obtained from a closed genus g surface by removing b disks.

This shows that $\mathcal{M}(X)_s \to \Omega^1(\partial X)_{s-\frac{1}{2}}$ is smooth. Since $\mathcal{A}_\flat(X)_s \to \mathcal{M}(X)_s, A \mapsto [A]$ is smooth the map $r_{\partial X} : \mathcal{A}_\flat(X)_s \to \Omega^1(\partial X)_{s-\frac{1}{2}}$ is smooth as well. It follows from (3) that $r_{\partial X}$ is a moment map. This proves (d), except for properness. (e) Let $*_1, \ldots, *_b$ be base points on the boundary components, and let $d_1, \ldots, d_b$ be the paths around the boundary. Choose paths $a_1, b_1, \ldots, a_g, b_g$ from $*_1$ to $*_1$ and $c_1, \ldots, c_{b-1}$ from $*_1$ to $*_2, \ldots, *_{b-1}$ so that the fundamental group of X is freely generated by a_i, b_i, $i = 1, \ldots, g$ and $\mathrm{Ad}(c_j)d_j, j = 1, \ldots, b-1$. Suppose $s > 2$. Then the holonomies around a_i, etc. are well-defined and (2) gives a map $\mathrm{Hol} : \mathcal{M}(X)_s \to K^{2(g+b-1)}$. The remainder of the proof is the same as in [23, Theorem 3.2]. Back to (d): For $s > 2$, properness of $r_{\partial X}$ follows from compactness of K^{2g} in the holonomy description. The extension to $s > 0$ follows from the symplectic cross-section theorem: For any face σ^b of $\mathfrak{A}^b$, let $\mathfrak{A}^b_\sigma$ denote the open subset of $\mathfrak{A}^b$ obtained by removing all faces τ whose closure $\overline{\tau}$ does not contain σ. Let $K(\partial X)_\sigma$ denote the stabilizer of any point in σ. This is a compact, connected subgroup of $K(\partial X)_s$, independent of the choice of s. Then [23, Section 4.2] $\Phi^{-1}(K(\partial X)_\sigma \mathfrak{A}_\sigma)$ is a finite dimensional symplectic submanifold of $\mathcal{M}(X)_s$ and there is a diffeomorphism

$$\Phi^{-1}(K(\partial X)_{s+\frac{1}{2}} \mathfrak{A}_\sigma) \to K(\partial X)_{s+\frac{1}{2}} \times_{K(\partial X)_\sigma} \Phi^{-1}(K(\partial X)_\sigma \mathfrak{A}_\sigma).$$

Any subset of $\Omega^1(\partial X; \mathfrak{k})_{s-\frac{1}{2}}$ can be written as a finite union of subsets of $K(\partial X)_{s+\frac{1}{2}} \mathfrak{A}_\sigma$, as σ ranges over faces of $\mathfrak{A}^b$. Therefore it suffices to show that the map

$$\Phi^{-1}(K(\partial X)_{s+\frac{1}{2}} \mathfrak{A}_\sigma) \to K(\partial X)_{s+\frac{1}{2}} \mathfrak{A}_\sigma$$

is proper. But this follows from properness of

$$K(\partial X)_{s+\frac{1}{2}} \times \Phi^{-1}(K(\partial X)_\sigma \mathfrak{A}_\sigma) \to K(\partial X)_{s+\frac{1}{2}} \times K(\partial X)_\sigma \mathfrak{A}_\sigma$$

and the fact that $K(\partial X)_\sigma$ is compact. (f) By (c), compactness for $s > 2$ implies compactness for $s > \frac{1}{4}$. (g) In general ${*_X}^2 = (-1)^{d(\dim(X)-d)}$ on $\Omega^d(X; \mathfrak{k})$. In this case $d = 1$ so ${*_X}^2 = -1$. (h) By Donaldson's theorem [10], $G(\partial X)_{s+\frac{1}{2}}$ acts transitively on $\mathcal{M}(X)_s$. The stabilizer of the trivial connection is $G_{\mathrm{hol}}(X)_{s+\frac{1}{2}}$; it follows that there is a homeomorphism (in fact, a diffeomorphism of Banach manifolds) $\mathcal{M}(X)_s \to G(\partial X)_{s+\frac{1}{2}}/G_{\mathrm{hol}}(X)_{s+\frac{1}{2}}$. (i) follows from the description in (h), since $G_{\mathrm{hol}}(X)_{s+\frac{1}{2}}$ is a complex Banach subgroup of $G(\partial X)_{s+\frac{1}{2}}$. □

A *marking* is an element $\mu \in \mathfrak{A}$. If $\mu_1, \ldots, \mu_b \in \mathfrak{A}$ then we define the *moduli space of flat bundles with fixed holonomies*

$$\mathcal{M}(X; \mu_1, \ldots, \mu_b) = K(\partial X)_{s+\frac{1}{2}} \backslash r_{\partial X}^{-1}(\mathcal{O}_1 \times \ldots \times \mathcal{O}_b)$$

where $\mathcal{O}_1, \ldots, \mathcal{O}_b$ are the orbits corresponding to $\mu_1, \ldots, \mu_b$ in (1).

3.1. **The determinant/Chern-Simons line bundle.** This is a Hermitian line bundle with connection $(\mathcal{L}(X)_s, \nabla) \to \mathcal{M}(X)_s$ whose curvature is $-2\pi i \omega$, see e.g [42, Section 2]. It may be constructed by symplectic reduction as follows [32], [23, Section 3.3]. The trivial line bundle $\mathcal{A}(X)_s \times \mathbb{C}$ with connection 1-form

$$T_A \mathcal{A}(X)_s \to \mathbb{R}, \quad a \mapsto \frac{1}{2} \int_X (a \wedge A).$$

has curvature equal to $-2\pi i \omega_{\mathcal{A}}$. The central $U(1)$-extension $\widehat{K(X)}_{s+1}$ defined by the cocycle

$$(k_1, k_2) \mapsto \exp\left(\pi i \int_X (k_1^{-1}\, \mathrm{d}k_1 \wedge \mathrm{d}k_2\, k_2^{-1}) \right)$$

($k_1^{-1}\, \mathrm{d}k_1$, resp. $\mathrm{d}k_2\, k_2^{-1}$ are the pull-backs of the left, resp. right Maurer-Cartan forms on K) acts on $\mathcal{A}(X)_s \times \mathbb{C}$ by connection preserving automorphisms by the formula

$$(k, z) \cdot (A, w) = \left(k \cdot A, \exp\left(\pi i \int_X (k^{-1}\, \mathrm{d}k \wedge A) \right) z\, w \right).$$

On the Lie algebra level, $\widehat{\mathfrak{k}(X)}_{s+1}$ is the central $\mathbb{R}$-extension of $\mathfrak{k}(X)_{s+1}$ defined by the cocycle

$$(\xi_1, \xi_2) \mapsto \int_X (d\xi_1 \wedge d\xi_2) = \int_{\partial X} (\xi_1\, d\xi_2). \tag{7}$$

One may use the Chern-Simons three-form to trivialize the restriction $\widehat{K(X)}_{s+1}$ to $K_\partial(X)_{s+1}$ [23, Section 3.3]. The quotient

$$\widehat{K(\partial X)}_{s+\frac{1}{2}} := \widehat{K(X)}_{s+1} / K_\partial(X)_{s+1}$$

is the unique central $U(1)$-extension of $K(\partial X)_{s+\frac{1}{2}}$ defined by the Lie algebra cocycle (7). Define the pre-quantum line bundle $\mathcal{L}(X)_s$ by

$$\mathcal{L}(X)_s = K_\partial(X)_{s+1} \backslash\!\backslash (\mathcal{A}(X)_s \times \mathbb{C}) := K_\partial(X)_{s+1} \backslash (A_\flat(X)_s \times \mathbb{C}).$$

The products $U_A \times \mathbb{C}$ are local slices for the $K_\partial(X)_{s+1}$- action, and equip $\mathcal{L}(X)_s$ with the structure of a $\widehat{K(\partial X)}_{s+\frac{1}{2}}$-equivariant Hermitian line bundle with connection. The total space $\mathcal{L}(X)_s$ has an almost complex structure $J_\mathcal{L}$ determined by the connection ∇ and the almost complex structure on $\mathcal{M}(X)_s$, derived from the splitting

$$T\mathcal{L}(X)_s \cong \pi^* T\mathcal{M}(X)_s \oplus \underline{\mathbb{C}},$$

where $\underline{\mathbb{C}}$ denotes the trivial line bundle.

Lemma 3.1.1. *For any $s \geq 1$, the almost complex structure $J_{\mathcal{L}}$ is integrable, that is, there exist local trivializations for which the transition maps for $\mathcal{L}$ are holomorphic.*

Proof. Define $\widehat{G(\partial X)}_{s+\frac{1}{2}}$ to be the pull-back of the central $\mathbb{C}^*$-extension

$$\mathbb{C}^* \to \operatorname{Aut}(\mathcal{L}(X)_s, J_{\mathcal{L}}) \to \operatorname{Aut}(\mathcal{M}(X)_s, J_{\mathcal{M}})$$

under the map $G(\partial X)_{s+\frac{1}{2}} \to \operatorname{Aut}(\mathcal{M}(X)_s, J_{\mathcal{M}})$. Uniqueness of the central extension with a given cocycle implies that $\widehat{G(\partial X)}_{s+\frac{1}{2}}$ is the *basic* central $\mathbb{C}^*$-extension of $G(\partial X)_{s+\frac{1}{2}}$; by [29, Chapter 6], $\widehat{G(\partial X)}_{s+\frac{1}{2}}$ is a complex Banach Lie group. Since the action of $G(\partial X)_{s+\frac{1}{2}}$ on $\mathcal{M}(X)_s$ is transitive, so is the action of $\widehat{G(\partial X)_{s+\frac{1}{2}}}$ on $\mathcal{L}(X)_s$. Fix as base point the trivial connection $[0] \in \mathcal{M}(X)_s$, and let $\mathcal{L}(X)_{[0],s}$ denote the fiber. The map

$$(\widehat{G(\partial X)_{s+\frac{1}{2}}} \times \mathcal{L}(X)_{[0],s})/\widehat{G_{\mathrm{hol}}(X)}_{s+\frac{1}{2}} \to \mathcal{L}(X)_s, \quad [\hat{g}, z] \mapsto \hat{g}z$$

is a diffeomorphism preserving the almost complex structure. Since $\widehat{G(\partial X)_{s+\frac{1}{2}}}$, $\widehat{G_{\mathrm{hol}}(X)}_{s+\frac{1}{2}}$ are complex Banach Lie groups, the almost complex structure on the total space of $\mathcal{L}(X)_s$ is integrable. Local holomorphic triviality follows from the existence of local slices. $\square$

3.2. **Gluing equals reduction.** Let $\overline{X}$ be a compact, connected Riemann surface, $S \subset \overline{X}$ an oriented embedded circle, and X the Riemann surface obtained by cutting $\overline{X}$ along S. Let $\pi : X \to \overline{X}$ denote the gluing map, and $S_\pm$ the component of $\pi^{-1}(S)$ whose orientation agrees (resp. is the opposite) of the orientation on S. Let $\pi_\pm$ denote the restriction of π to $S_\pm$. Consider the diagonal embedding

$$\delta: \; K(S)_{s+\frac{1}{2}} \to K(\partial X)_{s+\frac{1}{2}}, \qquad k \mapsto (\pi_+^* k, \pi_-^* k).$$

We denote by $r_\pm$ the pull-back (restriction to the boundary, a.k.a. trace map)

$$r_\pm: \; \Omega^1(X; \mathfrak{k})_s \to \Omega^1(S_\pm; \mathfrak{k})_{s-\frac{1}{2}}.$$

The moment map for $K(S)_{s+\frac{1}{2}}$ is

$$\Phi: \; \mathcal{M}(X)_s \to \Omega^1(S; \mathfrak{k})_{s-\frac{1}{2}}, \quad [A] \mapsto (r_- - r_+)A.$$

The reason for the minus sign is that the identification $S \to S_-$ is orientation reversing. The map $[A] \mapsto [\pi_X^* A]$ induces a homeomorphism

$$\mathcal{M}(\overline{X}) \to K(S)_{s+\frac{1}{2}} \backslash\backslash \mathcal{M}(X)_s;$$

in fact, an isomorphism of Kähler symplectic orbifolds on the quotient of the subset of $\Phi^{-1}(0)$ on which $K(S)_{s+\frac{1}{2}}$ acts with only finite stabilizers [23, Theorem 3.5].

3.3. **Estimates on the sizes of slices.** In this section, we bound the size of gauge slices (charts for $\mathcal{M}(X)$) from below. The notation $< c$ means less than a universal constant, $< c(R)$ means less than a constant depending on R, where $\frac{1}{2}\|(r_+ - r_-)A\|_0^2 < R$.

Proposition 3.3.1. *For any $s > 0$ and $A \in \mathcal{A}_{\flat,s}$ there exists a gauge transformation $k \in K(X)_{s+1}$ with $r_+k = r_-k$ such the open ball $V_{k\cdot A}$ defined in (5) has radius bounded from below by $c(R)$.*

The proof follows from several lemmas. Since $K(\partial X)_{s+\frac{1}{2}}\backslash\mathcal{M}(X)_s$ is compact, there exists a compact subset $\mathcal{A}_\circ(X)$ of $\mathcal{A}(X)_s$ such that any element of $\mathcal{M}(X)_s$ may be represented by an element of $\mathcal{A}_\circ(X)$ up to gauge transform, i.e.

$$\mathcal{M}(X)_s = \{k \cdot [A_\circ],\ k \in K(\partial X)_{s+1},\quad A_\circ \in \mathcal{A}_\circ(X)_s\}.$$

Let $[A] \in \mathcal{M}(X)_s$ and let $k_\circ \in K(X)_{s+1}$ and $A_\circ \in \mathcal{A}_\circ(X)$ be such that $[k_\circ \cdot A_\circ] = [A]$. Consider the operators

$$\mathrm{d}_A : \Omega^0(X;\mathfrak{k})_{s+1} \to \Omega^1(X;\mathfrak{k})_s, \quad \mathrm{d}_A^{-1} : \Omega^2(X;\mathfrak{k})_{s-1} \to \Omega^1(X;\mathfrak{k})_s.$$

Lemma 3.3.2. The norms of $\mathrm{d}_A, \mathrm{d}_A^{-1}$ can be bounded by a constant depending only on the norms of $k_\circ$, $k_\circ^{-1}$.

Proof. We have $\mathrm{d}_A^{-1} = \mathrm{d}_{k_\circ \cdot A_\circ}^{-1} = \mathrm{Ad}(k_\circ) \circ d_{A_0}^{-1} \circ \mathrm{Ad}(k_\circ^{-1})$. Since $\mathcal{A}_\circ$ is compact, $\|\mathrm{d}_A^{-1}\|, \|\mathrm{d}_A\|$ are bounded on $\mathcal{A}_\circ$. The claim follows. ☐

Lemma 3.3.3. Let $s \geq \frac{1}{2}$ and let $[A] \in \mathcal{M}(X)_s$. There is a differentiable path $k_t \in K(X)_{s+1}$ such that $A_t = k_t A_0$ satisfies

(a) $A_0 \in \mathcal{A}(X)_\circ$,
(b) $[A_1] = k[A]$ for some $k \in K(S)_{s+\frac{1}{2}}$ and
(c) for all $t \in [0,1]$, $\|k_t\|_{\frac{3}{2}}$, $\|k_t^{-1}\|_{\frac{3}{2}}$, $\|A_t\|_{\frac{1}{2}}$, and $\|\frac{\mathrm{d}}{\mathrm{d}t}A_t\|_{\frac{1}{2}}$ are all bounded by a constant depending only on R.

Proof. Choose $k \in K(X)_{s+1}$ with $r_+k = r_-k$ so that $r_+k \cdot A \in *_S\mathfrak{A}$. Let $c = \sup_{\xi\in\mathfrak{A}} \| *_S \xi\|_0$. Then

$$\|r_+k \cdot A\|_0 < c \ \text{ and } \ \|r_-k \cdot A\|_0 \leq \|r_+k \cdot A\|_0 + \|\Phi(k \cdot A)\|_0 < c(R).$$

Replacing A with $k \cdot A$, we may assume $\|r_{\partial X}A\|_0 < c(R)$. Let $A_0 \in \mathcal{A}_\circ(X)_s$ be a flat connection gauge equivalent to A, and $k_1 \in K(X)_{s+1}$ so that $k_1 \cdot A_0 = A$. Then $\|r_{\partial X}k_1\|_1 < c(R)$. Suppose that $r_{\partial X}k_1 = \exp(\zeta)$ for some $\zeta \in \mathfrak{k}(\partial X)_1$, with ζ taking values in

$$\mathfrak{A}^- := \{\beta \in \mathfrak{A}, \quad \alpha_0(\beta) < 1\}. \tag{8}$$

Then $\|\zeta\|_0 < c$, hence $\|\zeta\|_1 < c(R)$. In fact, the image of $\mathrm{Map}(\partial X, \mathfrak{A}^-)_1$ under the exponential map is dense, since $K(\mathfrak{A} - \mathfrak{A}^-)$ is codimension 2 in K. It follows that

$$r_{\partial X} k_1 = \exp(\zeta), \quad \text{for some } \zeta \text{ with } \|\zeta\|_1 < c(R).$$

Let

$$\mathcal{E} : \ \Omega^0(\partial X; \mathfrak{k})_1 \to \Omega^0(X; \mathfrak{k})_{\frac{3}{2}}, \quad r_{\partial X} \circ \mathcal{E} = \mathrm{Id}$$

be any extension operator and define

$$\xi := \mathcal{E}\zeta, \quad k_t = \exp(t\xi).$$

By Sobolev multiplication

$$\|k_t\|_{\frac{3}{2}} = \|\exp(t\xi)\|_{\frac{3}{2}} \le \exp(c\|t\xi\|_{\frac{3}{2}})$$

and similarly for k_t^{-1}. Define $A_t = k_t A_0$, so that A_1 and A have the same image in $\mathcal{M}(X)_s$. Then

$$\|A_t\|_{\frac{1}{2}} = \|e^{t\xi} A_0\|_{\frac{1}{2}} \le \|e^{t\xi} \mathrm{d} e^{-t\xi}\|_{\frac{1}{2}} + \|\operatorname{Ad}(e^{t\xi}) A_0\|_{\frac{1}{2}} < c(R)$$

and $\|\frac{\mathrm{d}}{\mathrm{d}t} A_t\|_{\frac{1}{2}} = \|\mathrm{d}_{A_t}\xi\|_{\frac{1}{2}} < c(R)$ as claimed. □

Proposition 3.3.1 follows from Lemmas 3.3.2,3.3.3 and 3.0.2(b).

3.4. **Digression on gauge slices.** Later we will need slices for groups of gauge transformations of various types. Define

$$T_A := \ker \mathrm{d}_A^* \oplus \mathrm{d}_A : \ \Omega^1(X; \mathfrak{k})_s \mapsto (\Omega^0 \oplus \Omega^2)(X; \mathfrak{k})_{s-1}.$$

Lemma 3.4.1. If $A \in \mathcal{A}_\flat(X)$ is smooth then there exist L^2-orthogonal splittings

(a) $T_A = (T_A \cap \ker(r_{\partial X} *_X)) \oplus (T_A \cap \operatorname{im}(d_A \mathcal{E}_A))$;
(b) $T_A = (T_A \cap \ker((r_- - r_+) *_X)) \oplus (T_A \cap \operatorname{im}(d_A \mathcal{E}_A \delta))$;
(c) $T_A = (T_A \cap \ker(((r_- - r_+) *_X) \times (r_- - r_+))) \oplus (T_A \cap \operatorname{im}((d_A \mathcal{E}_A \delta) + (*_X d_A \mathcal{E}_A \delta))$.

Proof. We will prove (c); the others are similar. The reader may wish to compare this with the standard argument [16, p. 194]. The adjoint of

$$(9) \qquad (\mathrm{d}_A \mathcal{E}_A \delta) + (*_X \mathrm{d}_A \mathcal{E}_A \delta) : \ \Omega^0(S; \mathfrak{k})^2_{s+\frac{1}{2}} \to \ker(\mathrm{d}_A \oplus \mathrm{d}_A^*)_s,$$

is

$$(10) \quad ((r_- - r_+) *_X) \times (r_- - r_+)) : \ \ker(\mathrm{d}_A \oplus \mathrm{d}_A^*)_{-s} \to \Omega^1(S; \mathfrak{k})^2_{-s-\frac{1}{2}}.$$

(10) is an elliptic boundary condition, so if a lies in the kernel of (10) with Sobolev class s', then a has Sobolev class $s' + 1$ by Sobolev multiplication and elliptic regularity. Hence the kernel of (10) for $s' = -s$

is identical to that for $s' = s$, and $\ker(\mathrm{d}_A \oplus \mathrm{d}_A^*)_s$ is the L^2-orthogonal sum of the image of (9) and the kernel of (10). □

Lemma 3.4.2. Let $A \in \mathcal{A}_\flat(X)$ be smooth.

(a) There exists a constant ϵ depending only on $\|\mathrm{d}_A^{-1}\|, \|\mathrm{d}_A\|$ such that for any connection $A' \in \mathcal{A}(X)_s$ satisfying $\|A' - A\|_s < \epsilon$, there exists a gauge transformation $k \in K(X)_{s+1}$ such that $\mathrm{d}_A^*(k \cdot A' - A) = 0$ and $r_{\partial X} *_X (k \cdot A' - A) = 0$.

(b) There exists a constant ϵ depending only on $\|\mathrm{d}_A^{-1}\|, \|\mathrm{d}_A\|$ such that for any connection $A' \in \mathcal{A}(X)_s$ satisfying $\|A' - A\|_s < \epsilon$, there exists a gauge transformation $k \in K(X)_{s+1}$ such that $r_+ k = r_- k$ and $k \cdot A' - A$ lies in

$$\ker \mathrm{d}_A^* \oplus (r_- - r_+) *_X .$$

(c) There exists a constant ϵ such that for any $A' \in \mathcal{A}(X)_s$ satisfying $\|A' - A\|_s < \epsilon$, there exists a $g \in G(X)_{s+1}$ with $r_+ g = r_- g$ depending smoothly on A' such that $g \cdot A' - A$ lies in

$$\ker \mathrm{d}_A \oplus \mathrm{d}_A^* \oplus (r_- - r_+) *_X \oplus (r_- - r_+). \tag{11}$$

Proof. By the implicit function theorem and Lemma 3.4.1. □

4. The heat flow: existence of trajectories

This section and the next one are modeled after Råde's treatment [31] of the Yang-Mills heat equation. For $s \geq \frac{1}{2}$ the norm-square of the moment map

$$f: \mathcal{M}(X)_s \to \mathbb{R}, \quad [A] \mapsto \frac{1}{2}\|(r_+ - r_-)A\|_0^2$$

is a $K(S)_{s+\frac{1}{2}}$-invariant smooth function, by Lemma 3.0.2 (d). The gradient flow for $-f$ has the following description in local slices U_A. By (4) we have

$$T_{A+a} U_A = \ker \mathrm{d}_A^* \oplus \mathrm{d}_{A+a}.$$

Let π_A^a be the projection of $T_{A+a} U_{A+a}$ along $\operatorname{Im} \mathrm{d}_{A+a}$ onto $T_{A+a} U_A$. For $a \in \Omega^1(X; \mathfrak{k})_s$ sufficiently small, the decomposition in Lemma 3.0.1 (b) implies that for $s > 0$

$$\Omega^1(X; \mathfrak{k})_s \cong \operatorname{Im}(\mathrm{d}_{A+a} | \Omega^0(X, \partial X; \mathfrak{k})_{s+1}) \oplus \operatorname{Ker}(\mathrm{d}_A^* | \Omega^1(X; \mathfrak{k})_s).$$

Hence for any $\xi \in \mathfrak{k}(\partial X)_{s+1/2}$ sufficiently small, ξ has a unique *harmonic extension*

$$\mathcal{E}_A^a \xi \in \mathfrak{k}(X)_{s+1}, \quad \mathrm{d}_A^* \mathrm{d}_{A+a} \mathcal{E}_A^a \xi = 0;$$

that is, $\mathrm{d}_{A+a}\mathcal{E}^a_A\xi$ is the representative for the generating vector field for ξ in the local chart near A. Define

$$Q^a_A = \pi^a_A \; *_X \; \mathrm{d}_{A+a}\, \mathcal{E}^0_{A+a}\delta *_S (r_- - r_+).$$

(Recall δ is the diagonal embedding.) The gradient flow

$$\frac{\mathrm{d}}{\mathrm{d}t}[A_t] = -\,\mathrm{grad}(f)([A_t]) \tag{12}$$

is given in the local chart by

$$\frac{\mathrm{d}}{\mathrm{d}t}(A + a_t) = -Q^a_A(A + a_t), \quad A + a_t \in U_A. \tag{13}$$

4.1. **The linear initial value problem for the boundary data.** Define linear operators

$$P^a_{A,\pm} = \pm *_S r_\pm \pi^a_A *_X \mathrm{d}_{A+a}\mathcal{E}^0_{A+a}\delta \tag{14}$$

and set $P_{A,\pm} = P^0_{A,\pm}$. The evolution equations for the boundary data

$$B_\pm := \frac{1}{2} *_S (r_+ \pm r_-)A \tag{15}$$

are

$$\left(\frac{\mathrm{d}}{\mathrm{d}t} + P^a_{A,-} + P^a_{A,+}\right) B_- = 0, \quad \frac{\mathrm{d}}{\mathrm{d}t}B_+ = (P^a_{A,-} - P^a_{A,+})B_-. \tag{16}$$

Lemma 4.1.1. For any smooth $A \in \mathcal{A}_\flat(X)$, the operators $P_{A,\pm}$ are elliptic pseudo-differential operators of order 1 with principal symbol the same as the square root of the Laplacian on S. The sum $P_{A,+}+P_{A,-}$ is non-negative and self-adjoint of order 1.

Proof. The Hodge star $*_X$ has the effect of exchanging tangent and normal directions to the boundary. The Dirichlet-to-Neumann operator for the generalized Laplacian $\mathrm{d}_A^*\mathrm{d}_A$ is an elliptic pseudo-differential operator of order one with principal symbol equal to the square root of that of $\mathrm{d}_A^*\mathrm{d}_A$, see [16, Chapter 21],[37]. The operators $r_\pm$ and δ are Fourier integral operators of order 0, whose composition is the identity. $P_{A,\pm}$ is the composition of the Dirichlet-to-Neumann operator with diagonal embedding and restriction to $S_\pm$, and is therefore a Fourier integral operator with diagonal canonical relation, that is, a pseudo-differential operator. The relation between the operators is shown in

Figure 1. $P_{A,+} + P_{A,-}$ is non-negative:

$$\begin{aligned}\int_S ((P_{A_+} + P_{A,-})b \wedge *_S b) &= -\int_{\partial X} (*_{\partial X} r_{\partial X} *_X \mathrm{d}_A \mathcal{E}^0_A \delta b \wedge *_{\partial X} \delta b) \\ &= -\int_X d(*_X \mathrm{d}_A \mathcal{E}^0_A \delta b \wedge \mathcal{E}^0_A \delta b) \\ &= -\int_X (*_X \mathrm{d}_A \mathcal{E}^0_A \delta b \wedge \mathrm{d}_A \mathcal{E}^0_A \delta b) \\ &\geq 0.\end{aligned}$$

The proof that $P_{A,+} + P_{A,-}$ is self-adjoint is similar. □

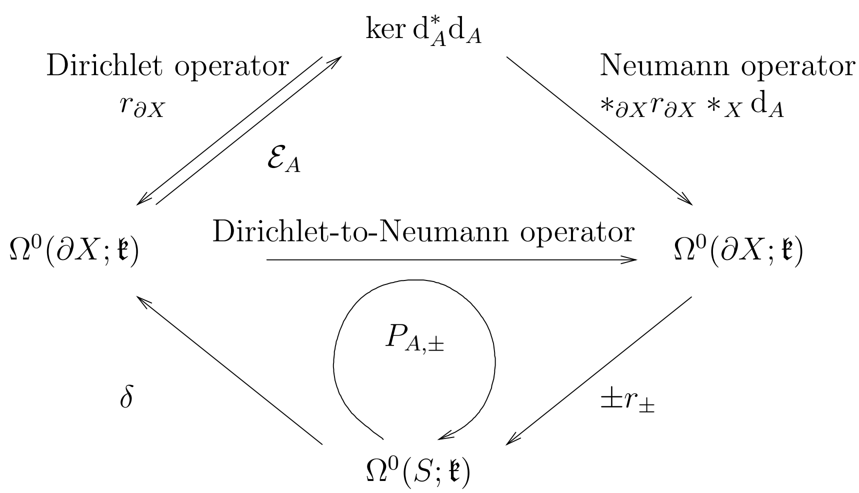

FIGURE 1. The operators $P_{A,\pm}$

Since $P_{+,A}$ and $P_{-,A}$ have the same principal symbol, $P_{+,A}+P_{-,A}$ is elliptic (and therefore Fredholm) and $P_{+,A}-P_{-,A}$ is a pseudo-differential operator of order 0. For A not smooth we will show the following properties of $P^a_{\pm,A}$.

Proposition 4.1.2. *For any $s > \frac{1}{4}$, flat $A \in \mathcal{A}(X)_s$ and $A + a \in U_A$,*

(a) $P^a_{A,+}+P^a_{A,-}$ *gives a Fredholm operator* $\Omega^0(\partial X; \mathfrak{k})_{s+\frac{1}{2}} \to \Omega^0(\partial X; \mathfrak{k})_{s-\frac{1}{2}}$;
(b) $P^a_{A,+}-P^a_{A,-}$ *gives an operator* $\Omega^0(\partial X; \mathfrak{k})_{s+\frac{1}{2}} \to \Omega^0(\partial X; \mathfrak{k})_{\min(2s-\frac{1}{2}, s+\frac{1}{2})}$.

This will be derived from:

Lemma 4.1.3. Let $s > \frac{1}{4}$.

(a) Let A_u be a differentiable path of flat connections. Then
    (i) $\frac{\mathrm{d}}{\mathrm{d}u} \mathrm{d}_{A_u} \mathcal{E}^0_{A_u}|_{u=v}$ gives an operator $\Omega^0(\partial X; \mathfrak{k})_{s+\frac{1}{2}} \to \Omega^1(X; \mathfrak{k})_{\min(2s, s+1)}$;
    (ii) $\frac{\mathrm{d}}{\mathrm{d}u} P^0_{A_u,\pm}|_{u=v}$ gives an operator $\Omega^0(\partial X; \mathfrak{k})_{s+\frac{1}{2}} \to \Omega^0(\partial X; \mathfrak{k})_{\min(2s-\frac{1}{2}, s+\frac{1}{2})}$.

(b) Let A_0+a_u be a differentiable path of flat connections such that a_u lies in U_{A_0}. Then
  (i) $\frac{\mathrm{d}}{\mathrm{d}u}\pi^{a_u}_{A_0}|_{u=v}$ gives an operator $(T_{A_u})_s \to \Omega^1(X;\mathfrak{k})_{\min(2s,s+1)}$;
  (ii) $\frac{\mathrm{d}}{\mathrm{d}u}P^{a_u}_{A_0,\pm}|_{u=v}$ gives an operator $\Omega^0(\partial X;\mathfrak{k})_{s+\frac{1}{2}} \to \Omega^0(\partial X;\mathfrak{k})_{\min(2s-\frac{1}{2},s+\frac{1}{2})}$.

Proof. We have

$$0 = \frac{\mathrm{d}}{\mathrm{d}u}\left((\mathrm{d}_{A_u}\oplus \mathrm{d}^*_{A_u})\mathrm{d}_{A_u}\mathcal{E}^0_{A_u}\right)b$$

and so

$$(\mathrm{d}_{A_u}\oplus \mathrm{d}^*_{A_u})(\frac{\mathrm{d}}{\mathrm{d}u}\mathrm{d}_{A_u}\mathcal{E}^0_{A_u}b) = -\left(\frac{\mathrm{d}}{\mathrm{d}u}(\mathrm{d}_{A_u}\oplus \mathrm{d}^*_{A_u})\right)\mathrm{d}_{A_u}\mathcal{E}^0_{A_u}b. \tag{17}$$

Also

$$0 = r_{\partial X}\left(\frac{\mathrm{d}}{\mathrm{d}u}\mathrm{d}_{A_u}\mathcal{E}^0_{A_u}b\right).$$

The right-hand side of (17) has norm $\min(s,2s-1)$. (a):(i) now follows from elliptic regularity for $d\oplus d^*$. (a):(ii) follows from (a):(i). (b):(i) By definition $\pi^{a_u}_{A_0}\alpha = \alpha - \mathrm{d}_{A_u}R_u\alpha$ where

$$R_u:\ (T_{A_u})_s \to \Omega^0(X,\partial X;\mathfrak{k})_{s+1}, \quad (\mathrm{d}_{A_u}\oplus \mathrm{d}^*_{A_0})(\alpha-\mathrm{d}_{A_u}R_u\alpha)=0.$$

Differentiating we obtain

$$(\mathrm{d}_{A_u}\oplus \mathrm{d}^*_{A_0})\frac{\mathrm{d}}{\mathrm{d}u}\mathrm{d}_{A_u}R_u\alpha = \left(-\operatorname{ad}\left(\frac{\mathrm{d}}{\mathrm{d}u}A_u\right)\oplus 0\right)\mathrm{d}_{A_u}R_u\alpha.$$

Since the right hand-side is order $\min(s,2s-1)$, $\frac{\mathrm{d}}{\mathrm{d}u}\pi^{a_u}_{A_0}\alpha$ is order $\min(s+1,2s)$. (ii) follows from (a), (b):(i). □

Lemma 4.1.2 follows from 4.1.3 by choosing a path A_u of connections from a smooth connection A_0 to A. We can also use 4.1.3 to derive bounds on the operator $\mathcal{E}_A$.

Lemma 4.1.4. For any $s>\frac{1}{4}$ and $\epsilon>0$ there exists $\delta>0$ such that if $A_u,\ u\in[0,1]$ is a differentiable path of flat connections and

$$\|A_u\|_s<\delta,\quad \|\frac{\mathrm{d}}{\mathrm{d}u}A_u\|_s<\delta \tag{18}$$

then
(a) $\mathcal{E}^0_{A_u}:\Omega^0(\partial X;\mathfrak{k})_{s+\frac{1}{2}}\to\Omega^1(X;\mathfrak{k})_{s+1}$ satisfies

$$\|\mathcal{E}^0_{A_u}\|<\epsilon\|\mathcal{E}^0_{A_0}\|.$$

(b) $\frac{\mathrm{d}}{\mathrm{d}u}\mathrm{d}_{A_u}\mathcal{E}^0_{A_u}:\Omega^0(\partial X;\mathfrak{k})_{s+\frac{1}{2}}\to\Omega^1(X;\mathfrak{k})_{\min(2s,s+1)}$ satisfies

$$\|\frac{\mathrm{d}}{\mathrm{d}u}\mathrm{d}_{A_u}\mathcal{E}^0_{A_u}\|<\epsilon.$$

(c) $P_{A_u,+} + P_{A_u,-} : \ \Omega^0(S;\mathfrak{k})_{s+\frac{1}{2}} \to \Omega^0(S;\mathfrak{k})_{s-\frac{1}{2}}$ satisfies

$$\|P_{A_u,+} + P_{A_u,-}\| < \epsilon \|P_{A_0,+} + P_{A_0,-}\|$$

and

(d) $P_{A_u,-} - P_{A_u,+} : \ \Omega^0(S;\mathfrak{k})_{s+\frac{1}{2}} \to \Omega^0(S;\mathfrak{k})_{\min(2s-\frac{1}{2},s+\frac{1}{2})}$ satisfies

$$\|P_{A_u,-} - P_{A_u,+}\| < \epsilon \|P_{A_0,-} - P_{A_0,+}\|.$$

Proof. By differentiating

$$\mathrm{d}_{A_u}^* \mathrm{d}_{A_u} \mathcal{E}^0_{A_u} b = 0$$

we obtain

$$\mathrm{d}_{A_u}^* \mathrm{d}_{A_u} \frac{\mathrm{d}}{\mathrm{d}u} \mathcal{E}^0_{A_u} b = (\frac{\mathrm{d}}{\mathrm{d}u} \mathrm{d}_{A_u}^* \mathrm{d}_{A_u}) \mathcal{E}^0_{A_u} b.$$

By Lemma 3.0.1, the operators $\mathrm{d}_{A_u}^* \mathrm{d}_{A_u}$ are surjective. Compactness of $\Omega(X;\mathfrak{k})_s \to \Omega(X;\mathfrak{k})_{s'}$, $s' \in (0,s)$ implies an upper bound for the norm of a right inverse $(\mathrm{d}_{A_u}^* \mathrm{d}_{A_u})^{-1}$. Hence

$$\|\frac{\mathrm{d}}{\mathrm{d}u} \mathcal{E}^0_{A_u} b\|_{s+1} < c \|\mathcal{E}^0_{A_u} b\|_{s+1}$$

for some $c > 0$ that approaches zero as δ does. Then

$$\frac{\mathrm{d}}{\mathrm{d}u} \ln \|\mathcal{E}^0_{A_u} b\|_{s+1} < \frac{\|\frac{\mathrm{d}}{\mathrm{d}u} \mathcal{E}^0_{A_u} b\|_{s+1}}{\|\mathcal{E}^0_{A_u} b\|_{s+1}} < c.$$

Integrating with respect to u gives

$$\|\mathcal{E}^0_{A_1} b\|_{s+1} < \|\mathcal{E}^0_{A_0} b\|_{s+1} \exp(c)$$

which implies (a). (c) is follows from (a) and the estimates on A_u. (b): By elliptic regularity for $\mathrm{d}_{A_u} \oplus \mathrm{d}_{A_u}^*$ we have

$$\begin{aligned} \|\frac{\mathrm{d}}{\mathrm{d}u} \mathrm{d}_{A_u} \mathcal{E}^0_{A_u} b\|_{\min(2s,s+1)} &< c_1 \|\frac{\mathrm{d}}{\mathrm{d}u} \mathrm{d}_{A_u} \mathcal{E}^0_{A_u} b\|_s + c_2 \|\mathrm{d}_{A_u} \mathcal{E}^0_{A_u} b\|_{\min(2s-1,s)} \\ &< (c_1 c + c_2) \|\mathrm{d}_{A_u} \mathcal{E}^0_{A_u} b\|_s; \end{aligned}$$

see (17). Using (a) this is less than ϵ for δ sufficiently small. (d) follows from (b). □

From Lemma 3.3.3 we get

Lemma 4.1.5. For any $s \geq \frac{1}{2}$, there exists a constant $c(R)$ such that for any $A \in \mathcal{A}_\flat(X)_s$ with $f([A]) < R$, there exists a gauge transformation $k \in K(X)_{s+1}$ such that $r_+ k = r_- k$ and

(a) $\|P_{+,k\cdot A} + P_{-,k\cdot A}\| < c(R)$ as an operator of order -1, and
(b) $\|P_{+,k\cdot A} - P_{-,k\cdot A}\| < c(R)$ as an operator of order $-\frac{1}{2}$.

4.2. **The linear initial-value problem in a local slice.** Let $s > \frac{1}{4}$. We solve the linear, time-independent boundary initial-value problem

$$(19)\qquad (\frac{\mathrm{d}}{\mathrm{d}t} + P_{A_0,+} + P_{A_0,-})B_- = 0, \qquad \frac{\mathrm{d}}{\mathrm{d}t}B_+ = (P_{A_0,-} - P_{A_0,+})B_-.$$

We denote by $\Omega^1(S;\mathfrak{k})_{r,s}$ the Sobolev space of $\mathfrak{k}$-valued one-forms on S, time-dependent on an interval $[0,T]$, with r derivatives in the time direction and s derivatives on X; see the appendix. We assume throughout that $T < 1$. By A.0.3 there is a solution operator for (19),

$$(20)\qquad M_{A_0,-} : \ \Omega^1(S;\mathfrak{k})_{s-\frac{1}{2}} \to \Omega^1(S;\mathfrak{k})_{(\frac{1}{2}-r,s-\frac{1}{2}+r)}, \quad B_-(0) \mapsto B_-$$

for any real r, with norm $\|M_{A_0,-}\| < c(R)\max(1,T^r)$. We have

$$(P_{A_0,-} - P_{A_0,+})B_- \in \Omega^1(S;\mathfrak{k})_{(\frac{1}{2}-r,2s+r-\frac{3}{2})}.$$

If $r \in (0,1)$, then by A.0.3(a) $\Omega^1(S;\mathfrak{k})_{(0,\frac{1}{2}-r,2s+r-\frac{3}{2})}$ is equal to $\Omega^1(S;\mathfrak{k})_{(\frac{1}{2}-r,2s+r-\frac{3}{2})}$. By A.0.3(g) integration gives a solution

$$B_+ \in \Omega^1(S;\mathfrak{k})_{0,\frac{3}{2}-r,2s+r-\frac{3}{2}}.$$

to (19). $\Omega^1(S;\mathfrak{k})_{(\frac{3}{2}-r,2s+r-\frac{3}{2})}$ embeds into $\Omega^1(S;\mathfrak{k})_{(\frac{3}{2}-r,s+r-\frac{3}{2})}$. Taking $r = -\epsilon$ and $r = 1-\epsilon$ gives a solution

$$B_- \in \Omega^1(S;\mathfrak{k})_{(\frac{1}{2}+\epsilon,s-\frac{1}{2}-\epsilon)\cap(-\frac{1}{2}+\epsilon,s+\frac{1}{2}-\epsilon)}, \quad B_+ \in \Omega^1(S;\mathfrak{k})_{\frac{1}{2}+\epsilon,2s-\frac{1}{2}-\epsilon}.$$

Lemma 4.2.1. Let $A_0 \in \mathcal{A}_\flat(X)$ be smooth. The image of $Q^0_{A_0}$ is perpendicular to $T_{A_0} \cap \ker r_{\partial X}$.

Proof. Suppose that $a \in T_{A_0} \cap \ker r_{\partial X}$. Integration by parts gives

$$\begin{aligned}
\int_X (Q^0_{A_0}A \wedge *_X a) &= \int_X (*_X \mathrm{d}_{A_0}\mathcal{E}_{A_0}\delta *_S (r_- - r_+)A \wedge *_X a) \\
&= \int_X (\mathrm{d}_{A_0}\mathcal{E}_{A_0}\delta *_S (r_- - r_+)A \wedge a) \\
&= \int_X (\mathcal{E}_{A_0}\delta *_S (r_- - r_+)A \wedge \mathrm{d}_{A_0}a) \\
&\quad + \int_{\partial X} (\delta *_S (r_- - r_+)A \wedge r_{\partial X}a) \\
&= 0.
\end{aligned}$$

□

We introduce norms on T_{A_0} corresponding to the choice of different Sobolev norms in the splitting

$$T_{A_0} \cong \mathrm{Ker}\, r_{\partial X}|_{T_{A_0}} \oplus \mathrm{Im}(r_+ - r_-)|_{T_{A_0}} \oplus \mathrm{Im}(r_- + r_+)|_{T_{A_0}}.$$

(To solve the Yang-Mills heat equation, one assumes $A \in H^s$ and $F_A \in H^{s-1}$.) The first summand is finite dimensional. Let $\epsilon \in (0, \frac{1}{4})$ and

$$(T_{A_0})'_s = (\ker r_{\partial X})_{\frac{1}{2}+\epsilon, s} \oplus (\operatorname{Im}(r_+ - r_-))_{(\frac{1}{2}+\epsilon, s-\frac{1}{2}-\epsilon)\cap(-\frac{1}{2}+\epsilon, s+\frac{1}{2}-\epsilon)} \oplus (\operatorname{Im}(r_+ + r_-))_{\frac{1}{2}+\epsilon, s-\frac{1}{2}-\epsilon} \tag{21}$$

where all operators are understood to be restricted to T_{A_0}. Similarly, define

$$(T_{A_0})'_{0,s} = (\ker r_{\partial X})_{0,\frac{1}{2}+\epsilon, s} \oplus (\operatorname{Im}(r_+ - r_-))_{0,(\frac{1}{2}+\epsilon, s-\frac{1}{2}-\epsilon)\cap(-\frac{1}{2}+\epsilon, s+\frac{1}{2}-\epsilon)} \oplus (\operatorname{Im}(r_+ + r_-))_{0,\frac{1}{2}+\epsilon, s-\frac{1}{2}-\epsilon}; \tag{22}$$

$$(T_{A_0})''_s = (\ker r_{\partial X})_{-\frac{1}{2}+\epsilon} \oplus (\operatorname{Im}(r_+ - r_-))_{-\frac{1}{2}+\epsilon, s-\frac{1}{2}-\epsilon} \oplus (\operatorname{Im}(r_+ + r_-))_{-\frac{1}{2}+\epsilon, s-\frac{1}{2}-\epsilon}. \tag{23}$$

The splitting of T_{A_0} induces an embedding $(T_{A_0})'_s \to (T_{A_0})_{\frac{1}{2}+\epsilon, s-\epsilon}$; there are similar embeddings for the other spaces.

Lemma 4.2.2. For $s > \frac{1}{4}$, solving the time-independent equation

$$(\frac{\mathrm{d}}{\mathrm{d}t} + Q^0_{A_0})a = 0, \quad a(0) = a_0 \tag{24}$$

defines an operator

$$M_{A_0} : \ A_0 + (T_{A_0})_s \to \ (T_{A_0})'_s, \quad A_0 + a_0 \mapsto a$$

with $\|M_{A_0}(A_0 + a)\| < c(R) \max(1, T^{-\epsilon}) \|A_0 + a\|$.

Proof. Let A denote the unique lift of the solution (B_+, B_-) such that $A - A_0$ is perpendicular to $T_{A_0} \cap \ker r_{\partial X}$. By Lemma 4.2.1, A solves (24). The estimate on the norm of M_{A_0} follows from that on $M_{A_0,-}$ in (20). □

4.3. **The non-linear initial value problem in a local slice.**

Theorem 4.3.1. *For any Sobolev class $s \geq \frac{1}{2}$ and $R > 0$, there exists a time $T = T(R)$ such that for all $A_0 \in \mathcal{A}_\flat(X)_s$ with $f(A_0) < R$ and sufficiently small $a_0 \in (T_{A_0})_s$, the initial value problem (13) has a unique solution $A = \varphi_{A_0}(a)$ on $[0, T]$ with a in $(T_{A_0})'_s$. The solution A lies in $C^0([0, T], (U_{A_0})_s)$ and depends smoothly on the initial condition a_0 in these topologies.*

The proof uses standard Sobolev space techniques. For any flat connection A define a bounded linear operator

$$L_A : (T_A)'_{0,s} \to (T_A)''_s \quad a \mapsto (\frac{\mathrm{d}}{\mathrm{d}t} + Q^0_A)a.$$

That is,

$$L_A(b_0, b_-, b_+) = (\frac{\mathrm{d}}{\mathrm{d}t}b_0, (\frac{\mathrm{d}}{\mathrm{d}t} + P_{A,+} + P_{A,-})b_-, \frac{\mathrm{d}}{\mathrm{d}t}b_+ - (P_{A,-} - P_{A,+})b_-).$$

Solving the inhomogeneous equation $(\frac{\mathrm{d}}{\mathrm{d}t} + P_{A_0,+} + P_{A_0,-})u = f$ defines a right inverse to the operator

$$\frac{\mathrm{d}}{\mathrm{d}t} + P_{A_0,+} + P_{A_0,-} : \ \Omega^1(S;\mathfrak{k})_{0,(\frac{1}{2}+\epsilon,s-\epsilon)\cap(-\frac{1}{2}+\epsilon,s+1-\epsilon)} \to \Omega^1(S;\mathfrak{k})_{-\frac{1}{2}+\epsilon,s-\epsilon},$$

with norm depending on $\|P_{A_0,+} + P_{A_0,-}\|$. The operators

$$\frac{\mathrm{d}}{\mathrm{d}t} : \ \Omega^1(S;\mathfrak{k})_{0,\frac{1}{2}+\epsilon,s-\frac{1}{2}-\epsilon} \to \Omega^1(S;\mathfrak{k})_{-\frac{1}{2}+\epsilon,s-\frac{1}{2}-\epsilon}$$

and

$$\frac{\mathrm{d}}{\mathrm{d}t} : \ (T_A \cap \ker r_{\partial X})_{0,\frac{1}{2}+\epsilon,s} \to (T_A \cap \ker r_{\partial X})_{-\frac{1}{2}+\epsilon,s}$$

are also invertible. Therefore, L_{A_0} has right inverse mapping (f_0, f_-, f_+) to

$$((\frac{\mathrm{d}}{\mathrm{d}t})^{-1}(f_0), (\frac{\mathrm{d}}{\mathrm{d}t} + P_{A,+} + P_{A,-})^{-1}(f_-), \\ (\frac{\mathrm{d}}{\mathrm{d}t})^{-1}f_+ + (\frac{\mathrm{d}}{\mathrm{d}t})^{-1}(P_{A,-} - P_{A,+})(\frac{\mathrm{d}}{\mathrm{d}t} + P_{A,+} + P_{A,-})^{-1}f_-) \tag{25}$$

giving the solution to the time-independent inhomogeneous problem

$$(\frac{\mathrm{d}}{\mathrm{d}t} + Q^0_{A_0})A = f, \quad A(0) = 0. \tag{26}$$

Lemma 4.3.2. For $s \geq \frac{1}{2}$ and $A_0 \in \mathcal{A}_\flat(X)_s$, there exists $k \in K(X)_{s+1}$ with $r_+k = r_-k$ and $\|L^{-1}_{k\cdot A_0}\| < c(R)$.

Proof. This follows from Lemma 4.1.5. □

The non-linear initial value problem is solved by perturbation. Working in the slice U_{A_0} near A_0 we write

$$A = \varphi_{A_0}(a_1 + a_2)$$

where $A_0 + a_1$ is the solution to the time-independent problem with initial condition $a_1(0) = a_0$. By Lemma 3.3.1, the size of V_{A_0} is bounded from below by $c(R)$. The problem (13) becomes

$$0 = (\frac{\mathrm{d}}{\mathrm{d}t} + Q^a_A)A = (\frac{\mathrm{d}}{\mathrm{d}t} + Q^0_{A_0} - Q^0_{A_0} + Q^{a_1+a_2}_{A_0})(A_0 + a_1 + a_2)$$

or using (24)

$$(\frac{\mathrm{d}}{\mathrm{d}t} + Q^0_{A_0})a_2 = (Q^0_{A_0} - Q^{a_1+a_2}_{A_0})(A_0 + a_1 + a_2). \tag{27}$$

Define

$$R_{A_0} = (Q^0_{A_0} - Q^{a_1}_{A_0})(A_0 + a_1)$$

and

$$N_{A_0} a_2 = (Q^0_{A_0} - Q^{a_1+a_2}_{A_0})(A_0 + a_1 + a_2) - R_{A_0}.$$

N_{A_0} is a non-linear operator with $N_{A_0}0 = 0$. We have to solve the initial value problem

$$(L_{A_0} - N_{A_0})a_2 = R_{A_0}, \quad a_2(0) = 0. \tag{28}$$

We will show that N_{A_0} and R_{A_0} have small norms for T small.

Lemma 4.3.3. For $s > \frac{1}{4}$, the operators $\frac{\mathrm{d}}{\mathrm{d}u} Q^{a_u}_{A_0}|_{u=v}$ and $Q^0_{A_0} - Q^a_{A_0}$ have order $\min(0, s-1)$.

Proof. We compute

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}u} Q^{a_u}_{A_0}|_{u=v} &= \frac{\mathrm{d}}{\mathrm{d}u}\left(\pi^{a_u}_{A_0} *_X \mathrm{d}_{A_u}\mathcal{E}_{A_u}\delta *_X (r_+ - r_-)\right)|_{u=v} \\ &= (\frac{\mathrm{d}}{\mathrm{d}u}\pi^{a_u}_{A_0}|_{u=v}) *_X \mathrm{d}_{A_u}\mathcal{E}_{A_u}\delta *_X (r_+ - r_-) \\ &+ \pi^{a_v}_{A_0} *_X \mathrm{d}_{A_v}(\frac{\mathrm{d}}{\mathrm{d}u}\mathrm{d}_{A_u}\mathcal{E}_{A_u}|_{u=v})(\pi_+ \times \pi_-)^* *_X (r_+ - r_-). \end{aligned}$$

The result follows from Lemma 4.1.3. For the second operator consider a path from A_0 to A and integrate. □

Lemma 4.3.4. For $s \geq \frac{1}{2}$ and $T < c(R)$, the map $L_{A_0} - N_{A_0}$ is a diffeomorphism of a neighborhood of 0 in $(T_{A_0})'_{0,s}$ onto a ball in $(T_{A_0})''_s$. Furthermore, there exists $k \in K(X)_{s+1}$ with $r_+k = r_-k$ such that after replacing A_0 with $k \cdot A_0$ the radius of the ball is at least $c(R)T^{-\frac{1}{2}+\epsilon}$ and the norm of R_{A_0} is at most $c(R)T^{\frac{1}{2}-2\epsilon}$. Equation (28) has a unique solution a_2 with $\|a_2\|''_s < c(R)T^{\epsilon-\frac{1}{2}}$. The solution a_2 depends smoothly on the initial condition a_0.

Proof. Estimate for R_{A_0}: By interpolation $(r_- - r_+)(A_0 + a_1)$ lies in $(T_{A_0})_{\epsilon,s-\epsilon}$. By Lemma 4.3.3 R_{A_0} lies in $(T_{A_0})_{\epsilon,\min(s,2s-1)-\epsilon}$. Since $s > 0$, this embeds in $(T_{A_0})_{-\frac{1}{2}+\epsilon,s-1-\epsilon}$, and the norm of the embedding is at most $cT^{\frac{1}{2}}$. By Lemma 4.2.2 the norm of $a_1 = M_{A_0}A_0$ is bounded by $c(R)$. Hence $\|R_{A_0}\|_{-\frac{1}{2}+\epsilon,s-1-\epsilon} \leq c(R)T^{\frac{1}{2}-\epsilon}$. Now consider $N_{A_0}a_2$. We have

$$(D_{a_2}N_{A_0})(a) = -\frac{\mathrm{d}}{\mathrm{d}u} Q^{a_1+a_2+ua}_{A_0}(A_0 + a_1 + a_2)|_{u=0} + (Q^0_{A_0} - Q^{a_1+a_2}_{A_0})(a).$$

The operators in this expression are again of order $\min(0, s-1)$, and the same argument as for R_{A_0} shows that

$$\|D_{a_2}N_{A_0}\| \leq c(R)(\|A_0\| + \|a_1\| + \|a_2\|)T^{\frac{1}{2}} < c(R)(T^{\frac{1}{2}-\epsilon} + \|a_2\|T^{\frac{1}{2}}).$$

Therefore, for $T < c(R)$ and $\|a_2\| \leq \frac{1}{4}c(R)T^{-\frac{1}{2}}$ we have

$$\|D_{a_2}N_{A_0}\| \leq \frac{1}{2}\|L_{A_0}^{-1}\|.$$

It follows that $D_{a_2}(L_{A_0} - N_{A_0})$ is invertible and

$$\|D_{a_2}(L_{A_0} - N_{A_0})^{-1}\| \leq 2\|L_{A_0}^{-1}\|. \tag{29}$$

We wish to show that the map $L_{A_0} - N_{A_0}$ is a diffeomorphism of a neighborhood of zero onto a ball of radius $c(R)T^{-\frac{1}{2}}$. Let b lie in $(T_{A_0})''_s$ with $\|b\| \leq c(R)T^{-\frac{1}{2}}$. For all $s \in [0,1]$ consider the equation

$$(L_{A_0} - N_{A_0})a_s = sb.$$

Solving this is equivalent to solving

$$\frac{\mathrm{d}}{\mathrm{d}\theta}a_\theta = (D(L_{A_0} - N_{A_0})^{-1})b.$$

By (29) this has a solution a_θ with

$$\|a_\theta\| \leq 2s\|L_{A_0}^{-1}\|\|b\|.$$

It remains to show that a_2 depends smoothly on a_0. This follows from the implicit function theorem for Banach spaces applied to the map $L_{A_0} - N_{A_0}$. □

Lemma 4.3.5. Let $s \geq \frac{1}{2}$. The solution $a_1 + a_2$ we have constructed actually lies in $(T_{A_0})_{\frac{1}{2}+\epsilon,s}$ and therefore by Sobolev embedding also in $C^0([0,T],(T_{A_0})_s)$.

Proof. $a_1 \in (T_{A_0})_{\frac{1}{2}+\epsilon,s-\epsilon}$ implies that $R_{A_0} = (Q^0_{A_0} - Q^{a_1}_{A_0})(A_0 + a_1) \in (T_{A_0})_{\frac{1}{2}+\epsilon,\min(2s-1,s)-\epsilon}$. Since $s \geq \frac{1}{2}$ and $\epsilon < \frac{1}{4}$ this embeds into $(T_{A_0})_{\frac{1}{2}+\epsilon,s-1}$. Similar arguments show $N_{A_0}a_2$ lies in the same space. This implies that $a_1 + a_2 \in (T_{A_0})_{\frac{1}{2}+\epsilon,s}$ and $a_1 + a_2$ depends smoothly on a_0 in this topology. □

4.4. **Uniqueness and long-time existence.**

Theorem 4.4.1. *For $s \geq \frac{1}{2}$, the initial value problem (12) has a unique solution $[A] \in C^0_{\mathrm{loc}}([0,\infty),\mathcal{M}(X)_s)$.*

Proof. Since $\mathcal{M}(X)$ is dense in $\mathcal{M}(X)_s$, we may after replacing A_0 with a gauge equivalent connection choose a flat, smooth A_0' arbitrarily close to A_0. By Theorem 4.3.1, the solution to the heat flow in the slice at A_0' exists for a time T depending only on an upper bound for $f([A_t])$, which is non-increasing. By iteration, a solution exists for all times. Let

$$a, a' \in (T_{A_0})_s' \cap C^0_{\mathrm{loc}}([0,\infty), (T_{A_0})_s)$$

be two solutions to (13), with the same initial connection $a_0 \in (T_{A_0})_s$. Suppose that $a \neq a'$. Let T_1 be the largest number such that the restrictions of a, a' to $[0, T_1]$ are equal. The restrictions of a, a' to $[T_1, T]$ solve the initial value problem (13) with initial data $a(T_1) = a'(T_1) \in H^s$. Without loss of generality we may assume that $T_1 = 0$. Let a_1, a_1' denote the solutions to the time-independent initial value problem (24), and let $a_2 = a - a_1$ and $a_2' = a' - a_1'$. Since the solution to the time-independent problem is unique, $a_1 = a_1'$. Since $\epsilon \in (0, \frac{1}{4})$, the space $(T_{A_0})_{0,s}'$ is the subspace of $(T_{A_0})_s'$ whose elements vanish at $t = 0$, see A.0.3 (a). It follows that $a_2, a_2' \in (T_{A_0})_{0,s}'$ so that a_2, a_2' solve (27). The norms of the restrictions $A|_{[0,T]}, A'|_{[0,T]}$ are uniformly bounded as $T \to 0$. By the proof of existence, the equation (27) has a unique solution of norm less than $c(R)T^{-\frac{1}{2}+\epsilon}$. Therefore, for T sufficiently small $a_2 = a_2'$ on $[0, T]$, which is a contradiction. □

It would be interesting to know whether negative-time trajectories exist, say for special $[A]$. A natural candidate is those $[A]$ which extend to an open neighborhood of X, in a Riemann surface X' containing X.

5. The heat flow: convergence at infinity

The purpose of this section is to prove the following.

Theorem 5.0.2. *For $s \geq \frac{1}{2}$ and any $[A_0] \in \mathcal{M}(X)_s$, the trajectory $[A_t]$ converges in $\mathcal{M}(X)_s$ to a critical point $[A_\infty]$ as $t \to \infty$. Let $\mathcal{C}$ be the set of connected components of* $\mathrm{crit}(f)$. *For any $C \in \mathcal{C}$, define*

$$\mathcal{M}(X)_C = \{[A] \in \mathcal{M}(X), \quad [A_\infty] \in C\}$$

so that $\mathcal{M}(X) = \bigcup_{C \in \mathcal{C}} \mathcal{M}(X)_C$. For any critical component C, the map

$$\rho_C : [0,\infty] \times \mathcal{M}(X)_C \to \mathcal{M}(X)_C, \quad [A] \mapsto [A_t]$$

is a deformation retract of $\mathcal{M}(X)_C$ onto C.

The critical points of f are represented by flat A such that

$$\mathrm{d}_A \, \mathcal{E}_A^0 \delta *_S (r_+ - r_-) A = 0; \tag{30}$$

this is the analog of the Yang-Mills equation.

Lemma 5.0.3. For any $s > \frac{1}{4}$ and $[A] \in \mathrm{crit}(f)$, the $K(S)_{s+\frac{1}{2}}$-orbit of any element $[A] \in \mathcal{M}(X)_{s,C}$ contains an element $[A'] \in \mathrm{crit}(f)$ such that A' is smooth and $r_{\partial X} A'$ is harmonic with values in $\mathfrak{t}$, that is,

$$r_- A' = *_S \xi_-, \quad r_+ A' = *_S \xi_+$$

for some $\xi_\pm \in \mathfrak{t}$. The pair (ξ_+, ξ_-) is uniquely defined up to the action of W_{aff} on $\mathfrak{t} \oplus \mathfrak{t}$.

Proof. By (1) we may after replacing A with a gauge-equivalent connection assume $r_+ A = *_S \xi_+$ for some $\xi_+ \in \mathfrak{A}$. Since $[A]$ is infinitesimally fixed by $\xi = *_S \Phi([A])$, $r_+ A$ and $r_- A$ are also fixed, so $\xi \in \mathfrak{k}(S)_{r_+ A}$. $K(S)_{r_+ A}$ is a compact Lie group, containing T as a maximal torus, and $\xi \in \mathfrak{k}(S)_{r_+ A}$. It follows that there exists $k \in K(S)_{r_+ A}$ such that $k \cdot \xi \in \mathfrak{t}$. Set $A' = k \cdot A$; then $r_- A'$ is of the form $*_S \xi_-$, for some $\xi_- \in \mathfrak{t}$. The intersection of the $K(S)$-orbit of $(r_- A', r_+ A')$ with $*_S \mathfrak{t} \oplus *_S \mathfrak{t}$ is an orbit of W_{aff}. It follows that (ξ_+, ξ_-) is unique up to the action of W_{aff}. Smoothness of A' follows from bootstrapping: $dA' = -\frac{1}{2}[A', A']$ and $d r_{\partial X} A' = 0$ imply that if A' has Sobolev class s, then A' has Sobolev class $\min(s, 2s-1) + 1$. □

Lemma 5.0.4. Let $[A_t]$ be a solution to the initial-value problem (12) in $\mathcal{M}(X)_s$. For any $s' \in (\frac{1}{4}, 1)$ there exists $t_i \in \mathbb{R}$ and $k_i \in K(S)_{s+\frac{1}{2}}$ for $i = 1, 2, \ldots$ such that $k_i [A_{t_i}] \to [A_\infty]$ in $\mathcal{M}(X)_{s'}$ for some $[A_\infty] \in \mathrm{crit}(f)$.

Proof. Since f is bounded from below, there exists a sequence t_i such that

$$\frac{\mathrm{d}}{\mathrm{d}t} f(t_i) = -\|Q_{A_i} A_i\|_0^2 \to 0 \text{ as } i \to \infty.$$

where $A_i := A_{t_i}$. Using Stokes theorem,

$$\|Q_{A_i} A_i\|_0^2 = ((P_{A_i,+} + P_{A_i,-}) B_{i,-}, B_{i,-});$$

see the proof of Lemma 4.1.1. Hence

$$\|(P_{A_i,+} + P_{A_i,-})^{\frac{1}{2}} B_{-,i}\|_0 \to 0 \quad \text{as } i \to \infty.$$

By Lemma 3.3.3, after modifying A_i by gauge transformations $k_i \in K(X)_{s+1}$ with $r_+ k_i = r_- k_i$ we may assume that $r_+ A_i \in *_S \mathfrak{A}$ and that $\|A_i\|_{\frac{1}{2}}$ is bounded. Let $s'' \in (\frac{1}{4}, \frac{1}{2})$. Since $\mathcal{A}(X)_{\frac{1}{2}} \to \mathcal{A}(X)_{s''}$ is compact, after passing to a subsequence we may assume that A_i converges in $\mathcal{A}(X)_{s''}$ to a limiting connection A_∞. By Lemma 4.1.4, $P_{A_i,-} + P_{A_i,-}$

converges to $P_{A_\infty,-} + P_{A_\infty,+}$. By elliptic regularity,

$$\begin{aligned}\|B_{-,i}\|_{1/2} &\leq c\left(\|B_{-,i}\|_0 + \|(P_{A_\infty,+}+P_{A_\infty,-})^{\frac{1}{2}}B_{-,i}\|_0\right)\\ &\leq c\left(\|B_{-,i}\|_0 + \|(P_{A_i,+}+P_{A_i,-})^{\frac{1}{2}}B_{-,i}\|_0\right)\\ &< c.\end{aligned}$$

(Here c denotes various constants.) Hence

$$\begin{aligned}\|r_{\partial X}A_i\|^2_{1/2} &= \|r_+A_i\|^2_{1/2} + \|r_-A_i\|^2_{1/2}\\ &\leq 2\|r_+A_i\|^2_{1/2} + \|B_{-,i}\|^2_{1/2}\\ &< c.\end{aligned}$$

For any $s' \in (\frac{1}{4},1)$, the embedding $\Omega^1(\partial X;\mathfrak{k})_{\frac{1}{2}} \to \Omega^1(\partial X;\mathfrak{k})_{s'-\frac{1}{2}}$ is compact. Hence after passing to a subsequence $r_{\partial X}A_i$ converges in $\Omega^1(\partial X;\mathfrak{k})_{s'-\frac{1}{2}}$. Let $\overline{I}$ denote the closure of the set $I = \{r_{\partial X}A_i\}$. Then $\overline{I}$ is compact (being the closure of the image of a convergent sequence) and since $r_{\partial X}$ is proper $r^{-1}_{\partial X}(\overline{I})$ is compact. Hence $[A_i]$ has a subsequence converging to an element $[A_\infty]$ in $\mathcal{M}(X)_{s'}$. □

Remark 5.0.5. The proof of the corresponding lemma in the Yang-Mills heat flow uses Uhlenbeck compactness, which is replaced here by properness of the moment map $r_{\partial X}$.

We prove that the trajectory A_t converges by showing $\int_0^\infty \|Q_{A_t}A_t\|_s \mathrm{d}t$ is finite. Note that

$$f(\infty) - f(0) = \int_0^\infty \frac{\mathrm{d}}{\mathrm{d}t}f(t)\mathrm{d}t = -\int_0^\infty \|Q_{A_t}A_t\|_0^2 \mathrm{d}t$$

is finite. We will give a lower bound for $\|Q_{A_t}A_t\|_0$.

Theorem 5.0.6. *Let $A \in \mathcal{A}_\flat(X)$ be a smooth connection with $[A] \in$ $\mathrm{crit}(f)$. There exists $\gamma \in [\frac{1}{2},1)$ such that for any $A+a \in U_A$ sufficiently close to A and $s \geq \frac{1}{2}$,*

$$\|Q_A^a(A+a)\|_{s-1} \geq c|f(A+a) - f(A)|^\gamma. \tag{31}$$

Proof. The proof involves several lemmas. Consider the L^2-splitting

$$T_{A+a}U_A = \ker(\mathrm{d}_{A+a} \oplus \mathrm{d}_A^* \oplus (r_- - r_+)*_X) \oplus \mathrm{Im}\, \mathrm{d}_A \mathcal{E}_A^a \delta;$$

as in Lemma 3.4.1 (b). We denote the projections by π^0, π^1 respectively. Define

$$\Sigma_A = U_A \cap \ker(r_- - r_+)*_X .$$

Σ_A is a local slice for the $K(S)_{s+\frac{1}{2}}$-action on $\mathcal{M}(X)_s$. The restriction of f has L^2-gradient

$$M:\ \Sigma_A \to T\Sigma_A,\ a \mapsto \pi^0 Q_A^a(A+a).$$

Lemma 5.0.7. If $\|a\|_s$ is small enough then $\|M(a)\|_{s-1} \geq \frac{1}{2}\|Q^a_A(A+a)\|_{s-1}$.

Proof. We have

$$\pi^1(Q^a_A(A+a)) = \mathrm{d}_{A+a}\mathcal{E}^a_A\delta\xi$$

for some $\xi \in \mathfrak{k}(S)_{s+\frac{1}{2}}$. Since $\mathrm{d}_{A+a} = \mathrm{d}_A + \mathrm{ad}(a)$, there exist $c' > 0$ such that

$$\|\mathrm{d}_{A+a}\| \leq c\|\mathrm{d}_{A+a}\mathrm{d}^*_{A+a}\|$$

for $\|a\|_s < c'$. So

$$\begin{aligned}
\|\pi^1 Q^a_A(A+a)\|_{s-1} &= \|\mathrm{d}_{A+a}\mathcal{E}^a_A\delta\xi\|_{s-1} \\
&\leq c\|\mathrm{d}^*_{A+a}\mathrm{d}_{A+a}\mathcal{E}^a_A\delta\xi\|_{s-2} \\
&\leq c\|\mathrm{d}^*_{A+a}Q^a_A(A+a)\|_{s-2} \\
&\leq c\|\mathrm{d}^*_A Q^a_A(A+a)\|_{s-2} + c\|\,\mathrm{ad}(a) *_X Q^a_A(A+a)\|_{s-2} \\
&\leq c\|a\|_s\|Q^a_A(A+a)\|_{s-1}.
\end{aligned}$$

For a sufficiently small we have

$$c\|a\|_s\|Q^a_A(A+a)\|_{s-1} \leq \frac{1}{2}\|Q^a_A(A+a)\|_{s-1}$$

which proves the lemma. □

It follows that $A+a$ is critical if and only if $M(a) = 0$. The derivative of $M(a)$ at $a = 0$ is the linear operator

$$L : \ (\ker \mathrm{d}_A \oplus \mathrm{d}^*_A \oplus (r_- - r_+) *_X)_s \to (\ker \mathrm{d}_A \oplus \mathrm{d}^*_A \oplus (r_- - r_+) *_X)_{s-1}$$

defined by

$$L(\alpha) = \pi^0 *_X \frac{\mathrm{d}}{\mathrm{d}\theta}(\mathrm{d}_{A+\theta\alpha}\mathcal{E}^{\theta\alpha}_A)\delta *_S (r_-{-}r_+)A + *_X\, \mathrm{d}_A\mathcal{E}^0_A\delta *_S (r_-{-}r_+)\alpha.$$

Lemma 5.0.8. For any $\alpha \in \ker \mathrm{d}_A \oplus \mathrm{d}^*_A \oplus (r_+ - r_-) *_X$,

$$(r_-{-}r_+) *_X \left(\frac{\mathrm{d}}{\mathrm{d}\theta}\mathrm{d}_{A+\theta\alpha}\mathcal{E}^{\theta\alpha}_A\delta *_S (r_-{-}r_+)A + \mathrm{d}_A\mathcal{E}^0_A\delta *_S (r_-{-}r_+)\alpha\right) = 0.$$

Proof. We have

$$(r_-{-}r_+)(\frac{\mathrm{d}}{\mathrm{d}\theta}\mathrm{d}_{A+\theta\alpha}\mathcal{E}^{\theta\alpha}_A)\delta *_S (r_-{-}r_+)A = \mathrm{ad}((r_-{-}r_+)\alpha) *_S (r_-{-}r_+)A$$

and

$$(r_-{-}r_+)\mathrm{d}_A\mathcal{E}^0_A(r_-{-}r_+)\delta *_S \alpha = \mathrm{ad}((r_-{-}r_+)A) *_S (r_-{-}r_+)\alpha$$

which cancel. □

It follows that

$$L(\alpha) = *_X (\frac{\mathrm{d}}{\mathrm{d}\theta}\mathrm{d}_{A+\theta\alpha}\mathcal{E}^{\theta\alpha}_A)(r_-{-}r_+)A + *_X\, \mathrm{d}_A\mathcal{E}^0_A(r_-{-}r_+)\alpha.$$

Lemma 5.0.9. *L is (1) self-adjoint and (2) Fredholm.*

Proof. (1) follows from

$$\begin{aligned}\int_X (\mathrm{ad}(\alpha_1)\mathcal{E}_A^0\delta *_S (r_- - r_+)A \wedge \alpha_2) &= \int_X (\mathcal{E}_A^0\delta *_S (r_- - r_+)A \wedge [\alpha_2,\alpha_1]) \\ &= -\int_X (\mathrm{ad}(\alpha_2)\mathcal{E}_A^0\delta *_S (r_- - r_+)A \wedge \alpha_1)\end{aligned}$$

by invariance of the inner product;

$$\begin{aligned}\int_X (\mathrm{d}_A \frac{\mathrm{d}}{\mathrm{d}\theta}\mathcal{E}_A^{\theta\alpha_1}\delta *_S (r_- - r_+)A \wedge \alpha_2) &= \int_{\partial X} (r_{\partial X}\frac{\mathrm{d}}{\mathrm{d}\theta}\mathcal{E}_A^{\theta\alpha_1}\delta *_S (r_- - r_+)A \wedge r_{\partial X}\alpha_2) \\ &= 0\end{aligned}$$

using Stokes' theorem; and

$$\begin{aligned}\int_X (\mathrm{d}_A\mathcal{E}_A\delta *_S (r_- - r_+)\alpha_1 \wedge \alpha_2) &= \int_{\partial X} (r_{\partial X}\mathcal{E}_A\delta *_S (r_- - r_+)\alpha_1 \wedge r_{\partial X}\alpha_1) \\ &= \int_S (*_S(r_- - r_+)\alpha_1 \wedge (r_- - r_+)\alpha_2)).\end{aligned}$$

(2) The operator $r_- - r_+$ is Fredholm on $T_0\Sigma_A$, so it suffices to show that $(r_- - r_+)L$ is Fredholm. We have

$$(r_- - r_+)L = (r_- - r_+) *_X (\frac{\mathrm{d}}{\mathrm{d}\theta}\mathrm{d}_{A+\theta\alpha}\mathcal{E}_A^{\theta\alpha})\delta *_S (r_- - r_+)A + P^0_{-,A}(r_- - r_+)\alpha.$$

Since the class of Fredholm operators is closed under composition and perturbation with compact operators, $(r_- - r_+)L$ is Fredholm. □

Let Σ_A^0 be the kernel of L, and Σ_A^1 its L^2-orthogonal complement. By the Lemma, Σ_A^0 is finite dimensional, so $\Sigma_A = \Sigma_A^0 \oplus \Sigma_A^1$, and L defines an invertible operator $(\Sigma_A^1)_s \to (\Sigma_A^1)_{s-1}$. Since L is the derivative of M at $a = 0$, it follows from the implicit function theorem that there exists $\epsilon_1, \epsilon_2 > 0$ and a real analytic map

$$l: \ B_{\epsilon_1}\Sigma_A^0 \to B_{\epsilon_2}\Sigma_A^1$$

such that $\pi^1 M(\alpha + l(\alpha)) = 0$. Define

$$f_0: \ B_\epsilon\Sigma_A^0 \to \mathbb{R}, \quad \alpha \mapsto f(\alpha + l(\alpha)).$$

For any $\alpha_1 \in B_{\epsilon_1}\Sigma_A^0$ and $\alpha_2 \in \Sigma_A^0$ we have

$$(\mathrm{grad}\, f_0(\alpha_1), \ \alpha_2)_0 = (M(\alpha_2 + l(\alpha_2)), \alpha_2 + Dl(\alpha_1)\alpha_2)_0.$$

Since $Dl(\alpha) \in \Sigma_A^1$, $(\mathrm{grad}\, f_0(\alpha_1), \ \alpha_2)_0 = (M(\alpha_2 + l(\alpha_2)), \alpha_2)_0$ which implies

$$\mathrm{grad}\, f_0(\alpha) = M(\alpha + l(\alpha)).$$

Therefore the set of critical connections $M(a) = 0$ near $a = 0$ is the set

$$\{a = a_0 + l(a_0), a_0 \in B_{\epsilon_1}\Sigma_A^0,\ \operatorname{grad} f_0(a_0) = 0\}.$$

For any a sufficiently small, we may write $a = a_0 + l(a_0) + a_1$, where $a_0 \in \Sigma_A^0, a_1 \in \Sigma_A^1$ and

$$\|a_0\|_s \le c\|a\|_s, \quad \|l(a_0)\|_s \le c\|a\|_s, \quad \|a_1\|_s \le c\|a\|_s. \tag{32}$$

Now we estimate the left-hand side of (31). We have

$$\begin{aligned}
\pi^0 Q_A^a(A+a) &= M(a) = M(a_0 + l(a_0) + a_1) \\
&= \operatorname{grad} f_0(a_0) + M(a_0 + l(a_0) + a_1) - M(a_0 + l(a_0)) \\
&= \operatorname{grad} f_0(a_0) + \int_0^1 DM(a_0 + l(a_0) + sa_1)a_1 \mathrm{d}s \\
&= \operatorname{grad} f_0(a_0) + La_1 + L_1 a_1
\end{aligned}$$

where

$$L_1 = \int_0^1 (DM(a_0 + l(a_0) + sa_1) - DM(0))a_1 \mathrm{d}s.$$

The spaces Σ_A^0 and Σ_A^1 are closed, disjoint subspaces of Σ_A. It follows that

$$\|\pi^0 Q_A^a(A+a)\|_{s-1} \ge c\left(\|\operatorname{grad} f_0(a_0)\|_{s-1} + \|La_1\|_{s-1}\right) - \|L_1 a_1\|_{s-1}.$$

From (32) and the smooth dependence of $DM(a)$ on a we see that for $\|a\|_s$ sufficiently small

$$\|\int_0^1 (DM(a_0 + l(a_0) + sa_1) - DM(0))\mathrm{d}s\| \le c\epsilon_1 \|a_1\|_s.$$

So for ϵ_1 sufficiently small we have

$$\|\pi^0 Q_A^a(A+a)\|_{s-1} \ge c\|\operatorname{grad} f_0(a_0)\|_{s-1} + c\|a_1\|_s.$$

Since M is the L^2-gradient of the restriction of f,

$$\begin{aligned}
f(A+a) &= f(A + a_0 + l(a_0) + a_1) \\
&= f_0(a_0) + f(A + a_0 + l(a_0) + a_1) - f_0(a_0) \\
&= f_0(a_0) + \int_0^1 (M(a_0 + l(a_0) + sa_1), a_1)_0 \mathrm{d}s \\
&= f_0(a_0) + (M(a_0 + l(a_0)), a_1)_0 \\
&\quad + \int_0^1 \int_0^1 (DM(a_0 + l(a_0) + sta_1)sa_1, a_1)_0 \mathrm{d}t\mathrm{d}s \\
&= f_0(a_0) + (\operatorname{grad} f_0(a_0), a_1)_0 + \frac{1}{2}(La_1, a_1)_0 + (L_2 a_1, a_1)_0,
\end{aligned}$$

where

$$L_2 = \int_0^1 \int_0^1 s(DM(a_0 + l(a_0) + sta_1) - DM(0))\mathrm{d}t\mathrm{d}s.$$

The second term vanishes since $\operatorname{grad} f_0(a_0) \in \Sigma^0_A,\ a_1 \in \Sigma^1_A$. The third term has norm at most $c\|a_1\|^2_{\frac{1}{2}}$, since L is a bounded linear operator. The norm of fourth term $(L_2 a_1, a_1)_0$ can be bounded in the same way as for L_1, by $c\epsilon_1\|a_1\|^2_{\frac{1}{2}}$. We conclude that for ϵ_1 sufficiently small

$$|f(A+a) - f(A)| \leq |f_0(a_0)| + c\|a_1\|^2_{\frac{1}{2}}.$$

Since f_0 is real analytic, Σ^0_A is finite-dimensional, $f_0(0) = 0$ and $\operatorname{grad} f_0(0) = 0$ we conclude by the Lojasiewicz gradient inequality [22] that there exists $\gamma \in [\frac{1}{2}, 1)$ such that for sufficiently small a_0,

$$\| \operatorname{grad} f_0(a_0)\|_{s-1} \geq c|f_0(a_0)|^\gamma.$$

This completes the proof of Theorem 5.0.6. □

Lemma 5.0.10. Let $s \geq \frac{1}{2}$ and A_∞ a representative of $[A_\infty] \in \operatorname{crit}(f)$. For any $\delta > 0$ and $T > \delta$, there exist constants $c, \epsilon > 0$ such that if A_t is a solution to the heat equation (13) in the slice U_{A_∞}, $0 < T_1 \leq T - \delta$ and $\|A_t - A_\infty\|_s \leq \epsilon$ for all $t \in [T_1, T]$ then

$$\int_{T_1+\delta}^T \|\frac{\mathrm{d}}{\mathrm{d}t} A\|_s \leq c\int_{T_1}^T \|\frac{\mathrm{d}}{\mathrm{d}t}A\|_0 \mathrm{d}t.$$

Proof. Let $A_t' = \frac{\mathrm{d}}{\mathrm{d}t}A_t = -Q^{A_t - A_\infty}_{A_\infty} A_t$. Then

$$\frac{\mathrm{d}}{\mathrm{d}t}A_t' = -(\frac{\mathrm{d}}{\mathrm{d}u}Q^{A_u - A_\infty}_{A_\infty})_{u=t}A_t + Q^{A_t-A_\infty}_{A_\infty}A_t'$$

so A_t' satisfies the parabolic equation

$$(\frac{\mathrm{d}}{\mathrm{d}t} - Q^0_{A_\infty})A_t' = -(\frac{\mathrm{d}}{\mathrm{d}u}Q^{A_u-A_\infty}_{A_\infty})_{u=t}A_t + (Q^{A_t-A_\infty}_{A_\infty} - Q^0_{A_\infty})A_t'.$$

In order to obtain estimates, we need to modify A_t' to obtain a function vanishing at $t = T_1$. Let $\eta(t)$ be a smooth cut-off function with $\eta = 0$ on $[T_1, T_1 + \delta/2]$ and $\eta = 1$ on $[T_1 + \delta, T]$. Then

$$(\frac{\mathrm{d}}{\mathrm{d}t} - Q^0_{A_\infty})(\eta A_t') = (\frac{\mathrm{d}}{\mathrm{d}u}Q^{A_u-A_\infty}_{A_\infty})_{u=t}(\eta A_t) + (Q^{A_t-A_\infty}_{A_\infty} - Q^0_{A_\infty})(\eta A_t') + (\eta' A_t').$$

Hence $\|\eta A_t'\|_{L^2([T_1,T],H^s)}$ is bounded by

$$\begin{aligned} &c\|-(\frac{\mathrm{d}}{\mathrm{d}u}Q^{A_u-A_\infty}_{A_\infty})_{u=t}(\eta A_t) + (Q^{A_t-A_\infty}_{A_\infty} - Q^0_{A_\infty})(\eta A_t') + (\eta' A_t')\|_{L^2([T_1,T],H^{s-1})} \\ &\leq c(R)\|A_t'\|_{L^2([T_1,T],H^{s-\frac{1}{2}})} + c(R)\|\eta A_t'\|_{L^2([T_1,T],H^{s-\frac{1}{2}})} + c(R)\|\eta' A_t'\|_{L^2([T_1,T],H^{s-\frac{1}{2}})} \end{aligned}$$

using 4.1.3 and 4.1.5. Using Hölder's inequality we get

$$\begin{aligned}\|A_t'\|_{L^1([T_1+\delta,T],H^s)} &\leq \|\eta A_t'\|_{L^1([T_1,T],H^s)}\\ &\leq (T-T_1)^{\frac{1}{2}}\|\eta A_t'\|_{L^2([T_1,T],H^s)}\\ &\leq c(R)(T-T_1)^{\frac{1}{2}}(\|A_t'\|_{L^2([T_1,T],H^{s-\frac{1}{2}})}+\\ &\quad \|\eta A_t'\|_{L^2([T_1,T],H^{s-\frac{1}{2}})}+c(R)\|\eta' A_t'\|_{L^2([T_1,T],H^{s-\frac{1}{2}})})\\ &\leq c(R)(T-T_1)^{\frac{1}{2}}(1+\delta^{-1})\|A_t'\|_{L^1([T_1,T],H^{s-\frac{1}{2}})}.\end{aligned}$$

For $\frac{1}{2} > s > 0$ the last expression is bounded from above by the L^2-norm. The case of arbitrary s follows by bootstrapping. □

Lemma 5.0.11. Let $[A_\infty] \in \mathrm{crit}(f)$ and $s \in [\frac{1}{2}, 1]$. There exist constants $c, \epsilon_2 > 0$ such that if $[A_t]$ is a solution to the evolution equation (12) and A_T is a representative of $[A_T]$ such that $\|A_T - A_\infty\|_s \leq \epsilon_2$ for some $T > 0$, then either $f([A_t]) < f([A_\infty])$ for some $t > T$ or $[A_t]$ is contained in the image of U_{A_∞} in $\mathcal{M}(X)$, for all $t \geq T$ and A_t converges to A'_∞ with $f(A'_\infty) = f(A_\infty)$, as $t \to \infty$. In the second case,

$$\|A'_\infty - A_\infty\|_s \leq c\|A_T - A_\infty\|_s. \tag{33}$$

Proof. Assume that $f(A_t) > f(A_\infty)$ for all $t \in [0, \infty)$. Since f is a smooth functional of A and A_∞ is a critical point, if we choose ϵ_2 small enough then

$$|f(A_T) - f(A_\infty)| \leq c\|A_T - A_\infty\|_s^2. \tag{34}$$

By Theorem 4.3.1, the solution to (12) in $C^0_{\mathrm{loc}}([0,\infty), H^s)$ depends smoothly on the initial data in H^s. It follows that if ϵ_2 is sufficiently small then

$$\|A_t - A_\infty\|_s \leq c\|A_T - A_\infty\|_s \tag{35}$$

for all $t \in [T, T+1]$. We claim that for ϵ_2 sufficiently small, $\|A_t - A_\infty\|_s < \epsilon_1$ for all $t \geq T$. Suppose the opposite. Let T_1 be the smallest number greater than T such that $\|A_{T_1} - A_\infty\|_s \geq \epsilon_1$. By (35) if we choose ϵ_2 small enough, then $T_1 > T + 1$. Since $s \leq 1$,

$$\|Q^a_A(A+a)\|_0 \geq \|Q^a_A(A+a)\|_{s-1}.$$

By Theorem 5.0.6 for all $t \in [T, T_1]$ we have

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}(f(A_t) - f(A_\infty))^{1-\gamma} &= -(1-\gamma)\|Q^a_A(A+a)\|_0^2\|(f(A_t) - f(A_\infty))^{-\gamma}\\ &\leq -c\|Q^a_A(A+a)\|_0 = -c\|\frac{\mathrm{d}}{\mathrm{d}t}A\|_0.\end{aligned}$$

Integrating with respect to t we get

$$\begin{aligned}\int_T^{T_1} \|\frac{\mathrm{d}}{\mathrm{d}t}A\|_0 &\leq c(f(A_T)-f(A_\infty))^{1-\gamma} \\ &\leq c\|A_T - A_\infty\|_s^{2(1-\gamma)} \\ &\leq c\epsilon_2^{2(1-\gamma)} \\ &\leq c\epsilon_2.\end{aligned}$$

using (34). On the other hand,

$$\begin{aligned}\int_T^{T_1} \|\frac{\mathrm{d}}{\mathrm{d}t}A\|_s \mathrm{d}t &\geq \|A_{T_1} - A_{T+1}\|_s \\ &\geq \|A_{T_1} - A_\infty\|_s - \|A_{T+1} - A_\infty\|_s \\ &\geq \epsilon_1 - c\epsilon_2.\end{aligned}$$

It follows from 5.0.10 that $\epsilon_1 - c\epsilon_2 \leq c\epsilon_2$. For ϵ_2 sufficiently small, this gives a contradiction.

We conclude that for ϵ_2 sufficiently small, $\|A_t - A_\infty\|_s < \epsilon_1$ for all $t \geq T$. Then

$$\int_{T+1}^{\infty} \|\frac{\mathrm{d}}{\mathrm{d}t}A\|_s \leq c\int_T^\infty \|\frac{\mathrm{d}}{\mathrm{d}t}A\|_0 \mathrm{d}t \leq c(f(A_T)-f(A_\infty))^{1-\gamma}. \tag{36}$$

It follows that A_t converges to A'_∞ as $t \to \infty$. By Lemma 5.0.3, the set of critical values of f is locally finite, so $f([A'_\infty]) = f([A_\infty])$ for ϵ_2 sufficiently small.

It remains to prove the estimate (33). Using (36) and (34)

$$\begin{aligned}\|A'_\infty - A_{T+1}\|_s &\leq \int_{T+1}^\infty \|\frac{\mathrm{d}}{\mathrm{d}t}A\|_s \\ &\leq c(f(A_T)-f(A_\infty))^{1-\gamma} \\ &\leq c\|A_T - A_\infty\|_s^{2(1-\gamma)} \\ &\leq c\|A_T - A_\infty\|_s.\end{aligned}$$

Using (35) completes the proof. □

Proof of Theorem 5.0.2. Let t_i, k_i be a sequence given by Proposition 5.0.4. Since the equation (12) is invariant under $K(S)$, the trajectory $k_n[A_{t_n+t}]$ is also a solution. For n sufficiently large, $k_n[A_{t_n+t}]$ satisfies the assumptions of Proposition 5.0.11. Therefore, $k_n[A_{t_n+t}]$ converges to some $[A'_\infty]$. It follows that $[A_t] \to k_n^{-1}[A'_\infty]$.

It remains to show that $[A_\infty]$ depends continuously on the initial data. Let $\epsilon_1 > 0$. Let $[A_t]$ be a solution to (12), and A_∞ a representative for $[A_\infty]$. By Proposition 5.0.11, there exists an $\epsilon_2 > 0$ such that if $[A'_t]$ is another solution to (12) such that $\|A'_T - A_\infty\|_s \leq \epsilon_2$ for some $T \geq 0$

and representative A'_T, and $f([A'_\infty]) = f([A_\infty])$, then $\|A'_\infty - A_\infty\|_s \leq \epsilon_1$. Choose T sufficiently large so that $\|A_T - A_\infty\|_s \leq \epsilon_2/2$, where A_T is the representative in the slice U_{A_∞}. By the first part of the theorem, there exists $\epsilon_3 > 0$ such that if $\|A'_0 - A_0\|_s < \epsilon_3$ then $\|A'_T - A_T\|_s < \epsilon_2/2$, and so $\|A'_\infty - A_\infty\|_s < \epsilon_1$. □

6. The stratification defined by the heat flow

From now on we drop the Sobolev subscripts $_s$; the statements that follow hold for any $s \geq \frac{1}{2}$, or $s \geq 1$ if we wish to use the action of the complex loop group.

6.1. **The critical set.**

Lemma 6.1.1. For any $R > 0$, there are a finite number of critical components C of $f : \mathcal{M}(X)_s \to \mathbb{R}$ such that $f(C) < R$.

Proof. For any subgroup $H \subset T$, let F be the fixed point set of the action of H on $\Phi^{-1}(B_R) \cap r_{\partial X}^{-1}(*_S \mathfrak{t} \oplus *_S \mathfrak{A})$. By the symplectic cross-section theorem for loop group actions [23] there are at most a finite number of components of F. Each contains at most a finite number of components of the critical set of ζ. By Lemma 5.0.3, any critical component of ζ contains elements of this form. Hence, the number is finite. □

We will give an explicit description of crit(f). By (30), any $[A] \in$ crit(f) is fixed by a one-parameter subgroup, which we may assume is generated by an element

$$\xi \in \mathfrak{t}, \quad *_S \xi = (r_- - r_+)A.$$

The centralizer $K_\xi \subset K$ is a connected subgroup of K, with Lie algebra $\mathfrak{k}_\xi$. Let $\mathcal{M}(X; K_\xi)$ denote the moduli space of flat K_ξ-bundles. The inclusion $\mathfrak{k}_\xi \to \mathfrak{k}$ induces an embedding

$$\iota_\xi : \ \mathcal{M}(X; K_\xi) \to \mathcal{M}(X)$$

whose image is the fixed point set $\mathcal{M}(X)^{U(1)_\xi}$ of $U(1)_\xi$. Let

$$\mathcal{M}(X; K_\xi; \xi) = \{[A] \in \mathcal{M}(X; K_\xi), \ \ (r_- - r_+)A = *_S \xi\}.$$

Then

$$\text{crit}(f) = \bigcup_\xi K(S) \cdot \iota_\xi \mathcal{M}(X; K_\xi; \xi). \tag{37}$$

The quotient of $K_\xi(S)\backslash\mathcal{M}(X; K_\xi; \xi)$ by the loop group $K_\xi(S)$ is homeomorphic to the moduli space $\mathcal{M}(\overline{X}; K_\xi; \xi)$ of bundles with constant central curvature ξ. The homeomorphism is constructed by subtracting a $Z(K_\xi)$-connection A_ξ with $(r_- - r_+)A_\xi = *_S \xi$; such a central connection exists after adding a boundary component with marking $\overline{\xi}$, the reflection of ξ into the fundamental alcove. This gives a homeomorphism to the moduli space of flat bundles on $\overline{X}$, with the additional marking. A similar discussion gives a homeomorphism between $\mathcal{M}(\overline{X}; K_\xi; \xi)$ and the same space. From (37) we obtain a homeomorphism

$$K(S)\backslash \operatorname{crit}(f) \to \bigcup_\xi \mathcal{M}(\overline{X}; K_\xi; \xi) \tag{38}$$

where the sum is over $\xi \in \mathfrak{t}_+$ such that ξ is a coweight for $K_\xi/[K_\xi, K_\xi]$. The same description holds for the critical set of the Yang-Mills functional on $\mathcal{A}(\overline{X})$ [3, Section 6].

6.2. **The semistable stratum.** For the following compare [24, p.36].

Definition 6.2.1. The *semistable* locus $\mathcal{M}(X)^{\mathrm{ss}}$ is the set of all points $[A]$ with $[A_\infty] \in \Phi^{-1}(0)$. The *stable* locus resp. $\mathcal{M}(X)^{\mathrm{s}}$ is the set of all points $[A] \in \mathcal{M}(X)$ with $[A_\infty] \in \Phi^{-1}(0)$, $[A_\infty] \in G(S)[A]$, and $[A]$ has finite stabilizer.

The aim of this section is to prove the following theorem.

Theorem 6.2.2. $\mathcal{M}(X)^{\mathrm{ss}}$ *is an open, $G(S)$-invariant subset of* $\mathcal{M}(X)$.

The theorem depends on the convexity of a certain function, which was introduced in Guillemin-Sternberg [13] and Kemp-Ness [17] and used by Donaldson [9] in infinite dimensions. For any $[A] \in \mathcal{M}(X)$, choose an element l in the fiber $\mathcal{L}(X)_{[A]}$ of the pre-quantum line bundle over $\mathcal{M}(X)$. The central $\mathbb{C}^*$ in $\widehat{G(S)}$ acts trivially on $\mathcal{L}(X)$, and so the action induces an action of $G(S)$. Consider the smooth function

$$\Psi : \ G(S) \to \mathbb{R}, \qquad g \mapsto -\ln \|gl\|^2.$$

Since $K(S)$ acts on $\mathcal{L}(X)$ preserving the metric, Ψ descends to a smooth function on $G(S)/K(S)$, which is diffeomorphic to $\mathfrak{k}(S)$ via the map

$$\mathfrak{k}(S) \to G(S)/K(S), \qquad \xi \mapsto [\exp(i\xi)].$$

We denote by ψ the induced function $\mathfrak{k}(S) \to \mathbb{R}$. As in the finite-dimensional case the function ψ satisfies

$$\frac{\partial}{\partial \xi}\psi = -4\pi(\Phi, \xi), \quad (\frac{\partial}{\partial \xi})^2\psi = 4\pi\|\xi_{\mathcal{M}(X)}\|^2. \tag{39}$$

ψ has the following properties:

Lemma 6.2.3. (a) ψ is convex.
(b) The only critical points of ψ are zeros of the pull-back of Φ.

(c) The gradient flow lines ζ_t of $-\psi$ map onto the gradient flows lines $[A_t]$ for $-4\pi f$ under the map $\zeta \mapsto \exp(i\zeta)[A]$.
(d) If $[A_\infty] \notin \Phi^{-1}(0)$ then $\psi(\zeta_t) \to -\infty$ as $t \to \infty$.

Proof. (a)-(c) follow immediately from (39). (d): We have

$$\frac{\mathrm{d}}{\mathrm{d}t}\psi(\zeta_t) \to -4\pi f([A_\infty]) \text{ as } t \to \infty$$

which implies $\psi(\zeta_t) \to -\infty$. □

Proof of Theorem 6.2.2. It follows from Theorem 5.0.2 that $\mathcal{M}(X)^{\mathrm{ss}}$ is open. Suppose that $G(S)[A] \cap \Phi^{-1}(0)$ is non-empty. By (a),(b) of the Lemma, $\Phi^{-1}(0)$ is a global minimum of ψ. By part (d) of the Lemma, $G(S)[A]$ is contained in $\mathcal{M}(X)^{\mathrm{ss}}$. More generally, suppose that $\overline{G(S)[A]} \cap \Phi^{-1}(0)$ is non-empty. Let ζ_t, ζ_t' be gradient trajectories for $-\psi$ and $[A_t], [A_t']$ the corresponding trajectories for $-4\pi f$, so that

$$\frac{\mathrm{d}}{\mathrm{d}t}\zeta_t \to 0, \quad \frac{\mathrm{d}}{\mathrm{d}t}\zeta_t' \to \xi_\infty \quad \text{as } t \to \infty. \tag{40}$$

Since ψ is convex, $\mathrm{grad}(\psi)$ is monotone, that is,

$$(\mathrm{grad}(\psi)(\zeta_t) - \mathrm{grad}(\psi)(\zeta_t'), \zeta_t - \zeta_t')_{L^2} \geq 0.$$

Hence $\|\zeta_t - \zeta_t'\|_{L^2}$ is non-increasing. If $\xi_\infty \neq 0$ then (40) implies $\|\zeta_t - \zeta_t'\|_{L_2} \to \infty$ which is a contradiction. Hence $0 = \xi_\infty = -4\pi\Phi([A_\infty])$. So $G(S)[A]$ is contained in $\mathcal{M}(X)^{\mathrm{ss}}$. □

6.3. **The unstable strata.** Let $C \in \mathcal{C}$. The purpose of this section is to prove

Theorem 6.3.1. *$\mathcal{M}(X)_C$ is a smooth Kähler $G(S)$-invariant submanifold of finite codimension.*

Let $\xi_\pm = *_S r_\pm([A])$ as described in Lemma 5.0.3 and

$$\xi_C = \xi_+ - \xi_-.$$

Let $\mathbb{C}^*_C$ denote the one-parameter subgroup of $G(S)$ generated by ξ_C,

$$\mathbb{C}^*_C = \{\exp(\tau\xi_C), \quad \tau \in \mathbb{C}\}.$$

Let $Z_C \subseteq \mathcal{M}(X)^{\mathbb{C}^*_C}$ be the component of the fixed point set of $\mathbb{C}^*_S$ containing $[A']$. Let P_C be the parabolic subgroup corresponding to the element ξ_C, so that

$$P_C = \{g \in G, \quad \lim_{\tau \to -\infty} \exp(\tau\xi_C) g \exp(-\tau\xi_C) \text{ exists}\}.$$

Let $P_C = L_C U_C$ be its standard Levi decomposition, so that L_C is the centralizer of ξ_C. Let K_C denote the maximal compact subgroup of L_C, that is, $K_C = L_C \cap K$. Let $P_C(S), L_C(S), K_C(S)$ etc. denote the identity components of the loop groups of maps $S \to P_C, L_C, K_C$ etc. Let $\pi_C : \mathfrak{k} \to \mathfrak{k}_C$ denote the projection. The group $L_C(S)$ acts on Z_C, and the action of the subgroup $K_C(S)$ is symplectic with moment map $\pi_C \Phi$. The same argument as in the proof of Theorem 6.2.2 shows that Z_C^{ss} is $L_C(S)$-invariant. Define

$$Y_C = \{[A] \in \mathcal{M}(X), \quad \exp(t\xi_C)[A] \to Z_C \text{ as } t \to -\infty \text{ in } \mathbb{C}_C^*\}.$$

Lemma 6.3.2. (a) Y_C is a $P_C(S)$-invariant complex submanifold.
(b) If $[A] \in G(S)Y_C$ then $f([A]) \geq f(C)$.

Proof. (a) Y_C is a stable manifold for the gradient flow of $-(\Phi, \xi_C)$. Since (Φ, ξ_C) is Morse-Bott, Y_C is an embedded complex submanifold by the stable manifold theorem [39, Theorem III.8]. To show Y_C is invariant under $P_C(S)$, suppose $\exp(t\xi_C)y \to z$ as $t \to \infty$ for some $z \in Z_C$. Then

$$\exp(t\xi_C)py = \exp(t\xi_C)p\exp(-t\xi_C)\exp(t\xi_C)y \to lz \in Z_C$$

since $L_C(S)$ commutes with ξ_C. (b) Since $G(S) = K(S)P_C(S)$ and Y_C is $P_C(S)$-invariant, $G(S)Y_C = K(S)Y_C$. Since f is $K(S)$-invariant we may assume $[A] \in Y_C$. Then $(\Phi(A), \xi_C) \geq (\xi_C, \xi_C)$ which implies (b). □

Let Y_C^{ss} denote the set of $[A]$ in Y_C such that $\exp(-t\xi_C)[A]$ converges to a point in Z_C^{ss} as $t \to \infty$.

Lemma 6.3.3. If $[A] \in G(S)Y_C^{\mathrm{ss}}$ then (i) ξ_C is the unique closest point to 0 in $\Phi(\overline{P_C(S)[A]})$ in the L^2-metric and (ii) $K(S)\xi_C$ is the set of points closest to 0 in $\Phi(\overline{G(S)[A]})$.

Proof. (i) We first show that ξ_C lies in $\Phi(\overline{P_C(S)[A]})$. Suppose $\exp(t\xi_C)[A]$ converges to $[A']$ as $t \to -\infty$. Then $\overline{P_C(S)[A]}$ contains $\overline{L_C(S)[A']}$, so it suffices to show that $\Phi(\overline{L_C(S)[A']})$ contains ξ_C. But this is the definition of Z_C^{ss}. Since $(\Phi(Z_C), \xi_C) \geq (\xi_C, \xi_C)$, ξ_C is the unique closest point. The argument for (ii) is similar. □

Lemma 6.3.4. Y_C^{ss} is the smallest open $P_C(S)$-invariant neighborhood of $Z_C \cap \Phi^{-1}(C)$ in Y_C.

Proof. Y_C^{ss} is the union of $P_C(S)$-orbits in Y_C whose closure intersects $\Phi^{-1}(C) \cap Z_C$. □

Lemma 6.3.5. If $\nu \in \mathfrak{k}(S)$ and $\nu_{[A]}$ is tangent to $T_{[A]}Y_C$ for some $[A] \in C$, then $\nu \in \mathfrak{k}_C(S)$.

Proof. We follow Kirwan [18, p.50]. We have

$$\Phi(\exp(t\nu)[A]) = \xi_C + t[\nu, \xi_C] + e(t)$$

where $e(t) = O(t^2)$ as $t \to 0$. This implies that

$$(\Phi(\exp(t\nu)[A]), \xi_C) = (\xi_C, \xi_C) + (e(t), \xi_C).$$

Since f is $K(S)$-invariant we also have

$$\|\xi_C\|_0^2 = \|\xi_C + t[\nu, \xi_C] + e(t)\|_0^2$$

which implies

$$2(\xi_C, e(t)) = -t^2 \|[\nu, \xi_C]\|_0^2 + O(t^3) \text{ as } t \to 0.$$

Therefore,

$$(\Phi(\exp(t\nu)[A]), \xi_C) = \|\xi_C\|_0^2 - \frac{1}{2} t^2 \|[\nu, \xi_C]\|_0^2 + O(t^3) \text{ as } t \to 0.$$

Since $T_{[A]}Y_C$ is the sum of the non-negative eigenspaces of the Hessian of (Φ, ξ_C), it follows that $[\nu, \xi_C] = 0$. □

Lemma 6.3.6. There exists an open neighborhood U of $C \cap Y_C^{\mathrm{ss}}$ in Y_C^{ss} such that if $k[A'] \in Y_C$ for some $k \in K(S)$ and $[A'] \in U$ then $k \in K_C(S)$.

Proof. It suffices to consider the case $s = \frac{1}{2}$. By the inverse function theorem, $K(S) \times_{K_C(S)} Y_C \to \mathcal{M}(X)$ is a local diffeomorphism onto its image in a neighborhood of $C \cap Y_C$. Hence there exist neighborhoods U of $C \cap Y_C$ in Y_C and V of $K_C(S)$ in $K(S)$ such that for all $[A'] \in U$ and $k \in K(S) \backslash K_C(S)$, $k \cdot [A'] \in Y_C$ implies $k \notin V$. On the other hand, for $A' \in U$, any $k \in K(S)$ such that $(\Phi(k[A']), \xi_C) \geq (\xi_C, \xi_C) - \delta$ is contained in V for U and δ sufficiently small. If $k[A'] \in Y_C$ then $(\Phi(k[A']), \xi) \geq (\Phi([A]), \xi)$. This forces $k \in V$ and $k \in K_C(S)$. □

Lemma 6.3.7. $G(S)Y_C^{\mathrm{ss}}$ is a smooth embedded complex submanifold isomorphic to $G(S) \times_{P_C(S)} Y_C^{\mathrm{ss}}$.

Proof. Suppose that $[A] \in Y_C^{\mathrm{ss}}$ and $g[A] \in Y_C^{\mathrm{ss}}$ for some $g \in G(S)$. By Lemma 6.3.3 for any neighborhood U of $\Phi^{-1}(\xi_C)$ in Y_C^{ss}, there exists an element $p \in P_C(S)$ such that $[A'] = p[A]$ lies in U. Since $G(S) = K(S)P_C(S)$, $gp^{-1} = p'k$ for some $p' \in P_C(S)$ and $k \in K(S)$. Since Y_C^{ss} is $P_C(S)$-invariant, we have $k[A'] \in Y_C^{\mathrm{ss}}$. By Lemma 6.3.6, for U sufficiently small this implies $k \in K_C(S)$, and so $g = p'kp \in P_C(S)$.

This shows that $G(S) \times_{P_C(S)} Y_C^{\mathrm{ss}} \to G(S)Y_C^{\mathrm{ss}}$ is a bijection. To show it is a diffeomorphism, we must show that the condition

$$\{\xi \in \mathfrak{g}(S),\ \xi_{[A]} \in T_{[A]}Y_C^{\mathrm{ss}}\} = \mathfrak{p}_C(S)$$

holds for all $[A] \in Y_C^{ss}$. This is open, and by the Lemma above, it holds in a neighborhood of $Z_C \cap \Phi^{-1}(C)$. The condition is also invariant under $P_C(S)$, hence by Lemma 6.3.4 it holds everywhere. □

We wish to show that $G(S)Y_C^{ss}$ is a minimizing submanifold for f in the sense of Kirwan.

Lemma 6.3.8. *For $[A] \in Y_C \cap C$, the L^2-orthogonal subspace $(T_{[A]}G(S)Y_C)^\perp$ to $T_{[A]}G(S)Y_C$ is a complement to $T_{[A]}G(S)Y_C$.*

Proof. Let $[A] \in C \cap Y_C$ and $\mathfrak{g}(S)[A] \subset T_{[A]}$ the span of the generating vector fields for $\mathfrak{g}(S)$. The L^2-orthogonal subspace to $\mathfrak{g}(S)[A]$ is a finite-dimensional complement to $\mathfrak{g}(S)[A]$ in $T_{[A]}$, by the proof of Lemma 3.4.1 (c). The Lemma follows since $T_{[A]}G(S)Y_C$ contains $\mathfrak{g}(S)[A]$. □

Lemma 6.3.9. The Hessian of f is negative definite on $(T_{[A]}G(S)Y_C)^\perp$, for any $[A] \in C \cap Y_C$.

Proof. We argue as in Kirwan [18, p. 55]. Let $[A_t]$ be a path with $[A_0] = [A]$ and $\frac{\mathrm{d}}{\mathrm{d}t}[A_t] \in (T_{[A]}G(S)Y_C)^\perp$. Since $\frac{\mathrm{d}}{\mathrm{d}t}[A_t]$ is perpendicular to the generating vector fields for the action,

$$\Phi([A_t]) = \Phi([A]) + e(t)$$

where $e(t) = O(t^2)$. Hence

$$2f([A_t]) = 2f([A]) + 2\int_S (*_S\Phi([A]) \wedge e(t)) + O(t^3).$$

On the other hand,

$$(\Phi([A_t]), \xi_C) = 2f([A]) + 2\int_S (*_S\Phi([A]) \wedge e(t)).$$

It follows that the Hessians of f and $(\Phi([A_t], \xi_C)$ agree up to a scalar on $(T_{[A]}G(S)Y_C)^\perp$. But $(T_{[A]}G(S)Y_C)^\perp$ is contained in $(T_{[A]}Y_C)^\perp$ on which the Hessian of (Φ, ξ_C) is negative, by definition of Y_C. □

Together with Lemma 6.3.3, this shows that $G(S)Y_C^{ss}$ is a minimizing manifold for f in the sense of Kirwan. It remains to show:

Theorem 6.3.10. *$G(S)Y_C^{ss}$ is equal to $\mathcal{M}(X)_C$.*

Proof. We follow Kirwan [18, p.91]. Let $[A] \in C$ as before and consider the splitting in Lemma 6.3.8. By the implicit function theorem there exists

(a) a neighborhood V_A of 0 in $\ker(\mathrm{d}_A \oplus \mathrm{d}_A^*)$,
(b) a neighborhood U_A of $[A]$ in $\mathcal{M}(X)$, and
(c) a diffeomorphism $\varphi_A : U_A \to V_A$

such that $U_A \cap G(S)Y_C$ is the pre-image of $V_A \cap T_{[A]}G(S)Y_C$. Let

$$H = \mathrm{diag}(H_+, H_-)$$

be the decomposition of the Hessian of f at $[A]$ into positive-semi-definite and negative definite components. After identifying $(T_A)_s$ with $(T_A)^*_{-s}$ using the L^2-metric, H becomes an operator $(T_A)_s \to (T_A)_{-s}$. H_- becomes an operator on a finite-dimensional space. In the slice U_A the trajectories $a_t = (a_{+,t}, a_{-,t})$ are solutions to

$$\frac{\mathrm{d}}{\mathrm{d}t} a_{+,t} = -H_+ a_t + F_+(a_t), \quad \frac{\mathrm{d}}{\mathrm{d}t} a_{-,t} = -H_- a_t + F_-(a_t)$$

where F_+, F_- have vanishing derivatives at $(a_+, a_-) = (0,0)$. It follows that

$$a_{+,t} = e^{-H_+ t} a_{+,0} + (\delta a_+)(t, a_0), \quad a_{-,t} = e^{-H_- t} a_{-,0} + (\delta a_-)(t, a_0)$$

where $\delta a_+, \delta a_-$ have vanishing first partial derivatives at the origin $a_0 = 0$. For any $\epsilon > 0$ we may reduce the size of U_A so that

$$(1+\epsilon)^{-1} \|a_-\| \le d(A, G(S)Y_C^{\mathrm{ss}}) \le (1+\epsilon)\|a_-\|$$

everywhere in U_A. (Since $T_{[A]}G(S)Y_C)^\perp$ is finite-dimensional, this holds for any Sobolev norm on a_-.)

Lemma 6.3.11. There exists a number $b > 1$, depending only on C, such that if U_A is taken sufficiently small then for any $a \in U_A$ we have $\|a_{-,1}\| \ge b\|a_{-,0}\|$.

Proof. Let c denote the minimum eigenvalue of e^{-H_-} on C. Choose $\theta > 0$ so that $c - \theta > 1$, and $b = c - \theta$. Since the partial Jacobian of δa_- vanishes at the origin, by shrinking U_A we may assume that $\|\partial_{a_-} \delta a_-(1, a_+, a_-)\| < \theta$ for all $a \in U_A$. It follows that $\|\delta a_-(1, a_+, a_-)\| \le \theta \|a_-\|$. Hence for any $a_0 = (a_+, a_-)$ we have

$$\begin{aligned} \|a_{-,1}\| &= \|e^{H_-} a_{-,0} + \delta a_-(1, a_{\ge,0}, a_{-,0})\| \\ &\ge c\|a_{-,0}\| - \theta\|a_{-,0}\| \\ &\ge b\|a_{-,0}\|. \end{aligned}$$

□

Lemma 6.3.12. $\mathcal{M}(X)_C = G(S)Y_C^{\mathrm{ss}}$ in a neighborhood of C.

Proof. By Lemma 6.3.11 there is a neighborhood U_C of C in $\mathcal{M}(X)$ such that if $[A_t]$ is a trajectory of (12) then

$$d([A_1], G(S)Y_C^{\mathrm{ss}}) \ge b(1+\epsilon)^{-1} d([A_0], G(S)Y_C^{\mathrm{ss}}).$$

Choose ϵ sufficiently small so that $b(1+\epsilon)^{-1} > 1$. By Lemma 5.0.11 there exists a neighborhood V_C of C in $\mathcal{M}(X)$ such that if $[A] \in V_C \cap \mathcal{M}(X)_C$ then $[A_t]$ lies in U_C for all $t \in [0,\infty)$. Then for any $n \geq 1$,

$$d([A_n], G(S)Y_C^{\mathrm{ss}}) \geq (b(1+\epsilon)^{-1})^n d([A_0], G(S)Y_C^{\mathrm{ss}}).$$

But we may assume without loss of generality that $d([A], G(S)Y_C^{\mathrm{ss}})$ is bounded on U_C. Hence $d([A], G(S)Y_C^{\mathrm{ss}}) = 0$. This shows that $\mathcal{M}(X)_C \subset G(S)Y_C^{\mathrm{ss}}$ in a neighborhood of C. The opposite inclusion follows from Lemma 6.3.3. □

We now complete the proof of Theorem 6.3.10. Suppose $[A] \in \mathcal{M}(X)_C$. Then $[A_\infty] \in C$ and so the trajectory $[A_t]$ intersects V_C. Since $[A_t] \in G(S)[A]$ this implies that $[A] \in G(S)Y_C^{\mathrm{ss}}$. Hence $\mathcal{M}(X)_C \subset G(S)Y_C^{\mathrm{ss}}$. By Proposition 6.3.3, the subsets $G(S)Y_C^{\mathrm{ss}}$ are disjoint. Since $\mathcal{M}(X)$ is the union of the stable manifolds $\mathcal{M}(X)_C$, we must have $\mathcal{M}(X)_C = G(S)Y_C^{\mathrm{ss}}$. □

Remark 6.3.13. Suppose that f is a Morse-Bott function, that is, the Hessian of $f|_C$ is non-degenerate along the normal bundle to S. The map given by the time-1 flow $A_0 \mapsto A_1$ is hyperbolic, using the estimates on the operator L above. It is then a consequence of the stable manifold theorem [39, Theorem III.8] that the strata $\mathcal{M}(X)_C$ are smooth.

7. Applications

7.1. **Kähler quantization commutes with reduction.** Recall that $\mathcal{L}(X) \to \mathcal{M}(X)$ is the Chern-Simons pre-quantum line bundle. Let

$$\iota : \mathcal{M}(X)_{\Phi^{-1}(0)} \to \mathcal{M}(X), \quad \pi : \mathcal{M}(X)_{\Phi^{-1}(0)} \to \mathcal{M}(\overline{X})$$

denote the inclusion, resp. the projection of the zero level set into $\mathcal{M}(X)$, resp. onto $\mathcal{M}(\overline{X})$.

Theorem 7.1.1. *Suppose $K(S)$ acts on $\Phi^{-1}(0)$ with finite stabilizers. (This happens only if X has markings.) There is an isomorphism of spaces of global sections*

$$H^0(\mathcal{M}(X), \mathcal{L}(X))^{G(S)} \to H^0(\mathcal{M}(\overline{X}), \mathcal{L}(\overline{X})), \quad s \mapsto \overline{s}, \quad \iota^* s = \pi^* \overline{s}.$$

Proof. The assumption on the stabilizers implies that the stable and semistable loci in $\mathcal{M}(X)$ coincide. Therefore, for any $[A] \in \mathcal{M}(X)^{\mathrm{ss}}$ there exists an element g_∞ such that $[A_\infty] = g_\infty[A]$. By Lemma 6.2.3 the image $[g_\infty] \in G(S)/K(S)$ of g_∞ is unique. An application of the implicit function theorem for Banach spaces shows that g_∞ depends holomorphically on $[A]$. Let $\mathcal{L}(X)^{\mathrm{ss}}$ denote the restriction of $\mathcal{L}(X)$ to

$\mathcal{M}(X)^{\mathrm{ss}}$, $\mathcal{L}(X)_0$ the restriction to $\Phi^{-1}(0)$, and $\pi : \mathcal{L}(X)_0 \to \mathcal{L}(\overline{X})$ the projection. The map

$$\mathcal{L}(X)^{\mathrm{ss}} \to \mathcal{L}(\overline{X}), \quad l \mapsto \pi(g_\infty l) \tag{41}$$

is a $G(S)$-invariant holomorphic map of line bundles. For any $s \in H^0(\mathcal{M}(\overline{X}), L(\overline{X}))$ define $\overline{s} \in H^0(\mathcal{M}(X), \mathcal{L}(X)^{\mathrm{ss}})^{G(S)}$ by pull-back of s under (41). Since (41) is holomorphic, $\overline{s}$ is a holomorphic section. $\|\overline{s}\|$ is bounded by $\|s\|$ by Lemma 6.2.3. By the Riemann extension theorem for complex Banach manifolds ([35, II.1.15]; a convenient reference is [14, Appendix]) $\overline{s}$ has a holomorphic extension to $\mathcal{M}(X)$. Conversely, given $\overline{s}$ the restriction $\iota^*\overline{s}$ is $K(S)$-invariant and so descends to $\mathcal{M}(X)$. □

7.2. **Analog of Kirwan surjectivity.** Let $H^\bullet_{K(S)}(\cdot, \mathbb{Q})$ denote $K(S)$-equivariant cohomology with rational coefficients. If $K(S)$ acts with finite stabilizers on $\Phi^{-1}(0)$, then $H^\bullet_{K(S)}(\Phi^{-1}(0), \mathbb{Q}) \cong H^\bullet(\mathcal{M}(\overline{X}), \mathbb{Q})$.

Theorem 7.2.1. *$f : \mathcal{M}(X) \to \mathbb{R}$ is an equivariantly perfect Morse function. In particular, the inclusion $\Phi^{-1}(0) \to \mathcal{M}(X)$ induces a surjection $H^\bullet_{K(S)}(\mathcal{M}(X), \mathbb{Q}) \to H^\bullet_{K(S)}(\Phi^{-1}(0), \mathbb{Q})$.*

Proof. This follows by the same argument as in Atiyah and Bott [3, 13.4]: For any critical component, the circle $U(1)_C$ acts non-trivially on the normal bundle ν_C to $\mathcal{M}(X)_C$ at Z_C. It follows that the Euler class of ν_C is invertible, and that the stratification is equivariantly perfect. □

The case that S bounds a disk Theorem 7.2.1 is a special case of a recent result of Bott, Tolman, and Weitsman [6]. In this case one has $X = X_+ \cup X_-$, say X_+ is homeomorphic to a disk, and

$$H^\bullet_{K(S)}(\mathcal{M}(X), \mathbb{Q}) = H^\bullet_{K(S)}(\mathcal{M}(X_-) \times \Omega K, \mathbb{Q}) = H^\bullet_K(\mathcal{M}(X_-), \mathbb{Q})$$

so $H^\bullet_K(\mathcal{M}(X_-), \mathbb{Q})$ surjects onto $H^\bullet(\mathcal{M}(\overline{X}), \mathbb{Q})$. Their proof uses the Morse theory of the energy functional on a space homotopy equivalent to $\mathcal{M}(X_-)$. Inductive formulas for the Poincaré polynomials of the moduli spaces of parabolic bundles using the Atiyah-Bott approach have been given by Nitsure [26] and Holla [15].

Example 7.2.2. To compare this approach with that of Atiyah and Bott, we compute the Poincaré polynomial of the simplest case $\mathcal{M}(\overline{X}; SU(2); \mu)$, where $\mu = \frac{1}{2}$ is the marking corresponding to holonomy -1 around a puncture. We identify $\mathfrak{t} \to \mathbb{R}$ so that $\mathfrak{A} = [0, \frac{1}{2}]$. Let S be a circle enclosing the puncture. Then $X = X_+ \cup X_-$, where X_+ is a surface of genus $2g$ with a single boundary and no markings, and X_- is a disk with a single marking. By the holonomy description Lemma 3.0.2 (e)

$$\mathcal{M}(X_+) = K^{2g} \times_K \Omega^1(S;\mathfrak{k}), \quad \mathcal{M}(X_-) = K(S)/K(S)_\mu$$

where $K(S)_\mu$ denotes the stabilizer of μ. (Note this is isomorphic to K but does *not* consist of constant loops.) Note that $\mathcal{M}(X_+)$ is a principal ΩK-bundle over K^{2g}. Because the commutator $K^{2g} \to K$ induces the trivial map in cohomology $H^\bullet(K) \to H^\bullet(K^{2g})$, the spectral sequence for this fibration collapses at the second term. Hence

$$P_{K(S)}(\mathcal{M}(X)) = P_{K(S)_\mu}(\mathcal{M}(X_+)) = \frac{P(\Omega K)P(K^{2g})}{1-t^4} = \frac{(1+t^3)^{2g}}{(1-t^2)(1-t^4)}.$$

The contributions from the critical components are given as follows. Each component contains a point with $\Phi(m_1,m_2) = *_S(\lambda,\mu)$, for some $\lambda \in \mathbb{Z}_{>0}$. The corresponding component C_λ is the $K(S)$-orbit of $(T^{2g} \times \{\lambda\}) \times \{\mu\}$ in the holonomy description of $\mathcal{M}(X_+) \times \mathcal{M}(X_-)$. The equivariant Poincaré polynomial of C_λ is therefore

$$P_{K(S)}(C_\lambda) = P_{U(1)}(T^{2g}) = \frac{(1+t)^{2g}}{1-t^2}.$$

Lemma 7.2.3. *The index of f at C_λ is $g + 2\lambda - 2$.*

Proof. The tangent space at any point in C_λ is isomorphic to

$$(\mathfrak{k}/\mathfrak{t})^{2g} \oplus \mathfrak{k}(S)/\mathfrak{k}(S)_\lambda \oplus \mathfrak{k}(S)/\mathfrak{k}(S)_\mu.$$

The moment map to second order is

$$\begin{aligned}\Phi(t(\xi_1,\ldots,\xi_{2g},\zeta_-,\zeta_+)) \;&=\; \lambda + t^2\sum_{i=1}^g[\xi_i,\xi_{g+i}] + t[\zeta_+,\lambda] + t^2[\zeta_+,[\zeta_+,\lambda]]/2 \\ &-\; \mu - t[\zeta_-,\mu] - t^2[\zeta_-,[\zeta_-,\mu]]/2 + O(t^3).\end{aligned}$$

The second order term in f is

$$\int_S (\lambda-\mu, \sum_{i=1}^g[\xi_i,\xi_{i+g}]) + \frac{([\zeta_-,\lambda],[\zeta_-,\mu]) + ([\zeta_+,\lambda],[\zeta_+,\mu])}{2} - ([\zeta_+,\lambda],[\zeta_-,\mu]).$$

It follows that the index of f on $(\mathfrak{k}/\mathfrak{t})^{2g}$ is g. Note that if $\xi = 0$ and $\zeta_- = \zeta_+$ then the Hessian vanishes; these are the directions tangent to the $K(S)$-orbit. On the other hand, suppose that $\zeta_- = -\zeta_+$ and is in the root space for an affine root α. Then the Hessian is negative if and only if α is negative on λ and positive on μ, or vice-versa. Therefore, the index of f on $\mathfrak{k}(S)/\mathfrak{k}(S)_\lambda \oplus \mathfrak{k}(S)/\mathfrak{k}(S)_\mu$ is the number of affine hyperplanes separating λ and μ, equal to $2\lambda - 2$ for $\lambda > 0$. □

Putting everything together, the Poincaré polynomial of $\mathcal{M}(\overline{X},\mu)$ is

$$\begin{aligned} P(\mathcal{M}(\overline{X},\mu)) &= P_{K(S)}(\mathcal{M}(X)) - \sum_{\lambda>0} t^{2(2\lambda+g-2)} P_{K(S)}(C_\lambda) \\ &= \frac{(1+t^3)^{2g} - (1+t)^{2g}t^{2g}}{(1-t^2)(1-t^4)} \end{aligned}$$

which agrees with [3].

7.3. **Another proof of Birkhoff factorization.** Let $\overline{X} = \mathbb{P}^1$ so that $\mathcal{M}(X) = \Omega K \times \Omega K$. The critical components for f are the orbits under $K(S)$ of pairs $(w_1 \cdot 0, w_2 \cdot 0)$, and therefore can be indexed by

$$W_{\text{aff}} \backslash (W_{\text{aff}} \times W_{\text{aff}})/(W \times W) \cong \Lambda_+$$

where Λ_+ is the set of dominant coweights. The orbit $G(S)(w_1 \cdot 0, w_2 \cdot 0)$ is a submanifold with the same codimension as $\mathcal{M}(X)_{[w_1,w_2]}$, and therefore equals $\mathcal{M}(X)_{[w_1,w_2]}$. Hence

$$\mathcal{M}(X) = \Omega K \times \Omega K = \bigcup_{W_{\text{aff}} \backslash (W_{\text{aff}} \times W_{\text{aff}})/(W \times W)} G(S)(w_1 \cdot 0, w_2 \cdot 0).$$

Using $\mathcal{M}(X_-) = \Omega K = G(S)/G_{\text{hol}}(X_-)$ we obtain (see [29, Ch. 8])

Theorem 7.3.1. *$G(S) = G_{\text{hol}}(X_-)\Lambda_+ G_{\text{hol}}(X_+)$.*

Appendix A. Sobolev spaces

A convenient reference for the basic material on Sobolev spaces is [16, Appendix]. The remaining results are variants of results in Lions-Magenes [21] and Palais [27, Chapter 9] which we learned from Råde [30, Section D].

For any $s \in \mathbb{R}$, the Sobolev space $H^s(\mathbb{R}^n;\mathbb{R}^r)$ is the completion of the space of smooth, compactly supported maps $u \in C_0^\infty(\mathbb{R}^n;\mathbb{R}^r)$ in the norm

$$\|u\|_s = (2\pi)^{-n/2} \|(1+\|\xi\|^2)^{s/2}\mathcal{F}(u)(\xi)\|_{L^2(\mathbb{R}^n;\mathbb{R}^r)}$$

where $\mathcal{F}$ denotes Fourier transform. Let M be a compact smooth manifold, and $E \to M$ a smooth Euclidean vector bundle of rank r. To define the Sobolev spaces of sections of E, we introduce a trivializing atlas $\{(U_i, V_i, \varphi_i : E|_{U_i} \to V_i \times \mathbb{R}^r)\}$ and a subordinate partition of unity ρ_i, for $i \in I$. For any real s, define $H^s(M;E)$ to be the completion of the space of smooth sections of E with respect to the norm

$$\|u\|_s = \left(\sum_{i \in I} \|\varphi_i \circ (\rho_i u)\|_s^2 \right)^{\frac{1}{2}}.$$

Suppose that M is a manifold with boundary, $\tilde{M}$ is a closed manifold of the same dimension containing M (for instance, the double of M), and $\tilde{E} \to \tilde{M}$ is a vector bundle whose restriction to M is isomorphic to E. Let $r: \ C_0^\infty(\tilde{M}; \tilde{E}) \to C_0^\infty(M; E)$ denote the restriction map, and $\overline{C_0^\infty}(M; E)$ the image of r; this is the space of *extendable* sections of E with compact support. The Sobolev space $H^s(M; E)$ is the completion of $\overline{C_0^\infty}(M; E)$ in the norm

$$\|u\|_s = \inf_{\tilde{u}} \|\tilde{u}\|_s$$

where the infimum is over $\tilde{u}$ that restrict to u. Similarly, $H^s_\partial(M; E)$ is the completion of the space $C_0^\infty(M \backslash \partial M; E)$, with respect to the Sobolev norm induced by the extension-by-zero map $C_0^\infty(M \backslash \partial M; E) \to C_0^\infty(\tilde{M})$. These spaces have the following properties:

Lemma A.0.2. (a) For $s > n/2$, the spaces $H^s(M; E)$ resp. $H^s_\partial(M; E)$ embed into the space $C(M; E)$ of continuous sections of E, resp. vanishing on the boundary.
(b) For any real s, there is a perfect pairing $H^s(M; E) \times H^{-s}_\partial(M; E) \to \mathbb{R}$.
(c) When $s > \frac{1}{2}$, the elements of $H^s(M; E)$ have boundary values (traces) in $H^{s-\frac{1}{2}}(\partial M; \partial E)$. More generally, choose a smooth vector field ν on M normal to the boundary. For $m \in \mathbb{N}$ and $s > m - \frac{1}{2}$, we have the Cauchy trace operator

$$H^s(M) \to \prod_{j=0}^{m-1} H^{s-j-\frac{1}{2}}(\partial M), \quad u \mapsto (u|_{\partial M}, \partial_\nu u|_{\partial M}, \ldots, \partial_\nu^m u|_{\partial M})$$

where ∂_ν denotes Lie derivative.
(d) For $s \in (m - \frac{1}{2}, m + \frac{1}{2})$, the kernel of the Cauchy trace map is equal to $H^s_\partial(M)$.
(e) For $s > \frac{1}{2}$, there is a continuous extension operator

$$\mathcal{E}: \ H^{s-\frac{1}{2}}(\partial M; \partial E) \to H^s(M; E)$$

which is a right inverse to $r_{\partial X}$, with the property that for any $f \in H^{s-\frac{1}{2}}(\partial M; \partial E)$, $\mathcal{E}(f)$ is smooth away from ∂M.
(f) For any real s_1, s_2, vector bundles $E_1, E_2 \to M$ and $s \le \min(s_1, s_2, s_1 + s_2 - n/2)$ (except for the borderline cases $s_2 = -s_1$ and $s = -n/2$; $s = s_1$ and $s_2 = n/2$; $s = s_2$ and $s_1 = n/2$) there is a continuous map $H^{s_1}(M; E_1) \times H^{s_2}(M; E_2) \to H^s(M; E_1 \otimes E_2)$.

We will also need Sobolev spaces of mixed order on $[0, T] \times M$. We assume throughout that $T \in (0, 1)$. Let $\underline{E}$ denote the pullback of $E \to M$ to $[0, T] \times M$. For any real r, s, the space $H^{r,s}([0, T] \times M; \underline{E})$

denotes the completion of the space of smooth time-dependent sections of E in the norm

$$\|u\|_{r,s} = \inf_{\tilde{u}} \|(\tau^2 + T^{-2})^{r/2}\mathcal{F}(\tilde{u})(\tau)\|_{L^2([0,T],H_s)}$$

where the infimum is over $\tilde{u} \in C^\infty(\mathbb{R}, H^s(M;E))$ that restrict to u on $[0,T]$. The space $H_0^{r,s}([0,T] \times M; \underline{E}) = H_0^{r,s}([0,T] \times M; \underline{E})$ is defined in the same way except that the infimum is taken over $\tilde{u} \in C^\infty(\mathbb{R}, H^s(M;E))$ that restrict to u on $[0,T]$ and vanish for $t < 0$. These spaces have the following properties:

Lemma A.0.3. (a) For any real r and s the identity map defines an operator $H_{0,r,s}([0,T] \times M; \underline{E}) \to H_{r,s}([0,T] \times M; \underline{E})$. For $r > -\frac{1}{2}$, this operator is injective. For $r < \frac{1}{2}$ it is onto.

(b) For $r_1 \geq r_2$ and $s_1 \geq s_2$, the identity map defines an operator $H^{r_1,s_1}([0,T] \times M; \underline{E}) \to H^{r_2,s_2}([0,T] \times M; \underline{E})$ of norm less than $cT^{r_1-r_2}$.

(c) For $r > \frac{1}{2}$ the map $f \mapsto f(0)$ extends to a restriction (trace) map $H^{r,s}([0,T] \times M; \underline{E}) \to H^s(M;E)$. More generally, for $m \in \mathbb{N}$ and $r > m - \frac{1}{2}$ we have a Cauchy trace operator

$$H^{r,s}([0,T] \times M; \underline{E}) \to \prod_{j=0}^{m-1} H^{r-j-\frac{1}{2},s}(M;E), \qquad u \mapsto \big(u(0), u'(0), \ldots, u^{(m)}(0)\big) \tag{42}$$

(d) For $r \in (m - \frac{1}{2}, m + \frac{1}{2})$, the kernel of the Cauchy trace map is equal to $H_0^{r,s}(M;E)$.

(e) Let $E_1 \to M$ and $E_2 \to M$ be restrictions of vector bundles $\tilde{E}_1 \to \tilde{M}$, $\tilde{E}_2 \to \tilde{M}$ to M. For real r_1, s_1, r_2, s_2, r, s with $r \leq \min(r_1, r_2, r_1 + r_2 - \frac{1}{2})$ and $s \leq \min(s_1, s_2, s_1 + s_2 - n/2)$ (except for the borderline cases $r_2 = -r_1$ and $r = -1/2$, $r = r_1$ and $r_2 = 1/2$; $r = r_2$ and $r_1 = 1/2$; $s_2 = -s_1$ and $s = -n/2$; $s = s_1$ and $s_1 = n/2$; $s = s_2$ and $s_1 = n/2$) there is a continuous map $H^{r_1,s_1}([0,T] \times M; \underline{E_1}) \times H^{r_2,s_2}([0,T] \times M; \underline{E_2}) \to H^{r,s}([0,T] \times M; \underline{E_1} \otimes \underline{E_2})$.

(f) For any real r, s, integration with respect to Lebesgue measure on $[0,T]$ defines a continuous map $H_0^{r,s}([0,T] \times M; \underline{E}) \to H_0^{r+1,s}([0,T] \times M; \underline{E})$.

(g) For real numbers r, s, r', s' the intersection $H^{r,s}([0,T] \times M; E) \cap H^{r',s'}([0,T] \times M; \underline{E})$ is a Banach space with norm $\| \quad \|_{(r,s)\cap(r',s')} = \left(\| \quad \|^2_{(r,s)} + \| \quad \|^2_{(r',s')}\right)^{1/2}$. Interpolation: For any $\theta \in [0,1]$, there is an embedding $H^{r,s}([0,T] \times M; \underline{E}) \cap H^{r',s'}([0,T] \times M; \underline{E}) \to H^{\theta r + (1-\theta) r', \theta s + (1-\theta) s'}([0,T] \times M; E)$.

(h) Let $P \in \Psi DO(M; E, E)$ be an self-adjoint, non-negative elliptic pseudodifferential operator on E of order m. For any real r, s, the operator $\frac{\mathrm{d}}{\mathrm{d}t} + P$ defines an invertible operator $H_0^{r+1,s}([0,T] \cap M; E) \cap H_0^{r,s+m}([0,T] \cap M; E) \to H^{r,s}(M; E)$.

(i) (Same assumptions on P) For any real s, r, solving the homogeneous initial value problem $(\frac{\mathrm{d}}{\mathrm{d}t} + P)u = 0, u(0) = v$ defines an operator $H^s(M; E) \to H^{\frac{1}{2} - r, s + mr}([0,T] \times M; \underline{E}), \quad v \mapsto u$ of order bounded by $c \max(1, T^r)$.

For short, we denote by $\Omega^k(M; E)_s$, resp. $\Omega^k(M; E)_{r,s}$, resp. $\Omega^k(M; E)_{0,r,s}$ the spaces $H^s(M; \Lambda^k(T^*M) \otimes E)$, resp. $H^{r,s}([0,T] \times M; \underline{\Lambda^k(T^*M) \otimes E})$, resp. $H_0^{r,s}([0,T] \times M; \underline{\Lambda^k(T^*M) \otimes E})$.

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