GAUGED FLOER THEORY OF TORIC MOMENT FIBERS

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Abstract. We investigate the small area limit of the gauged Lagrangian Floer cohomology of Frauenfelder [17]. The resulting cohomology theory, which we call *quasimap Floer cohomology*, is an obstruction to displaceability of Lagrangians in the symplectic quotient. We use the theory to reproduce the results of Fukaya-Oh-Ohta-Ono [21], [19] and Cho-Oh [12] on non-displaceability of moment fibers of not-necessarily-Fano toric varieties and extend their results to toric orbifolds, without using virtual fundamental chains. Finally we describe a conjectural relationship with Floer cohomology in the quotient.

Contents

1. Introduction

Let X be a symplectic manifold. A Lagrangian submanifold $L \subset X$ is (Hamiltonian) *displaceable* if there exists a Hamiltonian diffeomorphism $\phi : X \to X$ such that $L \cap \phi(L) = \emptyset$. Otherwise, L is called *non-displaceable*. Hamiltonian non-displaceability questions are among the most basic in symplectic topology. For example if $L = \{(x,x) | x \in X\} \subset X^- \times X$ is the diagonal and X is compact then Arnold conjectured (and Floer [15] later proved under certain conditions) that L is non-displaceable. Floer introduced a method for proving non-displaceability based on the study of a complex whose underlying vector space is generated by intersection points $L \cap \phi(L)$, if transversal, and whose differential counts finite-energy holomorphic strips with boundary values in $(L, \phi(L))$. The resulting *Floer cohomology*

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group $HF(L, \phi(L))$ is independent of the choice of ϕ, so that if L is displaceable then $HF(L, \phi(L))$ vanishes. On the other hand, one can sometimes compute $HF(L, \phi(L))$ by, for example, taking ϕ small and identifying it with the Morse homology.

In a series of papers [21], [19] Fukaya-Oh-Ohta-Ono used this strategy to prove non-displaceability results for certain moment fibers of toric varieties, generalizing earlier work of Cho-Oh [12] and Entov-Polterovich [14]. Most of these fibers have vanishing Floer cohomology, and many are displaceable by elementary means, which we will discuss further in a moment. Fukaya et al show that a moment fiber has non-vanishing Floer cohomology if there is a critical point of a *potential* obtained by counting holomorphic disks with boundary on the Lagrangian. Fukaya et al were able to write the potential as the sum of a *naive potential* plus quantum corrections arising from sphere bubbles, and show that the naive potential is given by an explicit formula which appeared in the paper of the physicists Hori-Vafa [29, 5.16].

One can take a different approach to this problem using the realization of a toric variety as a symplectic quotient. Let G be a compact connected group with Lie algebra $\mathfrak{g}$ and X a Hamiltonian G-manifold with moment map $\Phi : X \to \mathfrak{g}^\vee := \mathrm{Hom}(\mathfrak{g}, \mathbb{R})$. If G acts freely on the level set $\Phi^{-1}(0)$ then the *symplectic quotient* $X/\!\!/G = \Phi^{-1}(0)/G$ is a smooth symplectic manifold, or more generally a symplectic orbifold if the action has finite stabilizers. In particular, any smooth projective toric variety can be realized as the quotient $X/\!\!/G$ of a vector space X by the action of a torus G. Given a Lagrangian $L \subset X/\!\!/G$ we ask whether L is displaceable. The pre-image $\tilde{L}$ of L is a G-invariant Lagrangian in X. An approach to non-displaceability for Lagrangians in Hamiltonian G-manifolds was introduced by Frauenfelder [18]. At first sight, his theory looks even more complicated than that of Fukaya et al: he counts pairs (A, u) consisting of a connection $A \in \Omega^1(\Sigma, \mathfrak{g})$ on $\Sigma := \mathbb{R} \times [0, 1]$ together with a map $u : \Sigma \to X$ satisfying a pair of *vortex equations*: u is holomorphic with respect to the connection determined by A and the curvature F_A is equal to minus the pull-back $u^*\Phi$ of the moment map: $F_A = -u^*\Phi \,\mathrm{Vol}_\Sigma$. Here $\mathrm{Vol}_\Sigma \in \Omega^2(\Sigma)$ is a choice of area form, in this case a multiple of the standard area form, and constitutes a parameter in the theory that can be varied. In the limit $\mathrm{Vol}_\Sigma \to \infty$, the vortex equations become equivalent to the Cauchy-Riemann equation for a map to the quotient $X/\!\!/G$, so one expects the gauged Floer theory to be related to the Floer theory of the quotient [22]. In the case that X has no holomorphic spheres, the gauged Floer theory has better compactness properties than the theory in $X/\!\!/G$. Frauenfelder [17] used his gauged Floer theory to prove a version of the Arnold-Givental conjecture in this context.

In this paper we study the zero-area limit $\mathrm{Vol}_\Sigma \to 0$ of gauged Floer theory. The corresponding limit for gauged Gromov-Witten invariants was studied in Gonzalez-Woodward [26], and used to prove a version of the abelianization conjecture for Gromov-Witten invariants. In the zero-area limit the vortex equations reduce to the Cauchy-Riemann equation in X. We show that the resulting *quasimap Floer theory* retains the relationship to non-displaceability, that is, its non-vanishing obstructs displaceability, in cases where it is defined. For any toric moment fiber one obtains an A_∞ algebra, in a way that avoids one of the main technical complications of the theory of Fukaya et al: since all holomorphic disks are regular for the standard complex structure, there is no need for Kuranishi structures or virtual fundamental chains. A classification result of Cho-Oh [12] gives an explicit formula for the holomorphic disks, see Theorem 6.1. The classification leads to an explicit formula for the curvature of the resulting A_∞ algebra and an explicit criterion for the non-vanishing of the quasimap Floer cohomology. This allows us to reproduce the results of Fukaya et al [21], [19], as well as give extensions to orbifold quotients such as weighted projective spaces.

The main result Theorem 1.1 below is an explicit sufficient condition for a moment fiber of a toric orbifold to be non-displaceable; its form is the same as that of Fukaya et al [19]. Let $X \cong \mathbb{C}^N$ be Hermitian vector space, $H \cong U(1)^N$ the standard maximal torus of $\mathrm{Aut}(X)$, and $G \subset H$ a sub-torus. Choose a moment map for the H-action; this induces a moment map for the action of G. Suppose that the symplectic quotient $X/\!\!/G$ is locally free and so a symplectic orbifold. It has a residual action of a torus $T := H/G$, and the residual moment map $\Psi : X/\!\!/G \to \mathfrak{t}^\vee$ defines a homeomorphism of $(X/\!\!/G)/T$ onto its *moment polytope* $\Psi(X/\!\!/G)$. We denote by $v_1, \ldots, v_N \in \mathfrak{t}$ the images of minus the standard basis vectors $e_1, \ldots, e_N \in \mathfrak{h} \cong \mathbb{R}^N$ in $\mathfrak{t}$. The polytope $\Psi(X/\!\!/G)$ is given by linear inequalities

$$\Psi(X/\!\!/G) = \{\lambda \in \mathfrak{t}^\vee \mid l_i(\lambda) \geq 0\}, \quad l_i(\lambda)/2\pi := \langle \lambda, v_i \rangle - c_i, \quad i = 1, \ldots, N. \tag{1}$$

where $\langle \cdot, \cdot \rangle : \mathfrak{t}^\vee \times \mathfrak{t} \to \mathbb{R}$ is the canonical pairing and $c_1, \ldots, c_N$ are constants given by the choice of moment map. Let Λ be the *universal Novikov field* consisting of possibly infinite sums of real powers of a formal variable q,

$$\Lambda = \left\{ \sum_{n=0}^{\infty} a_n q^{d_n}, \quad a_n \in \mathbb{C}, d_n \in \mathbb{R}, \quad \lim_{n \to \infty} d_n = \infty \right\}.$$

Let Λ_+ resp. Λ_0 denote the subrings consisting of sums with only positive resp. non-negative powers. Any fiber $L_\lambda = \Psi^{-1}(\lambda)$ over an interior point $\lambda \in \mathrm{int}(\Psi(X/\!\!/G))$ is a Lagrangian torus, namely a single free T-orbit. We identify $H^1(L_\lambda, \Lambda_0) \cong H^1(T, \Lambda_0)^T \cong \mathfrak{t}^\vee \otimes \Lambda_0$, so that in particular for any $v \in \mathfrak{t}$ and $\beta \in H^1(L_\lambda, \Lambda_0)$ we have a pairing $\langle v, \beta \rangle \in \Lambda_0$ and an exponential $e^{\langle v, \beta \rangle} \in \Lambda_0$. Define the *Hori-Vafa potential*

$$W_\lambda^G : H^1(L_\lambda, \Lambda_0) \to \Lambda_0, \quad \beta \mapsto \sum_{i=1}^{N} e^{\langle v_i, \beta \rangle} q^{l_i(\lambda)};$$

as in [29, (5.16)].

Theorem 1.1. *If W_λ^G has a critical point, then $L_\lambda \subset X/\!\!/G$ is non-displaceable.*

In the case that $X/\!\!/G$ is compact and smooth and there exists a non-degenerate critical point this is due to Fukaya et al [21], and without the non-degeneracy condition in their second paper [19]. Overlapping results using different methods are given by Entov-Polterovich [14]. Related works include Alston [4], Alston-Amorim [5], and Abreu-Macarini, in progress, who compute the Floer cohomology of toric moment fibers with the real part of the ambient toric variety. The existence of λ in the interior such that W_λ^G has a critical point for compact $X/\!\!/G$ is proved in Fukaya et al [21, Proposition 4.7] for toric varieties with rational symplectic classes; the authors conjecture that the rationality condition is not necessary. For non-compact $X/\!\!/G$, there may not exist any such λ, for example if $X/\!\!/G = \mathbb{C}$ then every compact Lagrangian is displaceable. As far as the author knows, existence of a non-displaceable Lagrangian in an arbitrary compact symplectic manifold is an open question. We also prove a Theorem 7.1 which includes bulk deformations as Fukaya et al's second paper [19].

Our proof differs from Fukaya et al in several ways. We already mentioned that we count disks in X rather than in $X/\!\!/G$. The proof in Fukaya et al depends on a detailed study of the correction terms arising in their potential from sphere bubbling in $X/\!\!/G$, which is not necessary in our case since X has no holomorphic spheres. Furthermore we use the combined Morse-Fukaya approach to the construction of A_∞ algebras, in which one counts configurations consisting of gradient lines and holomorphic disks with Lagrangian boundary conditions; this avoids various difficulties with choices of generic chains or smoothness of moduli spaces of stable maps. A generic perturbation of the gradient flow equations gives an A_∞ structure on the space of Morse cochains of L. A disadvantage of this approach is that the source objects (tree disks etc.) have somewhat more complicated moduli spaces than the usual realizations of associahedra, multiplihedra etc. Furthermore the structure maps of the A_∞ algebras constructed this way do not satisfy a divisor equation of the type described by Cho [11], who noted that the perturbation scheme may destroy the necessary forgetful maps. The latter requires us to take a slightly different definition of the potential than Fukaya et al. Then we have to show that the special case of the divisor equation that we need does in fact hold, and implies that any critical point of the potential gives rise to non-vanishing Floer cohomology. For pairs of Lagrangians intersecting transversally in $X/\!\!/G$, we define an A_∞ bimodule by counting configurations of Floer trajectories in X, holomorphic disks, and gradient trees. The pre-image Lagrangians intersect only cleanly in X, and this requires several results (exponential decay, energy quantization, gluing) for Lagrangian clean intersections which are probably known to experts, but which do not seem to have completely appeared in the literature.

These results on non-displaceability should be contrasted with those of McDuff [41] who gives a method for *displacing* the fibers of a moment polytope of a toric variety, based on the observation that if λ is sufficiently close to the boundary of $\Psi(X/\!\!/G)$ then L_λ is small in some Darboux chart and so displaceable. More precisely a vector α in the coweight lattice $\mathfrak{t}_\mathbb{Z}$ is *integrally transverse* to an open facet F of the moment polytope $\Psi(X/\!\!/G)$ if $\{\alpha\}$ can be completed to a basis of $\mathfrak{t}_\mathbb{Z}$ by vectors parallel to F. A *probe* with direction $\alpha \in \mathfrak{t}_\mathbb{Z}$ and initial point $\mu \in F$ is the open half of the line segment lying in the direction of α from μ. If λ lies in a probe, then L_λ is displaceable.

Example 1.2. In the case $X/\!\!/G = \mathbb{P}^1$ with moment polytope $[0,1]$, $W_\lambda^G(\beta) = e^\beta q^\lambda + e^{-\beta} q^{1-\lambda}$ which has a critical point iff $\lambda = 1/2$. The fiber L_λ for $\lambda = 1/2$ is the unique non-displaceable moment fiber in $\mathbb{P}^1$, since any other fiber is displaced by a rotation, and L_λ is not displaceable since it separates $\mathbb{P}^1$ into two disks of equal area.

Example 1.3. This example is a case for which $X/\!\!/G$ is non-compact. Suppose that $X = \mathbb{C}^3$ and $G = \mathbb{C}^*$ acts with weights $-1, -1, 1$, and the moment map is chosen so that $X/\!\!/G$ is the blow-up of $X = \mathbb{C}^2$ at $(0,0)$, so that the moment polytope is $\{(\lambda_1, \lambda_2) \in \mathbb{R}^2_{\geq 0}, \lambda_1 + \lambda_2 \geq 1\}$. The gauged potential is

$$W^G_\lambda(\beta) = e^{\beta_1} q^{\lambda_1} + e^{\beta_2} q^{\lambda_2} + e^{\beta_1+\beta_2} q^{\lambda_1+\lambda_2-1}.$$

This has a critical point iff $(\lambda_1, \lambda_2) = (1,1)$, so the fiber over $(1,1)$ is not displaceable.

Example 1.4. This example appears in Fukaya et al [21, Example 4.7] and illustrates the dependence of the number of non-displaceable fibers on the choice of symplectic form: Suppose that $X/\!\!/G$ is the toric blow-up of $\mathbb{P}^1 \times \mathbb{P}^1$ with moment polytope

$$\Psi(X/\!\!/G) = \{(\lambda_1, \lambda_2) \in [-1,1]^2 \mid \lambda_1 + \lambda_2 \leq 1 + \alpha\}$$

for some real parameter $\alpha \in (-1, 1)$ describing the size of the blow-up.

Proposition 1.5. *The unique non-displaceable toric fibers for $X/\!\!/G$ the blow-up of $(\mathbb{P}^1)^2$ are described by the following three cases: (i) for $\alpha = 0$, the fiber over $\lambda = (0,0)$ (ii) for $\alpha > 0$, the fibers over $(0,0)$ and (α, α) (iii) for $\alpha < 0$, the fibers over $(-\alpha/3, -\alpha/3), (\alpha, \alpha+1)$, $(\alpha+1, \alpha)$.*

See Figure 1.

Proof. The gauged potential (which in this case is the same as the potential in Fukaya et al) is

$$W^G_\lambda(\beta) = e^{\beta_1} q^{1+\lambda_1} + e^{\beta_2} q^{1+\lambda_2} + e^{-\beta_1} q^{1-\lambda_1} + e^{-\beta_2} q^{1-\lambda_2} + e^{-\beta_1-\beta_2} q^{1+\alpha-\lambda_1-\lambda_2}. \tag{2}$$

The critical points are described in [21, Example 4.7]. The other fibers are displaceable by the technique of McDuff [41]. □

Note that the non-displaceable fibers "collide and scatter" at $\alpha = 0$. The number of non-displaceable fibers depends on the symplectic form, but the multiplicities in each case do add up to the dimension 5 of the cohomology $H(X/\!\!/G)$ [21]. The non-displaceable fibers for other choices of symplectic form are discussed in Fukaya et al [19]. The values of the potential, counted with multiplicity, match the eigenvalues for the quantum action of $c_1(TX)$ on the quantum cohomology $QH(X/\!\!/G)$, see [21].

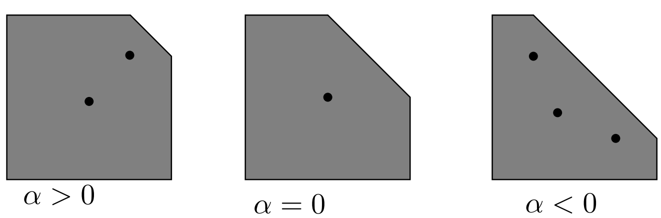

FIGURE 1. Non-displaceable fibers for the blow-up of $\mathbb{P}^1 \times \mathbb{P}^1$

Example 1.6. This example involves the orbifold case. Consider the weighted projective plane, $X/\!\!/G = \mathbb{P}(1, n_1, n_2)$ the symplectic quotient of $X = \mathbb{C}^3$ by the S^1 action with weights $(1, n_1, n_2)$. The two-torus T acts on $X/\!\!/G$ in Hamiltonian fashion with moment polytope $\Psi(X/\!\!/G)$ the convex hull of $(0,0), (n_1, 0), (0, n_2)$.

Proposition 1.7. *For $X/\!\!/G = \mathbb{P}(1, n_1, n_2)$ the fiber of Ψ over $\lambda = \operatorname{diag}((n_1 n_2)/(n_1 + n_2 + 1))$ is non-displaceable.*

Proof. The gauged potential is

$$W_\lambda^G(\beta) = e^{\beta_1} q^{\lambda_1} + e^{\beta_2} q^{\lambda_2} + e^{-\beta_1 n_2 - \beta_2 n_1} q^{n_1 n_2 - n_2 \lambda_1 - n_1 \lambda_2}.$$

The derivatives are given by

$$e^{-\beta_1} \partial_{\beta_1} W_\lambda^G(\beta) = q^{\lambda_1} - n_2 e^{-\beta_1 (n_2 + 1) - \beta_2 n_1} q^{n_1 n_2 - n_2 \lambda_1 - n_1 \lambda_2}$$

$$e^{-\beta_2} \partial_{\beta_2} W_\lambda^G(\beta) = q^{\lambda_2} - n_1 e^{-\beta_1 n_2 - \beta_2 (n_1 + 1)} q^{n_1 n_2 - n_2 \lambda_1 - n_1 \lambda_2}.$$

This has solutions in $\beta \in \Lambda_0$ if and only if

$$n_1 n_2 - (n_2 + n_1 + 1)\lambda_1 - (n_1 + n_2)\lambda_2 = 0, n_1 n_2 - (n_1 + n_2)\lambda_1 - (n_1 + n_2 + 1)\lambda_2 = 0.$$

The unique solution to these equations is $\lambda = \operatorname{diag}((n_1 n_2)/(n_1 + n_2 + 1))$ as claimed. $\square$

The case of $X/\!\!/G = \mathbb{P}(1, 3, 5)$ is shown below in Figure 2, taken from [41], with shaded regions displaceable by McDuff's probes. The non-displaceable fiber over $\lambda = (5/3, 5/3)$ is surrounded by an open subset of fibers for which displaceability is an open question. (The small line segment connecting $(1, 1)$ with $(3/2, 3/2)$ consist of fibers that are not displaceable by probes, either. In all the previous examples except this one, the combination of the McDuff method with the Floer theoretic methods completely resolved the question of non-displaceability of moment fibers.) We remark that homological mirror symmetry for the B-model on, in particular, $\mathbb{P}(1, 3, 5)$ is proved in Auroux et al [7]. It would be interesting to know whether the "twisted sectors" in the orbifold quantum cohomology of these weighted projective spaces play any role in the displaceability of moment fibers. Note that in this case there are two kinds of twisted sectors, coming from the two orbifold singularities, and these match the intersection of the "unknown region" with the boundary of the polytope.

Example 1.8. As a final example we consider non-displaceability for the weighted projective line $X/\!\!/G = \mathbb{P}(1, 2)$ with polytope $\Psi(X/\!\!/G) = [0, 1]$ and orbifold singularity the fiber over 1. The fiber over $2/3$ is non-displaceable, by the main Theorem 1.1 and a computation similar to the first example. However, any fiber over $\lambda \in [1/2, 1]$ is non-displaceable in this case, since any displacing flow would have to map the preimage of $[\lambda, 1]$ into the preimage of $[0, \lambda)$. But this is impossible, since any Hamiltonian vector field vanishes at the orbifold point, the pre-image of 1, which is therefore stationary under any Hamiltonian flow.

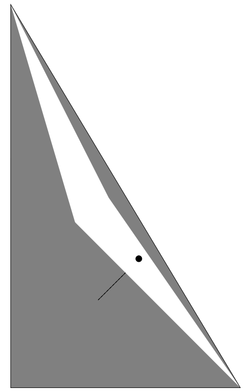

FIGURE 2. Displaceable and non-displaceable fibers for $\mathbb{P}(1,3,5)$.

The idea of studying curves in X instead of $X/\!\!/G$ is not new and fits into a long line of studies of *gauged sigma models* [62], [24], [29], [37]. This approach is the basis of Givental's study of mirror symmetry for complete intersections in toric varieties [24]. Givental's idea was to relate the invariants obtained by integration over moduli spaces of *quasimaps* to those associated to the moduli spaces of stable maps in the symplectic quotient, by algebraic arguments involving localization. By computing the (twisted) quasimap invariants, Givental obtained for example a formula for the descendent Gromov-Witten potential for the quintic three-fold. The space of quasimaps was identified with the moduli spaces of vortices by J. Wehrheim [58]. In the last section we explain how arguments similar to that of Gaio-Salamon [22] lead *heuristically* to relationship with Floer theory in the quotient. (However, as far as non-displaceability goes, there is no need to rigorously prove the correspondence between the two families of invariants, since the quasimap invariants already obstruct displaceability.) One naturally expects an A_∞-morphism from the gauged Fukaya algebra to the Fukaya algebra of the quotient, which we call the *open quantum Kirwan morphism*, and a relation between the *bulk-deformed* potential on $X/\!\!/G$ and the potential for the gauged theory on X. Carrying out this program would require not only the compactness and gluing results above but a proper development of virtual fundamental classes in this setting, which is why the arguments of the last section are only conjectural.

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2. Quasimap Floer cohomology

In this section we explain the definition of quasimap Floer cohomology, which associates to a pair of Lagrangians L_0, L_1 in a symplectic quotient $X/\!\!/G$ an abelian group $HQF(L_0, L_1)$ by counting strips in X with boundary in the pre-images $\tilde{L}_0, \tilde{L}_1$ of L_0, L_1 modulo the action of G. Actually, none of the results of this section will be used for the main Theorem 1.1, for the reason that the proof of Theorem 1.1 uses the cases that the two Lagrangians are equal (covered in Section 3) together with the vanishing of the resulting cohomology when the Lagrangian is displaceable (covered in Section 5). However, it seemed to the author that our Floer cohomology should be introduced before A_∞ algebras for expositional reasons.

2.1. **Symplectic vortices.** The definition of quasimap Floer cohomology is motivated by a gauged version of pseudoholomorphic curves introduced by Mundet and Salamon and collaborators, see for example Cieliebak-Mundet-Gaio-Salamon [13], which we now review. Readers not interested in the origin of quasimap Floer cohomology may skip to the following section, with the caveat that without reading this section the definition of quasimap Floer cohomology may seem rather miraculous. Let G be a compact connected group with Lie algebra $\mathfrak{g}$ and X a Hamiltonian G-manifold with symplectic form $\omega \in \Omega^2(X)$ and proper moment map $\Phi : X \to \mathfrak{g}^\vee$. Let $X/\!\!/G := \Phi^{-1}(0)/G$ denote the symplectic quotient. We assume that the action of G on $\Phi^{-1}(0)$ is locally free, that is, has only finite stabilizers, so that $X/\!\!/G$ is a symplectic orbifold. Furthermore we assume that the action has trivial generic stabilizer.

Let $\mathcal{J}(X)$ denote the set of compatible almost complex structures $J \in \operatorname{End}(TX)$ on X, $\mathcal{J}(X)^G$ the subset of invariants under the action of G by pull-back, and $\mathcal{J}(X/\!\!/G)$ the set of compatible almost complex structures on $X/\!\!/G$. There exists a map $\mathcal{J}(X)^G \to \mathcal{J}(X/\!\!/G)$, obtained by restricting J to $T\Phi^{-1}(0) \cap \mathfrak{g}^\perp \cong \pi^* T(X/\!\!/G)$, where $\pi : \Phi^{-1}(0) \to X/\!\!/G$ is the projection.

Let Σ be a compact Riemann surface. Holomorphic maps from Σ to $X/\!\!/G$ correspond to *gauged holomorphic maps* from Σ to X, as we now explain. Let $P \to \Sigma$ be a principal G-bundle. Denote by $\mathcal{A}(P) \subset \Omega^1(P, \mathfrak{g})^G$ the space of connections on P, and by $\mathcal{G}(P)$ the group of gauge transformations. Any connection $A \in \mathcal{A}(P)$ and $J \in \mathcal{J}(X)^G$ induces an almost complex structure on the associated fiber bundle $P \times_G X$. Let $\overline{\partial}_A$ be the associated Cauchy-Riemann operator, so that if in particular $\Gamma(P \times_G X)$ is the space of sections and $u \in \Gamma(P \times_G X)$ then $\overline{\partial}_A u \in \Omega^{0,1}(\Sigma, u^* T(P \times_G X))$. A *gauged holomorphic map* with bundle P is a pair $(A, u) \in \mathcal{A}(P) \times \Gamma(P \times_G X)$ satisfying $\overline{\partial}_A u = 0$. Let $\mathcal{H}(P, X)$ denote the space of gauged holomorphic maps with bundle P; in general this is a singular subset of $\mathcal{A}(P) \times \Gamma(P \times_G X)$. If J is integrable, then $\mathcal{H}(P, X)$ admits a formal symplectic structure depending on a choice of metric on $\mathfrak{g}$ and area form $\operatorname{Vol}_\Sigma \in \Omega^2(\Sigma)$, given by as follows: The pairing of two tangent vectors $(a_j, v_j) \in \Omega^1(\Sigma, P \times_G \mathfrak{g}) \oplus \Omega^0(\Sigma, u^*(P \times_G TX)), j = 0, 1$ is given by the integral over Σ

$$(a_0, v_0), (a_1, v_1) \mapsto \int_\Sigma (a_0 \wedge a_1) + u^*\omega(v_0, v_1) \operatorname{Vol}_\Sigma$$

where the first term uses the metric on $\mathfrak{g}$. By formal, we mean that each kernel of the linearized operator has a linear symplectic structure given by the above formula, so that where $\mathcal{H}(P, X)$ is smooth it is symplectic. The group $\mathcal{G}(P)$ acts on $\mathcal{H}(P, X)$ preserving the formal symplectic structure and a formal moment map is given by

$$\mathcal{H}(P, X) \to \Omega^2(\Sigma, P \times_G \mathfrak{g}), \quad (A, u) \mapsto F_A + u^*(P \times_G \Phi) \operatorname{Vol}_\Sigma$$

where $\Omega^2(\Sigma, P \times_G \mathfrak{g})$ is identified with a subset of the dual of the Lie algebra $\Omega^0(\Sigma, P \times_G \mathfrak{g})$ of the group of gauge transformations via the pairing given by integration and the metric on $\mathfrak{g}$. The formal symplectic quotient $M(P,X) := \mathcal{H}(P,X)/\!\!/\mathcal{G}(P)$ is the moduli space of *symplectic vortices*

$$M(P,X) = \left\{ \begin{array}{c} (A,u) \in \mathcal{A}(P) \times \Gamma(P \times_G X) \\ \overline{\partial}_A u = 0 \\ F_A + u^*(P \times_G \Phi)\,\mathrm{Vol}_\Sigma = 0 \end{array} \right\} / \mathcal{G}(P).$$

Define $M(\Sigma,X)$ to be the union of $M(P,X)$ over isomorphism classes of bundles P. Note the dependence on the choice of Vol_Σ. In the infinite area limit the second equation becomes $u^*(P \times_G \Phi) = 0$ and $M(\Sigma,X)$ is then the moduli space of holomorphic maps from Σ to $X/\!\!/G$. Indeed any solution descends to a holomorphic map to $X/\!\!/G$. Conversely any holomorphic map to $X/\!\!/G$ defines a pair (A,u), by choosing a connection on the bundle $\Phi^{-1}(0) \to X/\!\!/G$ and taking A to be the pull-back connection. In general one needs to compactify the moduli space in order to define invariants but in certain circumstances the moduli space is already compact, see for example [13]. For example, J. Wehrheim [58] shows that if $\Sigma = \mathbb{P}^1, X = \mathbb{C}^n, G = S^1$ acting diagonally, then $M(\Sigma,X)$ is diffeomorphic to $\bigcup_{d\geq 0} \mathbb{P}^{nd-1}$. Thus the moduli space of symplectic vortices is already compact while the moduli space of maps to $X/\!\!/G = \mathbb{P}^{n-1}$ has a natural compactification, the moduli space of stable maps, whose boundary is complicated.

Frauenfelder's thesis [17] exploited the better compactness properties of the vortex equations to prove a version of the Arnold-Givental conjecture. More precisely, suppose that $L_0, L_1 \subset X$ are compact invariant Lagrangian submanifolds. In good cases Frauenfelder constructed a *gauged Floer cohomology* by counting vortices on $\Sigma := \mathbb{R} \times [0,1]$. Since any bundle over Σ is trivial, a symplectic vortex consists of a pair $A \in \Omega^1(\Sigma,\mathfrak{g})$ of a connection A on $\Sigma := \mathbb{R} \times [0,1]$ together with a map $u : \Sigma \to X$ such that

$$\overline{\partial}_A u = 0, \quad u(s,j) \in L_j, j = 0,1, \forall s \in \mathbb{R}, \quad F_A + u^*\Phi\,\mathrm{Vol}_\Sigma = 0$$

modulo gauge transformation, where in this case $\mathrm{Vol}_\Sigma = \mathrm{d}s \wedge \mathrm{d}t$. The precise details will not concern us here, since we work in a slightly different set-up.

The gauged Floer cohomology (if everything is well-defined) is *independent of the choice of area form* Vol_Σ by a standard continuation argument, similar to the one giving independence of the width of the strip in [59]. (In the case of gauged Gromov-Witten invariants, the dependence on the choice of area form was studied in Gonzalez-Woodward [25].) Therefore, one expects an equivalent theory obtained by setting $\mathrm{Vol}_\Sigma = 0$, the opposite limit from the one related to the Floer cohomology on the quotient. In this case the theory drastically simplifies: the equation $F_A = 0$ implies that A is gauge equivalent to the trivial connection, in which case the other equation becomes the standard Cauchy-Riemann equation. However, since the trivial connection has automorphism group G, the resulting moduli space is that of the usual moduli space of holomorphic strips, modulo the action of G. Since the gauge theory becomes somewhat trivial in this case, we call the resulting Floer cohomology *quasimap Floer cohomology* in cases where it is defined, by analogy with Givental's use of the term *quasimaps*.

2.2. **Holomorphic quasistrips.** Having motivated the study of holomorphic disks modulo a group action, we now develop a Floer theory for Lagrangians that are inverse images of Lagrangians from the symplectic quotient. Let X be a Hamiltonian G-manifold with G acting locally freely on $\Phi^{-1}(0)$. We say that $L \subset X$ is *G-Lagrangian* if $\dim(L) = \dim(X)/2$ and the equivariant symplectic form vanishes on L, that is, the restriction of the symplectic form and moment map vanish on L. We will always assume that the action of G on L is free, so that $\tilde{L}/G$ is contained in the smooth locus of $X/\!\!/G$. The map $\tilde{L} \mapsto \tilde{L}/G$ defines a bijection between G-Lagrangians in X and Lagrangians in $X/\!\!/G$.

This correspondence extends to Lagrangian branes as follows. Suppose that X is equipped with a G-equivariant N-fold Maslov cover $\mathrm{Lag}^N(X) \to \mathrm{Lag}(X)$. One obtains an induced N-fold Maslov cover $\mathrm{Lag}^N(X/\!\!/G)$ on $X/\!\!/G$ by taking the quotient $\mathrm{Lag}^N(X)|\Phi^{-1}(0)/G$ and restricting to lifts of Lagrangian subspaces of $T(X/\!\!/G)$. A *G-Lagrangian brane* is an oriented Lagrangian submanifold equipped with a G-equivariant spin structure, a G-equivariant flat Λ-line bundle, and a G-equivariant grading, that is, a G-equivariant lift of the map $\tilde{L} \to \mathrm{Lag}(X)|\tilde{L}$ to $\mathrm{Lag}^N(X)|\tilde{L}$. There is a one-to-one correspondence between G-Lagrangian branes in X and Lagrangian branes in $X/\!\!/G$, given by $\tilde{L} \mapsto \tilde{L}/G$, the induced orientations and spin structure induced from a choice of orientation and the left invariant spin structure on G, induced by the trivialization $TG \cong G \times \mathfrak{g}$ via the action via right multiplication. In our example, G will be a torus, $N = 2$ and $\mathrm{Lag}^2(X) \to \mathrm{Lag}(X)$ is the double cover given by the choice of orientation. This Maslov cover is G-equivariant since G is connected and so acts trivially on the orientations.

The line bundles in our brane structures will arise as follows. Any flat Λ-line bundle on $L = \tilde{L}/G$ is determined by a *holonomy map* which, since the structure group is abelian, descends to a map on the underlying homology classes:

$$\mathrm{Hol}_L : H_1(L) \to \Lambda - \{0\}.$$

In particular if L is a torus then $H_1(L)$ is torsion-free and any cohomology class $b \in H^1(L, \Lambda_0)$ gives rise to a flat Λ-line bundle with holonomy around a loop representing a homology class $a \in H_1(L)$ is given by the pairing $e^{\langle a,b\rangle} \in \Lambda_0 \subset \Lambda$. (Note that the well-definedness of the exponential requires coefficients in Λ_0.)

The quasimap Floer cochain complex is freely generated by generalized intersections of transversally intersecting Lagrangians in the quotient. Let $L_0, L_1 \subset X/\!\!/G$ be Lagrangian submanifolds, and $\tilde{L}_0, \tilde{L}_1 \subset X$ their G-Lagrangian lifts to X. Let $H \in C^\infty([0,1] \times X)^G$, let $H/\!\!/G \in C^\infty([0,1] \times X/\!\!/G)$ the corresponding family of functions on the symplectic quotient $X/\!\!/G$ and let $(H/\!\!/G)_t^\# \in \mathrm{Vect}(X/\!\!/G), t \in [0,1]$ denote the corresponding Hamiltonian vector fields. Let $\mathcal{I}(L_0, L_1, H)$ denote the set of perturbed intersection points in $X/\!\!/G$,

(3)
$$\mathcal{I}(L_0, L_1, H) = \left\{ x : [0,1] \to X/\!\!/G, x(j) \in L_j, j = 0,1, \quad (\frac{d}{dt}x)(t) = (H/\!\!/G)_t^\#(x(t)) \right\}.$$

Let $\phi_t /\!\!/ G \in \operatorname{Aut}(X /\!\!/ G)$ denote the flow of $(H /\!\!/ G)^{\#}$. We require that $(\phi_1 /\!\!/ G)(L_1) \cap L_0$ is transverse, so that $\mathcal{I}(L_0, L_1, H)$ is finite and the intersection $\tilde{L}_0 \cap \tilde{L}_1$ is *clean*, that is, $T\tilde{L}_0 \cap T\tilde{L}_1 = T(\tilde{L}_0 \cap \tilde{L}_1)$, so that $\tilde{L}_0 \cap \tilde{L}_1$ is a finite union of orbits of G.

The differential in quasimap Floer cohomology is defined by counting holomorphic strips modulo the group action. Let $\Sigma := \mathbb{R} \times [0,1]$. Given a Hamiltonian perturbation $\tilde{H} \in \Omega^1(\Sigma, C^\infty(X))$, we say that a map $\Sigma \to X$ is *$(J, \tilde{H})$-holomorphic* if $\mathrm{d}u - \tilde{H}^{\#}(u)$ is a holomorphic map from $T_z\Sigma \to T_{u(z)}X$, for each $z \in \Sigma$, where $\tilde{H}^{\#} \in \Omega^1(\Sigma, \operatorname{Vect}(X))$ is the Hamiltonian vector field associated to $\tilde{H}$. For any $(J, \tilde{H})$-holomorphic map $u : \Sigma \to X$ the symplectic area resp. energy

$$A(u) := \int_\Sigma u^*\omega, \quad E_{\tilde{H}}(u) := \frac{1}{2}\int_\Sigma |\mathrm{d}u - \tilde{H}^{\#}(u)|^2 \mathrm{d}s \wedge \mathrm{d}t$$

are related by an identity involving the curvature of the connection determined by $\tilde{H}$ [42, 8.1.9]: Let $\tilde{H} = \tilde{H}_s \mathrm{d}s + \tilde{H}_t \mathrm{d}t$. The *curvature* of the connection on $\Sigma \times X$ defined by $\tilde{H}$ is

$$R_{\tilde{H}} = (\partial_s \tilde{H}_t - \partial_t \tilde{H}_s + \{\tilde{H}_s, \tilde{H}_t\}) \mathrm{d}s \wedge \mathrm{d}t.$$

Then the area-energy identity of $(J, \tilde{H})$-holomorphic maps is

$$E_{\tilde{H}}(u) = A(u) + \int_\Sigma R_{\tilde{H}}(u). \tag{4}$$

In particular, as long as the curvature $R_{\tilde{H}}$ is bounded, then any sequence of $(J, \tilde{H})$-holomorphic maps has bounded energy iff it has bounded symplectic area. Given a function $H \in C^\infty([0,1] \times X)$, a (J, H)-holomorphic strip is a $(J, \tilde{H})$-holomorphic strip for $\tilde{H} = H\mathrm{d}t$. For such strip, the energy is equal to the symplectic area, and depends only on its homotopy class.

Definition 2.1. Let $H \in C^\infty([0,1] \times X)^G$. A *$(J, H)$-holomorphic quasistrip* with boundary in $\tilde{L}_0, \tilde{L}_1$ is a (J, H)-holomorphic map $\Sigma \to X$ with boundary in $\tilde{L}_0, \tilde{L}_1$. An *isomorphism* of holomorphic quasistrips u_0, u_1 is an element $g \in G$ and an element $s_0 \in \mathbb{R}$ such that $u_1(s + s_0, t) = g u_0(s, t)$ for all $s \in \mathbb{R}, t \in [0,1]$.

That is, a quasistrip is the same as a strip, except that the notion of isomorphism is different.

Remark 2.2. If X is Kähler (that is, the almost complex structure is integrable) compact and the Hamiltonian H vanishes then then any holomorphic quasistrip defines a holomorphic strip in the quotient $X /\!\!/ G$ as follows. Let $G_{\mathbb{C}}$ be the complexification of G. Since X is compact, the action of G extends to an action of $G_{\mathbb{C}}$. The *semistable locus* X^{ss} of X is the smallest $G_{\mathbb{C}}$-invariant open set containing $\Phi^{-1}(0)$, and is equal to $G_{\mathbb{C}}\Phi^{-1}(0)$ if the action of G on $\Phi^{-1}(0)$ has finite stabilizers, see for example Kirwan [32]. Furthermore X^{ss} is the complement of a finite union of G-stable subvarieties of positive codimension. The composition of u with $X^{\mathrm{ss}} \to X /\!\!/ G$, where defined, defines a map from $\mathbb{R} \times [0,1]$ to $X /\!\!/ G$ on the complement of a finite set, and extends over $\mathbb{R} \times [0,1]$ by removal of singularities. Conversely, any map $v : \mathbb{R} \times [0,1] \to X/\!\!/G$ lifts to a quasistrip, since the holomorphic $G_{\mathbb{C}}$-bundle $v^*(X^{\mathrm{ss}} \to X/\!\!/G)$ is trivial. More generally, a similar discussion holds for non-compact X under the assumption that the G action extends to an action of $G_{\mathbb{C}}$.

Let $M(L_0, L_1; H)$ denote the moduli space of isomorphism classes of (J,H)-holomorphic quasimaps of finite energy with boundary in $\tilde{L}_0, \tilde{L}_1$.

The following lemmas on holomorphic strips with clean intersection Lagrangian boundary conditions were developed jointly with F. Ziltener several years ago, and are probably known to experts.

Lemma 2.3. *Let X be a compact or convex symplectic manifold equipped with a compatible almost complex structure J, and $L_0, L_1 \subset X$ compact Lagrangians intersecting cleanly. Then (i) there exists an open neighborhood U of the intersection $L_0 \cap L_1$ such that any finite energy (J,H)-holomorphic strip $u : \mathbb{R} \times [0,1] \to U$ with boundary in L_0, L_1 is trivial (ii) there exists a constant $\hbar > 0$ such that any (J,H)-holomorphic strip $u : \mathbb{R} \times [0,1] \to X$ with boundary in L_0, L_1 has energy $E(u)$ at least $\hbar$.*

Proof. (i) By the local model for clean intersections [30, Proposition C.3.1], there exists a neighborhood U of $L_0 \cap L_1$ and a strong deformation retract $\psi : [0,1] \times U \to U$ to $L_0 \cap L_1$ preserving L_0, L_1. Using the Cartan homotopy identity, one can construct $\alpha \in \Omega^1(U)$ with $\mathrm{d}\alpha = \omega$ so that α vanishes on L_0, L_1: Let $V_r \in \mathrm{Vect}(U)$ be the time-dependent vector field generating ψ, $V_r = (\mathrm{d}/\mathrm{d}r)\psi(x,r)$. The Poincaré formula

$$\alpha = \int_0^1 \psi_r^* \iota(V_r) \omega \mathrm{d}r$$

produces the required primitive since

$$\mathrm{d}\alpha = \int_0^1 \psi_r^* L_{V_r} \omega \mathrm{d}r = \int_0^1 \frac{d}{dt} \psi_r^* \omega \mathrm{d}r = \psi_1^* \omega - \psi_0^* \omega = \omega.$$

The pull-back $i_j^* L_j$ of α to $L_j, j = 0,1$ is

$$i_j^* \alpha = \int_0^1 \psi_r^* i_j^* \iota(V_r) \omega \mathrm{d}r = \int_0^1 \psi_r^* \iota(V_r) i_j^* \omega \mathrm{d}r = 0$$

since V_r is tangent to L_0, L_1. By Stokes' theorem,

$$E(u) = \lim_{s \to \infty} \int_{[-s,s] \times [0,1]} u^* \omega = \lim_{s \to \infty} \int_{\{s\} \times [0,1]} u^* \alpha - \int_{\{-s\} \times [0,1]} u^* \alpha.$$

The energy of u restricted to $[\pm(s-1), \pm(s+1)] \times [0,1]$ goes to zero as $s \to \infty$, since u is finite energy. It follows by the mean value inequality that $\sup_{t \in [0,1]} |\mathrm{d}u(s,t)| \to 0$ as $s \to \pm\infty$, so u has energy zero and must be trivial. (ii) Suppose otherwise that there exists a sequence u_ν of holomorphic strips with energy $E(u_\nu) \to 0$ but each $E(u_\nu)$ non-zero. A standard argument using compactness shows that there exists a number $\epsilon > 0$ such that any point in x within ϵ of both L_0 and L_1 lies in the open subset U from part (i). By the mean value inequality, for ν sufficiently large the image of u_ν is within distance ϵ of L_0 and L_1 (integrate the derivative over the segments $\{s\} \times [0,t]$ and $\{s\} \times [t,1]$) and so is contained in U. By part (i), u_ν is trivial. Hence $E(u_\nu)$ vanishes, which is a contradiction. □

Remark 2.4. The energy quantization lemma in McDuff-Salamon [42, 4.1.4] does not use a symplectic structure, while the proof above does since it uses the energy-area relation for holomorphic maps.

Lemma 2.5. *Suppose that L_0, L_1 are compact Lagrangians with clean intersection in a symplectic manifold X. There exist constants $\epsilon, \delta, C > 0$ such that if $u : [-S, S] \times [0,1] \to X$ is a holomorphic strip with boundary conditions L_0, L_1 with $E(u) < \epsilon$ then $E(u|_{[-S+s,S-s]\times[0,1]}) < Ce^{-\delta s}E(u)$ and $|\sup \mathrm{d}u|_{[-S+s,S-s]\times[0,1]} < Ce^{-\delta s/2}\sqrt{E(u)}$ for $s \in [1, S]$. Furthermore, if $u : \mathbb{R}\times[0,1] \to X$ is holomorphic with finite energy then u converges exponentially fast to limits $u(\pm\infty, t) \in \mathcal{I}(L_0, L_1)$ as $s \to \pm\infty$: there exist constants $C, \delta > 0$ such that $\mathrm{dist}(u(s,t), u(\pm\infty, t)) < Ce^{\mp\delta s}$ and $|\mathrm{d}u(s,t)| < Ce^{\mp\delta s}$ for $\pm s$ sufficiently large.*

Proof. Pozniak [47, Lemma 3.4.5] proves a relative version of the isoperimetric inequality for the relative action of paths for Lagrangian clean intersections: The *length* of a path $x : [0,1] \to X$ is $\ell(x) = \int_0^1 |\dot{x}| \mathrm{d}t$. for sufficiently small paths $x : [0,1] \to X, 0 \mapsto L_0, 1 \mapsto L_1$ the *relative action* of x is

$$\mathcal{A}_{L_0,L_1}(x) = -\int_{[0,1]^2} u^*\omega$$

where $u : [0,1]^2 \to X$ is a smooth map satisfying $u(0,t) \in L_0 \cap L_1, u(1,t) = x(t)$ for $t \in [0,1]$ and $u(s,i) \in L_i$ for $s \in [0,1], i \in \{0,1\}$ such that for each s, the path $u(s,\cdot)$ has sufficiently small length. (For a precise discussion of what sufficiently small means in the context of vortices, see Ziltener [65].) One then has a *relative isoperimetric inequality*: there exist constants $\delta, C > 0$ such that the following holds. If $x : [0,1] \to X$ is a path satisfying $x(i) \in L_i$, for $i = 0,1$ and $\ell(x) < \delta$ then the action is defined and $|\mathcal{A}_{L_0,L_1}(x)| \le C\|\dot{x}\|_2^2$. Furthermore, after possibly shrinking δ, for every pair $s_- \le s_+$ and every smooth map $u : \Sigma := [s_-, s_+] \times [0,1] \to X$ the following holds. If $u(s,i) \in L_i$, for $i = 0,1$, and $\ell(u(s,\cdot)) < \delta$, for every $s \in [s_-, s_+]$, then the actions of $u(s_-,\cdot)$ and $u(s_+,\cdot)$ are defined and one has an *area-action identity*:

$$\int_\Sigma u^*\omega = -\mathcal{A}_{L_0,L_1}(u(s_+,\cdot)) + \mathcal{A}_{L_0,L_1}(u(s_-,\cdot)).$$

Then the same convexity argument in [42, Lemma 4.7.3] proves the first claim for the energy. Using the mean value inequality one obtains an estimate for the first derivative $\mathrm{d}u$. The final claim follows by restricting u to $\pm[0,S] \times [0,1]$ and taking $S \to \infty$, deriving an estimate on the distance from the estimate on the first derivative. □

Remark 2.6. The constant in exponential decay cannot be chosen arbitrarily close to 1 as in McDuff-Salamon [42]; it depends on the geometry of intersection of the Lagrangians. The lemma also does not hold for non-compact Lagrangians in general, for a similar reason.

Remark 2.7. In the case considered in this paper, an alternative argument is possible: Suppose that X is Kähler and $\tilde{L}_0, \tilde{L}_1$ are Lagrangians in $\Phi^{-1}(0)$ that are inverse images of Lagrangians intersecting transversely in $X/\!\!/G$. Near any point $x \in \tilde{L}_0 \cap \tilde{L}_1$

we may write X holomorphically as the product of an open subset of $X/\!\!/G$ and $G_{\mathbb{C}}$, so that the Lagrangians $\tilde{L}_0$ resp. $\tilde{L}_1$ are the product of the Lagrangians L_0 resp. L_1 and G. Then the exponential decay estimates in Lemma 2.5 are a consequence of the corresponding exponential decay estimates for the transversely intersecting pair L_0, L_1 in $X/\!\!/G$, and for holomorphic strips in $G_{\mathbb{C}}$ with boundary in G.

For any finite energy holomorphic map $u : \Sigma := \mathbb{R} \times [0,1] \to X$ with Lagrangian boundary conditions in $\tilde{L}_0, \tilde{L}_1$, let $\partial_j u$ denote the restriction of u to $\mathbb{R} \times \{j\}$ and for $\alpha > 0$ define a *linearized Cauchy-Riemann operator*

$$D_u : \Omega^0(\Sigma, u^*TX, (\partial_0 u)^* T\tilde{L}_0, (\partial_1 u)^* T\tilde{L}_1)_{1,p,\alpha} \to \Omega^{0,1}(\Sigma, u^*TX)_{0,p,\alpha}, \quad \xi \mapsto \nabla_H^{0,1}\xi - \frac{1}{2}(\nabla_\xi J) J \partial_{J,H} u \tag{5}$$

c.f. McDuff-Salamon [42, p. 258]. The Sobolev spaces above are defined as follows. For integers $k \geq 0$ and $p \geq 1$ and ξ a j-form on Σ with values in u^*TX of class $W^{k,p}_{\text{loc}}$ set

$$\|\xi\|_{k,p,\alpha} = \sum_{i+j \leq k} \|e^{\alpha\gamma(s)s} \nabla_s^i \nabla_t^j \xi\|_p \tag{6}$$

where $\gamma(s) = -1, s < -1$ and $\gamma(s) = 1, s > 1$. For $k \geq 0$ and $p \geq 1$ let $\Omega^j(\mathbb{R} \times [0,1], u^*TX)^{\text{pre}}_{k,p,\alpha}$ denote the space of ξ with finite k, p, α norm. Let $\Omega^0(\mathbb{R} \times [0,1], u^*TX)^{\text{const}}$ be the space of smooth sections that are covariant constant in a neighborhood of infinity. Then

$$\Omega^0(\mathbb{R} \times [0,1], u^*TX)_{1,p,\alpha} := \Omega^0(\mathbb{R} \times [0,1], u^*TX)^{\text{pre}}_{1,p,\alpha} + \Omega^0(\mathbb{R} \times [0,1], u^*TX)^{\text{const}}$$

is the space of sections of class $1,p$ that differ in a neighborhood of infinity from a covariant constant section by a section of class $1, p, \alpha$; a norm on this space in a neighborhood of each end is given by the norm of the limit $\xi(\pm\infty)$ plus the norm of the element $\xi - \xi(\pm\infty)$ of $\Omega^0(\mathbb{R} \times [0,1], u^*TX)^{\text{pre}}_{1,p,\alpha}$ obtained by subtracting off the limit: $\|\xi\|_{1,p,\alpha} = \|\xi\|^{\text{pre}}_{1,p,\alpha} + \|\xi(\infty)\|$ for ξ supported on some $[0,\infty) \times [0,1]$. Patching together these norms with the $1,p$ norm on a compact subset of $\mathbb{R} \times [0,1]$ defines a norm on $\Omega^0(\mathbb{R} \times [0,1], u^*TX)_{1,p,\alpha}$, see e.g. [1, 4.7]. Let

$$\Omega^0(\mathbb{R} \times [0,1], u^*TX, (\partial_0 u)^* T\tilde{L}_0, (\partial_1 u)^* T\tilde{L}_1)_{1,p,\alpha} \subset \Omega^0(\mathbb{R} \times [0,1], u^*TX)_{1,p,\alpha}$$

denote the subspace with boundary values in $T\tilde{L}_0, T\tilde{L}_1$; in particular this means that any element ξ has exponential convergence on the ends to an element of $T\tilde{L}_0 \cap T\tilde{L}_1$. Let

$$\Omega^1(\Sigma, u^*TX)_{0,p,\alpha} := \Omega^1(\Sigma, u^*TX)^{\text{pre}}_{0,p,\alpha}$$

denote the space of one-forms with exponential decay of class $0, p, \alpha$; this space does not contain forms that are constant but non-zero on the ends. Because u has exponential decay, see Lemma 2.5, the map D_u of (5) is well-defined for sufficiently small $\alpha > 0$ and is a Fredholm operator by combining standard estimates for compactly-supported sections with totally real boundary conditions with arguments for manifolds with cylindrical ends as in Lockart-McOwen [36]. See also McDuff-Salamon [42, Section 3.1] and Abouzaid [1] who treats holomorphic strips with equal boundary conditions using weighted Sobolev spaces; the equality of the boundary conditions is only used to obtain exponential decay via removal of singularities, which we have obtained instead via Pozniak's relative isoperimetric inequality in Lemma 2.5. We say that u is *regular* if D_u is surjective.

Let $M^{\mathrm{reg}}(L_0, L_1; H)$ denote the moduli space of isomorphism classes of regular, finite energy J, H-holomorphic quasimaps with boundary in $\tilde{L}_0, \tilde{L}_1$.

Proposition 2.8. *The space $M^{\mathrm{reg}}(L_0, L_1; H)$ is a smooth finite dimensional manifold with tangent space at the isomorphism class $[u]$ of a quasimap u given by $T_{[u]}M^{\mathrm{reg}}(L_0, L_1; H) = \ker(D_u)/(\mathfrak{g} + \mathbb{R})$ for sufficiently small $\alpha > 0$.*

Proof. This is a standard implicit function theorem argument. Consider the map

$$\mathcal{F}_u : \Omega^0(\mathbb{R} \times [0,1], u^*TX, (\partial_0 u)^* T\tilde{L}_0, (\partial_1 u)^* T\tilde{L}_1)_{1,p,\alpha} \to \Omega^{0,1}(\mathbb{R} \times [0,1], u^*TX)_{0,p,\alpha}$$

$$\xi \mapsto \mathcal{T}_u(\xi)^{-1} \overline{\partial}_{J,H} \exp_u \xi$$

where $\mathcal{T}_u(\xi)$ denotes parallel transport along $\exp_u(\xi)$ using the complex-linear modification of the Levi-Civita connection $\tilde{\nabla} = \nabla - \frac{1}{2}(\nabla J)J$, and the exponential map is defined using metrics g_t so that g_j is totally geodesic with respect to $L_j, j = 0, 1$, so that TL_j maps to L_j. This means, however, that g_j is *not* the metric corresponding to the choice of almost complex structure. This difference gives rise to additional quadratic corrections in the map $\mathcal{F}_u$, which are explained in more detail in [60, Remark 2.2], [40, Section 4.3]. Sobolev multiplication and exponential decay estimates for u above imply that $\mathcal{F}_u$ is a smooth map of Banach manifolds. Elliptic regularity [42, B.4.1] implies that any solution ξ to $\mathcal{F}_u(\xi) = 0$ is smooth. (The regularity theorem there only applies to the case of a single Lagrangian, but the same proof holds when a Lagrangian is assigned to each component of the boundary.) If u is regular, the implicit function theorem implies that $\mathcal{F}_u^{-1}(0)$ is a smooth manifold modelled on $\ker D_u$. By the exponential decay Lemma 2.5 for α sufficiently small any nearby solution is of the form $\exp_u(\xi)$ for some $\xi \in \Omega^0(\mathbb{R} \times [0,1], u^*TX, (\partial_0 u)^* T\tilde{L}_0, (\partial_1 u)^* T\tilde{L}_1)_{1,p,\alpha}$, so any nearby holomorphic strip is represented by a point in $\mathcal{F}_u^{-1}(0)$. Since J is G-invariant, G acts by pull-back on the moduli space of holomorphic strips. The action of G is free and proper so the quotient, the moduli space of parametrized quasistrips, is a smooth manifold. The action of $\mathbb{R}$ by reparametrization on the resulting moduli space is also free and proper on the moduli space of non-constant trajectories: if $g_i u(s + s_i, t)$ converges to some $u(s,t)$ for some sequence $s_i \to \infty$ then we must have $u(s,t)$ constant, by exponential decay. It follows that the quotient has a smooth structure with the claimed tangent space. (Note that the image of $\mathbb{R}$ in $\ker(D_u)$ may be trivial, if the trajectory is constant, since it is obtained by differentiating the trajectory in the s direction.) □

To obtain a complex structure for which the moduli space is regular we follow the notation in Frauenfelder [17, p. 49], which itself follows closely Floer-Hofer-Salamon [16]. To set up the notation, for each $\xi \in \mathfrak{g}$ we denote by $\xi_X \in \mathrm{Vect}(X)$ the generating vector field. Let $J : TX \to TX$ be a G-invariant compatible almost complex structure. The infinitesimal action generates vector spaces

$$\mathfrak{g}(x) := \{\xi_X(x), \xi \in \mathfrak{g}\} \subset T_x X, \quad \mathfrak{g}_{\mathbb{C}}(x) = \mathfrak{g}(x) + J_x \mathfrak{g}(x).$$

In the case that the action of G admits an extension to an action of the complexified Lie group $G_{\mathbb{C}}$, the space $\mathfrak{g}_{\mathbb{C}}(x)$ is the subspace generated by the infinitesimal complexified group action. Each tangent space admits a splitting

$$T_x X = \mathfrak{g}_{\mathbb{C}}(x) \oplus \mathfrak{g}_{\mathbb{C}}(x)^{\perp} \tag{7}$$

using the metric on X induced by J; denote by $\pi_x : T_x X \to T_x X$ the projection onto the second factor. At a point $x \in \Phi^{-1}(0)$ we have $\mathfrak{g}_{\mathbb{C}}(x)^{\perp} \cong T_{p(x)}(X /\!\!/ G)$ where $p : \Phi^{-1}(0) \to X /\!\!/ G$ is the projection. Note that in the case of time-dependent J, the space $\mathfrak{g}_{\mathbb{C}}(x)$ as well as the splitting depend on t.

Definition 2.9. (c.f. [17, Definition 4.8]) A point $(s,t) \in \mathbb{R} \times [0,1]$ is *G-regular* for a Floer trajectory $u : \mathbb{R} \times [0,1] \to X$ joining Hamiltonian arcs $x^{\pm}(t)$ if

$$\pi_{u(s,t)} \partial_s u(s,t) \neq 0, \quad u(s,t) \notin G x^{\pm}(t),$$
$$u(s,t) \notin G u(\mathbb{R} - \{s\}, t), \quad G_{u(s,t)} = \{e\}$$

where $e \in G$ is the identity. Denote by $R(u) \subset \mathbb{R} \times [0,1]$ the set of G-regular points.

Theorem 2.10. *(c.f.* [17, Theorem 4.9]*) Let $u : \mathbb{R} \times [0,1] \to X$ be a Floer trajectory for $\tilde{L}_0, \tilde{L}_1$ such that $\pi_{u(s,t)} \partial_s u(s,t) \neq 0$ for some $s,t \in \mathbb{R} \times [0,1]$. Then there exists $S > 0$ such that $\{(s,t) \in R(u) | s \geq S\}$ is open and dense in $\{(s,t) \in \mathbb{R} \times [0,1] \mid s \geq S\}$. In particular, $R(u)$ is open and non-empty.*

Proof. The proof follows from a sequence of lemmas, taken almost word-for-word from Frauenfelder [17, Theorem 4.9].

Remark 2.11. It suffices to consider the case that the Hamiltonian perturbation H vanishes, by an observation in Floer-Hofer-Salamon [16, Discussion following (7)]: There is a bijection between Floer trajectories $u(s,t)$ for a pair (J_t, H_t) with Floer trajectories $\phi_t^{-1} u(s,t)$ for $((\phi_t^{-1})^* J_t, 0)$ where ϕ_t is the flow of $\hat{H}_t$. Indeed $\partial_s((\phi_t^{-1}) u(s,t)) = D\phi_t^{-1} \partial_s u(s,t)$ lies in $D\phi_t^{-1} \mathfrak{g}_{\mathbb{C}}(u(s,t)))$ iff $\partial_s u(s,t)$ lies in $\mathfrak{g}_{\mathbb{C}}(u(s,t)))$. On the other hand, since ϕ_t is G-equivariant, $D\phi_t^{-1} \mathfrak{g}_{\mathbb{C}}(\phi_t(x)) = D\phi_t^{-1} \mathfrak{g}(u(s,t)) + D\phi_t^{-1} J_t \mathfrak{g}(u(s,t)) = \mathfrak{g}(\phi_t^{-1}(u(s,t))) + (D\phi_t^{-1})^* J_t \mathfrak{g}(\phi_t^{-1}(u(s,t)))$.

Lemma 2.12. *$R(u)$ is open.*

Proof. Suppose otherwise. Then there exists a point $(s,t) \in R(u)$ which can be approximated by a sequence $(s_\nu, t_\nu) \notin R(u)$. Then $\pi_{u(s_\nu,t_\nu)} \partial_s u(s_\nu, t_\nu) \neq 0, u(s_\nu, t_\nu) \notin G x^{\pm}(t_\nu)$ and $G_{u(s_\nu,t_\nu)} = \{e\}$ for ν sufficiently large. Since $(s_\nu, t_\nu) \notin R(u)$ there exists a sequence $s'_\nu \in \mathbb{R}$ and $g(s'_\nu, t_\nu) \in G$ such that

$$u(s_\nu, t_\nu) = g(s'_\nu, t_\nu) u(s'_\nu, t_\nu), \quad s'_\nu \neq s_\nu.$$

If the sequence s'_ν is unbounded then, passing to a subsequence if necessary, we may assume $s'_\nu \to \pm\infty$. Then by exponential decay, $u(s'_\nu, t_\nu) \to x^{\pm}$ hence $g(s'_\nu, t_\nu)$ converges to some $h \in G$. Hence $u(s,t) = h x^{\pm}$ which contradicts the fact that $(s,t) \in R(u)$. Hence the sequence s'_ν is bounded and we may assume without loss of generality that $s'_\nu \to s'$. It follows that there exists $g \in G$ such that $u(s,t) = gu(s',t)$. Since $(s,t) \in R(u)$ we must have $s' = s$. Hence s'_ν and s_ν both converge to s and this contradicts the fact that $\pi_{u(s,t)}\partial_s u(s,t) \neq 0$. This proves that the set $R(u)$ is open. $\square$

Next we show that $R(u)$ is not empty, and in fact $R(u) \cap \{|s| \geq S\}$ is dense in $\{|s| \geq S\}$. It follows from the assumption that $\pi_{u(s,t)}\partial_s u(s,t)$ is somewhere non-zero that there exists a non-empty open subset

$$\Sigma := \{(s,t) \in \mathbb{R} \times [0,1],\ \pi_{u(s,t)}(\partial_s u(s,t)) \neq 0, u(s,t) \notin Gx^{\pm}, G_{u(s,t)} = \{e\}\}.$$

It remains to show that there exists $(s,t) \in \Sigma$ such that $u(s,t) \notin Gu(\mathbb{R} - \{s\}, t)$. To do that we show that the set of G-regular points in Σ is dense in Σ. Assume otherwise so that there exists $(s_0,t_0) \in \Sigma$ with $B_\epsilon(s_0,t_0) \cap R(u) = \emptyset$ for some $\epsilon > 0$.

Lemma 2.13. *There exists $\epsilon > 0$ so small and $S > 0$ so large so that the following holds:*

(a) $u(s,t) \notin Gu(B_\epsilon(s_0,t_0))$ *for* $|s| \geq S$ *and* $|t - t_0| \leq \epsilon$*; and*
(b) $\pi_{u(s,t)}(\partial_s u(s,t)) \neq 0$ *for every* $(s,t) \in B_\epsilon(s_0,t_0)$*, the map* $u : B_\epsilon(s_0,t_0) \to X$ *is an embedding and* $u(\overline{B}_\epsilon(s_0,t_0)) \cap gu(\overline{B}_\epsilon(s_0,t_0)) = \emptyset$ *for every* $g \in G \backslash \{e\}$.

Proof. The condition $u(s,t) \notin Gu(B_\epsilon(s_0,t_0))$ for $|s| \geq S$ and $|t-t_0| \leq \epsilon$ is achievable since $u(s_0,t_0) \notin Gx^{\pm}$. The condition $\pi_{u(s,t)}(\partial_s u(s,t)) \neq 0$ is achievable since it is open and $\pi_{u(s_0,t_0)}(\partial_s u(s_0,t_0)) \neq 0$. The latter condition implies that $\mathrm{d}u$ is injective and hence u is an embedding near (s_0,t_0). Finally the condition $\pi_{u(s,t)}(\partial_s u(s,t)) \neq 0$ implies that $u(\cdot,t)$ is transverse to the G-orbits and so $u(\overline{B}_\epsilon(s_0,t_0)) \cap gu(\overline{B}_\epsilon(s_0,t_0)) = \emptyset$ for every $g \in G \backslash \{e\}$ for ϵ sufficiently small. $\square$

The condition $B_\epsilon(s_0,t_0) \cap R(u) = \emptyset$ means that for all $(s,t) \in B_\epsilon(s_0,t_0)$ there exists an $s' \in \mathbb{R}$ such that $u(s,t) \in Gu(s',t)$ and $s' \neq s$.

Lemma 2.14. *There exists a point* $(\tilde{s}_0,\tilde{t}_0) \in B_\epsilon(s_0,t_0)$ *such that the set*

$$C := \{s \in \mathbb{R} : u(s,\tilde{t}_0) \in Gu(\tilde{s}_0,\tilde{t}_0)\}$$

is finite and for every $s \in C$ *we have* $\mathrm{rank}(\mathrm{d}u(s,\tilde{t}_0)) = 2$ *and* $\pi_{u(s,\tilde{t}_0)}(\partial_s u(s,\tilde{t}_0)) \neq 0$.

Proof. Choose $T \subset M$ a G-equivariant tubular neighborhood of $u(s_0,t_0)$ such that G acts freely on T. Define the function

$$\tilde{u} : u^{-1}(T) \to N := T/G, \quad (s,t) \mapsto Gu(s,t).$$

Since $\tilde{u}$ is an immersion at (s_0,t_0), by shrinking ϵ we may assume without loss of generality that $\tilde{u}(B_\epsilon(s_0,t_0)) \subset N$ and there exists an open set $\tilde{u}(B_\epsilon(s_0,t_0)) \subset V \subset N$ and a chart $\phi : V \to \mathbb{R}^m$ such that

$$\phi(\tilde{u}(B_\epsilon(s_0,t_0))) \subset \mathbb{R}^2 \times \{0\} \subset \mathbb{R}^2 \times \mathbb{R}^{m-2} \cong \mathbb{R}^m.$$

Here $m := \dim(X) - \dim(G)$ is the dimension of N. It follows from (a) of Lemma 2.13 that we can assume without loss of generality that there exists a compact set K such that $\tilde{u}^{-1}(V) \subset K \subset \mathbb{R} \times [0,1]$. Abbreviate

$$A := \tilde{u}(B_\epsilon(s_0,t_0)) = \tilde{u}(u^{-1}(u(B_\epsilon(s_0,t_0)))).$$

Let $\rho : \mathbb{R}^m \to \mathbb{R}^m$ be the linear projection on $\mathbb{R}^2 \times \{0\}^{m-2} \subset \mathbb{R}^m$. Define

$$\tilde{V} := \phi^{-1} \circ \rho|_{\phi(V)}^{-1}(\phi(A)) \subset V.$$

Note that $\tilde{V}$ and hence $B := \tilde{u}^{-1}(\tilde{V})$ are open. Moreover, $B \subset K \subset \mathbb{R} \times [0,1]$. We define further a map

$$v : B \to A, \quad v := (\phi|_{B_\epsilon(s_0,t_0)})^{-1} \circ \rho \circ \phi \circ \tilde{u}.$$

Observe that $u^{-1}(u(B_\epsilon(s_0,t_0))) \subset B$. Also, the maps v and $\tilde{u}$ have the same restriction to $u^{-1}(u(B_\epsilon(s_0,t_0)))$. Define the set

$$C(\tilde{u}) := \{z \in B_\epsilon(s_0,t_0) : \#\{\tilde{z} \in B : \tilde{u}(z) = \tilde{u}(\tilde{z})\} = \infty\} \subset B_\epsilon(s_0,t_0).$$

Because B is contained in the compact set K we may write

$$C(\tilde{u}) = \{z \in B_\epsilon(s_0,t_0) : \exists \{z_\nu\}_{\nu=1}^\infty, z_\nu \neq z, z_\nu \to \tilde{z} \in B, \tilde{u}(z_\nu) = \tilde{u}(z) = \tilde{u}(\tilde{z})\}.$$

It follows that $\tilde{u}(C(\tilde{u}))$ is contained in the set of critical points of $\tilde{u}$. Now the formula

$$\mathrm{d}v = (\mathrm{d}\phi)^{-1} \circ \mathrm{d}\rho \circ \mathrm{d}\phi \circ \mathrm{d}\tilde{u}$$

implies $\tilde{u}(C(\tilde{u})) \subset C_v(B)$ where $C_v(B)$ denotes the set of critical values of v in A. By Sard's theorem, the set $(A \backslash \tilde{u}(C(\tilde{u})) \supset (A \backslash C_v(B))$ is dense in A and in particular nonempty. Choose $q \in A \backslash \tilde{u}(C(\tilde{u}))$ and define

$$(\tilde{s}_0, \tilde{t}_0) := \tilde{u}^{-1}(q) \cap B_\epsilon(s_0,t_0).$$

Then $(\tilde{s}_0, \tilde{t}_0)$ has the required properties. This proves the Lemma. □

We investigate the failure of G-regularity in more detail. Fix a point $(s_0,t_0) = (\tilde{s}_0, \tilde{t}_0)$ given by the Lemma. Let $s_1, \ldots, s_N \in [-S, S]$ be the points with $\tilde{u}(s_0,t_0) = \tilde{u}(s_1,t_0) = \ldots = \tilde{u}(s_N,t_0)$. Let F_δ be the set of domain values which fail to be G-injective with one point near (s_0,t_0):

$$F_\delta := \{(s',t) \in \mathbb{R} \times [0,1] \mid \exists (s,t) \in B_{2\delta}(s_0,t_0) \mid \tilde{u}(s,t) = \tilde{u}(s',t)\} \backslash B_{2\delta(s_0,t_0)}.$$

Lemma 2.15. *For every constant $r > 0$ there exists a $\delta > 0$ such that $F_\delta \subset \bigcup_{j=1}^N B_r(s_j,t_0)$.*

Proof. Otherwise, there would exist $\rho > 0$ and a sequence $(s_\nu,t_\nu) \to (s_0,t_0)$ with $s'_\nu \neq s_\nu$ and $\tilde{u}(s_\nu,t_\nu) = \tilde{u}(s'_\nu,t_\nu)$ such that $(s'_\nu,t_\nu) \notin B_\rho(s_j,t_0)$ for every $j \geq 1$. By (b) of Lemma 2.13, there exists $\epsilon' > 0$ such that $|s_\nu - s'_\nu| > \epsilon'$. By (a) of Lemma 2.13, we have $|s'_\nu| \leq S$. Hence the sequence s'_ν has an accumulation point s' which must be distinct from all the points $s_0, \ldots, s_N$ but satisfies $\tilde{u}(s',t_0) = \tilde{u}(s_0,t_0)$. This contradiction proves the claim in the Lemma. □

Locally the set of domain values can be partitioned according to failure of G-regularity as follows. Fix an $r > 0$ and choose δ as in Lemma 2.15. Let Σ_j be the set of domain values for which the value at a point near (s_0,t_0) is repeated near (s_j,t_0), that is,

$$\Sigma_j := \{(s,t) \in \overline{B}_\delta(s_0,t_0) \mid \exists (s',t) \in \overline{B}_r(s_j,t_0), \ \tilde{u}(s',t) = \tilde{u}(s,t)\}$$

for $j = 1, \ldots, N$. The sets Σ_j are closed and $\overline{B}_\delta(s_0,t_0) = \Sigma_1 \cup \ldots \cup \Sigma_N$. Hence at least one of the sets Σ_j has a nonempty interior. Assume without loss of generality that $\operatorname{int}(\Sigma_1) \neq \emptyset$. Choose an open set $U \subset \Sigma_1$ and note that $U \cap B_{2r}(s_1, t_0) = \emptyset$. Furthermore, $u : B_{2r}(s_1, t_0) \to X$ is an embedding for r sufficiently small. Since $u(\cdot, t)$ is transverse to the G-orbits at s_1, we have

$$u(\overline{B}_{2r}(s_1, t)) \cap gu(\overline{B}_{2r}(s_1, t)) = \emptyset, \quad \forall g \in G \backslash \{e\}$$

provided $r > 0$ was chosen sufficiently small. On the other hand it follows from the definition of Σ_1 that for every $(s, t) \in U$ there exists an $s' \in \mathbb{R}$ such that $(s', t) \in B_{2r}(s_1, t_0)$ and $\tilde{u}(s, t) = \tilde{u}(s', t)$.

We show that there exist two regions of the domain on which the map is, up to gauge transformation, related by a diffeomorphism. Fix a point $(\hat{s}_0, \hat{t}_0) \in U$. Let $\sigma := \hat{s}_0' - \hat{s}_0 \neq 0$. Define

$$w : \tilde{W} := B_{2r}(s_1 - \sigma, t_0) \to X, \quad (s, t) \mapsto u(s + \sigma, t)$$

and similarly $\tilde{w} = \tilde{u}(u + \sigma, t)$. Note that $\tilde{w} : \tilde{W} \to N$ is an embedding for r sufficiently small. Moreover, $\tilde{w}(\hat{s}_0, \hat{t}_0) = \tilde{u}(\hat{s}_0, \hat{t}_0)$ and $\tilde{u}(U) \subset \tilde{w}(\tilde{W})$. Define

$$W := \tilde{w}^{-1}(\tilde{u}(U)).$$

Then $\tilde{u}^{-1} \circ \tilde{w} : W \to U$ is a diffeomorphism. Moreover, our assumptions assert that this map takes the form $\tilde{u}^{-1} \circ \tilde{w}(s, t) =: (\kappa(s, t), t)$. Differentiating the formula $\tilde{w}(s, t) = \tilde{u}(\kappa(s, t), t)$ we obtain

$$\begin{aligned} 0 &= \pi_{w(s,t)} \partial_s w(s, t) + J(t, w(s, t)) \pi_{w(s,t)} \partial_t w(s, t) \\ &= \pi_{u(\kappa,t)} (\partial_\kappa u(\kappa, t) \partial_s \kappa) + J(t, u(\kappa, t)) \pi_{u(\kappa,t)} (\partial_\kappa u(\kappa, t) \partial_t \kappa + \partial_t u(\kappa, t)) \\ &= \pi_{u(\kappa,t)} \partial_\kappa u(\kappa, t) \partial_s \kappa + \pi_{u(\kappa,t)} (\partial_t u(\kappa, t) \partial_t \kappa - \partial_\kappa u(\kappa, t)) \\ &= \pi_{u(\kappa,t)} (\partial_\kappa u(\kappa, t)(\partial_s \kappa - 1)) + \pi_{u(\kappa,t)} \partial_t u(\kappa, t) \partial_t \kappa. \end{aligned}$$

Since $\pi_{u(\kappa,t)} \partial_\kappa u(\kappa, t)$ and $\pi_{u(\kappa,t)} \partial_t u(\kappa, t)$ are linearly independent we deduce that $\partial_s \kappa = 1$ and $\partial_t \kappa = 0$. Hence $\kappa(s, t) = s + \hat{\sigma}$ for some $\hat{\sigma} \in \mathbb{R}$. Since $\tilde{u}(\hat{s}_0, \hat{t}_0) = \tilde{w}(\hat{s}_0, \hat{t}_0)$, we obtain $\hat{\sigma} = 0$ and hence $\kappa(s, t) = s$. This implies that $\tilde{w}$ and $\tilde{u}$ agree in the neighborhood $U = W$ of $(\hat{s}_0, \hat{t}_0)$. So $w(s, t) = g(s, t) u(s, t)$ for some $g : U \to G$. By holomorphicity $J(t, u(s, t))(\partial_t g)_X(u(s, t)) = (\partial_s g)_X(u(s, t))$, which is impossible unless g is constant. By unique continuation, $gu(s, t) = u(s + \sigma, t)$ for any s, t. By induction $g^k u(s, t) = u(s + k\sigma, t)$ for any $k \geq 0$. Without loss of generality $\sigma > 0$, and since $u(s + k\sigma, t)$ converges for $s \to \infty$ to x^+ this implies that u is constant in s. This contradiction completes the proof of the Theorem. □

It remains to show that we may assume that any Floer trajectory has somewhere non-trivial differential modulo the complexified infinitesimal group action. For this we use the following modification of a result of Xu [64], which says that for a suitable lift of a Hamiltonian for the symplectic quotient, any Floer trajectory has this property. It suffices to consider the case that the Hamiltonian perturbation on $X /\!/ G$ vanishes.

Theorem 2.16. *Suppose that* $\dim(X /\!/ G) > 0$*,* $L_0, L_1 \subset X /\!/ G$ *are transversally-intersecting Lagrangians and* $(J_t) \in \mathcal{J}(X)^G$ *is an invariant time-dependent almost complex structure. There exists* $H \in C^\infty([0, 1] \times X)^G$ *equal to* 0 *on* $\Phi^{-1}(0)$ *such* *that for every Floer trajectory* $u : \mathbb{R} \times [0,1] \to X$ *for* H*, there exists a point* $(s,t) \in \mathbb{R} \times [0,1]$ *with* $\pi_{u(s,t)} \mathrm{d}u(s,t) \neq 0$.

Proof. Let $H \in C^\infty([0,1] \times X)^G$. Recall that any Floer trajectory may be considered a gradient trajectory for the action functional on the universal cover $\tilde{P}(L_0, L_1)$ of the space of paths $P(L_0, L_1)$ from L_0 to L_1: for $\tilde{x} \in \tilde{P}(L_0, L_1)$ covering x,

$$\mathcal{A}_H(\tilde{x}) = -\int_{[0,1]^2} w^*\omega - \int_{[0,1]} x^* H \mathrm{d}t$$

where $w : [0,1]^2 \to X$ is a homotopy of x to a base point determined by the choice of lift $\tilde{x}$. The Hessian of the action functional is independent of the choice of lift and corresponds to the linear map

$$Q : \Omega^0(x^*TX) \mapsto \Omega^0(x^*TX), \quad \xi \mapsto J_t(\nabla_t \xi - \nabla_\xi \hat{H}_t). \tag{8}$$

According to a result of Robbin-Salamon [48, Theorem B], any Floer trajectory $u(s,t)$ is asymptotic to $\exp(-\lambda s)\xi(t)$ where ξ is an eigenvector of the Hessian (8) with non-zero eigenvalue λ, in the sense that $\xi(t) = \lim_{s \to \infty} e^{\lambda s} u(s,t)$. Furthermore there exist constants $s_0, c > 0$ such that $(1/c)e^{-\lambda s} \leq |\partial_s u| \leq c e^{-\lambda s}$ for all s such that $|s| \geq s_0$.

For a carefully chosen Hamiltonian all of the eigenvectors have the desired property. Let $Gx_1, \ldots, Gx_s = \phi_{1,H}\tilde{L}_0 \cap \tilde{L}_1$ be the finite set of orbits lifting the intersection points $\overline{x}_1, \ldots, \overline{x}_s$ of $\phi_{H,1} L_0 \cap L_1$. The map

$$\mathfrak{g} \times (T_{x_i(0)}(X/\!\!/G) \times \mathfrak{g}) \to X, \quad (\zeta, v, \theta) \mapsto \exp(\zeta) \exp_{x_i}(v + J_t(x_i)\theta_X(x_i)) \tag{9}$$

a local diffeomorphism onto its image on a neighborhood of 0. We let $v_1, \ldots, v_k$ resp. $\theta_1, \ldots, \theta_d$ resp. $\zeta_1, \ldots, \zeta_d$ be the coordinates near x_i arising from (9) with respect to orthonormal bases; these coordinates are invariant under the infinitesimal action of $\mathfrak{g}$ and so extend to G-invariant functions in a neighborhood of Gx_i. Choose cutoff-functions $\rho_1, \ldots, \rho_s \in C^\infty(X)^G$ equal to one on a neighborhood of $Gx_1, \ldots, Gx_s$ and with disjoint support. Let $\exp_{x_i} : T_{x_i}X \to X$ denote geodesic exponentiation.

We first deal with the case that $d := \dim(G) \leq \dim(X/\!\!/G) =: 2k$. Define an invariant time-dependent function F_i in a neighborhood of x_i by

$$F_i(v, \zeta, \theta) = \rho_i \sum_{l=1}^{d} \theta_l v_l.$$

Take the Hamiltonian perturbation to be the sum of the functions above, $H = \sum_{i=1}^s F_i$. We claim that the Hessian for H in (8) has no eigenvectors which map to zero under the projection to the tangent space to the symplectic quotient. Indeed suppose, by way of contradiction, there exists an eigenvector $\xi(t)$ with $\pi_{x_i}\xi(t) = 0$ for all $t \in [0,1]$ with eigenvalue λ. Then ξ has the form $(0, \zeta(t), \theta(t))$ for some functions $\zeta : [0,1] \to \mathfrak{g}, \theta : [0,1] \to \mathfrak{g}$. The eigenvector equation $J_t(\nabla_t \xi - \nabla_\xi \hat{F}_i) = \lambda \xi$ gives

$$J_t\left(\zeta'(t)_X + J_t\theta'(t)_X + [J_t\theta(t)_X, -\hat{F}_i]\right) = \lambda\zeta(t)_X + \lambda J_t\theta(t)_X. \tag{10}$$

Discounting terms that lie in $\mathfrak{g}_{\mathbb{C}}(x_i)$ one obtains that $[J_t\theta(t)_X, -\hat{F}_i]$ vanishes. This implies that $\theta(t)$ vanishes as well, by the explicit choice of F_i above. So $J_t\zeta'(t)_X = \lambda\zeta(t)_X$. Since $\lambda \neq 0$, $\zeta(t)_X$ vanishes. Since the action is locally free near x_i, ζ vanishes as well. This shows that there do not exist eigenvectors with values contained in $\mathfrak{g}_{\mathbb{C}}(x_i)$ for all $t \in [0,1]$.

Now we deal with the case $d > 2k$, in which we must choose the Hamiltonian in a more complicated way. Given a family of matrices $a_{jl}(t)$ for $1 \leq l \leq d, 1 \leq j \leq 2k$, define using the coordinates v, θ, ζ above a function $F_i \in C^\infty([0,1] \times X)^G$ given by

$$F_i(t, v, \theta, \zeta) = \rho_i \sum_{1 \leq l \leq d} \sum_{1 \leq j \leq 2k} a_{jl}(t)\theta_l v_j.$$

Consider the map

$$Q_{i,t} : \mathfrak{g} \to T_{x_i}X, \quad \zeta \mapsto J_t[J_t\zeta_X, \hat{F}_i](x_i). \tag{11}$$

Its image satisfies

$$\operatorname{Im}(Q_{i,t}) \subset T_{\overline{x}_i}(X/\!\!/G) \subset T_{x_i}X \tag{12}$$

in the splitting (7), where $\overline{x}_i$ is the image of x_i in $X/\!\!/G$. The matrix of $Q_{i,t}$ with respect to the given bases of $\mathfrak{g}, T_{\overline{x}_i}(X/\!\!/G)$ is $a_{jl}(t)$. Choose the matrices $(a_{jl}(t))$ so that the kernel of (11) is spanned by $\epsilon_l + \cos(2l\pi t)\epsilon_d,\ l = 1, \ldots, d-2k$. Suppose that $\xi(t) = \zeta(t)_X + J_t\theta(t)_X$ is an eigenvector for the Hessian with non-zero eigenvalue λ. Then by (10) and (12) $\theta(t) \in \operatorname{span}\{\epsilon_l(t) \mid l = 1, \ldots, d-2k\}$. Hence there exist functions θ_l such that

$$\theta(t) = \sum_{l=1}^{d-2k} \theta_l(t)(\epsilon_l + \cos(2\pi lt)\epsilon_d). \tag{13}$$

The eigenvalue equation (10) gives $\zeta'(t) = \lambda\theta(t),\ \theta'(t) = -\lambda\zeta(t)$. Together with the Lagrangian boundary conditions this implies $\theta''(t) = -\lambda^2\theta(t),\ \theta(0) = \theta(1) = 0$. Hence there exist $\alpha \in \mathfrak{g}$ such that $\theta(t) = \alpha\sin(2\pi\lambda t)$. By pairing (13) with ϵ_l we obtain $\theta_l(t) = \langle \alpha, \epsilon_l \rangle \sin(2\pi\lambda t),\ l = 1, \ldots, d-2k$. Pairing (13) with ϵ_d gives

$$\langle \alpha, \epsilon_d \rangle \sin(2\pi\lambda t) = \sum_{l=1}^{d-2k} \langle \alpha, \epsilon_l \rangle \sin(2\pi\lambda t)\cos(2\pi lt). \tag{14}$$

Comparing Fourier coefficients in (14) implies that all coefficients $\langle \alpha, \epsilon_l \rangle$ vanish. It follows that $\theta(t)$ and also $\zeta(t)$ vanish. □

We now show that for a generic choice of almost complex structure, the moduli spaces of Floer trajectories are smooth manifolds. The proof is based on the Sard-Smale theorem. We denote by

$$D_u : \Omega^0(\mathbb{R} \times [0,1], u^*TX; u^*_{\mathbb{R}\times\{0,1\}}TL_0 \sqcup TL_1) \to \Omega^{0,1}(\mathbb{R} \times [0,1], u^*TX)$$

the linearized Cauchy-Riemann operator associated to the almost complex structure J, acting on sections of the pull-back bundle u^*TX with boundary conditions in TL_0, TL_1.

Theorem 2.17. *There exists a comeager subset $\mathcal{J}^* \subset \mathcal{J}$ consisting of those $J \in \mathcal{J}$ such that D_u is surjective for every Floer trajectory u with $\pi_{u(s,t)}\partial_s u$ not identically zero.*

Proof. Let $\mathcal{B} = \mathrm{Map}_{k,p,\alpha}(\mathbb{R} \times [0,1], X, L_0, L_1)$ denote maps locally of Sobolev class k, p, for some $k \geq 1, p > 2$, completed with respect to a metric with exponential decay with weight α with boundary in L_0, L_1. Let $\mathcal{E}$ denote the vector bundle over $\mathcal{B}$ with fiber $\mathcal{E}_u = T^{*,0,1}(\mathbb{R} \times [0,1]) \otimes u^*TX$ and

$$\overline{\partial} : \mathcal{B} \to \mathcal{E}, \quad u \mapsto \overline{\partial}_J u$$

the section given by the Cauchy-Riemann operator. Let $\mathcal{J}^\ell$ denote the space of invariant compatible almost complex structures of class C^ℓ for $\ell \geq 1$ and consider the space

$$\widetilde{\mathcal{MW}}^{\mathrm{univ},\ell}(L_0, L_1) := \{(u, J) \in \mathcal{B} \times \mathcal{J}^\ell : \overline{\partial}_J u = 0\}.$$

The universal moduli space

$$\mathcal{MW}^{\mathrm{univ},\ell}(L_0, L_1) := \widetilde{\mathcal{MW}}^{\mathrm{univ},\ell}(L_0, L_1)/G$$

is a separable Banach manifold of class C^l. To see this, one verifies that the map

$$D_{u,J} : T_u\mathcal{B} \times T_J\mathcal{J}^\ell \to \mathcal{E}_u, \quad (\zeta, Y) \mapsto D_u(\zeta) + Y(u)\partial_t u$$

is surjective. Since D_u is a Fredholm operator, it has closed range and finite dimensional kernel. Hence $D_{u,J}$ has closed range and finite dimensional cokernel and it only remains to prove that its range is dense. To see this, let $\eta \in W^{-k+1,q}_{-\alpha}(\mathbb{R} \times [0,1], u^*TX)$ for $1/p + 1/q = 1$ such that η vanishes on the range of $D_{u,J}$, that is,

$$\int_{\mathbb{R}\times[0,1]} \langle \eta, D_u\zeta \rangle \mathrm{d}s\mathrm{d}t = 0, \quad \int_{\mathbb{R}\times[0,1]} \langle \eta, Y(u)\partial_t u \rangle \mathrm{d}s\mathrm{d}t = 0 \tag{15}$$

for every $\zeta \in T_u\mathcal{B}$ and for every $Y \in T_J\mathcal{J}^\ell$. Equation (15) says that $D_u^*\eta = 0$ where D_u^* is the formal adjoint of D_u. An argument using elliptic regularity shows that $\eta \in W^{\ell+1,p}_{loc}(\mathbb{R} \times [0,1], u^*TX)$. Let $(s,t) \in \mathbb{R} \times [0,1]$ be a G-regular point for u. We claim that $\eta(s,t) = 0$. Otherwise, one could construct as in [16] a $Y \in T_J\mathcal{J}^\ell$ with small support around $(t, u(s,t))$ such that

$$\int_{\mathbb{R}\times[0,1]} \langle \eta, Y(u)\partial_t u \rangle \mathrm{d}s\mathrm{d}t > 0$$

contradicting the second equation in (15). This implies that η vanishes at every G-regular point. Unique continuation for the first order elliptic operator D_u^* implies that $\zeta = 0$. Hence $D_{u,J}$ is onto for every $(u, J) \in \mathcal{B} \times \mathcal{J}^\ell$. It follows from the implicit function theorem that $\mathcal{MW}^{\mathrm{univ},\ell}(L_0, L_1)$ is a Banach manifold of class C^q for any $q > \ell - k$. The projection $p^\ell : \mathcal{MW}^{\mathrm{univ},\ell}(L_0, L_1) \to \mathcal{J}^\ell$ is a Fredholm map since the kernel and cokernel of Dp^ℓ at (u, J) may be identified with that of D_u. By the Sard-Smale theorem, after restricting to maps of bounded index, for ℓ sufficiently large the set $\mathcal{J}^{\ell,*}$ of regular values of p^ℓ is a countable intersection of open and dense sets in $\mathcal{J}^\ell$. Note that $J \in \mathcal{J}^\ell$ is a regular value of p^ℓ exactly if D_u is surjective for every $u \in \mathcal{F}_J^{-1}(0)$. A standard Taubes argument (see [42, Chapter 3]) implies that the space of smooth almost complex structures such that the moduli space is regular is also comeager. □

2.3. **Compactification.** In general, a sequence of quasistrips can have bubbling or breaking behavior, namely disk bubbling with boundary values in $\tilde{L}_0, \tilde{L}_1$, sphere bubbling, and breaking of Floer trajectories. Here we restrict to the case that (X, ω) is *aspherical*, that is, $\int_{S^2} u^*\omega = 0$ for any $u : S^2 \to X$, which rules out sphere bubbling.

Recall that a *nodal disk* Σ is a contractible space obtained from a collection $D_1, \ldots, D_k$ of disks by identifying points in a distinct set $\{\{w_1^+, w_1^-\}, \ldots, \{w_m^+, w_m^-\}\}$ of *nodes* on the boundary. An *isomorphism* of nodal disks Σ, Σ' is a homeomorphism $\Sigma \to \Sigma'$ holomorphic on each disk component. A *nodal strip* is a nodal disk Σ with 2 markings z_+, z_- on the boundary, distinct from each other and the nodal points. For each component between the markings z_- and z_+, the complement of the markings and the nodes is biholomorphic to a strip $\mathbb{R} \times [0, 1]$ uniquely up to translation. We call these components the *strip components* and the other components *disk bubbles*. See Figure 3, where the vertical dotted line represents a node connecting strip components. The *combinatorial type* of a nodal strip is the ribbon tree Γ with two semi-infinite edges corresponding to $z_\pm$, obtained by replacing each disk or strip component with a vertex, each node with a finite edge, and each marking with a semiinfinite edge, with the ribbon structure (cyclic order of edges at each vertex) given by the ordering of the nodes and markings around the boundary of each strip or disk component.

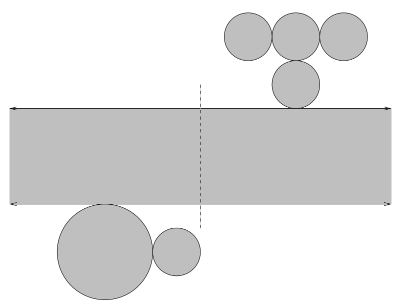

FIGURE 3. A nodal strip

For any nodal strip Σ let $\tilde{\Sigma}$ denote the disjoint union of the components, so that Σ is obtained by identifying pairs of points $w_j^\pm \in \tilde{\Sigma}, j = 1, \ldots, m$. The nodes are of three types: they either connect strip components, disk components, or a disk component to a strip component. If Γ is the combinatorial type of Σ, let $\mathrm{Def}_\Gamma(\Sigma)$ denote the direct sum of the tangent spaces $T_{w_j^\pm}\partial\Sigma$ over nodes that do *not* connect strip components. Each element ζ of $\mathrm{Def}_\Gamma(\Sigma)$ represents a deformation $w_j^\pm(\zeta)$ of the attaching points used to construct Σ, and so gives rise to a deformed nodal strip Σ^ζ for sufficiently small ζ. A *collection of gluing parameters* is an m-tuple $\delta = (\delta_1, \ldots, \delta_m)$ of non-negative real numbers. For any collection δ, we denote by

Σ^δ the nodal strip obtained by removing half-disks in a neighborhood of attaching nodes and gluing together via the map $z_{+,j} \sim \delta_j / z_{-,j}, j = 1, \ldots, m$ where $z_{\pm,j}$ are the coordinates on the half-disks near $w_j^\pm$; if the disks represent strip components then this means that the strips are glued together using a neck of length $-\log(\delta_j)$, each of which we identify with a strip $[-2S_j, 2S_j] \times [0,1]$. The length S_j is determined by radii of the small half-balls $\epsilon_j^+, \epsilon_j^-$ used in gluing and the gluing parameter δ_j by the relation $4S_j = \log(\epsilon_j^- \epsilon_j^+ / \delta_j)$. The *deformation space* $\mathrm{Def}(\Sigma)$ is the sum of the deformations $\mathrm{Def}_\Gamma(\Sigma)$ preserving the combinatorial type and the gluing parameters

$$\mathrm{Def}(\Sigma) := \mathrm{Def}_\Gamma(\Sigma) \times [0, \infty)^m$$

with the last factor representing the deformation parameters. For any sufficiently small $(\zeta, \delta) \in \mathrm{Def}(\Sigma)$ we denote by $\Sigma^{\zeta,\delta}$ the nodal strip obtained by applying the gluing construction with parameters δ to the deformed nodal strip Σ^ζ.

Definition 2.18. A *nodal (J,H)-holomorphic quasistrip* with Lagrangian boundary conditions L_0, L_1 consists of a nodal strip Σ together with a continuous map $u : \Sigma \to X$ with boundary in $\tilde{L}_0$ resp. $\tilde{L}_1$ such that u is (J,H)-holomorphic with finite energy on each strip component and J-holomorphic on any other disk component. An *isomorphism* of nodal holomorphic quasistrips $u_j : \Sigma_j \to X$ is an isomorphism $\phi : \Sigma_0 \to \Sigma_1$ and an element $g \in G$ such that $gu_0 = \phi^* u_1$. A nodal holomorphic quasistrip is *stable* if each constant strip component has at least one nodal point on the boundary, and each constant disk component has at least three nodal points on the boundary, or equivalently, there are no non-trivial automorphisms.

The *energy* of a nodal holomorphic quasistrip is the sum of the energies of the strip components and the energies of the holomorphic disk components. Let $\overline{M}(L_0, L_1; H)$ denote the moduli space of isomorphism classes of stable holomorphic quasistrips of finite energy. There is a natural notion of *Gromov convergence* of a sequence of stable holomorphic quasistrips, generalizing the usual notion for holomorphic maps but incorporating the action of G, which we will not spell out.

One possible set-up which guarantees compactness not only of spaces of holomorphic maps with bounded energy but also of symplectic vortices is that of Cieliebak-Mundet-Gaio-Salamon [13] or Frauenfelder [17]:

Definition 2.19. Let X be a Hamiltonian G-manifold. A *convex structure* on X is a pair (f, J) where $J \in \mathcal{J}(X)^G$ and $f \in C^\infty(X)^G$ satisfies

$$\langle \nabla_\xi \nabla f(x), \xi \rangle \geq 0, \quad \mathrm{d} f(x) J_x \Phi(x)^\#(x) \geq 0 \tag{16}$$

for every $x \in X$ and $\xi \in T_x X$ outside of a compact subset of X.

For example, if X is a Hermitian vector space and G is a torus acting with proper moment map then the standard complex structure J together with the function $f(z) = |z|^2/2$ defines a convex structure. In our situation, it suffices for the first equation in (16) to hold, by a standard argument involving the maximum principle.

Theorem 2.20. *If X is compact or convex and $\tilde{L}_0, \tilde{L}_1 \subset X$ are compact Lagrangian submanifolds with clean intersection, then any sequence of stable holomorphic quasistrips with bounded energy has a convergent subsequence.*

Proof. This is a version of Gromov compactness which combines arguments in McDuff-Salamon [42, Chapter 4] (which proves energy quantization for holomorphic disks with Lagrangian boundary conditions) with energy quantization for holomorphic strips with clean intersection Lagrangian boundary conditions Lemma 2.3 and exponential decay estimates explained above in Lemma 2.5 for Floer trajectories for clean intersection, used to show that bubbles connect. □

With a little more work as in [42], one can show that for any $C > 0$ the subset of the moduli space $\overline{M}(L_0, L_1, H)$ with energy at most C is compact; this requires showing that in the topology defined by Gromov convergence, convergence is equivalent to that of Gromov convergence. However, we will not need this result.

We denote by $M_\Gamma(L_0, L_1; H)$ the stratum of stable holomorphic quasistrips of combinatorial type Γ. Near the equivalence class defined by a map $u : \Sigma \to X$ the space $M_\Gamma(L_0, L_1; H)$ is a quotient of the zero set of the Fredholm map on weighted Sobolev spaces

$$\mathcal{F}_u : \mathrm{Def}_\Gamma(\Sigma) \times \Omega^0(\tilde{\Sigma}, u^*TX, (\partial_0 u)^* T\tilde{L}_0, (\partial_1 u)^* T\tilde{L}_1)_{1,p,\alpha} \to \Omega^{0,1}(\tilde{\Sigma}, u^*TX)_{0,p,\alpha} \oplus \bigoplus_{j=1}^m T_{u(w_j^\pm)} I_{\delta(j)} \tag{17}$$

where $I_{\delta(j)}$ is either $\tilde{L}_0$, for a node attaching to the bottom of the strip, $\tilde{L}_1$, for a node attaching to the top of the strip, or $\tilde{L}_0 \cap \tilde{L}_1$, for a node connecting two strip components, and the map $\mathcal{F}_u$ is given by $(\zeta, \xi) \mapsto \mathcal{T}_u(\xi)^{-1} \overline{\partial}_{J,H} \exp_u \xi$ on the strip components, $(\zeta, \xi) \mapsto \mathcal{T}_u(\xi)^{-1} \overline{\partial}_J \exp_u \xi$ on the disk components and the differences

$$(\zeta, \xi) \mapsto \left(\exp^{-1}_{u(w_j^\pm)} \exp_{u(w_j^+(\zeta))}(\xi(w_j^+(\zeta))) - \exp^{-1}_{u(w_j^\pm)} \exp_{u(w_j^-(\zeta))}(\xi(w_j^-(\zeta))) \right)_{j=1}^m \tag{18}$$

at the nodes, where the position of the nodes $w_j^\pm(\zeta)$ depends on the deformation parameter ζ. Here weighted Sobolev spaces with weight α are used on the strip components and ordinary Sobolev spaces on the disk components. $M_\Gamma(L_0, L_1; H)$ is given near $u : \Sigma \to X$ as the zero set of $\mathcal{F}_u$, quotiented by the action of $G \times \mathrm{Aut}(\Sigma)$. Define the *linearized operator*

$$D_u : \mathrm{Def}_\Gamma(\Sigma) \times \Omega^0(\tilde{\Sigma}, u^*TX, (\partial_0 u)^* T\tilde{L}_0, (\partial_1 u)^* T\tilde{L}_1)_{1,p,\alpha} \to \Omega^{0,1}(\tilde{\Sigma}, u^*TX)_{0,p,\alpha} \oplus \bigoplus_{j=1}^m T_{u(w_j^\pm)} I_{\delta(j)} \tag{19}$$

given by the linearized Cauchy-Riemann operator on the disk components and the differential of the evaluation at the nodes,

$$(\zeta, \xi) \mapsto \prod_{j=1}^m \xi(w_j^+(0)) - \xi(w_j^-(0)) + \frac{d}{dt}|_{t=0} u(w_j^+(t\zeta_j^+)) - \frac{d}{dt}|_{t=0} u(w_j^-(t\zeta_j^-)). \tag{20}$$

We say that u is *regular* if D_u is surjective. Let $M_\Gamma^{\mathrm{reg}}(L_0, L_1; H)$ denote the locus of regular stable holomorphic quasistrips of combinatorial type Γ.

Theorem 2.21. *$M_\Gamma^{\rm reg}(L_0, L_1; H)$ is a smooth manifold with tangent space at $[u]$ isomorphic to the quotient of $\ker(D_u)$ by $\operatorname{aut}(\Sigma) \times \mathfrak{g}$.*

Proof. The zero set $\mathcal{F}_u^{-1}(0)$ is a smooth manifold with tangent space $\ker(D_u)$ at 0 by the implicit function theorem for Banach manifolds. Since G acts freely on $\tilde{L}_0, \tilde{L}_0$ and is compact, it acts properly on $\mathcal{F}_u^{-1}(0)$ and the quotient $\mathcal{F}_u^{-1}(0)/G$ is a smooth manifold. It remains to check that $\operatorname{Aut}(\Sigma)$ acts freely and properly on $\mathcal{F}_u^{-1}(0)/G$. It suffices to consider the case that Σ is a disk with one or two markings, by considering each component separately. (There are no automorphisms permuting components, because there are no automorphisms of a disk permuting the points on the boundary.) The automorphism group of any disk Σ with one resp. two markings on the boundary can naturally be identified with the group of translations and dilations resp. dilations. Suppose that ϕ is a non-trivial automorphism fixing Gu, and let ψ denote its (non-trivial) restriction to the boundary. Since ψ fixes as least one point, for any other point $z \in \partial\Sigma$ the sequence $\phi^n(z)$ converges to some limit z_∞. Then $g^{-n}u(z) = u(\phi^n(z)) \to u(z_\infty)$ for some element $g \in G$. Since G acts freely on $\tilde{L}_k$, g is the identity. If ϕ is a non-trivial translation or dilation and $\phi^* u = u$ then u must be constant, so $\operatorname{Aut}(\Sigma)$ acts freely. Similarly if $g_n \phi_n^* u$ converges to some map v for some sequence $\phi_n \in \operatorname{Aut}(\Sigma)$ of automorphisms going to infinity and some sequence $g_n \in G$, then u, v are constant. This shows that $\operatorname{Aut}(\Sigma)$ acts properly. □

We denote by $M_\Gamma(L_0, L_1; H)_d$ the subset of stable holomorphic quasistrips u with $\operatorname{Ind}(D_u) - \dim(\operatorname{aut}(\Sigma)) - \dim(\mathfrak{g}) = d$. Thus if u is regular, then the $M_\Gamma(L_0, L_1; H)_d$ of stable holomorphic quasistrips with the same combinatorial type Γ as u is a smooth manifold of dimension d in a neighborhood of u, by Theorem 2.21.

Let $M_1(\tilde{L}_j)$ denote the moduli space of holomorphic disks $u : (D, \partial D) \to (X, \tilde{L}_j)$, modulo automorphisms fixing a point $1 \in \partial D$. For any such u we denote by D_u its usual linearized Cauchy-Riemann operator, and by $M_1(\tilde{L}_j)_d$ the subset of $M_1(\tilde{L}_j)_d$ with $\operatorname{Ind}(D_u) = d + 2$. For $k = 0, 1$ we denote by $M_{k,1}(\tilde{L}_0, \tilde{L}_1; H)$ the moduli space of (J, H)-holomorphic strips for $\tilde{L}_0, \tilde{L}_1, H$ with a single marking on the boundary component $\mathbb{R} \times \{k\} \subset \mathbb{R} \times [0, 1]$, and $M_{k,1}(\tilde{L}_0, \tilde{L}_1; H)_d$ the subset whose quotient $M_{k,1}(\tilde{L}_0, \tilde{L}_1; H)_d/G$ has formal dimension d.

Theorem 2.22. *If X is convex aspherical and J, H are such that every non-constant stable holomorphic quasistrip with index 1 or 0 is regular then the one-dimensional component $\overline{M}(L_0, L_1; H)_1$ is a compact one-manifold with boundary given by*

$$\begin{aligned} (21) \quad \partial\overline{M}(L_0, L_1; H)_1 = M(L_0, L_1; H)_0 \times_{\mathcal{I}(L_0, L_1)} M(L_0, L_1; H)_0 \\ \cup \bigcup_{k=0,1} (M_{k,1}(\tilde{L}_0, \tilde{L}_1; H)_0 \times_{\tilde{L}_k} M_1(\tilde{L}_k)_0)/G. \end{aligned}$$

Proof. Except for the quotient by the group action and the clean rather than transversal intersection, this is a standard combination of compactness and gluing theorems, c.f. Oh [43]. Compactness was proved in Theorem 2.20. We sketch the proof of the gluing theorem for clean intersections which follows Fukaya et al [20] and Abouzaid [1], who deal with holomorphic strips with equal Lagrangian boundary conditions $\tilde{L}_0 = \tilde{L}_1$. However, equality of the Lagrangian boundary in these references is only used to obtain exponential decay, which we proved in Lemma 2.5. Let $\delta = (\delta_1, \ldots, \delta_m)$ be a collection of gluing parameters, and Σ^δ the glued strip. We assume in order to simplify notation that all gluing parameters are non-zero. We denote by $\nu_j^\pm : \pm[-\infty, 2S_j] \times [0,1] \to \Sigma$ the embeddings given by the local coordinates near the nodes, so that $\{0\} \times [0,1]$ maps to the half-circle of radius $\sqrt{\delta_j}$. (That is, take the logarithm of the local coordinate and shift by $\log(\delta_j)/2$.) The glued surface is obtained by cutting off $\nu_j^\pm(\pm[2S_j, \infty) \times [0,1])$ and identifying the remaining finite strips. Let $\nu_j : [-2S_j, 2S_j] \times [0,1] \to \Sigma^\delta$ be the coordinates described above on the j-th neck given by this procedure. Using cutoff functions one defines an *approximate trajectory* $G_\delta^{\text{approx}} u : \Sigma \to X$ given by u away from the neck, and on the j-th neck as follows: Let $\rho : \mathbb{R} \to \mathbb{R}$ be a cutoff function with $\rho(s) = 1$ for $s \geq 1$ and $\rho(s) = 0$ for $s \leq 0$, and $\xi_j^\pm \in \Omega^0(\pm[-\infty, 2S_j], u^*TX)$ are defined by requiring $\exp_{u(w_j)^\pm}(\xi_j^\pm) = (\nu_j^\pm)^* u$ on the component containing $w_j^\pm$. Then we define

$$\nu_j^* G_\delta^{\text{approx}} u(s,t) = \exp_{u(w_j)^\pm}(\rho(1-s)\xi_j^-(s,t) + \rho(s+1)\xi_j^+(s,t)).$$

An *approximate right inverse* to $D_{G_\delta^{\text{approx}} u}$ is constructed on Sobolev spaces obtained by weighting the standard Sobolev norm on the neck. More precisely, choose a function $\kappa : \Sigma^\delta \to [1, \infty)$ such that κ is the previously defined weight function (6) outside of the neck and $\kappa(s,t) = \exp((2S_j - |s|)p\alpha)$ on the neck ν_j. Then

$$\|\eta\|_{0,p,\delta,\alpha}^p = \int_{\Sigma^\delta} \kappa \|\eta\|^p$$

defines a δ-dependent norm on $\Omega^{0,1}(\Sigma^\delta, u^*TX)$. Similarly for any $\xi \in \Omega^0(\Sigma^\delta, u^*TX)$ we define a norm as follows: Let $z_j \in \Sigma^\delta$ be the point in the middle of the neck given by $\nu_j(0, 1/2)$. Let ξ_j^{const} denote the section obtained by parallel transport of $\xi(z_j)$ using the Levi-Civita connection from $u(0,t)$ to $u(s,t)$. Let

$$\|\xi\|_{1,p,\delta,\alpha}^p = \|\xi - \xi^{\text{const}}\|_{0,p,\delta,\alpha}^p + \|\nabla(\xi - \xi^{\text{const}})\|_{0,p,\delta,\alpha}^p + \|\xi^{\text{const}}(z)\|^p.$$

Then a patching argument [1, (5.50)] constructs from a one-form η on Σ^δ a zero-form $\xi = T_u^\delta \eta$ which satisfies $\|D_{G_\delta^{\text{approx}} u} T_u^\delta \eta - \eta\|_{0,p,\delta,\alpha} \leq \sum_j e^{-2S_j\alpha} \|\eta\|_{0,p,\delta,\alpha}$. We denote by $\Omega^0(\Sigma, u^*TX, (\partial_0)u^*T\tilde{L}_0, (\partial_1 u)^*T\tilde{L}_1)_{1,p,\alpha,\delta}$ the subset of $\Omega^0(\Sigma, u^*TX)_{1,p,\alpha,\delta}$ with boundary values on $\tilde{L}_0, \tilde{L}_1$. The explicit formula for the patching of the element ξ obtained by applying the right inverse to D_u is in terms of the coordinates given by ν_j

$$\begin{aligned}(22)\quad \nu_j^* \xi^\delta(s,t) = \mathcal{T}\xi(w_j^\pm) + \rho(S_j - s)(\mathcal{T}\xi_-(s,t) - \mathcal{T}\xi(w_j^\pm)) \\ + \rho(s + S_j)(\mathcal{T}\xi_+(s,t) - \mathcal{T}\xi(w_j^\pm))\end{aligned}$$

where $\mathcal{T}$ denotes suitable parallel transports. The proof that it is an approximate right inverse requires uniform quadratic estimates, see [1, p. 89-96], that are the same in this situation as in [1] (except for the additional corrections discussed in [40, Section 4.3]) and we will not reproduce here; these depend on the fact that the weighted Sobolev norms introduced above control the usual Sobolev norms, and these in turn control by C^0 norm by estimates that are uniform in the gluing parameter δ, because the cone angle of the metric on Σ^δ is uniformly bounded from below. Because u decays exponentially on the ends by some constant α_0 given by Lemma 2.5, there exists a constant C so that

$$\|\mathcal{F}_{G_\delta^{\text{approx}}u}(0)\|_{0,p,\delta,\alpha} < Ce^{-S(\alpha_0-\alpha)}.$$

The implicit function theorem then produces for $0 < \alpha < \alpha_0$ and S sufficiently large, a solution (ζ, ξ) to $\mathcal{F}_{G_\delta^{\text{approx}}(u)}(\xi) = 0$ with $\zeta \in \operatorname{Def}(\Sigma^\delta)$ in the image of $\operatorname{Def}_\Gamma(\Sigma)$, and we set $G_\delta(u) = \exp_{G_\delta^{\text{approx}}(u)}(\xi)$. One expects that the map $(\delta, u) \mapsto G_\delta(u)$ is actually injective, but this is not needed: rather one shows that each configuration on the right-hand side of (21) is the limit of a unique one-parameter family of elements of $M(L_0, L_1; H)_1$, using the uniqueness of the solution given by the implicit function theorem. □

Orientations on the moduli spaces of stable quasistrips can be constructed as follows. Recall [20],[61] that a *relative spin structure* on an oriented Lagrangian submanifold $L \subset X$ is a lift of the class of TL defined in the first relative Čech cohomology group for the inclusion $i : L \to X$ with values in $\mathrm{SO}(\dim(L))$ to first relative Čech cohomology with values in $\mathrm{Spin}(\dim(L)))$. The set of such objects naturally forms a category, equivalent to the category of trivializations of the image of the second Stiefel-Whitney class $w_2(TL)$ in the relative cohomology $H^2(X, L; \mathbb{Z}_2)$. In particular, the set of isomorphism classes $\mathrm{Spin}(TL, X)$ of relative spin structures is non-empty iff $w_2(TL) \in H^2(L; \mathbb{Z}_2)$ is in the image of $H^2(X; \mathbb{Z}_2)$, and if non-empty has a faithful transitive action of $H^1(X, L; \mathbb{Z}_2)$. Any relative spin structure on L induces orientations on the moduli spaces of stable holomorphic quasistrips as in, for example, [61], by deforming each linearized operator so that index is identified with a complex vector space plus a tangent space to the Lagrangian; the relative spin structure implies that the resulting orientation is independent of the choice of deformation.

Proposition 2.23. *Relative spin structures on L_0, L_1 induce orientations on the regular parts of the moduli spaces so that the inclusion of the components in* (21) *is orientation preserving for the broken strips, and orientation preserving resp. reversing for bubbles in L_0 resp. L_1.*

The proof is similar to that for nodal disks and strips given in [61], and omitted. We denote by

$$\epsilon : M(L_0, L_1; H)_0 \to \{\pm 1\}$$

the map comparing the constructed orientations with the canonical orientation of a point and by $\epsilon(u)$ its value at $[u] \in M(L_0, L_1; H)_0$.

2.4. **Quasimap Floer cohomology.** For any $x_\pm \in \mathcal{I}(L_0, L_1; H)$ we denote by $\tilde{x}_\pm \subset \operatorname{Map}(I, X)$ the set of trajectories of $H_t^\#$ covering $x_\pm : I \to X/\!\!/G$ (any two lifts are related by the G-action) and by $M(x_+, x_-) := M(x_+, x_-, L_0, L_1; H)$ the subset of $M(L_0, L_1; H)$ the set of (J, H)-holomorphic quasistrips such that

$$(t \mapsto \lim_{s \to \pm\infty} u(s, t)) \in \tilde{x}_\pm.$$

For each $x \in \mathcal{I}(L_0, L_1; H)$ we fix a trivialization of the Λ-line bundle along x and denote by

$$\mathrm{Hol}_{L_0,L_1} : M(L_0, L_1; H) \to \Lambda$$

the product of parallel transport maps, that is, parallel transport in L_0 from x_+ to x_- along $u|_{\mathbb{R}\times\{0\}}$ and then parallel transport from x_- to x_+ along $u|_{\mathbb{R}\times\{1\}}$; we think of this as a holonomy around the loop in $L_0 \cup L_1$, although it is not because of the Hamiltonian perturbation. Let $CQF(L_0, L_1; H)$ the chain complex generated by $\mathcal{I}(L_0, L_1; H)$,

$$CQF(L_0, L_1; H) = \bigoplus_{x \in \mathcal{I}(L_0,L_1;H)} \Lambda <x> .$$

Define

$$\partial_{L_0,L_1,H} <x_+>= \sum_{[u]\in M(x_+,x_-)_0} \epsilon(u)\,\mathrm{Hol}_{L_0,L_1}(u) q^{A(u)} <x_-> .$$

For $k = 0, 1$ we denote by $\mu_0(L_k)$ the count

$$\mu_0(L_k) = \sum_{[u]\in M_1(\tilde{L}_k,\tilde{x}_k)_0} \epsilon(u) q^{A(u)}\,\mathrm{Hol}_{L_k}(u)$$

of holomorphic disks $u : D \to X$ with boundary in $\tilde{L}_k$ passing through a generic point $\tilde{x}_k \in \tilde{L}_k$, where $\epsilon(u) \in \{\pm 1\}$ is the orientation induced by the relative spin structure; assuming every such disk is regular these disks are all Maslov index two.

Theorem 2.24. *(c.f.* [17, Section 5.2]*) Suppose that X is convex aspherical, L_0, L_1, J, H are such that every stable holomorphic quasistrip is regular and every non-trivial holomorphic disk has positive Maslov index. Then $\partial^2_{L_0,L_1,H} = \mu_0(L_1) - \mu_0(L_0)$.*

Proof. By Theorem 2.22, the additivity of the energy, the multiplicativity of the holonomies, and orientations Proposition 2.23. □

If $\mu_0(L_1) = \mu_0(L_0)$ then the *quasimap Floer cohomology* is

$$HQF(L_0, L_1) := HQF(L_0, L_1; H) := H(\partial_{L_0,L_1,H});$$

if L_0, L_1 are equipped with N-Maslov gradings then this is $\mathbb{Z}_{2N}$-graded group. In our situation we will only have $N = 1$, so our Floer cohomologies will be only $\mathbb{Z}_2$-graded.

Theorem 2.25. *In the situation of Theorem 2.24, if J is such that every stable holomorphic disk is regular and has positive Maslov index, and $H_0, H_1 \in C^\infty_c([0, 1] \times X)^G$ are two Hamiltonians so that every stable holomorphic quasistrip is regular, then $HQF(L_0, L_1; H_0)$ is isomorphic to $HQF(L_0, L_1; H_1)$. If $L_1 \cap \phi(L_0)$ is empty for some Hamiltonian diffeomorphism ϕ on $X/\!\!/G$ then $HQF(L_0, L_1; H)$ is trivial for any H.*

Proof. The first statement is a standard continuation argument counting $(J, \tilde{H})$-holomorphic strips where $\tilde{H}$ is a generic homotopy between $H_0 \mathrm{d}t$ and $H_1 \mathrm{d}t$. Since we give a more general argument in the A_∞ setting in Section 5, we omit the proof. To prove the second statement, let $H_0 \in C^\infty_c([0, 1] \times X/\!\!/G)$ be a time-dependent Hamiltonian whose flow ϕ satisfies $L_1 \cap \phi(L_0) = \emptyset$. If $H \in C^\infty_c([0, 1] \times X)^G$ is any lift of H_0 then we have $\mathcal{I}(L_0, L_1; H) = \emptyset$ hence $HQF(L_0, L_1; H)$ vanishes. □

The homotopy argument works with Λ coefficients but not Λ_0-coefficients: because of the energy-area identity (4), the symplectic area of a $(J, \tilde{H})$-strip is possibly negative. That is, quasimap Floer cohomology, as well as Floer cohomology, is defined using Λ_0 coefficients but as such is not an invariant of Hamiltonian isotopy.

Notice that we have not said anything about independence from J. There is probably no hope of achieving regularity for an arbitrary family of almost complex structures, and so the independence from J falls outside of the techniques considered in this paper.

3. Quasimap A_∞ algebra for a Lagrangian

In order to understand the structure of the Floer cohomology groups defined in the previous section, it is helpful to understand how they arise via A_∞ algebras and bimodules. This will take the next several sections; in this section we use a standard trick which combines Morse theory and Floer cohomology via *treed disks*, and which in particular gives an A_∞ algebra associated to a Lagrangian without using Kuranishi structures if every stable holomorphic disk is regular and positive Maslov index. While this is well-known (see for example Seidel [53]) there is unfortunately no complete writeup in the literature, and the version we need replaces disks in the quotient with quasidisks. We begin with some generalities about A_∞ algebra.

3.1. A_∞ **algebras.** An A_∞ *algebra* over Λ consists of a $\mathbb{Z}$-graded Λ-module A and a sequence of *higher composition maps* $(\mu_n : A^{\otimes n} \to A[2-n])_{n \geq 0}$ satisfying the A_∞ *associativity relation*

$$0 = \sum_{i+j \leq n} (-1)^{\aleph} \mu_{n-i+1}(a_1, \ldots, a_j, \mu_i(a_{j+1}, \ldots, a_{j+i}), a_{j+i+1}, \ldots, a_n) \tag{23}$$

where $a_1, \ldots, a_n$ are homogeneous elements of degree $|a_1|, \ldots, |a_n|$,

$$\aleph = \sum_{k=1}^{j} (|a_k| - 1) \tag{24}$$

is the sum of the *reduced degrees* of the elements to the left of the inner operation, see Seidel [55], and we adopt the standard convention of writing commas instead of tensor products to save space. The map $\mu_0 : \Lambda \to A$ is the *curvature* of A and is determined by a single element $\mu_0(1) \in \Lambda$. If $\mu_0 = 0$ then A is *flat*. The signs in the A_∞ associativity relation only depend on the induced $\mathbb{Z}_2$-grading, and this means that one can replace the assumption of a $\mathbb{Z}$-grading with a $\mathbb{Z}_2$-grading, which will be the case in our example. We also work with A_∞ algebras defined over the Novikov ring Λ_0, which means that A is a Λ_0-module and the maps μ_n are Λ_0-module morphisms.

A *morphism* of A_∞ algebras $(A_1, (\mu_{1,i})), (A_2, (\mu_{2,i}))$ is a collection $(\phi_n : A_1^{\otimes n} \to A_2)_{n \geq 0}$ satisfying a relation

$$0 = \sum_{i,j} (-1)^{\aleph} \phi_{n-j+1}(a_1, \ldots, a_i, \mu_{j,1}(a_{i+1}, \ldots, a_{i+j}), a_{i+j+1}, \ldots, a_n) + \sum_{i_1, \ldots, i_r} \mu_{r,2}(\phi_{i_1}(a_1, \ldots, a_{i_1}), \ldots, \phi_{i_r}(a_{i_1 + \ldots + i_{r-1}+1}, \ldots, a_n)). \tag{25}$$

The map ϕ_0 is the *curvature* of the morphism; if $\phi_0 = 0$ then the morphism is *flat*.

3.2. **Treed disks.** Stasheff's *associahedron* K_n is a cell complex whose vertices correspond to total bracketings of n variables $x_1, \ldots, x_n$ [57]. For example, K_3 is an interval, with vertices corresponding to the expressions $(x_1x_2)x_3$ and $x_1(x_2x_3)$. The associahedron has several geometric realizations. The first, well-known description is as the moduli space of stable marked nodal disks. A nodal disk is *stable* if it has no automorphisms, or equivalently, each disk component has at least three nodal or marked points. Let $\overline{M}_n$ denote the moduli space of stable nodal $n+1$-marked disks.

A second and older description of the associahedron due to Boardman-Vogt [9] is the moduli space of stable ribbon metric trees: A *ribbon metric tree* is a ribbon tree

$$\Gamma = (V(\Gamma), E(\Gamma), O(\Gamma))$$

consisting of a ribbon tree $(V(\Gamma), E(\Gamma))$ equipped with a labelling of the semiinfinite edges by integers $0, \ldots, n$, a cyclic ordering $O(\Gamma)$ of the edges at each vertex together with a *metric* $l : E(\Gamma) \to [0, \infty]$. The semiinfinite edges are required to have infinite length. Two metric trees are *isomorphic* if the ribbon metric trees obtained by collapsing all edges of length zero are isomorphic, that is, there is a bijection between the vertex sets preserving the edges, the cyclic orderings, and their lengths. Equivalently, a metric tree can be taken to be a tree with a metric on each one-dimensional segment, and an isomorphism of metric trees is required to preserve the metric on each edge. A metric ribbon tree is *stable* if each vertex has valence at least two, any edge containing a vertex of valence two has infinite length, and any edge of infinite length contains at least one vertex of valence three. (This is a somewhat obscure way of saying that we allow an edge of finite length to degenerate to a broken edge of infinite length; the standard method would just be to allow edges of infinite length but this convention would cause notational problems when we talk about gradient trees later on.) There is a natural topology on the set of isomorphism classes $\overline{W}_n$ of stable ribbon metric trees with $n+1$ leaves, which allows each edge with length approaching infinite to degenerate to a broken edge, that is, a pair of infinite length edges joined by a vertex. We refer to [39] for a review. The edges of each tree are oriented so that the positive orientation is in the direction of the zero-th semiinfinite edge.

Both $\overline{W}_n$ and $\overline{M}_n$ have natural structures as manifolds with corners, isomorphic as such to Stasheff's associahedron K_n. In the case of $\overline{M}_n$ the boundary components have interiors consisting of configurations with exactly two components containing prescribed markings. In the case of $\overline{W}_n$ the boundary components have interiors whose trees have a single edge of infinite length. The cell structure of $\overline{M}_n$ is identical to that of K_n while the cell structure of $\overline{W}_n$ is not.

The following definition combines the two constructions: A *treed disk* is a collection of a collection $D_1, \ldots, D_k$ of holomorphic disks, segments-with-metric S_i which for simplicity we take to be intervals $S_i = [\epsilon_i^-, \epsilon_i^+], i = 1, \ldots, l$ where $\epsilon_i^- < \epsilon_i^+$ and $\epsilon_i^\pm \in [-\infty, \infty]$, together with a collection

$$\{\{w_1^+, w_1^-\}, \ldots, \{w_m^+, w_m^-\}\}$$

of *nodes* in the disjoint union $\partial D_1 \sqcup \ldots \sqcup \partial D_k \sqcup \partial S_1 \sqcup \ldots \sqcup \partial S_l$ such that the set $w_1^+, \ldots, w_m^-$ is distinct and *markings* $z_1, \ldots, z_n$ contained in $\partial S_1 \sqcup \ldots \sqcup \partial S_l$ disjoint from the nodes. Gluing the nodes gives a topological space

$$\Sigma := (D_1 \sqcup \ldots \sqcup D_k \sqcup S_1 \sqcup \ldots \sqcup S_l) / (w_j^+ \sim w_j^-)_{j=1}^m$$

which is required to be contractible. The ordering of the markings is required to agree with the ordering of the leaves of the underlying ribbon tree, that is, the tree obtained by collapsing the disks to vertices and taking the ribbon structure induced from the ordering of the nodes on the boundary of the disks. We say that a node $w_j^\pm = \epsilon_j^\pm$ contained in the boundary of $S_j = [\epsilon_j^-, \epsilon_j^+]$ is *infinite* if $\epsilon_j^\pm = \pm\infty$, and *finite* otherwise. An *isomorphism* of treed nodal disks Σ, Σ' is a homeomorphism $\Sigma \to \Sigma'$ holomorphic on each disk component and preserving the metric on each line segment, mapping the markings $z_1, \ldots, z_n$ to $z_1', \ldots, z_n'$. A treed nodal disk is *stable* if it has no automorphisms, or equivalently, each disk component has at least three nodal or marked points, and in addition each node connecting two line segments is infinite. More explicitly, this means that each disk component has at least three nodal points, and each sequence of line segments connecting two disk components is a *broken line segment*: all nodes of finite type attach to disk components. Let $\overline{MW}_n$ denote the moduli space of connected stable tree disks with $n+1$ semiinfinite edges. See Figure 4 for the example $n = 3$. The combinatorial type

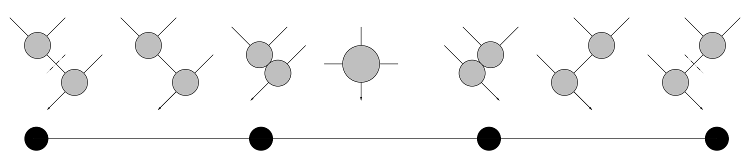

FIGURE 4. Moduli space of stable treed disks

of a treed disk is a ribbon tree $\Gamma = (V(\Gamma), E(\Gamma), O(\Gamma))$ with a partition of the set of vertices $V(\Gamma)$ into two subsets $V^1(\Gamma), V^2(\Gamma)$, corresponding to the two-dimensional and one-dimensional components being glued together. We denote by $MW_{\Gamma,n}$ the subset of $\overline{MW}_n$ of combinatorial type Γ. A codimension one stratum $MW_{\Gamma,n}$ is a *true boundary component* if it consists of points in the topological boundary of $\overline{MW}_n$, considered as a topological manifold with boundary, and is a *fake boundary component* otherwise. For any (not necessarily stable) treed disk Σ we denote by $\tilde{\Sigma}$ denote the disjoint union of the disk components of Σ, together with a copy of the real line for each one-dimensional component of Σ. Denote by $\mathrm{Def}_\Gamma(\Sigma)$ the space of infinitesimal deformations of Σ preserving the combinatorial type, that is, the direct sum of the tangent spaces of the boundaries of the disks or tangent spaces to the lines at the nodes and markings (representing deformations of the attaching points and markings)

$$\mathrm{Def}_\Gamma(\Sigma) = \bigoplus_{j=1,\pm}^{m} T_{w_i^\pm}\partial\tilde{\Sigma} \oplus \bigoplus_{j=1}^{n} T_{z_i}\partial\tilde{\Sigma} \tag{26}$$

where in the case of one-dimensional components we define the boundary to be the entire component. Over a neighborhood of 0 in $\mathrm{Def}_\Gamma(\Sigma)$ we have a family of treed disks, given by varying the attaching points using a collection of exponential maps $T_{w_i^\pm}\partial\tilde{\Sigma} \to \partial\Sigma$ defined using a metric on $\partial\tilde{\Sigma}$. This is a smoothly trivial family in the sense that the fibers are all smoothly diffeomorphic, and any choice of identification identifies the deformations with deformations of the complex structure on the disk components and deformations of the metric on the line segments, but our construction of the gluing map will not use such a trivialization. Let $\mathrm{Aut}(\Sigma) = \mathrm{Aut}(\tilde{\Sigma})$ denote the product of the automorphism groups $SL(2,\mathbb{C})$ for each disk component together the automorphism groups $\mathbb{R}$ of each one-dimensional component, and $\mathrm{aut}(\Sigma)$ the Lie algebra of $\mathrm{Aut}(\Sigma)$. The action of $\mathrm{Aut}(\Sigma)$ on $\tilde{\Sigma}$ induces a map $\mathrm{aut}(\Sigma) \to \mathrm{Def}_\Gamma(\Sigma)$. If Σ is stable, then a neighborhood of $[\Sigma]$ in $MW_{\Gamma,n}$ is homeomorphic to a neighborhood of 0 in $\mathrm{Def}_\Gamma(\Sigma)/\mathrm{aut}(\Sigma)$. The full deformation space is obtained by taking the direct sum

$$\mathrm{Def}(\Sigma) = \mathrm{Def}_\Gamma(\Sigma) \oplus ((-\infty,0) \cup \{0\} \cup (0,\infty))^z \oplus [0,\infty)^i$$

where z is the number of edges of zero length (that is, nodes connecting disks) and i is the number of infinite nodes connecting segments of infinite length. A neighborhood of $[\Sigma]$ in $\overline{MW}_n$ is isomorphic as a cell complex to a neighborhood of 0 in $\mathrm{Def}(\Sigma)/\mathrm{aut}(\Sigma)$, by a map $(\zeta,\delta) \mapsto [\Sigma^{\zeta,\delta}]$ obtained by combining the deformations above with, in the case of a negative gluing parameter associated to an edge of zero length, carrying out the gluing procedure for disks mentioned above, see e.g. [39]. Thus in particular the true boundary components of $\overline{MW}_n$ correspond to configurations with an edge of infinite length.

$\overline{MW}_n$ admits *two* natural forgetful maps, one which forgets the disk components $f_M : \overline{MW}_n \to \overline{W}_n$ and takes the ribbon structure to be the one induced from the ordering of the points on the boundaries of the disks, and one that forgets the edges $f_W : \overline{MW}_n \to \overline{M}_n$. The fiber of f_M over each stratum $M_{n,\Gamma}$ is a product of intervals $[0,\infty]$ corresponding to the lengths of the edges, while the fiber over a stratum $W_{n,\Gamma}$ is a product of associahedra corresponding to the valences at the vertices. The product $f_M \times f_W : \overline{MW}_n \to \overline{M}_n \times \overline{W}_n$ defines an injection into a product of compact spaces, so that $\overline{MW}_n$ has a natural topology for which it is compact. With respect to this topology each stratum $MW_{\Gamma,n}$ has a neighborhood in $\overline{MW}_n$ homeomorphic to the product of $MW_{\Gamma,n}$ with a product of intervals $[0,\infty)$, one for each node connecting segments of infinite length, together with a product of intervals $(-\infty,\infty)$, one for each node connecting disk components. From the first forgetful map one sees that $\overline{MW}_n$ is again homeomorphic to Stasheff's associahedron, but with a more refined cell structure.

The strata of $\overline{MW}_n$ can be oriented as follows. First, M_n is oriented via the identification with $\{0 < z_2 < z_2 < \ldots < z_{n-1} < 1\} \subset \mathbb{R}^{n-2}$ which maps the first marked point to ∞, the second to 0, the last to 1, and the remaining points to the interval $(0,1) \cong \mathbb{R}$. The orientations of the boundary strata can be determined as follows. Any stratum $M_{\Gamma,n} \subset \overline{M}_n$ corresponding to a tree with m nodes has a neighborhood homeomorphic to a neighborhood of $M_{\Gamma,n}$ in $M_{\Gamma,n} \times [0,\infty)^m$. In the case $m = 1$ so that $M_{\Gamma,n} \cong M_i \times M_{n-i+1}$ for some i, this homeomorphism has an inverse given by a *gluing map* of the form

$$\begin{aligned}(27)\quad & \mathbb{R} \times M_i \times M_{n-i+1}, (\delta, (w_2, \ldots, w_{i-1}), (z_2, \ldots, z_{n-i})) \to \\ & \qquad (z_2, \ldots, z_j, z_j + \delta w_2, \ldots, z_j + \delta w_{i-1}, z_j + \delta, z_{j+1}, \ldots, z_{n-i}).\end{aligned}$$

It follows that the inclusion $M_i \times M_{n-i+1} \to M_n$ has orientation sign $ij + 1 - j - i$. Next, each stratum $MW_{\Gamma,n}$ corresponding to a combinatorial type Γ of marked stable disks is isomorphic to

$$M_{\Gamma,n} \times \prod_{e \in E_{<\infty}(\Gamma)} \mathbb{R} \tag{28}$$

where $M_{\Gamma,n}$ is the corresponding stratum of $\overline{M}_n$ and $E_{<\infty}(\Gamma)$ is the set of finite edges. Here the order is taken first by distance to the root edge of Γ, and then by cyclic order determined by the ribbon structure on Γ. Thus $MW_{\Gamma,n}$ inherits an orientation from that on $M_{\Gamma,n}$, which in turn is induced by an orientation on $\overline{M}_n$ induced from that on M_n. On the other hand, the boundary of $\overline{MW}_n$ may be identified with the union over types Γ of trees with one infinite edge of the product $\overline{MW}_{n-i+1} \times \overline{MW}_i$ where $n+1$ is the number of semi-infinite edges of the component tree not containing the root. Thus

$$\partial \overline{MW}_n \cong \bigcup_{i+j \leq n+1} MW_{n-i+1} \times MW_i$$

where the sum is over edges i of the tree with $n - i + 1$ leaves other than the root. Then by construction, the inclusion of $MW_{n-i+1} \times MW_i$ has sign that of the inclusion of $M_{n-i+1} \times M_n$.

Over $\overline{MW}_n$ there is a *universal treed disk* $\overline{UMW}_n \to \overline{MW}_n$ with the property that the fiber over $[\Sigma]$ is isomorphic to the treed disk Σ. This is again a cell complex, with cells indexed by pairs consisting of a combinatorial type of tree disk together with a cell in the corresponding treed disk.

3.3. **Unstable trees.** Standard perturbation schemes for Fukaya categories, as in Seidel [52], use systems of perturbations depending on the underlying stable disk. We need a different perturbation scheme which depends on the underlying (possibly unstable) tree, similar to that used for the construction of the Morse A_∞ category via perturbations in e.g. Abouzaid [2, Section 2.2], which has non-vanishing perturbations on edges that would normally be collapsed by the forgetful morphism to the moduli space of stable trees. This section may be skipped at first reading.

Let $\overline{W}_{n,v}$ denote the moduli space of connected rooted metric ribbon trees with $n+1$ leaves, at least one vertex, and at most v vertices, and edges of possibly zero or infinite length, where equivalence is generated by collapsing edges of zero length. Given a fixed underlying ribbon tree Γ, the corresponding stratum $W_{n,v,\Gamma}$ is homeomorphic to a product of intervals $(0,\infty)$ describing the lengths of the edges, and so the decomposition

$$\overline{W}_{n,v} = \bigcup_{\Gamma} W_{n,v,\Gamma}$$

gives $\overline{W}_{n,v}$ the structure of a finite cell complex, hence a compact Hausdorff space. A neighborhood of $W_{n,v,\Gamma}$ is homeomorphic to a neighborhood of $W_{n,v,\Gamma} \times \{0\}$ in the product of $W_{n,v,\Gamma}$ with a subset of $\prod_{v \in V(\Gamma)} W_{n(v),v}$ with at most v vertices in total, where $n(v)$ is the valence of v; the homeomorphism is obtained by replacing each vertex v by the corresponding tree. Over $\overline{W}_{n,v}$ there is a *universal tree* $\overline{UW}_{n,v} \to \overline{W}_{n,v}$ with the property that the fiber over $[\Sigma]$ is isomorphic to the treed disk Σ. The moduli space $\overline{W}_{1,3}$ is the square with edges identified shown in Figure 5.

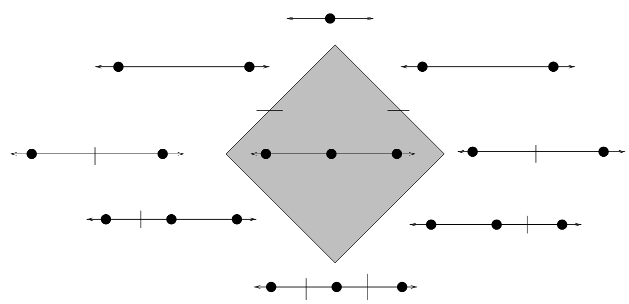

FIGURE 5. Strata of $\overline{W}_{1,3}$

Note that $\overline{W}_{1,3}$ is not combinatorially a manifold with corners.

3.4. **Holomorphic treed quasidisks.** Let X be a Hamiltonian G-manifold equipped with a compatible almost complex structure $J \in \mathcal{J}(X)^G$, as above. Let $L \subset X /\!\!/ G$ be a compact oriented Lagrangian submanifold equipped with a brane structure, that is, a relative spin structure and grading. Let $(B : TL^{\otimes 2} \to \mathbb{R}, F : L \to \mathbb{R})$ be a Morse-Smale pair, that is, B is a Riemannian metric, F is Morse, and for each pair of critical points $x_\pm \in \mathrm{crit}(F)$, the intersection of the stable resp. unstable manifolds $W^+(x_+) \cap W^-(x_-)$ is transverse. In particular, the moduli space of gradient trajectories from x_- to x_+ is naturally isomorphic to $(W^+(x_+) \cap W^-(x_-))/\mathbb{R}$, which is a smooth finite-dimensional manifold. The stable resp. unstable manifolds $W^\pm(x)$ of $x \in \mathrm{crit}(F)$ are diffeomorphic to the positive resp. negative parts $T_x^\pm L$ of the tangent space $T_x L$ with respect to the Hessian of F; we denote by $I(x) = \dim(T_x^- L)$ the *index* of x so that the dimension of the space of gradient trajectories $(W^+(x_+) \cap W^-(x_-))/\mathbb{R}$ is $I(x_+) - I(x_-) - 1$. For simplicity, we assume that L is connected and there is a unique critical point $x_{\min} \in \mathrm{crit}(F)$ of index 0. In addition, we will need to lift gradient trajectories of F in L to the pre-image $\tilde{L}$

in X. For this purpose we assume that we have fixed a G-invariant metric on $\tilde{L}$ so that the induced metric on F is obtained from the quotient $F = \tilde{F}/G$. We may then identify the gradient trajectories of F on L with those of its G-invariant lift $\tilde{F}$ on $\tilde{L}$, up to the action of G.

Definition 3.1. A *holomorphic treed quasidisk* for L consists of a treed disk Σ together with a map $u : \Sigma \to X$ such that on each disk component in Σ, u is holomorphic and maps the boundary to $\tilde{L}$, and on each edge in Σ, u is a gradient flow line of the function $\tilde{F} : \tilde{L} \to \mathbb{R}$. An *isomorphism* of treed quasidisks $u_j : \Sigma_j \to X, j = 0, 1$ consists of an isomorphism $\phi : \Sigma_0 \to \Sigma_1$ and an element $g \in G$ such that $\phi^* u_1 = g u_0$. A holomorphic treed quasidisk is *stable* if u has no automorphisms (equivalently, any disk on which u is constant has at least three marked or nodal points and u is non-constant on each segment of each broken line) and if two line segments are joined at a node, then the node maps to a critical point of F. (This means that any node of finite type is an attaching point to a disk.)

Because the semiinfinite edges have infinite length, the limit of u along the j-th semiinfinite edge maps to a critical point $x_j \in \mathrm{crit}(F)$ under the projection $\tilde{L} \to L$. A picture of a holomorphic treed disk is shown in Figure 6.

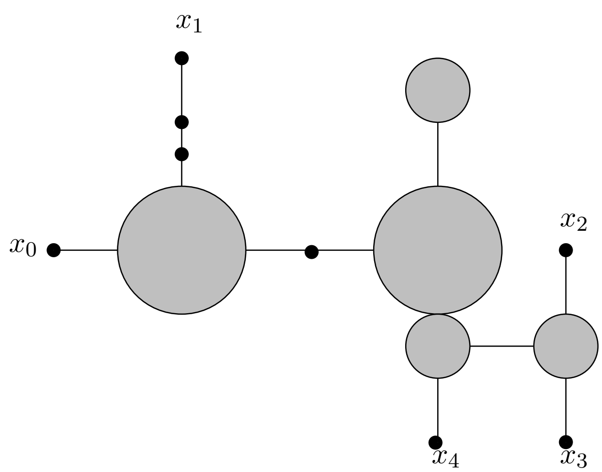

FIGURE 6. A holomorphic treed disk

Our perturbation scheme depends on choosing generic perturbations of the gradient flow equations on the edges, as in the construction of the Morse A_∞ algebra by perturbation. For any treed disk Σ and v sufficiently large we denote by $\Sigma_{(1)} \in \overline{W}_{n,v}$ the (possibly unstable) metric ribbon tree obtained by collapsing the disk components of Σ to vertices. Let $F_{n,v} \in C^\infty(UW_{n,v} \times L)$ resp. $B_{n,v} \in C^\infty(UW_{n,v} \times TL^{\otimes 2})$ be a function equal to F resp. metric equal to B on the complement of a compact subset of the union of open edges. An $(F_{n,v}, B_{n,v})$-*perturbed treed disk* is a treed nodal disk Σ together with a continuous map $u : \Sigma \to X$ such that u satisfies the

Cauchy-Riemann equation on the disks and the gradient equation

$$(\frac{d}{dt}u)(t) = \operatorname{grad}(\tilde{F}_{n,v}(\Sigma_{(1)}, t, u(t))) \tag{29}$$

for the lift $\tilde{F}_{n,v}$ of $F_{n,v}$ from L to $\tilde{L}$ at the points t in the segments of Σ, which are identified with the segments of $\Sigma_{(1)}$. That is, $(\Sigma_{(1)}, t)$ represents a point on the universal tree $\overline{UW}_{n,v}$. The parameters t on the segments are well-defined up to translation, because of the choice of metric. The *energy* of a holomorphic treed disk is the sum of the energies of the disk components. In particular, there is no contribution to the energy from the gradient flow lines. Given a bound on the energy, we obtain a bound on the number of disk components in any holomorphic treed disk by energy quantization for disks, hence a bound on the number of vertices in the underlying tree. We always suppose that $v = v(E)$ has been chosen sufficiently large so that if $u : \Sigma \to X$ is a holomorphic treed disk of energy at most E then $\Sigma_{(1)}$ has at most v vertices, and so defines a point in $\overline{W}_{n,v}$.

Theorem 3.2. *If X is aspherical convex and L is compact, then any sequence of stable holomorphic treed disks with bounded energy and number of semiinfinite edges has a convergent subsequence.*

Proof. The number of segments in any broken gradient trajectory is bounded by $\dim(X)$ by the Morse-Smale condition. Furthermore the number of disk components is bounded by the energy divided by the constant in energy quantization for holomorphic disks, proved in McDuff-Salamon [42, Proposition 4.1.4]. It follows that the number of possible combinatorial types is finite, and so we may assume after passing to a subsequence that the combinatorial type is constant.

By Gromov compactness for marked disks, each sequence of disk components converges to a nodal disk, after passing to a subsequence. Similarly after passing to a subsequence the sequence of gradient trees converges to a limiting gradient tree. The limiting treed disk is obtained by attaching the limiting gradient segments to their attaching points in the limiting holomorphic disks, treating the attaching points as markings. □

For each homology class $\gamma \in H_2(X, \tilde{L})$ let $v(\gamma)$ be such that any holomorphic treed disk $u : \Sigma \to X$ has at most $v(\gamma)$ vertices or disk components. Choose a function $F_{n,v(\gamma)} \in C^\infty(UW_{n,v(\gamma)} \times L)$ and let $MW_n(L, \gamma)$ denote the moduli space of stable perturbed holomorphic treed quasidisks with boundary in L of class γ using the perturbation $F_{n,v(\gamma)}$. Given two moduli spaces $MW_{n-i+1}(L, \gamma), MW_i(L, \gamma')$ defined using perturbations $F_{n-i+1,v(\gamma)}, F_{i,v(\gamma')}$, a gluing procedure produces a perturbation system for $MW_n(L, \gamma + \gamma')$ in a neighborhood of $MW_{n-i+1}(L, \gamma) \times_L MW_i(L, \gamma')$, with the function on the collapsing segment equal to F in a neighborhood of the infinite node in the case that the length goes to infinity. Similarly, for any stratum Γ corresponding to an edge of zero length, any perturbation datum for $MW_{\Gamma,n}(L)$ produces a perturbation datum in a neighborhood by taking the perturbation to vanish on the edge whose length is going to zero. We say that a system of perturbations $F_* = (F_{n,v(\gamma)})_{n \geq 0, \gamma \in H_2(X, \tilde{L})}$ has been *chosen compatibly* if the function

$F_{n,v(\gamma'')}$ is equal to the function induced by gluing in a neighborhood of the image of $UW_{n-i+1,v(\gamma)} \times UW_{i,v(\gamma')}$ in $UW_{n,v}$ whenever $\gamma'' = \gamma + \gamma'$, and similarly in the case of perturbing an edge length from zero.

Each stratum $MW_{\Gamma,n}(L)$ is given locally as the quotient of a zero set of a Fredholm map of Banach manifolds. For each $\zeta \in \operatorname{Def}_\Gamma(\Sigma)$ let $(w_j^\pm(\zeta) \in \tilde{\Sigma})_{j=1}^m$ denote the corresponding nodes. Let $\tilde{\Sigma}_{(d)}, d = 1,2$ the union of 1 resp. 2-dimensional components of $\tilde{\Sigma}$, and by $u_d, d = 1,2$ the restriction of u to $\tilde{\Sigma}_d$. Let

$$\begin{aligned} \mathcal{F}_u : \operatorname{Def}_\Gamma(\Sigma) \oplus \Omega^0(\tilde{\Sigma}_{(2)}, (u_2)^*TX, (\partial u_2)^* T\tilde{L})_{1,p} \oplus \Omega^0(\tilde{\Sigma}_{(1)}, (u_1)^* T\tilde{L})_{1,p,\alpha} \\ \to \Omega^{0,1}(\tilde{\Sigma}_{(2)}, (u_2)^*TX)_{0,p} \oplus \Omega^1(\tilde{\Sigma}_{(1)}, (u_1)^* T\tilde{L})_{0,p,\alpha} \oplus \bigoplus_{j=1}^m T_{u(w_j^\pm)} \tilde{L} \end{aligned} \tag{30}$$

denote the map given by $(\zeta,\xi) \mapsto \mathcal{T}_u(\xi)^{-1} \overline{\partial}_J \exp_u \xi$ on the components $\tilde{\Sigma}_{(2)}$ of dimension 2, the gradient operator

$$(\zeta,\xi) \mapsto \mathcal{T}_u(\xi)^{-1}(\frac{d}{dt} \exp_u(\xi) - \operatorname{grad}(\tilde{F}_{n,v})(\Sigma_{(1)}(\zeta), t, \exp_u(\xi)))\mathrm{d}t$$

on the components $\tilde{\Sigma}_{(1)}$ of dimension 1, together with the differences in (18) where the position of the nodes $w_j^\pm(\zeta)$ depends on the deformation parameter ζ; the weighted Sobolev spaces are defined as in (6) but on $\mathbb{R}$ rather than $\mathbb{R} \times [0,1]$. The space $MW_{\Gamma,n}(L)$ is given near $u : \Sigma \to X$ as the zero set of $\mathcal{F}_u$, quotiented by the action of $G \times \operatorname{Aut}(\Sigma)$. The *linearized operator* associated to $\mathcal{F}_u$ is the operator formed by combining (5) with the linearized gradient operator

$$(\zeta,\xi) \mapsto (\nabla_t \xi - \nabla_\xi \operatorname{grad}(\tilde{F}_{n,v}))\mathrm{d}t$$

on the one-dimensional components, and finally for the third factor the differential of the third component of $\mathcal{F}_u$:

$$(\zeta,\xi) \mapsto \prod_{j=1}^m \xi(w_j^+(0)) - \xi(w_j^-(0)) + \frac{d}{dt}|_{t=0} u(w_j^+(t\zeta_j^+)) - \frac{d}{dt}|_{t=0} u(w_j^-(t\zeta_j^-)). \tag{31}$$

The first two components depend only on $\xi \in \Omega^0(\Sigma, u^*TX)$. We say that u is *regular* if D_u is surjective. Let $MW_{\Gamma,n}^{\mathrm{reg}}(L)$ denote the locus of regular maps of type Γ.

Theorem 3.3. *$MW_{\Gamma,n}^{\mathrm{reg}}(L)$ is a smooth manifold in a neighborhood of $[u]$ with tangent space at $[u]$ given by the quotient of $\ker(D_u)$ by the subspace generated by the Lie algebra $\mathfrak{g}$ and the infinitesimal automorphisms $\operatorname{aut}(\Sigma)$ of Σ.*

Proof. An argument similar to that used in the proof of Theorem 2.21 shows that $\mathcal{F}_u$ is a smooth map of Banach manifolds. The linearized Cauchy-Riemann resp. gradient operator is elliptic so the first two components of D_u define a Fredholm operator. Then D_u is itself Fredholm, since $\operatorname{Def}_\Gamma(\Sigma)$ is finite dimensional. The claim of the theorem then follows by the implicit function theorem for Banach manifolds, and properness of the action of $\operatorname{Aut}(\Sigma) \times G$, which holds for the same reasons as in the proof of Theorem 2.21. □

Denote the component of $\overline{MW}_n(L)$ with homotopy class $\gamma \in \pi_2(X, \tilde{L})$ by $\overline{MW}_n(L, \gamma)$. We suppose that a system $B_{n,v}$ of tree-dependent metrics on L, agreeing with the given metrics on the ends, has been fixed.

Theorem 3.4. *Suppose that $F^0_{n,v(\gamma)} \in C^\infty(UW_{n,v(\gamma)} \times L)$ is such that every treed disk with boundary in $\tilde{L}$ is regular, over an open neighborhood V of the boundary $\partial W_{n,v(\gamma)}$. Let $\rho \in C^\infty_c(UW_{n,v(\gamma)})$ be a compactly supported function which on the complement of V is non-vanishing somewhere on each edge. Then there exists a comeager subset $\mathcal{P}^{\mathrm{reg}}(L) \subset \mathcal{P}(L) := C^\infty_b(UW_{n,v(\gamma)} \times L)$ with the property that if $F^1_{n,v(\gamma)} \in \mathcal{P}^{\mathrm{reg}}(L)$ and $F_{n,v(\gamma)} = F^0_{n,v(\gamma)} + \rho F^1_{n,v(\gamma)}$ then every perturbed holomorphic treed quasidisk of class γ is regular.*

Proof. The reader may wish to compare with the argument in Seidel's book [52] of the Fukaya A_∞ algebra in the exact case, or in the construction of the Morse A_∞ algebra, see Abouzaid [2, Section 2.2]. The *universal moduli space*

$$
\begin{aligned}
(32)\quad MW^{\mathrm{univ}}_n(L) = \{&(F_{n,v(\gamma)}, u, (w^\pm_j)^m_{j=1}) \in C^l_b(UW_{n,v(\gamma)} \times L) \times \mathrm{Map}(\tilde{\Sigma}, X, \tilde{L})_{1,p} \times \\
&(\partial\tilde{\Sigma}_{(2)} \cup \tilde{\Sigma}_{(1)})^{2m} \, |\overline{\partial} u = 0,\ (29)\ ,\ u(w^+_j) = u(w^-_j), j = 1, \ldots, m\}
\end{aligned}
$$

is a smooth Banach manifold in a neighborhood of any tree disk, since the linearized operators on the disk components are surjective by definition, the linearized operators on the one-dimensional components of $\tilde{\Sigma}$ are surjective, and the evaluation maps at the nodes are transversal. To explain the last point, for each segment $S_i = [\epsilon^-_i, \epsilon^+_i]$ we can find a perturbation $f_{n,v(\gamma)}$ whose image under the linearized operator on S_i has arbitrary end points in $T_{u(\epsilon_i)^\pm} L$, and hence the image of $T_{u(\epsilon^-_i)} L \oplus T_{u(\epsilon^+_i)} L \oplus \mathfrak{g}$ in $T_{u(\epsilon^-_i)} \tilde{L} \oplus T_{u(\epsilon^+_i)} \tilde{L}$ is contained in the image of the linearized operator; the first factor is omitted if the segment is a semi-infinite edge of the tree. Keeping in mind that each node is contained in two components, the extra factor of $\mathfrak{g}$ is also in the image by an inductive argument, beginning with the semi-infinite edges corresponding to the markings $z_1, \ldots, z_n$. The claim now follows from Sard-Smale applied to each component with fixed index for sufficiently large l, and a Taubes-type argument to pass to smooth perturbations. □

It follows from Theorem 3.4 that we can construct perturbations $F_{n,v(\gamma)}$ so that every treed disk is regular, by induction on n. We denote by

$$MW_n(L)_d = \{[u] \in MW_n(L) | \operatorname{Ind}(D_u) - \dim(\mathrm{aut}(\Sigma) \oplus \mathfrak{g}) = d\}$$

the component of dimension d.

Theorem 3.5. *Suppose that X is aspherical convex and compatible perturbations have been chosen so that every holomorphic treed disk is regular. Then $\overline{MW}_n(L)_1$ has the structure of a compact one-manifold with boundary*

$$(33)\qquad \partial \overline{MW}_n(L)_1 = \cup_{k+i \le n} \left(MW_{n-i+1}(L)_0 \times_{\mathrm{ev}_k \times \mathrm{ev}_0} MW_i(L)_0 \right).$$

Proof. Compactness is Theorem 3.2, while smoothness of the strata is Theorem 3.3. The local structure at the boundary follows from a gluing construction: For any collection δ of *gluing parameters*, that is, real numbers attached to the edges of zero length and nodes of infinite type, denote by Σ^δ the treed disk obtained by (in the case of edges of zero length) removing half-disks in a neighborhood of attaching nodes and gluing together via the map $z_{j,+} \sim \delta_j / z_{-,j}$ where $z_{j,\pm}$ are the coordinates on the half-disks near $w_j^\pm$. Using cutoff functions one defines an *approximate treed holomorphic map* $G_\delta^{\text{approx}} u : \Sigma \to X$, see the proof of Theorem 2.22. The implicit function theorem then produces a solution (ζ, ξ) to $\mathcal{F}_{G_\delta^{\text{approx}}(u)}(\zeta, \xi) = 0$ with $\zeta \in \operatorname{Def}_\Gamma(\Sigma)$, and we set $G_\delta(u) = \exp_{G_\delta^{\text{approx}}(u)}(\xi)$, using uniform error estimates, uniformly bounded right inverse, and uniform quadratic estimates that combine those used in the construction of the Morse and Fukaya A_∞ categories given in Abouzaid [1]. These estimates involve the gluing at clean intersection ends which we already described in the proof of Theorem 2.22. The estimates for the Morse A_∞ algebra are special cases of the corresponding estimates for those of Floer trajectories, in the special case that the Floer trajectory comes from a Morse trajectory. Uniformity of the estimates and the exponential decay Lemma 2.5 then implies that for every sequence $u_t : \Sigma_t \to X$ converging to u, there exists a T such that for $t > T$, u_t is in the image of the gluing map. □

In fact, for a regular system of perturbations, $\overline{MW}_n(L)$ is a topological manifold with corners, with a non-canonical C^1-structure, by using arguments as in [27]. However, this requires a very complicated study of the gluing construction and we will not need it.

Any relative spin structure on L induces orientations on $MW_n(L)$ by combining the orientations on the moduli spaces of stable treed disks MW_n (if $n \geq 3$) in (28) with those on the linearized operators D_u described in [61], see also Katic-Milinkovic [31]. Let $CQF(L)$ denote the formal sum over critical points (that is, the same vector space appearing in Morse complex with Λ coefficients)

$$CQF(L) = \bigoplus_j CQF^j(L), \quad CQF^j(L) = \bigoplus_{x \in \operatorname{crit}(F)^j} \Lambda <x>$$

where $\operatorname{crit}(F)^j \subset \operatorname{crit}(F)$ denotes the subset of critical points of index j. We denote by $1_L \in CQF^0(L)$ the class of the critical point $x_{\min}$ of degree zero (or more generally, the sum of critical points of degree zero, if the Morse function F has more than one.) For any disk $u : (D, \partial D) \to (X, L)$, we denote by $\operatorname{Hol}_L(u) = \operatorname{Hol}_L([\partial u]) \in \Lambda$ the holonomy of the flat line bundle Λ around the boundary ∂u of u. Denote by $\overline{MW}_n(x_0, \ldots, x_n) \subset \overline{MW}_n(L)$ the subset with markings mapping to $x_0, \ldots, x_n$.

Theorem 3.6. *For any choice of perturbation system so that all treed holomorphic disks are regular, the formula*

$$\mu_n(<x_1>, \ldots, <x_n>) = \sum_{[u] \in \overline{MW}_n(x_0, \ldots, x_n)_0} (-1)^\heartsuit \epsilon(u) q^{A(u)} \operatorname{Hol}_L(u) <x_0>$$

where $\heartsuit = \sum_{i=1}^n i|x_i|$ *define the structure of a* $\mathbb{Z}_2$*-graded strictly* A_∞*-algebra on* $CQF(L)$.

Proof. The A_∞ axiom up to signs follows from Theorem 3.5, the additivity of the area and the multiplicativity of the holonomies. The signs are painfully justified as follows. The determinant line $\det(D_u)$ of the linearized operator for the holomorphic disks admits a canonical orientation, by deforming to a connect sum of holomorphic spheres and constant disks as in [20], [61]. Consider the gluing map

$$MW_i(y, x_{j+1}, \ldots, x_{j+i})_0 \times MW_{n-i+1}(x_0, x_1, \ldots, y, \ldots, x_n)_0 \to MW_n(x_0, \ldots, x_n)_1.$$

$$TMW_i(y, x_{j+1}, \ldots, x_{j+i}) \to TMW_i \oplus TL \oplus T_y L^+ \oplus T_{x_{j+1}} L^- \oplus \ldots \oplus T_{x_{j+i}} L^-$$

plus a complex vector space. Here $T_{x_i}L^-$ is the negative part of the tangent space $T_{x_i}L$ with respect to the Hessian of F. Similarly the orientation on the second factor is determined by an isomorphism

$$TWM_{n-i+1}(x_0, x_1, \ldots, y, \ldots, x_n) \to TMW_{n-i+1} \oplus TL \oplus TL^+_{x_0} \oplus TL^-_{x_1} \oplus \ldots \oplus TL^-_{x_n}$$

plus a complex vector space. A sign arises from a switch of TMW_{n-i+1} with $TL \oplus \ldots \oplus T_{x_{j+i}}L^-$, that is, a sign of (-1) to the power $i(n-i)$, plus the sign arising from a switch of $TL \oplus T_y L^+ \oplus T_{x_{j+1}}L^- \oplus \ldots \oplus T_{x_{j+i}}L^-$ with $TL \oplus TL^+_{x_0} \oplus TL^-_{x_1} \oplus \ldots \oplus TL^-_{x_j}$, giving (-1) to the power $i\left(|y| + \sum_{k=1}^{j} |x_j|\right)$, for a total of (-1) to the power $i\left(\sum_{k=j+i+1}^{n} |x_k|\right)$. The gluing map $MW_i \times MW_{n-i+1} \to MW_n$ is the same as that of $M_i \times M_{n-i+1} \to M_n$ by construction, which has sign (-1) to the power $ij + 1 - j - i$, see (27). Comparing the contributions from $(-1)^{\heartsuit}$ with an overall sign of $(-1)^{\square}$, where $\square = \sum_{k=1}^{n} k|x_k|$, contributes (-1) to the power

$$\begin{aligned}(34)\quad & \sum_{k=1}^{n} k|x_k| + \sum_{k=1}^{i} k|x_{j+k}| + \sum_{k=1}^{j} k|x_k| + \sum_{k=1}^{n-i-j} (j+k)|x_{j+i+k}| + (j+1)|y| \\ &\equiv_2 j(|y| + i) + (i-1)\Big(\sum_{k=1}^{n-i-j} |x_{j+i+k}|\Big) + (j+1)|y| \\ &\equiv_2 |y| + ji + (i-1)\Big(\sum_{k=1}^{n-i-j} |x_{j+i+k}|\Big)\end{aligned}$$

while the sign in the A_∞ axiom contributes $\sum_{k=1}^{j}(|x_k| - 1)$. Combining the signs one obtains in total

(35)
$$\begin{aligned} & i\left(\sum_{k=i+j+1}^{n} |x_k|\right) + ij + 1 - j - i + ji + (i-1)\left(\sum_{k=i+j+1}^{n} |x_k|\right) + |y| + \sum_{k=1}^{j}(|x_k| - 1) \\ &= \sum_{k=1}^{j}(|x_k| - 1) + |y| + 1 + \sum_{k=i+j+1}^{n} |x_k| \equiv_2 1 + \sum_{k=1}^{n} |x_k| \end{aligned}$$

which is independent of i, j. The A_∞ -associativity relation follows. □

If the line bundle is defined over Λ_0 resp. is trivial then the A_∞ algebra is defined over Λ_0 resp. $\mathbb{Z}$. The combined gradient trees/holomorphic disks have appeared in many places, for example the Piunikhin-Salamon-Schwarz [46], Biran-Cornea [8], Albers [3], and Seidel [53]. See also the homological perturbation lemma in Kontsevich-Soibelman [33], which involves a sum over trees.

3.5. **Quasimap Floer cohomology.** We wish to achieve units in the quasimap Fukaya algebra, in a suitable weak sense. Let $1_L =< x_{\min} >$ denote the generator corresponding to the critical point of index zero $x_{\min}$ of $F : L \to \mathbb{R}$, assuming it is unique. We denote by $\mathcal{MW}_\Gamma(L)$ the moduli space of stable treed holomorphic disks of combinatorial type Γ and by $\mathcal{P}^*_\Gamma(L)$ the space of perturbation data for which every element of $\mathcal{MW}_\Gamma(L)$ is regular.

Proposition 3.7. *Suppose that J is such that every non-trivial stable holomorphic disk in L is regular and has positive Maslov index. Then for sufficiently small perturbation data,*

$$\mu_0(1) = \sum_{I(u)=2} q^{A(u)} \operatorname{Hol}_L(u) 1_L$$

where the sum is over equivalence classes of Maslov index two holomorphic disks with boundary in $\tilde{L}$ such that the boundary of u maps to a generic point under the projection $X \to X /\!\!/ G$. Furthermore, the element 1_L satisfies

$$(-1)^{|c|}\mu_2(1_L, c) = \mu_2(c, 1_L) = c, \quad \forall c \in CF(L).$$

Proof. The assertion on $\mu_0(1)$ follows from a dimension count and the assumption of no disks of non-positive Maslov index. By the transversality assumption $\overline{M}_1(x_{\min})$ consists of a gradient trajectory to $x_{\min}$ attached to a holomorphic disk of index two. Since $x_{\min}$ is index zero, the gradient trajectory must be length zero, so that the disk actually passes through $x_{\min}$; the perturbation has the effect of replacing this with a generic point.

To prove the second assertion, let $[u : C \to X] \in \mathcal{MW}_\Gamma(L)$ be an isolated element with limit along an incoming semi-infinite edge mapping to the maximum $x_0 \in \operatorname{crit}(f)$, the other incoming edge mapping to $x_1 \in \operatorname{crit}(f)$ and the outgoing edge mapping to $x_2 \in \operatorname{crit}(f)$. The domain C of u is obtained by replacing the vertices v of a tree T with disks $D(v)$. By assumption all non-constant stable disks are assumed regular and positive Maslov index. By transversality, all edges in the domain of u have finite length. We claim that C consists of a single disk $D(v)$ on which u is constant, and three semi-infinite edges attached to the boundary. We first show that there are no disks $D(v)$ in the domain of u with a single special point. Indeed if there existed such a disk $D(v) \subset C$ then by varying the length of the connecting edge d to $D(v)$ one would obtain a one-dimensional family of configurations as follows. Let $t(d), h(d) \in C$ denote the endpoints of d. We have $t(d) = \varphi_{\ell(d)} h(d)$ where $\ell(d)$ is the length of d and $\phi_{\ell(d)}$ is the time $\ell(d)$ perturbed gradient flow. By transversality the moduli space $\mathcal{M}^2_1(L)$ of holomorphic disks with boundary on L and one marked point on the boundary of Maslov index 2 is smooth and evaluation on the boundary is a local diffeomorphism. Replacing $\ell(d)$ by a small perturbation and adjusting $t(d)$ produces a one-dimensional family in $\mathcal{MW}_\Gamma(L)$ containing $[u]$ which is a contradiction. Thus disks with only one special point are involved only in the definition of μ_0, and not in the composition maps $\mu_n, n > 0$.

Secondly, any disk $D(v)$ meeting the semi-infinite edge labelled $x_{\min}$ must be constant. Let $z \in D(v)$ be the point on the boundary connecting to the semi-infinite edge labelled $x_{\min}$. We have $\lim_{t\to-\infty} \varphi_t(u(z)) = x_{\min}$ which is an open condition. If u is non-constant on $D(v)$ one could vary z to obtain a one-parameter family of elements of $\mathcal{MW}_\Gamma(L)$ containing $[u]$, which is a contradiction.

Thirdly, the domain has a single disk component. Let $D(v)$ denote the first disk component attached to the semi-infinite edge labelled $x_{\min}$. By the previous paragraph, u is constant on $D(v)$ with value $u(D(v))$. After replacing $D(v)$ with a point, one obtains a perturbed treed holomorphic disk with only one incoming edge with $\mathrm{Ind}(D_u)-\dim(\mathrm{aut}(C)) = -1$ where D_u is the associated Fredholm operator and $\mathrm{aut}(C)$ the space of infinitesimal automorphisms of the domain. By assumption we have chosen (f, g) so that the moduli space of holomorphic treed disks with a single incoming edge is regular, that is, there are no non-constant holomorphic treed disks $u : C \to X$ with $\mathrm{Ind}(D_u)-\dim(\mathrm{aut}(C)) = -1$. The same is true for perturbations in a neighborhood of (f, g), in particular, for (f_Γ, g_Γ) in a neighborhood of (f, g), there are no such configurations. Hence all disks in C are constant. Since each constant disk between the semi-infinite edges labelled x_1 and x_2 corresponds to a semi-infinite edge different from the one labelled x_1 and x_2, and there are only three semi-infinite edges, there must be a single disk component.

Collapsing the disk $D(v)$ and forgetting the incoming trajectory from $x_{\min}$, one obtains a perturbed gradient trajectory u' joining x_1 to x_2 with a distinguished point $z \in C$. For any critical points x_1, x_2 of equal index, there is a parametrized Morse trajectory for (f, g) of index zero connecting x_1, x_2 if and only if x_1, x_2 are equal. The same holds after any sufficiently small perturbation f_Γ, g_Γ. It follows that $\mathcal{MW}_\Gamma(L)$ is a point if $x_1 = x_2$, and is empty otherwise. The orientation is induced by the orientation on the moduli space of parametrized trajectories from x_1 to x_2, which is positive. Thus

$$(-1)^{|x_1|}\mu_2(<x_{\min}>, <x_1>) = \mu_2(<x_1>, <x_{\min}>) = <x_1>, \quad \forall x_1 \in \mathrm{crit}(f).$$

The statement for Floer cochains follows by taking linear combinations. □

Remark 3.8. If one wishes an A_∞ algebra with strict units may be obtained by the method of Fukaya-Oh-Ohta-Ono [20], see also Ganatra [23] and Sheridan [56] as we now explain. Let A be an A_∞ algebra over Λ. A *homotopy unit* for A is an A_∞ structure on the Λ-module A^+ generated by A and additional generators $1_L^-, 1_L^+$, that is, $A^+ = A \oplus \Lambda 1_L^-[1] \oplus \Lambda 1_L^+$ so that the A_∞ structure maps $\mu_n, n \geq 1$ coincide with the previous A_∞ structure on A and in addition the relations on the maps $\mu_n, n \geq 0$ hold as follows:

$$\begin{aligned} \mu_1(1_L^-) &= 1_L^+ - 1_L \\ \mu_1(1_L^+) &= 0 \\ (-1)^{|a|}\mu_2(1_L^+, a) &= \mu_2(a, 1_L^+) = a \\ \mu_n(\ldots, 1_L^+, \ldots) &= 0 \text{ for } n \geq 3 \end{aligned}$$

for all homogeneous elements $a \in A$. In particular A^+ is a strictly unital A_∞ category with strict unit 1_L^+.

A homotopy unit on the quasimap Fukaya A_∞ algebra may be obtained following Fukaya-Oh-Ohta-Ono [20, Remark 10.3], Ganatra [23] and Sheridan [56] by homotoping the perturbation data for insertions involving the unit to perturbation data for which one has forgetful maps. Since the composition maps involving the generators of $CF(L)$ and the strict unit only are already determined, the problem is to define the composition maps involving at least one input 1_L^-. These are defined by counts of *weighted treed disks*, by which mean a treed disks equipped with a subset $I \subset \{1, \ldots, n\}$ of the semi-infinite edges corresponding to the inputs marked 1_L^-, and for each $i \in I$ a *weight* $\rho_i \in [0, 1]$. A weighted treed disk is stable if the underlying treed disk is stable. A collection (f_Γ^+, g_Γ^+) of perturbation data for weighted treed disks is *coherent* if

(a) in a neighborhood of any boundary stratum $\mathcal{T}_{\Gamma'} \subset \mathcal{T}_\Gamma$, f_Γ^+, g_Γ^+ are pulled back from the boundary perturbation data $f_{\Gamma'}^+, g_{\Gamma'}^+$ under the gluing construction;
(b) whenever a weight parameter ρ_i is equal to 0, f_Γ^+, g_Γ^+ are pulled back under the forgetful map forgetting the i-th semi-infinite edge and stabilizing; and
(c) whenever a weight parameter ρ_i is equal to 1, then the perturbation data f_Γ^+, g_Γ^+ is equal to the perturbation data f_Γ, g_Γ used to define the composition maps for $CF(L)$.

Given coherent collections of perturbation data, counting isolated points in the moduli spaces of stable weighted treed holomorphic disks defines the higher composition maps for inputs involving 1_L^-. In particular, the A_∞ relations involving $\mu_1(1_L^-) = 1_L^+ - 1_L$ hold for the following reason: Weighted disks with some $\rho_i \in \{0, 1\}$ are boundary components of the moduli space of weighted disks for which the weight ρ_i is allowed to vary freely. When $\rho_i = 0$, the perturbation data is pulled back under the forgetful map. So the higher compositions vanish after insertion of 1_L^+ while for $\rho_i = 1$ one obtains the corresponding higher composition map with insertion of 1_L. The A_∞ relations are proved in the same way as before and yield a strictly unital A_∞ algebra $CF^+(L)$. This ends the remark.

The A_∞ algebra $CQF(L)$ might be called the *gauged* or *equivariant Fukaya* algebra of $\tilde{L}$. However, the definition of the structure maps do not involve any connection. Note that the vortex equations are not conformally invariant, that is, they depend on a choice of area form. In the next section we investigate the A_∞ bimodule associated to a pair of Lagrangians given by counting holomorphic strips. Since the automorphism group of a strip *does* preserve the standard area form, one should expect an A_∞ bimodule defined using the vortex equations. The rigorous construction of such a bimodule, however, has not been carried out.

Because of Proposition 3.7 we have $\mu_0(1) = w 1_L$ for some $w \in \Lambda_0$ and the square of the first structure map $\mu_1 : CQF(L) \to CQF(L)$ satisfies

$$(\mu_1)^2(a) = -\mu_2(a, w1_L) + (-1)^{|a|}\mu_2(w1_L, a) = w(-(-1)^{|a|}a + (-1)^{|a|}a) = 0.$$

It follows that the *quasimap Floer cohomology*

$$HQF(L) := H(\mu_1)$$

is well-defined.

The higher composition maps have an obvious expansion by energy, such that the leading order term is the higher composition map in the A_∞ Morse algebra defined by the perturbations $F_{n,v}$. In particular the leading order term in μ_1 is the Morse-Smale-Witten operator counting isolated gradient flow lines, and one has a Morse-to-Floer spectral sequence (compare e.g. Oh [44], Fukaya et al [20], and especially Buhovsky [10], who uses the same combined Morse-Fukaya framework.)

Proposition 3.9. *Suppose that perturbations have been chosen above so that every holomorphic treed quasidisks are regular. There is a spectral sequence $(E^j, \mu_1^j)_{j\geq 1}$ converging to $HQF(L,\Lambda_0)$ with first page $E^1 = HM(L,\Lambda_0)$ the Morse cohomology.*

Proof. The holomorphic treed disks with zero energy have all holomorphic disks constant. These can be isolated only if each disk has exactly three markings, in which case they can be collapsed to stable gradient trees. If any disk occurs then the resulting tree has at least three semiinfinite edges. Hence no disks occur in the zero energy contributions to μ_1, which are therefore those of the Morse differential. Filtering the chain complex by energy intervals

$$CQF(L,\Lambda_0) = \bigcup_{n\in\mathbb{Z}_{\geq 0}} CQF(L)_{\geq n\delta}, \quad CQF(L)_{\geq n\delta} = \bigoplus_{x\in \mathrm{crit}(F)} q^{n\delta}\Lambda_0 <x>$$

for sufficiently small $\delta > 0$ and taking the associated spectral sequence gives the result. □

Note that here we have *assumed* that all stable holomorphic quasidisks are regular, and explicitly constructed perturbations achieving transversality inductively using moduli spaces of not-necessarily-stable metric ribbon trees.

3.6. **The divisor equation.** A feature of the combined Morse-Fukaya moduli spaces is that there is *not* a forgetful morphism of the form $f_i : \overline{MW}_n(L) \to \overline{MW}_{n-1}(L)$ forgetting the i-th marking. If there was such a morphism, then one would have an analog of the divisor equation in Gromov-Witten theory, as explained by Cho [11]: if $b \in CQF(L)$ represents a cycle of codimension one in Morse cohomology then

$$\sum_{i=0}^{n} \mu_{n+1}(a_1,\ldots,a_i,b,a_{i+1},\ldots,a_n;\gamma) \overset{?}{=} \langle [\partial\gamma],[b]\rangle \mu_n(a_1,\ldots,a_n;\gamma)$$

where $\mu_n(\cdots;\gamma)$ is the contribution to $\mu_n(\cdot)$ from disks of class $\gamma \in \pi_2(X,\tilde{L})$, and $[\partial\gamma] \in H_1(L)$ is the boundary homology class of γ. Indeed, given a holomorphic disk with the first n markings mapping to $a_1,\ldots,a_n$ and class γ, the boundary of the disk traces out a circle of class $[\partial\gamma] \in H_1(L)$, and so there are $\langle [\partial\gamma],[b]\rangle$ possibilities for the placement of an additional marked point on the disk so that it lies on the cycle given by b.

Unfortunately the equations on the segments are *perturbed* gradient flows, and it is not possible to choose the perturbation so that it vanishes on every segment that might be collapsed after forgetting a marked point and stabilization. So there is no forgetful map, for the perturbation system constructed above. However, we have the following special case of the divisor equation:

Proposition 3.10. *Suppose that all non-trivial stable holomorphic treed disks are regular and have positive Maslov index. For any Morse cocycle $b \in CQF^1(L)$ and $\gamma \in \pi_2(X, \tilde{L})$ with $I(\gamma) = 2$, we have $\mu_1(b;\gamma) = \langle [b], [\partial\gamma]\rangle \mu_0(1;\gamma)$.*

Proof. The only contributing configuration is that of a single disk of index two and two gradient trajectories: all configurations with disks whose degrees sum to more than two vanish for reasons of degree, while the underlying tree has only one vertex since the Morse product of b and a point vanishes, so there are no contributing configurations with constant disks. Suppose that $b = \sum_{x \in \mathrm{crit}(F)^1} n_x <x>$ for some coefficients n_x. The contributions to $\mu_1(b;\gamma)$ correspond to choices for the position of the point on the boundary of the disk component connecting via a gradient trajectory to some x appearing in b, that is, with $n_x \neq 0$, since by assumption the moduli spaces are regular.

To check that the sign is the expected one, note that for any x_1 such that $W^-_{x_1}$ meets $u(\partial D)$, the orientation on $\overline{MW}_1(x_0, x_1)$ is ± 1 depending on whether $\partial_\theta u(e^{i\theta})$ maps positively or negatively onto $T^-_{x_1}\tilde{L}$: The orientation on $T_u MW(x_0, x_1)_0$ is induced from an identification of the determinant line with the determinant line of $\ominus\mathbb{R} \oplus \mathbb{C} \ominus T^-_{x_1}\tilde{L}$ where $\mathbb{C}$ is a factor arising from the complex linear kernel of a Cauchy-Riemann operator on a sphere, with value prescribed at a marked point. If u is an element of the kernel of the Cauchy-Riemann operator vanishing at the marking, then the orientation on the complex line is determined by $\partial_v u \wedge \partial_w u$, where v, w are vector fields on the sphere $\mathbb{C} \cup \{\infty\}$ such that $v \wedge w$ is positive on $\mathbb{C}$. This property transfers via the gluing construction, so that the orientation is given as follows. Consider an identification $D - \{-1, 1\} \to \mathbb{R} \times [0, 1]$ mapping $\{-1, 1\}$ to the point at $\mp\infty$. Translation on the strip induces vector fields $v, w \in \mathrm{Vect}(D)$. Since $v \wedge w$ is positive, $\partial_v u \wedge \partial_w u$ is the orientation on $\mathrm{ev}(1)^{-1}(0) \cap \ker(D_u) \cong \mathbb{C}$. Since the translation $\mathbb{R}$ identifies with span of $\partial_v u$, the sign of the contribution is given by 1 resp. -1 if $-\partial_\theta u(e^{i\theta})|_{\theta=\pi}$ gives the orientation resp. minus the orientation of $T^-_{x_1}\tilde{L}$. Combining with the factor $(-1)^{|x_1|} = -1$ in the definition of μ_1 gives the claim. Thus the total contribution from each such disk is the sum of these contributions times n_x, for each x with $n_x \neq 0$, or equivalently, $\langle [b], [\partial\gamma]\rangle q^{A(u)} \mathrm{Hol}_L(u)$. Summing over such disks proves the Proposition. □

Combining Proposition 3.7 with Proposition 3.10 we obtain:

Corollary 3.11. *Suppose that the moduli space of treed quasidisks is regular and $\beta \in CQF^1(L)$ is a Morse cocycle. Then*

$$\mu_1(\beta) = \sum_{I(u)=2} \langle [\beta], [\partial\gamma]\rangle q^{A(u)} \mathrm{Hol}_L(u) 1_L$$

where the sum is over equivalence classes of index two holomorphic disks with boundary in $\tilde{L}$ and a boundary point mapping to a generic point in L.

Suppose that L is a torus and consider $H^1(L, \Lambda_0)$ as parametrizing a family of brane structures on a compact oriented Lagrangian L equipped with the standard spin structure. For any $b \in H^1(L, \Lambda_0)$ we denote by μ^b_n the structure coefficients for

L^b, that is, the Lagrangian L with brane structure given by b. The *potential* for L is the function

$$W : H^1(L, \Lambda_0) \to \Lambda_0, \quad \mu_0^b(1) = W(b)1_L.$$

Any β in $CQF^1(L)$ on which the Morse differential vanishes defines a class $[\beta] \in H^1(L, \Lambda_0)$.

Proposition 3.12. *For $\beta \in CQF^1(L)$ with $\mu_1^b(\beta; 0) = 0$ we have $\mu_1^b(\beta) = \partial_{[\beta]} W(b)$.*

Proof. By Proposition 3.11, Corollary 3.7, and the formula $\mathrm{Hol}_L(u) = \exp\langle [b], [\partial\gamma] \rangle$. □

3.7. A_∞ **homotopy invariance.** The homotopy type of the quasimap Fukaya algebra defined above is independent of all choices. The argument uses moduli spaces of *quilted treed disks*, which are a particular realization of Stasheff's *multiplihedron* L_n [57]. This is a cell complex whose vertices correspond to total bracketings of $x_1, \ldots, x_n$, together with the insertion of expressions $f(\cdot)$ so that every x_j is contained in an argument of some f. For example, L_2 is an interval with vertices $f(x_1)f(x_2)$ and $f(x_1 x_2)$. A geometric realization of this polytope can be given as follows: A *quilted metric ribbon tree* is a rooted metric ribbon tree

$$\Gamma = (V(\Gamma), E(\Gamma), O(\Gamma), l : E(\Gamma) \to [0, \infty])$$

together with a subset $V^{\mathrm{col}}(\Gamma)$ of the vertices $V(\Gamma)$, satisfying the condition the length of the finite part of the path $P(z_0, v)$ from the root vertex z_0 to any *colored vertex* $v \in V^{\mathrm{col}}$ given by $\prod_{e \in P(z_0, v)} l(e)$ is independent of the choice of colored vertex $v \in V^{\mathrm{col}}(\Gamma)$, see Ma'u-Woodward [39]. The set of finite resp. semiinfinite edges is denoted $E_{<\infty}(\Gamma)$ resp. $E_\infty(\Gamma)$; the latter are equipped with a labelling by integers $0, \ldots, n$. A quilted tree is *stable* if each colored vertex has valence at least two, any non-colored vertex has valence at least three or connects two edges of infinite length, and each edge contains either a trivalent vertex or a colored vertex.

There is a natural notion of convergence of quilted trees, in which edges whose length approaches zero are contracted and edges whose lengths go to zero are replaced by broken edges. Let $\overline{W}_{n,1}$ denote the moduli space of quilted metric ribbon trees; this is naturally homeomorphic to L_n, see [39], although not isomorphic as a cell complex.

There is a different realization of the multiplihedron given in Ma'u-Woodward [39] which gives the correct cell structure. Namely in [39] a *quilted disk* was defined as a marked disk $(D, z_0, \ldots, z_n \in \partial D)$ (the points are required to be in cyclic order) together with a circle $C \subset D$ tangent to the 0-th marking z_0. An *isomorphism* of quilted disks from $(D, C, z_0, \ldots, z_n)$ to $(D', C', z_0', \ldots, z_n')$ is an isomorphism of holomorphic disks $D \to D'$ mapping C to C' and $z_0, \ldots, z_n$ to $z_0', \ldots, z_n'$. The moduli space $M_{n,1}$ of quilted disks admits a compactification $\overline{M}_{n,1}$, isomorphic to L_n as a cell complex, which allows the interior circle C to "bubble out" into the extra disk bubbles, or disk bubbles without interior circles to form when the points come together. (The homotopy invariance of the A_∞ algebra of a Lagrangian is proved in Fukaya et al [20] using a moduli space of *weighted* stable disks; we find the used of quilted stable disks more natural because it reproduces exactly Stasheff's cell structure, although the quilting has no geometric meaning in the current paper.) The open stratum $M_{n,1}$ may be identified with the set of sequences $0 = w_1 < \ldots < w_n$; the bubbles form either when the points come together, in which case a disk bubble forms, or when the markings go to infinity, in which case one rescales to keep the maximum distance between the markings constant. One has a quilted disk bubbles for each group of markings that come together after re-scaling, but not before.

There is a combined moduli space $\overline{MW}_{n,1}$ that combines both objects: A *quilted treed disk* Σ is given by (i) a quilted rooted metric ribbon tree (ii) for each non-colored vertex of the tree with valence at least three, a disk with markings whose number is the valence of the given vertex (iii) for each colored vertex, a quilted disk with number of markings equal to the valence that of the colored vertex. From this datum one defines a space obtained by attaching the endpoints of the segments of the quilted metric tree to the marked points on the disks corresponding to the vertices (except the uncolored vertices of valence two, which are not attached to disks). A quilted treed disk is *stable* if it has no automorphisms, that is, each quilted disk resp. unquilted disk has at least two resp. three marked or nodal points, and in addition any one-dimensional component is attached to at least one disk.

Thus in particular each disk in a stable quilted treed disk is connected by a sequence of one resp. two edges, which are of finite resp. infinite length. Let $\overline{MW}_{n,1}$ denote the moduli space of stable quilted treed disks. See Figure 7 for a picture of $\overline{MW}_{2,1}$. The quilted disks are those with two shadings; while the ordinary disks have either light or dark shading depending on whether they can be connected to the zero-th edge without passing a colored vertex. The hashes on the line segments indicate nodes connecting segments of infinite length, that is, broken segments. The space

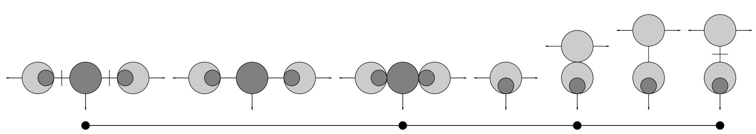

FIGURE 7. Moduli space of stable quilted treed disks

$\overline{MW}_{n,1}$ admits two forgetful maps $f_M : \overline{MW}_{n,1} \to \overline{M}_{n,1}, \quad f_W : \overline{MW}_{n,1} \to \overline{W}_{n,1}$ obtained by forgetting the edges resp. disks and taking the ribbon structure induced by the order of the markings on the boundary of the disk. The product $f_M \times f_W$ defines an injection into $\overline{M}_{n,1} \times \overline{W}_{n,1}$, defining a topology on $\overline{MW}_{n,1}$ for which it is compact and a sequence converges iff the underlying sequence of quilted trees and quilted disks converges. If a stratum $MW_{n,1,\Gamma}$ is contained in the closure of another stratum $MW_{n,1,\Gamma'}$, then there is an explicit description of the normal cone as a real toric singularity, explained in [39] for quilted trees or disks.

The cells of $\overline{MW}_{n,1}$ can be oriented by identifying each with a corresponding cell of $\overline{M}_{n,1}$ times a product of real lines corresponding to the edges. The boundary is naturally isomorphic to a union of moduli spaces with lower numbers of markings:

$$\partial \overline{MW}_{n,1} \cong \bigcup_{i,j} \left(\overline{MW}_{n-i+1,1} \times \overline{MW}_i \right) \cup \bigcup_{i_1,\ldots,i_r} \left(\overline{MW}_r \times \prod_{j=1}^r \overline{MW}_{i_j,1} \right).$$

By construction the sign of the inclusions of boundary strata are the same as that for the corresponding inclusion of boundary facts of $\overline{M}_{n,1}$, that is, $(-1)^{ij+n-j-i}$ for the facet of the first type. For facets of the second type, the gluing map is

$$\mathbb{R} \oplus M_r \oplus \bigoplus_{j=1}^{r} M_{|I_j|,1} \to M_n$$

(36) $(\delta, z_1 = -1, z_2, \ldots, z_{r-1}, z_r = 0, (w_{1,j} = 0, w_{2,j}, \ldots, w_{|I_j|,j})_{j=1}^r)$
$\to (-\delta^{-1}, -\delta^{-1}+w_{2,1}, \ldots, -\delta^{-1}+w_{|I_1|,1}, \delta^{-1}z_2, \delta^{-1}z_2+\delta w_{2,2}, \ldots, w_{|I_r|,1}, \ldots, \delta w_{|I_r|,r})$

and so changes orientations by $\sum_{j=1}^r (r-j)(|I_j|-1)$. Finally, there is a *universal quilted tree disk* $\overline{UMW}_{n,1} \to \overline{MW}_{n,1}$ which is a cell complex whose fiber over Σ is isomorphic to Σ.

Theorem 3.13. *Suppose that $J \in \mathcal{J}(X)^G, L \subset X/\!/G$ are such that every stable disk with boundary in $\tilde{L}$ is regular. Then the A_∞ algebra $CQF(L)$ is independent up to A_∞ homotopy of the choice of perturbation system used to construct it.*

To prove this, one considers two systems of perturbations $(B_{*,0}, F_{*,0}), (B_{*,1}, F_{*,1})$ and extends them to a set of perturbations resp. connections for the moduli space of the following:

Definition 3.14. A *holomorphic quilted treed quasidisk* to X is a quilted treed disk Σ together with a map $u : \Sigma \to X$ such that u is holomorphic on each disk component, and u is a gradient trajectory for $\tilde{F}_{n,v} \in C^\infty(UW_{n,v} \times \tilde{L})$ on each edge, where $v = v(\gamma)$ depends on the homology class $\gamma = \gamma(u)$ of u. An *isomorphism* of holomorphic quilted treed quasidisks $u_j : \Sigma_j \to X$ is an isomorphism $\phi : \Sigma_0 \to \Sigma_1$ together with an element $g \in G$ such that $\phi^* u_1 = g u_0$; in particular this means that ϕ preserves the quilting on each quilted disk. A holomorphic quilted treed quasidisk $u : \Sigma \to X$ is *stable* if it has no infinitesimal automorphisms (each disk component on which u is constant has at least three marked or nodal points, and u is non-constant on each edge of positive length) and each node connecting two line segments maps to a critical point.

The additional data of the quilting on each quilted disk is not used to modify the Cauchy-Riemann equation (as opposed to the situation in [38] where the quilting has a geometric meaning.) Let $\overline{MW}_{n,1}(L)$ denote the moduli space of holomorphic quilted treed quasidisks with boundary in L, and perturbation system (B_*, F_*). Perturbations can be constructed inductively in a neighborhood of the boundary components, by taking the perturbation system $(B_{*,0}, F_{*,0})$ above the quilted disk components (that is, on the components not connected to the root by unquilted components) and $(B_{*,1}, F_{*,1})$ below, and zero on the edges of small length. Assuming that the perturbations have been constructed inductively, the boundary components of $\overline{MW}_{n,1}(L)$ are the combinatorial types with at least one segment of infinite length. (There are also "fake boundary component" with edges of zero length, but these are not part of the "true boundary".) In particular, the boundary components of dimension one consist of two types: configurations where a collection of quilted treed disks has broken off, or an unquilted treed disk has broken off as shown in Figure 7:

$$\partial\overline{MW}_{d,1}(L) = \bigcup_{i_1+\ldots+i_r=d} \overline{MW}_{r,1}(L) \times_{\mathrm{crit}(F)^r} \prod_{j=1}^{r} \overline{MW}_{i_j}(L; B_{*,1}, F_{*,1}) \cup \bigcup_{i+k\le d} \overline{MW}_i(L; B_{*,0}, F_{*,0}) \times_{\mathrm{crit}(F)} \overline{MW}_{d-i+1,1}(L). \tag{37}$$

Theorem 3.15. *If every stable holomorphic disk is regular then there exist compatible systems of perturbations so that every quilted treed quasidisk is regular.*

Proof. This is similar to the argument given for the unquilted case above in Theorem 3.4, and is left to the reader. □

Assuming that a perturbation system has been chosen as in Theorem 3.15 define

$$\begin{aligned} \phi_n : CQF(L; B_{*,0}, F_{*,0})^{\otimes n} &\to CQF(L; B_{*,1}, F_{*,1}) \\ (<x_1>, \ldots, <x_n>) &\mapsto \sum_{[u]\in\overline{MW}_{n,1}(L;x_0,\ldots,x_n)_0} (-1)^{\heartsuit}\epsilon(u)q^{A(u)}\,\mathrm{Hol}_L(u) <x_0> . \end{aligned} \tag{38}$$

Theorem 3.16. *$\phi = (\phi_n)_{n\ge 0}$ is an A_∞ morphism.*

Proof. For $\mathbb{Z}_2$ coefficients, this follows from the description of the boundary in (37). The signs for the terms (25) of the first type is similar to those for the A_∞ axiom and will be omitted. For terms of the second type we need to determine the sign of the isomorphism

$$\mathbb{R} \oplus T\overline{MW}_r(y_0, \ldots, y_r) \oplus \bigoplus_{j=1}^{r} T\overline{MW}_{|I_j|,1}(y_i, x_{I_j}) \to T\overline{MW}_n(y_0, x_1, \ldots, x_n).$$

The former is determined by an isomorphism with

$$\mathbb{R} \oplus TL \oplus T^+_{y_0} \oplus T\overline{MW}_r \oplus T^-_{y_1} \ldots \oplus T^-_{y_r} \oplus \bigoplus_{j=1}^{r} \left(TL \oplus T^+_{y_j} \oplus T\overline{MW}_{|I_j|,1} \oplus \bigoplus_{k\in I_j} T^-_{x_k} \right)$$

(where $T^+_{y_0}$ denotes $T^+_{y_0}L$ etc.) except that this differs by signs

$$|y_0|(r-2) + \sum_{j=1}^{r} (|I_j| - 1)|y_j| \tag{39}$$

from our previous convention. Equivalently since each moduli space has formal dimension zero the determinant line is given by

$$\mathbb{R} \oplus TL \oplus T^+_{y_0} \oplus T\overline{MW}_r \oplus \bigoplus_{j} \left(T^-_{y_j} \oplus TL \oplus T^+_{y_j} \oplus T\overline{MW}_{|I_j|,1} \oplus \bigoplus_{k\in I_j} T^-_{x_k} \right).$$

Using $T^-_{y_j} \oplus T^+_{y_j} \cong TL$ these three factors disappear. Moving each $T\overline{MW}_{|I_j|,1}$ past $T^-_{x_k}$ for $k < \min I_j$ contributes a number of signs $(|I_j| - 1)\sum_{k<\min I_j} |x_k|$. Moving

$TL \oplus T^+_{y_0}$ past $\mathbb{R}$ gives $|y_0|$ additional signs. Finally we have a contribution from the signs in the definition of $\phi_{|I_j|}$ and the sign from the definition of μ_r

$$\sum_{j=1}^{r} \sum_{i=1}^{|I_j|} i|x_i| + \sum_{j=1}^{r} j|y_j|$$

and the overall sign used in the proof the A_∞ axiom, $1+\sum_{k=1}^n (k+1)|x_k|$. The gluing map has sign (36). In total, the number of signs is

$$\sum_{j=1}^{r} \left((|I_j|-1) \sum_{k<\min I_j} |x_k| \right) + |y_0| + \sum_{j=1}^{r} \sum_{i=1}^{|I_j|} i|x_i| + \sum_{j=1}^{r} j|y_j| + 1 + \sum_{k=1}^{n} (k+1)|x_k| + \sum_{j=1}^{r} (r-j)(|I_j|-1). \tag{40}$$

The difference is

$$\sum_{k=1}^{n} k|x_k| - \sum_{j=1}^{r} \sum_{i=1}^{|I_j|} i|x_i| = \sum_{j=1}^{r} \sum_{i\in I_j} (|I_1| + \ldots + |I_{j-1}|)|x_i|,$$

so (40) equals

$$\sum_{j=1}^{r} (|I_j|-1) \sum_{k<\min I_j} |x_k| + |y_0| + \sum_{j=1}^{r} |I_j| \left(\sum_{k>\max(I_j)} |x_k| \right) + 1 + \sum_{k=1}^{n} |x_k| + \sum_{j=1}^{r} j|y_j| + \sum_{j=1}^{r} (r-j)(|I_j|-1). \tag{41}$$

Now

$$\sum_{j=1}^{r} |I_j| \sum_{k\notin I_j} |x_k| = \sum_{j=1}^{r} |I_j| (\sum_k |x_k| - \sum_{k\in I_j} |x_k|) = n(|y_0|+n-2) - \sum_{j=1}^{r} |I_j|(|y_j|+|I_j|-1).$$

So (40) equals

$$\left(\sum_{j=1}^{r} \sum_{k<\min I_j} |x_k| \right) + |y_0| + 1 + (n-2+|y_0|) + \left(\sum_{j=1}^{r} j|y_j| \right) + n(|y_0|+n-2) - \sum_{j=1}^{r} (|I_j|(|y_j|+|I_j|-1)) + \sum_{j=1}^{r} (r-j)(|I_j|-1).$$

The first term is

$$\sum_{j=1}^{r} \sum_{k\in I_j} (r-j)|x_k| = \sum_{j=1}^{r} (|y_j|+|I_j|-1)(r-j) = \sum_{j=1}^{r} (|y_j|r - |y_j|j + (|I_j|-1)(r-j).$$

So (40) equals

(42)
$$\left(\sum_{j=1}^r |y_j|r - |y_j|j + (|I_j|-1)(r-j)\right) + |y_0| + 1 + (n-2+|y_0|) + \left(\sum_{j=1}^r j|y_j|\right) + n(|y_0|+n-2) - \sum_{j=1}^r (|I_j|(|y_j|+|I_j|-1)) + \sum_{j=1}^r (r-j)(|I_j|-1).$$

Cancelling and simplifying gives

$$(|y_0|+r)r + n|y_0| + 1 - \sum_{j=1}^r |I_j|(|y_j|+|I_j|-1)$$

which is congruent mod 2 to $(n-r)|y_0| + (1-r) - \sum_{j=1}^r |I_j||y_j|$. Combining with (39) and switching $T^+_{y_j} \oplus TL$ with MW_n we obtain

$$(n-r)|y_0| + (1-r) - \sum_{j=1}^r |I_j||y_j| + |y_0|(r-2) + \sum_{j=1}^r (|I_j|-1)|y_j| + (n-1)|y_0|$$

which equals $(1-r) + |y_0| + \sum_{j=1}^r |y_j| = (1-r) + (r-1) = 0$ as claimed. □

If two perturbation systems are equal then we may take as perturbation system for quilted strips the one given by pullback under the map that forgets the quilting. In this case the only isolated holomorphic quilted treed disks are the constant ones, since then $\overline{MW}_{n,1}(L)$ is a $[0,1]$-fiber bundle over the subset of $\overline{MW}_n(L)$ consisting of non-constant holomorphic treed disks. The morphism ϕ is then the identity morphism.

Returning to the general case, the morphism ϕ is a homotopy equivalence of A_∞ algebras, by an argument using *twice-quilted disks* similar to the argument for A_∞ bimodules given later in Section 5.3. We show in Figure 8 the moduli space of twice-quilted stable disks $\overline{M}_{n,2}$, in the case $n=2$ which is a pentagon whose vertices correspond to the expressions

$$f(g(x_1x_2)), f(g(x_1)g(x_2)), f(g(x_1))f(g(x_2)), ((fg)(x_1))((fg)(x_2)), (fg)(x_1x_2).$$

There is a similar treed version $\overline{MW}_{n,2}$, whose description we omit. Given a triple of perturbation systems $(B_{*,0},F_{*,0}),(B_{*,1},F_{*,1}),(B_{*,2},F_{*,2})$ and morphisms

$$\phi_{ij} : CQF(L; B_{*,i}, F_{*,i}) \to CQF(L; B_{*,j}, F_{*,j}), \quad 0 \le i < j \le 2$$

defined by extension of these over the moduli space of quilted disks, we wish to compare the composition $\phi_{12} \circ \phi_{01}$ with ϕ_{02}. Over $\overline{MW}_{n,2}$ we consider the moduli space $\overline{MW}_{n,2}(L)$, using a perturbation system (B_*,F_*) equal to the given perturbation systems equal to the given perturbation systems in a neighborhood of the boundary. (That is, $(B_{*,0},F_{*,0})$ on the edges above the medium shaded region, $(B_{*,1},F_{*,1})$ on edges between the beginning of the medium shaded region and the darkly shaded region, and $(B_{*,2},F_{*,2})$ after the beginning of the darkly shaded region.) The facets of $\overline{MW}_{n,2}$ correspond to either to terms in the definition of composition of A_∞ maps

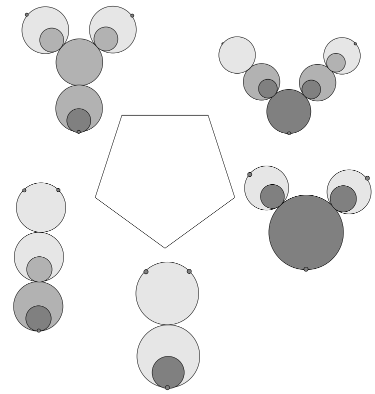

FIGURE 8. Twice-quilted disks

$\phi_{12}\circ\phi_{01} : CQF(L; B_{*,0}, F_{*,0}) \to CQF(L; B_{*,2}, F_{*,2})$, to the components contributing to $\phi_{02} : CQF(L; B_{*,0}, F_{*,0}) \to CQF(L; B_{*,2}, F_{*,2})$, or to terms corresponding to the bubbling off of some markings on the boundary which define a homotopy operator for the difference $\phi_{12} \circ \phi_{01} - \phi_{02}$. Since we do not need homotopy invariance for any of our results (we do need homotopy invariance of the A_∞ bimodule) we omit the proof; see [38] for a related argument for A_∞ functors associated to Lagrangian correspondences. In case $(B_{*,0}, F_{*,0}) = (B_{*,2}, F_{*,2})$ this produces a homotopy between $\phi_{12} \circ \phi_{01}$ and the identity, and hence ϕ_{12}, ϕ_{01} define a homotopy equivalence between $CQF(L; B_{*,0}, F_{*,0})$ and $CQF(L; B_{*,1}, F_{*,1})$. In particular homotopy invariance of $CQF(L)$ implies that the quasimap Floer cohomologies $HQF(L)$ defined using different perturbation systems are isomorphic.

4. Quasimap A_∞ bimodule for Lagrangian pair

We describe here a version of Floer theory for pairs (L_0, L_1) of Lagrangian submanifold, which counts *perturbed holomorphic treed strips* giving rise to an A_∞ *bimodule*, invariant up to A_∞ homotopy of the choice of perturbation. In particular, if L_1 is displaceable from L_0 by a Hamiltonian diffeomorphism then the cohomology of this A_∞ bimodule vanishes.

4.1. A_∞ **bimodules.** Let A_0, A_1 be A_∞ algebras. An A_∞ *bimodule* is a $\mathbb{Z}$-graded vector space M equipped with operations

$$\mu_{d|e} : A_0^{\otimes d} \otimes M \otimes A_1^{\otimes e} \to M[1-d-e]$$

satisfying the relations

$$\sum_{i,k} (-1)^{\aleph} \mu_{d-i+1|e}(a_{0,1}, \ldots, \mu_{0,i}(a_{0,k}, \ldots, a_{0,k+i-1}), a_{0,k+i}, \ldots, a_{0,d}, m, a_{1,1}, \ldots, a_{1,e}))$$
$$+ \sum_{j,k} (-1)^{\aleph} \mu_{d|e-j+1}(a_{0,1}, \ldots, a_{0,d}, m, a_{1,1}, \ldots, \mu_{1,j}(a_{1,k}, \ldots, a_{1,k+j-1}), \ldots, a_{1,e})$$
$$+ \sum_{i,j} (-1)^{\aleph} \mu_{d-i|e-j}(a_{0,1}, \ldots, a_{0,d-i}, \mu_{i|j}(a_{0,d-i+1}, \ldots, a_{0,d}, m, a_{1,1}, \ldots, a_{1,j}), \ldots, a_{1,e}) = 0$$

where we follow Seidel's convention [55] of denoting by $(-1)^{\aleph}$ the sum of the reduced degrees to the left of the inner expression, except that m has ordinary (unreduced) degree. As before, the $\mathbb{Z}$-grading is not necessary and a $\mathbb{Z}_2$-grading suffices.

A *morphism* ϕ of A_∞ -bimodules M_0 to M_1 of degree $|\phi|$ is a collection of maps

$$\phi_{d|e} : A_0^{\otimes d} \otimes M_0 \otimes A_1^{\otimes e} \to M_1[|\phi| - d - e]$$

satisfying a splitting axiom

(43)
$$\sum_{i,j}(-1)^{|\phi|\aleph}\mu_{1,d|e}(a_{0,1},\ldots,a_{0,d-i},\phi_{i|j}(a_{0,d-i+1},\ldots,a_{0,d},m,a_{1,1},\ldots,a_{1,j}),a_{1,j+1},\ldots,a_{1,e})$$
$$+\sum_{i,j}(-1)^{|\phi|+1+\aleph}\phi_{d|e}(a_{0,1},\ldots,a_{0,d-i},\mu_{0,i|j}(a_{0,d-i+1},\ldots,a_{0,d},m,a_{1,1},\ldots,a_{1,j}),a_{1,j+1},\ldots,a_{1,e})$$
$$+\sum_{i,j}(-1)^{|\phi|+1+\aleph}\phi_{d|e}(a_{0,1},\ldots,a_{0,j-1},\mu_{0,i}(a_{0,j},\ldots,a_{0,j+i-1}),\ldots,a_{0,d},m,a_{1,1},\ldots,a_{1,e})$$
$$+\sum_{i,j}(-1)^{|\phi|+1+\aleph}\phi_{d|e}(a_{0,1},\ldots,a_{0,e},m,a_{1,1},\ldots,\mu_{1,j}(a_{1,j},\ldots,a_{j+i-1}),\ldots,a_{1,e}) = 0.$$

Composition of morphisms $\phi : M_0 \to M_1, \psi : M_1 \to M_2$ is defined by

(44)
$$(\phi \circ \psi)_{d|e}(a_{0,1},\ldots,a_{0,d},m,a_{1,1},\ldots,a_{1,e})$$
$$= \sum_{i,j}(-1)^{|\psi|\aleph}\phi_{i|j}(a_{0,1},\ldots,a_{0,i},\psi_{d-i|e-j}(a_{0,i+1},\ldots,a_{0,d},m,a_{1,1},\ldots,a_{1,e-j}),a_{1,e-j+1},\ldots,a_{1,e}).$$

A *homotopy* of morphisms $\psi_0, \psi_1 : M_0 \to M_1$ of degree zero is a collection of maps $(\phi_{d|e})_{d,e\geq 0}$ such that the difference $\psi_1 - \psi_0$ is given by the expression on the left hand side of (43). Any A_∞ algebra A is an A_∞ bimodule over itself with operations

(45)
$$\mu_{d|e}(a_{0,1},\ldots,a_{0,d},m,a_{1,1},\ldots,a_{1,e})$$
$$= (-1)^{1+\diamondsuit}\mu_{d+e+1}(a_{0,1},\ldots,a_{0,d},m,a_{1,1},\ldots,a_{1,e}), \quad \diamondsuit = \sum_{j=1}^{e}(|a_{1,j}|+1),$$

see Seidel [54, 2.9].

4.2. **Treed strips.** In this section we describe the combinatorics of the objects used to define our A_∞ bimodule. A *treed strip with* (d,e) *markings* is a treed disk with $d+e+2$ markings, such that the edges connecting the 0-th marking with the $d+1$-marking all have length zero. The disk components in between z_0 and z_{d+1} have canonical embeddings of holomorphic strips $\mathbb{R} \times [0,1]$, whose complements are the nodes or markings connecting (eventually) to z_0 or z_{d+1}. We denote by $\overline{MW}_{d|e}$ the moduli space of *stable* (d,e)-marked treed strips. See Figure 9 for the case $d = 1, e = 2$. The orientation on $\overline{MW}_{d|e}$ can be chosen via its identification with $\overline{MW}_{d+e+1}$. Denote by $\overline{UMW}_{d|e} \to \overline{MW}_{d|e}$ the *universal treed strip*. An *infinitesimal deformation* of a treed strip of fixed type Σ is an infinitesimal deformation of the nodal points or lengths of the line segments as in (26). Let $\operatorname{Def}_\Gamma(\Sigma)$ denote the space of deformations preserving the combinatorial type. We denote by z the

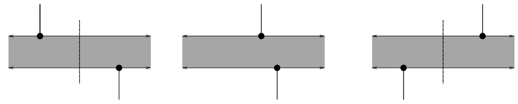

FIGURE 9. Moduli space of stable treed strips with $d = 1, e = 2$

number of nodes corresponding to edges of zero length not connecting strips, and by i the number of infinite nodes connected edges of infinite length or strips. Define

$$\mathrm{Def}(\Sigma) = \mathrm{Def}_\Sigma(\Sigma) \times ((-\infty, 0) \cup \{0\} \cup (0, \infty))^z \times [0, \infty)^i$$

with the second two factors representing *gluing parameters* for line segments of length zero resp. infinity. The gluing construction described above defines a homeomorphism from a neighborhood of zero in $\mathrm{Def}(\Sigma)/\mathrm{aut}(\Sigma)$ to a neighborhood of Σ in $\overline{MW}_{d|e}$.

4.3. **Holomorphic treed quasistrips.** Let (X, ω) be a compact or convex Hamiltonian G-manifold equipped with a G-invariant compatible almost complex structure $J \in \mathcal{J}(X, \omega)^G$ and $L_0, L_1 \subset X/\!\!/G$ compact Lagrangian submanifolds. We suppose that the $L_i, i = 0, 1$ are equipped with metrics induced from G-invariant metrics on $\tilde{L}_i, i = 0, 1$, and Morse-Smale functions $F_i \to \mathbb{R}$. Denote by $\tilde{F}_i : \tilde{L}_i \to X$ the lifts of F_i to $\tilde{L}_i$.

Definition 4.1. A *holomorphic treed quasistrip* for $L_0, L_1 \subset X/\!\!/G$, $J \in \mathcal{J}(X)^G$ and $H \in C_c^\infty([0, 1] \times X)^G$ consists of a treed strip Σ, a continuous map $u : \Sigma \to X$ such that (i) on each strip, u is (J, H)-holomorphic strip with boundary in $(\tilde{L}_0, \tilde{L}_1)$; (ii) on each disk connecting to $\mathbb{R} \times \{j\}$, u is a J-holomorphic disk with boundary in $\tilde{L}_j$, and (iii) on each edge connecting to $\mathbb{R} \times \{j\}, j = 0, 1$, u is a gradient trajectory of $\tilde{F}_j$ on $\tilde{L}_j$. An *isomorphism* of treed quasistrips $u_j : \Sigma_j \to X, j = 0, 1$ is a morphism of treed strips $\phi : \Sigma_0 \to \Sigma_1$ together with an element $g \in G$, such that $gu_0 = \phi^* u_1$. A treed quasistrip is *stable* if it has no automorphisms (that is, each strip or disk on each u is constant has at least three nodal or marked points) and each node connecting two edges maps to a critical point.

See Figure 10, in which the dotted line indicates a broken trajectory. For each treed disk we have a metric tree $\Sigma_{(1)}$ obtained by forgetting the disk and strip components. As in the construction of the Fukaya algebra we allow perturbation systems $F_* = (F_{n,v} \in C^\infty(UW_{n,v} \times (L_0 \cup L_1)))$ depending on $\Sigma_{(1)}$. We require that $F_{n,v}$ is equal to F on the complement of a compact subset of the union of open edges. Let $\overline{MW}_{d|e}(L_0, L_1; H, F_*)$ denote the moduli space of isomorphism classes of finite energy stable treed perturbed holomorphic quasistrips with boundary in $(\tilde{L}_0, \tilde{L}_1)$ with d left resp. e right markings.

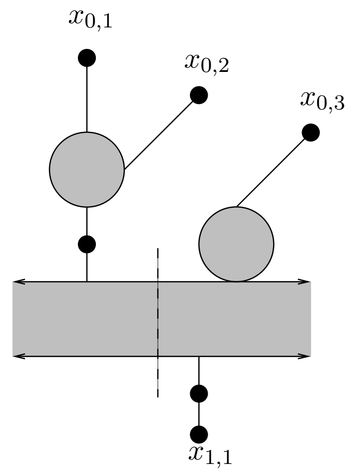

FIGURE 10. A holomorphic treed strip

Given a holomorphic quasistrip u we denote by D_u the linearization of the map

(46)
$$\mathcal{F}_u : \mathrm{Def}_\Gamma(\Sigma) \oplus \Omega^0(\tilde{\Sigma}_{(2)}, u_2^* TX, (\partial u_2)^* T(\tilde{L}_0 \sqcup \tilde{L}_1))_{1,p,\alpha} \oplus \Omega^0(\tilde{\Sigma}_{(1)}, u_1^* T(\tilde{L}_0 \sqcup \tilde{L}_1))_{1,p,\alpha}$$
$$\to \Omega^{0,1}(\tilde{\Sigma}_{(2)}, u^* TX)_{0,p,\alpha} \oplus \Omega^1(\tilde{\Sigma}_{(1)}, u_1^* T(\tilde{L}_0 \sqcup \tilde{L}_1))_{0,p,\alpha} \oplus \bigoplus_{j=1}^m T_{u(w_j^\pm)}(I_{\delta(j)})$$

where $T(\tilde{L}_0 \sqcup \tilde{L}_1)$ denotes either boundary conditions in $T\tilde{L}_0$ or $T\tilde{L}_1$, depending on the boundary component, $I_{\delta(j)} = \tilde{L}_0, \tilde{L}_1$ or $\tilde{L}_0 \cap \tilde{L}_1$ depending on the type of node, given by $\xi \mapsto \mathcal{T}_u(\xi)^{-1} \overline{\partial}_{J,H} \exp_u(\xi)$ on the strip components of dimension 2, by the Cauchy-Riemann operator on the disk components, the gradient operator on the one-dimensional components, and the difference of evaluation maps at the nodes $w_j^\pm$, see (18). If H is chosen so that the set of generalized intersection points $\mathcal{I}(L_0, L_1; H)$ of (3) is transverse, then D_u is Fredholm, by the discussion in Section 2. We say that u is *regular* if D_u is surjective. Let $MW^{\mathrm{reg}}_{d|e,\Gamma}(X, L_0, L_1; H)$ denote the moduli space of regular stable treed quasistrips with the combinatorial type Γ. Then $MW^{\mathrm{reg}}_{d|e,\Gamma}(X, L_0, L_1; H)$ is a smooth manifold, with tangent space at u equal to the kernel of D_u modulo the subspace generated by $\mathfrak{g}$ and $\mathrm{aut}(\Sigma)$, by an application of the implicit function theorem for Banach spaces similar to that of Theorem 2.21.

As in the previous section, choices of perturbations used to define the spaces $\overline{MW}_{d'|e'}(L_0, L_1), \overline{MW}_i(L_0), \overline{MW}_j(L_1)$ for $d' < d, e' < e$ and all i, j induce perturbations $F^{pre}_{n,v}$ on $\overline{MW}_{d|e}(L_0, L_1)$ in a neighborhood of the image of the strata

$$\overline{MW}_{d_0|e_0}(L_0, L_1)_0 \times_{\mathcal{I}(L_0, L_1)} \overline{MW}_{d_1|e_1}(L_0, L_1)_0$$

and

$$\overline{MW}_{d-i+1|e}(L_0, L_1)_0 \times_{\mathrm{crit}(F_0)} \overline{MW}_i(L_0)_0, \quad \overline{MW}_{d|e-i+1}(L_0, L_1)_0 \times_{\mathrm{crit}(F_1)} \overline{MW}_i(L_1)_0$$

under the gluing construction. These extend to a system of perturbations over all of $\overline{MW}_{d'|e'}(L_0, L_1)$, since the space of perturbations is convex.

Theorem 4.2. *Suppose that every stable holomorphic disk with boundary in $\tilde{L}_0, \tilde{L}_1$ is regular, and a Hamiltonian perturbation H has been chosen so that every quasistrip is regular. For any d, e, γ, given a system of perturbations F_* for strata with $d' < d, e' < e, E(\gamma') < E(\gamma)$ extending the given pairs on the semi-infinite ends such that every holomorphic treed strip of class γ' is regular, there exists a comeager subset of perturbations $\mathcal{P}^{\mathrm{reg}}(L_0, L_1)$ in $C^\infty_b(UW_{d+e+1,v} \times (L_0 \sqcup L_1))$ such that if $F_1 \in \mathcal{P}^{\mathrm{reg}}(L_0, L_1)$ and ρ is a cutoff function on $UW_{d+e+1,v}$ with sufficiently large compact support on the union of open edges, every holomorphic treed quasistrip $u : \Sigma \to X$ for $F_{n,v} := F^{pre}_{n,v} + \rho F_1$ with homotopy class γ is regular.*

Proof. As in Theorem 3.4; the only issue is to make the evaluation maps at the nodes connecting edges with disks transverse. This holds for generic perturbations F_1 by Sard-Smale, by choosing sufficiently large perturbations on the interiors of the edges so that the evaluation maps at the endpoints are submersions, for perturbations of class C^l, and then for smooth perturbations by density. □

Theorem 4.3. *Suppose that L_0, L_1 are such that all holomorphic disks with boundary in $\tilde{L}_0, \tilde{L}_1$ are regular; a compactly supported Hamiltonian perturbation H has been chosen as above so that all (J, H)-holomorphic strips are regular; and perturbations $F_{n,v}$ have been chosen inductively as above so that all treed quasistrips are regular. Then $\overline{MW}_{d|e}(L_0, L_1)_1$ is a compact one-manifold with boundary given by the union of broken treed trajectories*

$$\overline{MW}_{d_0|e_0}(L_0, L_1)_0 \times_{\mathcal{I}(L_0,L_1)} \overline{MW}_{d_1|e_1}(L_0, L_1)_0$$

and configurations

$$\overline{MW}_{d-i+1|e}(L_0, L_1)_0 \times_{\mathrm{crit}(F_0)} \overline{MW}_i(L_0)_0, \quad \overline{MW}_{d|e-i+1}(L_0, L_1)_0 \times_{\mathrm{crit}(F_1)} \overline{MW}_i(L_1)_0$$

of a treed strip and a treed disk.

Proof. Compactness is Gromov compactness as in [42] combined with compactness for Floer trajectories for Lagrangian pairs with clean intersection, discussed in Section 2, and for broken gradient trajectories; note that compact support of the Hamiltonian perturbation H allows us to apply the maximum principal outside of a compact set given by the convexity condition. Let u represent a regular point in $\overline{MW}_{d|e}(L_0, L_1)$. Define the *preglued treed holomorphic strip* $G^{\mathrm{approx}}_\delta(u) : \Sigma^\delta \to X$ by combining the gluing procedures for Morse trajectories for pairs of Lagrangians with clean intersections (discussed above in the proof of Theorem 2.22) and gluing for disks (see Abouzaid [1]) and gluing for Morse trajectories (see Schwarz [50]). The proofs of existence of a uniformly bounded right inverse, error estimate and uniform quadratic bound are the same as in those situations, since the third term in $\mathcal{F}_{G^{\mathrm{approx}}_\delta(u)}$ does not involve the gluing parameter. By the implicit function theorem there exists a unique solution (ζ, ξ) to $\mathcal{F}_{G^{\mathrm{approx}}_\delta(u)}(\zeta, \xi)) = 0$ in a sufficiently small neighborhood of 0 in $\mathrm{Def}_\Gamma(\Sigma) \oplus \Omega^0(\Sigma^\delta_{(2)}, (G^{\mathrm{approx}}_\delta(u))^* TX)_{1,p,\delta,\alpha} \oplus \Omega^0(\Sigma^\delta_{(1)}, (G^{\mathrm{approx}}_\delta(u))^* T(\tilde{L}_0 \sqcup \tilde{L}_1))_{1,p,\delta,\alpha}$, and we set $G_\delta(u) := \exp_{G^{\mathrm{approx}}_\delta(u)}(\xi) : \Sigma^{\zeta,\delta} \to$

X, where $\mathcal{F}_{G_\delta^{\mathrm{approx}}(u)}$ is as in the proof of Theorem 2.22 but with the perturbed Cauchy-Riemann operator on the strips. □

Remark 4.4. In the proof above we required weighted Sobolev spaces on $\Sigma_{(1)}$ because the function $\tilde{F}$ is Morse-Bott but not Morse. However, there is bijection between gradient trajectories of $\tilde{F}$ modulo G and gradient trajectories of F, and by working with gradient trajectories of F instead of $\tilde{F}$ the use of weighted Sobolev spaces on $\Sigma_{(1)}$ may be avoided.

Suppose that every element of $\overline{MW}_{d|e}(L_0, L_1)$ is regular. Orientations are constructed by choosing for each such end an orientation on the disk one with marking, and boundary given by a path from TL_0 to TL_1, see [61]. Define operations

$$\mu_{d|e} : CQF(L_0)^{\otimes d} \otimes CQF(L_0, L_1) \otimes CQF(L_1)^{\otimes e} \to CQF(L_0, L_1)$$

by counting treed Floer trajectories;

$$\begin{aligned} (47) \quad & <x_{0,1}> \otimes \ldots \otimes <x_{0,d}> \otimes <x> \otimes <x_{1,1}> \otimes \ldots \otimes <x_{1,e}> \\ & \mapsto \sum_{[u] \in \overline{MW}_{d|e}(x_{0,1},\ldots,x_{0,d},x,x_{1,1},\ldots,x_{1,e},y)_0} (-1)^{\heartsuit + \diamondsuit} \epsilon(u) \, \mathrm{Hol}_{L_0,L_1}(u) q^{A(u)} <y> \end{aligned}$$

where $\diamondsuit$ is defined in (45)

Theorem 4.5. *Suppose that perturbations have been chosen as in Theorem 4.3. Then the maps $(\mu_{d|e})_{d,e \geq 0}$ induce on $CQF(L_0, L_1)$ the structure of a $\mathbb{Z}_2$-graded A_∞ $(CQF(L_0), CQF(L_1))$-bimodule.*

Proof. By the previous Theorem 4.3 and the definition in (43). The sign computation is equivalent to that in Theorem 3.6. □

The A_∞ bimodule $CQF(L_0, L_1)$ is independent of the choice of Hamiltonian perturbation H, metrics B_j on $\tilde{L}_j, j = 0, 1$, and perturbation system F_* used to construct it, up to homotopy of A_∞ bimodules over Λ. (Independence of F_* holds over Λ_0, while independence from H holds only over Λ.) However, we will not give the argument, since it is very similar to the A_∞ bimodule map given in the next section. Note we are assuming that all holomorphic disks are regular, and H is such that (J, H)-holomorphic strips are regular.

Lemma 4.6. *Suppose that $J \in \mathcal{J}(X)^G$ is such that every non-constant stable holomorphic disk with boundary in L is regular and has positive Maslov index. Then for sufficiently small perturbations we have $\mu_{0,0}^2 = 0$ and hence the quasimap Floer cohomology of the pair*

$$HQF(L_0, L_1) := H(\mu_{0,0})$$

is well-defined.

Proof. The proof is similar to that of Proposition 3.7. The A_∞ relation $\mu_{0,0}^2(\cdot) = \mu_{1,0}(\mu_0^{L_0}(1), \cdot) \pm \mu_{0,1}(\cdot, \mu_0^{L_1}(1))$ and the fact that $\mu_0^{L_j}(1)$ is a multiple of the generator corresponding to the maximum of the Morse function implies that $\mu_{0,0}^2(\cdot) = 0$. □

5. A_∞ -BIMODULE ISOMORPHISM IN THE CASE OF AN EQUAL PAIR

We show that the A_∞ algebra constructed in Section 3 for a Lagrangian L, considered as an A_∞ bimodule over itself, is isomorphic to the A_∞ bimodule constructed in Section 4, for the pair (L, L).

5.1. **Quilted strips and quilted treed quasistrips.** A (d, e)*-marked quilted strip* is the same as a (d, e)-marked strip, that is, a strip with markings on the boundary, except that isomorphisms do not include translations. Equivalently, a (d, e)-marked quilted strip is a (d, e)-marked strip $(\mathbb{R} \times [0, 1], z_{0,1}, \ldots, z_{0,d}, z_{1,1}, \ldots, z_{1,e})$ with an additional interior marking at some point $y = (s, 1/2)$, and an isomorphism from $(\Sigma, z_{0,1}, \ldots, z_{0,d}, z_{1,1}, \ldots, z_{1,e}, y)$ to $(\Sigma', z'_{0,1}, \ldots, z'_{0,d}, z'_{1,1}, \ldots, z'_{1,e}, y')$ is an isomorphism $\Sigma \to \Sigma'$ mapping $z_{i,j}$ to $z'_{i,j}$ and y to y'. Geometrically the interior marking y represents the point at which we "turn on" a Hamiltonian perturbation, and we will draw a quilted strip by drawing a shaded region beginning at y and extending to the right.

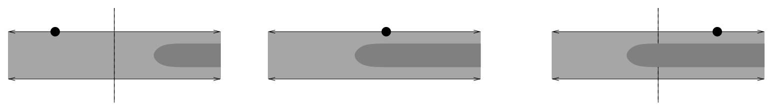

FIGURE 11. Quilted strips

The space $M_{d|e,1}$ of $(d|e)$-marked quilted strips has a compactification $\overline{M}_{d|e,1}$ allowing disk bubbles to develop on the boundary and strips to bubble off on both ends. In the pictures, we find it convenient to indicate the quilted strip (as opposed to the strip bubbles) by adding a shaded region, signifying that the Cauchy-Riemann equation is given a Hamiltonian perturbation on this region. See Figure 11 for the moduli space of quilted $(1, 0)$-marked strips. A *quilted component* of the quilted strip is a component with some shading, otherwise the component is *unquilted*. Each quilted component has either one or two *quilted ends*, that is, ends with some shaded region. The cells of $\overline{M}_{d|e,1}$ are in one-to-one correspondence with expressions in d variables $x_1, \ldots, x_d$, a variable m, and variables $y_1, \ldots, y_e$, and a function f obtained by adding parentheses to the expression $x_1 \ldots x_d m y_e \ldots y_1$ and then adding $f(\ldots)$ around some expression including m. For example, for the case $(d, e) = (1, 0)$ the possible expressions are $f((x_1 m)), f(x_1 m), x_1 f(m)$. This parametrization of cells is similar to that of the associahedron, but the combinatorial types correspond to $d + e + 2$-leaved trees together with a choice of vertex on the path connecting the 0-th to the $d + 1$-st leaf.

As before, there is also a treed version. A (d, e)*-marked quilted treed strip* is the same as a quilted strip, but now allowing components of dimension one of possibly infinite or zero length between disk components or between disk and strip components or between unquilted strip components or between unquilted strip components and quilted strip components, but not between quilted strip components. In addition, there are one-dimensional components attached to the markings on the boundary, and to the point at infinity at the left of the strip. Two quilted treed strips are *isomorphic* if they are related by automorphism. A quilted treed strip is *stable* if it has no infinitesimal automorphisms, and any node connecting two components of dimension one is of infinite type, that is, attaching ends of the segments of infinite length. Let $\overline{MW}_{d|e,1}$ denote the moduli space of stable quilted treed strips. The case $d = 2, e = 0$ is shown in Figure 12. We have not drawn the line segments connecting the strips, as this makes the picture even more complicated; thus dotted lines in the left-hand configurations mean that the strip has broken and is connected by a segment of infinite length. Each picture shows the configuration represented by a nearby stratum. The moduli spaces $\overline{MW}_{d|e,1}$ are complicated even for low values of d, e, as can be expected from the definition of morphism of A_∞ bimodules and the fact that we are adding trees.

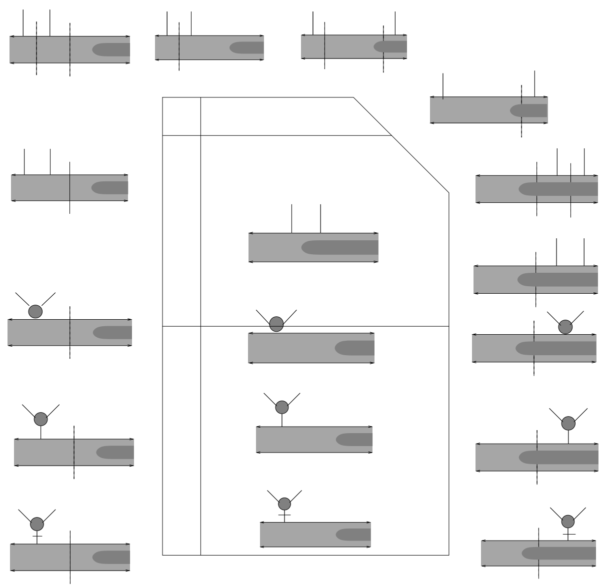

FIGURE 12. The moduli space $\overline{MW}_{2|0,1}$ of $2|0$-marked treed quilted strips

The forgetful map $\overline{MW}_{d|e,1} \to \overline{MW}_{d|e}$ has one-dimensional fibers that are canonically oriented so that the positive orientation corresponds to moving the quilting to the left, and so the orientation of $\overline{MW}_{d|e}$ induces an orientation on $\overline{MW}_{d|e,1}$.

5.2. A_∞ **bimodule morphisms.** We construct a morphism of A_∞ bimodules $CQF(L) \to CQF(L, L)$ via a continuation argument which counts *quilted treed holomorphic quasistrips.* Let $H \in C_c^\infty((0,1) \times X)^G$ be a regular perturbation for the pair (L, L). Given a quilted strip with shaded region beginning at s_- and shaded region ending at s_0 we consider a Hamiltonian perturbation $H_1 = H_{1,s}\mathrm{d}s + H_{1,t}\mathrm{d}t$ equal to $H\mathrm{d}t$ for $s \gg 0$ and vanishing for $s \ll 0$. For any (J, H_1)-holomorphic u, we denote by D_u the associated linearized operator, using weighted Sobolev spaces on the strip-like ends. Let $\tilde{F}_{n,v} \in C^\infty(UW_{n,v} \times \tilde{L})^G$ be a perturbation of $\tilde{F}$ depending on the underlying tree $\Sigma_{(1)}$ as above.

Definition 5.1. A (perturbed) *holomorphic quilted treed quasistrip* to X consists of a quilted treed strip Σ and a continuous map $u : \Sigma \to X$ such that (i) on each quilted strip with one quilted end, u is a (J, H_1)-holomorphic map (ii) on each quilted strip with two quilted ends, u is (J, H)-holomorphic map (iii) on each unquilted disk or strip, u is a J-holomorphic map (iv) on each edge, u is a gradient trajectory of $\tilde{F}_{n,v}$. An *isomorphism* of holomorphic quilted treed quasistrips $u_j : \Sigma_j \to X$ is an isomorphism $\phi : \Sigma_0 \to \Sigma_1$ and an element $g \in G$ such that $\phi^* u_1 = g u_0$; in particular ϕ must preserve the quilting on the quilted strip component. A holomorphic quilted treed quasistrip is *stable* if it has no automorphisms, and every node connecting two edges maps to a critical point of F.

See Figure 13 where the unquilted components are represented as disks, the component where the quilted regions begins is represented as a disk with a single strip-like end, and the quilted components are represented by strips.

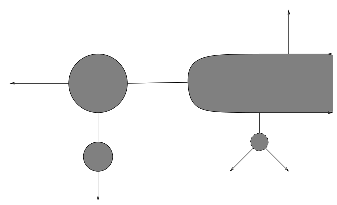

FIGURE 13. A holomorphic quilted treed quasistrip

Let $\widetilde{\mathcal{P}}(X)$ be the subset of $\Omega^1(\Sigma, C^\infty(X)^G)$ consisting of forms $H_s \mathrm{d}s + H_t \mathrm{d}t$ with $H_s, H_t \in C^\infty_b(X)^G$ having all derivatives bounded. Let $\rho \in C^\infty(\Sigma)$ be a function with $\rho(s) = 0, s \leq 0$ and $\rho(s) = 1, s \geq 1$. Let $\rho_1 \in C^\infty_c(\Sigma \times X)$ be non-zero on $[0,1] \times [0,1] \times (L_0 \cup L_1)$. For any element $H_1' \in \widetilde{\mathcal{P}}(X)$, we form an admissible perturbation $H_1 = \rho H \mathrm{d}t + \rho_1 H_1'$. The proof of the following is similar to that of Theorem 3.4 and is omitted:

Proposition 5.2. *There exists a comeager subset* $\widetilde{\mathcal{P}}^{\mathrm{reg}}(X, L) \subset \widetilde{\mathcal{P}}(X)$ *such if* $H_1' \in \widetilde{\mathcal{P}}^{\mathrm{reg}}(X, L)$ *then every* (J, H_1)*-holomorphic strip is regular.*

The moduli space $\overline{MW}_{d|e,1}(L)$ of holomorphic quilted treed quasistrips with boundary in $\tilde{L}$ has a natural evaluation map

$$\mathrm{ev} : \overline{MW}_{d|e,1}(L) \to \mathrm{crit}(F)^d \times \mathrm{crit}(F) \times \mathrm{crit}(F)^e \times \mathcal{I}(L, L)$$

where $\mathcal{I}(L, L) \cong L \cap \phi_1(L)$ is the set of generalized intersection points of L with itself. Let $\overline{MW}_{d|e,1}(z_1, \ldots, z_{d+e+1}, \ldots, y)$ denote the moduli space of holomorphic quilted treed disks with limits $z_1, \ldots, z_{d+e+1}, y$ along the semi-infinite edges. Assuming that every stable holomorphic disk is regular and H, H_1 have been chosen as above, perturbations of the functions on the edges can be constructed inductively so that every quilted treed holomorphic strip is regular. Consider $CQF(L)$ as an A_∞ bimodule over itself (note the change of signs (45)), and let $CQF(L, L)$ denote the A_∞ bimodule defined by the perturbation H. Using the trivialization of the Λ-line bundle over the paths in $\mathcal{I}(L,L)$ we let $\mathrm{Hol}_L(u) \in \Lambda$ denote the holonomy around the line bundle around the boundary of u. Define

$$\begin{aligned} \phi_{d|e} : CQF(L)^{\otimes d} \otimes CQF(L;\Lambda) \otimes CQF(L)^{\otimes e} &\to CQF(L,L) \\ \langle z_{0,1}\rangle \otimes \ldots \otimes \langle z_{0,d}\rangle \otimes \langle z\rangle \otimes \langle z_{1,1}\rangle \otimes \ldots \otimes \langle z_{1,e}\rangle \\ \mapsto \sum_{[u]\in \overline{MW}_{d|e,1}(z_{0,1},\ldots,z_{0,d},z,z_{1,1},\ldots,z_{1,e},y)_0} (-1)^{\heartsuit+\diamondsuit} \epsilon(u)\,\mathrm{Hol}_L(u) q^{A(u)} \langle y\rangle \end{aligned} \tag{48}$$

where the cochain groups involved are defined using Λ coefficients; the values $A(u)$ are not necessarily positive because of the additional term in the energy-area relation (4).

Proposition 5.3. *Assuming perturbations have been chosen so that all holomorphic treed quasistrips are regular, the collection $\phi = (\phi_{d|e})_{d,e\geq 0}$ is a morphism of A_∞ bimodules from $CQF(L)$ to $CQF(L,L)$. If all non-trivial holomorphic disks have positive Maslov index then $\phi_{0|0}$ is a chain map: $\phi_{0,0} \circ \mu_{1,L} = \mu_{1,L,L} \circ \phi_{0,0}$.*

Proof. The boundary components of $\overline{MW}_{d|e,1}(z_{0,1},\ldots,z_{0,d},z,z_{1,1},\ldots,z_{1,e},y)_1$ consist of configurations where a treed strip has broken off or a treed disk has broken off. The former case corresponds to one of the first two terms in (43) while the latter corresponds to the last two terms. The signs are similar to that of Theorem 3.6: The first term in (43) (in which μ appears before ϕ) has an additional sign from coming from the definition of the orientation on $\overline{M}_{d|e,1}$ as an $[0,1]$-bundle over $\overline{M}_{d|e}$, so that the orientation of the fiber corresponds to moving the quilting to the right; this means that the positive orientation on the gluing parameter for the boundary components corresponding to terms of the first type becomes identified with the negative orientation on these fibers, giving rise to the additional sign. The degree of the morphism is zero, which causes those contributions to the sign in (43) to vanish. This leaves the contributions from $\aleph$, which are similar to those dealt with before and left to the reader. The proof of the assertion on $\phi_{0|0}$ is similar to that of Lemma 4.6: in the absence of holomorphic disks of non-positive index, the additional terms in the A_∞ relation vanish. □

On the other hand, counting strips with given limits on the *right* defines a similar morphism of A_∞ bimodules $\psi = (\psi_n)_{n\geq 0}$ from $CQF(L,L)$ to $CQF(L)$, by the same argument but reading everything from right to left.

5.3. **Homotopies of morphisms of A_∞ bimodules.** To show that the composition of $\psi \circ \phi$ is homotopic to the identity we introduce a moduli space of *twice* quilted treed quasistrips. This means that each strip $\Sigma = \mathbb{R} \times [0,1]$ has two distinguished lines, represented by values $s_-, s_+ \in \mathbb{R}$ where the quilted region starts and where it ends; the corresponding components of Σ are the quilted components. Let $\overline{MW}_{d|e,2}$ denote the moduli space of twice quilted strips with d markings on the first resp. second boundary component. The moduli space $\overline{M}_{0|0,2}$ is shown in Figure 14; the shading represents the region where a Hamiltonian perturbation is allowed. As before there is a notion of *treed twice-quilted strip* which allows one-dimensional segments between the disk and strip components. Let $\overline{MW}_{d|e,2}$ denote the moduli

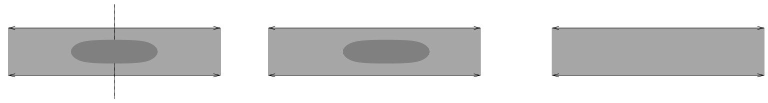

FIGURE 14. Twice quilted quasistrips

space of stable treed twice-quilted strips. The reader is encouraged to draw the picture of $\overline{MW}_{1|0,2}$ for his or herself.

The structure of a twice-quilting defines a region where the Hamiltonian perturbation is non-zero. Suppose we are given a compactly supported Hamiltonian perturbation $H \in C_c^\infty([0,1] \times X)^G$ that is a regular perturbation for the pair L, L, and a one-form $H_1 \in \Omega_1(\Sigma, C_c^\infty([0,1] \times X)^G)$ that makes the moduli space of quilted strips regular. Given a twice quilted strip with shaded region beginning at s_- and shaded region ending at s_+ consider a Hamiltonian-valued one-form H_2 depending on $s_+ - s_-$ with support on $[s_-, s_+] \times [0,1]$ which "turns on and off the Hamiltonian perturbation". That is, if $s_+ - s_- > 2$ then H_2 is equal to $H\mathrm{d}t$ between $s_- + 1$ and $s_+ - 1$, is equal to the translation of H_1 by s_- on $(s_-, , s_- + 1)$ and the reflection and translation of H_1 on $(s_+ - 1, s_+)$, and if $s_+ - s_- < 1$ then H_2 vanishes. Let $F_{n,v} \in C^\infty(UW_{n,v} \times L)$ be a perturbation of F depending on the underlying tree $\Sigma_{(1)}$ as above. A (perturbed) *holomorphic twice-quilted treed quasistrip* to X consists of a twice-quilted treed disk Σ and a continuous map $u : \Sigma \to X$ such that (i) on each twice-quilted strip, u is a (J, H_2)-holomorphic map (ii) on each quilted strip, u is a (J, H_1)-holomorphic map (recall H_1 is the homotopy from 0 to $H\mathrm{d}t$ used in 5.2) (iii) on each unquilted strip between two quilted components, u is a (J, H)-holomorphic map (iv) on each unquilted strip not between quilted components, u is J-holomorphic (v) on each edge, u is a gradient trajectory of $\tilde{F}_{n,v}$. An *isomorphism* of holomorphic twice-quilted treed quasistrips $u_j : \Sigma_j \to X$ is an isomorphism $\phi : \Sigma_0 \to \Sigma_1$ and an element $g \in G$ such that $\phi^* u_1 = g u_0$ and the quilt structures are preserved. A holomorphic twice-quilted treed quasistrip is *stable* if it has no automorphisms, and every node connecting two edges maps to a critical point of F. Let $\overline{MW}_{d|e,2}(L)$ denote the moduli space of stable holomorphic (d, e)-marked treed twice quilted quasistrips. Assume that all holomorphic quasidisks with boundary in $\tilde{L}$ are regular. By a generic choice of perturbations on the twice-quilted strips (that is, a generic perturbation of H_2 depending on $s_+ - s_-$ that is equal to the given function in a neighborhood of the boundary, and a generic perturbation of the function $\tilde{F}$ equal to the given function on a neighborhood of the boundary) one obtains transversality as in the previous discussion in the proof of Proposition 5.2. For generic choices of perturbations chosen inductively, the moduli spaces of treed strips $\overline{MW}_{d|e,2}(L)$ consists entirely of regular elements and the boundary consists of configurations where either (i) the shaded strip has separated into two components of a broken strip (ii) a strip has broken off at $\pm\infty$ (iii) a treed disk has bubbled off along an edge of infinite length. Define

$$\tau_{d|e} : CQF(L)^{\otimes d} \otimes CQF(L) \otimes CQF(L)^{\otimes e} \to CQF(L)[-1] \tag{49}$$
$$(z_{0,1}, \ldots, z_{0,d}, z, \ldots, z_{1,1}, \ldots, z_{1,e}, y)$$
$$\mapsto \sum_{[u] \in \overline{MW}_{d|e,2}(z_{0,1},\ldots,z_{0,d},z,z_{1,1},\ldots,z_{1,e},y)_0} (-1)^{\heartsuit + \diamondsuit} \epsilon(u) \operatorname{Hol}_{L_0,L_1}(u) q^{A(u)} <y> .$$

Theorem 5.4. *Suppose that every stable holomorphic disk with boundary in L is regular, and perturbations have been chosen so that all treed strips and treed quilted strips are regular leading to maps of A_∞ bimodules $\phi : CQF(L) \to CQF(L, L)$ resp. $\psi : CQF(L, L) \to CQF(L)$. Then $(\tau_{d|e})_{d,e \geq 0}$ defines a homotopy of morphisms of A_∞ bimodules from the identity to $\psi \circ \phi$ in* $\operatorname{Hom}(CQF(L), CQF(L, L))$. *If every non-trivial holomorphic disks has positive Maslov index then $\tau_{0|0}$ is a chain homotopy of the identity to $\psi_{0|0} \circ \phi_{0|0}$. In particular, $HQF(L; \Lambda)$ is isomorphic to $HQF(L, L)$.*

Proof. The components of the boundary of $\overline{MW}_{d|e,2}(z_{0,1}, \ldots, z_{0,d}, z, z_{1,1}, \ldots, z_{1,e}, y)_1$ which do *not* involve the shaded region breaking, or the shaded region disappearing, correspond the terms on the left-hand-side of (43). The components where the shaded region breaks correspond to the contributions to the composition $\psi \circ \phi$, while the components where the shaded region vanishes give the identity morphism of A_∞ modules. The latter follows from a standard symmetry argument: since we have chosen the perturbation so that regularity has been achieved, and the equation on the strip with no shading is translation invariant, the only solutions are the constant strips. The first assertion follows. The proof of the second assertion is similar to that of Proposition 5.3: In the case that every non-trivial holomorphic disk has positive Maslov index, $\mu_{0,L}(1)$ resp. $\mu_{0,L,L}(1)$ is a multiple of the generator 1_L resp. $1_{L,L}$ and the additional terms in the A_∞ relation vanish. □

Corollary 5.5. *Suppose that $L \subset X /\!\!/ G$ is a Lagrangian such that every non-trivial holomorphic disk in X with boundary in $\tilde{L}$ is regular and positive Maslov index. If L is displaceable, then $HQF(L; \Lambda) = 0$.*

Proof. Since $HQF(L; \Lambda) \cong HQF(L, L)$ and the latter vanishes if L is displaceable, since $CQF(L, L)$ does for the function H whose flow displaces L. □

6. Gauged potentials for toric moment fibers

In this section we prove the main Theorem 1.1. Let $X \cong \mathbb{C}^N$, $G \subset H := U(1)^N$ a sub-torus, acting on X with moment map $\Phi : X \to \mathfrak{g}^\vee$. Suppose that the G action on $\Phi^{-1}(0)$ has only finite stabilizers, so that $X /\!\!/ G$ is a presentation of a quasiprojective toric orbifold with residual action of $T = H/G$ and moment map $\Psi : X /\!\!/ G \to \mathfrak{t}^\vee$. Recall that the moment polytope $\Psi(X /\!\!/ G)$ is given by a finite set of linear inequalities (1). Let λ lie in the interior of $\Psi(X /\!\!/ G)$. Recall that $L_\lambda = \Psi^{-1}(\lambda) \subset X /\!\!/ G$ is a Lagrangian torus, and $\tilde{L}_\lambda \subset X$ its inverse image in X. By definition $\langle v_i, \cdot \rangle - c_j$ is the function on $\mathfrak{t}^\vee$ defined by the i-th coordinate function on $\mathfrak{h}^\vee \cong \mathbb{R}^N$, so the moment map for the action of H on $\mathbb{C}^N$ takes value $l_j(\lambda)$ on

$\tilde{L}_\lambda$. Hence $\tilde{L}_\lambda$ is obtained from the standard torus $(S^1)^N \subset \mathbb{C}^N$ by re-scaling the factors:

$$\tilde{L}_\lambda = \{(z_1, \dots, z_N) \mid |z_j|^2/2 = l_j(\lambda)/2\pi, j = 1, \dots, N\} = \prod_{j=1}^{N} (l_j(\lambda)/\pi)^{1/2} S^1.$$

The following classification result of Cho-Oh [12, Theorem 5.3] for disks with boundary in $\tilde{L}_\lambda$ allows the computation of the gauged potential:

Proposition 6.1. *Any holomorphic disk in $X = \mathbb{C}^N$ with boundary in $\tilde{L}_\lambda$ is given by a Blaschke product*

$$u(z) = \left((l_j(\lambda)/\pi)^{1/2} \prod_{k=1}^{d_j} \frac{z - \alpha_{j,k}}{1 - \overline{\alpha}_{j,k} z} \right)_{j=1}^{N}$$

for some constants $\alpha_{j,k}$ of norm less than one and some non-negative integers d_j.

Proof. For completeness we reproduce the proof. Since the complex structure and Lagrangian are split, it suffices to consider the case that $X = \mathbb{C}$ and $\tilde{L}_\lambda = S^1$ is the unit circle. Let $u : D \to X$ be a holomorphic disk with boundary in $\tilde{L}_\lambda$ with d zeroes counted with multiplicity. Let v be the Blaschke product whose zeroes are those of u with boundary in $\tilde{L}_\lambda$. Then u/v is a map without zeroes, whose boundary lies in $\tilde{L}_\lambda$. But any such map must be constant (by e.g. the maximum principal for the components of u/v and v/u) so v is equal to u up to scalars of unit norm. □

Corollary 6.2. *For the standard complex structure on X, every stable disk with boundary in $\tilde{L}_\lambda$ is regular and has positive Maslov index.*

Proof. For smooth domain this is proved by Cho-Oh [12, Section 6]. An alternative argument involves doubling the disk to obtain a sum of rank one, non-negative index Fredholm problems on the sphere, see e.g. [61, p.5]; these always have trivial cokernel. Because of the H-action on space of holomorphic disks with the given boundary condition the evaluation map at a marked point on the boundary is a submersion, since H acts transitively on $\tilde{L}_\lambda$. Regularity for stable disks whose combinatorial type is a tree with k components, follows by induction on k, as in Fukaya et al [21, Theorem 11.1]: Suppose that Σ is a marked disk such that the component containing the root z_0 is attached to d other stable disks. By the inductive hypothesis the d stable disks are regular and the evaluation map at the attaching point is submersive. It follows that the evaluation maps at all nodes are transverse to the diagonal, and in addition the evaluation map at the root is a submersion. The assertion on the Maslov index is immediate from the explicit description. □

By Theorem 3.4 we have

Corollary 6.3. *For $X = \mathbb{C}^n$ and $L \subset X /\!/ G$ a toric moment fiber as above, there exist compatible systems of perturbations so that every holomorphic treed quasidisk is regular.*

Corollary 6.4. *The* gauged potential *for L_λ, that is, the potential associated to the A_∞ algebra $CQF(L_\lambda)$, is given by*

$$\mu_0(1) = W^G_\lambda(\beta) 1_L, \quad W^G_\lambda(\beta) = \sum_{i=1}^N e^{\langle v_i, \beta\rangle} q^{l_i(\lambda)}.$$

Proof. For degree reasons, only classes γ of index two contribute, in which case the classification theorem shows there is exactly one up to isomorphism for each class γ corresponding to a divisor, given by $z \mapsto (l_i(\lambda)/\pi)^{1/2} z$ in the i-th component and constant on the other components. The holonomy of the brane structure around the boundary of u is $\exp(\langle v_i, \beta\rangle)$ by definition, while the area is $l_i(\lambda)$, hence the claim. □

Remark 6.5. The proof in Fukaya et al [21], uses a stronger version of the divisor equation, which we have avoided by taking a more direct (but less general) definition of the potential.

Theorem 6.6. *If W^G_λ has a critical point $b \in H^1(L_\lambda, \Lambda_0)$ then $HQF(L^b_\lambda; \Lambda_0) = HM(L_\lambda; \Lambda_0)$.*

Proof. The argument is the same as in Cho-Oh [12], Fukaya et al [21, Theorem 13.1]. Suppose that $b \in \operatorname{crit}(W^G_\lambda)$, and consider the first page of the Morse-to-Floer spectral sequence of Proposition 3.9, so that the differential μ_1^b is defined on $HM(L_\lambda)$. If the partial derivatives of the potential W vanish at b then $\mu_1(\beta) = 0$ for all classes β of degree one, by Corollary 3.12. In general we induce on the degree of a class a. Suppose that the coefficient $\mu^b_{1,\gamma}(a)$ of q^γ in $\mu^b_1(a)$ vanishes for any $a \in HM^{|a|}(L_\lambda)$ with $|a| \le k$ and any $\gamma \in \pi_2(X, \tilde{L})$. For elements a_1, a_2 of degree at most k, consider the Leibniz rule arising from the A_∞ structure equation,

$$\mu^b_{1,\gamma}(a_1 \cup a_2) = \sum_{\gamma_1+\gamma_2=\gamma} \pm \mu^b_{2,\gamma_1}(\mu^b_{1,\gamma_2}(a_1), a_2) \pm \mu^b_{2,\gamma_1}(a_1, \mu^b_{1,\gamma_2}(a_2)) \pm \sum_{\gamma_1+\gamma_2=\gamma, \gamma_2\neq 0} \mu^b_{1,\gamma_1}(\mu^b_{2,\gamma_2}(a_1,a_2)) \tag{50}$$

where $\cup$ denotes the leading order term in the product structure, that is, defined by gradient trees; there are no contributions from $\mu_0(1)$, by unitarity. The right-hand side vanishes by the inductive hypothesis on the degree, since $\mu^b_{2,\gamma_2}(a_1, a_2)$ has lower degree than $a_1 \cup a_2$ for $\gamma_2 \neq 0$. Since $HM(L_\lambda)$ is generated by degree one classes, μ^b_1 vanishes on all classes, hence the higher differentials in the spectral sequence vanish and $HQF(L^b_\lambda; \Lambda_0) = HM(L_\lambda; \Lambda_0)$. □

See also Biran-Cornea [8] for further discussion of this technique. As explained in Fukaya et al [20], non-vanishing of the free part of $HQF(L_\lambda; \Lambda_0)$ is an invariant of Hamiltonian isotopy. In particular if $HQF(L_\lambda; \Lambda_0)$ is a non-trivial free Λ_0-module then $HQF(L_\lambda; \Lambda) = HQF(L_\lambda; \Lambda_0) \otimes_{\Lambda_0} \Lambda$ is non-vanishing. (In general, passing to Λ coefficients may kill the torsion, but non-vanishing of the free part is preserved.) Hence if W^G_λ has a critical point $b \in H^1(L_\lambda; \Lambda_0)$ then L_λ is non-displaceable, which concludes the proof of Theorem 1.1 except for the following technical point in the case that $X/\!/G$ is non-compact: So far we have assumed that the Hamiltonian perturbations are compactly supported. However, suppose that $\tilde{L}_\lambda$ is displaced by a Hamiltonian flow ϕ_t generated by arbitrary Hamiltonians $H_t \in C^\infty(X)^G$. Since $\tilde{L}_\lambda$ is compact, the union of images $K = \cup_{t\in[0,1]}\phi_t(\tilde{L}_\lambda)$ is compact, so there exists a cutoff function ρ equal to one on K, and vanishing outside a compact set. Let ϕ_t^ρ denote the flow of ρH_t. Then ϕ_t^ρ is equal to ϕ_t on K for all $t \in [0,1]$ and so $\phi_1^\rho(\tilde{L}_\lambda) \cap \tilde{L}_\lambda = \phi_1(\tilde{L}_\lambda) \cap \tilde{L}_\lambda = \emptyset$.

We have the following improvement of Theorem 1.1, under the same assumptions.

Proposition 6.7. *Suppose the* gauged potential W_λ^G *has a critical point. Then for any compactly supported Hamiltonian diffeomorphism* $\phi \in \mathrm{Diff}^h(X/\!/G)$ *such that* $\phi(L_\lambda)$ *intersects* L_λ *transversally, the number of intersection points is at least* $2^{\dim(T)}$.

Proof. Suppose that ϕ is the flow of $H \in C_c^\infty((0,1) \times X/\!/G)$. A generic small perturbation H' of H has the property that every (J,H')-holomorphic quasistrip is regular by Theorem 5.2. Let $\phi' \in \mathrm{Diff}^h(X/\!/G)$ denote the flow of H'. For H' sufficiently small, the number of points in $\phi'(L_\lambda) \cap L_\lambda$ is the same as $\phi(L_\lambda) \cap L_\lambda$. By Theorem 6.6, this number is at least $2^{\dim(T)}$, which proves the proposition. □

The following characterization of the class of non-compact symplectic toric manifolds admitting presentations as symplectic quotients of vectors spaces (so that the main result Theorem 1.1 applies) resulted from discussions with E. Lerman and M. Abouzaid, and appears in [34, Theorem 1.14].

Proposition 6.8. *A non-compact symplectic toric orbifold* Y *with action of a torus* T *and moment map* $\Psi : Y \to \mathfrak{t}^\vee$ *admits a presentation as a symplectic quotient of a vector space* X *by a torus* G *if and only if the following three conditions are satisfied: (i)* Ψ *is proper, (ii)* $\Psi(Y)$ *has finitely many facets, and (iii)* $\Psi(Y)$ *has at least one vertex.*

Indeed, any Y which can be realized as $X/\!/G$ satisfies these conditions: (i) and (ii) follow from the corresponding property the N-torus action on $X \cong \mathbb{C}^N$, while (iii) follows from the fact that $f : X \to \mathbb{R}, (z_1,\ldots,z_N) \mapsto \sum_{j=1}^N |z_j|^2$ is proper and descends to a proper function $f/\!/G$ on Y and generates a Hamiltonian circle action. Its minimum is a compact symplectic toric manifold and as such automatically contains a T-fixed point, which is then a vertex of $\Psi(Y)$. Conversely, any symplectic toric manifold satisfying these conditions has the property that $\Psi(Y)$ is contained in the interior of the moment image of some Hamiltonian T-action on a vector space Z, see [34], and so can be constructed by symplectic cutting.

Theorem 1.1 can be generalized to the case of an arbitrary symplectic toric manifold with proper moment map such that the moment polytope has finitely many facets (not necessarily a symplectic quotient) as follows. Any convex polyhedron P is a prism over a convex polyhedron Q containing a vertex, that is, isomorphic to $Q \times R$ for some vector space R. By the classification in [34] it follows:

Corollary 6.9. *Suppose that* Y *is a symplectic toric orbifold* Y *with proper moment map* Ψ *such that* $\Psi(Y)$ *has finitely many facets. Then* Y *is symplectomorphic to the* *product of a quotient $X/\!/G$ of a vector space X by a torus G with a cotangent bundle $T^*(S^1)^r$, for some $r \geq 0$.*

Proposition 6.10. *Let Y be as in Corollary 6.9. Then the statement of Theorem 1.1 holds.*

Proof. The Floer homology of a toric moment fiber in Y is the tensor product of Floer homologies of a moment fiber $(S^1)^r$ and a moment fiber in $X/\!/G$. Consider the quasimap Floer theory for the action of G on $X \times T^*(S^1)^r$ trivial on the second factor. The various moduli spaces used to defined quasimap Floer cohomology are compact, since any holomorphic maps in $X \times T^*(S^1)^r$ projects to holomorphic maps in X and $T^*(S^1)^r$, which are convex; one may then apply the maximum principle on the factors. The quasimap Floer cohomology of $L_\lambda \times (S^1)^r$ is the tensor product of quasimap Floer cohomology of L_λ with the Morse homology of $(S^1)^r$, since there are no non-trivial holomorphic disks in $T^*(S^1)^r$ with boundary on $(S^1)^r$. Non-vanishing obstructs displaceability as before by the same argument involving the quasimap Floer cohomology of the pair. □

Remark 6.11. The statement of Theorem 1.1, and so the Proposition, depends on the choice of a realization of $X/\!/G$ as a symplectic quotient, of which there are several; it seems from computations to be the case that the minimal presentation of $X/\!/G$ (that is, involving only irredundant inequalities) gives the same list of non-displaceable fibers as the other presentations.

Remark 6.12. The existence of a non-displaceable torus can be rephrased as follows for those readers interested mainly in Hamiltonian dynamics: Recall that a *quasi-fixed point* of a diffeomorphism φ of a symplectic T-orbifold Y is an orbit $O \subset Y$ such that $\varphi(O) \cap O \neq \emptyset$. Any compact completely integrable torus action is a projective toric variety, by Delzant's theorem and its extension to orbifolds [35]. Theorem 1.1 then implies: For any completely integrable Hamiltonian torus action on a compact symplectic orbifold Y with rational symplectic class and possibly with orbifold singularities, there exists at least one orbit $O \subset Y$ that is a quasifixed point for *any* Hamiltonian diffeomorphism.

7. Non-displaceability via bulk deformations

In this section we briefly describe an extension of quasimap Floer cohomology to include *bulk-deformations* as explored in [19]. These groups give additional obstructions to displaceability.

7.1. **Bulk deformations via twists.** We first give a somewhat naive definition of bulk-deformed A_∞ algebra for degree two classes by twisting the coefficients of the composition maps. Let X be a Hamiltonian G-manifold so that $X/\!/G$ is locally free, $L \subset X/\!/G$ a Lagrangian contained in the free locus, and $\tilde{L} \subset X$ its inverse image in X. For $\alpha \in H^2_G(X, \tilde{L}; \Lambda_0)$, define an *$\alpha$-twisted* A_∞ algebra $CQF^\alpha(L)$ by

$$\mu_n^\alpha(<x_1>, \ldots, <x_n>) = \sum_{[u] \in \overline{MW}_n(x_0,\ldots,x_n)_0} (-1)^{\heartsuit} \epsilon(u) e^{\langle \alpha, [u] \rangle} q^{A(u)} \operatorname{Hol}_L(u) <x_0>$$

well-defined via perturbations as above if every holomorphic disk with boundary in $\tilde{L}$ is regular. The proof of A_∞ associativity is the same as in Theorem 3.6. We denote by

$$W_\alpha^G : H^1(L,\Lambda_0) \to \Lambda_0, \quad \beta \mapsto \sum_{I(u)=2} e^{\langle \alpha,[u]\rangle} q^{A(u)} \operatorname{Hol}_{L^\beta}(u)$$

the a-deformed gauged potential corresponding to the family of brane structures L^β on L determined by $\beta \in H^1(L,\Lambda_0)$. Similarly given Lagrangians $L_0, L_1 \subset X/\!\!/G$ and a class $\alpha \in H^2_G(X, \tilde{L}_0 \cup \tilde{L}_1)$ there is, if every holomorphic disk with boundary in $\tilde{L}_0$ and $\tilde{L}_1$ is regular, an α-deformed A_∞ bimodule $CQF(L_0,L_1)$, isomorphic to $CQF(L)$ in the case $L_0 = L_1 = L$. In particular, if the free part of $HQF(L)$ is non-vanishing then $\tilde{L}$ is not displaceable by $H \in C_c^\infty([0,1]\times X)^G$. This involves no new analysis, but only a check that the A_∞ associativity relations are unchanged by the twisting above.

Suppose that $X/\!\!/G$ is a (possibly orbifold) projective toric variety and L_λ is a toric moment fiber. Then $H^2_G(X,\mathbb{C}) = \bigoplus_{i=1}^N \mathbb{C} c_1^G(\mathbb{C}_i)$ where $\mathbb{C}_i$ is the i-th component of $\mathbb{C}^N$, and $c_1^G(\mathbb{C}_i)$ denotes the equivariant first Chern class, and each of these classes vanishes on $\tilde{L}_\lambda$. Suppose that $\alpha = \sum_{i=1}^N \alpha_i c_1^G(\mathbb{C}_i)$. Then for $\beta \in H^1(L_\lambda;\Lambda_0)$ we have

$$W^G_{\lambda,\alpha}(\beta) = \sum_{i=1}^N \exp(\alpha_i + \langle v_i, \beta\rangle) q^{l_i(\lambda)}$$

using the fact that the pull-back of α to L vanishes, that is, α is in the image of $H^2_G(X,\tilde{L}_\lambda) \to H^2_G(X)$, since each $c_1^G(\mathbb{C}_i)$ is a Thom class for the i-th coordinate.

Theorem 7.1. *If $W^G_{\alpha,\lambda}$ has a critical point at $\beta \in H^1(L_\lambda;\Lambda_0)$, then $HF^{\alpha,\beta}(L_\lambda,\Lambda_0)$ is isomorphic to $H(L_\lambda,\Lambda_0)$, and $\tilde{L}_\lambda$ is not displaceable by the flow of any invariant time-dependent Hamiltonian.*

The proof is the same as in the untwisted case. We refer to Fukaya et al [19] for examples and computations.

7.2. **Bulk deformations via insertions.** We now explain the connection with "bulk insertions". Let $\overline{M}_{d;e}$ denote the moduli space of stable disks with d markings on the boundary and e markings in the interior. The boundary strata are formed when markings on the boundary come together to form disks, interior markings in the interior go to the boundary to form disks, or interior markings come together to form spheres.

Similarly let $\overline{MW}_{d;e}$ denote the moduli space of stable treed disk with d markings on the boundary and e markings in the interior; this means that there are, in addition to disk and sphere components, dimension one components attached to the boundary. See Figure 16 for the case $(d;e) = (2,1)$, where the edges attached to boundary markings have been omitted to save space. The boundary of $\overline{MW}_{d;e}$ is the same as that for $\overline{MW}_{d;e}$, but there are also codimension two strata corresponding to sphere bubbles. Standard arguments imply that $\overline{MW}_{d;e}$ is compact and local descriptions of the moduli space may be obtained by combining the local descriptions

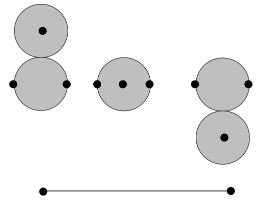

FIGURE 15. Moduli space of stable 2; 1 marked disks

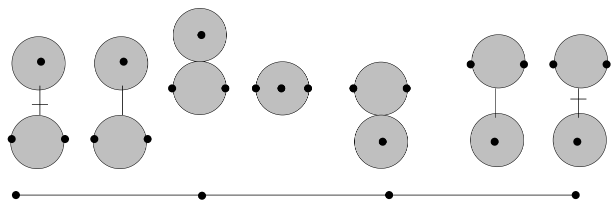

FIGURE 16. Moduli space of treed disks with interior marking

of $\overline{MW}_d$ with local descriptions of the moduli space of stable marked curves; these are left to the reader.

Let X and $L \subset X/\!\!/G$ be as above. A *treed holomorphic* $(d;e)$*-marked quasidisk* to $(X, \tilde{L})$ consists of a treed $(d;e)$-marked disk together with a continuous map $u : \Sigma \to X$ which is holomorphic on each disk and sphere component and a gradient trajectory on each edge. A treed holomorphic quasidisk is *stable* if it has no infinitesimal automorphisms, and each node connecting two segments maps to a critical point. Let $\overline{MW}_{d;e}(L)$ denote the moduli space of stable tree marked quasidisks. Let $\overline{MW}^{\mathrm{fr}}_{d;e}(X,L)$ denote the moduli space of *framed* quasidisks: this means that we do not mod out by the G-action, so that $\overline{MW}_{d;e}(X,L) = \overline{MW}^{\mathrm{fr}}_{d;e}(X,L)/G$. Associated to any treed holomorphic $(d;e)$-marked disk is a *linearized operator* D_u as before. The map u is *regular* if D_u is surjective.

Theorem 7.2. *The moduli space* $\overline{MW}^{\mathrm{reg}}_{d;e}(X,L)$ *resp.* $\overline{MW}^{\mathrm{fr,reg}}_{d;e}(X,L)$ *of* regular *holomorphic* $(d;e)$*-marked treed disks resp. framed* $(d;e)$*-marked treed disks has the structure of a topological manifold with boundary given by the union over strata corresponding to types where a tree has broken off, resp. principal* G*-bundle over* $\overline{MW}^{\mathrm{reg}}_{d;e}(X,L)$*.*

This theorem requires a slightly more detailed analysis of the gluing construction, see the review section of [27] which constructs C^1-compatible charts for the moduli space of holomorphic maps on surfaces without boundary. (Since we do not give a proof for the case with boundary, the reader should feel free to consider this an axiom for the following discussion.) The claim on $\overline{MW}^{\mathrm{fr,reg}}_{d;e}(X,L)$ follows from the fact that any free differentiable G-action has the structure of a principal G-bundle.

The moduli space admits combined evaluation maps to $\operatorname{crit}(F)^{d+1}$, by evaluation at the boundary markings, and an evaluation map at the interior markings

$$\operatorname{ev}^{\mathrm{fr}} : \overline{MW}^{\mathrm{fr}}_{d;e}(X,L) \to X^e.$$

By the second part of the theorem, there is a classifying map to the Borel construction $\overline{MW}^{\mathrm{fr,reg}}_{d;e}(X,L) \to EG$. Combining this with the framed evaluation maps we obtain evaluation maps at the interior markings

$$\operatorname{ev} : \overline{MW}^{\mathrm{reg}}_{d;e}(X,L) \to X^e_G.$$

Assuming every point in the moduli space is regular, "integration" over these moduli spaces defines operations

$$\begin{aligned} (51) \quad \mu_{d;e} : CQF(L)^{\otimes d} \otimes Z_G(X)^{\otimes e} &\to CQF(L) \\ <x_1> \otimes \ldots \otimes <x_d> \otimes a_1 \otimes \ldots \otimes a_e & \\ \mapsto \sum_{\gamma \in \pi_2(X,\tilde{L})} q^{A(\gamma)} \operatorname{Hol}_L(\gamma)(\overline{MW}_{d;e}(X,L;x_0,\ldots,x_d;\gamma), &\operatorname{ev}_1^* a_1 \cup \ldots \cup \operatorname{ev}_e^* a_e) <x_0> \end{aligned}$$

where $A(\gamma)$ denotes the symplectic area, depending only on on the homotopy class γ of u, $(\cdot,\cdot)$ denotes the pairing of the cochain $\operatorname{ev}^* a = \operatorname{ev}_1^* a_1 \cup \ldots \cup \operatorname{ev}_e^* a_e$ with a chain representing the relative fundamental class, obtained by for example triangulating $\overline{MW}_{d;e}(X,L;x_0,\ldots,x_d;\gamma)$; since $\operatorname{ev}^* a$ is closed the pairing is independent of the choice of triangulation. One can also define the pairing using equivariant de Rham theory. In this setting, the form $\operatorname{ev}^* a$ is obtained by pull-back of an equivariant form $a \in \Omega_G(X)$ [28] to $\overline{MW}^{\mathrm{fr}}_{d;e}(X,L;x_0,\ldots,x_d;\gamma)$ and then obtaining an ordinary form on the quotient $\overline{MW}_{d;e}(X,L;x_0,\ldots,x_d;\gamma)$ using the Cartan construction.

The operations satisfy a relation similar to that of A_∞ associativity, (continuing to assume that every stable marked quasimap is regular)

$$\begin{aligned} (52) \quad 0 = \sum_{k,i,J \subset \{1,\ldots,e\}} &\mu_{d-i;e-|J|}(x_1,\ldots,x_j \\ &\mu_{i;|J|}(x_{j+1},\ldots,x_{j+i};a_j, j \in J),\ldots,x_n;a_j, j \notin J). \end{aligned}$$

For any cycle $\alpha \in Z_G(X,\Lambda_+)$ we obtain an A_∞ algebra with operations denotes μ^α_n called the *bulk-deformed* A_∞ algebra of L, by summing over all possible insertions:

$$\mu^\alpha_d(x) = \sum_{e \geq 0} \mu_{d;e}(x;\alpha,\ldots,\alpha)/e!.$$

Any cohomologous choice $\alpha' \in Z_G(X,\Lambda_+)$ gives a homotopic A_∞ algebra, so that the *bulk-deformed Floer cohomology* $HQF^\alpha(L) = H(\mu^\alpha_1)$ depends only on the class $[\alpha] \in H_G(X)$, up to isomorphism. The bulk-deformed invariants satisfy a divisor relation for $\alpha \in Z^2_G(X)$ representing a class in the image of $H^2_G(X,\tilde{L}) \to H^2_G(X)$

$$\mu_{\alpha,n}(b;\gamma) = e^{\langle [\alpha],\gamma\rangle} \mu_n(b;\gamma)$$

obtained from the forgetful map $\overline{MW}_{d;e}(L,\gamma) \to \overline{MW}_{d;e-1}(L,\gamma)$. In particular, this divisor relation implies that the bulk-deformed A_∞ algebra is equal to that defined by twisting in the previous section for classes of degree 2.

One also has a bulk-deformed A_∞ bimodule $CQF^\alpha(L_0, L_1)$ for a pair $L_0, L_1 \subset X/\!\!/G$ of Lagrangians, obtained by adding bulk insertions to the holomorphic strips, and an isomorphism of A_∞ bimodules $CQF^\alpha(L_0, L_1) \to CQF^\alpha(L)$ in the case of an equal pair $L = L_0 = L_1$. In particular, if the free part of $HF^\alpha(L, \Lambda_0)$ is non-trivial then $\tilde{L}$ is not displaceable by any flow generated by a G-invariant family $H \in C_c^\infty([0,1] \times X)^G$. However, except for α in $H_G^2(X, \tilde{L})$ these seem difficult to compute.

8. Relationship to Floer cohomology in the quotient

In this final section we outline a formal argument which relates the potential of a symplectic quotient to the gauged potential by a combined bulk-boundary quantum Kirwan map. Let Σ be the unit disk equipped with standard area form Vol_Σ. For any $\epsilon \in [0, \infty)$ let $\overline{M}_{d;e}(x_0, \ldots, x_d)_\epsilon$ denote the moduli space of solutions to the vortex equations on Σ with area form $\epsilon \,\mathrm{Vol}_\Sigma$ with d markings on the boundary, e markings in the interior, and boundary in $\tilde{L}$, compactified as for the previous moduli spaces, with boundary markings mapping to $x_0, \ldots, x_d \in \mathrm{crit}(F)$, modulo holomorphic automorphisms of Σ which preserve Vol_Σ. Let $\mathrm{ev} : \overline{M}_{d;e}(x_1, \ldots, x_d)^\epsilon \to X_G^e$ denote the evaluation maps at the interior markings (obtained by combining with a classifying map as above). Integration over $\overline{M}_{d;e}(x_1, \ldots, x_n)_\epsilon$ should define invariants

$$\tau_{d;e}^\epsilon : CQF(L_\lambda, \Lambda_0)^{\otimes d} \otimes Z_G(X, \Lambda_0)^{\otimes e} \to \Lambda_0$$

via the formula

$$\tau_{d;e}^\epsilon(x_1, \ldots, x_d, \alpha) = \sum_\gamma q^{A(\gamma)} \,\mathrm{Hol}_L(\gamma)([\overline{M}_{d;e}(x_1, \ldots, x_d; \gamma)_\epsilon], \mathrm{ev}^* \alpha).$$

If $\epsilon = 0$ then these moduli spaces are quasimap moduli spaces, but with parametrized domain. Define a *vortex potential*

$$W_\lambda^\epsilon(\beta, \alpha) = \sum_{e \geq 1} (1/e!) \tau_{d;e}(\alpha, \ldots, \alpha)$$

where $\beta \in H^1(L, \Lambda_0)$ corresponds to the local system and $\alpha \in Z_G(X, \Lambda_0)$. Since the vortex moduli spaces are always reducible free one expects these potentials to be independent of ϵ. We study the large area limit $\epsilon \to \infty$. We consider, as in Gaio-Salamon [22], a sequence (A_ν, u_ν) of vortices on $\Sigma = \mathbb{R} \times [0,1]$ with boundary in $\tilde{L}$ with area form $\epsilon_\nu \,\mathrm{Vol}_\Sigma$ with $\epsilon_\nu \to \infty$. Suppose that

$$c_\nu := |\mathrm{d}_{A_\nu} u_\nu(z_\nu)| := \sup |\mathrm{d}_{A_\nu} u_\nu|.$$

We denote by $\mathrm{dist}(z, \partial\Sigma)$ the distance of a point $z \in \Sigma$ to the boundary $\partial\Sigma$. Six types of bubbling are possible (we do not exclude the possibility of sphere bubbling in X in the following discussion:)

(a) If $c_\nu/\epsilon_\nu \to \infty$ and $c_\nu \,\mathrm{dist}(z_\nu, \partial\Sigma) \to \infty$ as $\nu \to \infty$ then after re-scaling and passing to a subsequence one obtains a sphere bubble in X;
(b) If $c_\nu/\epsilon_\nu \to \infty$ and $c_\nu \,\mathrm{dist}(z_\nu, \partial\Sigma)$ has a finite limit as $\nu \to \infty$ then after re-scaling and passing to a subsequence one obtains a disk bubble in $(X, \tilde{L})$;

(c) If c_ν/ϵ_ν has a finite limit and $c_\nu \operatorname{dist}(z_\nu, \partial\Sigma) \to \infty$ as $\nu \to \infty$ then after re-scaling and passing to a subsequence one obtains a vortex on $\mathbb{C}$ with values in X;

(d) If c_ν/ϵ_ν has a finite limit and $c_\nu \operatorname{dist}(z_\nu, \partial\Sigma)$ has a finite limit as $\nu \to \infty$ then after re-scaling and passing to a subsequence one obtains a vortex on $\mathbb{H} = \{\operatorname{Im}(z) \geq 0\}$ with boundary in $\tilde{L}$;

(e) If $c_\nu/\epsilon_\nu \to 0$ and $c_\nu \operatorname{dist}(z_\nu, \partial\Sigma) \to \infty$ as $\nu \to \infty$ then after re-scaling and passing to a subsequence one obtains a sphere bubble in $X /\!\!/ G$;

(f) If $c_\nu/\epsilon_\nu \to 0$ and $c_\nu \operatorname{dist}(z_\nu, \partial\Sigma)$ has a finite limit as $\nu \to \infty$ then after re-scaling and passing to a subsequence one obtains a disk bubble in $X /\!\!/ G$ with values in L.

One expects the potential $W_\lambda^\infty(\beta, \alpha)$ counting quasimap invariants to be related to the parametrized potential on the quotient after incorporating vortex bubbles. Let $MW_{d;e}(\mathbb{H}, X, L)$ denote the moduli space of treed vortices on $(\mathbb{H}, \mathbb{R})$ with values in $(X, \tilde{L})$ with d markings on the boundary and e markings in the interior, and $\overline{MW}_{d,e}(\mathbb{H}, X, L)$ its compactification allowing disk and sphere bubbles. The codimension one boundary strata consist of configurations where a holomorphic disk with values in $\tilde{L}$ has bubbled off

$$(\overline{MW}^{\mathrm{fr}}_{d_0;e_0}(\mathbb{H}, X, L) \times_{\tilde{L}} \overline{MW}_{d_1;e_1}(X, \tilde{L}))/G^n$$

and configurations

$$\overline{MW}^{\mathrm{fr}}_{r;e_0}(X /\!\!/ G, L) \times_{L^r} \prod_{j=1}^{r} \overline{MW}_{|I_j|;e_j}(\mathbb{H}, X, L)$$

where r vortices on $\mathbb{H}$ have bubbled off, leaving as the "main component" a holomorphic disk in $X /\!\!/ G$ with values in L. Let $CQF(L)$ denote the quasimap Fukaya algebra defined above and suppose that the Fukaya algebra $CF(L)$ of $L \subset X /\!\!/ G$ using holomorphic disks in $X /\!\!/ G$ has been rigorously defined. If we assume that all moduli spaces are regular and compact, then one can define an *open quantum Kirwan morphism*

$$Q\kappa_{X,L} : CQF(L)^{\otimes d} \to CF(L)$$

$$(<x_1>, \ldots, <x_d>) \mapsto \sum_{[u] \in \overline{M}_d(\mathbb{H}, X, L; x_0, \ldots, x_d)} (-1)^{\heartsuit} \epsilon(u) q^{A(u)} \operatorname{Hol}_L(u) <x_0> .$$

Adding bulk insertions produces maps

$$Q\kappa_{X,L} : CQF(L)^{\otimes d} \otimes Z_G(X) \to CF(L)$$

$$(<x_1>, \ldots, <x_d>; \alpha) \mapsto \sum_{\gamma, e} (1/e!)(-1)^{\heartsuit} q^{A(\gamma)} \operatorname{Hol}_L(\gamma) \\ (\overline{M}_{d;e}(\mathbb{H}, X, L; x_0, \ldots, x_d), \operatorname{ev}_1^* \alpha \cup \ldots \cup \operatorname{ev}_e^* \alpha) <x_0> . \tag{53}$$

Formally, $Q\kappa_{X,L}$ without bulk insertions satisfies the axioms of an A_∞ -morphism, because the boundary components of the one-dimensional strata correspond to facets of the multiplihedra as in Ma'u-Woodward [39]. Similarly we have a version for vortices on $\mathbb{C}$, which was already discussed in for example Woodward-Ziltener [63] and formally produces a map

$$Q\kappa_X : Z_G(X) \otimes \Lambda \to Z(X /\!\!/ G) \otimes \Lambda.$$

Since the bubbling that appears in the limit have been incorporated into the bulk and boundary quantum Kirwan morphisms, one expects

Conjecture 8.1. *With $X, X/\!\!/G$ and $L \subset X$ any G-Lagrangian brane,*

$$W_\lambda^\infty(Q\kappa_{L,X}(\beta,\alpha), Q\kappa_X(\alpha)) = W_\lambda^0(\beta,\alpha).$$

This conjecture is closely related to a question of Auroux (related to the discussion in [6]), who asked whether W_λ is related to W_λ^G by a change of coordinates. In particular we conjecture that the "bulk" part of the necessary change of coordinates is equivalent to that appearing in Givental's work as the "mirror map". There are at least two explicit examples which provide evidence for this conjecture, besides the conceptual framework provided by the large area limit explained above: first, Auroux's computation of the potential for the second Hirzebruch surface [6, Proposition 3.1] shows that the corrected potential is related to the naive potential by a transformation equivalent to Givental's mirror map. Secondly, the computation in [45] relates a disk potential defined without markings on the boundary (thus, not the potential that appears in A_∞ setting) to a naive potential via a coordinate change that is the same as in the closed case. However, it is not clear in general how to fit the invariants of [45] into the above framework, in general; the relation between the various disk potentials seems an interesting topic for further investigation.

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