Floer cohomology and flips

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Abstract

We show that blow-ups or reverse flips (in the sense of the minimal model program) of rational symplectic manifolds with point centers create Floer-non-trivial Lagrangian tori. These results are part of a conjectural decomposition of the Fukaya category of a compact symplectic manifold with a singularity-free running of the minimal model program, analogous to the description of Bondal-Orlov [**19**] and Kawamata [**66**] of the bounded derived category of coherent sheaves on a compact complex manifold.

Received by the editor July 2, 2022.

2010 *Mathematics Subject Classification.* Primary 53D40.

This work was partially supported by NSF grants DMS 1207194 and DMS 1711070 and a Simons Fellowship.

CHAPTER 1

Introduction

Lagrangian Floer cohomology was introduced in [**40**] as a version of Morse theory for the space of paths from a Lagrangian submanifold to itself. Although the theory was introduced decades ago, it has been far from clear which Lagrangian submanifolds have well-defined or non-trivial Floer cohomology. Similarly, one would like to know whether a compact symplectic manifold contains Lagrangians with non-trivial Floer cohomology. The purpose of this paper is to describe a method for producing Floer-non-trivial Lagrangians via the surgeries that appear in the minimal model program, namely *flips* and *blow-ups*. In particular, if one has knowledge of a sufficiently nice minimal model program (mmp) for the symplectic manifold (in a suitably modified sense) then one can read off a list of Floer non-trivial Lagrangians that conjecturally generate the Fukaya category. At each stage in the mmp running the Lagrangians that break off have the same slope, in the sense of ratio of the Maslov class to the disks of minimal area. In this sense any such mmp running provide a filtration of the Fukaya category analogous to the Harder-Narasimhan filtration of a complex vector bundle. The results are analogous to those of Bondal-Orlov [**19**] and Kawamata [**66**] on the behavior of the bounded derived category of coherent sheaves on a compact complex manifold under flips. The approach taken here is inspired by the work of Fukaya-Oh-Ohta-Ono [**47**], [**48**] on the toric case. Fukaya et al [**47**, Theorem 1.4] prove the existence of a Floer non-trivial torus in any rational toric manifold by shrinking the moment polytope. Our results describe a similar mechanism that produces a Floer non-trivial torus in birationally-Fano smooth projective varieties, by shrinking the manifold via a topological version of the Kahler-Ricci flow. The connection of the location of the Floer non-trivial tori in Fukaya et al [**47**] to the minimal model program was noted in work with González [**53**].

To describe the results more precisely, recall that a running of the mmp for a smooth birationally-Fano projective variety X is a sequence of birational varieties $X = X_0, X_1, \ldots, X_k$ such that each X_{i+1} is obtained from X_i by an *mmp transition*. The simplest mmp transition is a blow-down of a divisor, or more generally a divisorial contraction. The other operations are *flips* which are birational isomorphisms on the complement of codimension four exceptional loci. The minimal model program ends (in the birationally-Fano case) with a *Mori fibration* which here means a fibration with Fano fiber. Each blow-up or flip $X_i \dashrightarrow X_{i+1}$ has a *center* Z_i which is a subquotient of both X_i and X_{i+1}. A flip, in the cases considered here, replaces a weighted-projective bundle $\mathbb{P}(N_i^+) \to Z_i$ over the center with another weighted-projective bundle $\mathbb{P}(N_{i+1}^-) \to Z_i$:

$$\begin{array}{ccccccc} X_i & \longleftarrow & \mathbb{P}(N_i^+) & & \mathbb{P}(N_{i+1}^-) & \longrightarrow & X_{i+1} \\ & & & \searrow \quad \swarrow & & & \\ & & & Z_i & & & \end{array} .$$

A *weighted-projective bundle* means a bundle whose fibers are the git quotient of a vector space by an action of the multiplicative group $\mathbb{C}^\times = \mathbb{C} - \{0\}$. Each weighted-projective bundle $\mathbb{P}(N_i^\pm)$ is obtained by removing the zero section 0 from a vector bundle $N_i^\pm$ over the center Z_i and taking the quotient of $N_i^\pm - 0$ by a fiber-wise linear $\mathbb{C}^\times$-action with only the zero section 0 as fixed point set. In the case of a divisorial contraction, the map $\mathbb{P}(N_{i+1}^-) \to Z_i$ would be an isomorphism, while in the case of a Mori fibration, one may think of the center as the result of the transition: $\mathbb{P}(N_i^+) \cong X_i, Z_i \cong X_{i+1}$. Sometimes singularities in the spaces X_i are unavoidable; however, in good cases (such as toric varieties) one may assume that the X_i's are smooth orbifolds. In this case we say that X has a *smooth running* of the mmp. Existence of mmp runnings is known for varieties of low dimension and in many explicit examples. For toric varieties, mmp runnings exist by work of Reid [**110**].

Each of the minimal model transitions has a symplectic analog. The local change in the symplectic manifold follows the algebraic model, and the symplectic class varies in the anticanonical direction. The symplectic analog of a blow-up was studied in McDuff [**93**], without the anticanonical variation. Note that the anticanonical variation of the symplectic class in our definition of blow-up allows blow-ups to be infinitely large, that is, there exist reverse symplectic runnings of the mmp with a single blow-up transition and infinite duration. The definitions of symplectic flip and symplectic Mori fibration are similar, see Definition 2.7. We say that a symplectic manifold X_+ is obtained from a symplectic manifold X_- *by a symplectic blow-up resp. flip* if there exists a symplectic running of the mmp with a single transition given by a blow-up resp. flip with point center. In this case we say that the *multiplicity* of the transition is

$$m = \dim(QH(X_+)) - \dim(QH(X_-)).$$

A simple argument using Mayer-Vietoris implies that m is always positive in the cases considered in this paper.

The version of Floer theory that we use requires the notion of *bounding cochain* of Fukaya-Oh-Ohta-Ono [**46**]. We begin with a brief discussion of what we mean by the Fukaya algebra $CF(L)$ of a Lagrangian $L \subset X$. There are several foundational systems which at the moment are not known to be equivalent (or in some cases, completely written down.) Working in the Morse cochain model, the structure coefficients in these Fukaya algebras count rigid elements $u : C \to X$ in the moduli space of *treed holomorphic disks* with Lagrangian boundary condition given by L. We regularize these moduli spaces using a *stabilizing divisor* as in Cieliebak-Mohnke [**28**], which means a codimension two symplectic hypersurface $D \subset X$ which meets any non-constant holomorphic sphere in X in finitely many but at least three points, and making the Lagrangian L exact in the complement of the hypersurface D. The intersections $u^{-1}(D)$ of any pseudoholomorphic disk or sphere $u : C \to X$ stabilize the domain. Domain-dependent almost complex structures and Morse functions suffice to make all moduli spaces of low expected dimension regular for generic perturbations. While this foundational scheme is somewhat less general than the other approaches, it requires no discussion of virtual fundamental classes and so makes the necessary foundational arguments substantially shorter.

The Fukaya algebra also depends on the choice of local system with values in the units in the Novikov ring. Let Λ denote the universal Novikov field and $\Lambda^\times \subset \Lambda$ the group of elements with vanishing valuation with respect to the formal variable. For a connected Lagrangian L denote by $\mathcal{R}(L) = \operatorname{Hom}(\pi_1(L), \Lambda^\times)$ the group of $\Lambda^\times$-local systems. For any compact oriented spin Lagrangian submanifold L equipped with a local system $y \in \mathcal{R}(L)$ there is a *strictly unital Fukaya algebra* $CF(L, y)$, defined by counting *weighted treed pseudoholomorphic disks*, independent of all choices up to homotopy equivalence. The resulting moduli space of solutions $MC(L, y)$ to the *projective Maurer-Cartan equation* is independent of all choices and for each solution $b \in MC(L, y)$ the *Floer cohomology group* $HF(L, y, b)$ is independent of the choice of perturbations up to isomorphism. The main result of this paper is the existence of Floer non-trivial Lagrangians near mmp transitions:

Theorem 1.1. *Suppose that X_+ is a compact rational symplectic manifold obtained from a compact rational symplectic manifold X_- by a reverse simple flip or blow-up with trivial center with multiplicity m, with sufficiently small exceptional locus. In a contractible neighborhood of the exceptional locus there exists a Lagrangian torus $L \subset X_+$ with m distinct local systems and Maurer-Cartan solutions*

$$y_k \in \mathcal{R}(L), \quad b_k \in MC(L, y_k), \quad k = 1 \ldots, m,$$

with non-trivial Floer cohomology

$$HF(L, b_k, y_k) \cong H(L, \Lambda) \neq \{0\}.$$

Existence of Floer non-trivial Lagrangians in compact symplectic manifolds is known only in a special cases. For symplectic manifolds with anti-symplectic involutions, Floer cohomology of the fixed point sets is studied in Fukaya-Oh-Ohta-Ono [**49**]. Cho-Oh [**31**] and Fukaya-Oh-Ohta-Ono [**47**] show that any compact toric manifold contains at least one moment fiber with non-trivial Floer cohomology. In semi-Fano hypersurfaces, at least one Lagrangian has non-trivial Floer cohomology by work of Seidel [**118**] and Sheridan [**122**]. Related discussions of blow-ups can be found in Smith [**126**].

We conjecture that the Lagrangians appearing in Theorem 1.1, applied iteratively, split-generate the Fukaya category. Denote by $\operatorname{Fuk}(X)$ the Fukaya category of X, independent of all choices up to homotopy equivalence of curved A_∞ algebras. For any element w in the universal Novikov ring denoted by $\operatorname{Fuk}(X, w)$ the category whose objects are pairs (L, b) where L is an object of X and b is a weakly bounding cochain satisfying the projective Maurer-Cartan equation with value w and whose morphism spaces are explained in [**46**]. Let $D^\pi \operatorname{Fuk}(X, w)$ be the idempotent completion of the derived Fukaya category and define

$$D^\pi \operatorname{Fuk}(X) = \bigsqcup_w D^\pi \operatorname{Fuk}(X, w).$$

Conjecture 1.2. *Suppose $X = X_0, X_1, \ldots, X_k = X'$ is a sequence of compact symplectic manifolds such that each X_{i+1} is obtained from X_i by an mmp transition with center Z_i. Then the idempotent-closure of the derived Fukaya category $D^\pi \operatorname{Fuk}(X_0)$ is isomorphic to the disjoint union of categories of centers of the mmp* *transitions:*

$$D^\pi \operatorname{Fuk}(X) \cong D^\pi \operatorname{Fuk}(X') \sqcup \bigsqcup_{i=1}^{k} D^\pi \operatorname{Fuk}(Z_i)^{m_i} \tag{1.1}$$

where

$$m_i = \dim(QH(X_i)) - \dim(QH(X_{i+1}))$$

is the multiplicity *of the* i*-th mmp transition given as the difference of the quantum cohomology rings* $QH(X_{i+1}), QH(X_i)$*.*

By the disjoint union of categories we mean the category whose objects are the disjoint union, and whose morphism groups between elements of different sets in the disjoint union are trivial. In the case of toric manifolds, the Lagrangians have been announced to generate the Fukaya category by Abouzaid-Fukaya-Oh-Ohta-Ono, based on the technique in Abouzaid [**2**]. Since our evidence is mostly in the birationally-Fano case, it is possible that the conjecture 1.2 needs some similar restriction. A special case is shown in Venugopalan-Woodward-Xu [**132**].

The decomposition (1.1) should be related to the decomposition by quantum multiplication of the first Chern class in the following sense: Recall that quantum multiplication by the first Chern class $c_1(X) \in H^2(X)$ induces an endomorphism

$$c_1(X)\star : QH(X) \to QH(X), \quad \alpha \mapsto c_1(X) \star \alpha.$$

The eigenvalues $\lambda_1, \ldots, \lambda_k \in \Lambda$ of $c_1(X)\star$ that are non-zero have a well-defined q-valuation $\operatorname{val}_q(\lambda_i) \in \mathbb{R}$, given by the exponent of the leading order term. We expect that these valuations (when non-zero) are the transition times in the minimal model program. The conjecture 1.2 provides a method of attack on another conjecture of Kontsevich, that the Fukaya category $\operatorname{Fuk}(X)$ of a symplectic manifold X is expected to be a categorification of the quantum cohomology $QH(X)$, at least in many cases, in the following sense [**72**]: there should be an isomorphism

$$H(\operatorname{Fuk}(X)) := \bigoplus_w H(\operatorname{Fuk}(X, w)) \cong QH(X) \tag{1.2}$$

from the Hochschild cohomology $H(\operatorname{Fuk}(X))$ of the Fukaya category to the quantum cohomology $QH(X)$; here $\operatorname{Fuk}(X, w)$ is the summand of the Fukaya category with curvature w. If the centers of the mmp transitions have the property (1.2) then the manifold should have the same property and hence also the quantum cohomology decomposes into summands corresponding to transitions.

The idea that the quantum cohomology of a symplectic manifold should behave well under minimal model transitions is not new; see for example Ruan [**112**], Lee-Lin-Wang [**76**], Bayer [**11**], Acosta-Shoemaker [**4**]; we also heard related results in talks of H. Iritani. See Li [**80**] for related result in the case of open symplectic manifolds. Including bulk deformations one should obtain a filtration rather than a splitting of the Fukaya category, analogous to the situation in complex geometry where mmp transitions provide a semi-orthogonal decomposition.

The analogy with results of Bondal-Orlov [**19**] and Kawamata [**66**] is somewhat mysterious, since the mirror of the minimal model program on the algebraic side is not expected to be the symplectic version of the minimal model program considered here. Rather, the minimal model program on the symplectic side should correspond under mirror symmetry to a deformation of the mirror potential by a change of variables in the potential

$$W \mapsto \phi_t^* W, \ \phi_t(y) := yq^{-tc_1(X)}.$$

Because the mirror should be understood as a formal completion at $q = 0$, such a deformation changes the mirror by eliminating some critical loci via the formal completion. This flow should be related to the renormalization group flow in the physics of non-linear sigma models as discussed in, for example, Hori-Vafa [**62**].

The proof of Theorem 1.1 combines a neck-stretching argument with local toric computations. Near the exceptional locus the reverse flip is toric. The results in the toric case imply the existence of a Floer-non-trivial torus in the corresponding toric variety. This Lagrangian torus collapses at the singularity of a running of the minimal model program; in this sense the Lagrangian is a "vanishing cycle". The exceptional locus of the flip is separated from the rest of the symplectic manifold by a coisotropic submanifold fibered over a toric variety. Stretching the neck, as in symplectic field theory, produces a homotopy-equivalent *broken Fukaya algebra* associated to the Lagrangian which counts maps to the pieces combined with Morse trajectories on the toric variety. Similar arguments are common in the literature, for example, in the work of Iwao-Lee-Lin-Wang [**76**], [**75**]. One computes explicitly, using a Morse function arising as component of a moment map, that the resulting broken Fukaya algebra is weakly unobstructed and that the broken Floer cohomology of the Lagrangian is non-vanishing. Moduli spaces of pseudoholomorphic disks in toric varieties with invariant constraints are never isolated, and this implies the unobstructedness of Floer cohomology. The classification of disks of small area implies the existence of a critical point of the potential.

As applications of Theorem 1.1 we show in Chapter 8.6 that various symplectic manifolds contain Hamiltonian non-displaceable Lagrangian tori. For example, in the case of toric manifolds we reproduce in Chapter 2.2 some of the results of Fukaya-Oh-Ohta-Ono [**47**]. We also show existence of Floer non-trivial tori in symplectic quotients of products of two-spheres (Lemma 8.20) and moduli spaces of rank two parabolic bundles in genus zero (Lemma 8.21).

We thank Kai Cieliebak, Octav Cornea, Sheel Ganatra, François Lalonde, Dusa McDuff, and Klaus Mohnke for helpful discussions. We also thank Sushmita Venugopalan and Guangbo Xu for pointing out mistakes in earlier versions.

CHAPTER 2

Symplectic flips

The goal of the minimal model program (mmp for short) is to classify algebraic varieties by finding a *minimal model* in each birational equivalence class, see [**71**]. In the birationally-Fano case, the hoped-for minimal model is a *Mori fibration*: a fibration with Fano fiber. In general, singularities play an important role in the minimal model program. Here we assume that the variety admits a *singularity-free* running of the mmp. While this case is considered somewhat trivial by algebraic geometers, it includes a number of smooth projective varieties whose symplectic geometry is poorly understood. We introduce a symplectic version of the mmp given as a path (with singularities corresponding to surgeries) of symplectic manifolds. By a suggestion of Song-Tian and others [**125**], an mmp running is conjecturally equivalent to running Kähler-Ricci flow with surgery; the paths in our symplectic mmp are a "topological version" of the Kähler-Ricci flow.

2.1. Symplectic mmp runnings

Each mmp transition is a special kind of *birational transformation.* For the sake of completeness we recall the terminology. Let $X_\pm$ be normal projective varieties. A *rational map* from X_+ to X_- is a Zariski dense subset $U \subset X_+$ and a morphism $\phi : U \to X_-$; we write $\phi : X_+ \dashrightarrow X_-$. A rational map ϕ is a *birational equivalence* if ϕ has a rational inverse, that is, a rational map $(V \subset X_-, \psi : V \to X_+)$ to X_+ such that the compositions $\phi \circ \psi|_{\psi^{-1}(U)}, \psi \circ \phi|_{\phi^{-1}(V)}$ are the identity on the domains of definition. Birational equivalences have a natural notion of composition, making birational equivalence into an equivalence relation.

The minimal model program involves the following types of morphisms, see for example Hacon-McKernan [**58**]. The first two types are birational equivalences.

Definition 2.1. A *minimal model transition* of X_+ is one of the following three types:

(a) (Divisorial contractions) A morphism $\tau : X_+ \to X_-$ that is the contraction of a Cartier divisor (codimension one subvariety $Y \subset X_+$); a typical example is a blow-down.
(b) (Flips) Let $\tau_+ : X_+ \to X_0$ be a birational morphism. The morphism τ_+ is *small* if and only if τ_+ does not contract a divisor. The morphism τ_+ is a *flip contraction* if and only if the relative Picard number of τ_+ is one. The *flip* of X_+ is another small birational morphism $\tau_- : X_- \to X_0$ of relative Picard number one.
(c) (Mori fibrations) A Mori fibration is a fibration $\tau : X_+ \to X_-$ with Fano fiber, that is, a fibration whose relative anticanonical bundle $K_{X_+}^{-1}/\tau^* K_{X_-}^{-1}$ is very ample on the fibers $\tau^{-1}(x) \subset X_+$.

Here the *relative Picard number* is the difference in Picard numbers, that is, the difference in dimensions

$$\mathrm{Pic}(\tau_+) = \dim(\mathrm{Pic}(X_+)) - \dim(\mathrm{Pic}(X_0))$$

in the moduli spaces $\mathrm{Pic}(X_+), \mathrm{Pic}(X_0)$ of line bundles. The relative Picard number is one if and only if every two curves contracted by τ_- are numerical multiples of each other, that is, define proportional linear functions on the space of degree two cohomology classes.

DEFINITION 2.2. (a) A *running* of the mmp is a sequence of smooth projective varieties

$$X = X_0, \ldots, X_{k+1}$$

such that for $i = 0, \ldots, k$, X_{i+1} is obtained from X_i by a divisorial contraction or a flip, and X_{k+1} is obtained from X_k by a fibration with Fano fiber (sometimes called Mori fibration). The variety X_k may be called the *minimal model* of X_0. The different minimal models that occur are related by the *Sarkisov program* [**59**].

(b) An *extended running* of the mmp is a sequence of smooth projective varieties $X_0, \ldots, X_{k_1}, X_{k_1+1}, \ldots, X_{k_l+1}$, where $k_1, \ldots, k_l$ is an increasing sequence of integers, such that X_{i+1} is obtained from X_i by a divisorial contraction or flip for $k \neq k_i$ and X_{k_i+1} is obtained from X_{k_i} by a Mori fibration.

It is expected that runnings of the mmp exist for all smooth projective varieties. This existence has proved up to dimension three [**71**]. Existence of Mori fibration models is known, in the birationally Fano case, by [**12**, Corollary 1.3.2], but the existence of runnings in higher dimension seems to be unknown.

EXAMPLE 2.3. The simplest example of the mmp occurs for toric surfaces. Recall that any toric surface X corresponds to a two-dimensional fan $\mathcal{C} = \{C \subset \mathbb{R}^2\}$ of strictly convex cones. Elementary combinatorics shows that if $\mathcal{C}$ is complete and has more than four vertices then there always exists some invariant prime divisor D_i with $D_i^2 = -1$; blowing down D_i one eventually reaches a Hirzebruch surface or projective plane which is the minimal model, see Audin [**7**, Theorem VIII.2.9]. Note that toric surfaces often admit several runnings of the toric mmp, depending on the order in which the -1-curves are blown down.

EXAMPLE 2.4. Del Pezzo surfaces also provide elementary examples. Let dP_n denote a del Pezzo surface given by the blow up of $\mathbb{P}^2$ at $9 - n$ generic points. For $1 \leq n \leq 9$, dP_n is Fano and admits a running of the mmp with (in our notation) one transition, since dP_n is itself a Mori fibration over a point. However, dP_n for $n < 8$ admits multiple mmp runnings. For example, for dP_6 one can blow down a the exceptional curves $E_i \subset dP_6, E_i.E_i = -1, i = 1, 2, 3$ created by the blow-ups in any order; on the other hand, one may view dP_6 as the thrice-blow-up of the *dual* projective plane via the Cremona transformation. Blowing down the exceptional curves $E_j', E_j'.E_j' = -1$ in any order gives another six runnings of the mmp. Taking the blow-ups to respect the toric structure dP_6 is toric with moment polytope a hexagon shown in Figure 2.1. The 12 runnings above correspond to the ways of "restoring the corners" to make a triangle. The dots in Figure 2.1 correspond to the locations of the Floer-non-trivial Lagrangians produced by the main result.

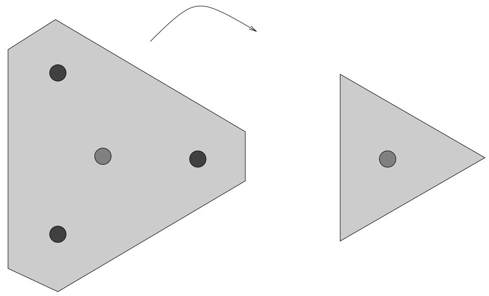

FIGURE 2.1. Blowing down dP_6 to $\mathbb{P}^2$

REMARK 2.5. Runnings of the mmp can often be obtained from geometric invariant theory as follows, as in Reid [**111**] and Thaddeus [**131**]; see also [**24**] and [**38**]. Let G be a complex reductive group and X a smooth projective G-variety. Recall that a *linearization* of X is an ample G-line bundle $\widetilde{X}$. Given such a linearization the geometric invariant theory quotient $X/\!\!/G$ is the quotient of the semistable locus, defined as the set of points where an invariant section is non-vanishing,

$$X^{\mathrm{ss}} := \{x \in X \mid \exists k > 0,\ s \in H^0(\widetilde{X}^k)^G, s(x) \neq 0\},$$

by the orbit equivalence relation

$$x_1 \sim x_2 \iff \overline{Gx_1} \cap \overline{Gx_2} \cap X^{\mathrm{ss}} \neq 0.$$

The git quotient $X/\!\!/G$ depends only on the ray $\{\widetilde{X}^t, t > 0\} \subset \mathrm{Pic}^G(X)$ generated by $\widetilde{X}$ in the equivariant Picard group $\mathrm{Pic}^G(X)$. That is, tensor powers of $\widetilde{X}$ define the same git quotient, since if some section of $\widetilde{X}^k$ is non-vanishing at $x \in X$ then so are all its tensor powers. In particular, git quotients are defined for ample elements of the rational Picard group $\mathrm{Pic}^G_{\mathbb{Q}}(X) = \mathrm{Pic}^G(X) \otimes_{\mathbb{Z}} \mathbb{Q}$.

An path of linearizations gives rise to a path of git quotients with a discrete set of transition times. Let $\widetilde{X} = \widetilde{X}_0$ be as above, and let $\widetilde{X}_1$ be another G-equivariant line bundle. The tensor powers

$$\widetilde{X}_t = \widetilde{X}_0 \otimes \widetilde{X}_1^t \in \mathrm{Pic}_{\mathbb{Q}}(X), t \in \mathbb{Q}$$

are ample for $t \in [0, T] \cap \mathbb{Q}$ for T sufficiently small. We call $\tilde{X}_0$ the *base point* of the variation of linearization and $\tilde{X}_1$ the *direction*. Let

$$X/\!\!/_t G := X_t^{\mathrm{ss}} / \sim$$

be the family of geometric invariant theory quotients obtained from the linearizations $\widetilde{X}_t$. As the polarization varies the git quotients undergo a sequence of divisorial contractions, flips without the ampleness condition, and fibrations each obtained as follows: The *transition times* are the set of values of t where the quotient $X/\!\!/_t G$ has stabilizer groups of positive dimension:

$$\mathcal{T} := \{t \in \mathbb{R},\ \exists x \in X_t^{\mathrm{ss}},\ \#G_x = \infty\}.$$

Dolgachev-Hu [**38**] and Thaddeus [**131**] reduce the study of the wall-crossings to the case of circle actions as follows. Define the *master space*

$$X_{t_1,t_2} := \mathbb{P}(\widetilde{X}^{t_1} \oplus \widetilde{X}^{t_2})/\!\!/G \tag{2.1}$$

for some $t_1, t_2 \in [0,T]$ not in the set of transition times $\mathcal{T}$. The times t_1, t_2 may be taken to be integers after replacing $\widetilde{X}$ with a tensor power, which does not affect the computation. The space X_{t_1,t_2} is equipped with a natural linearization, obtained by tensoring the relative hyperplane bundle with the pull-back of $\widetilde{X}_0$. The $\mathbb{C}^\times$-action on the fibers induces a $\mathbb{C}^\times$-action on the quotient.

$$X_{t_1,t_2} /\!\!/ \mathbb{C}^\times \cong X /\!\!/_t G$$

for $t \in (t_1, t_2)$.

One can now check that variation of quotient produces a flip, with the ampleness condition satisfied if the direction of the variation $\widetilde{X}_1$ is the canonical bundle K. In the circle group case $G = \mathbb{C}^\times$ let $F \subset X_t^{\mathrm{ss}}$ be a component of the fixed point set that is stable at time t. Let $\mu_1, \ldots, \mu_k \in \mathbb{Z}$ denote the weights of $\mathbb{C}^\times$ on the normal bundle N to F, and N_i the weight space for weight $\mu_i, i = 1, \ldots, k$. Let

$$N_\pm := \bigoplus_{\pm \mu_i > 0} N_i \subset N$$

denote the positive resp. negative weight subbundles of N. For $t_- < t < t_+$ with $t_\pm$ close to t the Hilbert-Mumford criterion and Luna slice theorem imply that the semistable locus changes by replacing a variety isomorphic to N_+ with one isomorphic to N_-, see [**131**]. Hence $X /\!\!/_{t_+} G$ is obtained from $X /\!\!/_{t_-} G$ by replacing the (weighted)-projectivized bundle $N_+^\times / G$ of with $N_-^\times / G$:

$$(X /\!\!/_{t_-} G) \backslash (N_-^\times / G) \cong (X /\!\!/_{t_+} G) \backslash (N_+^\times / G).$$

One checks easily from the local model that these morphisms are relatively K-ample resp. $-K$-ample over the center. Thus, in the absence of singularities (that is, if G acts freely for generic t) the spaces $X /\!\!/_t G$ yield a smooth running of the minimal model program.

Remark 2.6. The symplectic version of these transitions can be described in terms of Morse theory of the moment map, as explained in Guillemin-Sternberg [**55**]. Let (X, ω) be a compact symplectic manifold equipped with Hamiltonian $U(1)$-action with proper moment map $\Phi : X \to \mathbb{R}$. Let

$$\mathrm{Crit}(\Phi) = \{x \in X \;| \mathrm{d}\Phi(x) = 0\}, \quad \mathrm{Critval}(\Phi) = \Phi(\mathrm{Crit}(\Phi))$$

denote the set of critical points resp. critical values of Φ. Given a critical value $c \in \mathrm{Critval}(\Phi)$, we denote by $c_\pm \in \mathbb{R}$ regular values on either side of c, so that c is the unique critical value in (c_-, c_+). We suppose for simplicity that $\Phi^{-1}(c)$ contains a unique critical point $x_0 \in X$, so that

$$\mathrm{Critval}(\Phi) \cap (c_-, c_+) = \{c\}, \quad \mathrm{Crit}(\Phi) \cap \Phi^{-1}(c) = \{x_0\}.$$

By the equivariant Darboux theorem, there exist Darboux coordinates $z_1, \ldots, z_n$ near x_0 and *weights* $\mu_1, \ldots, \mu_n \in \mathbb{Z}$ so that

$$\Phi(z_1, \ldots, z_n) = c - \sum_{j=1}^n \mu_j |z_j|^2 / 2.$$

In particular, Φ is Morse and we denote by $W_{x_0}^\pm$ the stable and unstable manifolds of $-\mathrm{grad}(\Phi)$ with respect to some invariant metric. The time t gradient flow

$\phi_t : X \to X$ of $-\operatorname{grad}(\Phi)$ induces a diffeomorphism between level sets on the complement of the stable and unstable manifolds:

$$\Phi^{-1}(c_+)\backslash W^+_{x_0} \to \Phi^{-1}(c_-)\backslash W^-_{x_0}, \quad x \mapsto \phi_t(x),\ t \text{ such that } \ \Phi(\phi_t(x)) = c_-.$$

Assuming the gradient vector field is defined using an invariant metric one obtains an identification of symplectic quotients except on the symplectic quotients of the stable and unstable manifolds:

$$(X/\!\!/_{c_+}U(1))\backslash(W^+_{x_0}/\!\!/_{c_+}U(1)) \to (X/\!\!/_{c_-}U(1))\backslash(W^+_{x_0}/\!\!/_{c_-}U(1)).$$

By the description from equivariant Darboux, one sees that the symplectic quotients of the stable and unstable manifolds are weighted projective spaces with weights given by the positive resp. negative weights:

$$(W^+_{x_0}/\!\!/_{c_+}U(1)) \cong \mathbb{P}[\pm\mu_i, \pm\mu_i > 0].$$

Thus $X/\!\!/_{c_-}U(1)$ is obtained from $X/\!\!/_{c_+}U(1)$ by replacing one weighted projective space with another.

The change in the symplectic class under variation of symplectic quotient is described by Duistermaat-Heckman theory [**37**]. Let $c_0 \in \mathbb{R} - \operatorname{Critval}(\Phi)$ be a regular value of Φ. Consider the product $\Phi^{-1}(c_0) \times [c_-, c_+]$ for $c_\pm$ close to c_0. Let $\pi_C, \pi_\mathbb{R}$ be the projections on the factors of $\Phi^{-1}(c_0) \times [c_-, c_+]$. Choose a connection one-form and denote its curvature

$$\alpha \in \Omega^1(\Phi^{-1}(c_0))^{U(1)}, \quad \operatorname{curv}(\alpha) \in \Omega^2(X/\!\!/_{c_0}U(1)).$$

Define a closed two-form on $\Phi^{-1}(c_0) \times [c_-, c_+]$ by

$$\omega_0 = \pi_C^*\omega_c + \mathrm{d}(\alpha, \pi_\mathbb{R} - c_0) \in \Omega^2(\Phi^{-1}(c_0) \times [c_-, c_+]).$$

For c_-, c_+ sufficiently close to c_0, ω_0 is symplectic and has moment map given by $\pi_\mathbb{R}$. By the coisotropic embedding theorem, for c_-, c_+ sufficiently small there exists an equivariant symplectomorphism $\psi : U \to \Phi^{-1}(c_0) \times [c_-, c_+]$ of a neighborhood U of $\Phi^{-1}(c_0)$ in X with $\Phi^{-1}(c_0) \times [c_-, c_+]$. Hence the symplectic quotients $X/\!\!/_c U(1)$ for c close to c_0 are diffeomorphic to $X/\!\!/_{c_0}C$, with symplectic form $\omega_c + (c - c_0)\operatorname{curv}(\alpha)$. This ends the Remark.

The following describes a symplectic version of an mmp running. It requires that the family of symplectic manifolds be given locally by variation of symplectic quotient, and globally the change in symplectic class is the canonical class.

Definition 2.7. (a) (Simple symplectic flips) Two symplectic manifolds V_+, V_- of dimension $2n$ are related by a *simple symplectic flip* if there exists a symplectic vector space $\tilde{V} \cong \mathbb{C}^{n+1}$ with a Hamiltonian action of the circle S^1 with moment map $\Psi : \tilde{V} \to \mathbb{R}$ such that $V_\pm = \tilde{V}/\!\!/_\pm U(1)$ are the symplectic quotients at small values $\pm\epsilon$ for some $\epsilon > 0$, with the following properties:

(i) the sum of the weights is positive:

$$\sum_{i=1}^{n+1} \mu_i > 0;$$

(ii) all of the weights are non-zero:

$$\mu_i \neq 0, \quad \forall i = 1, \ldots, n+1;$$

(iii) at least two weights are positive, and at least two weights are negative:

$$n_{\pm} := \#\{\mu_i > 0\} \geq 2.$$

We write $\tilde{V}_{\pm}$ for the sum of the positive resp. negative weight spaces in $\tilde{V}$, so that

$$\tilde{V} = \tilde{V}_{+} \oplus \tilde{V}_{-}.$$

The semi-stable loci are then

$$\tilde{V}^{\text{ss},-} = (\tilde{V}_{-} - \{0\}) \times \tilde{V}_{+}, \quad \tilde{V}^{\text{ss},+} = \tilde{V}_{-} \times (\tilde{V}_{+} - \{0\}). \tag{2.2}$$

In particular, V_{+} is obtained from V_{-} by replacing a weighted-projective space of dimension $n_{-} - 1$ with a weighted-projective space of dimension $n_{+} - 1$, as in Remark 2.5, where $n_{\pm}$ are the number of positive resp. negative weights.

(b) (Fibered flips) More generally, suppose that $\tilde{V}$ is a vector bundle over a base Z equipped with a symplectic structure on the total space and a Hamiltonian $U(1)$-action with the same properties as in the simple case. Then the quotients $\tilde{V} /\!\!/_{+} U(1)$ and $\tilde{V} /\!\!/_{-} U(1)$ are related by a symplectic flip *with center* Z.

(c) (Symplectic flips) Let $X_{\pm}$ be non-empty symplectic manifolds. We say X_{+} is obtained from X_{-} by a *symplectic flip* if X_{+} is obtained *locally* by a symplectic flip: that is, there exist open covers

$$X_{\pm} = U_{\pm} \cup V_{\pm}$$

such that the following hold:

(i) U_{+} is diffeomorphic to U_{-} and admits a family of symplectic forms

$$\omega_{U,t} \in \Omega^2(U_{\pm}), t \in [-\epsilon, \epsilon]$$

and embeddings

$$i_{\pm} : U_{\pm} \to X_{\pm}, \quad i_{\pm}^{*} \omega_{\pm} = \omega_{U,\pm\epsilon}.$$

(ii) The manifolds $V_{\pm}$ are related by a simple flip as in part 2.7 (b).

(iii) The difference between symplectic classes on X_{+} and X_{-} is a positive multiple of the first Chern class in the following sense: Under the canonical identification $H^2(X_{-}) \to H^2(X_{+})$ induced by the diffeomorphism in codimension at least four the first Chern class $c_1(X_{-})$ maps to $c_1(X_{+})$. Then for some $\epsilon > 0$

$$[\omega_{+}] - [\omega_{-}] = 2\epsilon c_1(X_{-}).$$

(d) (Symplectic divisorial contraction) Symplectic divisorial contractions are defined in the same way as symplectic flips, but in this case all but one weight is positive. In this case there exists a projection $\pi_{+} : X_{+} \to X_{-}$ with exceptional locus $E \subset X_{+}$ a (weighted)-projective bundle. We require

$$[\omega_{+}] = \pi_{+}^{*}[\omega_{-}] + 2\epsilon \pi_{+}^{*} c_1(X_{-}) + \epsilon(n-1)[E]^{\vee}$$

where $[E]^{\vee} \in H^2(X_{+})$ is the Poincaré dual of the exceptional divisor. Note that this differs from the usual definition of symplectic blow-down because of the presence of the additional change in symplectic class $c_1(X_{-})$. The term $\epsilon[E]^{\vee}/(n-1)$ guarantees that the path $[\omega_{+} + (t-\epsilon)c_1(X_{+})] = [\omega_{+} + (t-\epsilon)\pi_{+}^{*} c_1(X_{-}) + (t-\epsilon)(n-1)[E]^{\vee}]$ becomes singular at time $t = 0$.

(e) (Symplectic Mori fibration) By a *symplectic Mori fibration* we mean a symplectic fibration $\pi : X_+ \to X_-$ such that the fiber $\pi^{-1}(x), x \in X_-$ is a monotone symplectic manifold (the definition of Guillemin-Lerman-Sternberg [**56**] is more general.) In all our examples flips X_- will be obtained from X_+ by a variation of symplectic quotient using a global Hamiltonian circle action. Symplectic Mori fibrations will then be Mori fibrations in the usual sense.

(f) (Symplectic mmp running) A *symplectic mmp running* we mean a sequence $X = X_0, X_1, \ldots, X_k$ together with, for each $i = 0, \ldots, k$, a path of symplectic forms

$$\omega_{i,t} \in \Omega^2(X_i), t \in [t_i^-, t_i^+], \quad \frac{d}{dt}[\omega_{i,t}] = c_1(X_i) \tag{2.3}$$

and each $(X_i, \omega_{t_i^-})$ is obtained from $(X_{i-1}, \omega_{t_{i-1}^+})$ by a symplectic mmp transition for $i = 1, \ldots, k$.

Example 2.8. (a) (Disjoint unions) If (X_t', ω_t') and (X_t'', ω_t'') are symplectic mmp runnings with distinct transition times then so is $(X_t' \sqcup X_t'', \omega_t' \sqcup \omega_t'')$. The set of centers of the running for $X_t' \sqcup X_t''$ is the union of the centers for X_t' and X_t''.

(b) (Products) If (X_t', ω_t') and (X_t'', ω_t'') are symplectic mmp runnings with distinct transition times then so is $(X_t' \times X_t'', \omega_t' \times \omega_t'')$. The set of centers of the running for $X_t' \times X_t''$ is the union of the products $Z_{t_i'}' \times X_{t_i'}''$ and $X_{t_i''}' \times Z_{t_i''}''$ where $Z_{t_i'}', Z_{t_i''}''$ are the centers for the mmp runnings for X_t', X_t''.

(c) (Equivariant mmp runnings) Let G be a compact Lie group and (X_t, ω_t) be a symplectic mmp running equipped with a Hamiltonian G-action with moment maps $\Phi_t : X_t \to \mathfrak{g}^\vee$. A family (X_t, ω_t) is a *G-equivariant symplectic mmp running* if the local models $\tilde{V}_i$ are $G \times U(1)$-Hamiltonian manifolds, so that the mmp transition is given locally by variation of $U(1)$ quotient, the variation formula (2.3) holds in equivariant cohomology, and at each transition the identifications $U_\pm, V_\pm \to X_\pm$ are identifications of Hamiltonian G-manifolds.

(d) (Quotients) If X_t is an equivariant symplectic mmp running, then the family of symplectic quotients $X_t /\!\!/ G$ is a symplectic mmp running, with local model given by $\tilde{V} /\!\!/ G$ and centers given by the quotients of the centers of X_t, assuming that these are free. Indeed, the family (X_t, ω_t) be represented as a family of $U(1)$ quotients by the master space construction (2.1) which realizes $X_t /\!\!/ G$ as a family of $U(1)$-quotients. Note that $X_t /\!\!/ G$ may have mmp transitions even if X_t does not.

2.2. Runnings for toric manifolds

In this and the following subsections we give a sequence of examples of mmp runnings. Each of these examples is in some sense obtained by variation of quotient as in Remark 2.5. The minimal model program for toric varieties was established by Reid [**110**]. Here we describe mmp runnings given by shrinking the moment polytope, see González-Woodward [**53**] and Pasquier [**103**]. First we recall the following.

Proposition 2.9. (Delzant [**34**, Section 3.2], Lerman-Tolman [**79**], Coates et al [**29**, Section 3.1].) *Any smooth toric Deligne-Mumford stack with projective coarse moduli space has a presentation as a geometric invariant theory quotient.*

We review the construction. Take V be a Hermitian vector space of dimension k with an action of a torus G. Denote the weights of the action

$$\mu_1, \dots, \mu_k \in \mathfrak{g}_{\mathbb{Z}}^\vee = \operatorname{Hom}(\mathfrak{g}_{\mathbb{Z}}, \mathbb{Z}).$$

A linearization of V is determined by an equivariant Kähler class

$$\omega_{V,G} \in H^2_G(V) \cong \mathfrak{g}_{\mathbb{Z}}.$$

The geometric invariant theory $V /\!\!/ G$ is, if locally free, the quotient of the semistable locus

$$V^{\mathrm{ss}} = \{(v_1, \dots, v_k) \in V, \quad \operatorname{span}\{\mu_k | v_k \neq 0\} \ni \omega_{V,G}\} \tag{2.4}$$

by the action of G. Suppose G is contained in a maximal torus H of the unitary group of V; then the residual torus $T = H/G$ acts on $X = V /\!\!/ G$ making X into a toric variety. The moment polytope for the action of the residual torus on the quotient can be computed from the moment polytope for the original action. The class $\omega_{V,G} \in H^2_G(V)$ has a lift to $H^2_H(V)$ where $H = (\mathbb{C}^\times)^k$. In terms of the standard basis vectors $\epsilon_i^\vee \in \mathfrak{h}^\vee$ the cohomology class of such a lift is given by

$$\omega_{V,H} = \sum c_i \epsilon_i^\vee \in \mathfrak{h}^\vee \cong H^2_H(V).$$

Choose coordinates $z_1, \dots, z_k$ on V so that the symplectic form is

$$\omega_V = (1/2i) \sum_{j=1}^k \mathrm{d} z_j \wedge \mathrm{d} \overline{z}_j.$$

A moment map for the H-action is given by the formula

$$(z_j)_{j=1}^k \mapsto (-c_j - |z_j|^2/2)_{j=1}^k.$$

Let ν_j denote the image of the j-standard basis vector $-\epsilon_j \in \mathfrak{h}$ under the projection $\mathfrak{h} \to \mathfrak{g}$. The residual action of the torus $T = H/G$ on $X = V /\!\!/ G$ has moment image

$$P = \{\lambda \in \mathfrak{t}^\vee \mid \langle \lambda, \nu_j \rangle \geq c_j, \quad j = 1, \dots, k\}. \tag{2.5}$$

The description of a toric manifold as a git quotient leads to the following description of mmp runnings.

Proposition 2.10. *Suppose that the constants $c_j, j = 1, \dots, k$ from (2.5) are generic. A running of the mmp for the toric variety X is given by the sequence of toric varieties X_t corresponding to the sequence of polytopes*

$$P_t = \{\mu \in \mathfrak{t}^\vee \mid \langle \mu, \nu_j \rangle \geq c_j + t \quad j = 1, \dots, k\}.$$

The transition times are the set of times

$$\mathcal{T} := \{\, t \;\mid\; \exists \mu \in \mathfrak{t}^\vee, \ \{\nu_j \mid \langle \mu, \nu_j \rangle = c_j + t\} \text{ is linearly dependent} \,\}.$$

Every symplectic manifold $X_t, t \notin \mathcal{T}$ is an orbifold. If the set of normal vectors ν_j to facets stay the same (that is, if the inequalities $\langle \mu, \nu_j \rangle = c_j + t$ continue to have solutions for every $j = 1, \dots, k$) then the transition is a flip; if a facet disappears then the transition is a divisorial contraction (where the divisor is the preimage of the disappearing face).

PROOF. The statement of the proposition is a special case of 2.8 (d). Indeed suppose that X is a symplectic quotient of $\mathbb{C}^k$ with respect to the equivariant symplectic class determined by the constants $(c_1, \ldots, c_N) \in \mathbb{R}^N \cong H_2^G(\mathbb{C}^k)$. Then $(c_1 - t, \ldots, c_N - t)$ represents a canonical variation of symplectic class, and so a G-equivariant mmp running on $\mathbb{C}^k$. By Remark 2.8 (4.10) this descends by the quotient construct to an mmp running for X. The description of the types of the transitions follows from the fact that the boundary divisors defined by $z_i = 0$ intersect the stable loci (2.4) on both sides exactly if the unstable locus $V - V^{\rm ss}$ is complex codimension at least two. □

For example, in Figure 2.2 there are two divisorial contractions, occurring at the dots shaded in the diagram. Finally if a generic point μ in P_{t_0} satisfies $\langle \mu, \nu_j \rangle =$

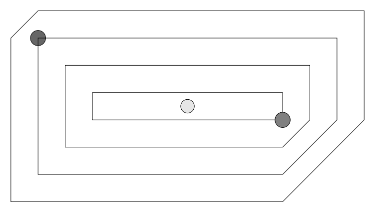

FIGURE 2.2. Moment images of regular tori for the twice blow-up of $\mathbb{P}^1 \times \mathbb{P}^1$

$c_j + t_0$ then X_t undergoes a Mori fibration at t_0 with base given by the toric variety X_{t_0} with polytope P_{t_0}; one can then continue the running with X_{t_0} to obtain an extended running.

2.3. Runnings for polygon spaces

The moduli space of polygons is the quotient of a product of two-spheres by the diagonal action of the group of Euclidean rotations in three-space. This moduli space is often used as one of the primary examples of geometric invariant theory and symplectic quotients, see for example Kirwan [**68**, Example 2.2]. This space is also a special case of the moduli space of flat bundles on a Riemann surface, which appears in a number of constructions in mathematical physics.

First recall the Hamiltonian structure of the two-sphere via its realization as a coadjoint orbit. Let $S^2 \subset \mathbb{R}^3$ the unit two-sphere equipped with the $SO(3)$-invariant symplectic form

$$\omega \in \Omega^2(S^2), \quad \int_{S^2} \omega = 1.$$

The action of $SO(3)$ on S^2 is naturally Hamiltonian. Viewing $SO(3)$ as a coadjoint orbit in $\mathfrak{so}(3)^\vee \cong \mathbb{R}^3$, the moment map is the inclusion of S^2 in $\mathbb{R}^3$, as a special case of the Kirillov-Kostant-Souriau construction [**127**].

Consider the diagonal action on the product of spheres in the previous paragraph. Let $n \geq 1$ be an integer, and $\lambda_1, \ldots, \lambda_n > 0$ a sequence of positive real numbers. The product

$$\widetilde{X} = (S^2 \times \ldots \times S^2, \lambda_1 \pi_1^* \omega + \ldots + \lambda_n \pi_n^* \omega)$$

is naturally a symplectic manifold of dimension $2n$. The group $SO(3)$ acts diagonally on $\widetilde{X}$ with moment map

$$\Psi : \widetilde{X} \to \mathbb{R}^3, \quad (v_1, \ldots, v_n) \mapsto v_1 + \ldots + v_n.$$

The symplectic quotient $X = \widetilde{X} /\!\!/ SO(3)$ is the *moduli space of n-gons*

$$P(\lambda_1, \ldots, \lambda_n) = \left\{ (v_1, \ldots, v_n) \in (\mathbb{R}^3)^n \;\middle|\; \|v_i\| = \lambda_i, \forall i, \quad \sum_{i=1}^n v_i = 0 \right\}.$$

The moduli space may be alternatively realized from the geometric invariant theory perspective as a quotient by the complexified group. We view each v_i as a point in the projective line $\mathbb{P}^1$. A tuple $(v_1, \ldots, v_n) \in \mathbb{P}^1$ is semistable if and only if for each $w \in \mathbb{P}^1$, the *slope inequality*

$$\sum_{v_i = w} \lambda_i \leq \sum_{v_i \neq w} \lambda_i$$

holds [**100**]. Then $P(\lambda_1, \ldots, \lambda_n)$ is the quotient of the semistable locus by the action of $SL(2, \mathbb{C})$ by a special case of the Kempf-Ness theorem [**67**].

A running of the mmp for the moduli space of polygons is given by varying the lengths in a uniform way. The first Chern class of the product of spheres $\widetilde{X}$ is the class

$$c_1(\widetilde{X}) = \sum_{j=1}^n \pi_j^* c_1(\mathbb{P}^1)$$

where π_j is projection onto the j-th factor. It follows from Remark 2.5 that the sequence of moduli spaces $P(\lambda_1 - t, \ldots, \lambda_n - t)$ is a mmp (after rescaling the time parameter by 2) for $P(\lambda_1, \ldots, \lambda_n)$. Transitions occur whenever there are one-dimensional polygons. That is,

$$\mathcal{T} := \left\{ t \;\middle|\; \exists \epsilon_1, \ldots, \epsilon_n \in \{-1, 1\}, \; \sum_{j=1}^n \epsilon_j (\lambda_j - t) = 0 \right\}. \tag{2.6}$$

We may assume that the number of positive signs is greater at least the number of minus signs, by symmetry. For example, if the initial configuration is $\lambda = (10, 10, 12, 13, 14)$ then there are three transitions, at $t = 5, 7, 9$, corresponding to the equalities

$$5 + 5 + 7 = 8 + 9, \; 3 + 3 + 6 = 5 + 7, \; 1 + 1 + 5 = 3 + 4.$$

There is a final transition when the smallest edge acquires zero length.

Each flip or blow-down replaces a projective space of dimension equal to the number of plus signs in (2.6), minus one, with a projective space of dimension equal to the number of minus signs in (2.6), minus one. One can explicitly describe the projective spaces involved in the flips as follows. For $\pm \in \{+, -\}$ let

$$I_\pm = \{i, \epsilon_i = \pm 1\}$$

denote the set of indices with positive resp. negative signs. Let $t_+ > t$ resp. $t_- < t$ and let

$$S_\pm = \{ \, (v_1, \ldots, v_n) \in P(\lambda_1 - t_\pm, \ldots, \lambda_n - t_\pm) \;\mid\; \mathbb{R}_{>0} v_i = \mathbb{R}_{>0} v_j, \; \forall i, j \in I_\pm \, \}$$

denote the locus where the vectors with indices in $I_\pm$ point in the same direction. (Since $\pm$ is a variable, this is a requirement for I_+ or I_- but not both.) Thus $S_\pm$

is the symplectic quotient of a submanifold $\tilde{S}_\pm$ of the product of two-spheres with only positive resp. negative weights for the circle action. So the space S_+ is a projective space and the flip replaces S_+ with S_-:

$$\begin{array}{ccccc} P(\lambda_1 - t_-, \ldots, \lambda_n - t_-) & & & & P(\lambda_1 - t_+, \ldots, \lambda_n - t_+) \\ \uparrow & & & & \uparrow \\ S_- & \longrightarrow & \text{pt} & \longleftarrow & S_+ \end{array} .$$

For example, in the case of lengths $10, 10, 12, 13, 14$ for the transition at $t = 5$, the configurations with edge lengths $5+\epsilon, 5+\epsilon, 7+\epsilon, 8+\epsilon, 9+\epsilon$ with $9+\epsilon, 8+\epsilon$ edges approximately colinear are replaced with configurations with edges with lengths $5-\epsilon, 5-\epsilon, 7-\epsilon$ approximately colinear. The first set of configurations is a two-sphere, a moduli space of quadrilaterals, while the latter set of configurations is a point, a moduli space of triangles. It follows that the transition is a blow-down.

There are two situations in which one obtains a Mori fibration. First, in the case that one of the λ_i's becomes zero, say $\lambda_i - t$ is small in relation to the other weights, there is a fibration

$$P(\lambda_1 - t, \ldots, \lambda_n - t) \to P(\lambda_1 - t, \ldots, \lambda_{i-1} - t, \lambda_{i+1} - t, \ldots, \lambda_n - t).$$

Symplectically, this is a special case of the results of Guillemin-Lerman-Sternberg [**56**, Section 4]. From the algebraic point of view, in this case the value of v_i does not affect the semistability condition, and forgetting v_i defines the fibration. In the case that the moduli space becomes empty before one of the λ_i's reaches zero, the moduli space is a projective space at the last stage, by the same discussion as in the case of flips. For example, in the case of lengths $10, 10, 12, 13, 14$ one obtains a fibration $P(.1, .1, 2.1, 3.1, 4.1) \to P(2, 3, 4)$ when $t = 10$ over the moduli space $P(2, 3, 4)$ which is a point. Therefore, the moduli space at $t = 10 - \epsilon$ is diffeomorphic to a product of two-spheres: $P(\epsilon, \epsilon, 2+\epsilon, 3+\epsilon, 4+\epsilon) \cong S^2 \times S^2$. The conclusion is that $P(10, 10, 12, 13, 14)$ is a thrice-blow-up of $S^2 \times S^2$, that is, a del Pezzo surface.

The second way that one may obtain a Mori fibration is that one can reach a chamber in which the moduli space is empty, because one of the lengths, say $\lambda_i - t$ is so long compared to the others that the sum of the remaining lengths is smaller than the first length:

$$\sum_{j \neq i} \lambda_j - t < \lambda_i - t.$$

In this case, the last non-empty moduli space admits the structure of a fiber bundle over the space of reducible polygons corresponding to the transition, which form a point. The fiber is diffeomorphic to a projective space, by a repeat of the arguments above. See Moon-Yoo [**99**] for more details.

2.4. Runnings for moduli spaces of flat bundles

The moduli space of flat bundles on a Riemann surface is an example of an infinite-dimensional symplectic quotient, and studied in for example Atiyah-Bott [**6**]:

Proposition 2.11. *Let Σ be a compact Riemann surface and G a simply-connected compact Lie group. The moduli space $\mathcal{R}(\Sigma)$ of flat G-bundles on Σ has a presentation as an infinite-dimensional symplectic quotient of the affine space $\mathcal{A}(\Sigma)$ of G-connections by the Hamiltonian action of the group of gauge transformations $\mathcal{G}(\Sigma)$.*

Recall of construction. The trivial G-bundle $P = \Sigma \times G$ has space of connections $\mathcal{A}(P)$ canonically identified with the space of $\mathfrak{g}$-valued one-forms which are G-invariant and induce the identity on the vertical directions:

$$\mathcal{A}(P) := \{\alpha \in \Omega^1(\Sigma \times G)^G, \quad \alpha(\xi_P) = \xi, \forall \xi \in \mathfrak{g}\}$$

where $\xi_P \in \mathrm{Vect}(P)$ is the vector field generated by ξ. The space $\mathcal{A}(P)$ is an affine space modelled on $\Omega^1(\Sigma, \mathfrak{g})$, A symplectic structure is given by wedge product and integration:

$$\Omega^1(\Sigma, \mathfrak{g})^2 \to \mathbb{R}, \quad (a_1, a_2) \mapsto \int_\Sigma (a_1 \wedge a_2).$$

Here $(a_1 \wedge a_2) \in \Omega^2(\Sigma)$ is the result of composition

$$\Omega^1(\Sigma, \mathfrak{g})^{\otimes 2} \to \Omega^2(\Sigma, g^{\otimes 2}) \to \Omega^2(\Sigma, \mathbb{R})$$

where the latter is induced by an invariant inner product $\mathfrak{g} \times \mathfrak{g} \to \mathbb{R}$. The action of the group $\mathcal{G}(P)$ of gauge transformations on $\mathcal{A}(P)$ by pullback is Hamiltonian with moment map given by the curvature:

$$\mathcal{A}(P) \to \Omega^2(\Sigma, P(\mathfrak{g})), \quad \alpha \mapsto F_\alpha.$$

The symplectic form on $\mathcal{A}(P)$ descends to a closed two-form on the symplectic quotient

$$\mathcal{R}(\Sigma) := \mathcal{A}(P) /\!\!/ \mathcal{G}(P) = \{\alpha \in \mathcal{A}(P) \mid F_\alpha = 0\} / \mathcal{G}(P)$$

the moduli space of flat connections on the trivial bundle. The tangent space to $\mathcal{R}(\Sigma)$ at the isomorphism class of a connection α has a natural identification

$$T_{[\alpha]} \mathcal{R}(\Sigma) \cong H^1(\mathrm{d}_\alpha)$$

with the cohomology $H^1(\mathrm{d}_\alpha)$ of the associated covariant derivative d_α in the adjoint representation. The Hodge star furnishes a Kähler structure on the moduli space. □

Definition 2.12. (Moduli spaces of bundles on surfaces with boundary) Extensions of the construction to the case that the surface has boundary are given in Mehta-Seshadri [**96**]. Suppose Σ is a compact oriented surface of genus g with n boundary components. That is, Σ is obtained from a closed compact oriented surface of genus g by removing n disjoint disks as in Figure 2.3. Let $Z_k \subset \Sigma$ be the k-th boundary circle, and $[Z_k] \in \pi_1(\Sigma)$ the class defined by a small loop around the k-th boundary component for $k = 1, \ldots, n$. Let $G = SU(2)$ denote group of special unitary 2×2 matrices. The space of conjugacy classes $G/\operatorname{Ad}(G)$ is naturally parametrized by an interval:

$$[0, 1/2] \to G/\operatorname{Ad}(G), \quad \lambda \mapsto \mathrm{diag}(\exp(\pm 2\pi i \lambda)).$$

Let $\lambda_1, \ldots, \lambda_n \in [0, 1/2]$ be *labels* attached to the boundary components. Choose a base point in Σ and let $\pi_1(\Sigma)$ denote the fundamental group of homotopy classes of based loops with respect to some base point. Each loop Z_k defines an element $[Z_k] \in \pi_1(\Sigma)$, by connecting Z_k to the base point, and is well-defined up to conjugacy. For numbers μ_1, μ_2 we denote by $\mathrm{diag}(\mu_1, \mu_2)$ the diagonal 2×2 matrix

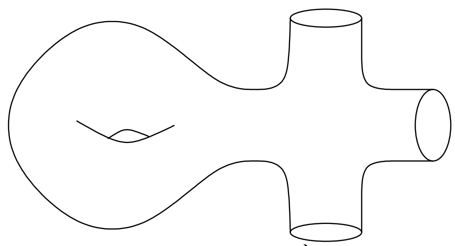

FIGURE 2.3. Compact surface with labelled boundary

with diagonal entries μ_1 and μ_2. Let $\mathcal{R}(\lambda_1, \ldots, \lambda_n)$ denote the moduli space of isomorphism classes of flat bundles with holonomy around the boundary circles given by $\exp(2\pi i \operatorname{diag}(\lambda_k, -\lambda_k)), k = 1, \ldots, n$. Since any flat bundle is described up to isomorphism by the associated holonomy representation of the fundamental group, we have the explicit description

$$\mathcal{R}(\lambda_1, \ldots, \lambda_n) = \left\{ \begin{array}{l} \varphi \in \operatorname{Hom}(\pi_1(\Sigma), SU(2)) \\ \varphi([Z_k]) \in SU(2) \exp(2\pi i \operatorname{diag}(\lambda_k, -\lambda_k)) \\ k = 1, \ldots, n \end{array} \right\} / SU(2). \tag{2.7}$$

By Mehta-Seshadri [**96**], the moduli space of flat bundles may be identified with the moduli space of *parabolic bundles* with weights $\lambda_1, \ldots, \lambda_n$. Here a parabolic bundle means a holomorphic bundle $E \to \Sigma$ on closed Riemann surface Σ with markings $z_1, \ldots, z_n \in \Sigma$ with the additional datum of one-dimensional subspace $L_i \subset E_{z_i}$ in the fiber E_{z_i} at each marking, together with the weights $\lambda_1, \ldots, \lambda_n$.

REMARK 2.13. (Moduli of spherical polygons) In the case of rank two bundles there is a simple interpretation of these moduli spaces in terms of *spherical polygons*. Namely $\pi_1(\Sigma)$ is generated by homotopy classes of paths

$$\gamma_1, \ldots, \gamma_n \in \pi_1(\Sigma), \quad \gamma_1 \ldots \gamma_n = 1.$$

Thus a representation of the fundamental group corresponds to a tuple

$$g_1, \ldots, g_n \in SU(2), \quad g_1 \ldots g_n = e$$

where $e \in SU(2)$ is the identity. Consider the polygon in $SU(2)$ with vertices

$$(e,\ g_1,\ g_1 g_2, \ldots,\ g_1 \ldots g_{n-1}) \in SU(2)^n.$$

Choose a metric on $SU(2) \cong S^3$ invariant under the left and right actions. Because the metric on $SU(2)$ is invariant under the right action, the distance between the $j-1$-th and j-th vertices is the distance between e and g_j. Using invariance again it suffices to assume that $g_j = \operatorname{diag}(2\pi i(\lambda_j, -\lambda_j))$. The distance is then λ_j, once the metric is normalized so the maximal torus has volume one. Via the identification of S^3 with $SU(2)$, any representation gives rise to a polygon in S^3 with edge lengths $\lambda_1, \ldots, \lambda_n$. Conversely, any closed polygon determines a representation assigning the edge elements to the generators of $\pi_1(\Sigma)$, and this correspondence is bijective up to isometries of the three-sphere.

For generic weights the moduli space of flat bundles has a smooth running of the mmp given by varying the labels in a uniform way. First, a result of Boden-Hu [**18**] and Thaddeus [**131**, Section 7] shows that varying the labels leads to a generalized flips in the sense that all conditions are satisfied except the condition that the morphisms to the singular quotient are relatively ample. For this the variation of Kähler class should be in the canonical direction as we now explain. In the case without boundary, the anticanonical class was computed by Drezet-Narasimhan [36] and in the case of parabolic bundles by Biswas-Raghavendra [13]; see Meinrenken-Woodward [85] for a symplectic perspective. The anticanonical class is expressed in terms of the symplectic class and the line bundles L_j associated to the eigenspaces of the holonomy around the boundary components by

$$c_1(\mathcal{R}(\lambda_1,\ldots,\lambda_n)) = 4[\omega_{\mathcal{R}(\lambda_1,\ldots,\lambda_n)}] - \sum_{j=1}^{n}(4\lambda_j - 1)2c_1(L_j).$$

In particular, if all weights $\lambda_i = \frac{1}{4}$ then the moduli space is Fano.

PROPOSITION 2.14. *The moduli space $\mathcal{R}(\lambda,\ldots,\lambda_n)$ has a smooth running of the mmp given by the sequence of moduli spaces*

$$\mathcal{R}\left(\frac{\lambda_1 - t}{1-4t},\ldots,\frac{\lambda_n - t}{1-4t}\right). \tag{2.8}$$

PROOF. This family can be produced as a variation of symplectic quotient using the construction of [85] as follows: Let LG denote the loop group of $G = SU(2)$, $\mathcal{R}$ denote the moduli space of flat G-connections with framings on the boundary equipped with its natural Hamiltonian action of LG^n, and $\mathcal{O}_{\lambda_1},\ldots,\mathcal{O}_{\lambda_n}$ the LG-coadjoint orbits through $\lambda_1,\ldots,\lambda_n$. Then $\mathcal{R}(\lambda_1,\ldots,\lambda_n)$ has a realization as a symplectic quotient

$$\mathcal{R}(\lambda_1,\ldots,\lambda_n) = (\mathcal{R}\times\mathcal{O}_{\lambda_1}\times\ldots\times\mathcal{O}_{\lambda_n})/\!\!/ LG^n.$$

Consider the product of anticanonical bundles

$$K_{\mathcal{R}}^\vee \boxtimes K_{\mathcal{O}_{\lambda_1}}^\vee \boxtimes \ldots \boxtimes K_{\mathcal{O}_{\lambda_n}}^\vee \to \mathcal{R}\times\mathcal{O}_{\lambda_1}\times\ldots\times\mathcal{O}_{\lambda_n} \tag{2.9}$$

in the sense of [85]. Its total space minus the zero section has closed two form given by

$$\pi^*\omega_{\mathcal{R}\times\mathcal{O}_{\lambda_1}\times\ldots\times\mathcal{O}_{\lambda_n}} + \mathrm{d}(\alpha,\phi) \in \Omega^2((K_{\mathcal{R}}^\vee \boxtimes K_{\mathcal{O}_{\lambda_1}}^\vee \boxtimes \ldots \boxtimes K_{\mathcal{O}_{\lambda_n}}^\vee) - \{0\})$$

where α is a connection one-form and ϕ is the logarithm of the norm on the fiber. This two-form is non-degenerate on the region defined by $(\lambda_i - \phi)/(1-4\phi) \in (0,1/2)$ for each i. The action of S^1 by scalar multiplication in the fibers is Hamiltonian with moment map ϕ. The quotient

$$\tilde{\mathcal{R}}(\lambda_1,\ldots,\lambda_n) := (K_{\mathcal{R}}^\vee \boxtimes K_{\mathcal{O}_{\lambda_1}}^\vee \boxtimes \ldots \boxtimes K_{\mathcal{O}_{\lambda_n}}^\vee - \{0\})/\!\!/ LG^n$$

has a residual action of S^1 whose quotients are the family given above:

$$\tilde{\mathcal{R}}(\lambda_1,\ldots,\lambda_n)/\!\!/_t S^1 = \mathcal{R}\left(\frac{\lambda_1 - t}{1-4t},\ldots,\frac{\lambda_n - t}{1-4t}\right).$$

At any fixed point the action of S^1 on the anticanonical bundle has positive weight, by definition. Note that the case $\lambda_1 = \ldots = \lambda_n = \frac{1}{4}$ has a trivial mmp. One should think of the markings as moving away from the "center" $\frac{1}{4}$ of the Weyl alcove $[0,1/2]$ under the mmp. □

The flips or blow-downs occur at transition times at which there are reducible (abelian) bundles. More precisely, the set of transition times is

$$\mathcal{T} = \left\{ t \;\middle|\; \exists \epsilon_1, \ldots, \epsilon_n \in \{-1, 1\}, \quad \sum_{i=1}^{n} \epsilon_i \frac{\lambda_i - t}{1 - 4t} \in \frac{1}{2}\mathbb{Z} \right\}. \tag{2.10}$$

The projective bundles involved in the flip can be explicitly described as follows using the description of the moduli space as loop space quotient in [**84**]; we focus on the genus zero case and omit the proofs. Let

$$I_\pm = \{i, \epsilon_i = \pm 1\}.$$

Fix a decomposition of the curve Σ into Riemann surfaces with boundary Σ_+, Σ_- such that $\Sigma_\pm$ contains the markings in $I_\pm$. Let $S_\pm$ denote the moduli space of bundles that are abelian on $\Sigma_\pm$:

$$S_\pm = \left\{ [A] \in \mathcal{R}\left(\frac{\lambda_1 - t_\pm}{1 - 4t}, \ldots, \frac{\lambda_n - t_\pm}{1 - 4t} \right) \;\middle|\; \dim(\operatorname{Aut}(A)) = 1 \right\}.$$

Then the flip replaces S_+ with S_-:

$$\begin{array}{ccccc} \mathcal{R}\left(\left(\frac{\lambda_j - t_-}{1 - 4t} \right)_{j=1}^{n} \right) & & & & \mathcal{R}\left(\left(\frac{\lambda_j - t_+}{1 - 4t} \right)_{j=1}^{n} \right) \\ \uparrow & & & & \uparrow \\ S_- & \longrightarrow & \mathcal{R}^{\mathrm{ab}}\left(\left(\frac{\lambda_j - t_-}{1 - 4t} \right)_{j=1}^{n} \right) & \longleftarrow & S_+ \end{array}$$

where

$$\mathcal{R}^{\mathrm{ab}}\left(\left(\frac{\lambda_j - t_-}{1 - 4t} \right)_{j=1}^{n} \right) = \left\{ \rho \in \mathcal{R}\left(\frac{\lambda_1 - t}{1 - 4t}, \ldots, \frac{\lambda_n - t}{1 - 4t} \right) \;\middle|\; \rho(\pi_1(\Sigma)) \in T \right\}$$

is the moduli space of representations in the maximal torus $T = \{\operatorname{diag}(e^{i\theta}, e^{-i\theta})\} \subset SU(2)$. Thus a projective bundle over the Jacobian gets replaced with another projective bundle.

As in the case of polygon spaces, there are two ways of obtaining Mori fibrations: First, fibrations with $\mathbb{P}^1$-fiber occur whenever one of the markings $\lambda_i - t$ reaches 0 or 1/2, with base the moduli space of flat bundles with labels

$$\frac{\lambda_1 - t}{1 - 4t}, \ldots, \frac{\lambda_{i-1} - t}{1 - 4t}, \frac{\lambda_{i+1} - t}{1 - 4t}, \ldots, \frac{\lambda_n - t}{1 - 4t}$$

resp.

$$\frac{\lambda_1 - t}{1 - 4t}, \ldots, \frac{\lambda_{i-1} - t}{1 - 4t}, 1/2, \frac{\lambda_{i+1} - t}{1 - 4t}, \ldots, \frac{\lambda_n - t}{1 - 4t}$$

if the marking λ_i reaches 0 resp. 1/2. Second, in genus zero the moduli space can become empty before any of the markings reach 0 or 1/2. By a special case of Agnihotri-Woodward [**5**], proved earlier by Treloar [**130**] we have

$$\mathcal{R}(\lambda_1, \ldots, \lambda_n) = \emptyset \iff \exists I = \{i_1 \neq \ldots \neq i_{2k+1}\}, \quad \sum_{i \in I} \lambda_i > k + \sum_{i \notin I} \lambda_i. \tag{2.11}$$

Thus in the last stage there is either a fibration over a moduli space with one less parabolic weight, with two-sphere fiber, or in genus zero one can also have a projective space at the last stage if the moduli space becomes empty. One can then continue with the base to obtain an extended running. The mmp of this moduli space is discussed in greater detail in Moon-Yoo [**99**] as well as Boden-Hu [**18**] and Thaddeus [**131**, Section 7].

Remark 2.15. (Runnings for flag varieties) Flag varieties admit mmp runnings given by fibrations over partial flag varieties. Let X be the variety of complete flags in a vector space of dimension n with symplectic class $[\omega] \in H^2(X) \cong \mathbb{R}^n$ corresponding to an n-tuple $\lambda = (\lambda_1 \geq \ldots \geq \lambda_n) \in \mathbb{Z}^n$. The space X has a natural transitive action of the unitary group which induces a diffeomorphism $X \cong U(n)/U(1)^n$. The unique polarization of X is $\tilde{X} = (U(n) \times \mathbb{C})/U(1)^n$ where $U(1)^n$ acts on $U(n)$ from the right and on the left on $\mathbb{C}$ with weight λ. We identify the Lie algebra with $\mathbb{R}^n$, the weight lattice with $\mathbb{Z}^n$ and let $\epsilon_1, \ldots, \epsilon_n \in \mathbb{Z}^n$ denote the standard basis of weights for $U(1)^n$. We identify the set of positive weights with the intersection of $\mathbb{Z}^n$ with

$$\mathfrak{t}_+^\vee = \{(\mu_1 \geq \ldots \geq \mu_n)\} \subset \mathbb{R}^n.$$

The tangent bundle of X is the associated fiber bundle

$$TX \cong U(n) \times_{U(1)^n} \bigoplus_{1 \leq i < j \leq n} \mathbb{C}_{\epsilon_i - \epsilon_j}$$

where $\mathbb{C}_{\epsilon_i - \epsilon_j}$ is the space on which $U(1)^n$ acts with weight $\epsilon_i - \epsilon_j$. Hence the canonical bundle of X is

$$\Lambda^{\mathrm{top}} TX \cong U(n) \times_{U(1)^n} \mathbb{C}_{2\rho}$$

where $\mathbb{C}_{2\rho}$ denotes the one-dimensional representation of $U(1)^n$ with weight

$$2\rho := ((n-1)\epsilon_1 + (n-3)\epsilon_2 + \ldots + (1-n)\epsilon_n).$$

An extended running of the mmp is the sequence of partial flag varieties obtained as follows. Consider the piecewise linear path λ_t starting with $\lambda_t = \lambda - \rho t$ and continuing as follows: Whenever λ_t hits a wall $\sigma \subset \mathfrak{t}_+^\vee$ of the positive chamber the mmp running continues with the path

$$\lambda_t = \lambda_{t_i} - (t - t_i)\pi_\sigma \rho$$

where π_σ is the projection onto σ. Each transition is a Mori fibration with Grassmann fiber and base the partial flag variety corresponding to the element λ_{t_i}.

A simple example is the variety of complex flags in a three-dimensional complex vector space which admits the structure of a Mori fibration in two ways. For example, let

$$X = \mathrm{Fl}(\mathbb{C}^3) := \{V_1 \subset V_2 \subset \mathbb{C}^3 \mid \dim(V_k) = k, k = 1, 2\}$$

be the variety of full flags in $\mathbb{C}^3$. A running of the mmp is given by $X, \mathbb{P}^2, \mathrm{pt}$. There are no flips or divisorial contractions in this case, so no regular Lagrangians.

CHAPTER 3

Lagrangians associated to flips

We introduce a class of Lagrangians associated to minimal model transitions which we call *regular*; based on the local models studied in Fukaya et al. [**47**]. Regularity refers to the fact that the Maslov index two disks in a toric neighborhood have equal area. These will be shown in Theorem 7.22 below to be Floer non-trivial.

3.1. Regular Lagrangians

Each regular Lagrangian is given locally in a toric model and bounds a collection of holomorphic disks of equal area. We introduce the following terminology.

Definition 3.1. Let X be obtained by a reverse flip or blow-up with center Z. The reverse flip or blow-up replaces a projective bundle $\mathbb{P}(N_-) \to Z$ with a projective bundle $\mathbb{P}(N_+) \to Z$. By the constant rank embedding theorem [**83**], a neighborhood U of $\mathbb{P}(N_+)$ in X is symplectomorphic to a neighborhood V of the zero section in a symplectic vector bundle E_+ over $\mathbb{P}(N_+)$. Let $\phi : U \to \mathbb{R}_{\geq 0}$ be the canonical moment map for the Hamiltonian S^1-action on the fibers of E_+, so that $\mathbb{P}(N_+) = \phi^{-1}(0)$. Let J denote an almost complex structure equal to the toric complex structure on the fibers of E_+.

(a) A Lagrangian $L \subset \mathbb{P}(N_+)$ is *toric* if the restriction of $\pi : \mathbb{P}(N_+) \to Z$ to L fibers over a submanifold L_Z of Z with fiber a standard Lagrangian (in the fiber) torus orbit $L_F \subset \pi^{-1}(z)$ for some $z \in Z$. A Lagrangian L in E_+ is *toric* if the restriction of $\pi : E_+ \to \mathbb{P}(N_+)$ fibers over a toric Lagrangian in $\mathbb{P}(N_+)$ with fiber a standard torus L_F in the fiber of E_+.

(b) A Lagrangian L in X is *regular* if there exists a constant $c > 0$ such that
   (i) L is a toric Lagrangian in $\phi^{-1}([0, c))$;
   (ii) the holomorphic disks $u : (D, \partial D) \to (X, L)$ of Maslov index two contained in $\phi^{-1}([0, c))$ all have equal area A_0;
   (iii) any non-constant holomorphic disk $u : (D, \partial D) \to (X, L)$ meeting the complement of U has area greater than A_0.

Example 3.2. (Blow-up at a point) Let $X = \mathrm{Bl}_0(\mathbb{C}^n)$ be the symplectic blow-up of $\mathbb{C}^n$ at a point. The symplectic manifold X admits a Hamiltonian torus action with all weights -1 and moment map $\Phi : X \to \mathfrak{t}^\vee \cong \mathbb{R}^n$ with image

$$\Phi(X) = \{(x_1, \ldots, x_n) \in \mathbb{R}^n_{\geq 0} \mid x_1 + \ldots + x_n \geq 1.\}.$$

The inverse image

$$L = \Phi^{-1}(1, \ldots, 1)$$

is the unique regular torus. Indeed, X can be realized as the quotient $\mathbb{C}^{n+1} /\!\!/ \mathbb{C}^\times$ of $\mathbb{C}^{n+1}$ by the $\mathbb{C}^\times$-action with weights $-1, \ldots, -1, 1$. The only disks of Maslov index

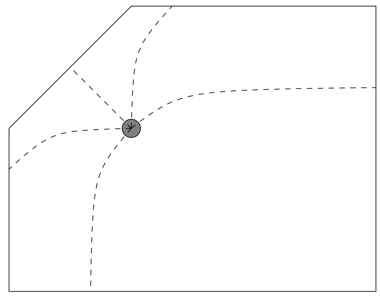

FIGURE 3.1. Representation of Maslov index two disks

two are given by

$$u_i : B \to \mathbb{C}^{n+1}, \quad z \mapsto (1, \dots, 1, z, 1, \dots, 1).$$

All disks $u_i, i = 1, \dots, n+1$ have equal area. A similar local model applies to toric blow-ups in general; see Chapter 3.2 for more details. A regular Lagrangian in a blow-up of a product of projective lines is shown in Figure 3.1.

3.2. Regular Lagrangians for toric manifolds

The follow classification of disks in the toric case is used in the definition. We continue with the notation from Chapter 2.2.

PROPOSITION 3.3. (Cho-Oh [**31**, Section 4]) *Let X be a compact symplectic toric manifold equipped with the action of a torus T, whose moment polytope $\Phi(X)$ has k facets. Holomorphic disks with boundary in a Lagrangian torus orbit are classified by k-tuples of non-negative integers $\underline{d} \in \mathbb{Z}^k_{\geq 0}$. The area and index of a disk corresponding to $\underline{d}$ are*

$$A(\underline{d}) = \sum_{j=1}^k \langle \lambda, \nu_j \rangle - c_j, \quad I(\underline{d}) = 2 \sum_{j=1}^k d_j.$$

PROOF. We recall the classification. Suppose X is realized as a symplectic quotient of a vector space $V \cong \mathbb{C}^k$ by the action of a torus G, and let $L \subset V$ be a Lagrangian orbit of T. Let $\tilde{L} \cong (S^1)^k \subset V$ denote the lift of L to V, given as the orbit of a point $(\tilde{\mu}_1, \dots, \tilde{\mu}_k)$ under the standard torus action:

$$\tilde{L} = \{(e^{i\theta_1}\tilde{\mu}_1, \dots, e^{i\theta_k}\tilde{\mu}_k) \mid \theta_1, \dots, \theta_k \in \mathbb{R}\}. \tag{3.1}$$

Let

$$C = \{z \in \mathbb{C} \mid |z| \leq 1\}$$

denote the unit disk. A *Blaschke product* of degree $(d_1, \dots, d_k)$ is a map from C to $\mathbb{C}^k$ with boundary in a toric Lagrangian prescribed by coefficients

$$a_{i,j} \in \mathbb{C}, \quad |a_{i,j}| < 1, \quad i \leq n, \quad j \leq d_i :$$

$$u : C \to \mathbb{C}^k, \quad z \mapsto \left(\prod_{j=1}^{d_i} \frac{z - a_{i,j}}{1 - z\overline{a_{i,j}}} \right)_{i=1,\dots,n}. \tag{3.2}$$

As in Cho-Oh [**31**], the products (3.2) are a complete description of holomorphic disks with boundary in L. Since the image of $\tilde{u}(z)$ is disjoint from the semistable locus, the Blaschke products descend to disks $u : (C, \partial C) \to (X, L)$. We compute their Maslov index using the splitting $\tilde{u}^* TV \cong u^* TX \oplus \mathfrak{g}$. Since the Maslov index of bundle pairs is additive, and the second factor has Maslov index zero, the Maslov index of the disk is given by

$$I(u) = \sum_{i=1}^{k} 2d_j = 2\#u(D).(\sum_{i=1}^{i}[D_i])$$

twice the sum of the intersection number with the anticanonical divisor, which is the disjoint union of prime invariant divisors. In particular the disks of index two are those maps u_i with lifts of the form

$$\tilde{u}_i(z) = (\tilde{\mu}_1, \ldots, \tilde{\mu}_{i-1}, \tilde{\mu}_i z, \tilde{\mu}_{i+1}, \ldots, \tilde{\mu}_k).$$

The area of this disk is

$$A(u_i) = A(\tilde{u}_i) = \langle \lambda, \nu_i \rangle - c_i$$

by a standard computation in Darboux coordinates. The homology class of higher index Maslov disks $u : C \to X, I(u) > 2$ is a sum of these homology classes, and so the area $A(u)$ of such a disk u is the sum of the areas of disks of index two. □

The moment fibers that are regular Lagrangians in Definition 3.1 are described as follows. Let $\mu \in P$ and

$$t(\mu) = \min_j \langle \mu, \nu_j \rangle - c_j.$$

The real number $t(\mu)$ is the time at which μ "disappears" under the mmp. Suppose the set of normal vectors of facets meeting the singular point

$$N(\mu) := \{\nu_j \mid \langle \mu, \nu_j \rangle - c_j = t\}$$

is linearly dependent. Then $L_\mu := \Phi^{-1}(\mu)$ satisfies the first two parts of the definition of regularity in Definition 3.1 (b). To see this we first compute the areas of disks with boundary on the Lagrangian. Suppose that X is realized as the git quotient of a vector space $V \cong \mathbb{C}^k$ by a torus G. We may assume that $\dim(X) > 1$ so that the real codimension of the unstable locus is at least four. Let $\tilde{L}_\mu$ denote the preimage of L_μ in C^k. The Lagrangian $\tilde{L}_\mu$ is a Lagrangian torus orbit of the group $U(1)^k$ acting on $\mathbb{C}^k$, that is, $\tilde{L}_\mu = U(1)^k(\tilde{\mu}_1, \ldots, \tilde{\mu}_k)$ for some constants $(\tilde{\mu}_1, \ldots, \tilde{\mu}_k)$. Each map of a holomorphic disk to $\mathbb{C}^k$ with boundary in $\tilde{L}_\mu$ corresponds to a collection of maps from disks to $\mathbb{C}$ with boundary on $U(1)\tilde{\mu}_j$. If L_μ is a regular Lagrangian then the minimal areas $A(u)$ of the holomorphic disks $u : C \to X$ with Lagrangian boundary condition L_μ are t. These correspond to disks in the components corresponding to facets distance t from μ, and each of these disks u has Maslov index $I(u)$ two. The last assumption in Definition 3.1 (b) holds if the other facets are sufficiently far away. For example, in Figure 2.2 we have two regular Lagrangians, given as the inverse images of the darkly shaded dots in the Figure under the moment map.

3.3. Regular Lagrangians for polygon spaces

A natural family of Lagrangian tori is generated by the *bending flows* studied in Klyachko [**70**] and Kapovich-Millson [**98**]. Fix a subset $I \subset \{1, \ldots, n\}$ of the edges of the polygon and define a *diagonal length* function

$$\tilde{\Psi}_I : S^2 \times \ldots \times S^2 \to \mathbb{R}_{\geq 0}, \quad (v_1, \ldots, v_n) \mapsto \|v_I\|, \quad v_I := \sum_{i \in I} v_i.$$

The diagonal length is smooth on the locus where it is positive. The Hamiltonian flow of the diagonal length function is given by rotating part of the polygon around the diagonal. The following is, for example, explained in [**98**, Section 3]:

LEMMA 3.4. *The function $\tilde{\Psi}_I$ generates on $\tilde{\Psi}_I^{-1}(\mathbb{R}_{>0})$ a Hamiltonian circle action given by rotating the vectors $v_i, i \in I$ around the axis spanned by $v_I := \sum_{i \in I} v_i$:*

$$v_i \mapsto R_{\theta, v_I} v_i, \quad i = 1, \ldots, n \tag{3.3}$$

where R_{θ,v_I} is rotation by angle θ around the span of v_I. Furthermore if $\mathcal{T}$ is a collection of subsets such that for all $I, J \in \mathcal{T}$ either $I \subseteq J$ or $J \subseteq I$ then the associated flows $R_{\theta,v_I}, R_{\theta',v_J}$ commute.

PROOF. We provide a proof for completeness. By the symplectic cross-section theorem [**57**, Theorem 26.7], the inverse of the interior of the positive Weyl chamber $\mathbb{R}_{>0}$ under the moment map Ψ is a symplectic submanifold $\Psi^{-1}(\mathbb{R}_{>0} \times \{0\} \times \{0\}) \subset \widetilde{X}$. This inverse image is the locus

$$\widetilde{X}_I = \left\{ (v_1, \ldots, v_n) \in \widetilde{X} \;\middle|\; v_I \in \mathbb{R}_{>0} \times \{0\} \times \{0\} \subset \mathbb{R}^3 \right\}.$$

Since the stabilizer of any point in $\widetilde{X}_I$ is contained in $SO(2)$, the flow-out of $\widetilde{X}_I$ is $SO(3)\widetilde{X}_I = SO(3) \times_{SO(2)} \widetilde{X}_I$. As a result, any $SO(2)$-equivariant Hamiltonian diffeomorphism of $\widetilde{X}_I$ extends uniquely to a Hamiltonian diffeomorphism of $SO(3)\widetilde{X}_I$ which is equivariant with respect to the $SO(3)$-action. Now on $\widetilde{X}_I$, the function $\tilde{\Psi}_I$ is the first component of the moment map and so the flow of $\tilde{\Psi}_I$ is rotation around the first axis. It follows that the flow of $\tilde{\Psi}_I$ is rotation around the line spanned by $\sum_{i \in I} v_i$, as long as this vector is non-zero. In particular the flow of $\tilde{\Psi}_I$ is $SO(3)$-equivariant and so descends to a function Ψ_I generating a circle action on a dense subset of $P(\lambda_1, \ldots, \lambda_n) = \widetilde{X} /\!\!/ SO(3)$. If $I \subset J$ then we may assume that the ordering of vectors $v_1, \ldots, v_n$ is such that I, J consist of adjacent indices. The vectors v_I and v_J break the polygon into three pieces, and the flows of Ψ_I and Ψ_J are rotation of the first and third pieces around the diagonals v_I, v_J respectively. In particular, these flows commute. □

Given a triangulation of a polygon we associate a moment map for a densely-defined torus action as follows. A triangulation $\mathcal{E} = \{T \subset \{1, \ldots, n\}\}$ of the abstract n-gon with edges $v_1, \ldots, v_n$ is a collection of subsets, called triangles. Each triangle $T \in \mathcal{E}$ is a set of size three I_1, I_2, I_3 whose elements indicate which sums $v_{I_j} := \sum_{i \in I_j} v_i$ form the edge. If I_j has size more than one, the corresponding vector v_{I_j} is a *diagonal*. See Figure 3.2. Any triangulation $\mathcal{E}$ triangulation gives rise to a map

$$\Psi_{\mathcal{E}} : P(\lambda_1, \ldots, \lambda_n) \to \mathbb{R}^{n-3}_{\geq 0}, \quad [v_1, \ldots, v_n] \to (\|v_{I_j}\|)_{j=1}^{n-3}$$

given by taking the edge lengths of the diagonals. By Lemma 3.4, the map $\Psi_{\mathcal{E}}$ is, where smooth, a moment map for the action of an $n-3$-dimensional *bending torus* $G \cong U(1)^{n-3}$ which acts as follows: Let $(\exp(i\theta_1), \ldots, \exp(i\theta_{n-3})) \in G$ and $[v_1, \ldots, v_n]$ be an equivalence class of polygons. For each diagonal v_I, divide the

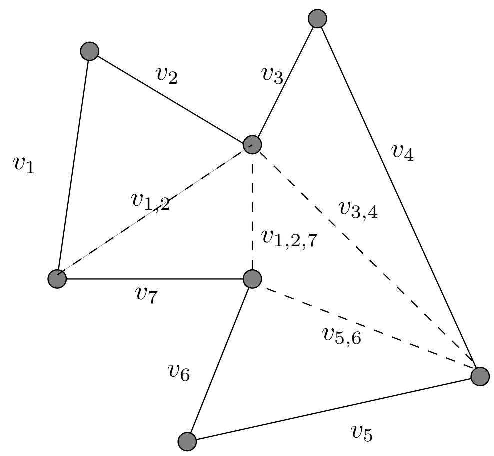

FIGURE 3.2. Triangulated polygon

polygon into two pieces along v_I, and rotate one of those pieces, say $(v_i)_{i\in I}$ around the diagonal by the given angle θ_I. The resulting polygon is independent of the choice of which piece is rotated, since polygons related by an overall rotation define the same point in the moduli space. See [**98**, Section 3] for more details.

The regular Lagrangian tori are described as follows as fibers of the Goldman map for which all triangles have the same "looseness". Suppose that $P(\lambda_1 - t, \dots, \lambda_n - t)$ is an mmp running for $P(\lambda_1, \dots, \lambda_n)$. As noted in Example 2.2, mmp transitions correspond to partitions

$$\{1, \dots, n\} = I_+ \cup I_-, \quad \sum_{i \in I_+} \lambda_i - t = \sum_{i \in I_-} \lambda_i - t.$$

For each triangle T in the triangulation with labels μ_1, μ_2, μ_3 we denote by $l(T)$ the *looseness* of the triangle

$$l(T) := \min_{i,j,k \text{ distinct}} (\mu_i + \mu_j - \mu_k)/2.$$

In other words, the looseness measures the failure of the triangle to be degenerate. A labelling $\mu \in \mathbb{R}^{n-3}_{\geq 0}$ is called *regular* if $l(T)$ is independent of T and less than $\min_i \lambda_i$. For example, if the edge lengths are $2, 3, 4, 7$, and the triangulation separates the first two edges from the last two, then a regular triangulation is obtained by assigning 4 to the middle edge. The looseness of each triangle is $1 = 2 + 3 - 4 = 4 + 4 - 7$. See Figure 3.3.

PROPOSITION 3.5. *Let $\mu \in \mathbb{R}^{n-3}_{\geq 0}$ be a regular labelling, and $\mu(t)$ the family of labellings obtained by replacing each μ_i with $\mu_i - t$ which becomes singular at first time $t = t_i$. Then for $\epsilon > 0$ sufficiently small and $t \in (t_i - \epsilon, t_i)$, any labelling $\mu(t)$ has the property that $\Psi^{-1}(\mu)$ is regular.*

PROOF. A local toric structure on $P(\lambda_1, \dots, \lambda_n)$ is given by choosing a triangulation compatible with the partition into positive and negative edges, and the action of the bending torus T above. Let $v \in P(\lambda_1 - t, \dots, \lambda_n - t)$ denote the one-dimensional polygon corresponding to the transition time. Each triangle in the triangulation is degenerate for v and so for each T, $\mu_i + \mu_j = \mu_k$ for some edges

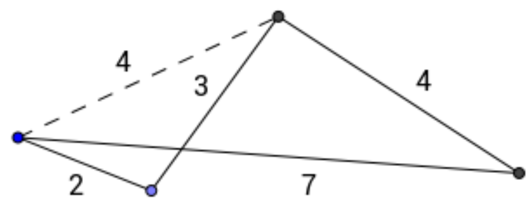

FIGURE 3.3. A regular triangulation of a quadrilateral

i, j, k of T. The inequalities defining $\Psi_t(T)$ near v are of the form $\mu_i + \mu_j \geq \mu_k$ as i, j, k range over all possible indices. It follows that the polytope defining the image of the map Ψ is given locally by the triangle inequalities, $l(T) \geq 0$ for each of the $n-2$ triangles in the triangulation:

$$\begin{aligned} (3.4) \quad & \Psi_{\mathcal{E}}(P(\lambda_1, \ldots, \lambda_n)) \\ &= \left\{ (\mu_1, \ldots, \mu_{n-3}) \in \mathbb{R}^{n-3}_{\geq 0} \;\middle|\; \forall T \in \mathcal{E}, (T = \{v_i, v_j, v_k\}) \implies \mu_i + \mu_j \geq \mu_k \right\}. \end{aligned}$$

The Maslov index two disks $u : C \to X$ with respect to this structure whose areas go to zero at the transition time have areas $A(u)$ given by the differences $\mu_i + \mu_j - \mu_k$, and by assumption the non-constant disks with lowest area all have equal area. On the other hand, for times close to the transition time, any other Maslov index two disk $u : C \to X$ has larger area, since otherwise it would be contained in the toric piece $X_\subset$ by the diameter estimate in Sikorav [**123**, 4.4.1]. □

3.4. Regular Lagrangians for moduli spaces of flat bundles

The analog of the bending flow was introduced by Goldman [**51**]. First one constructs a densely defined circle action on the moduli space of bundles associated to any circle on the surface. Given any circle $C \subset \Sigma$ disjoint from the boundary, the holonomy $\varphi(C)$ of the flat bundle P around C is given by an element $\exp(\mathrm{diag}(\pm 2\pi i \mu))$ up to conjugacy. After gauge transformation, the holonomy is $\exp(\mathrm{diag}(\pm 2\pi i \mu))$. Given an element $\exp(2\pi i \tau) \in U(1)$, one may construct a bundle P_τ by cutting Σ along C into pieces Σ_+, Σ_- and gluing back the restrictions $P|\Sigma_+, P|\Sigma_-$ together using the transition map $e(\tau) := \mathrm{diag}(\exp(\pm 2\pi i \tau))$:

$$P_\tau := (P|\Sigma_+) \bigcup_{e(\tau)} (P|\Sigma_-). \tag{3.5}$$

See Figure 3.4.
The automorphism given by $\mathrm{diag}(\exp(\pm 2\pi i \tau))$ commutes with the holonomy so the resulting bundle has a canonical flat structure, whose holonomies around loops Σ_+, Σ_- are equal, but parallel transport from Σ_+ to Σ_- is twisted by $\mathrm{diag}(\exp(\pm 2\pi i \tau))$. Let $\mathcal{R}(\lambda_1, \ldots, \lambda_n)^C$ denote the locus where $\mu \notin \{0, 1/2\}$, for which the construction $[P] \mapsto [P_\tau]$ is well-defined and independent of all choices. The map

$$\mathcal{R}(\lambda_1, \ldots, \lambda_n)^C \to \mathcal{R}(\lambda_1, \ldots, \lambda_n)^C, \quad [P] \mapsto [P_\tau]$$

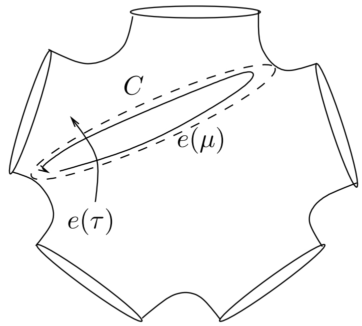

FIGURE 3.4. Twisting a bundle along a circle

defines a circle action. Furthermore, if C_1, C_2 are disjoint circles then the circle actions defined above commute on the common locus

$$\mathcal{R}(\lambda_1, \ldots, \lambda_n)^{C_1} \cap \mathcal{R}(\lambda_1, \ldots, \lambda_n)^{C_2}.$$

Recall that a *pants decomposition* of a surface is a decomposition into three-holed spheres. Any compact oriented Riemann surface with boundary admits a finite pants decomposition, by choosing sufficiently many separating surfaces so that each piece has Euler characteristic one. Choose a pants decomposition $\mathcal{P}$ that refines the decomposition into pieces Σ_+, Σ_-. Given a pants decomposition, one repeats the construction for each interior circle in the pants decomposition to obtain a moment map

$$\Psi_{\mathcal{P}} : \mathcal{R}(\lambda_1, \ldots, \lambda_n) \to [0, 1/2]^{n-3} \tag{3.6}$$

for a densely-defined torus action, see [**51**, Section 4] or, for a summary, Jeffrey-Weitsman [**65**]. In the genus zero case, the generic fibers are Lagrangian tori. For each pairs of pants P in the pants decomposition with labels μ_1, μ_2, μ_3, define the *looseness* of P by

$$l(P) := \min\left(\min_{i \neq j \neq k}(\mu_i + \mu_j - \mu_k), 1 - \mu_1 - \mu_2 - \mu_3\right).$$

A labelling $\mu \in [0, 1/2]^{n-3}$ is *regular* if the looseness $l(P)$ is the same for each pair of pants $P \in \mathcal{P}$,

$$\#\{l(P) | P \in \mathcal{P}\} = 1$$

and if the first fibration in the running occurs at a time greater than $l(P)$. See Figure 8.1 for two examples in the case $n = 5$.

The regular Lagrangians are described as follows. Consider an mmp running as in (2.8) $\mathcal{R}\left(\frac{\lambda_1 - t}{1-4t}, \ldots, \frac{\lambda_n - t}{1-4t}\right)$. The transition times $\mathcal{T}$ are the times t for which there is an abelian representation. Given such a representation with holonomies $\operatorname{diag}(\exp(\pm\epsilon_j \mu_j))$ define a partition of the surface Σ into pieces Σ_+, Σ_- containing the markings μ_j for which ϵ_j is positive resp. negative. We claim that if μ is regular and $l(\mu)$ is sufficiently small then the Goldman fiber $L_\mu := \Psi_{\mathcal{P}}^{-1}(\mu)$ is regular. The Goldman bending flow (3.5) induces a toric structure on $\mathcal{R}(\lambda_1, \ldots, \lambda_n)$ in a neighborhood of the exceptional locus. The image of the Goldman map (3.6) is given by

$$\Psi_{\mathcal{P}}(\mathcal{R}(\lambda_1, \ldots, \lambda_n)) = \left\{ \mu \in [0, 1/2]^{n-3} \;\middle|\; \forall P \in \mathcal{P}, l(P) \geq 0 \right\};$$

that is, for each pair of pants in the pants decomposition the looseness is non-negative [**65**]. It follows that if l is a regular labelled pants decomposition then $\Psi^{-1}(l)$ is toric and there are $n-2$ homotopy classes of disks with Maslov index two and boundary in $\Psi^{-1}(l)$, of equal area $l(\mu)$ while (for sufficiently small looseness) the remaining disks of Maslov index two have area $A(u) > l(\mu)$. Figure 8.1 gives an example of a labelled pants decomposition giving rise to a regular Lagrangian, corresponding to an mmp transition at time $t = .06$.

CHAPTER 4

Fukaya algebras

The Fukaya algebra is a homotopy-associative algebra whose higher composition maps are counts of configurations involving perturbed pseudoholomorphic disks with boundary in the Lagrangian [**45**]. Because the moduli spaces of disks involved in the construction are usually singular, there are technical issues involved in its construction similar to those involved in the construction of virtual fundamental classes for moduli spaces of pseudoholomorphic curves. Fukaya-Oh-Ohta-Ono [**46**] introduced a method of solving these issues using *Kuranishi structures* in which one first constructs local thickenings of the moduli spaces and then introduces perturbations constructed locally. In this section we construct Fukaya algebras of Lagrangians in a compact *rational* symplectic manifold using a perturbation scheme that we find particularly convenient for various computations: the *stabilizing divisors* scheme introduced by Cieliebak-Mohnke [**28**]. We also incorporate Morse gradient trees introduced by Fukaya [**45**] and Cornea-Lalonde [**32**], see also Seidel [**117**] and Charest [**26**]. Stasheff's homotopy-associativity equation follows from studying the boundary strata in the moduli space of treed disks as in Figure 4.1. This construction

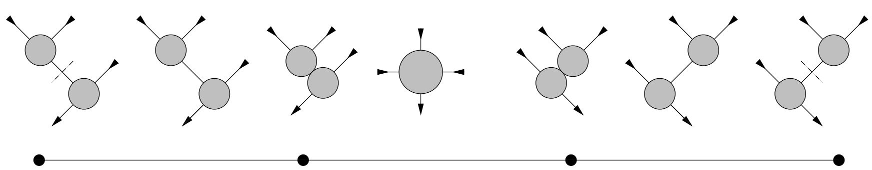

FIGURE 4.1. Moduli space of stable treed disks

allows us to take our Floer cochain spaces to be finite-dimensional. The structure constants for the Fukaya algebras in the stabilizing divisors approach count pseudoholomorphic disks with Lagrangian boundary conditions and Morse gradient trajectories on the Lagrangians with domain-dependent almost complex structures and Morse functions depending on the position of additional markings mapping to a stabilizing divisor. Because the additional marked points must be ordered in order to obtain a domain without automorphisms, this scheme gives a multi-valued perturbation. The resulting structure maps

$$\mu^n : CF(L)^{\otimes n} \to CF(L), \quad n \geq 0$$

for the Fukaya algebra are defined only using rational coefficients. We also equip Fukaya algebras with strict units so that disk potentials are defined. To achieve this we incorporate a slight enhancement, similar to that of homotopy units in Fukaya-Oh-Ohta-Ono [**46**, (3.3.5.2)], in which perturbation systems compatible with breakings are homotoped to perturbation systems that admit forgetful maps.

Given the construction of strictly unital Fukaya algebras described above, the Floer cohomology is defined over a space of projective Maurer-Cartan solutions. Let $e_L \in CF(L)$ denote the resulting strict unit and $CF(L)^{\text{odd},+} \subset CF(L)$ the subset of odd elements whose coefficients all have positive q-valuation. The Maurer-Cartan map

$$\mu : CF(L)^{\text{odd},+} \to CF(L), \quad b \mapsto \sum_{n \geq 0} \mu^n \left(\underbrace{b, \ldots, b}_{n} \right) \tag{4.1}$$

has solution space

$$MC(L) := \mu^{-1}(\Lambda e_L) \subset CF(L)$$

also denoted $MC(L, y)$ if we wish to emphasize the dependence on the local system y. The *Floer cohomology* is the fiber-wise cohomology of operator μ_b^1 defined below in Chapter 4.8

$$HF(L) = \cup_{b \in MC(L)} HF(L, b), \quad HF(L, b) := \ker(\mu_b^1) / \operatorname{im}(\mu_b^1), \quad b \in MC(L).$$

The Floer cohomology $HF(L)$ is said to be non-vanishing if the fiber $HF(L, b)$ is non-vanishing for some $b \in MC(L)$. We prove the following:

THEOREM 4.1. *Let (X, ω) be a compact symplectic manifold with rational symplectic class $[\omega] \in H^2(X, \mathbb{Q})$ and $L \subset X$ a compact rational embedded Lagrangian submanifold equipped with a relative spin structure and grading. For a comeager subset of perturbation data, counting weighted treed pseudoholomorphic disks defines a convergent A_∞ structure with strict unit independent of all choices up to convergent strictly-unital A_∞ homotopy. Furthermore, for any $b \in MC(L)$ the Floer cohomology $HF(L, b)$ is independent of all choices up to gauge equivalence (to be explained below).*

Theorem 4.1 is a combination of Theorem 4.32 and Corollary 5.13 below.

4.1. A_∞ algebras

Homotopy-associative algebras were introduced by Stasheff [**128**] in order to capture algebraic structures on the space of cochains on loop spaces. We follow the sign convention in Seidel [**119**]. Let $g > 0$ be an even integer. A *$\mathbb{Z}_g$-graded A_∞ algebra* consists of a $\mathbb{Z}_g$-graded vector space A together with for each $d \geq 0$ a multilinear degree zero *composition map*

$$\mu^d : \ A^{\otimes d} \to A[2 - d]$$

satisfying the A_∞ *-associativity equations* [**119**, (2.1)]

$$0 = \sum_{\substack{n,m \geq 0 \\ n+m \leq d}} (-1)^{n + \sum_{i=1}^n |a_i|} \mu^{d-m+1}(a_1, \ldots, a_n, \mu^m(a_{n+1}, \ldots, a_{n+m}), a_{n+m+1}, \ldots, a_d) \tag{4.2}$$

for any tuple of homogeneous elements $a_1, \ldots, a_d$ with degrees $|a_1|, \ldots, |a_d| \in \mathbb{Z}_g$. The signs are the *shifted Koszul signs*, that is, the Koszul signs for the shifted grading in which the structure maps have degree one as in Kontsevich-Soibelman [**74**]. The notation $[2 - d]$ denotes a degree shift by $2 - d$, so that without the shifting μ^1 has degree 1, μ^2 has degree 0 etc. The element $\mu^0(1) \in A$ (where $1 \in \Lambda$

is the unit) is called the *curvature* of the algebra. The A_∞ algebra A is *flat* if the curvature vanishes. A *strict unit* for A is an element $e_A \in A$ such that

$$\mu^2(e_A, a) = a = (-1)^{|a|}\mu^2(a, e_A), \quad \mu^n(\ldots, e_A, \ldots) = 0, \forall n \neq 2. \tag{4.3}$$

A *strictly unital* A_∞ algebra is an A_∞ algebra equipped with a strict unit. The *cohomology* of a flat A_∞ algebra A is defined by

$$H(\mu^1) = \frac{\ker(\mu^1)}{\operatorname{im}(\mu^1)}.$$

The algebra structure on $H(\mu^1)$ is given by

$$[a_1 a_2] = (-1)^{|a_1|}[\mu^2(a_1, a_2)]. \tag{4.4}$$

An element $e_A \in A$ is a *cohomological unit* if $[e_A]$ is a unit for $H(\mu^1)$. A result of Seidel [**120**, Corollary 2.14] implies that any flat A_∞ algebra with a cohomological unit is equivalent to an A_∞ algebra with strict unit. However, we will construct strict units by a *homotopy unit* construction.

4.2. Associahedra

The combinatorics of the A_∞ associativity axiom (4.2) is closely related to a sequence of cell complexes introduced by Stasheff [**128**] under the name *associahedra*. One realization of these spaces is as *moduli spaces of metric ribbon trees*. An *oriented tree* is a connected, cycle-free graph given by a pair

$$\Gamma = (\operatorname{Edge}(\Gamma), \operatorname{Vert}(\Gamma))$$

where $\operatorname{Vert}(\Gamma)$ is the set of vertices and $\operatorname{Edge}(\Gamma)$ is the set of edges equipped with *head* and *tail* maps

$$h, t : \operatorname{Edge}(\Gamma) \to \operatorname{Vert}(\Gamma) \cup \{\infty\}.$$

The *valence* $|v|$ of any vertex $v \in \operatorname{Vert}(\Gamma)$ is the number of edges $e \in h^{-1}(v) \cup t^{-1}(v)$ meeting the vertex v. An edge $e \in \operatorname{Edge}(\Gamma)$ is *combinatorially finite* if $\infty \notin \{h^{-1}(e), t^{-1}(e)\}$ *semi-infinite* if $\{h^{-1}(e), t^{-1}(e)\} = \{v, \infty\}$ for some $v \in \operatorname{Vert}(\Gamma)$ and *infinite* if $\operatorname{Vert}(\Gamma) = \emptyset$ and $\operatorname{Edge}(\Gamma)$ has a single element e. If $\operatorname{Vert}(\Gamma)$ is non-empty, then we denote by $\operatorname{Edge}_-(\Gamma)$ resp. $\operatorname{Edge}_\to(\Gamma)$ the set of combinatorially finite resp. semi-infinite edges. In the special case $\operatorname{Vert}(\Gamma)$ is empty we denote by $\operatorname{Edge}_\to(\Gamma)$ two copies e_+, e_- of the single edge $e \in \operatorname{Edge}(\Gamma)$ (so that there is a single incoming e_- and single outgoing e_+ semi-infinite edge). A *ribbon tree* is a tree Γ equipped with a *ribbon structure*: a cyclic ordering $o_v : \{e \in \operatorname{Edge}(\Gamma), e \ni v\} \to \{1, \ldots, |v|\}$ of the edges incident to each vertex $v \in \operatorname{Vert}(\Gamma)$; a cyclic ordering is an equivalence class $[o_v]$ of orderings where two orderings o_v, o_v' are equivalent if they are related by a cyclic permutation $o_v'(\tau(e)) = o_v(\tau(e)) + k \mod |v|$. A single edge $e_0 \in \operatorname{Edge}_\to(\Gamma)$ is outgoing (with head at ∞) and called the *root* of the tree. All the edges of Γ are oriented towards the root e_0. Incoming semi-infinite edges $e \in \operatorname{Edge}_\to(\Gamma), e \neq e_0$ are called *leaves*.

A moduli space of metric ribbon trees is obtained by allowing the finite edges to acquire lengths. A *metric ribbon tree* is a pair $T = (\Gamma, \ell)$ consisting of a ribbon tree Γ equipped with a *metric* ℓ. By definition a metric is a labelling

$$\ell : \operatorname{Edge}_-(\Gamma) \to [0, \infty]$$

of its combinatorially finite edges $e \in \mathrm{Edge}_-(\Gamma)$ by elements $\ell(e)$ of $[0, \infty]$ called *lengths*. We think of T as the topological space obtained by joining together intervals T_e of length $\ell(e)$ at the vertices. From this point of view there is a natural equivalence relation on ribbon metric trees defined by collapsing edges of length zero: Given a tree T with an edge e of length $\ell(e)$ zero, removing the interior $\mathrm{int}(e)$ of the edge e from T and identifying its head $h(e)$ and tail $t(e)$ gives an equivalent metric tree

$$T' = (T - \mathrm{int}(e))/(h(e) \sim t(e)). \tag{4.5}$$

To obtain a compactification of the moduli space we also allow the lengths of the edges to go to infinity in which case the edge becomes a *broken edge*: The interior of the edge e with length $\ell(e) = \infty$ is equipped with a finite number of points $b_1, \ldots, b_k \subset \mathrm{int}(e)$ called *breakings*. A *broken metric tree* is obtained from a finite collection of metric trees by gluing roots to leaves as follows: given two metric trees T_1, T_2 and semi-infinite root edge $e_2 \in \mathrm{Edge}_\to(\Gamma_2)$ and leaf edge $e_1 \in \mathrm{Edge}_\to(\Gamma_2)$, let $\overline{T}_1$ resp. $\overline{T}_2$ denote the space obtained by adding a point ∞_2 resp. ∞_1 at the open end of e_2 resp. e_1. The space

$$T := \overline{T}_1 \cup_{\infty_1 \sim \infty_2} \overline{T}_2 \tag{4.6}$$

is a broken metric tree, the point $\infty_1 \sim \infty_2$ being called a *breaking*. See Figure 4.2. In general, broken metric trees T are obtained by from broken metric trees

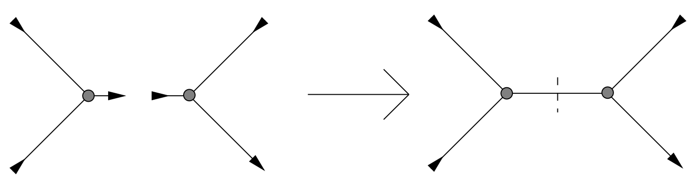

FIGURE 4.2. Creating a broken tree

T_1, T_2 as in (4.6) in such a way that the resulting space T is connected and has no non-contractible cycles, that is, $\pi_0(T)$ is a point and $\pi_1(T)$ is the trivial group. We think of the gluing points as breakings rather than vertices, so that there are no new vertices in the glued treed T. If a combinatorially finite edge $e \in \mathrm{Edge}_-(T)$ has infinite length $\ell(e) = \infty$ then one attaches an additional positive integer $b(e)$ to that edge indicating its number of breakings, see [**26**].

In order to obtain a compact moduli space of broken trees a stability condition is imposed. A broken metric tree $T = (\Gamma, \ell, b)$ is *stable* if and only if each combinatorially semi-infinite edge $e \in \mathrm{Edge}_\to(T)$ is unbroken, that is, $b(e) = 0$; each combinatorially finite edge $e' \in \mathrm{Edge}_-(T)$ is broken at most once, that is, $b(e') \leq 1$; and the valence $|v|$ of each vertex $v \in \mathrm{Vert}(T)$ is at least 3. The moduli space of stable metric ribbon trees with a fixed number of semiinfinite edges is a finite cell complex studied in, for example, Boardman-Vogt [**16**]. This moduli space is the first realization of Stasheff's associahedron as a moduli space of geometric objects. However, the natural cell structure on this moduli space is a *refinement* of the canonical cell structure on the associahedra.

A second realization of the associahedron that reproduces the canonical cell structure involves nodal disks with boundary markings.

DEFINITION 4.2. (a) A *holomorphic disk* is a complex surface with boundary diffeomorphic to the complex unit disk $B^2 = \{ z \in \mathbb{C} \mid \|z\| \leq 1 \}$. A *nodal disk* with a single boundary node is a topological space S obtained from a disjoint union of holomorphic disks S_1, S_2 by identifying pairs of boundary points $w_{12} \in S_1, w_{21} \in S_2$ on the boundary of each component so that

$$S = S_1 \cup_{w_{12} \sim w_{21}} S_2. \tag{4.7}$$

See Figure 4.3. The image of w_{12}, w_{21} in the space S is the *nodal point.* A

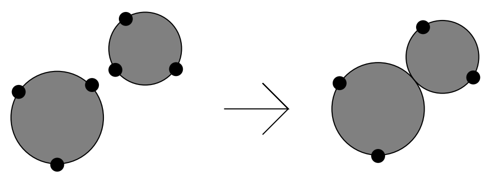

FIGURE 4.3. Creating a nodal disk

nodal disk S with multiple nodes $w_{ij}, i, j \in \{1, \ldots, k\}, i \neq j$ is obtained by repeating this construction (4.7) with S_1, S_2 nodal disks with fewer nodes, and w_{12}, w_{21} distinct from the previous nodes. For an integer $n \geq 0$ a *nodal disk with* $n+1$ *boundary markings* is a nodal disk S equipped with a finite ordered collection of points $\underline{x} = (x_0, \ldots, x_n)$ on the boundary ∂S, disjoint from the nodes, in counterclockwise cyclic order around the boundary ∂S.

(b) (Stable nodal disks) An $(n + 1)$-marked nodal disk $(S, \underline{z})$ is *stable* if each component S_v has at least three special (nodal or marked) points. The moduli space of $(n+1)$-marked stable disks forms a compact cell complex, isomorphic as a cell complex to the associahedron from Stasheff [**128**].

(c) (Sphere components and interior markings) A *holomorphic sphere* is a complex surface biholomorphic to the projective line $S^2 \cong \mathbb{P}^1$. We allow sphere components $\mathbb{P}^1 \cong S_v \subset S$ and interior markings $z_1, \ldots, z_n \in \text{int}(S)$ in the definition of marked nodal disks S. A nodal disk S with a single interior node $w \in S$ is defined similarly to that of a boundary node by using the construction (4.7), except in this case S is obtained by gluing together a nodal disk S_1 with a holomorphic marked sphere S_2 with w_{12}, w_{21} points in the interior $\text{int}(S)$.

A combination of the above constructions involves both trees and disks, as in Oh [**106**], Cornea-Lalonde [**32**], Biran-Cornea [**15**], and Seidel [**117**]. A *treed disk* C is obtained from a nodal disk S by replacing each node w with a (possibly broken) edge e of some length $\ell(e)$ and $b(e) \geq 0$ breakings; that is, by replacing w with two copies w_{12}, w_{21} and gluing to the endpoints of T_e. We also allow a (possibly broken) edge e with no disks in which case $C \cong \mathbb{R}$ has two semi-infinite edges e_+, e_- and an arbitrary number of breakings $b(e)$. Let $\Gamma(C)$ be the *combinatorial type* of C, equal to the combinatorial type of the nodal disk S but equipped with the additional data of a number of breakings $b : \text{Edge}(C) \to \mathbb{Z}_{\geq 0}$. Thus a treed disk C consists of a surface part

$$S = (S_v, \underline{x}_v, \underline{z}_v)_{v \in \text{Vert}(\Gamma)}$$

(where $\underline{x}_v$ resp. $\underline{z}_v$ denotes the ordered set of boundary resp. interior markings) a tree part

$$T = (T_e, \ell(e), b(e))_{e \in \mathrm{Edge}(\Gamma)}$$

and an ordering

$$o : \mathrm{Edge}_{\bullet,\to}(\Gamma) \to \{1, \ldots, n\}$$

of the set of interior leaves. Denote by

$$z_e = T_e \cap S, e \in \mathrm{Edge}_{\bullet,\to}(\Gamma)$$

the attaching points of the interior leaves and call them *interior markings*. A treed disk $C = S \cup T$ is *stable* if and only if the nodal disk S is stable, each combinatorially-finite edge is broken at most once, and each semi-infinite edge is unbroken. The set of vertices $\mathrm{Vert}(\Gamma)$ is equipped with a partition

$$\mathrm{Vert}(\Gamma) = \mathrm{Vert}_\circ(\Gamma) \sqcup \mathrm{Vert}_\bullet(\Gamma)$$

into vertices corresponding to disks and vertices correspond to spheres. Similarly, the set of edges is partitioned into sets of edges

$$\mathrm{Edge}(\Gamma) = \mathrm{Edge}_\circ(\Gamma) \sqcup \mathrm{Edge}_\bullet(\Gamma)$$

representing boundary nodes resp. interior nodes. For each $v \in \mathrm{Vert}_\circ(\Gamma)$, the edges $e \in \mathrm{Edge}_\circ(E)$ incident to v are equipped with a cyclic ordering o_v induced by the orientation on the boundary of the disk S_v. We call the resulting tree Γ equipped with a length function $\ell : \mathrm{Edge}_-(\Gamma) \to [0, \infty]$ also a *metric ribbon tree*, although only some of the incident edges $e \in (h \times t)^{-1}(v)$ are equipped with a cyclic ordering. See Figure 4.4. An *isomorphism* of treed disks $\phi : C \to C'$ with

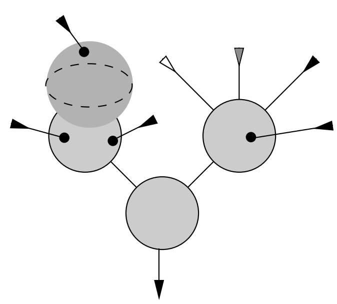

FIGURE 4.4. A treed disk with three disk components and one sphere component

positive edge lengths is a collection of isomorphisms $\phi_v : S_v \to S'_v$ of nodal disks preserving the markings and a collection of isomorphisms $T_e \to T'_e$ of broken edges (that is, preserving the number of breakings and lengths). These data combine to a homeomorphism $\phi : C \to C'$ that is an isometry $\phi|_T : T \to T'$ on the tree part and a biholomorphism $\phi|_S : S \to S'$ on the surface part. A treed disk C with an edge length $\ell(e)$ zero is declared equivalent to the treed disk C' where the edge e is replaced by a node $w \in C'$.

In order to obtain Fukaya algebras with strict units, we attach additional parameters to certain of the semi-infinite edges called *weightings* as in Ganatra [50]. When the weighting of an edge is infinite, we will assume that the perturbation data is pulled back under the forgetful map forgetting that edge and stabilizing.

For this reason, the edges where the weightings are forced to be infinite are called *forgettable*.

DEFINITION 4.3. (Weightings) A *weighting* of a treed disk $C = S \cup T$ of type Γ is

(a) (Weighted, forgettable, and unforgettable edges) a partition of the boundary semi-infinite edges

$$\mathrm{Edge}^{\blacktriangledown}(\Gamma) \sqcup \mathrm{Edge}^{\triangledown}(\Gamma) \sqcup \mathrm{Edge}^{\blacktriangledown}(\Gamma) = \mathrm{Edge}_{\circ,\to}(\Gamma)$$

into *weighted* resp. *forgettable* resp. *unforgettable* edges, and

(b) (Weighting) a map

$$\rho : \mathrm{Edge}_{\circ,\to}(\Gamma) \to [0, \infty]$$

satisfying the property: each of the semi-infinite e edges is assigned a *weight* $\rho(e)$ such that

$$\rho(e) \in \begin{cases} \{0\} & e \in \mathrm{Edge}^{\blacktriangledown}(\Gamma) \\ [0,\infty] & e \in \mathrm{Edge}^{\blacktriangledown}(\Gamma) \\ \{\infty\} & e \in \mathrm{Edge}^{\triangledown}(\Gamma) \end{cases}.$$

If the outgoing edge $e_0 \in \mathrm{Edge}_{\to}(\Gamma)$ is unweighted (forgettable or unforgettable) then an isomorphism $\psi : (C, \rho) \to (C', \rho')$ of weighted treed disks is an isomorphism of treed disks $C \to C'$ that preserves the types of semi-infinite edges $e \in \mathrm{Edge}_{\to}(\Gamma) \cong \mathrm{Edge}_{\to}(\Gamma')$ and weightings: $\rho(e) = \rho'(e')$ for all corresponding edges $e \in \mathrm{Edge}_{\circ,\to}(\Gamma)$, $e' \in \mathrm{Edge}_{\circ,\to}(\Gamma')$. This ends the definition.

The case that the outgoing edge $e_0 \in \mathrm{Edge}_{\to}(\Gamma)$ is weighted $\rho(e_0) > 0$ is rare in our examples and should be considered an exceptional case. There is an additional notion of equivalence in this case: If the outgoing edge e_0 is weighted then an isomorphism of weighted treed disks $C \to C'$ is an isomorphism of treed disks preserving the types of semi-infinite edges $e \in \mathrm{Edge}_{\circ,\to}(\Gamma)$ and the weights $\rho(e), e \in \mathrm{Edge}_{\circ,\to}(\Gamma)$ up to scalar multiples:

$$\exists \lambda \in (0,\infty),\ \forall e \in \mathrm{Edge}_{\circ,\to}(\Gamma), e' \in \mathrm{Edge}_{\circ,\to}(\Gamma'),\ \rho(e) = \lambda \rho'(e'). \tag{4.8}$$

In particular, any weighted tree T such that $\mathrm{Vert}(\Gamma) = \emptyset$ and a single edge $e \in \mathrm{Edge}_{\circ,\to}(\Gamma)$ that is weighted $\rho(e) \in (0, \infty)$ is isomorphic to any other such configuration T' with a different weight $\rho(e') \in (0,\infty), e \in \mathrm{Edge}(\Gamma')$.

The *combinatorial type* of any weighted treed disk is the tree associated to the underlying nodal disk with additional data recording which lengths resp. weights are zero or infinite. Namely if $C = S \cup T$ is a weighted treed disk then its combinatorial type is the tree $\Gamma = \Gamma(C)$ obtained by gluing together the combinatorial types $\Gamma(S_v)$ of the disks S_v along the edges corresponding to the edges of T; and equipped with the additional data of

(a) the subsets

$$\mathrm{Edge}^{\blacktriangledown}(\Gamma) \text{ resp. } \mathrm{Edge}^{\triangledown}(\Gamma) \text{ resp. } \mathrm{Edge}^{\blacktriangledown}(\Gamma) \subset \mathrm{Edge}_{\circ,\to}(\Gamma)$$

of weighted, resp. forgettable, resp. unforgettable semi-infinite edges;

(b) the subsets

$$\mathrm{Edge}_{-}^{\infty}(\Gamma) \text{ resp. } \mathrm{Edge}_{-}^{0}(\Gamma) \text{ resp. } \mathrm{Edge}_{-}^{(0,\infty)}(\Gamma) \subset \mathrm{Edge}_{-}(\Gamma)$$

of combinatorially finite edges of infinite resp. zero length resp. non-zero finite length;

(c) the subset

$$\mathrm{Edge}^{\blacktriangledown,\infty}(\Gamma) \text{ resp. } \mathrm{Edge}^{\blacktriangledown,0}(\Gamma) \subset \mathrm{Edge}^{\blacktriangledown}(\Gamma)$$

of weighted edges with infinite resp. zero weighting.

A well-behaved moduli space of weighted treed disks is obtained after imposing a stability condition.

Definition 4.4. A weighted treed disk $C = S \cup T$ of type Γ is *stable* if either

(a) (At least one disk) there is at least one disk component $S_v, v \in \mathrm{Vert}_\circ(\Gamma)$, and the following conditions hold:
   (i) each disk component $S_v, v \in \mathrm{Vert}_\circ(\Gamma)$ has at least three edges $e \in \mathrm{Edge}(\Gamma)$ attached to the boundary ∂S_v or at least one edge attached to the boundary ∂S_v and one edge to the interior $\mathrm{int}(S_v)$;
   (ii) each sphere component $S_v, v \in \mathrm{Vert}_\bullet(\Gamma)$ has at least three edges $e \in \mathrm{Edge}(\Gamma)$ attached;
   (iii) each combinatorially-finite edge $e \in \mathrm{Edge}_{-}(\Gamma)$ is broken at most once, and each semi-infinite edge $e \in \mathrm{Edge}_{\rightarrow}(\Gamma)$ is unbroken;
   (iv) if the outgoing edge is weighted $e_0 \in \mathrm{Edge}^{\blacktriangledown}(\Gamma)$ then at least one leaf $e_i \in \mathrm{Edge}_{\circ,\rightarrow}(\Gamma), i > 0$ is also weighted, that is, $e_i \in \mathrm{Edge}^{\blacktriangledown}(\Gamma)$.

(b) (No disks) if there are no disks, so that $\mathrm{Vert}(\Gamma) = \emptyset$, there is a single weighted leaf $e_1 \in \mathrm{Edge}^{\blacktriangledown}(\Gamma)$ and an unweighted (forgettable or unforgettable) root $e_0 \in \mathrm{Edge}^{\triangledown}(\Gamma) \cup \mathrm{Edge}^{\blacktriangledown}(\Gamma)$.

These conditions guarantee that the moduli space $\mathcal{M}_\Gamma$ of stable weighted treed disks of each combinatorial type Γ is expected dimension, see Remark 4.5 below. Because a configuration $C \cong \mathbb{R}$ with no disks is allowed, the stability condition for weighted treed disks is not equivalent to the absence of non-trivial automorphisms, that is, the triviality $\mathrm{Aut}(C) \cong \{1\}$ of the group $\mathrm{Aut}(C)$ of automorphisms of C.

The moduli spaces of stable weighted treed disks are naturally cell complexes with multiple cells of top dimension. For integers $n, m \geq 0$ denote by $\overline{\mathcal{M}}_{n,m}$ the moduli space of isomorphism classes of stable weighted treed disks C with n boundary leaves and m interior leaves. For each combinatorial type Γ denote by $\mathcal{M}_\Gamma \subset \overline{\mathcal{M}}_{n,m}$ the set of isomorphism classes of weighted stable treed disks of type Γ. The dimension of $\mathcal{M}_\Gamma$ is equal to $n + 2m - 2$, in the absence of weighted edges. In particular, if Γ has no vertices $v \in \mathrm{Vert}(\Gamma)$ then the dimension $\dim(\mathcal{M}_\Gamma)$ of $\mathcal{M}_\Gamma$ is zero. The moduli spaces decomposes into strata of fixed type

$$\overline{\mathcal{M}}_{n,m} = \bigcup_\Gamma \mathcal{M}_\Gamma.$$

Let $\overline{\mathcal{M}}_\Gamma$ denote the closure of $\mathcal{M}_\Gamma$ in $\overline{\mathcal{M}}_{n,m}$. In Figure 4.5 a subset of the moduli space $\overline{\mathcal{M}}_{2,1}$ with one interior marking is shown, where the interior marking is constrained to lie on the line half-way between the special points on the boundary. We denote by $\mathcal{M}_{n,m} \subset \overline{\mathcal{M}}_{n,m}$ the interior, by which we mean the (disconnected) union of top-dimensional strata.

Remark 4.5. The moduli spaces of weighted treed disks are related to unweighted moduli spaces by taking products with intervals: If Γ has at least one vertex $v \in \mathrm{Vert}(\Gamma)$ and Γ' denotes the combinatorial type of Γ obtained by setting the weights $\rho(e)$ to zero and the outgoing edge e_0 of Γ is unweighted $e_0 \in \operatorname{Edge}^{\blacktriangledown}_{\to}(\Gamma)$ then

$$\mathcal{M}_\Gamma \cong \mathcal{M}_{\Gamma'} \times (0,\infty)^{|\operatorname{Edge}^{\triangledown,(0,\infty)}(\Gamma)|}.$$

If the outgoing edge e_0 is weighted $e_0 \in \operatorname{Edge}^{\triangledown}_{\to}(\Gamma)$ and at least one leaf $e \in \operatorname{Edge}^{\triangledown}_{\to}(\Gamma)$ is weighted then

$$\mathcal{M}_\Gamma \cong \mathcal{M}_{\Gamma'} \times (0,\infty)^{|\operatorname{Edge}^{\triangledown,(0,\infty)}(\Gamma)|-2}$$

since only the ratios of the weightings of leaves must be preserved by the isomorphisms, see (4.8). In particular, if Γ is a type with a single weighted leaf $e \in \operatorname{Edge}^{\triangledown}_{\to}(\Gamma)$ and no vertices, $\operatorname{Vert}(\Gamma) = \emptyset$, the outgoing edge may be unforgettable or forgettable, the weightings on the leaf are irrelevant and $\mathcal{M}_\Gamma$ is a point.

In general moduli spaces of stable curves only admit universal curves in an orbifold sense. In the setting here orbifold singularities are absent and the moduli spaces of stable treed disks admit honest universal curves. For any stable combinatorial type Γ let $\overline{\mathcal{U}}_\Gamma$ denote *universal treed disk* consisting of isomorphism classes of pairs (C,z) where C is a treed disk of type Γ and z is a point in C, possibly on a disk component $S_v \cong \{|z| \le 1\}$, a sphere component $S_v \cong \mathbb{P}^1$, or one of the edges e of the tree part $T \subset C$. The map

$$\overline{\mathcal{U}}_\Gamma \to \overline{\mathcal{M}}_\Gamma, \quad [C,z] \to [C]$$

is the universal projection. Because of the stability condition, there is a natural bijection

$$\overline{\mathcal{U}}_\Gamma = \bigcup_{[C]\in\overline{\mathcal{M}}_\Gamma} C.$$

Denote by

$$\overline{\mathcal{S}}_\Gamma = \{[C = S \cup T, z] \in \overline{\mathcal{U}}_\Gamma \mid z \in S\}$$

the locus where z lies on a disk or sphere of C. Denote by

$$\overline{\mathcal{T}}_\Gamma = \{[C = S \cup T, z] \in \overline{\mathcal{U}}_\Gamma \mid z \in T\}$$

the locus where z lies on an edge of C. Hence

$$\overline{\mathcal{U}}_\Gamma = \overline{\mathcal{S}}_\Gamma \cup \overline{\mathcal{T}}_\Gamma$$

and $\overline{\mathcal{S}}_\Gamma \cap \overline{\mathcal{T}}_\Gamma$ is the set of points on the boundary of the disks meeting the edges of the tree. In case Γ has no vertices we define $\overline{\mathcal{U}}_\Gamma$ to be the real line, considered as a fiber bundle over the point $\overline{\mathcal{M}}_\Gamma$. The tree part splits into *interior* and *boundary* tree parts depending on whether the edge is attached to an interior point or a boundary point of a disk or sphere:

$$\overline{\mathcal{T}}_\Gamma = \overline{\mathcal{T}}_{\circ,\Gamma} \cup \overline{\mathcal{T}}_{\bullet,\Gamma}. \tag{4.9}$$

Later we will need local trivializations of the universal treed disk and the associated families of complex structures and metrics on the domains. For a stable combinatorial type Γ let

$$\tau^i : \mathcal{U}^i_\Gamma \to \mathcal{M}^i_\Gamma \times C, \quad i = 1, \ldots, l$$

be a collection of local trivializations of the universal treed disk. The trivialization τ^i identifies each fiber C' with the fixed treed disk C. The complex structures on the fibers of induce a family

$$\mathcal{M}^i_\Gamma \to \mathcal{J}(S), \quad m \mapsto j(m) \tag{4.10}$$

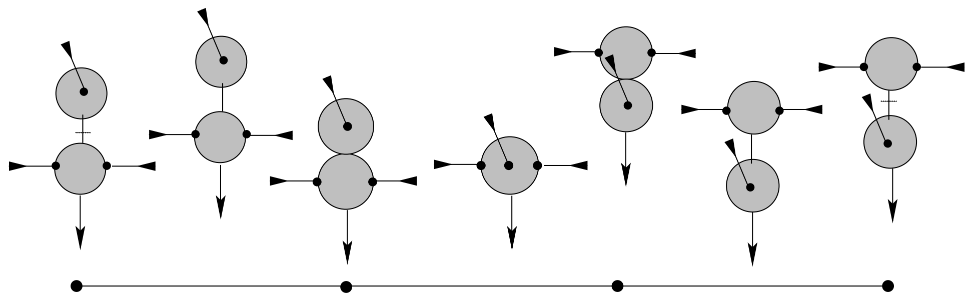

FIGURE 4.5. Treed disks with interior leaves

of complex structures on the two-dimensional locus $S \subset C$.

The following operations on treed disks will be referred to in the coherence conditions on perturbation data.

DEFINITION 4.6. (Morphisms of graphs) A *morphism* of graphs $\Upsilon : \Gamma \to \Gamma'$ is a surjective morphism of the set of vertices $\mathrm{Vert}(\Gamma) \to \mathrm{Vert}(\Gamma')$ obtained by combining the following *elementary morphisms*:

(a) (Cutting edges) Υ *cuts an edge with infinite length* with a single breaking if there exists

$$e \in \mathrm{Edge}_-(\Gamma'), \quad \ell(e) = \infty$$

so that the map $\mathrm{Vert}(\Gamma) \to \mathrm{Vert}(\Gamma')$ on vertices is a bijection, and

$$\mathrm{Edge}_\circ(\Gamma) \cong \mathrm{Edge}_\circ(\Gamma') - \{e\} + \{e_+, e_-\}$$

where $e_\pm \in \mathrm{Edge}_{\circ,\to}(\Gamma)$ are attached to the vertices contained in e. Since our graphs are trees, Γ is disconnected with pieces Γ_-, Γ_+ that are types of stable treed disks. The ordering on $\mathrm{Edge}_{\bullet,\to}(\Gamma')$ is required to agree with the ordering on $\mathrm{Edge}_{\bullet,\to}(\Gamma_\pm)$ by viewing the latter as a subset of the former.

The weighting and type of the cut edges are defined as follows. Suppose that Γ_- is the component of Γ not containing the root edge. If Γ_- has any interior leaves, set $\rho(e_\pm) = 0$ and $e_\pm \in \mathrm{Edge}^{\blacktriangledown}(\Gamma)$. Otherwise (and these are relatively rare exceptional cases in our examples and used only for the construction of strict units) if there are no interior leaves let $e_1, \ldots, e_k$ denote the leaves of Γ_-.

(i) If any of $e_1, \ldots, e_k$ are unforgettable then $e_\pm \in \mathrm{Edge}^{\blacktriangledown}(\Gamma)$ are also unforgettable.

(ii) If none of $e_1, \ldots, e_k$ are unforgettable and at least one of $e_1, \ldots, e_k$ is weighted then $e_\pm \in \mathrm{Edge}^{\triangledown}(\Gamma)$ are also weighted.

(iii) If $e_1, \ldots, e_k$ are forgettable then $e_\pm \in \mathrm{Edge}^{\triangledown}(\Gamma)$ are also forgettable.

See Figure 4.6 for an example. Define the weighting on the cut edges $\rho(e_\pm) = \min(\rho(e_1), \ldots, \rho(e_k))$. In particular if Γ_- has all zero weights $\rho(e_l) = 0, l = 1, \ldots, k$ then $\rho(e_\pm) = 0$.

(b) (Collapsing edges) Υ *collapses an edge* if the map on vertices is a bijection except for a single vertex $v' \in \mathrm{Vert}(\Gamma')$

$$\mathrm{Vert}(\Upsilon) : \mathrm{Vert}(\Gamma) \to \mathrm{Vert}(\Gamma'), \quad \mathrm{Vert}(\Upsilon)^{-1}(v') = \{v_-, v_+\}$$

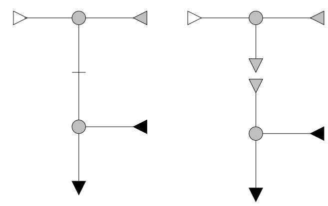

FIGURE 4.6. Cutting an edge

that has two pre-images $v_\pm \in \mathrm{Vert}(\Gamma)$. The vertices v_-, v_+ are connected by an edge $e \in \mathrm{Edge}(\Gamma)$ so that $\mathrm{Edge}(\Gamma') \cong \mathrm{Edge}(\Gamma) - \{e\}$. See Figure 4.7.

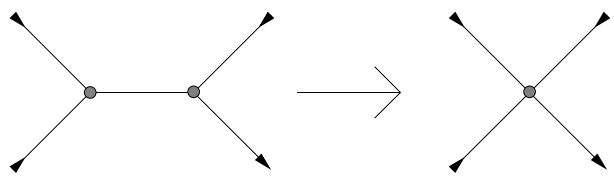

FIGURE 4.7. Collapsing an edge

(c) (Making an edge length finite or non-zero) Υ *makes an edge finite resp. non-zero* if Γ' is the same graph as Γ with the same orderings and the lengths of the edges of Γ' and Γ are the same except for a single edge e:

$$\ell|_{\mathrm{Edge}_-(\Gamma)-\{e\}} = \ell'|_{\mathrm{Edge}_-(\Gamma')-\{e\}}.$$

For the edge e we require

$$\ell(e) = \infty \text{ resp. } 0, \quad \ell(e') \in (0, \infty).$$

(d) (Forgetting tails) $\Upsilon : \Gamma \to \Gamma'$ *forgets a tail* (semi-infinite edge) and collapses edges to make the resulting combinatorial type stable. The ordering on $\mathrm{Edge}_{\bullet,\to}(\Gamma)$ is required to agree with the one on $\mathrm{Edge}_{\bullet,\to}(\Gamma')$ viewing the latter as a subset. See Figure 4.8.

(e) (Making an edge weight finite or non-zero) Υ *makes a weight finite or non-zero* if Γ' is the same graph as Γ and the weights of the edges $\rho(e), e \in \mathrm{Edge}^{\blacktriangledown}(\Gamma)$ are the same with orderings except for a single edge e,

$$\rho|_{\mathrm{Edge}_-(\Gamma)-\{e\}} = \rho'|_{\mathrm{Edge}_-(\Gamma')-\{e\}}.$$

For the edge e we have $\rho(e) = \infty$ resp. 0 and $\rho'(e) \in (0, \infty)$.

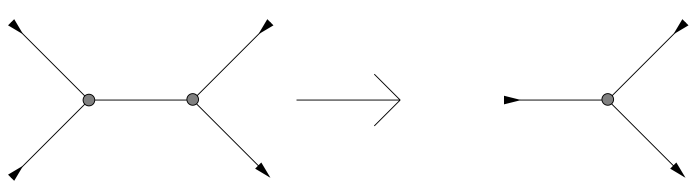

FIGURE 4.8. Forgetting a tail

The operations of cutting edges commute. For example, if Γ' is obtained from Γ by cutting two edges, say $e', e'' \in \mathrm{Edge}_-(\Gamma)$, then the induced weighting on Γ' is independent of the order of the cutting. This follows from the identity

$$\min(\rho(e_1), \ldots, \rho(e_j), \min(\rho(e_{j+1}), \ldots, \rho(e_{j+k})), \ldots, \rho(e_i)) = \min(\rho(e_1), \ldots, \rho(e_i)).$$

Each of the above operations on graphs corresponds to a map of moduli spaces of stable marked treed disks.

Definition 4.7. (Morphisms of moduli spaces)

(a) (Cutting edges) Suppose that Γ' is obtained from Γ by cutting an edge e. There are diffeomorphisms

$$\overline{\mathcal{M}}_\Gamma \to \overline{\mathcal{M}}_{\Gamma'}, \quad [C] \to [C']$$

obtained as follows. Given a treed disk C of type Γ', let z_+, z_- denote the endpoints at infinity of the edge corresponding to e. Form a treed disk C' by identifying $z_+ \sim z_-$ and choosing the labelling of the interior leaves to be that of Γ'.

(b) (Collapsing edges) Suppose that Γ is obtained from Γ' by collapsing an edge. There is an embedding $\iota_\Gamma^{\Gamma'} : \overline{\mathcal{M}}_\Gamma \to \overline{\mathcal{M}}_{\Gamma'}$. In the case of an edge of $\mathrm{Edge}_-^0(\Gamma)$, the image of $\iota_\Gamma^{\Gamma'}(\overline{\mathcal{M}}_\Gamma)$ is a 1-codimensional corner of $\overline{\mathcal{M}}_\Gamma$. In the case of an edge of $\mathrm{Edge}_{\bullet,\to}(\Gamma)$ the image $\iota_\Gamma^{\Gamma'}(\overline{\mathcal{M}}_\Gamma)$ is a 2-codimensional submanifold of $\overline{\mathcal{M}}_{\Gamma'}$.

(c) (Making an edge or weight finite resp. non-zero) If Γ is obtained from Γ' by making an edge finite resp. non-zero then $\overline{\mathcal{M}}_\Gamma$ also embeds in $\overline{\mathcal{M}}_{\Gamma'}$ as the 1-codimensional corner. The image is the set of configurations where the edge e reaches infinite resp. zero length $\ell(e)$ or weight $\rho(e)$.

(d) (Forgetting tails) Suppose that Γ' is obtained from Γ by forgetting a tail (either in $\mathrm{Edge}_{\bullet,\to}(\Gamma)$ or $\mathrm{Edge}_{\circ,\to}(\Gamma')$). Collapsing the unstable vertices $v \in \mathrm{Vert}(\Gamma')$ and combining the lengths $\ell(e)$ (if any) into the new metric $\ell(e') = \sum_{e \mapsto e'} \ell(e)$ defines a map $\overline{\mathcal{M}}_\Gamma \to \overline{\mathcal{M}}_{\Gamma'}$. Each weighted semi-infinite edge e for Γ defines a weighted semi-infinite edge e' for Γ' with the same weight $\rho(e) = \rho(e')$.

Each of the maps involved in the operations (Collapsing edges/Making edges or weights finite or non-zero), (Forgetting tails), (Cutting edges) extends to a smooth map of universal treed disks. In the case that the type is disconnected we have

$$\Gamma = \Gamma_1 \sqcup \Gamma_2 \quad \Longrightarrow \quad \overline{\mathcal{M}}_\Gamma \cong \overline{\mathcal{M}}_{\Gamma_1} \times \overline{\mathcal{M}}_{\Gamma_2}.$$

In this case the universal disk $\overline{\mathcal{U}}_\Gamma$ is the disjoint union of the pullbacks of the universal disks $\overline{\mathcal{U}}_{\Gamma_1}$ and $\overline{\mathcal{U}}_{\Gamma_2}$: If π_1, π_2 are the projections on the factors above then

$$\overline{\mathcal{U}}_\Gamma = \pi_1^* \overline{\mathcal{U}}_{\Gamma_1} \sqcup \pi_2^* \overline{\mathcal{U}}_{\Gamma_2}.$$

In the case of forgetting a tail, we denote by $f_\Gamma^{\Gamma'} : \overline{\mathcal{U}}_\Gamma \to \overline{\mathcal{U}}_{\Gamma'}$ the map of universal curves obtained by collapsing components that becomes unstable after forgetting the tail, and by $\overline{\mathcal{U}}_\Gamma^{\mathrm{st}} \subset \overline{\mathcal{U}}_\Gamma$ the union of components *not* collapsed by the forgetful morphism. The restriction of $f_\Gamma^{\Gamma'}$ to $\overline{\mathcal{U}}_{\Gamma'}^{\mathrm{st}}$ has finite fibers, and injective except at the points $w \in T$ where edges $T_{e'}, T_{e''}$ are glued together after a disk component $S_v, v \in \mathrm{Vert}(\Gamma)$ collapses.

Orientations on the main strata (i.e. of maximal dimension) of the moduli space of (non-weighted) treed disks may be constructed as follows.

Definition 4.8. (Orientations on moduli of treed disks)

(a) (A single disk) Consider a stratum $\mathcal{M}_\Gamma$ of treed disks having a single disk corresponding to a single vertex $\operatorname{Vert}(\Gamma) = \{v\}$. One can identify a smooth disk S with $n+1$ boundary attaching points $x_0, \ldots, x_n$ and $m \geq 1$ attaching points of interior leaves $z_1, \ldots, z_m$ with the positive half-space $\mathbb{H} \subset \mathbb{C}$ by a map $\phi : S - \{x_0\} \to \mathbb{H}, \quad z_1 \mapsto i$ so that the boundary attaching points x_i, $i \geq 1$ map to an ordered tuple in $\mathbb{R} \subset \mathbb{C}$. For $m = 0$ interior leaves, there are $n+1 \geq 3$ boundary leaves. We identify $\phi : S - \{x_0\} \to \mathbb{H}$ so that $x_1 \mapsto 0, \quad x_2 \mapsto 1$. The remaining boundary points x_i, $i \geq 3$ map to an ordered tuple of $]1, \infty[\subset \mathbb{R} \subset \mathbb{C}$. The moduli space $\mathcal{M}_\Gamma$ of disks of this type then inherits an orientation $O_\Gamma : \mathcal{M}_\Gamma \to \det(T\mathcal{M}_\Gamma)$ from the canonical orientation on $\mathbb{R}^{n-2} \times \mathbb{C}^m$.

(b) (Multiple disks) Consider a stratum $\mathcal{M}_\Gamma$ treed disks having more than a single disk. The closure $\overline{\mathcal{M}}_\Gamma$ contains strata $\overline{\mathcal{M}}_{\Gamma'}$ with fewer edges with finite non-zero lengths

$$\#\{e \in \operatorname{Edge}(\Gamma') \mid \ell(e) \in (0,\infty)\} < \#\{e \in \operatorname{Edge}(\Gamma) \mid \ell(e) \in (0,\infty)\}.$$

The inclusion identifies treed disks $C \to C'$ by identifying a boundary node $w \in C$ with an edge $e \subset C'$ of length $\ell(e)$ zero. The addition of an edge e of finite non-zero length $\ell(e) > 0$ corresponds to identifying the closures of two main strata on a 1-codimensional corner strata $\mathcal{M}_{\Gamma'}$. Choose an orientation O_Γ so that the induced orientations on the boundary strata $\mathcal{M}_{\Gamma'}$ corresponding to a zero length are opposite. The orientations O_Γ then glue together to an orientation $O_{n,m}$ on $\overline{\mathcal{M}}_{n,m}$.

4.3. Treed pseudoholomorphic disks

The composition maps in the Fukaya algebra will be obtained by counting treed pseudoholomorphic disks, which we now define.

Definition 4.9. (a) (Gradient flow lines) Let L be a compact connected smooth manifold. Denote by

$$\mathcal{G}(L) \subset \operatorname{Map}(TL^{\otimes 2}, \mathbb{R})$$

the space of smooth Riemannian metrics on L. Fix a metric $G \in \mathcal{G}(L)$ and a Morse function $F : L \to \mathbb{R}$ having a unique maximum $x_M \in L$. Let $I \subset \mathbb{R}$ be a connected subset containing at least two elements, that is, an open or closed interval. The *gradient vector field* of F is defined by

$$\operatorname{grad}_F : L \to TL, \quad G(\operatorname{grad}_F, \cdot) = \mathrm{d}F \in \Omega^1(L).$$

A *gradient flow line* for $-F$ is a map

$$u : I \to L, \quad \frac{d}{ds} u = -\operatorname{grad}_F(u)$$

where s is a unit velocity coordinate on I. Given a time $s \in \mathbb{R}$ let

$$\phi_s : L \to L, \quad \frac{d}{ds}\phi_s(x) = -\operatorname{grad}_F(\phi_s(x)), \quad \forall x \in L$$

denote the time s gradient flow of $-F$.

(b) (Stable and unstable manifolds) Denote by

$$\mathcal{I}^{\text{geom}}(L) := \text{crit}(F) \subset L$$

the space of critical points of F. Taking the limit of the gradient flow determines a discontinuous map

$$L \to \text{crit}(F), \quad y \mapsto \lim_{s \to \pm\infty} \phi_t(y).$$

By the stable manifold theorem each $x \in \mathcal{I}^{\text{geom}}(L)$ determines stable and unstable manifolds

$$W_x^{\pm} := \left\{ y \in L \;\middle|\; \lim_{s \to \pm\infty} \phi_s(y) = x \right\} \subset L$$

consisting of points whose downward resp. upwards gradient flow converges to x. Denote by

$$i : \mathcal{I}^{\text{geom}}(L) \to \mathbb{Z}_{\geq 0}, \quad x \mapsto \dim(W_x^-) \tag{4.11}$$

the *index map.* The pair (F, G) is *Morse-Smale* if the intersections

$$W_{x_-}^+ \cap W_{x_+}^- \subset L$$

are transverse for $x_+, x_- \in \mathcal{I}^{\text{geom}}(L)$, and so a smooth manifold of dimension $i(x_+) - i(x_-)$. The additive group $\mathbb{R}$ acts on the intersection $W_{x_-}^+ \cap W_{x_+}^-$ by the flow of $-\text{grad}(F)$, and the quotient

$$\mathcal{M}(x_+, x_-) = (W_{x_-}^+ \cap W_{x_+}^-)/\mathbb{R}$$

is canonically identified with the space of Morse trajectories from x_+ to x_-.

(c) (Almost complex structures) Let (X, ω) be a symplectic manifold. An almost complex structures on (X, ω) given by

$$J : TX \to TX, \quad J^2 = -I$$

is *tamed* if and only if $\omega(\cdot, J\cdot)$ is positive definite and *compatible* if in addition $\omega(\cdot, J\cdot)$ is symmetric, hence a Riemannian metric on X. Denote by $\mathcal{J}_\tau(X)$ the space of smooth tamed almost complex structures and $\mathcal{J}_\tau(X)_l$ the space of tamed almost complex structures of class C^l. The space $\mathcal{J}_\tau(X)_l$ has a natural Banach manifold structure modelled on the space of C^l sections δJ of the endomorphism bundle $\text{End}(TX) \to TX$ satisfying the linearized condition

$$\delta J : X \to \text{End}(TX), \quad J(\delta J) = -(\delta J)J.$$

In order to obtain the necessary transversality our Morse functions and almost complex structures must be allowed to depend on a point in the domain. Fix a compact *thick part* of the universal tree $\overline{\mathcal{T}}_\Gamma^{\text{thick}} \subset \overline{\mathcal{T}}_\Gamma$ with the following property: Its interior $\text{int}(\overline{\mathcal{T}}_\Gamma^{\text{thick}})$ contains at least one point on each edge:

$$\text{int}(\overline{\mathcal{T}}_\Gamma^{\text{thick}}) \cap \text{int}(e) \neq \emptyset, \quad \forall \text{ edges } e.$$

Also fix a compact subset

$$\overline{\mathcal{S}}_\Gamma^{\text{thick}} \subset \overline{\mathcal{S}}_\Gamma - \{w_e \in \overline{\mathcal{S}}_\Gamma, \ e \in \text{Edge}_-(\Gamma)\}$$

disjoint from the fiber-wise boundary $\partial\overline{\mathcal{S}}_\Gamma$ and spherical nodes, with the property: $\overline{\mathcal{S}}_\Gamma^{\text{thick}}$ contains in its interior $\text{int}(\overline{\mathcal{S}}_\Gamma)$ at least one point on each sphere and disk component $S_v \subset S, v \in \mathrm{Vert}(\Gamma)$ in each fiber $S \subset \overline{\mathcal{S}}_\Gamma$. Thus the complement $\mathcal{T}_\Gamma^{\mathrm{thin}} = \overline{\mathcal{T}}_\Gamma - \overline{\mathcal{T}}_\Gamma^{\mathrm{thick}} \subset \overline{\mathcal{T}}_\Gamma$ is a neighborhood of infinity on each edge. Furthermore, the complement $\mathcal{S}_\Gamma^{\mathrm{thin}} = \overline{\mathcal{S}}_\Gamma - \overline{\mathcal{S}}_\Gamma^{\mathrm{thick}} \subset \overline{\mathcal{S}}_\Gamma$ is a neighborhood of the boundary and nodes.

DEFINITION 4.10. (a) (Domain-dependent Morse functions) Let Γ be a type of stable treed disk. Let $\overline{\mathcal{T}}_\Gamma \subset \overline{\mathcal{U}}_\Gamma$ be the tree part of the universal treed disk, and $\overline{\mathcal{T}}_{\circ,\Gamma}$ its boundary part as in (4.9). Let (F, G) be a Morse-Smale pair. For an integer $l \geq 0$ a *domain-dependent perturbation* of F of class C^l is a C^l map

$$F_\Gamma : \overline{\mathcal{T}}_{\circ,\Gamma} \times L \to \mathbb{R} \tag{4.12}$$

equal to the given function F away from the compact part:

$$F_\Gamma | (\overline{\mathcal{T}}_{\circ,\Gamma} - \overline{\mathcal{T}}_{\circ,\Gamma}^{\mathrm{thick}}) = \pi_2^* F$$

where π_2 is the projection on the second factor in (4.12).

(b) (Domain-dependent almost complex structure) Let $J \in \mathcal{J}_\tau(X)$ be a tamed almost complex structure. Let $l \geq 0$ be an integer. A *domain-dependent almost complex structure* of class C^l for treed disks of type Γ and base J is a map

$$J_\Gamma : \ \overline{\mathcal{S}}_\Gamma \times X \to \mathrm{End}(TX).$$

We require that J_Γ is equal to the given J away from the compact part:

$$J_\Gamma | (\overline{\mathcal{S}}_\Gamma - \overline{\mathcal{S}}_\Gamma^{\mathrm{thick}}) = \pi_2^* J$$

where π_2 is the projection on the second factor in (4.12).

(c) (Perturbation data) A perturbation datum for a type Γ of holomorphic treed disk is a pair $P_\Gamma = (F_\Gamma, J_\Gamma)$ consisting of a domain-dependent Morse function F_Γ and a domain-dependent almost complex structure J_Γ.

The following are three operations on perturbation data related to the morphisms of weighted graphs in Definition 4.6.

DEFINITION 4.11. (a) (Collapsing edges/making an edge or weight finite or non-zero) Suppose that Γ' is obtained from Γ by collapsing an edge or making an edge length/weight finite/non-zero. Any perturbation datum $P_{\Gamma'}$ for Γ' induces a datum for Γ by pullback of $P_{\Gamma'}$ under $\iota_\Gamma^{\Gamma'} : \overline{\mathcal{U}}_\Gamma \to \overline{\mathcal{U}}_{\Gamma'}$.

(b) (Cutting edges) Suppose that Γ' is a combinatorial type obtained by cutting an edge of Γ. A perturbation datum for Γ' gives rise to a perturbation datum for Γ by pushing forward $P_{\Gamma'}$ under the map $\pi_\Gamma^{\Gamma'} : \overline{\mathcal{U}}_\Gamma \to \overline{\mathcal{U}}_{\Gamma'}$. That is, define

$$J_\Gamma(z, x) = J_{\Gamma'}(z', x), \quad \forall z \in (\pi_\Gamma^{\Gamma'})^{-1}(z').$$

The definition is independent of the choice of lift z by the (Constant near the nodes and markings) axiom. The definition for F_Γ is similar.

(c) (Forgetting tails) Suppose that Γ' is a combinatorial type of stable treed disk is obtained from Γ by forgetting a semi-infinite edge. Consider the map of universal disks $f_{\Gamma'}^\Gamma : \overline{\mathcal{U}}_\Gamma \to \overline{\mathcal{U}}_{\Gamma'}$ given by forgetting the edge and stabilizing. Any perturbation datum $P_{\Gamma'}$ induces a datum P_Γ by pullback of $P_{\Gamma'}$.

We are now ready to define coherent collections of perturbation data. These are data that behave well with each type of operation in Definition 4.11. Given a type Γ, denote by $\Gamma_\circ$ the combinatorial type obtained by collapsing all vertices $v \in \text{Vert}_\bullet(\Gamma)$ corresponding to spherical vertices to markings, and let $\Gamma(v) \subset \Gamma$ denote the subgraph with vertex v and adjacent edges.

DEFINITION 4.12. (Coherent families of perturbation data) A collection of perturbation data $\underline{P} = (P_\Gamma)$ is *coherent* if it is compatible with the morphisms of moduli spaces $\mathcal{M}_\Gamma \to \mathcal{M}_{\Gamma'}$ induced by morphisms of weighted treed disks $\Gamma \to \Gamma'$ in the sense that

(a) (Collapsing edges/making an edge or weight finite or non-zero) if Γ' is obtained from Γ by collapsing an edge or making an edge/weight finite/non-zero, then P_Γ is the pullback of $P_{\Gamma'}$;

(b) (Cutting edges) if Γ' is obtained from Γ by cutting an edge of infinite length, then $P_{\Gamma'}$ is the pullback of P_Γ. If Γ' is the union of types Γ_1, Γ_2 obtained by cutting an edge of Γ, then $P_{\Gamma'}$ is obtained from P_{Γ_1} and P_{Γ_2} as follows: Let

$$\pi_k : \overline{\mathcal{M}}_{\Gamma'} \cong \overline{\mathcal{M}}_{\Gamma_1} \times \overline{\mathcal{M}}_{\Gamma_2} \to \overline{\mathcal{M}}_{\Gamma_k}$$

denote the projection on the kth factor. The universal curve $\overline{\mathcal{U}}_{\Gamma'}$ is the union of $\pi_1^*\overline{\mathcal{U}}_{\Gamma_1}$ and $\pi_2^*\overline{\mathcal{U}}_{\Gamma_2}$. We require that $P_{\Gamma'}$ is equal to the pullback of P_{Γ_k} on $\pi_k^*\overline{\mathcal{U}}_{\Gamma_k}$ for each $k \in \{1, 2\}$:

$$P_{\Gamma'}|\overline{\mathcal{U}}_{\Gamma_k} = \pi_k^* P_{\Gamma_k}. \tag{4.13}$$

In particular suppose that Γ_1 corresponds to a configuration $u : C \to X$ with a single unmarked disk $S_v \subset C$ and two leaves $T_{e'}, T_{e''} \subset C$, one of which, say e' is weighted resp. forgettable as in the bottom row in Figure 4.11. Then by our conventions for (Cutting Edges) the corresponding leaf e of Γ_2 is weighted resp. forgettable, with the same weight $\rho(e) = \rho(e')$ of the leaf e' of Γ_1, and we require (4.13).

(c) (Locality axiom) For any spherical vertex v in a type Γ, the perturbation datum P_Γ restricts to the pull-back of a perturbation $P_{\Gamma,v}$ on the image of $\pi^*\mathcal{U}_{\Gamma(v)}$ in $\mathcal{U}_\Gamma$. (Note that the perturbation $P_{\Gamma,v}$ is allowed to depend on Γ, not just $\Gamma(v)$). Furthermore, the restriction of P_Γ to the disk components and boundary edges in $\mathcal{U}_\Gamma$ is the pull-back of a perturbation datam $P_{\Gamma_\circ}$ from $\mathcal{U}_{\Gamma_\circ}$.

(d) (Forgettable edges) If some weight parameter $\rho(e), e \in \text{Edge}_\to(\Gamma')$ of a type Γ' is equal to infinity and Γ is the type obtained by (Forgetting tails) then $P_{\Gamma'} = (f_\Gamma^{\Gamma'})^* P_\Gamma$ is pulled back under the forgetful map $f_\Gamma^{\Gamma'}$ forgetting the first forgettable leaf $e_i \in \text{Edge}_\to^\triangledown(\Gamma)$ and stabilizing from the perturbation datum P_Γ given by (Forgetting tails). The last sentence of the previous item guarantees that this condition is compatible with the product axiom, in the case that forgetting an edge e with infinite weight $\rho(e) = \infty$ leads to a collapse of a disk component.

REMARK 4.13. The (Locality Axiom) implies that on any disk component $S_v, v \in \text{Vert}_\circ(\Gamma)$ that the perturbation $J_\Gamma|S_v$ depends only on special points attached to disk components and the lengths $\ell(e)$ of the edges $e \in \text{Edge}(\Gamma)$.

Domain-dependent perturbations are obtained by pull-back from inclusions into the universal curve. Let C be a stable treed disk of type Γ. Since C is stable, C

admits a unique identification with a fiber of a universal treed disk $\mathcal{U}_\Gamma$. Given a perturbation datum P_Γ for type Γ, we obtain a domain-dependent almost complex structure and Morse function for C, still denoted J_Γ, F_Γ, by pull-back under the map $C \to \mathcal{U}_\Gamma$. If C is an arbitrary treed disk of type Γ with at least one vertex, let $C \to C^{\text{st}}$ denote the stabilization obtained by collapsing unstable sphere and disk component and collapsing trajectories of infinite length (e.g. in the case of combinatorially finite edges with more than one breaking or sphere or disk components with too few special points) to points. Let Γ^{st} denote the combinatorial type of C^{st}. By pull-back, one obtains domain-dependent perturbations for C from those for C^{st}. A perturbation datum for type Γ is a perturbation for type Γ^{st}. In the case that C has only one leaf, so that C is a string of disks $S_v \cong B^2, v \in \text{Vert}(\Gamma)$ and segments $T_e, e \in \text{Edge}(\Gamma(C))$, we take $J_\Gamma = J$ and $F_\Gamma = F$ to be domain-independent.

Definition 4.14. (Perturbed pseudoholomorphic treed disks) Given a perturbation datum P_Γ for weighted treed disks of type Γ, a pseudoholomorphic treed disk in X with boundary in L of type Γ consists of a treed disk $C = S \cup T$ of type Γ and a continuous map $u = (u_S, u_T) : C \to X$ such that the following holds: Let $T = T_\circ \cup T_\bullet$ be the splitting into boundary and interior parts as in (4.9).

(a) (Lagrangian boundary condition) On the boundary $u(\partial S \cup T_\circ) \subset L$.
(b) (Surface equation) On the surface part S of C the map u is J_Γ-holomorphic for the given domain-dependent almost complex structure: if j denotes the complex structure on S then

$$J_{\Gamma,u(z),z}\ \mathrm{d}u_S = \mathrm{d}u_S\ j.$$

(c) (Boundary tree equation) On the boundary tree part $T_\circ \subset C$ the map u is a collection of gradient trajectories:

$$\frac{d}{ds}u_T = -\operatorname{grad}_{F_{\Gamma,(s,u(s))}} u_T$$

where s is a coordinate on the segment so that the segment has the given length. Thus for each edge $e \in \text{Edge}\_(\Gamma)$ the length of the trajectory is given by the length $u|_{e\subset T}$ is equal to $\ell(e)$.
(d) (Interior tree condition) On the interior tree $T_\bullet \subset T$ the map u is constant.

The (Interior tree condition) means that the interior parts $T_\bullet$ of the tree T are essentially irrelevant from our point of view. However, from a conceptual viewpoint if one is going to replace boundary markings with edges then one should also replace interior markings with leaves; this conceptual point will become important in the proof of homotopy invariance in Chapter 5.

A compact Hausdorff moduli space of treed disks is obtained by requiring an energy bound and the following stability condition. The moduli space that we define will, however, not be smooth; we will achieve smoothness for a perturbation of these moduli spaces later using Cieliebak-Mohnke perturbations. By a *node* of a treed disk $C = S \cup T$ we mean an intersection point $S_{v'} \cap S_{v''}$ of two components $S_{v'}, S_{v''}$ of the surface part S or the intersection $e \cap S_v$ of an edge $e \subset T$ with a surface component $S_v \subset S$.

Definition 4.15. (Stable pseudoholomorphic treed disks) A pseudoholomorphic weighted treed disk $u : C \to X$ with interior nodes $z_1, \ldots, z_k$ and boundary nodes $w_1, \ldots, w_m$ is *stable* if

(a) each disk component on which u is constant has at least three boundary nodes or one boundary node and one interior node:

$$\mathrm{d}u(S_v) = 0 \quad S_v \text{ disk} \implies 2\#\{z_k, w_k \in \mathrm{int}(S_v)\} + \#\{w_k \in \partial S_v\} \geq 3;$$

(b) each sphere component $S_v \subset C$ component on which u is constant has at least three special points:

$$\mathrm{d}u(S_v) = 0, \quad S_v \text{ sphere} \quad \implies \quad \#\{z_k, w_k \in S_v\} \geq 3$$

and

(c) each infinite line on which u is constant has a weighted leaf and an unforgettable or forgettable root:

$$\mathrm{d}u(C) = 0, \quad C \text{ line} \quad \implies e_0 \in \mathrm{Edge}^{\blacktriangledown}(\Gamma) \text{ and } \; e_1 \in \mathrm{Edge}^{\blacktriangledown}(\Gamma) \cup \mathrm{Edge}^{\triangledown}(\Gamma).$$

Note that we allow a configuration $u : C \to X$ with no disks or spheres, so that $S = \emptyset$, and a single edge $T_e \cong \mathbb{R}$ equipped with a non-constant Morse trajectory $u : T_e \to L$.

The stability condition in Definition 4.15 is not quite the same as a map $u : C \to X$ having no non-trivial automorphisms $\phi : C \to C, \phi^* u = u$. Indeed given a configuration $u : C \cong \mathbb{R} \to X$ on which u is constant as in (c), we have $\mathrm{Aut}(u) \cong \mathbb{R}$, corresponding to translations of the line $C \cong T \cong \mathbb{R}$. However, the moduli space of such maps $u : C \to X$ is still of expected dimension 0 because the weighting $\rho(e) \in (0, \infty)$ raises the expected dimension of the moduli space $\mathcal{M}_\Gamma$.

There is a further notion of equivalence of pseudoholomorphic weighted treed disks related to attaching constant trajectories. Given a non-constant pseudoholomorphic treed disk $u : C \to X$ with leaf e_i for which the weighting $\rho(e_i) = \infty$ resp. 0, we declare u to be *equivalent* to the pseudoholomorphic treed disk $u' : C \to X$ obtained by attaching to e_i a constant trajectory $u'' : \mathbb{R} \to L$ with weighted incoming e_i^- and forgettable resp. unforgettable outgoing edge e_i^+. See Figure 4.9. Conversely, removing a constant segment $u'' : \mathbb{R} \to X$ and relabelling gives equiva-

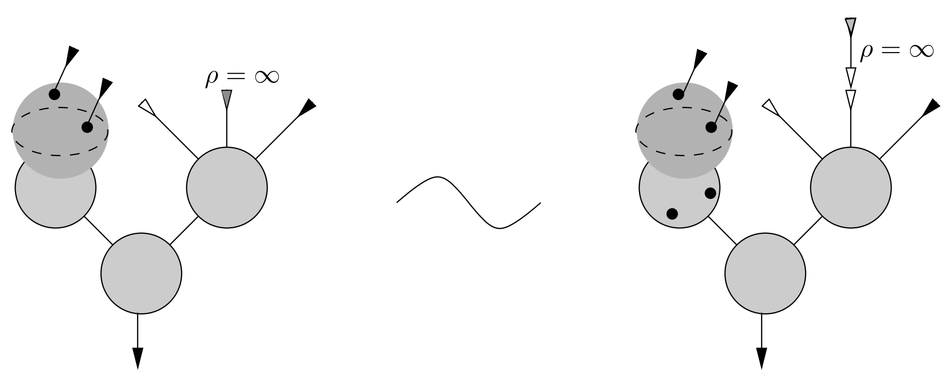

FIGURE 4.9. Equivalent weighted treed disks

lent holomorphic treed disks. Also, any two configurations $u : C \to X, u' : C' \to X$ with an outgoing weighted edge e_0 with the same underlying tree Γ are considered equivalent. See Figure 4.10.

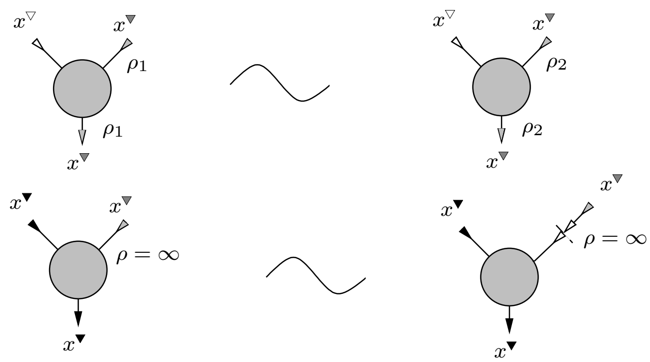

FIGURE 4.10. Equivalent weighted treed disks, ctd.

We introduce notation for various moduli spaces of equivalence classes of stable weighted treed disks. For non-negative integers n, m denote by $\overline{\mathcal{M}}_{n,m}(L)$ the (possibly empty) moduli space of equivalence classes of stable treed pseudoholomorphic disks with n boundary leaves and m interior leaves. A natural extension of the Gromov topology on $\overline{\mathcal{M}}_{n,m}(L)$ is Hausdorff as in [**95**, Section 5.6]. For any connected combinatorial type Γ of treed pseudoholomorphic disk, denote by $\mathcal{M}_\Gamma(L)$ the subset of type Γ so that

$$\overline{\mathcal{M}}_{n,m}(L) = \bigcup_{\Gamma} \mathcal{M}_\Gamma(L).$$

Denote by $\overline{\mathcal{M}}_\Gamma(L)$ the union of strata $\mathcal{M}_{\Gamma'}(L)$ such that there exists a morphism $\Gamma' \to \Gamma$. After the regularization below, $\overline{\mathcal{M}}_\Gamma(L)$ is the closure of $\mathcal{M}_\Gamma(L)$, at least for the strata we consider. The moduli space of treed disks decomposes further into components depending on the limits along the semi-infinite edges. Define

$$\mathcal{I}(L) = (\mathcal{I}^{\text{geom}}(L) - \{x_M\}) \cup \{x^{\triangledown}, x^{{\color{gray}\blacktriangledown}}, x^{\blacktriangledown}\}.$$

Thus $\mathcal{I}(L)$ is the set of critical points of F, with the maximum x_M replaced by three copies $x^{\triangledown}, x^{{\color{gray}\blacktriangledown}}, x^{\blacktriangledown}$. We extend the index map on $\mathcal{I}^{\text{geom}}(L)$ to $\mathcal{I}(L)$ by

$$i(x^{\triangledown}) = i(x^{\blacktriangledown}) = 0, \quad i(x^{{\color{gray}\blacktriangledown}}) = -1.$$

Define the set of generators of degree d

$$\mathcal{I}_d(L) = \{x \in \mathcal{I}(L) \mid i(x) = d\}. \tag{4.14}$$

An *admissible labelling* of a (non-broken) weighted treed disk C with leaves $e_1, \dots, e_n$ and outgoing edge e_0 is a sequence $\underline{x} = (x_0, \dots, x_n) \in \mathcal{I}(L)$ satisfying:

(a) (Label axiom) If $x_i = x^{{\color{gray}\blacktriangledown}}$ resp. $x^{\triangledown}$ resp. $x^{\blacktriangledown}$ then the corresponding semi-infinite edge e_i is required to be weighted resp. forgettable resp. unforgettable, that is,

$$\begin{aligned} &x_i = x^{{\color{gray}\blacktriangledown}} \text{ resp. } x^{\triangledown} \text{ resp. } x^{\blacktriangledown} \\ &\qquad \implies e_i \in \operatorname{Edge}^{{\color{gray}\blacktriangledown}}(\Gamma) \text{ resp. } e_i \in \operatorname{Edge}^{\triangledown}(\Gamma) \text{ resp. } e_i \in \operatorname{Edge}^{\blacktriangledown}(\Gamma). \end{aligned} \tag{4.15}$$

Furthermore, in this case the limit along the i-th semi-infinite edge is required to be x_M:

$$\lim_{s \to \infty} u(\varphi_e(s)) = x_M.$$

In every other case the semi-infinite edge is required to be unforgettable:

$$x_i \notin \{x^{\color{gray}\blacktriangledown}, x^{\triangledown}, x^{\blacktriangledown}\} \implies e_i \in \mathrm{Edge}^{\blacktriangledown}(\Gamma).$$

(b) (Outgoing edge axiom)
   (i) The outgoing edge e_0 is weighted, $e_0 \in \mathrm{Edge}^{\color{gray}\blacktriangledown}(\Gamma)$ only if there are two leaves $e_1, e_2 \in \mathrm{Edge}_{\circ,\to}(\Gamma)$, exactly one of which, say e_1 is weighted with the same weight $\rho(e_1) = \rho(e_0)$, and the other, say e_2 is forgettable with weight $\rho(e_2) = \infty$, and there is a single disk S with no interior leaves, that is, $T \cap \mathrm{int}(S) = \emptyset$.
   (ii) The outgoing edge e_0 can only be forgettable, that is, $e_0 \in \mathrm{Edge}^{\triangledown}(\Gamma)$ if either
      - there are two forgettable leaves $e_1, e_2 \in \mathrm{Edge}^{\triangledown}(\Gamma)$, or
      - there is a single leaf $e_1 \in \mathrm{Edge}^{\color{gray}\blacktriangledown}(\Gamma)$ that is weighted and the configuration $u : C \to X$ has no interior leaves, that is, $\mathrm{Edge}_{\bullet,\to}(\Gamma) = \emptyset$.

   See Figure 4.11.

The (Outgoing edge axiom) treats the case of constant treed disks. As is typical in Floer theory, constant configurations must be treated with great care. Denote by

$$\mathcal{M}_\Gamma(L, \underline{x}) \subset \mathcal{M}_\Gamma(L)$$

the moduli space of isomorphism classes of pseudoholomorphic treed disks of type Γ with boundary in L and admissible labelling $\underline{x} = (x_0, \ldots, x_n)$.

Remark 4.16. (Constant trajectories) If $x_1 = x^{\color{gray}\blacktriangledown}$ and $x_0 = x^{\blacktriangledown}$ resp. $x_0 = x^{\triangledown}$ then the moduli space $\mathcal{M}(L, x_0, x_1)$ contains a configuration with no disks and single edge on which u is constant, corresponding to a weighted leaf $e \in \mathrm{Edge}^{\color{gray}\blacktriangledown}(\Gamma)$ and a root edge $e_0 \in \mathrm{Edge}(\Gamma)$ that is unforgettable resp. forgettable. These trajectories are pictured in Figure 4.11.

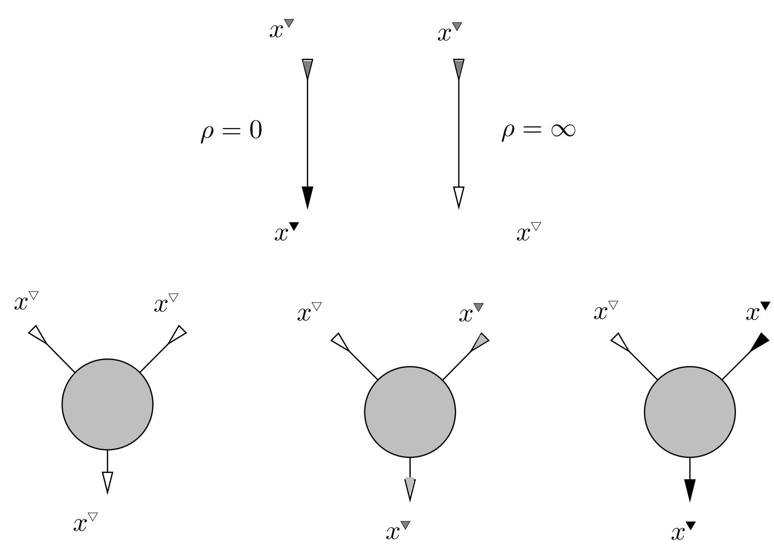

Figure 4.11. Unmarked treed disks

The moduli spaces of stable pseudoholomorphic treed disks of any fixed combinatorial type are cut out locally by Fredholm maps in suitable Sobolev completions. Let $p \geq 2$ and $k \geq 1$ be integers and let $C = S \cup T$ be a treed disk. Denote by $\mathrm{Map}^{k,p}(C, X, L)$ the space of continuous maps u from C to X of Sobolev class $W^{k,p}$ on each disk, sphere and edge component such that the boundaries $\partial C := \partial S \cup T$ of the disks and edges map to L. In each local chart for each component of C and X the map u is given by a collection of continuous functions with k partial derivatives of class L^p. Let $\exp : TX \to X$ denote geodesic exponentiation with respect to a metric on X is chosen for which L is totally geodesic, that is, TL maps to L. The space $\mathrm{Map}^{k,p}(C, X, L)$ has the structure of a smooth Banach manifold, with a local chart at $u \in \mathrm{Map}^{k,p}(C, X, L)$

$$W^{k,p}(C, u^*TX, (u|_{\partial C})^*TL) \to \mathrm{Map}^{k,p}(C, X, L), \quad \xi \mapsto \exp_u(\xi).$$

Denote by

$$\mathrm{Map}^{k,p}_\Gamma(C, X, L) \subset \mathrm{Map}^{k,p}(C, X, L) \tag{4.16}$$

the subset of maps $u : C \to X$ such that u has the prescribed homology class $[u|S_v]$ on each component $S_v \subset S, v \in \mathrm{Vert}(\Gamma)$. For each local trivialization of the universal tree disk $\mathcal{U}_\Gamma$ as in (4.10) we consider the ambient moduli space defined as follows. Let $\mathrm{Map}^{k,p}_\Gamma(C, X, L)$ denote the space of maps of Sobolev class $k \geq 1, p \geq 2, kp > 2$ mapping the boundary of C into L, defined using a Sobolev norm induced by a choice of connections on C and X, independent of such choice. The base of the universal bundle in the local trivialization is

$$\mathcal{B}^i_{k,p,\Gamma} := \mathcal{M}^i_\Gamma \times \mathrm{Map}^{k,p}_\Gamma(C, X, L).$$

Consider the map given by the local trivialization

$$\mathcal{M}^i_\Gamma \to \mathcal{J}(S), \ m \mapsto j(m).$$

Consider the smooth Banach vector bundle $\mathcal{E}^i = \mathcal{E}^i_{k,p,\Gamma}$ over $\mathcal{B}^i_{k,p,\Gamma}$ whose fibers are given by

$$(\mathcal{E}^i_{k,p,\Gamma})_{m,u,J} \subset \Omega^{0,1}(C, u^*TX)_{k-1,p} := \Omega^{0,1}_{j,J,\Gamma}(S, u_S^*TX)_{k-1,p} \oplus \Omega^1(T, u_T^*TL)_{k-1,p}$$

the space of $0,1$-forms with respect to $j(m), J$ that vanish to order $m_\pm(e)-1$ at the nodes corresponding to each edge e. Local trivializations of $\mathcal{E}^i_{k,p,\Gamma}$ are defined by geodesic exponentiation from u and parallel transport using the Hermitian connection defined by the almost complex structure

$$\Phi_\xi : \Omega^{0,1}(S, u_S^*TX)_{k-1,p} \to \Omega^{0,1}(S, \exp_{u_S}(\xi)^*TX)_{k-1,p}$$

see for example [**95**, p. 48]. The Cauchy-Riemann and shifted gradient operators may be applied to the restrictions u_S resp. u_T of u to the two resp. one dimensional parts of $C = S \cup T$. These define a section

$$\overline{\partial}_\Gamma : \mathcal{B}^i_{k,p,\Gamma} \to \mathcal{E}^i_{k,p,\Gamma}, \quad (m, u) \mapsto \left(\overline{\partial}_{j(m),J} u_S, \left(\frac{d}{ds} + \mathrm{grad}_F \right) u_T \right) \tag{4.17}$$

where

$$\overline{\partial}_{j(m),J} u := \frac{1}{2}(\mathrm{d}u_S + J \mathrm{d}u_S j(m)), \tag{4.18}$$

and s is a local coordinate on T with unit speed with respect to the given metric. Let $\mathrm{Aut}(C)$ be the group of automorphisms of C. The *local moduli space* is

$$\mathcal{M}_\Gamma^{\mathrm{univ},i}(L) = \overline{\partial}^{-1}\mathcal{B}^i_{k,p,\Gamma}/\,\mathrm{Aut}(C)$$

where $\mathcal{B}^i_{k,p,\Gamma}$ is embedded as the zero section. Define a non-linear map on Banach spaces using the local trivializations

(4.19)

$$\mathcal{F}_u : \mathcal{M}^i_\Gamma \times \Omega^0(C, u^*(TX,TL))_{k,p} \to \Omega^{0,1}(S, u_S^*TX)_{k-1,p} \oplus \Omega^1(, u_T^*TL)_{k-1,p},$$
$$(\xi, C) \mapsto \Phi_\xi^{-1}\overline{\partial}_\Gamma \exp_u(\xi).$$

denote the linearization of $\mathcal{F}_u$ (c.f. Floer-Hofer-Salamon [**42**, Section 5])

$$\tilde{D}_u = D_0\mathcal{F}_u = \frac{d}{dt}|_{t=0}\mathcal{F}_u(t\xi, tm). \tag{4.20}$$

Standard arguments show that the operator $\tilde{D}_u$ is Fredholm.

4.4. Transversality

In this section we regularize the moduli space of stable pseudoholomorphic treed disks with boundary in a Lagrangian submanifold using domain-dependent almost complex structures and metrics. The domains are stabilized using Donaldson hypersurfaces as in Charest-Woodward [**27**]; this construction extends that of Cieliebak-Mohnke [**28**]. For a submanifold $L \subset X$, let

$$h_2 : \pi_2(X,L) \to H_2(X,L), \quad \text{resp.} \quad [\omega]^\vee : H_2(X,L) \to \mathbb{R}$$

be the degree two relative Hurewicz morphism, resp. the map induced by pairing with $[\omega] \in H^2(X,L)$. A symplectic manifold X with two-form $\omega \in \Omega^2(X)$ will be called *rational* if the class $[\omega] \in H^2(X,\mathbb{R})$ is rational, that is, in the image of $H^2(X,\mathbb{Q}) \to H^2(X,\mathbb{R})$. Equivalently, (X,ω) is rational if there exists a *linearization* of X: a line bundle $\widetilde{X} \to X$ with a connection whose curvature is $(2\pi k/i)\omega$ for some integer $k > 0$. A Lagrangian $L \subset X$ of a rational symplectic manifold X with linearization $\widetilde{X}$ will be called *strongly rational* if $\widetilde{X}|L$ has a covariant constant section $L \to \widetilde{X}|L$. A Lagrangian submanifold $L \subset X$ is *rational* if and only if the set of areas of disks is discrete:

$$\exists e > 0, \quad [\omega]^\vee \circ h_2(\pi_2(X,L)) = \mathbb{Z}\cdot e \subset \mathbb{R}.$$

These rationality assumptions guarantee that we can find *stabilizing divisors* in the following sense. A *divisor* in X is a closed codimension two symplectic submanifold $D \subset X$. An almost complex structure $J : TX \to TX$ is *adapted* to a divisor D if D is an almost complex submanifold of (X,J). A divisor $D \subset X$ is *strongly stabilizing* for a Lagrangian submanifold $L \subset X$ if and only if D is disjoint from L and any disk $u : (C,\partial C) \to (X,L)$ with non-zero area $\omega([u]) > 0$ intersects D in at least one point. A divisor $D \subset X$ is *weakly stabilizing* for a Lagrangian submanifold $L \subset X$ if and only if D is disjoint from L and there exists an almost-complex structure $J_D \in \mathcal{J}(X,\omega)$ adapted to D such that any non-constant J_D-holomorphic disk $u : (C,\partial C) \to (X,L)$ intersects D in at least one point.

The existence of stabilizing divisors is an application of the theory of Donaldson-Auroux-Gayet-Mohsen stabilizing divisors [**35**], [**10**]. Roughly speaking one chooses an approximately holomorphic section σ of $\tilde{X}$ concentrated on the Lagrangian L; then a generic perturbation defines the desired divisor by $D = \sigma^{-1}(0)$.

THEOREM 4.17. [**27**, Section 4] *There exists a divisor $D \subset X - L$ that is weakly stabilizing for L representing $k[\omega]$ for some sufficiently large integer k. Moreover, if L is rational resp. strongly rational then there exists a divisor $D \subset X - L$ that is weakly stabilizing for L resp. strongly stabilizing for L representing $k[\omega]$ for some large k and such that L is exact in $(X - D, \omega|_{X-D})$.*

In the case that X is a smooth projective algebraic variety, stabilizing divisors may be obtained using a result of Borthwick-Paul-Uribe [**17**]. There is also a time-dependent version of this result which will be used later to prove independence of the homotopy type of the Fukaya algebra from the choice of base almost complex structure: If $J^t, t \in [0,1]$ is a smooth path of compatible almost complex structures on X then (see [**27**, Lemma 4.20]) there exists a path of J^t-stabilizing divisors $D^t, t \in [0,1]$ connecting D^0, D^1.

DEFINITION 4.18. (Adapted treed disks) Let L be a compact Lagrangian and D a codimension two submanifold disjoint from L. A stable pseudoholomorphic treed disk $u : C \to X$ with boundary in L is *adapted* to D if and only if

(a) (Stable surface axiom) The domain $C = S \cup T$ has stable or empty surface part S; and
(b) (Leaf axiom) each interior semi-infinite leaf $e \subset C$ maps to D and each connected component of $u^{-1}(D)$ contains an interior semi-infinite leaf e.

The (Stable surface axiom) implies that the domain is stable except that arbitrary breakings of each edge are allowed; in particular, the case that S is empty corresponds to a broken edge with an arbitrary number of breakings.

The moduli space of adapted treed pseudoholomorphic disks is stratified by combinatorial type as follows. Denote by $\Pi(X)$ resp. $\Pi(X, L)$ the set of homotopy classes of maps from the two sphere resp. disk with boundary in L. The multiplicities of the intersection with the divisor D at $z_e \in \operatorname{int}(S) \cap T$ are the winding numbers $m_\pm(e)$ of small loop around z_e in the complement of the zero section in a tubular neighborhood of D in X. The combinatorial type of a pseudoholomorphic treed disk $u : C \to X$ adapted to D consists of

(a) the combinatorial type $\Gamma = (\operatorname{Vert}(\Gamma), \operatorname{Edge}(\Gamma))$ (orderings etc. omitted from the notation) of its domain C together with
(b) the labelling
$$d : \operatorname{Vert}(\Gamma) \to \Pi(X) \cup \Pi(X, L)$$
of each vertex v of Γ corresponding to a disk or sphere component with the corresponding homotopy class and
(c) the labelling
$$m = (m_+, m_-) : \operatorname{Edge}_\bullet(\Gamma) \to \mathbb{Z}^2_{\geq 0} \tag{4.21}$$
recording the intersection multiplicity of the map u to the divisor D at each of the nodes $z_e^\pm$ at the end of each edge $e \in \operatorname{Edge}_{\bullet,\to}(\Gamma)$; we assign $m_\pm(e) = 0$ if the point $z_e^\pm$ does not map to D or the entire component maps to D.

We introduce the following notation. Let $\overline{\mathcal{M}}(L,D)$ the moduli space of adapted treed marked disks in X with boundary in L and $\mathcal{M}_\Gamma(L,D)$ the locus of combinatorial type Γ. For $\underline{x} \in \mathcal{I}(L)^n$ let

$$\mathcal{M}_\Gamma(L,D,\underline{x}) \subset \mathcal{M}_\Gamma(L,D)$$

denote the adapted subset made of pseudoholomorphic treed disks of type Γ adapted to D with limits $\underline{x} = (x_0,\ldots,x_n) \in \mathcal{I}(L)$ along the root and leaves.

The expected dimension of the moduli space is given by a formula involving the Maslov indices of the disks and the indices and number of semi-infinite edges. Denote the components $u_v = u|S_v$ of u, denote by the Maslov index $I(u_v)$ resp. twice the Chern number, if S_v is a sphere. The expected dimension of $\mathcal{M}_\Gamma(L,D,\underline{x})$ at $[u : C \to X]$ is given by

(4.22)

$$i(\Gamma,\underline{x}) := i(x_0) - \sum_{i=1}^n i(x_i) + \sum_{v\in \operatorname{Vert}(\Gamma)} I(u_v) + n - 2 - |\operatorname{Edge}_-^0(\Gamma)| - |\operatorname{Edge}_-^\infty(\Gamma)|$$
$$- 2|\operatorname{Edge}_{\bullet,-}(\Gamma)| - \sum_{e\in \operatorname{Edge}_{\bullet,\to}(\Gamma)} 2m(e) - \sum_{e\in \operatorname{Edge}_{\bullet,-}(\Gamma)} 2m(e)$$

where $m(e) = m_+(e) + m_-(e)$ is the intersection multiplicity of the map with the stabilizing divisor at each end of the edge, see (4.21). Denote by

$$\mathcal{M}_\Gamma(L,D,\underline{x})_d \subset \mathcal{M}_\Gamma(L,D,\underline{x})$$

the locus of expected dimension d. We denote by

$$\mathcal{M}(L,D,\underline{x}) = \bigcup_\Gamma \mathcal{M}_\Gamma(L,D,\underline{x})$$

the union over strata of top dimension, that is, for which the edge lengths in Γ connecting disk and sphere components are finite and non-zero. Finally, denote by

$$\mathcal{M}(L,D,\underline{x})_d \subset \mathcal{M}(L,D,\underline{x})$$

the locus of expected dimension d. We call elements of $\mathcal{M}(L,\underline{x})_0$ *rigid*. The elements of $\mathcal{M}_\Gamma(L,D,\underline{x})_0$ for types Γ corresponding to strata that are not of maximal dimension in $\overline{\mathcal{M}}(L,D,\underline{x})$ are stratum-wise rigid, but at (formally) may be deformed in the direction normal to the stratum.

The energy and the number of interior leaves are related as follows. Let Γ be a type of stable treed disk. Disconnecting the components that are connected by boundary nodes with positive length one obtains types $\Gamma_1,\ldots,\Gamma_l$, and a decomposition of the universal curve $\mathcal{U}_\Gamma$ into components $\overline{\mathcal{U}}_{\Gamma_1},\ldots,\overline{\mathcal{U}}_{\Gamma_l}$. Let $n(\Gamma_i)$ denote the number of interior leaves on $\overline{\mathcal{U}}_{\Gamma_i}$. We assume that D has Poincaré dual given by $k[\omega]$. Exactness of L in the complement of D implies the following:

Proposition 4.19. *Any stable treed disk $u : C \to X$ with domain of type Γ and only transverse intersections with the divisor has energy equal to*

$$E(u|C_i) \le n(\Gamma_i,k) := \frac{n(\Gamma_i)}{k}$$

on the component $C_i \subset C$ contained in $\overline{\mathcal{U}}_{\Gamma_i}$.

In order to obtain transversality we begin by fixing an open subset of the universal curve on which the perturbations will vanish. Let $\overline{\mathcal{U}}_\Gamma^{\text{thick}} \subset \overline{\mathcal{U}}_\Gamma$ be a compact subset disjoint from the nodes and attaching points of the edges such that the interior of $\overline{\mathcal{U}}_\Gamma^{\text{thick}}$ in each two and one-dimensional component is open. Suppose that perturbation data $P_{\Gamma'}$ for all boundary types $\mathcal{U}_{\Gamma'} \subset \overline{\mathcal{U}}_\Gamma$ have been chosen. Let

$$\mathcal{P}_\Gamma^l(X,D) = \{P_\Gamma = (F_\Gamma, J_\Gamma)\}$$

denote the space of perturbation data $P_\Gamma = (F_\Gamma, J_\Gamma)$ of class C^l that are

- equal to the given pair (F,J) on $\overline{\mathcal{U}}_\Gamma - \overline{\mathcal{U}}_\Gamma^{\text{thick}}$, and such that
- the restriction of P_Γ to $\overline{\mathcal{U}}_{\Gamma'}$ is equal to $P_{\Gamma'}$, for each boundary type Γ', that is, type of lower-dimensional stratum $\overline{\mathcal{M}}_{\Gamma'} \subset \overline{\mathcal{M}}_\Gamma$.

The second condition will guarantee that the resulting collection satisfies the (Collapsing edges/Making edges or weights finite or non-zero) axiom of the coherence condition Definition 4.12. Let $\mathcal{P}_\Gamma(X,D)$ denote the intersection of the spaces $\mathcal{P}_\Gamma^l(X,D)$ for $l \geq 0$.

One cannot expect, using stabilizing divisors, to obtain transversality for all combinatorial types. The reason is an analog of the multiple cover problem: once one has a ghost bubble $S_v \subset S$ mapping to the divisor D then one has configurations with arbitrary number of interior leaves $T_e \subset C$ meeting the component S_v. The expected dimension $\dim \mathcal{M}_\Gamma(L,D)$ of the component $\mathcal{M}_\Gamma(L,D)$ of the moduli space containing these configurations has limit minus infinity but each $\mathcal{M}_\Gamma(L,D)$ is non-empty. A type Γ will be called *uncrowded* if each maximal ghost component contains at most one endpoint of an interior leaf. Write $\Gamma' \prec \Gamma$ if and only if Γ is obtained from Γ' by (Collapsing edges/making edge lengths or weights finite/non-zero) or Γ is obtained from Γ' by (Forgetting a forgettable tail).

Theorem 4.20. (Transversality) *Suppose that* Γ *is an uncrowded type of adapted pseudoholomorphic treed disk of expected dimension* $i(\Gamma, \underline{x}) \leq 1$*, see* (4.22)*. Suppose admissible perturbation data for types of adapted treed marked disk* $\Gamma' \prec \Gamma$ *are given. Then there exists a comeager subset*

$$\mathcal{P}_\Gamma^{\text{reg}}(X,D) \subset \mathcal{P}_\Gamma(X,D)$$

of regular perturbation data *for type* Γ *coherent with the previously chosen perturbation data such that if* $P_\Gamma \in \mathcal{P}_\Gamma^{\text{reg}}(X,D)$ *then*

(a) (Smoothness of each stratum) *the stratum* $\mathcal{M}_\Gamma(L,D)$ *is a smooth manifold of expected dimension;*

(b) (Tubular neighborhoods) *if* Γ *is obtained from* Γ' *by collapsing an edge of* $\text{Edge}_{\circ,-}(\Gamma')$ *or making an edge or weight finite/non-zero or by gluing* Γ' *at a breaking then the stratum* $\mathcal{M}_{\Gamma'}(L,D)$ *has a tubular neighborhood in* $\overline{\mathcal{M}}_\Gamma(L,D)$*; and*

(c) (Orientations) *there exist orientations on* $\mathcal{M}_\Gamma(L,D)$ *compatible with the morphisms* (Cutting an edge) *and* (Collapsing an edge/Making an edge/weight finite/non-zero) *in the following sense:*

   (i) *If* Γ *is obtained from* Γ' *by* (Cutting an edge) *then the isomorphism* $\mathcal{M}_{\Gamma'}(L,D) \to \mathcal{M}_\Gamma(L,D)$ *is orientation preserving.*

   (ii) *If* Γ *is obtained from* Γ' *by* (Collapsing an edge) *or* (Making an edge/weight finite/non-zero) *then the inclusion* $\mathcal{M}_{\Gamma'}(L,D) \to \mathcal{M}_\Gamma(L,D)$

has orientation (using the decomposition

$$T\mathcal{M}_\Gamma(L,D)|\mathcal{M}_{\Gamma'}(L,D) \cong \mathbb{R} \oplus T\mathcal{M}_{\Gamma'}(L,D)$$

and the outward normal orientation on the first factor) given by a universal sign depending only on Γ, Γ'.

Proof. We discuss only (a); the item (b) is a combination of standard gluing theorems, c.f. Schmäschke [**113**, Section 7], while orientations (c) are discussed further in Remark 4.24 below. (a) is an application of the Sard-Smale theorem to a universal moduli space. In the first part of the proof we assume that Γ has no forgettable leaves and construct a perturbation datum P_Γ by extending the given perturbation data on the boundary of the universal moduli space. Let $p \geq 2$ and $k \geq 1$ be integers with $kp > 2$. As in the discussion before (4.16) denote by $\mathrm{Map}^{k,p}(C,X,L)$ the space of continuous maps u from C to X of Sobolev class $W^{k,p}$ on each disk, sphere and edge component such that the boundaries $\partial C := \partial S \cup T$ of the disks and edges mapping to L. That is, the restriction of u to each component is in class $W^{k,p}_{\mathrm{loc}}$ (which is independent of the choice of Sobolev norm) and has finite $W^{k,p}$ norm with respect to some choice of connections on the tangent bundles of the domain and target (which depends strongly on the choice of norm; however in the end the moduli spaces will be independent of such choices.) In each local chart for each component of C and X the map u is given by a collection of continuous functions with k partial derivatives of class L^p. The space $\mathrm{Map}^{k,p}(C,X,L)$ has the structure of a Banach manifold, with a local chart at $u \in \mathrm{Map}^{k,p}(C,X,L)$ given by the geodesic exponential map

$$W^{k,p}(C, u^*TX, (u|_{\partial C})^*TL) \to \mathrm{Map}^{k,p}(C,X,L), \quad \xi \mapsto \exp_u(\xi)$$

where we assume that the metric on X is chosen so that L is totally geodesic, that is, preserved by geodesic flow. For any combinatorial type Γ with vertices labelled by homology classes denote by $\mathrm{Map}^{k,p}_\Gamma(C,X,L) \subset \mathrm{Map}^{k,p}(C,X,L)$ the subset of maps such that u has the prescribed homology class on each component.

For each local trivialization of the universal tree disk as in (4.10) we define a Banach manifold that combines variations in the domain with variations in the map, depending on a choice of Sobolev norm. Let $\mathrm{Map}^{k,p}_\Gamma(C,X,L,D)$ denote the subspace of maps mapping the boundary of C into L, the interior markings into D, and constant on each disk with no interior marking, and with the prescribed order of vanishing at the intersection points with D. Let $l \gg k$ be an integer and

$$\mathcal{B}^i_{k,p,l,\Gamma} := \mathcal{M}^i_\Gamma \times \mathrm{Map}^{k,p}_\Gamma(C,X,L,D) \times \mathcal{P}^l_\Gamma(X,D).$$

Consider the map given by the local trivialization

$$\mathcal{M}^{\mathrm{univ},i}_\Gamma \to \mathcal{J}(S), \ m \mapsto j(m).$$

Consider the Banach bundle $\mathcal{E}^i_{k,p,l,\Gamma}$ over $\mathcal{B}^i_{k,p,l,\Gamma}$ given by

$$(\mathcal{E}^i_{k,p,l,\Gamma})_{m,u,J} \subset \Omega^{0,1}_{j,J,\Gamma}(S,(u_S)^*TX)_{k-1,p} \oplus \Omega^1(T,(u_T)^*TL)_{k-1,p}$$

the space of $0,1$-forms with respect to $j(m), J$ that vanish to order $m_\pm(e)-1$ at the nodes corresponding to the edge e, if $m_\pm(e)$ is positive. The space $\mathcal{E}^i_{k,p,l,\Gamma}$ is then a C^q-Banach vector bundle for $q < l-k$. The Cauchy-Riemann and shifted gradient operators may be applied to the restrictions u_S resp. u_T of u to the two resp. one dimensional parts of $C = S \cup T$. These define a C^q section

$$\overline{\partial}_\Gamma : \mathcal{B}^i_{k,p,l,\Gamma} \to \mathcal{E}^i_{k,p,l,\Gamma}, \quad (m, u, J, F) \mapsto \left(\overline{\partial}_{j(m),J} u_S, (\frac{d}{ds} + \operatorname{grad}_F) u_T \right) \tag{4.23}$$

where

$$\overline{\partial}_{j(m),J} u := \frac{1}{2}(\mathrm{d}u_S + J \mathrm{d}u_S j(m)), \tag{4.24}$$

and s is a local coordinate with unit speed. The *local universal moduli space* is

$$\mathcal{M}^{\mathrm{univ},i}_\Gamma(L, D) = \overline{\partial}^{-1} \mathcal{B}^i_{k,p,l,\Gamma} \tag{4.25}$$

where $\mathcal{B}^i_{k,p,l,\Gamma}$ is embedded as the zero section.

To show the local universal moduli space is cut out transversally, we show that an element of the cokernel of the linearized operator vanishes everywhere. Suppose that

$$\eta = (\eta_2, \eta_1) \in \Omega^{0,1}_{j,J,\Gamma}(S, (u_S)^* TX)_{k-1,p} \oplus \Omega^1(T, (u_T)^* TL)_{k-1,p}$$

is in the cokernel of derivative $\tilde{D}_u$ of (4.23), introduced in (4.20) but now with an additional term arising from the variation of almost complex structure and Morse function, with 2-dimensional part η_2 and one-dimensional part η_1. Variation of (4.23) with respect to the section ξ_1 on the one-dimensional part T gives

$$0 = \int_T (D_{u_1} \xi_1, \eta_1) \mathrm{d}s = \int_T (\xi_1, D^*_{u_1} \eta_1) \mathrm{d}s, \quad \forall \xi \in \Omega^0_c(u_1^* TL)$$

where D_{u_1} is the operator of (4.20). Hence

$$\nabla_\xi * \eta_1 = 0. \tag{4.26}$$

On the other hand, the linearization of (4.23) with respect to the Morse function is pointwise surjective:

$$\{ - \operatorname{grad}_{F_\Gamma} u_T(s) \mid F_\Gamma \in C^\ell_c(\overline{\mathcal{T}}_{\circ,\Gamma} \times L) \} = T_{u_T(s)} L.$$

This surjectivity implies that

$$\eta_1(u_T(s)) = 0, \quad \forall s \in \overline{\mathcal{U}}_\Gamma - \overline{\mathcal{U}}^{\mathrm{thin}}_\Gamma. \tag{4.27}$$

Combining (4.26) and (4.27) implies that $\eta_1 = 0$. Similarly the two-dimensional part η_2 satisfies

$$(D_u \xi_2, \eta_2) = 0, \quad \int_S (Y \circ \mathrm{d}u \circ j) \wedge \eta_2 = 0 \tag{4.28}$$

for every $\xi_2 \in \Omega^0(S, u^* TX)$ with given orders of vanishing at the intersection points with D and variation of almost complex structure $Y \in \Omega^0(S, u^* \operatorname{End}(TX))$ as in [**95**, Chapter 3]. Hence in particular $D_u^* \eta_2 = 0$ away from the intersection points with D. So if $\mathrm{d}u(z) \neq 0$ for some $z \in S$ not equal to an intersection point then η_2 vanishes in an open neighborhood of z. By unique continuation η_2 vanishes everywhere on the component of S containing z except possibly at the intersection points with D. At these points η_2 could in theory be a sum of derivatives of delta functions. That η_2 also vanishes at these intersection points follows from [**28**, Lemma 6.5, Proposition 6.10].

It remains to consider components of the two-dimensional part on which the map is constant. On such components variation of the almost complex structure

J_Γ produces no variation of the one-form $\overline{\partial}_\Gamma u$ obtained by applying the Cauchy-Riemann operator. Let $S' \subset S$ be a union of disk and sphere components of $S \subset C$. If $u : C \to X$ is a map that is constant on $S' \subset C$ the linearized operator is also constant is surjective by standard arguments, c.f. Oh [**105**]. However, we also must check that the matching conditions at the nodes are cut out transversally. Let S'' denote the normalization of S', obtained by replacing each nodal point in S' with a pair of points in S''. Since the combinatorial type of the component S' is a connected subgraph Γ' of a tree, the combinatorial type Γ' must itself be a tree. Denote by $T_u L$ the tangent space at the constant value of u on S'. Taking the differences of the maps at the nodes defines a map

$$\delta : \ker(D_u | S'') \cong T_u L^k \to T_u^m, \quad \xi \mapsto (\xi(w_i^+) - \xi(w_i^-))_{i=1}^m. \tag{4.29}$$

An explicit inverse to δ is given by defining recursively as follows. Consider the orientation on the combinatorial type $\Gamma' \subset \Gamma$ induced by the choice of outgoing semi-infinite edge of Γ. For $\eta \in T_u^m$ define an element $\xi \in T_u L^k$ by

$$\xi(t(e)) = \xi(h(e)) + \eta(e)$$

whenever $t(e), h(e)$ are the head and tail of an edge e corresponding to a node. This element may be defined recursively starting with the edge e_0'' of Γ' closest to the outgoing edge of Γ and taking $\xi(v) = 0$ for v the vertex corresponding to the disk component closest to outgoing edge. The matching conditions at the nodes connecting S' with the complement $S - S'$ are also cut out transversally. Indeed on the adjacent components the linearized operator restricted to sections vanishing at the node is already surjective as in [**28**, Lemma 6.5]. This implies that the conditions at the boundary nodes cut out the moduli space transversally. A similar discussion for collections of sphere components on which the map is constant implies that the matching conditions at the spherical nodes are also cut out transversally. This completes the proof that the parametrized linearized operator is surjective. The surjectivity of the parametrized linearized and the implicit function theorem implies that the local universal moduli space is a Banach manifold. More precisely, $\mathcal{M}_\Gamma^{\mathrm{univ},i}(L,D)$ is a Banach manifold of class C^q, and the forgetful morphism $\varphi_i : \mathcal{M}_\Gamma^{\mathrm{univ},i}(L,D) \to \mathcal{P}_\Gamma(L,D)_l$ is a C^q Fredholm map. Let $\mathcal{M}_\Gamma^{\mathrm{univ},i}(L,D)_d \subset \mathcal{M}_\Gamma^{\mathrm{univ},i}(L,D)$ denote the component on which φ_i has Fredholm index d. □

In order to obtain compactness later it will also be necessary to obtain a certain kind of transversality for *crowded* combinatorial types.

Definition 4.21. (Regularity for crowded types) For each such crowded combinatorial type Γ let Γ' denote the combinatorial type obtained by forgetting all but one marking on the ghost components. The same proof shows that there exists a comeager subset $\mathcal{P}_\Gamma^{\mathrm{reg}}(X,D)$ such that the moduli space $\mathcal{M}_{\Gamma'}(L,D)$ is a smooth manifold whose dimension is the sum of the expected dimension, necessarily empty if the expected dimension of the crowded type was at most one. We call a perturbation P_Γ *regular* if the moduli space $\mathcal{M}_{\Gamma'}(L,D)$ is cut out transversally.

Definition 4.22. (Regular perturbations for types of disks) For any combinatorial type Γ of adapted treed disk there exist (up the multiple breakings of edges) finitely many combinatorial types Γ_X of *pseudoholomorphic* treed disks whose underlying type of treed disk is Γ. Indeed, if a component $u|S_v : S_v \to X$ is non-constant, then its energy $E(u|S_v)$ is controlled by the number of interior leaves as in Proposition 4.19. On the other hand, if $u|S_v$ is constant then its energy is zero. Thus the energy of the map $u(C)$ is controlled by the number $\#\,\mathrm{Edge}_{\bullet,\to}$ of interior leaves. This implies that there are a finite number of combinatorial types Γ_X that correspond to any particular Γ. Given a type Γ of domain, define

$$\mathcal{P}^{\mathrm{reg}}_{\Gamma}(X,D) = \cap_{\Gamma_X} \mathcal{P}^{\mathrm{reg}}_{\Gamma_X}(X,D)$$

where Γ_X ranges over types of pseudoholomorphic treed disks with type of domain Γ. We call perturbation data $P_\Gamma \in \mathcal{P}^{\mathrm{reg}}_{\Gamma}(X,D)$ *regular.* Since countable intersections of comeager sets are comeager, Theorem 4.20 implies the regular perturbation data form a comeager subset. This ends the definition.

REMARK 4.23. The strata $\mathcal{M}_{\Gamma'}(L,D,\underline{x})$ of expected dimension one, with $\mathcal{M}_{\Gamma'}$ of maximal dimension, have boundary strata $\mathcal{M}_{\Gamma}(L,D,\underline{x})$ that are of two possible types.

(a) The first possibility is that Γ has a broken edge, as in Figure 4.12. Note

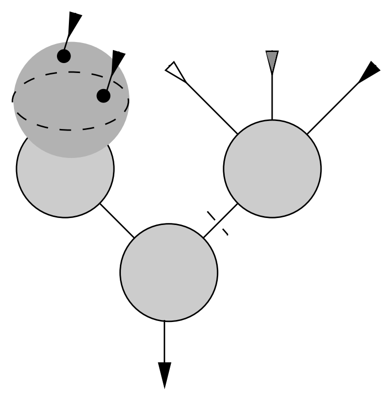

FIGURE 4.12. Treed disk with a broken edge

that the broken edge e may be a leaf and that leaf may have label $x^{\blacktriangledown}$ and weight 0 or ∞ as in Remark 4.16. In this case any configuration u of type Γ is constant on the first segment in the leaf and the normal bundle is again isomorphic to $\mathbb{R}_{\geq 0}$. The normal direction corresponds to deformations that replace that segment with an unbroken segment labelled $x^{\blacktriangledown}$ and deform the weight $\rho(e)$ away from 0 resp. ∞.

(b) The second possibility is that Γ corresponds to a stratum with a boundary node: either T has an edge of length zero or equivalently S has a disk with a boundary node. See Figure 4.13.

We call $\mathcal{M}_\Gamma(L,D)$ in the first resp. second case a *true* resp. *fake boundary stratum.* A fake boundary stratum $\mathcal{M}_\Gamma(L,D)$ is not a part of the boundary $\partial\overline{\mathcal{M}}(L,D)$ of the moduli space of adapted treed disks in the sense that $\overline{\mathcal{M}}(L,D)$ is homeomorphic to an open ball, rather than a half-ball near a $\mathcal{M}_\Gamma(L,D)$ if regular. Indeed any such treed disk $u : C \to X$ may be deformed either by deforming the disk node $z \in C$, or deforming the length $\ell(e)$ of the edge corresponding to the node to a positive real number $\ell(e) > 0$.

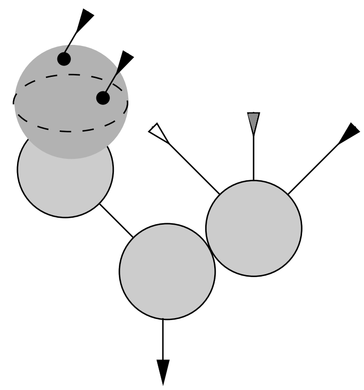

FIGURE 4.13. Treed disk with a zero-length edge

REMARK 4.24. (Orientations) Orientations on moduli spaces of treed pseudoholomorphic disks are defined in the presence of relative spin structures on the Lagrangian as in Fukaya-Oh-Ohta-Ono [**46**, Chapter 10], [**133**]. A *relative spin structure* for an n-dimensional oriented Lagrangian $L \subset X$ is a lift of the Čech class of its tangent bundle TL to relative non-abelian cohomology with values in $\mathrm{Spin}(n)$ relative to the map $i : L \to X$; equivalently, a collection of lifts

$$\tilde{\psi}_{\alpha\beta} : U_\alpha \cap U_\beta \to \mathrm{Spin}(n)$$

for TL of the transition maps $\psi_{\alpha\beta} : U_\alpha \cap U_\beta \to SO(n)$ for TL with respect to some open cover $\{U_\alpha, \alpha \in I\}$ of L to $\mathrm{Spin}(n)$ satisfying the cocycle condition

$$\tilde{\psi}_{\alpha\beta}\tilde{\psi}_{\alpha\gamma}^{-1}\tilde{\psi}_{\beta\gamma} = i^*\phi_{\alpha\beta\gamma}$$

where $(\phi_{\alpha\beta\gamma} \in \{\pm 1\})_{\alpha\beta\gamma}$ is a 2-cycle on X. Given a relative spin structure, the determinant line on the treed pseudoholomorphic disk is oriented as follows: Fix coherent orientations on the stable and unstable manifolds of the Morse function on L. Let $(u : C \to X) \in \mathcal{M}_\Gamma(L, D)$ be an adapted pseudoholomorphic treed disk of combinatorial type Γ of expected dimension 0. Assuming that regularity has been achieved, the linearization $D\overline{\partial}_\Gamma|_{P_\Gamma}$ of the section $\overline{\partial}_\Gamma$ of (4.23) restricted to the perturbation P_Γ is an isomorphism. Choose orientations

$$O_{x_i} : W^\pm(x_i) \to \det(TW^{\pm(x_i)})$$

on the stable and unstable manifolds of the Morse function $W^\pm(x_i)$ for $x_i \in \mathcal{I}(L)$ so that the map

$$T_{x_i}W^-(x_i) \oplus T_{x_i}W^+(x_i) \to T_{x_i}L$$

induces an orientation-preserving isomorphism of determinant lines

$$\det(T_{x_i}W^-(x_i) \oplus T_{x_i}W^+(x_i)) \to \det(T_{x_i}L). \tag{4.30}$$

In case $x_i = x^{\blacktriangledown}$ we define $W^\pm(x^{\blacktriangledown}) = W^\pm(x_M) \times \mathbb{R}$ and choose orientations similarly. One naturally obtains an orientation of the determinant line $\det(\tilde{D}_u)$ of the linearized operator $\tilde{D}_u$ of (4.20) from the isomorphism

$$\det(\tilde{D}_u) \to \det(T\mathcal{M}_\Gamma) \otimes \det(TL) \otimes \det(TW^+(x_0)) \otimes \bigotimes_{i=1}^{n} \det(TW^-(x_i)), \tag{4.31}$$

the orientations on the stable and unstable manifolds $W^{\pm}(x_i)$ and the orientation on the underlying moduli space $\mathcal{M}_\Gamma$ of treed disks. See [26], [133] for similar discussions. In particular, denote by

$$\epsilon : \mathcal{M}(L, D)_0 \to \{\pm 1\} \tag{4.32}$$

the map assigning to any rigid treed disk u the associated sign. The case of the type Γ of a trivial trajectory $u : C \cong \mathbb{R} \to X$ with no disks is treated separately: In the case of a trajectory connecting $x^{\blacktriangledown}$ with $x^{\triangledown}$ resp. $x^{\blacktriangledown}$ the moduli space is a point and we define the orientation to agree resp. disagree with the standard orientation. This ends the Remark.

4.5. Compactness

In this section we show that the subset of the moduli space satisfying an energy bound is compact for suitable perturbation data.

Definition 4.25. For $E > 0$, an almost complex structure $J_D \in \mathcal{J}_\tau(X)$ is *E-stabilized* by a divisor D if and only if

(a) (Non-constant spheres) D contains no non-constant J_D-holomorphic spheres of energy less than E; and
(b) (Sufficient intersections) each non-constant J_D-holomorphic sphere $u : C \to X$ resp. J_D-holomorphic disk $u : (C, \partial C) \to (X, L)$ with energy less than E has at least three resp. one intersection points resp. point with the divisor D:

$$E(u) < E \quad \Longrightarrow \quad \#u^{-1}(D) \geq 1 + 2(\chi(C) - 1)$$

where $\chi(C)$ is the Euler characteristic.

Denote by

$$H_2(X, \mathbb{Z})_\bullet \subset H_2(X, \mathbb{Z}), \quad \text{resp.} \quad H_2(X, L, \mathbb{Z})_\circ \subset H_2(X, L, \mathbb{Z})$$

the set of classes representing non-constant J_D-holomorphic spheres, resp. non-constant J_D-holomorphic disks with boundary in L. A divisor D with Poincaré dual $[D]^\vee = k[\omega]$ for some $k \in \mathbb{N}$ has *sufficiently large degree* for an almost complex structure J_D if and only if

$$\begin{array}{lll} ([D]^\vee, \alpha) & \geq 2(c_1(X), \alpha) + \dim(X) + 1 & \forall \alpha \in H_2(X, \mathbb{Z})_\bullet \\ ([D]^\vee, \beta) & \geq 1 & \forall \beta \in H_2(X, L, \mathbb{Z})_\circ. \end{array} \tag{4.33}$$

This ends the definition.

We introduce the following notation for spaces of almost complex structures sufficiently close to the given one. Given $J \in \mathcal{J}_\tau(X, \omega)$ denote by

$$\mathcal{J}_\tau(X, D, J, \theta) = \{J_D \in \mathcal{J}_\tau(X, \omega) \mid \|J_D - J\| < \theta, \quad J_D(TD) = TD\}$$

the space of tamed almost complex structures close to J in the sense of [28, p. 335] and preserving TD. The following lemma on existence of stabilizing almost complex structures is a special case of the results of [28].

Lemma 4.26. (Cieliebak-Mohnke [28, Proposition 8.14, Corollary 8.20].) *For θ sufficiently small, suppose that D has sufficiently large degree for an almost complex structure θ-close to J. For each energy $E > 0$, there exists an open and dense subset*

$$\mathcal{J}^*(X, D, J, \theta, E) \subset \mathcal{J}_\tau(X, D, J, \theta)$$

such that if $J_D \in \mathcal{J}^*(X, D, J, \theta, E)$, *then* J_D *is* E*-stabilized by* D. *Similarly, if* $D = (D^t)$ *is a family of divisors for* J^t, *then for each energy* $E > 0$, *there exists a dense and open subset of time-dependent almost complex structures*

$$\mathcal{J}^*(X, D^t, J^t, \theta, E) \subset \mathcal{J}_\tau(X, D^t, J^t, \theta)$$

such that if $J_{D^t} \in \mathcal{J}^*(X, D^t, J^t, \theta, E)$, *then* J_{D^t} *is* E*-stabilized for all* t.

We restrict to perturbation data taking values in $\mathcal{J}^*(X, D, J, \theta, E)$ for a (weakly or strictly) stabilizing divisor D having sufficiently large degree for an almost-complex structure θ-close to J. Let $J_D \in \mathcal{J}_\tau(X, D, J, \theta)$ be an almost complex structure that is stabilized for all energies, for example, in the intersection of $\mathcal{J}^*(X, D, J, \theta, E)$ for all E. For each energy E, there is a contractible open neighborhood $\mathcal{J}^{**}(X, D, J_D, \theta, E)$ of J_D in $\mathcal{J}^*(X, D, J, \theta, E)$ that is E-stabilized.

Definition 4.27. A perturbation datum $P_\Gamma = (F_\Gamma, J_\Gamma)$ for a type of stable treed disk Γ is *stabilizing* with respect to D if J_Γ takes values in $\mathcal{J}^*(X, D, J, \theta, n(\Gamma_i, k))$ on $\overline{\mathcal{U}}_{\Gamma_i}$ (in particular, if J_Γ takes values in $\mathcal{J}^{**}(X, D, J, \theta, n(\Gamma_i, k))$.)

To save space we call a perturbation system $\underline{P} = (P_\gamma)$ *admissible* if it is coherent, regular, and stabilizing.

Theorem 4.28. (Compactness for fixed type) *For any collection* $\underline{P} = (P_\Gamma)$ *of admissible perturbation data and any uncrowded type* Γ *of expected dimension at most one, the moduli space* $\overline{\mathcal{M}}_\Gamma(L, D)$ *of adapted treed marked disks of type* Γ *is compact and the closure of* $\mathcal{M}_\Gamma(L, D)$ *contains only configurations with disk bubbling.*

Proof. (c.f. [**27**, Proof of Proposition 4.10].) Because of the existence of local distance functions, similar to [**95**, Section 5.6], it suffices to check sequential compactness. Let $u_\nu : C_\nu \to X$ be a sequence of stable treed disks of type Γ, necessarily of fixed energy $E(\Gamma)$. After passing to a subsequence we may assume that the sequence of stable treed disks $[C_\nu]$ converges to a limiting stable disk $[C]$ in $\overline{\mathcal{M}}_\Gamma$. Considering each disk or sphere component or Morse trajectory of u_ν separately one sees the sequence of maps $u_\nu : C_\nu \to X$ has subsequence that admits a stable Gromov limit $u : \hat{C} \to X$, where $\hat{C} = \hat{S} \cup \hat{T}$ is a possibly unstable sphere or disk with stabilization C of type Γ_∞.

We show that the limit constructed in the previous paragraph is adapted, that is, interior leaves correspond to intersections with the Donaldson hypersurface. By definition we have $J_\Gamma(x, z) = J_D(x), \ \forall x \in D$. Now $J_D \in \mathcal{J}^*(X, D, J, \theta, n(\Gamma_i, k))$ was chosen so that D contains no non-constant J_D-holomorphic spheres. To show the (Leaf axiom) property, note that each connected component C_i of $u^{-1}(D)$ has surface part $S_i = S \cap C_i$ either a point z_i or a union of sphere and disk components. In the first case, the intersection multiplicity $m(z_i)$ with the divisor at the point z_i is positive while in the second, the intersection multiplicity $m(w)$ at each node $w \in S_i$ connecting the component S_i with the rest of the domain $S - S_i$ is positive. In the first case, by topological invariance of the intersection multiplicity there exists a sequence z_ν of isolated points in $u_\nu^{-1}(D)$ with positive intersection multiplicity converging to z_i. By the (Leaf axiom) property of u_ν, the points z_ν must be markings. Hence z_i is a marking as well. In the second case, the positivity of the intersection multiplicity implies that there exists a sequence of points $z_{\nu,i} \in u_\nu^{-1}(D) \subset C_\nu$ converging to C_i. Either $z_{\nu,i}$ are isolated in $u_\nu^{-1}(D)$, or lie on some sequence of connected components $C_{\nu,i}$ of $u_\nu^{-1}(D) \subset C_\nu$ on which u_ν is constant in which case any limit point of $C_{\nu,i}$ lies in C_i. Since each $C_{\nu,i}$ contains a marking by the (Leaf axiom) property, so does C_i. Note that if $u_\nu(z_{i,\nu}) \in D$ then $u(z_i) \in D$, by convergence on compact subsets of complements of the nodes. This shows the (Leaf axiom) property.

We next show the (Stable surface axiom). Note that since D is stabilizing for L, any non-constant disk component $\hat{S}_i$ of $\hat{S}$ must have at least one interior intersection point, call it z, with $u^{-1}(D)$. Since z lies in the interior of S_i, and the boundary ∂S_i has at least one special point the component $\hat{S}_i$ is stable. Next consider a spherical component $\hat{S}_i$ of $\hat{S}$ attached to a component of S. First suppose that u is non-constant on $\hat{S}_i$. Suppose that $\hat{S}_i$ is attached at a point of S contained in $\overline{\mathcal{U}}_{\Gamma_i}$ for some type Γ_i, so that the energy of $u|_{\hat{S}_i}$ is at most $n(\Gamma_i, k)$. Since J_Γ is constant and equal to an element of $\mathcal{J}^*(X, D, J, \theta, n(\Gamma_i, k))$ on $\hat{S}_i$, the restriction of u to $\hat{S}_i$ has at least three intersection points resp. one intersection point with D. By definition these points must be markings, which contradicts the instability of $\hat{S}_i$. Hence the stable map u must be constant on $\hat{S}_i$, and thus $\hat{S}_i$ must be stable. This shows that $\hat{S}$ is equal to S and the (Stable surface axiom) follows.

It remains to rule out sphere components in the limit. Suppose the limiting domain C has a spherical component $S_v \subset S$. By the (Locality Axiom), forgetting all but one marking on maximal ghost components gives a configuration $u_{\text{red}} : C_{\text{red}} \to X$ in an uncrowded moduli space $\mathcal{M}_{\Gamma_{\text{red}}}(L, D)$ for some type Γ_{red} with respect to some perturbation datum $P^{\text{red}}_{\Gamma_{\text{red}}}$ given by the perturbation datum $P_{\Gamma_\circ}$ on disk components and p perturbations $P^{\text{red}}_{\Gamma_v}$ for sphere components S^{red}_v of C_{red}. For generic choices of P_Γ, the moduli space $\mathcal{M}_{\Gamma_{\text{red}}}(L, D)$ with respect to this induced perturbation datum is of expected dimension. Since the configuration u_{red} has a spherical node, this expected dimension is at least two less than the expected dimension of $\mathcal{M}_\Gamma(L, D)$, which is at most one. Hence the dimension $\dim \mathcal{M}_{\Gamma_{\text{red}}}(L, D)$ is negative, a contradiction. □

4.6. Composition maps

In this section we use pseudoholomorphic treed disks to define the structure coefficients of the Fukaya algebra. Let q be a formal variable and Λ the *universal Novikov field* of formal sums with complex coefficients

$$\Lambda = \left\{ \sum_i c_i q^{\alpha_i} \;\middle|\; c_i \in \mathbb{C},\ \alpha_i \in \mathbb{R}, \quad \alpha_i \to \infty. \right\} \tag{4.34}$$

Denote by $\Lambda_{\geq 0}$ resp. $\Lambda_{>0}$ the subalgebra with only non-negative resp. positive exponents. Denote by

$$\Lambda^\times = \left\{ c_0 + \sum_{i>0} c_i q^{\alpha_i} \in \Lambda_{\geq 0} \;\middle|\; c_0 \neq 0 \right\}$$

the subgroup of formal power series with invertible leading coefficient. Although our Fukaya algebras will be defined over the rationals, allowing complex number provides additional solutions to the Maurer-Cartan equation.

Lagrangians will be equipped with additional data called *brane structures*. Let $\text{Lag}(X)$ denote the fiber bundle over X whose fiber $\text{Lag}(X)_x$ at x consists of Lagrangian subspaces of $T_x X$. Let g be an even integer. A *Maslov cover* is an g-fold cover $\mathrm{Lag}^g(X) \to \mathrm{Lag}(X)$ such that the induced two-fold cover

$$\mathrm{Lag}^2(X) := \mathrm{Lag}^g(X)/\mathbb{Z}_{g/2} \to \mathrm{Lag}(X)$$

is the oriented double cover. A Lagrangian submanifold L is *admissible* if L is compact and oriented; we also assume for simplicity that L is connected. A *grading* on L is a lift of the canonical map

$$L \to \mathrm{Lag}(X), \quad l \mapsto T_l L$$

to $\mathrm{Lag}^g(X)$, see Seidel [**116**]. A *brane structure* on a compact oriented Lagrangian is a relative spin structure together with a grading and a $\Lambda_\times$ local system. (The grading will be irrelevant until we discuss Fukaya bimodules.) An *admissible Lagrangian brane* is an admissible Lagrangian submanifold equipped with a brane structure. For L an admissible Lagrangian brane define the space of Floer cochains

$$CF(L) = \bigoplus_{d \in \mathbb{Z}_g} CF_d(L), \quad CF_d(L) = \bigoplus_{x \in \mathcal{I}_d(L)} \Lambda x$$

where the degree d generators $\mathcal{I}_d(L)$ are as in (4.14). Let $CF^{\mathrm{geom}}(L) \subset CF(L)$ the subspace generated by $x \in \mathcal{I}^{\mathrm{geom}}(L)$.

The composition maps are defined in the following. Given a Lagrangian brane L and a tree disk $u : C \to X$ with boundary in L, we denote by $y(u) \in \Lambda^\times$ the evaluation of the local system $y \in \mathcal{R}(L)$ on the homotopy class of loops $[\partial u] \in \pi_1(L)$ defined by going around the boundary of each disk component $S_v \subset C$ in the treed disk C once. Denote by $\sigma(u) \in \mathbb{Z}_{\geq 0}$ the number of interior leaves of $u \in \overline{\mathcal{M}}(L, D, \underline{x})$. Let $E(u)$ denote the energy, or equivalently area, of surface part of u, c.f. (6.12) below.

Definition 4.29. (Composition maps) For admissible perturbation data (P_Γ) define

$$\mu^n : CF(L)^{\otimes n} \to CF(L)$$

on generators by

$$\mu^n(x_1, \ldots, x_n) = \sum_{x_0, u \in \mathcal{M}(L,D,\underline{x})_0} (-1)^{\heartsuit} (\sigma(u)!)^{-1} y(u) q^{E(u)} \epsilon(u) x_0 \tag{4.35}$$

where $\heartsuit = \sum_{i=1}^n i|x_i|$ and $\epsilon(u)$ is the sign of (4.32).

Remark 4.30. (Zero-th composition map is a quantum correction) Any configuration $u : C \to X$ with no boundary leaves $T_e \subset T$ must have at least one non-constant disk component $S_v \subset S$. Indeed, since each vertex has valence at least three, any configuration must have leaves but in this case all leaves must be interior. Hence $\mu^0(1)$ has $\mathrm{val}_q(\mu^0(1)) > 0$.

Remark 4.31. (Leading order term in the first composition map) The constant trajectories at the maximum x_M with weighted leaf and outgoing unweighted root are elements of $\mathcal{M}_\Gamma(L, D, x_0, x^{\blacktriangledown})_0$ for some type Γ, as in Remark 4.5. The orientations on these trajectories are determined by the orientation on $\mathcal{M}_\Gamma$. By the discussion after (4.31) the orientation is negative resp. positive for $x_0 = x^{\blacktriangledown}$ resp. $x_0 = x^{\triangledown}$. Hence

$$\mu^1(x^{\blacktriangledown}) = x^{\triangledown} - x^{\blacktriangledown} + \sum_{x_0, [u] \in \mathcal{M}(L,D,x^{\blacktriangledown},x_0)_0, E(u)>0} (-1)^{\heartsuit} (\sigma(u)!)^{-1} q^{E(u)} \epsilon(u) y(u) x_0. \tag{4.36}$$

This formula is similar to that in Fukaya-Oh-Ohta-Ono [**46**, (3.3.5.2)]. Presumably the discussion here is a version of their treatment of homotopy units.

THEOREM 4.32. (A_∞ algebra for a Lagrangian) *For any admissible perturbation system $\underline{P} = (P_\Gamma)$ the maps $(\mu^n)_{n\geq 0}$ satisfy the axioms of a (possibly curved) A_∞ algebra $CF(L)$ with strict unit. The subspace $CF^{\mathrm{geom}}(L)$ is a subalgebra without unit.*

PROOF. By Theorems 4.20 and 4.28, the one-dimensional component of the moduli space with energy bound $\overline{\mathcal{M}}^{<E}(L)_1$ is a finite union of compact oriented one manifolds $\mathcal{M}_\Gamma^{\leq E}(L)$ for some combinatorial types Γ with a single broken edge. Hence the boundary points of the moduli space come in pairs. This produces the identity mod 2

$$0 = \sum_{\Gamma\in\mathcal{T}_{n,m}} \sum_{[u]\in\partial\overline{\mathcal{M}}_\Gamma(L,D,\underline{x})_1} \epsilon(u)(\sigma(u)!)^{-1} q^{E(u)} y(u) \tag{4.37}$$

where $\mathcal{T}_{n,m}$ denotes the set of types of treed disks with n boundary and m interior edges. In case the moduli spaces do not involve weightings then each combinatorial type $\Gamma \in \mathcal{T}_{n,m}$ with a single interior edge $e \in \mathrm{Edge}_-(\Gamma)$ of infinite length $\ell(e) = \infty$ is obtained by gluing together graphs Γ_1, Γ_2 with $n - n_2 + 1$ and n_2 leaves along semi-infinite edges e_-, e_+, say with m_1 resp. m_2 interior markings. By the (Cutting edges axiom) we have an isomorphism

$$\overline{\mathcal{M}}_\Gamma(L,D,\underline{x}) = \bigcup_{w\in\mathcal{I}(L)} \overline{\mathcal{M}}_{\Gamma_1}(L,D,x_0,x_1,\ldots,x_{i-1},w,x_{i+n_2},\ldots,x_n) \times \overline{\mathcal{M}}_{\Gamma_2}(L,D,w,x_i,\ldots,x_{i+n_2-1}). \tag{4.38}$$

Since there are m choose m_1, m_2 ways of distributing the interior leaves to the two component graphs, mod 2 we have

$$0 = \sum_{\substack{i,m_1+m_2=m \\ [u_1]\in\mathcal{M}_{\Gamma_1}(L,D,x_0,x_1,\ldots,x_{i-1},y,x_{i+n_2},\ldots,x_n)_0 \\ [u_2]\in\mathcal{M}_{\Gamma_2}(L,D,y,x_i,\ldots,x_{i+n_2-1})_0}} (m!)^{-1} \begin{pmatrix} m \\ m_1 \end{pmatrix} q^{E([u_1])+E([u_2])} \epsilon([u_1])(\epsilon[u_2])y(u_1)y(u_2). \tag{4.39}$$

This gives the A_∞ axiom (4.2) up to signs. We refer to [**91**] for the sign computation. In the case of a weighted leaf e one has additional boundary terms corresponding to types where the weighting $\rho(e)$ becomes zero or infinity. Those configurations correspond to splitting off a constant Morse trajectory with weighted leaf $e \in \mathrm{Edge}_\to^{\blacktriangledown}(\Gamma)$ and outgoing forgettable or unforgettable root $e_0 \in \mathrm{Edge}_\to^{\blacktriangledown}(\Gamma) \cup \mathrm{Edge}_\to^{\triangledown}(\Gamma)$. These correspond to the terms $x^\triangledown$ and $x^\blacktriangledown$ in $\mu^1(x^\blacktriangledown)$.

The existence of strict units follows from the (Forgettable edges axiom). We claim that a strict unit is given by the element $e_L = x^\triangledown \in CF(L)$. By the (Forgettable edges) axiom, the perturbation data P_Γ used to define $\overline{\mathcal{M}}_\Gamma(L,D,\ldots,x_{i-1},x_i = x^\triangledown,x_{i+1},\ldots,)$ is pulled back, on the stable part of the universal curve, from data $P_{\Gamma'}$ for $\overline{\mathcal{M}}_{\Gamma'}(L,D,\ldots,x_{i-1},x_{i+1},\ldots,)$ for the type Γ' obtained from Γ by forgetting the corresponding leaf e_i. As long as Γ' has at least one vertex, the moduli space $\overline{\mathcal{M}}_{\Gamma'}(L,D,\ldots,x_{i-1},x_{i+1},\ldots,)$ is one dimension lower, and so empty. Thus the compositions $\mu^n(\ldots,x^\triangledown,\ldots)$ vanish except for the case that the resulting type has no stabilization: In the case $n = 2$, the underlying configuration $u : C' \to X$ has no non-constant disks $S_i \subset C$, as in Figure 4.11. One obtains from a configuration with type Γ' and constant values the identity

$$\mu^2(x, x^\triangledown) = (-1)^{\deg(x)}\mu^2(x^\triangledown, x) = x, \quad \forall x \in \mathcal{I}(L). \qquad \square$$

4.7. Divisor equation

The divisor equation [**73**, 2.2.4] expresses the fact that the number of pseudoholomorphic curves mapping a point on the domain to a codimension two submanifold (divisor) is the pairing of the homology class of the map with the class of the divisor. In the case of pseudoholomorphic curves with boundary, the resulting equation [**30**, Proposition 6.3] gives an equality once one sums over all possible places of the insertion (since the theory is not invariant under permutations of the insertions.) In any particular perturbation scheme, the existence of such an equation relies on the existence of forgetful maps. The Morse moduli spaces above do not admit forgetful morphisms in general, hence the divisor axiom is not satisfied. However, there is a weak version for the case of a single leaf. Let Γ be a combinatorial type with a single boundary leaf and at least one interior leaf and $f(\Gamma)$ the type obtained by forgetting the leaf. We consider perturbations of the form

$$P_\Gamma = f^* P_{f(\Gamma)} \tag{4.40}$$

for type Γ pulled back under the map of universal curves $f : \mathcal{U}_\Gamma \to \mathcal{U}_{f(\Gamma)}$.

LEMMA 4.33. *Let Γ be a combinatorial type of adapted pseudoholomorphic treed disk with a single boundary leaf of expected dimension at most one and suppose that $J_D \in \mathcal{J}_\tau(X)$ is such that all non-constant J_D-holomorphic disks u in X with boundary in L have positive Maslov index $I(u) > 0$. For a comeager set of perturbations $\mathcal{P}^{\mathrm{reg,div}}_{f(\Gamma)}$, if $P_{f(\Gamma)} \in \mathcal{P}^{\mathrm{reg,div}}_{f(\Gamma)}$ then the moduli space $\mathcal{M}_\Gamma(X, L)$ is smooth of expected dimension.*

PROOF. Let C be a tree disk of type Γ with single boundary leaf e and $f(C)$ the treed disk obtained by forgetting e and stabilizing. In the construction of the local universal moduli space (4.25) perturbations of the form $f^* P_{f(\Gamma)}$ suffice to achieve transversality. Indeed, at most one disk component of C is collapsed under the forgetful map (depending on whether the disk containing $e \cap S$ is constant or not). If the disk S_v containing $e \cap S$ is not collapsed, then variations $\xi \in \Omega^0(S_v, (u|S_v)^* TX)$ that vanish at the point $S \cap e$ are enough to force an element η in the cokernel of the linearized operator to satisfy $D_u^* \eta = 0$ on S_v. The same argument as in that construction produces a comeager subset of perturbations. If S_v is collapsed, then it maps to a point on an edge of $f(C)$ which is necessarily combinatorially semi-infinite in at most one direction, since C must contain at least one non-constant disk. Let $z = f(S \cap e)$ be the collapsed disk in $f(C)$. If z lies on an edge T_e not diffeomorphic to $\mathbb{R}$ (that is, a semi-infinite rather than infinite interval) then domain-dependent perturbations F_Γ of the Morse function suffice to make the moduli space of maps $v : f(C) \to X$ with $v(z) \in W^-_{l_1}$ transverse, and this moduli space is isomorphic to $\mathcal{M}_\Gamma(X, L)$. The final case is that z maps to a broken segment $e \cong \mathbb{R}$ of a semi-infinite edge. The latter corresponds to a broken configuration $u : C \to X$ that splits into pieces $u_1 : C_1 \to X$ with no leaves, a configuration $u_2 : C_2 \to X$ with two leaves but only non-constant disks, and a Morse trajectory $u_3 : \mathbb{R} \to L$. Since the configuration u_1 must be non-negative expected dimension, the breaking must be at a critical point $l_2 \in \mathcal{I}(L)$ of index zero. There are no such configurations, since u_3 must flow to a critical point of positive index. $\square$

Note that in general one may not choose perturbations pulled back under forgetful maps: Configurations $u \in \mathcal{M}_\Gamma$ with constant disks correspond to intersections of the unstable manifolds $W_x^-, x \in \operatorname{crit}(m)$ of the unperturbed Morse function $m : L \to \mathbb{R}$. The stable manifolds W_x^-, W_y^- are not transversally intersecting for obvious reasons (e.g. they may be equal if $x = y$.)

Perturbations satisfying the forgetful property (4.40) give the following weak form of the divisor equation. Let $c \in \ker(\mu^{1,0})$ be a cocycle for the Morse operator $\mu^{1,0}$ and $[c] \in H(L)$ the Morse homology class, represented by the sum of unstable manifolds $W^-(x)$ of the critical points x appearing in c. That is, if $c = \sum c_x x$ then

$$[c] = \left[\sum c_x W^-(x)\right].$$

As in Fukaya-Oh-Ohta-Ono [**47**, Lemma 13.1] we write

$$\mu^d = \sum_\beta q^\beta \mu^{d,\beta} \tag{4.41}$$

for some complex numbers $\mu^{d,\beta}$.

Proposition 4.34. (Weak divisor equation) *Suppose that an admissible perturbation system $\underline{P} = (P_\Gamma)$ is such that $P_\Gamma = f^* P_{f(\Gamma)}$ for any type Γ with one leaf. For any Morse cycle $c \in \ker(\mu^{1,0}) \cap CF^1(L)$, we have*

$$\mu^1(c) = \sum_{u \in \mathcal{M}_1(l_0)} q^{A(u)} \epsilon(u)\sigma(u)([\partial u], [c]) y(u) \tag{4.42}$$

$$= \sum (\partial\beta, [c]) q^\beta \mu^{0,\beta}(c) \tag{4.43}$$

where $\partial\beta$ is the image of $\beta \in H_2(X, L)$ in $H_1(L)$.

Proof. By assumption there is a forgetful morphism

$$f : \mathcal{M}_\Gamma(X, L, D, x_1, x_0) \to \mathcal{M}_{f(\Gamma)}(X, L, D, x_0) \tag{4.44}$$

obtained by forgetting the map on the leaf. The fiber over an element

$$[u : C \to X] \in \mathcal{M}_{\Gamma'}(x_0)$$

is the set of points in the boundary ∂C mapping to $W^-(x_1)$,

$$f^{-1}(u) \cong (u|\partial C)^{-1}(W^-(x_1))$$

and is cut out transversally. The areas, numbers of interior leaves, and holonomies of u and $f(u)$ are equal. Furthermore, by construction the sign $\epsilon(u)$ is equal to $\epsilon(f(u))$ times the sign of the corresponding intersection of $(u|\partial C)^{-1}(W^-(x_1))$. Indeed in the case that $\{z\} = S \cap e$ is a point on a constant disk with one other leaf $e' \subset T$, attaching the edge e at the disk with the leaves reversed gives a contribution with opposite sign; thus we may ignore such contributions. In the case that $\{z\} = S \cap e$ attaches to a non-constant disk $S_v \subset S$, the construction of orientations in (4.31) defines the orientation on these moduli spaces as that on $\det(Tf^{-1}(C)) \otimes \det(TW^-(x_1))$ which is identified with the trivial determinant line $\mathbb{R}$ via the determinant line of $\det(D_z u) \mod TW^-(x_1)$. This identification is positive if the intersection of ∂u with $W^-(x_1)$ is positive. The claimed equality follows. $\square$

The following argument similar to that Fukaya-Oh-Ohta-Ono [**47**, Section 13] in the toric case implies that critical points of the disk potential determine Floer-non-trivial brane structures: Let

$$\mathcal{R}(L) := \operatorname{Hom}(\pi_1(L), \Lambda^\times)$$

denote the space of isomorphism classes of local systems on L. Any representation descends to the abelianization $H_1(L)$ of $\pi_1(L)$. Coordinates on $\mathcal{R}(L)$ are given by exponentiation:

$$\exp : H^1(L, \Lambda_{\geq 0}) \to \mathcal{R}(L), \quad f \mapsto \exp(f).$$

In particular we have an isomorphism of tangent spaces

$$T_y \mathcal{R}(L) \cong H^1(L, \Lambda_{\geq 0})$$

at any $y \in \mathcal{R}(L)$. Any element of $\mathcal{R}(L)$ is given by an element $[f] \in H^1(L, \Lambda_{\geq 0})$ in the sense that the map is given by

$$\pi_1(L) \to \Lambda^\times, \quad [\gamma] \mapsto y_{[f]}([\gamma]) = \exp([\gamma], [f]).$$

The derivative of evaluation at a class $[\gamma] \in \pi_1(L)$ is the evaluation of the cohomology class on the homology class of the loop:

$$\partial_{[c]} y_{[f]}([\gamma]) = ([c], [\gamma]).$$

The argument for the Floer non-triviality involves variation of the local system. Suppose that $0 \in MC(L, y)$ for every choice of local system $y \in \mathcal{R}(L)$. Set

$$\mathcal{W} : \mathcal{R}(L) \to \Lambda_{\geq 0}, \quad \mathcal{W}(y) e_L = \mu^0(1). \tag{4.45}$$

Continuing the computation from (4.42) we identify a first cohomology class with a tangent vector to the space of local systems. Then

$$\begin{aligned}
\mu^1(c) &= \sum_{\beta > 0} q^\beta (\partial\beta, [c]) \mu^{0,\beta} \\
&= \sum_{x_0, u \in \mathcal{M}_1(L, D, x_0)_0} (-1)^\heartsuit ([\partial u], [c]) (\sigma(u)!)^{-1} y(u) q^{E(u)} \epsilon(u) x_0 \\
&= \partial_{[c]} \mu^0(1) = \partial_{[c]} \mathcal{W}(y) e_L.
\end{aligned}$$

In the following proposition, similar to Fukaya-Oh-Ohta-Ono [**47**, Lemma 13.1], we have in mind the case that the Lagrangian is a torus and the Morse function is the standard one obtained from taking the product of the standard height function on the component circles, so that the Morse coboundary vanishes. A related result in the monotone case is in Biran-Cornea [**14**, Theorem 1.2.2].

Proposition 4.35. *Suppose that the Morse operator $\mu^{1,0} = \mu^1|_{q=0}$ vanishes, $0 \in MC(L)$ and for some $y \in \mathcal{R}(L)$, for all $b \in H^1(L, \Lambda_{\geq 0})$, we have $\partial_b \mathcal{W}(y) = 0$. Then the Floer cohomology is isomorphic to the ordinary cohomology:*

$$H(\mu^1_y, \Lambda_{\geq 0}) = H(L, \Lambda_{\geq 0}).$$

Proof. For completeness we include the proof is a double induction on energy and classical degree. Let $\hbar > 0$ be the energy quantization constant so that $E(\beta) > \hbar$ for all classes $\beta \in H_2(\mathbb{X}, L)$ represented by non-constant treed holomorphic disks. Suppose that $m_1^\beta = 0$ for all β with $E(\beta) \leq E$. We claim that $m_1^\beta = 0$ for all β with $E(\beta) \leq E + \hbar$. The base step is furnished by the assumption that the Morse function is standard, so the Morse differential vanishes. By induction on degree we may assume that $m_1^\beta(c) = 0$ for degree $\deg(c) \leq d$. The inductive step is furnished by the assumption that $\mu^1(c) = 0$ for $\deg(c) = 1$ via the computation (4.46). We claim that $m_1^\beta(c_1 \cup c_2) = 0$ for all classes c_1, c_2 with $1 \le \deg(c_1), \deg(c_2) \le d$. Using the decomposition (4.41) and comparing terms with coefficient q^β in the A_∞ associativity relation we obtain

$$(4.46) \quad m_1^\beta(m_2^0(c_1, c_2)) = \sum_{\beta_1+\beta_2=\beta} \pm m_2^{\beta_1}(m_1^{\beta_2}(c_1), c_2) + \sum_{\beta_1+\beta_2=\beta} \pm m_2^{\beta_1}(c_1, m_1^{\beta_1}(c_2)) + \sum_{\beta_1+\beta_2=\beta, \beta_2 \neq 0} \pm m_1^{\beta_1}(m_2^{\beta_2}(c_1, c_2)).$$

The first two terms on the right vanish by the inductive hypothesis for degree. The last term vanishes by the inductive hypothesis for energy since $E(\beta_1) < E(\beta) - \hbar \le E$. Since any class c of degree $d+1$ is a linear combination of classes $c_1 \cup c_2$ with $1 \le \deg(c_1), \deg(c_2) \le d$ this shows that $\mu^1(c) = 0$ for classes of degree $d+1$. $\square$

EXAMPLE 4.36. (Potential for the projective line) Suppose that $X = S^2$ with area A and $L \cong S^1$ separates X into pieces of areas A_1, A_2. Thus the Maslov-index-two disks have areas A_1, A_2 with opposite boundary homotopy classes. The disk potential is

$$\mathcal{W}(y) = q^{A_1} y + q^{A_2}/y.$$

The Floer differential vanishes if and only if

$$0 = y \partial \mathcal{W}/\partial y = q^{A_1} y - q^{A_2}/y$$

for some y. The equation $y^2 = q^{A_2 - A_1}$ has a solution in $\mathcal{R}(L) \cong \Lambda^\times$ if and only if $A_1 = A_2$. Thus only in this case (the case that the Lagrangian is Hamiltonian isotopic to the equator) one has non-trivial Floer cohomology.

Brane structures corresponding to non-degenerate critical points are particularly well-behaved. A Lagrangian torus L equipped with brane structure is *non-degenerate* if the leading order part $\mathcal{W}_0$ of the potential $\mathcal{W}$ (that is the sum of terms with lowest q-degree) has a non-degenerate critical point. Existence of a non-degenerate critical point is invariant under perturbation by Fukaya-Oh-Ohta-Ono [**47**, Theorem 10.4]. We reproduce the proof here for the sake of completeness. Since $\mathcal{R}(L) = \operatorname{Hom}(H_1(L), \Lambda^\times)$, $\mathcal{R}(L)$ is a smooth manifold at any $y_0 \in \mathcal{R}(L)$ in the sense that there exists a coordinate chart $\exp_{y_0} : \Lambda_{\ge 0}^{\otimes N} \supset U \to \mathcal{R}(L)$, where U is an open neighborhood of 0. The notion of critical points of $\mathcal{W}$ and the Hessian at those critical points is therefore well-defined.

THEOREM 4.37. *Suppose that $\mathcal{W} : \mathcal{R}(L) \to \Lambda_{\ge 0}$ is a function of the form $\mathcal{W} = \mathcal{W}_0 + \mathcal{W}_1$ where $\mathcal{W}_0$ consists of the terms of lowest q-valuation. Suppose that $y_0 \in \operatorname{Crit}(\mathcal{W}_0)$ and $\mathcal{W}_0$ has non-degenerate Hessian $D^2_{y_0}\mathcal{W}_0$ at y_0. Then there exists*

$$y_1 \in \operatorname{Hom}(\pi_1(L), \Lambda_{>0}), \quad y = y_0 \exp(y_1) \in \operatorname{Crit}(\mathcal{W})$$

with the terms in y_1 having higher q-valuation than those in y_0.

PROOF. The proof is an application of a formal implicit function theorem adapted to the setting of Novikov rings. Given a critical point y_0 of $\mathcal{W}_0$ we solve for a critical point y of $\mathcal{W}$ order by order, using non-degeneracy of the Hessian of $\mathcal{W}$ at y_0. Suppose that

$$\mathcal{W} = q^{\epsilon_0}\mathcal{W}_0 + q^{\epsilon_1}\mathcal{W}_1$$

where $\mathcal{W}_0$ has vanishing q-valuation and $\epsilon_1 > \epsilon_0$. Suppose that y_0 is a critical point of $\mathcal{W}$ mod terms divisible by q^δ for some δ with $\delta > \epsilon_0$. Taking a Taylor expansion of $D_y\mathcal{W}$ at $y = y_0$ we solve

$$0 = D_{y_0 \exp(y_1)}\mathcal{W} = D_{y_0}\mathcal{W} + D^2_{y_0}\mathcal{W}(y_1) + F_{y_0}(y_1) \tag{4.47}$$

$$= q^\delta G(y_0) + D^2_{y_0}\mathcal{W}(y_1) + F_{y_0}(y_1) \tag{4.48}$$

where $F_{y_0}(y_1)$ is a linear map satisfying a quadratic bound for small y_1. Here the Hessian $D^2_{y_0}\mathcal{W}$ is viewed as a map

$$D^2_{y_0}\mathcal{W} : T_{y_0}\mathcal{R}(L) \to \operatorname{Hom}(T_{y_0}\mathcal{R}(L), \Lambda_{\geq 0}).$$

We solve

$$D^2_{y_0} q^{-\epsilon_0}\mathcal{W}(y_1) = -q^{\delta-\epsilon_0}G(y_0).$$

Then (4.47) fails to hold only because of the term $F_{y_0}(y_1)$. Since $\mathcal{W}$ is divisible by q^{ϵ_0} and y_1 is divisible by $q^{\delta-\epsilon_0}$, $F_{y_0}(y_1)$ is divisible by $q^{2(\delta-\epsilon_0)+\epsilon_0}$. Replacing y_0 with $y_0\exp(y_1)$ and continuing by induction one obtains a solution to all orders. □

Since any critical point of $\mathcal{W}$ is, to leading order, a critical point of $\mathcal{W}_0$, we have the following corollary:

COROLLARY 4.38. *If every critical point of $\mathcal{W}_0$ is non-degenerate then there exists a bijection* $\operatorname{Crit}(\mathcal{W}_0) \cong \operatorname{Crit}(\mathcal{W})$ *between the set* $\operatorname{Crit}(\mathcal{W}_0)$ *of critical points of $\mathcal{W}_0$ and critical points* $\operatorname{Crit}(\mathcal{W})$ *of $\mathcal{W}$.*

4.8. Maurer-Cartan moduli space

This section contains a version of the results of Fukaya-Oh-Ohta-Ono [**46**, Section 3.6] on solutions to the projective Maurer-Cartan equation, adapted to the Morse setting. In particular we define a *cohomology complex* associated to an A_∞ algebra satisfying suitable convergence conditions.

The Floer complex is a complex of vector bundles over a space of solutions to a *projective Maurer-Cartan equation.* Let A be an A_∞ algebra free and finitely generated over the Novikov ring $\Lambda_{\geq 0}$. Such A is *convergent* if $\operatorname{val}_q(\mu^0(1)) > 0$. Let

$$A^+ = \{a \in A \mid \operatorname{val}_q(a) > 0\}$$

denote the subalgebra of terms with positive q-valuation.

LEMMA 4.39. *For $b \in A^+$ the sum*

$$\mu^0_b(1) := \mu^0(1) + \mu^1(b) + \mu^2(b, b) + \ldots \tag{4.49}$$

is well-defined.

PROOF. Since b has positive q-valuation, we may write $b = q^E b_0$ for some $b_0 \in A$ and $E > 0$. Then the coefficient of each generator of A in the composition $\mu^n(b, \ldots, b)$ has q-valuation $\operatorname{val}_q(\mu^n(b, \ldots, b)) > nE$, if non-zero. This inequality implies that the sum (4.49) converges. □

More generally the same argument implies convergence of the *deformed composition map*

$$\mu^n_b(a_1, \ldots, a_n) = \sum_{i_1,\ldots,i_{n+1}} \mu^{n+i_1+\ldots+i_{n+1}}(\underbrace{b, \ldots, b}_{i_1}, a_1, \underbrace{b, \ldots, b}_{i_2}, a_2, b, \ldots, b, a_n, \underbrace{b, \ldots, b}_{i_{n+1}})$$

over all possible combinations of insertions of the element $b \in A^+$ between (and before and after) the elements $a_1, \ldots, a_n$. For b odd the maps μ_b^n define an A_∞ structure on A. In particular

$$(\mu_b^1)^2(a_1) = \mu_b^2(\mu_b^0(1), a_1) - \mu_b^2(a_1, \mu_b^0(1)).$$

The *projective Maurer-Cartan equation* for $b \in A^+$ is

$$\mu_b^0(1) = \mu^0(1) + \mu^1(b) + \mu^2(b,b) + \ldots \in \Lambda e_A. \tag{4.50}$$

Denote by

$$MC(A) = \{b \in A^{+,\mathrm{odd}} \mid \mu_b^0(1) \in \Lambda e_A\} \tag{4.51}$$

the space of odd solutions to the projective Maurer-Cartan equation (4.50). Any solution to the projective Maurer-Cartan equation defines an A_∞ algebra such that $(\mu_b^1)^2 = 0$ and so has a well-defined cohomology

$$H(\mu_b^1) = \frac{\ker(\mu_b^1)}{\mathrm{im}(\mu_b^1)}.$$

An A_∞ algebra is *weakly unobstructed* if there exists a solution to the projective Maurer-Cartan equation.

We introduce a notion of gauge equivalence for solutions to the projective Maurer-Cartan equation, so that cohomology is invariant under gauge equivalence. For $b_0, \ldots, b_n \in A^+$ of odd degree define

$$\begin{aligned} &\mu^n_{b_0,b_1,\ldots,b_n}(a_1,\ldots,a_n) = \\ &\sum_{i_1,\ldots,i_{n+1}} \mu^{n+i_1+\ldots+i_{n+1}}(\underbrace{b_0,\ldots,b_0}_{i_1}, a_1, \underbrace{b_1,\ldots,b_1}_{i_2}, a_2, b_2, \ldots, b_2, \ldots, a_n, \underbrace{b_n,\ldots,b_n}_{i_{n+1}}). \end{aligned} \tag{4.52}$$

Two projective Maurer-Cartan solutions b_0, b_1 are *gauge equivalent* if and only if

$$\exists h \in A^+, \ b_1 - b_0 = \mu^1_{b_0,b_1}(h), \quad \deg(h) \text{ even}.$$

We then write $b_0 \sim_h b_1$.

Gauge equivalence is an equivalence relation, by a discussion parallel to that in Seidel [**120**, Section 1h]. To show transitivity, suppose that $b_0 \sim_{h_{01}} b_1$ and $b_1 \sim_{h_{12}} b_2$. Then $b_0 \sim_{h_{02}} b_2$ where

$$h_{02} = h_{01} + h_{12} - \mu^2_{b_0,b_1,b_2}(h_{01}, h_{12}).$$

Indeed, the first composition maps involving b_0, b_1, b_2 are related by

$$\mu^1_{b_0,b_1} = \mu^1_{b_0,b_2} - \mu^2_{b_0,b_1,b_2}(\cdot, b_2 - b_1), \quad \mu^1_{b_1,b_2} = \mu^1_{b_0,b_2} + \mu^2_{b_0,b_1,b_1}(b_1 - b_0, \cdot). \tag{4.53}$$

It follows that

$$\begin{aligned} \mu^1_{b_0,b_2}(h_{02}) &= \mu^1_{b_0,b_2}(h_{01} + h_{12} + \mu^2_{b_0,b_1,b_2}(h_{01},h_{12})) \\ &= \mu^1_{b_0,b_1}(h_{01}) + \mu^2_{b_0,b_1,b_2}(h_{01}, b_2 - b_1) + \mu^1_{b_1,b_2}(h_{12}) \\ &\quad - \mu^2_{b_0,b_1,b_2}(b_1 - b_0, h_{12}) + \mu^1_{b_0,b_2}(\mu^2_{b_0,b_1,b_2}(h_{01},h_{12}))) \\ &= (b_1 - b_0) + \mu^2_{b_0,b_1,b_2}(h_{01}, \mu^1_{b_1,b_2}(h_{12})) + (b_2 - b_1) \\ &\quad - \mu^2_{b_0,b_1,b_2}(\mu^1_{b_0,b_1}(h_{01}), h_{12}) + \mu^1_{b_0,b_2}(\mu^2_{b_0,b_1,b_2}(h_{01},h_{12})) \\ &= b_2 - b_0 \end{aligned}$$

where the cancellation of the terms involving $\mu^2_{b_0,b_1,b_2}$

$$\mu^2_{b_0,b_1,b_2}(h_{01}, \mu^1_{b_1,b_2}(h_{12})) - \mu^2_{b_0,b_1,b_2}(\mu^1_{b_0,b_1}(h_{01}), h_{12}) - \mu^1_{b_0,b_2}(\mu^2_{b_0,b_1,b_2}(h_{01}, h_{12}))$$

follows from the A_∞ axiom for the deformed maps (4.52). To prove symmetry suppose that $b_0 \sim_{h_{01}} b_1$. Define

$$\phi(h_{10}) = h_{10} - \mu^2_{b_1,b_0,b_1}(h_{10}, h_{01}) \quad \psi(h_{11}) = h_{11} + \mu^2_{b_1,b_1,b_0}(h_{01}, h_{11}).$$

The identities (4.53) imply that ϕ, ψ are chain maps: Indeed,

$$\begin{aligned}
\phi(\mu^1_{b_1,b_0}(h_{10})) &= \mu^1_{b_1,b_0}(h_{10}) - \mu^2_{b_1,b_0,b_1}(\mu^1_{b_1,b_0}(h_{10}), h_{01})) \\
&= \mu^1_{b_1,b_1}(h_{10}) + \mu^2_{b_1,b_0,b_1}(h_{10}, \mu^1_{b_0,b_1}(h_{01})) - \mu^2_{b_1,b_0,b_1}(\mu^1_{b_1,b_0}(h_{10}), h_{01})) \\
&= \mu^1_{b_1,b_1}(h_{10}) - \mu^1_{b_1,b_1}(\mu^2_{b_1,b_0,b_1}(h_{10}, h_{01})) \\
&= \mu^1_{b_1,b_1}(\phi(h_{10}))
\end{aligned}$$

and similarly for ψ. Since the $q = 0$ part of $\mu^2_{b_1,b_0,b_1}(\cdot, h_{01})$ resp. $\mu^2_{b_1,b_1,b_0}(h_{01}, \cdot)$ has negative degree, the maps ϕ, ψ are invertible. Furthermore,

$$\phi(b_0 - b_1) = \mu^1_{b_1,b_1}(h_{01}) \tag{4.54}$$

$$\psi(\mu^1_{b_1,b_1}(h_{01})) = \mu^1_{b_0,b_1}\mu^2_{b_0,b_1,b_1}(h_{01}, h_{01}) + b_0 - b_1. \tag{4.55}$$

Hence if we define

$$h_{10} := (\psi \circ \phi)^{-1}(-h_{01} - \mu^2_{b_0,b_1,b_1}(h_{01}, h_{01}))$$

then $b_1 \sim_{h_{10}} b_0$:

$$\begin{aligned}
\mu^1_{b_1,b_0}(h_{10}) &= \mu^1_{b_1,b_0}\phi^{-1}\psi^{-1}(-h_{01} - \mu^2_{b_0,b_1,b_1}(h_{01}, h_{01})) \\
&= \phi^{-1}\psi^{-1}\mu^1_{b_0,b_1}(-h_{01} - \mu^2_{b_0,b_1,b_1}(h_{01}, h_{01})) \\
&= \phi^{-1}\psi^{-1}(\mu^1_{b_1,b_1}\mu^2_{b_0,b_1,b_1}(h_{01}, h_{01}) + b_0 - b_1) \\
&= b_0 - b_1.
\end{aligned}$$

Also $b \sim_0 b$ for any b, hence $\sim$ is reflexive.

We define the potential of the algebra as a function on the moduli space of solutions to the projective Maurer-Cartan equation. Denote by $\overline{MC}(A)$ the quotient of $MC(A)$ by the gauge equivalence relation,

$$\overline{MC}(A) = MC(A)/\sim .$$

The quotient $\overline{MC}(A)$ is the *moduli space* of solutions to the projective Maurer-Cartan equation. Define a *potential*

$$W : MC(A) \to \Lambda$$

on the space of solutions to the projective Maurer-Cartan equation by

$$W(b)1_A = \mu^0_b(1).$$

We remark that W would be related to the potential $\mathcal{W}$ defined in (4.45) via the divisor equation, but the divisor equation in the perturbation system given here only holds in weak form (4.42).

Lemma 4.40. *The potential W is gauge-invariant and so descends to a potential $\overline{W} : \overline{MC}(A) \to \Lambda$.*

PROOF. Suppose $b_0 \sim_h b_1$ so $b_1 - b_0 = \mu^1_{b_0,b_1}(h)$. We have

$$\begin{aligned}
\mu^0_{b_1}(1) - \mu^0_{b_0}(1) &= \sum_{i,j} \mu^{i+j+1}(\underbrace{b_0,\ldots,b_0}_{i}, b_1 - b_0, \underbrace{b_1,\ldots,b_1}_{j}) \\
&= \sum_{i,j} \mu^{i+j+1}(\underbrace{b_0,\ldots,b_0}_{i}, \mu^1_{b_0,b_1}(h), \underbrace{b_1,\ldots,b_1}_{j}) \\
&= \sum_{i,j,k} \mu^{i+j+k+2}(\underbrace{b_0,\ldots,b_0}_{i}, \mu^0_{b_0}(1), \underbrace{b_0,\ldots,b_0}_{j}, h, \underbrace{b_1,\ldots,b_1}_{k}) \\
&- \sum_{i,j,k} \mu^{i+j+k+1}(\underbrace{b_0,\ldots,b_0}_{i}, h, \underbrace{b_1,\ldots,b_1}_{j}, \mu^0_{b_1}(1), \underbrace{b_1,\ldots,b_1}_{k}) \\
&= \mu^2(W(b_0)e_A, h) - \mu^2(h, W(b_1)e_A) \\
&= (W(b_0) - W(b_1))h.
\end{aligned}$$

It follows that $(e_A + h)W(b_1) = (e_A + h)W(b_0)$. Since $h \in A^+$ the sum $e_A + h$ is non-zero. Hence $W(b_1) = W(b_0)$ as claimed. □

COROLLARY 4.41. *If $b_0 \sim_h b_1$, then $\mu^1_{b_0,b_1}$ is a differential.*

PROOF. Using the A_∞ relations and strict unitality we have

$$\begin{aligned}
(\mu^1_{b_0,b_1})^2(a) &= \mu^2(\mu^0_{b_0}(1), a) - (-1)^{|a|}\mu^2(a, \mu^0_{b_1}(1)) \\
&= (W(b_1) - W(b_0))a = 0. \quad \square
\end{aligned}$$

The cohomology of a curve A_∞ algebra is a collection of vector spaces over the space of solutions to the projective Maurer-Cartan equation For any $b \in MC(A)$ define

$$H(b) := H(\mu^1_b) = \frac{\ker(\mu^1_b)}{\operatorname{im}(\mu^1_b)}.$$

The *cohomology complex* is the resulting complex of sheaves over $MC(A)$:

$$A \times MC(A) \to A \times MC(A), \quad (a, b) \mapsto (\mu^b_1(a), b) \tag{4.56}$$

The "stalks" of this complex fit together to the cohomology

$$H(A) := \bigcup_{b \in MC(A)} H(b). \tag{4.57}$$

LEMMA 4.42. *The cohomology $H(A)$ is gauge-invariant in the sense that if $b_0 \sim_{h_{10}} b_1$, then $H(b_0) \cong H(b_1)$.*

PROOF. One can verify that

$$\mu^2_{b_1,b_0,b_0}(h_{10}, a) = \sum_{n_1,n_2,n_3} \mu^{2+n_1+n_2+n_3}(\underbrace{b_1,\ldots,b_1}_{n_1}, h_{10}, \underbrace{b_0,\ldots,b_0}_{n_2}, a, \underbrace{b_0,\ldots,b_0}_{n_3})$$

satisfies

$$\mu^2_{b_1,b_0,b_0}(h_{10}, \mu^1_{b_0}(a)) - \mu^1_{b_1,b_0}(\mu^2_{b_1,b_0,b_0}(h_{10}, a)) = \mu^1_{b_0}(a) - \mu^1_{b_1,b_0}(a).$$

Hence the operator

$$\mu^2_{b_1,b_0,b_0}(h_{10}, \_) - \operatorname{Id}(\_) : (A, \mu^1_{b_0}) \to (A, \mu^1_{b_1,b_0}),$$

is a chain morphism, see (4.54). For the same reasons,

$$\mu^2_{b_0,b_1,b_0}(h_{10},\_) - \mathrm{Id}(\_) : (A, \mu^1_{b_1,b_0}) \to (A, \mu^1_{b_0}).$$

is a chain morphism. Consider the map

$$A \to A, \quad a \mapsto H_1^{b_0,b_1,b_0,b_0}(h_{01}, h_{10}, a)$$

where

$$H_1^{b_0,b_1,b_0,b_0}(h_{01}, h_{10}, a) := \sum_{n_1,n_2,n_3,n_4} \mu^{3+n_1+n_2+n_3+n_4}(\underbrace{b_0,\dots,b_0}_{n_1}, h_{01}, \underbrace{b_1,\dots,b_1}_{n_2}, h_{10}, \underbrace{b_0,\dots,b_0}_{n_3}, a, \underbrace{b_0,\dots,b_0}_{n_4}). \tag{4.58}$$

This map is a chain homotopy between

$$(\mu^2_{b_0,b_1,b_0}(h_{10},\_) - \mathrm{Id}(\_)) \circ (\mu^2_{b_1,b_0,b_0}(h_{10},\_) - \mathrm{Id}(\_))$$

and the chain map

$$\Phi(a) = a + \mu^2_{b_0}(h_{01} + h_{10} + \mu^2_{b_0,b_1,b_0}(h_{01}, h_{10}), a).$$

The latter can be seen to induce an automorphism of $H(b_0)$. In the same way, the reverse composition of these maps is homotopic to a map inducing an automorphism of

$$H(b_1, b_0) := H(A, \mu^1_{b_1,b_0}).$$

Similarly, $\mu_2^{b_1,b_1,b_0}(\_, h_{10}) - \mathrm{Id}(\_)$ and $\mu_2^{b_1,b_0,b_1}(\_, h_{01}) - \mathrm{Id}(\_)$ define a chain morphism from $(A, \mu^1_{b_1})$ to $(A, \mu^1_{b_1,b_0})$ and $(A, \mu^1_{b_1,b_0})$ to $(A, \mu^1_{b_1})$, respectively. This shows that $H(b_1, b_0) \cong H(b_1)$. Hence $H(b_0) \cong H(b_1, b_0) \cong H(b_0)$ as claimed. □

We consider the special case of the projective Maurer-Cartan solutions for a Fukaya algebra. The following Lemma will be used in the proof of Theorem 7.22 below to show that Lagrangians are weakly unobstructed.

LEMMA 4.43. *Suppose that $\mu^0(1) \in \Lambda x^{\blacktriangledown}$ and every non-constant disk has positive Maslov index. Then $MC(L)$ is non-empty.*

PROOF. Suppose $\mu^0(1) = W x^{\blacktriangledown}$ and the condition in the Lemma holds. Equation (4.36) becomes $\mu^1(x^{\triangledown}) = x^{\triangledown} - x^{\blacktriangledown}$. Hence with notation as in (4.1) we have

$$\mu(W x^{\triangledown}) = \mu^0(1) + W\mu^1(x^{\triangledown}) = W x^{\blacktriangledown} + W(x^{\triangledown} - x^{\blacktriangledown}) = W x^{\triangledown} \in \Lambda x^{\triangledown}.$$

Thus $W x^{\triangledown} \in MC(L)$. □

We also mention that the addition of homotopy units does not affect the Floer cohomology:

LEMMA 4.44. *Suppose that $b \in MC(L)$. Then $HF(L, b)$ is isomorphic to the cohomology $HF^{\mathrm{geom}}(L, b)$ of μ^1_b acting on $CF^{\mathrm{geom}}(L)$.*

PROOF. Choose $\hbar$ so that the first term in μ^1_b with non-vanishing q-valuation has valuation at least $\hbar$. Consider the first page in the spectral sequence for $CF(L)$ induced by the filtration $q^{n\hbar}CF(L)$, $n \in \mathbb{Z}_{\geq 0}$. The differential $\mu^1_{b,1}$ on the first page E^1 of this spectral sequence comes from trajectories with no disks, and in particular (4.36) becomes $\mu^1_{b,1}(x^{\triangledown}) = x^{\triangledown} - x^{\blacktriangledown}$. It follows that $H(\mu^1_{b,1})$ is the cohomology of the Morse differential $\mu^1_b|_{q=0}$ on $CF^{\mathrm{geom}}(L)$. The claim follows. □

CHAPTER 5

Homotopy invariance

In this and following sections we show that the Fukaya algebra constructed above is independent, up to A_∞ homotopy invariance, of the choice of perturbation system. The argument uses moduli spaces of *quilted treed disks*, introduced without trees in Ma'u-Woodward [**92**].

5.1. A_∞ morphisms

Recall the definition of A_∞ morphisms and homotopies.

Definition 5.1. (a) (A_∞ morphisms) Let A_0, A_1 be A_∞ algebras. An A_∞ *morphism* $\mathcal{F}$ from A_0 to A_1 consists of a sequence of linear maps

$$\mathcal{F}^d: \; A_0^{\otimes d} \to A_1[1-d], \quad d \geq 0$$

such that the following holds:

(5.1)
$$\sum_{i+j\leq d} (-1)^{i+\sum_{j=1}^{i}|a_j|} \mathcal{F}^{d-j+1}(a_1,\ldots,a_i,\mu^j_{A_0}(a_{i+1},\ldots,a_{i+j}),a_{i+j+1},\ldots,a_d) = \sum_{i_1+\ldots+i_m=d} \mu^m_{A_1}(\mathcal{F}^{i_1}(a_1,\ldots,a_{i_1}),\ldots,\mathcal{F}^{i_m}(a_{i_1+\ldots+i_{m-1}+1},\ldots,a_d))$$

where the first sum is over integers i, j with $i + j \leq d$, the second is over partitions $d = i_1 + \ldots + i_m$. An A_∞ morphism $\mathcal{F}$ is *unital* if and only if

$$\mathcal{F}^1(e_0) = e_1, \quad \mathcal{F}^k(a_1,\ldots,a_i,e_0,a_{i+2},\ldots,a_k) = 0$$

for every $k \geq 2$ and every $0 \leq i \leq k-1$, where e_0 resp. e_1 is the strict unit in A_0 resp. A_1.

(b) (Composition of A_∞ morphisms) The *composition* of A_∞ morphisms $\mathcal{F}_0, \mathcal{F}_1$ is defined by

$$(5.2) \quad (\mathcal{F}_0 \circ \mathcal{F}_1)^d(a_1,\ldots,a_d) = \sum_{i_1+\ldots+i_m=d} \mathcal{F}_0^m(\mathcal{F}_1^{i_1}(a_1,\ldots,a_{i_1}), \ldots,\mathcal{F}_1^{i_m}(a_{d-i_m+1},\ldots,a_d)).$$

(c) (A_∞ natural transformations) Let $\mathcal{F}_0, \mathcal{F}_1 : A_0 \to A_1$ be A_∞ morphisms. A *pre-natural transformation* $\mathcal{T}$ from $\mathcal{F}_0$ to $\mathcal{F}_1$ consists of for each $d \geq 0$ a multilinear map

$$\mathcal{T}^d: \; A_0^d \to A_1.$$

Let $\mathrm{Hom}(\mathcal{F}_0, \mathcal{F}_1)$ denote the space of pre-natural transformations from $\mathcal{F}_0$ to $\mathcal{F}_1$. Define a differential on $\mathrm{Hom}(\mathcal{F}_0, \mathcal{F}_1)$ by

(5.3)
$$(\mu^1_{\mathrm{Hom}(\mathcal{F}_0,\mathcal{F}_1)}\mathcal{T})^d(a_1,\ldots,a_d) = \sum_{k,m}\sum_{i_1,\ldots,i_m}(-1)^{\dagger}\mu^m_{A_2}(\mathcal{F}_0^{i_1}(a_1,\ldots,a_{i_1}),\mathcal{F}_0^{i_2}(a_{i_1+1},\ldots,a_{i_1+i_2}),\ldots,$$
$$\mathcal{T}^{i_k}(a_{i_1+\ldots+i_{k-1}+1},\ldots,a_{i_1+\ldots+i_k}),\mathcal{F}_1^{i_{k+1}}(a_{i_1+\ldots+i_k+1},\ldots,),\ldots,\mathcal{F}_1^{i_m}(a_{d-i_m},\ldots,a_d))$$
$$-\sum_{i,e}(-1)^{i+\sum_{j=1}^i|a_j|+|\mathcal{T}|-1}\mathcal{T}^{d-e+1}(a_1,\ldots,a_i,\mu^e_{A_1}(a_{i+1},\ldots,a_{i+e}),a_{i+e+1},\ldots,a_d)$$

where

$$\dagger = (|\mathcal{T}|-1)(|a_1|+\ldots+|a_{i_1+\ldots+i_{k-1}}|-i_1-\ldots-i_{k-1}).$$

A *natural transformation* is a closed pre-natural transformation. An A_∞ (pre)natural transformation $\mathcal{T}$ from a unital morphism $\mathcal{F}_0$ to a unital morphism $\mathcal{F}_1$ is *unital* if and only if

$$\mathcal{T}^k(a_1,\ldots,a_i,e_0,a_{i+2},\ldots,a_k) = 0 \quad \forall k\geq 1,\ 0\leq i\leq k-1$$

where e_0 is the unit in A_0.

(d) (Composition of natural transformations) Given two pre-natural transformations

$$\mathcal{T}_1 : \mathcal{F}_0 \to \mathcal{F}_1, \quad \mathcal{T}_2 : \mathcal{F}_1 \to \mathcal{F}_2$$

define $\mu^2(\mathcal{T}_1, \mathcal{T}_2)$ by

(5.4)
$$(\mu^2(\mathcal{T}_1,\mathcal{T}_2))^d(a_1,\ldots,a_d) = \sum_{m,k,l}\sum_{i_1,\ldots,i_m}(-1)^{\ddagger}\mu^m_{A_2}(\mathcal{F}_0^{i_1}(a_1,\ldots,a_{i_1}),\ldots,\mathcal{F}_0^{i_{k-1}}(\ldots),$$
$$\mathcal{T}_1^{i_k}(a_{i_1+\ldots+i_{k-1}+1},\ldots,a_{i_1+\ldots+i_k}),\mathcal{F}_1^{i_{k+1}}(\ldots),\ldots,\mathcal{F}_1^{i_{l-1}}(\ldots),$$
$$\mathcal{T}_2^{i_l}(a_{i_1+\ldots+i_{l-1}+1},\ldots,a_{i_1+\ldots+i_l}),\mathcal{F}_2^{i_{l+1}}(\ldots),\ldots,\mathcal{F}_2^{i_m}(a_{d-i_m},\ldots,a_d))$$

where

$$\ddagger = \sum_{i=1}^{i_1+\ldots+i_{k-1}}(|\mathcal{T}_1|-1)(|a_i|-1) + \sum_{i=1}^{i_1+\ldots+i_{l-1}}(|\mathcal{T}_2|-1)(|a_i|-1).$$

Let $\mathrm{Hom}(A_0, A_1)$ denote the space of A_∞ morphisms from A_0 to A_1, with morphisms given by pre-natural transformations. Compositions give $\mathrm{Hom}(A_0, A_1)$ the structure of an A_∞ category [**43**, 10.17], [**77**, 8.1], [**120**, Section 1d].

(e) (A_∞ homotopies) Suppose that $\mathcal{F}_1, \mathcal{F}_2 : A_0 \to A_1$ are morphisms. A *homotopy* from $\mathcal{F}_1$ to $\mathcal{F}_2$ is a pre-natural transformation $\mathcal{T} \in \mathrm{Hom}(\mathcal{F}_1, \mathcal{F}_2)$ such that

$$\mathcal{F}_1 - \mathcal{F}_2 = \mu^1_{\mathrm{Hom}(\mathcal{F}_1,\mathcal{F}_2)}(\mathcal{T}). \tag{5.5}$$

where $\mu^1_{\mathrm{Hom}(\mathcal{F}_1,\mathcal{F}_2)}(\mathcal{T})$ is defined in (5.3). If a homotopy exists we say that $\mathcal{F}_1$ is homotopic to $\mathcal{F}_2$ and write $\mathcal{F}_1 \equiv \mathcal{F}_2$. As shown in [**120**, Section 1h], homotopy of A_∞ morphisms is an equivalence relation.

(f) (Composition of homotopies) Given homotopies $\mathcal{T}_1$ from $\mathcal{F}_0$ to $\mathcal{F}_1$, and $\mathcal{T}_2$ from $\mathcal{F}_1$ to $\mathcal{F}_2$, the sum

$$\mathcal{T}_2 \circ \mathcal{T}_1 := \mathcal{T}_1 + \mathcal{T}_2 + \mu^2(\mathcal{T}_1, \mathcal{T}_2) \in \mathrm{Hom}(\mathcal{F}_0, \mathcal{F}_2) \tag{5.6}$$

is a homotopy from $\mathcal{F}_0$ to $\mathcal{F}_2$.

(g) (Composition of morphisms) Let A_0, A_1, A_2 be A_∞ algebras. Given a morphism $\mathcal{F}_{12} : A_1 \to A_2$ resp. $\mathcal{F}_{01} : A_0 \to A_1$, right composition with $\mathcal{F}_{12}$ resp. left composition with $\mathcal{F}_{01}$ define A_∞ morphisms

$$\begin{aligned} \mathcal{R}_{\mathcal{F}_{12}} &: \mathrm{Hom}(A_0, A_1) \to \mathrm{Hom}(A_0, A_2) \\ \mathcal{L}_{\mathcal{F}_{01}} &: \mathrm{Hom}(A_0, A_2) \to \mathrm{Hom}(A_1, A_2). \end{aligned}$$

The action on pre-natural transformations is given as follows [**120**, Section 1e]: Let $\mathcal{F}_{01}', \mathcal{F}_{01}'' : A_0 \to A_1$ be A_∞ morphisms and T_{01} a pre-natural transformation from $\mathcal{F}_{01}'$ to $\mathcal{F}_{01}''$. Define

$$\begin{aligned} &(\mathcal{R}_{\mathcal{F}_{12}}(T_{01}))_d(a_1, \ldots, a_d) \\ &= \sum_{r,j} \sum_{i_1 + \ldots + i_r = d} (-1)^{\ddagger} \mathcal{F}_{12}(\mathcal{F}_{01}'(a_1, \ldots, a_{i_1}), \ldots, \mathcal{F}_{01}'(\ldots), \\ &\qquad T_{01}(a_{i_1+\ldots,+i_{j-1}+1}, \ldots, a_{i_1+\ldots+i_j}), \mathcal{F}_{01}''(\ldots), \\ &\qquad\qquad \ldots, \mathcal{F}_{01}''(a_{i_1+\ldots+i_{r-1}+1}, \ldots, a_d)). \end{aligned} \tag{5.7}$$

(h) (Homotopy equivalence of A_∞ algebras) Two A_∞ algebras A_0, A_1 are *homotopy equivalent* if there exist morphisms $\mathcal{F}_{01} : A_0 \to A_1$ and $\mathcal{F}_{10} : A_1 \to A_0$ such that $\mathcal{F}_{01} \circ \mathcal{F}_{10}$ and $\mathcal{F}_{10} \circ \mathcal{F}_{01}$ are homotopic to the respective identities. Homotopy equivalence of A_∞ algebras is an equivalence relation: Symmetry and reflexivity are immediate. For transitivity note that by the previous item, if $\mathcal{F}_{01}'$ and $\mathcal{F}_{01}''$ are homotopy equivalent then so are $\mathcal{F}_{12} \circ \mathcal{F}_{01}'$ and $\mathcal{F}_{12} \circ \mathcal{F}_{01}''$; repeating the argument for left composition, if $\mathcal{F}_{01}'$ and $\mathcal{F}_{01}''$ are homotopic and also $\mathcal{F}_{12}'$ and $\mathcal{F}_{12}''$ then $\mathcal{F}_{12}' \circ \mathcal{F}_{01}'$ is homotopic to $\mathcal{F}_{12}'' \circ \mathcal{F}_{01}''$. Hence if

$$\mathcal{F}_{01} : A_0 \to A_1, \mathcal{F}_{10} : A_1 \to A_0, \quad \mathcal{F}_{12} : A_1 \to A_2, \mathcal{F}_{21} : A_2 \to A_1$$

are homotopy equivalences then

$$\begin{aligned} (F_{10} \circ \mathcal{F}_{21}) \circ (\mathcal{F}_{12} \circ \mathcal{F}_{01}) &= \mathcal{F}_{10} \circ (\mathcal{F}_{21} \circ \mathcal{F}_{12}) \circ \mathcal{F}_{01} \\ &\cong \mathcal{F}_{10} \circ \mathcal{F}_{01} \cong \mathrm{Id} \end{aligned}$$

and similarly for $(\mathcal{F}_{12} \circ \mathcal{F}_{01}) \circ (F_{10} \circ \mathcal{F}_{21})$.

The following is a version of the material on Maurer-Cartan moduli spaces in [**46**, Chapter 4]. Let A_0, A_1 be convergent A_∞ algebras. Let $\mathcal{F} : A_0 \to A_1$ be an A_∞ morphism or pre-natural transformation. Such $\mathcal{F}$ is *convergent* if and only if the zero-th term $\mathcal{F}^0(1)$ lies in A^+.

Lemma 5.2. (Map between Maurer-Cartan moduli spaces) *Suppose that A_0, A_1 are convergent strictly-unital A_∞ algebras and $\mathcal{F} : A_0 \to A_1$ is a convergent unital A_∞ morphism. Then*

$$\phi_{\mathcal{F}}(b) = \mathcal{F}^0(1) + \mathcal{F}^1(b) + \mathcal{F}^2(b,b) + \mathcal{F}^3(b,b,b) + \ldots \tag{5.8}$$

defines a map from A_0^+ to A_1^+ and restricts to a map $MC(A_0) \to MC(A_1)$ of the moduli spaces of solutions to the projective Maurer-Cartan equation and descends to *a map* $\overline{MC}(A_0)$ *to* $\overline{MC}(A_1)$*. That is: For every* $b \in MC(A_0)$*,* $\phi_{\mathcal{F}}(b) \in MC(A_1)$*; and*

$$(b \sim b') \implies (\phi_{\mathcal{F}}(b) \sim \phi_{\mathcal{F}}(b')), \quad \forall b, b' \in MC(A_0).$$

Moreover, if $\mathcal{F}_0, \mathcal{F}_1 : A_0 \to A_1$ *are unital* A_∞ *morphisms that are homotopic by an* A_∞ *homotopy* $\mathcal{T} : A_0 \to A_1$ *then* $\mathcal{F}_0, \mathcal{F}_1$ *induce the same map on Maurer-Cartan moduli spaces, that is,* $\phi_{\mathcal{F}_0} = \phi_{\mathcal{F}_1}$*. In particular, if* $\mathcal{F} : A \to A$ *is convergent and convergent-homotopic to the identity then* $\mathcal{F}$ *induces the identity on* $\overline{MC}(A)$*. Hence the set* $\overline{MC}(A)$ *is an invariant of the homotopy type of* A*.*

Proof. The proof that the sum (5.8) converges is essentially the same as that for Lemma 4.39 and left to the reader. Regarding gauge invariance, first notice that for every $b \sim b' \in MC(A_0)$ with $h \in A_0$ such that $b - b' = \mu^1_{b,b'}(h)$,

$$\begin{aligned}
\phi_{\mathcal{F}}(b) - \phi_{\mathcal{F}}(b') &= \sum_{n_0,n_1} \mathcal{F}^{n_0+n_1+1}(\underbrace{b,\dots,b}_{n_0}, b - b', \underbrace{b',\dots,b'}_{n_1}) \\
&= \sum_{n_0,n_1,n_2,n_3} \mathcal{F}^{n_0+n_1+1}(\underbrace{b,\dots,b}_{n_0}, \mu^{n_2+n_3+1}(\underbrace{b,\dots,b}_{n_2}, h, \\
&\qquad \underbrace{b',\dots,b'}_{n_3}), \underbrace{b',\dots,b'}_{n_1}) \\
&= \mu^1_{\phi_{\mathcal{F}}(b),\phi_{\mathcal{F}}(b')} \left(\sum_{n_4,n_5} \mathcal{F}^{n_4+n_5+1}(\underbrace{b,\dots,b}_{n_4}, h, \underbrace{b',\dots,b'}_{n_5}) \right)
\end{aligned}$$

so that $\phi_{\mathcal{F}}(b) \sim \phi_{\mathcal{F}}(b')$. Note that the third equality above uses the fact that $b, b' \in MC(A_0)$ and the unitality of $\mathcal{F}$.

We now prove the second part of the claim. Suppose that $\mathcal{F}_0 - \mathcal{F}_1 = \mu^1_{\mathrm{Hom}(\mathcal{F}_0,\mathcal{F}_1)}(\mathcal{T})$ for some unital pre-natural transformation $\mathcal{T}$. Then

$$\begin{aligned}
\phi_{\mathcal{F}_0}(b) - \phi_{\mathcal{F}_1}(b) &= \sum_{k \geq 0} (\mu^1_{\mathrm{Hom}(\mathcal{F}_0,\mathcal{F}_1)}\mathcal{T})^k(b,\dots,b) \\
&= \sum_{k\geq 0}\sum_{m}\sum_{i_1+\ldots+i_m=k} (-1)^{\dagger}\mu^m_{A_2}(\mathcal{F}_0^{i_1}(b,\dots,b), \mathcal{F}_0^{i_2}(b,\dots,b),\dots, \\
&\qquad \mathcal{T}^{i_k}(b,\dots,b), \mathcal{F}_1^{i_{k+1}}(b,\dots,b),\dots,\mathcal{F}_1^{i_m}(b,\dots,b)) \\
&\qquad - \sum_{i,e} (-1)^{i+\sum_{j=1}^{i}|b|+|\mathcal{T}|-1}\mathcal{T}^{k-e+1}(\underbrace{b,\dots,b}_{i}, \mu^e_{A_1}(b,\dots,b), b,\dots,b) \\
&=: \clubsuit.
\end{aligned}$$

Using that b is a projective Maurer-Cartan solution we continue

$$\begin{aligned} \clubsuit \quad = & \sum_{k\geq 0}\sum_{m}\sum_{i_1+\ldots+i_m=k}(-1)^{\dagger}\mu^m_{A_2}(\mathcal{F}_0^{i_1}(b,\ldots,b),\mathcal{F}_0^{i_2}(b,\ldots,b),\ldots, \\ & \mathcal{T}^{i_k}(b,\ldots,b),\mathcal{F}_1^{i_{k+1}}(b,\ldots,b),\ldots,\mathcal{F}_1^{i_m}(b,\ldots,b)) \\ & -\sum_{i,e}(-1)^{i+\sum_{j=1}^{i}|b|+|\mathcal{T}|-1}\mathcal{T}^{k-e+1}(b,\ldots,b,\lambda e_{A_1},b,\ldots,b) \\ = & \sum_{k\geq 0}\sum_{m}\sum_{i_1+\ldots+i_m=k}(-1)^{\dagger}\mu^m_{A_2}(\mathcal{F}_0^{i_1}(b,\ldots,b),\mathcal{F}_0^{i_2}(b,\ldots,b),\ldots, \\ & \mathcal{T}^{i_k}(b,\ldots,b),\mathcal{F}_1^{i_{k+1}}(b,\ldots,b),\ldots,\mathcal{F}_1^{i_m}(b,\ldots,b)) \\ = & \mu^1_{\phi_{\mathcal{F}_0}(b),\phi_{\mathcal{F}_1}(b)}\left(\sum_k \mathcal{T}^k(b,\ldots,b)\right). \end{aligned}$$

Since $b\in A^+$ the sum

$$h_{01} := \sum_k \mathcal{T}^k(b,\ldots,b)$$

exists in A. Furthermore since $\mathcal{T}^0(1)\in A^+$ we have $h_{01}\in A^+$. Hence $\phi_{\mathcal{F}_0}(b)\sim_{h_{01}} \phi_{\mathcal{F}_1}(b)$ as claimed. □

Similarly one has a homotopy invariance property of the cohomology vector bundle introduced in (4.57):

LEMMA 5.3. (Maps of cohomology bundles) *Any convergent A_∞ morphism $\mathcal{F}: A_0\to A_1$ induces a morphism $H(\mathcal{F}): H(A_0)\to H(A_1)$, that is, a morphism $H(b)\to H(\mathcal{F}(b))$ for each $b\in MC(A_0)$. If $\mathcal{F}_0,\mathcal{F}_1: A_0\to A_1$ are convergent morphisms related by a convergent homotopy then $H(\mathcal{F}_0)$ is equal to $H(\mathcal{F}_1)$ up to gauge transformation. In particular, if there exist convergent A_∞ maps $\mathcal{F}_{01}: A_0\to A_1$ and $\mathcal{F}_{10}: A_1\to A_0$ such that $\mathcal{F}_{01}\circ\mathcal{F}_{10}$ and $\mathcal{F}_{10}\circ\mathcal{F}_{01}$ are homotopic to the identities via convergent homotopies then $H(A_0)$ is isomorphic to $H(A_1)$ up to gauge equivalence in the sense that*

$$H(b_0)\cong H(\phi_{\mathcal{F}_{01}}(b_0)),\quad H(b_1)\cong H(\phi_{\mathcal{F}_{10}}(b_1))$$

for any $b_0\in MC(A_0), b_1\in MC(A_1)$.

The proof is similar to that of Lemma 4.42 and omitted. Thus having a non-trivial cohomology is an invariant of the homotopy type of a convergent, strictly-unital A_∞ algebra.

5.2. Multiplihedra

The terms in the A_∞ morphism axiom correspond to codimension one cells in a cell complex called the multiplihedron introduced by Stasheff [**128**]. Stasheff's definition identifies the n-multiplihedron as the cell complex whose vertices correspond to expressions involving bracketings of formal variables $x_1,\ldots,x_n$, together with expressions $f(\cdot)$ so that every x_j is contained in an argument of some f. For example, the second multiplihedron is an interval with vertices $f(x_1)f(x_2)$ and $f(x_1x_2)$. A geometric realization of this polytope was given by Boardman-Vogt [**16**] in terms of what we will call *quilted (metric ribbon) trees.* A quilted metric ribbon tree is a ribbon metric tree

$$T=(\Gamma, e_0\in \mathrm{Edge}_{\circ,\to}(\Gamma), \ell:\mathrm{Edge}_{-}(\Gamma)\to[0,\infty])$$

(ribbon structure omitted from the notation) together with a subset

$$\mathrm{Vert}^1(\Gamma) \subset \mathrm{Vert}(\Gamma)$$

of *colored vertices*. This set is required to satisfy the condition that every simple path of edges $e_0, e_1, \ldots, e_k$ from the root e_0 to a leaf e_k meets precisely one colored vertex $v \in \mathrm{Vert}^1(\Gamma)$, and the metric ℓ is required to satisfy the following:

(Balanced lengths condition) For any two *colored vertices* $v_1, v_2 \in \mathrm{Vert}^1(\Gamma)$,

$$\sum_{e \in P_+(v_1,v_2)} \ell(e) = \sum_{e \in P_-(v_1,v_2)} \ell(e) \tag{5.9}$$

where $P(v_1, v_2)$ is the (finite length) oriented non-self-crossing path from v_1 to v_2 and $P_+(v_1, v_2)$ resp. $P_-(v_1, v_2)$ is the part of the path towards resp. away from the root edge, see Ma'u-Woodward [**92**].

The set of combinatorially finite resp. semi-infinite edges is denoted $\mathrm{Edge}_-(\Gamma)$ resp. $\mathrm{Edge}_\to(\Gamma)$; the latter are equipped with a labelling by integers $0, \ldots, n$. A quilted tree is *stable* if each colored vertex $v \in \mathrm{Vert}^1(\Gamma)$ has valence $|v|$ at least two and any non-colored vertex $v \in \mathrm{Vert}(\Gamma) - \mathrm{Vert}^1(\Gamma)$ has valence $|v|$ at least three. Broken quilted trees are defined as in the non-quilted case, but requiring that any simple path from the root of the broken tree to a leaf still meets only one colored vertex. There is a natural notion of convergence of quilted trees $T_i \to T_\infty$, in which edges e_i of Γ_i whose length $\ell(e_i)$ approaches zero are contracted and edges e_i whose lengths $\ell(e_i)$ go to infinity are replaced by broken edges.

A different realization of the multiplihedron is the moduli space of stable *quilted* disks in Ma'u-Woodward [**92**]. In this realization, one obtains Stasheff's cell structure on the multiplihedron naturally.

DEFINITION 5.4. (a) A *quilted disk* is a datum $(S, Q, x_0, \ldots, x_n \in \partial S)$ consisting of a marked complex disk $(S, x_0, \ldots, x_n \in \partial S)$ (the points are required to be in counterclockwise cyclic order) together with a circle *seam* $Q \subset S$ (here we take S to be a ball in the complex plane, so the notion of circle makes sense) tangent to the 0-th marking x_0. An *isomorphism* of quilted disks from $(S, Q, x_0, \ldots, x_n)$ to $(S', Q', x_0', \ldots, x_n')$ is an isomorphism of pseudoholomorphic disks $S \to S'$ mapping Q to Q' and $x_0, \ldots, x_n$ to $x_0', \ldots, x_n'$.

(b) An *affine structure* on a disk S with boundary point $z_0 \in \partial S$ is an isomorphism with a half-space $\phi : S - \{z_0\} \to \mathbb{H}$. Two affine structures $\phi, \phi' : S - \{z_0\} \to \mathbb{H}$ are considered equivalent if $\phi(z) = \phi'(z) + \zeta$ for some $\zeta \in \mathbb{R}$. A quilting is equivalent to an affine structure, by taking the quilting to be $Q = \{\mathrm{Im}(z) = 1\}$.

(c) In this context the notion of quilted disk admits a natural generalization to the notion of a *quilted sphere*: a marked sphere $(C, (z_0, \ldots, z_n))$ equipped with an isomorphism ϕ from $C - \{z_0\} \to \mathbb{C}$ to the affine line $\mathbb{C}$. Again, two such isomorphisms ϕ, ϕ' are considered equivalent if they differ by a translation: $\phi(z) = \phi'(z) + \zeta$ for some $\zeta \in \mathbb{C}$.

(d) The *combinatorial type* Γ of a quilted nodal marked disk $(S, Q, \underline{x})$ is defined similar to the combinatorial type of a nodal marked disk disk. The set of vertices $\mathrm{Vert}(\Gamma)$ has a distinguished subset $\mathrm{Vert}^1(\Gamma)$ of *colored vertices* corresponding to the quilted components. Thus the unique non-self-crossing path γ_e from the root edge e_0 of the tree to any leaf e is required to pass through exactly one colored vertex $v \in \mathrm{Vert}^1(\Gamma)$.

(e) A *nodal quilted disk* of type Γ is a union of disks $S_v, v \in \mathrm{Vert}_\circ(\Gamma) - \mathrm{Vert}_\circ^1(\Gamma)$, spheres $S_v, v \in \mathrm{Vert}_\bullet(\Gamma) - \mathrm{Vert}_\bullet^1(\Gamma)$, quilted disks $S_v, v \in \mathrm{Vert}_\circ^1(\Gamma)$, and quilted spheres $S_v, v \in \mathrm{Vert}_\bullet^1(\Gamma)$, with the property that along a non-self-crossing path of components $S_v, v \in \mathrm{Vert}(\Gamma)$ from the root edge e_0 to any other leaf e_i, a single colored vertex $v \in \mathrm{Vert}^1(\Gamma)$ occurs. A nodal quilted disk S is *stable* if there are no non-trivial automorphisms $\mathrm{Aut}(S) - \{1\}$, or equivalently, each component $S_v, v \in \mathrm{Vert}(\Gamma)$ has no automorphisms. In the case of sphere components $S_v, v \in \mathrm{Vert}_\bullet(\Gamma)$, this means that S_v has at least three special points if it is unquilted, or at least two special points if it is quilted.

The moduli space of stable quilted disks with interior and boundary markings is a compact cell complex. As the interior and boundary markings go to infinity, they bubble off onto either quilted disks or quilted spheres. The case of combined boundary and interior markings is a straight-forward generalization of the boundary and interior cases treated separately in [**92**].

There is a combined moduli space which includes both quilted disk, spheres, and possibly broken segments. A quilted treed disk C is obtained from a quilted nodal disk S by replacing each node $w(e) \in S, e \in \mathrm{Edge}(\Gamma(S))$ with a (possibly broken) segment T_e by attaching the endpoints of T_e to two copies of $w(e)$. The *combinatorial type* $\Gamma(C)$ of a quilted treed disk C is the combinatorial type $\Gamma(S)$ of the surface with the additional labellings of the number of breakings $b(e)$ of each edge T_e. A quilted treed disk is *stable* if the underlying quilted disk is stable. Let $\overline{\mathcal{M}}_{n,m,1}$ denote the moduli space of stable marked quilted treed disks $u : C \to X$ with n boundary leaves and m interior leaves. See Figure 5.1 for a picture of $\overline{\mathcal{M}}_{2,0,1}$. A picture of $\overline{\mathcal{M}}_{1,1,1}$ is shown in Figure 5.2. In the picture the quilted disks $S_v \subset S, v \in \mathrm{Vert}^1(\Gamma)$ are those with two shadings; while the ordinary disks $S_v, v \notin \mathrm{Vert}^1(\Gamma)$ have either light or dark shading depending on whether they can be connected to the zero-th edge without passing a colored vertex. The hashes on the line segments T_e indicate breakings. Any non-self-crossing path from the

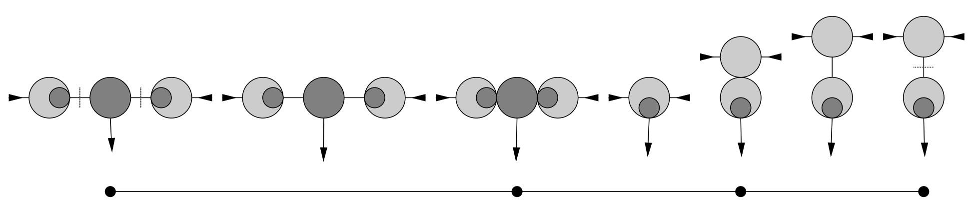

FIGURE 5.1. Moduli space of stable quilted treed disks

root edge e_0 to a leaf e must pass through exactly one colored vertex $v \in \mathrm{Vert}^1(\Gamma)$ correspond to either a quilted disk or quilted sphere $S_v \subset C$. See Figure 5.2 for the combinatorics of the top-dimensional cells in the case of one boundary leaf $e \in \mathrm{Edge}_{\circ,\to}(\Gamma)$ and one interior leaf $e \in \mathrm{Edge}_{\bullet,\to}(\Gamma)$; the s indicates a quilted sphere component $S_v \cong \mathbb{P}^1, v \in \mathrm{Vert}_\bullet(\Gamma)$. Orientations of the stratum $\mathcal{M}_\Gamma$ of the moduli space of quilted treed disks can be defined from the moduli space $\mathcal{M}_{\Gamma'}$ of unquilted disks with the same number of markings via the morphism $\mathcal{M}_\Gamma \to$

$\mathbb{R} \times \mathcal{M}_{\Gamma'}$ where the extra factor $\mathbb{R}$ prescribes the position of the seam in Definition 5.4; this orientation extends over the other top-dimensional cells as in the unquilted case.

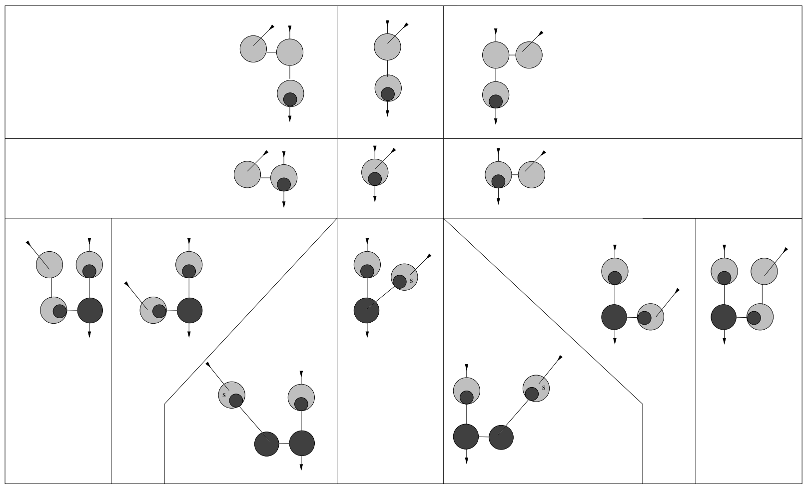

FIGURE 5.2. Moduli space of stable quilted treed disks with a boundary leaf and an interior leaf

The boundary of the moduli space of quilted treed disks is a union of moduli spaces involving quilted and unquilted disks with fewer markings. For any integers $n, m \geq 1$ we have

(5.10)
$$\partial\overline{\mathcal{M}}_{n,m,1} \cong \bigcup_{\substack{i,j \\ m_1+m_2=m}} (\overline{\mathcal{M}}_{n-i+1,m_1,1} \times \overline{\mathcal{M}}_{i,m_2}) \cup \bigcup_{\substack{i_1,\ldots,i_r \\ m_0+\sum m_j = m}} \left(\overline{\mathcal{M}}_{r,m_0} \times \prod_{j=1}^{r} \overline{\mathcal{M}}_{i_j,m_j,1} \right).$$

In the first sum j is the index of the attaching leaf in the quilted tree and so ranges from 1 to $n-i+1$, while $i_1, \ldots, i_r$ in the second union range over partitions $i_1 + \ldots + i_r = n$. By construction, for the facet of the first type, the sign of the inclusions of boundary strata are the same as that for the corresponding inclusion of boundary facets of $\overline{\mathcal{M}}_{n,m,1}$, that is, $(-1)^{i(n-i-j)+j}$. For facets of the second type, the gluing map $(0,\infty) \times \mathcal{M}_{r,m_0} \times \Pi_{j=1}^{r} \mathcal{M}_{|I_j|,m_j,1} \to \mathcal{M}_{n,m,1}$ is for boundary markings

$$(5.11) \quad (\delta, x_1, \ldots, x_r, (x_{1,j} = 0, x_{2,j}, \ldots, x_{|I_j|,j})_{j=1}^{r}) \mapsto$$
$$(x_1, x_1 + \delta^{-1}x_{2,1}, \ldots, x_1 + \delta^{-1}x_{|I_1|,1}, \ldots, x_r, x_r + \delta^{-1}x_{2,r}, \ldots, x_r + \delta^{-1}x_{|I_r|,r}).$$

This map changes orientations by $\sum_{j=1}^{r}(r-j)(|I_j|-1)$; in case of non-trivial weightings, $|I_j|$ should be replaced by the number of incoming markings or non-trivial weightings on the j-th component.

The combinatorial type of a quilted disk is the tree obtained as in the unquilted case. The resulting tree Γ has vertices $\mathrm{Vert}(\Gamma)$ equipped with a distinguished subset $\mathrm{Vert}^1(\Gamma)$ of colored vertices v corresponding to quilted components $S_v \subset C$. Morphisms of quilted trees (Cutting infinite length edges, collapsing edges, making

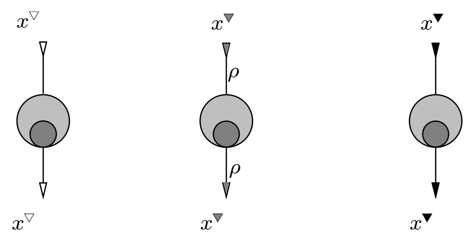

FIGURE 5.3. Unmarked stable treed quilted disks

lengths/weights finite/non-zero, and forgetting tails) induce morphisms of moduli spaces of stable quilted treed disks as in the unquilted case. In the special case of cutting an edge e of infinite length $\ell(e) = \infty$, one of the pieces Γ_1 will be a (possibly disconnected) quilted type and the other Γ_2 an unquilted type. The uncolored vertices $v \notin \mathrm{Vert}^1(\Gamma)$ admit a $\{0,1\}$-labelling $\mathrm{Vert}(\Gamma) - \mathrm{Vert}^1(\Gamma) \to \{0,1\}$ with value 0 resp. 1 if component is further away from the root resp. closer to the root edge $e \subset \mathrm{Edge}_{\to}(\Gamma)$ than the quilted components with respect to any non-self-crossing path of components. For any combinatorial type Γ of quilted disk there is a *universal quilted treed disk* $\overline{\mathcal{U}}_\Gamma \to \overline{\mathcal{M}}_\Gamma$ which is a cell complex whose fiber over C is isomorphic to C, and splits into surface and tree parts $\overline{\mathcal{U}}_\Gamma = \overline{\mathcal{S}}_\Gamma \cup \overline{\mathcal{T}}_{\circ,\Gamma} \cup \overline{\mathcal{T}}_{\bullet,\Gamma}$, where the last two sets are the boundary and interior parts of the tree respectively. Labels and weights on the semi-infinite ends $e \in \mathrm{Edge}_{\to}(\Gamma)$ of types Γ of quilted tree disks are defined as in the case of treed disks. We suppose there is a partition of the boundary semi-infinite edges

$$\mathrm{Edge}^{\blacktriangledown}(\Gamma) \sqcup \mathrm{Edge}^{\triangledown}(\Gamma) \sqcup \mathrm{Edge}^{\blacktriangledown}(\Gamma) = \mathrm{Edge}_{\circ,\to}(\Gamma)$$

into *weighted* resp. *forgettable* resp. *unforgettable* edges as in the unquilted case, except that now the root of the quilted tree with one weighted leaf and no marking is weighted with the same weight as the leaf, see Figure 5.3. The moduli space with a single weighted leaf $e \in \mathrm{Edge}_{\to}(\Gamma), \rho(e) \in (0,\infty)$ and no markings is then a point.

5.3. Quilted pseudoholomorphic disks

In the next few sections we prove invariance of the homotopy type of the Fukaya algebra under a change of divisors of the same degree, built from homotopic sections of the same line bundle; the general case is treated in Chapter 5.6 below. We introduce the following notation:

(a) (Paths of divisors) Suppose that $D^0, D^1 \subset X$ are stabilizing divisors for L built from homotopic unitary sections of $\tilde{X}$ of the same degree that are asymptotically holomorphic with respect to compatible almost complex structures $J^0, J^1 \in \mathcal{J}(X)$. Choose a path $J^t \in \mathcal{J}(X)$ from J^0 to J^1. By Lemma 4.26 above, there exists a path of J^t-stabilizing divisors $D^t \subset X, t \in [0,1]$ connecting D^0, D^1, and a path $J_{D^t} \in \mathcal{J}(X, D^t)$ of compatible almost complex structures such that D^t contains no J_{D^t}-holomorphic spheres.

(b) (Distance-to-seam function) In order to specify which divisor of the above family to use at a given point of a quilted domain, define a function as follows. For every point $z \in C$ of a quilted treed disk C, let

$$d(z) := \pm \sum_{e \in P(z)} \ell(e) \in [-\infty, \infty]$$

be the distance of z to the quilted components of C, that is, the sum of the lengths of the edges e in the path $P(z)$ between z and the quilted components times 1 resp. -1 if z is above resp. below the quilted components (that is, further from resp. closer to the root than the quilted components).

(c) (Perturbation morphisms) Given perturbation data $\underline{P}^0$ and $\underline{P}^1$ with respect to metrics $G^0, G^1 \in \mathcal{G}(L)$ over unquilted treed disks for D^0 resp. D^1, a *perturbation morphism* $\underline{P}^{01}$ from $\underline{P}^0$ to $\underline{P}^1$ for the quilted combinatorial type Γ consists of
   (i) a smooth function $\delta^{01}_\Gamma : [-\infty, \infty] \to [0,1]$ (to be composed with d)
   (ii) a smooth domain-dependent choice of metric
   $$G^{01}_\Gamma : \overline{\mathcal{T}}_{\circ,\Gamma} \to \mathbb{R}$$
   constant to G^0 resp. G^1 on the neighborhood of the points at infinity of the semi-infinite edges
   $$\overline{\mathcal{T}}_{\circ,\Gamma} - \overline{\mathcal{T}}^{\text{thick}}_{\circ,\Gamma}$$
   for which $d = -\infty$ resp. $d = \infty$,
   (iii) a domain-dependent Morse function
   $$F^{01}_\Gamma : \overline{\mathcal{T}}_{\circ,\Gamma} \to \mathbb{R}$$
   constant to F^0 resp. F^1 on the neighborhood $\overline{\mathcal{T}}_{\circ,\Gamma} - \overline{\mathcal{T}}^{\text{thick}}_{\circ,\Gamma}$ of the endpoints for which $d = -\infty$ resp. $d = \infty$ and equal to $F^0_{\Gamma_0}$ resp. $F^1_{\Gamma_1}$ on the (unquilted) treed disks components of type Γ_0, Γ_1 for which $d = -\infty$ resp. $d = \infty$, and
   (iv) a domain-dependent almost complex structure
   $$J^{01}_\Gamma : \overline{\mathcal{S}}_\Gamma \to \mathcal{J}_\tau(X)$$
   given as the product of maps pulled back from
   $$J^{01}_{\Gamma(v)} : \overline{\mathcal{S}}_{\Gamma(v)} \times X \to \operatorname{End}(TX)$$
   with the property that for any surface component C_i of C, J^{01}_Γ is
      (i) equal to the given J away from the compact part:
      $$J_\Gamma | \overline{\mathcal{S}}_\Gamma - \overline{\mathcal{S}}^{\text{thick}}_\Gamma = \pi_2^* J$$
      where π_2 is the projection on the second factor in (4.12) and
      (ii) equal to the complex structures $J^0_{\Gamma_0}$ resp. $J^1_{\Gamma_1}$ on the (unquilted) treed disks components of type Γ_0, Γ_1 for which $d = -\infty$ resp. $d = \infty$: Let $\iota_k : \overline{\mathcal{S}}_{\Gamma_k} \to \overline{\mathcal{S}}_\Gamma$ denote the inclusion of the unquilted components. Then we require
      $$J_\Gamma | \iota_k : \overline{\mathcal{S}}_{\Gamma_k} = J^k_{\Gamma_k}, \quad k \in \{0,1\}.$$

(d) (Quilt-independence) If the perturbations $\underline{P}^=(P^k_\Gamma)$ are equal one can also require the following independence property: A perturbation system for quilts $\underline{P}^{01} = (P^{01}_\Gamma = (F^{01}_\Gamma, J^{01}_\Gamma, G^{01}_\Gamma))$ is *quilt-independent* if G^{01}_Γ, F^{01}_Γ, and

J^{01}_Γ are pull-backs under the forgetful morphism forgetting the quilting $Q_v \subset S_v$ on the quilted disk components $S_v, v \in \mathrm{Vert}^1(\Gamma)$.

Pseudoholomorphic quilted disks with respect to domain-dependent structures are defined as follows. Let C be a stable quilted treed disk of type Γ. By the stability condition C may be identified with a fiber $\pi^{-1}[C]$ of the universal curve $\pi : \mathcal{U}_\Gamma \to \mathcal{M}_\Gamma$. By pullback we obtain a perturbation on C, still denoted $(\delta^{01}_\Gamma, J^{01}_\Gamma, F^{01}_\Gamma, G^{01}_\Gamma)$. If $C = S \cup T$ is obtained from a stable quilted treed disk C' by attaching an infinite segment $e \subset T$ at any of the semi-infinite ends of C'; choose perturbation data on C such that the Morse function F^{01}_Γ on C is constant on the infinite segment e. A *pseudoholomorphic quilted treed disk* $u : C \to X$ of combinatorial type Γ is a continuous map u from a quilted treed disk C that is smooth on each component $S_v, T_e \subset C$, J^{01}_Γ-holomorphic on the surface components $S_v, v \in \mathrm{Vert}(\Gamma)$, a $-\mathrm{grad}(F^{01}_\Gamma)$-Morse trajectory with respect to the metric G^{01}_Γ on each boundary tree segment $T_e \subset C, e \in \mathrm{Edge}_\circ(\Gamma)$, and constant on the tree segments of sphere type $T_e \subset C, e \in \mathrm{Edge}_\bullet(\Gamma)$. The interior leaves of the tree are irrelevant for our purposes; but they do affect the combinatorics of the boundary of the moduli spaces because of the balanced condition, see Figure 5.2. A stable holomorphic quilted tree disk is *adapted* if and only if (Stable surface axiom) the surface part is stable and (Leaf axiom) each interior leaf T_e maps to $D^{\delta^{01}_\Gamma \circ d(z_i)}$ and for each $t \in [0,1]$, each component of $u^{-1}(D^t) \cap (\delta^{01}_\Gamma)^{-1}(t)$ contains an interior leaf e. Two adapted weighted disks $u_0 : C_0 \to X, u_1 : C_1 \to X$ are *isomorphic* if there exists an isomorphism of weighted disks $\phi : C_0 \to C_1$ so that $u_0 \circ \phi = u_1$. Note that if C_0, C_1 have a single unmarked quilted disk component $S_{v,k}, v \in \mathrm{Vert}^1(\Gamma), k \in \{0,1\}$ and a single leaf $e_{1,k} \in \mathrm{Edge}_\to(\Gamma)$, the weightings $\rho(e_{1,0}), \rho(e_{1,1})$ are not required to be equal. Given a non-constant pseudoholomorphic quilted treed disk $u : C \to X$ with leaf $e_i \in \mathrm{Edge}^\blacktriangledown(\Gamma)$ on which there is a weighting $\rho(e_i) = 0$ resp. ∞, we declare $u : C \to X$ to be *equivalent* to the pseudoholomorphic quilted treed disk $u' : C' \to X$ obtained by replacing the label $x^{\blacktriangledown}$ at e_i by $x^{\blacktriangledown}$ resp. $x^{\triangledown}$ and adding a segment $C' = C \cup [-\infty, \infty)$ so that u is constant on $[-\infty, \infty)$, and so represents a trajectory from $x^{\blacktriangledown}$ to $x^{\blacktriangledown}$ resp. $x^{\triangledown}$ above that leaf as in the unquilted case in Figure 4.9. In other words, forgetting constant infinite segments and replacing them by the appropriate weights gives equivalent pseudoholomorphic treed disks. For any combinatorial type Γ of quilted disks we denote by $\overline{\mathcal{M}}_\Gamma(L, D)$ the compactified moduli space of equivalence classes of adapted quilted pseudoholomorphic treed disks.

The moduli space of adapted pseudoholomorphic quilted disks breaks into components depending on the limits along the root and leaf edges. Denote by $\mathcal{M}_\Gamma(L, D, \underline{x}) \subset \overline{\mathcal{M}}_\Gamma(L, D)$ the moduli space of isomorphism classes of adapted pseudoholomorphic quilted treed disks with boundary in L and limits $\underline{x}$ along the root and leaf edges, where $\underline{x} = (x_0, \ldots, x_n) \in \mathcal{I}(L)$ satisfies the requirement:

(a) (Label axiom)
   (i) (Incoming unit) If $x_0 = x^{\blacktriangledown}$ resp. $x_0 = x^{\triangledown}$, then there is a single leaf reaching $x^{\blacktriangledown}$ resp. $x^{\triangledown}$ and no interior marking (in which case the moduli space will be a point).
   (ii) (Outgoing unit) If $x_i = x^{\blacktriangledown}$ resp. $x^{\triangledown}$ resp. $x^{\blacktriangledown}$ for some $i \geq 1$ then the i-th leaf is required to be weighted resp. forgettable resp. unforgettable and the limit along this leaf is required to be x_M.
   (iii) (Non-units) If $x_i \notin \{x^{\blacktriangledown}, x^{\triangledown}, x^{\blacktriangledown}\}$, then the i-th leaf is required to be unforgettable.

(b) (Outgoing edge axiom) The outgoing edge e_0 is weighted (resp. forgettable) only if there is a single leaf e_1, which is weighted (resp. forgettable) with the same weight $\rho(e_0) = \rho(e_1)$ and the domain C has no interior leaves (so there is a single quilted disk $S \subset C$ with no interior leaves.)

In order to obtain moduli spaces with the expected boundary, we introduce a coherence condition. A collection $\underline{P}^{01} = (P^{01}_\Gamma)$ of perturbation morphisms is *coherent* if P^{01}_Γ is compatible with the morphisms of moduli spaces as before:

(a) (Collapsing edges/making an edge or weight finite or non-zero) If Γ' is obtained from Γ by collapsing an edge or edges, then P^{01}_Γ is the pullback of $P^{01}_{\Gamma'}$.

(b) (Cutting edges) If Γ' is obtained from Γ by cutting an edge or a collection of edges of infinite length, then P^{01}_Γ is the pushforward of $P^{01}_{\Gamma'}$. Suppose that Γ' is the union of a quilted type Γ_1 and a non-quilted type Γ_0, then $P^{01}_{\Gamma'}$ is obtained from $P^{01}_{\Gamma_1}$ and $P^0_{\Gamma_0}$ as follows: Let $\pi_k : \overline{\mathcal{M}}_{\Gamma'} \cong \overline{\mathcal{M}}_{\Gamma_1} \times \overline{\mathcal{M}}_{\Gamma_0} \to \overline{\mathcal{M}}_{\Gamma_k}$ denote the projection on the k-factor, so that $\overline{\mathcal{U}}_{\Gamma'} \cong \pi_1^*\overline{\mathcal{U}}_{\Gamma_1} \cup \pi_0^*\overline{\mathcal{U}}_{\Gamma_0}$. Then we require that $P_{\Gamma'}$ is equal to the pullback of $P^{01}_{\Gamma_1}$ on $\pi_1^*\overline{\mathcal{U}}_{\Gamma_1}$ and to the pullback of $P^0_{\Gamma_0}$ on $\pi_0^*\overline{\mathcal{U}}_{\Gamma_0}$.

Similarly, if Γ' is the union of a non-quilted type Γ_1 and quilted types $\cup_i\Gamma_{0,i}$ and quilted types with interior markings, then $P^{01}_{\Gamma'}$ is equal to the pullback of $P^1_{\Gamma_1}$ on $\pi_1^*\overline{\mathcal{U}}_{\Gamma_1}$ and to the pullback of $P^{01}_{\Gamma_{0,i}}$ on $\pi_{0,i}^*\overline{\mathcal{U}}_{\Gamma_{0,i}}$.

The case of constant quilted types requires special treatment. If any of the types $\Gamma_{0,i}$ have no interior markings and a single weighted leaf $e \in \text{Edge}(\Gamma), \rho(e) \in (0,\infty)$ then we label the corresponding leaf of Γ_1 with the same weight, by our (Cutting edges) construction. This guarantees that the moduli spaces are of expected dimension.

(c) (Locality axiom) For any spherical vertex v, P_Γ restricts to the pull-back of a perturbation $P_{\Gamma,v}$ on the image of $\pi^*\mathcal{U}_{\Gamma(v)}$ in $\mathcal{U}_\Gamma$. (Note that the perturbation $P_{\Gamma,v}$ is allowed to depend on Γ, not just $\Gamma(v)$.) Furthermore, the restriction of P_Γ to the disk components and boundary edges in $\mathcal{U}_\Gamma$ is the pull-back of a perturbation datam $P_{\Gamma_\circ}$ from $\mathcal{U}_{\Gamma_\circ}$.

(d) (Forgettable edges) Whenever some weight parameter $\rho(e)$ attached to a leaf $e \in \text{Edge}_\to(\Gamma)$ is equal to infinity, then the P^{01}_Γ is pulled back under the forgetful map $f : \mathcal{U}_\Gamma \to \mathcal{U}_{f(\Gamma)}$ forgetting the first leaf $e_i \in \text{Edge}^\triangledown_\to(\Gamma)$ and stabilizing from the perturbation morphism $P^{01}_{f(\Gamma)}$ given by (Forgetting tails).

The moduli space of pseudoholomorphic adapted quilted treed disks has compactness and transversality properties similar to those for unquilted disks. A perturbation morphism $\underline{P}^{01} = (P^{01}_\Gamma)$ is *stabilizing* if it satisfies a condition analogous to that in Definition (4.27): the divisors contain no non-constant pseudoholomorphic spheres and the intersection of any pseudoholomorphic sphere with the divisor contains at least three points. For a comeager subset of perturbation morphisms $\underline{P}^{01}$ extending those chosen for unquilted disks, the uncrowded moduli spaces $\mathcal{M}_\Gamma(L, D)$ of expected dimension at most one are smooth and of expected dimension. For sequential compactness, it suffices to consider a sequence $u_\nu : C_\nu \to X$ of quilted treed disks of fixed combinatorial type Γ_ν constant in ν. Coherence of the perturbation morphism implies the existence of a stable limit $u : C \to X$ which we claim is adapted. In particular, the (Leaf Property) is justified as follows. For each component $C_i \subset C$, the almost complex structure $J_\Gamma | C_i$ is constant near the almost complex submanifold $D_{\delta_\Gamma^{01} \circ d(C_i)}$. Suppose that C_i is a component of the limit of some sequence of components $C_{i,\nu}$ of C_ν. Coherence for the parameter δ_{01}^ν implies that $D_{\delta_\Gamma^{01} \circ d(C_i)}$ is the limit of the divisors $D_{\delta_{\Gamma_\nu}^{01} \circ d(C_{i,\nu})}$. Then local conservation of intersection degree[1] implies that any component of $u^{-1}(D_{\delta_\Gamma^{01} \circ d(C_i)})$ contains a limit point of some markings $z_{i,\nu} \in u_\nu^{-1}(D_{\delta_{\Gamma_\nu}^{01} \circ d(C_{i,\nu})})$. For types of index at most one, each component of $u^{-1}(D_{\delta_\Gamma^{01} \circ d(C_i)})$ is a limit of a unique component of $u_\nu^{-1}(D_{\delta_{\Gamma_\nu}^{01} \circ d(C_{i,\nu})})$, otherwise the intersection degree would be more than one which is a codimension two condition. Since non-trivial sphere bubbling is a codimension two condition and ghost bubbling is impossible unless two markings come together, this implies that $u^{-1}(D_{\delta_\Gamma^{01} \circ d(C_i)}) = \{z_i\}$ is also a marking. As before, we call a perturbation morphism $\underline{P}^{01}$ admissible if it is coherent, regular, and stabilizing.

The condition (d) implies the following properties of the moduli spaces of constant quilted disks. By the argument in the proof of Proposition 5.8, in the 0-dimensional strata all of the quilted disks are mapped to points. The 1-dimensional strata will be of two types: First, there are 1-dimensional families of quilted trajectories where every quilted disk is constant. Second, there may be 1-dimensional families for which only the quilting parameter on a single non-constant quilted disk varies from $-\infty$ to ∞.

5.4. Morphisms of Fukaya algebras

Morphisms of Fukaya algebras are defined using the moduli spaces of quilted disks in the previous section. Given an admissible perturbation morphism $\underline{P}^{01}$ from $\underline{P}^0$ to $\underline{P}^1$, define for each $n \geq 0$

$$\begin{aligned} \phi^n : CF(L; \underline{P}^0)^{\otimes n} &\to CF(L; \underline{P}^1) \\ (x_1, \ldots, x_n) &\mapsto \sum_{x_0, u \in \mathcal{M}_\Gamma(L, D, x_0, \ldots, x_n)_0} (-1)^{\heartsuit} \epsilon(u) (\sigma(u)!)^{-1} q^{E(u)} y(u) x_0 \end{aligned} \tag{5.12}$$

where the sum is over quilted disks in strata of dimension zero with $x_1, \ldots, x_n$ incoming labels.

Remark 5.5. (Lowest energy terms) For $x \in \mathcal{I}(L)$, the element $\phi^1(x^{\blacktriangledown})$ resp. $\phi^1(x^{\blacktriangledown})$ resp. $\phi^1(x^{\triangledown})$ has a term containing $x^{\blacktriangledown}$ resp. $x^{\blacktriangledown}$ resp. $x^{\triangledown}$ coming from the count of quilted treed disks $u : C \to X$ with no interior leaves. Such a configuration consists of a quilted treed disk $C = S \cup T$ with only one disk component $S \cong B$ that is quilted and mapped to a point $u(S) = \{x_M\}$.

Remark 5.6. The codimension one strata $\mathcal{M}_\Gamma(L, D)$ in the moduli spaces of pseudoholomorphic quilted treed disks $\overline{M}(L, D)$ are of several possible types: either there is one (or a collection of) edge $e \subset T$ of length $\ell(e)$ infinity, there is one (or a collection of) edge e of length $\ell(e)$ zero, or equivalently, boundary nodes, or there is an edge e with zero or infinite weight $\rho(e)$. The case of an edge of zero or infinite

[1] Since δ_{01}^Γ is constant on each disk or sphere the union

$$D^{\delta_{01}^\Gamma} = \cup_{z \in \mathcal{S}_\Gamma} \left(\{z\} \times D^{\delta_{01}^\Gamma \circ d(z)} \right)$$

is an almost complex submanifold of $\mathcal{S}_\Gamma \times X$. In particular, the intersection multiplicity of $u : C \to X$ with $D^{\delta_{01}^\Gamma}$ at z_i is positive.

weighting $\rho(e)$ is equivalent to breaking off a constant trajectory, and so may be ignored. In the case of edges of infinite length(s) $l(e)$, then either Γ is

(a) (Breaking off an uncolored tree) a pair $\Gamma_1 \sqcup \Gamma_2$ consisting of a colored tree Γ_1 and an uncolored tree Γ_2 as in Figure 5.4; necessarily the breaking must be a leaf of Γ_1; or

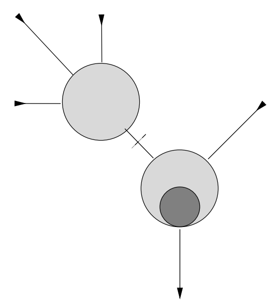

FIGURE 5.4. Breaking off an unquilted treed disk

(b) (Breaking off colored trees) a collection consisting of an uncolored tree Γ_0 containing the root e_0 and a collection $\Gamma_1, \ldots, \Gamma_r$ of colored trees attached to each of its r leaves as in Figure 5.5. Such a stratum $\mathcal{M}_\Gamma$ is codimension one because of the (Balanced Condition) which implies that if the length $\ell(e)$ of any edge e between e_0 to e_i is infinite for some i then the path from e_0 to e_i for any i has the same property.

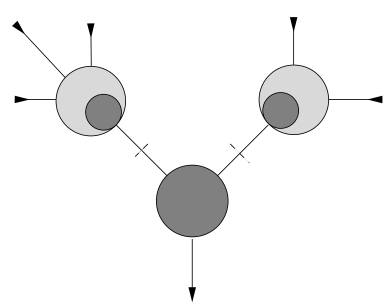

FIGURE 5.5. Breaking off a collection of quilted disks

In the case of a zero length(s), one obtains a fake boundary component with normal bundle $\mathbb{R}$, corresponding to either deforming the edge(s) e to have non-zero length $\ell(e)$ or deforming the node(s). This ends the Remark.

THEOREM 5.7. (A_∞ morphisms via quilted disks) *For any admissible collection* $\underline{P}^{01}$ *of perturbations morphisms from* $\underline{P}^0$ *to* $\underline{P}^1$*, the collection of maps* $\phi = (\phi^n)_{n \geq 0}$ *constructed in* (5.12) *is a convergent unital* A_∞ *morphism from* $CF(L, \underline{P}^0)$ *to* $CF(L, \underline{P}^1)$.

PROOF. The statement is an algebraic consequence of the description of the boundary of $\overline{\mathcal{M}}_{n,1}(L, D)_1$ in Remark 5.6. By counting the ends of the one-dimensional moduli spaces we obtain the relation (4.37) but with the union over type replaced by the set of types of quilted treed disks with n leaves and m interior markings. The true boundary strata $\mathcal{M}_\Gamma(L, D)$ are those described in Remark 5.6 and correspond to the terms in the axiom for A_∞ morphisms (5.1). We refer to [**91**] for the sign computation in a slightly different, but equivalent, context.

The assertion on the strict units is a consequence of the existence of forgetful maps for infinite values of the weights. By assumption the ϕ^n products of type Γ that involve $x^\triangledown$ as inputs involve counts of quilted treed disks $u : C \to X$ using perturbations P_Γ that are pulled back under the forgetful morphisms $\overline{\mathcal{U}}_\Gamma \to \overline{\mathcal{U}}_{f(\Gamma)}$ forgetting the first leaf e_i labelled with the strict unit $x^\triangledown$. Since forgetting that semi-infinite edge gives a configuration $u' : C' \to X$ of negative expected dimension, if non-constant, the only configurations contributing to these terms must be the constant maps $u : C \to X$ with a single quilted disk S on which u is constant and a single leaf T_{e_1} mapping to x_M and root edge T_{e_0} mapping to x_M as well. Hence $\phi^1(x^\triangledown) = x^\triangledown$ and $\phi^n(\dots, x^\triangledown, \dots) = 0, n \geq 2$. □

In the following we show that in the case that the incoming and outgoing perturbations are the same, the corresponding morphism may be taken to be the identity. This fact is used for the proof of invariance of the homotopy type of the Fukaya algebra in Corollary 5.11 below.

Proposition 5.8. *Suppose that $\underline{P}^0 = \underline{P}^1$ is an admissible perturbation datum for treed disks. For each type Γ of quilted treed disk, let Γ' denote the corresponding type of unquilted treed disk obtained by forgetting the quilting and collapsing unstable components. Pulling back $P_{\Gamma',0} = P_{\Gamma',1}$ to a perturbation morphism P_Γ^{01} for quilted treed disks gives an admissible perturbation morphism for quilted disks such that the corresponding A_∞ morphism is the identity.*

Proof. The claim follows by choosing quilt-independent perturbations $\underline{P} = (P_\Gamma)$ as in (d) on page 90. Given a non-constant quilted treed disk $u : C \to X$ contributing to ϕ^n in the moduli space $\mathcal{M}_\Gamma(L, D)_0$ of expected dimension 0, one obtains an unquilted treed disk $u' : C' \to X$ in a stratum $\mathcal{M}_{\Gamma'}(L, D)_{-1}$ where Γ' is the type obtained from Γ by forgetting to quilting, which is a contradiction. Therefore, the only configurations contribution to ϕ^n are the constant configurations. Hence $\phi^1 = \mathrm{Id} : CF(L) \to CF(L)$ is the identity and all other maps $\phi^n : CF(L)^{\otimes n} \to CF(L), n > 0$ vanish. □

5.5. Homotopies

The morphism of Fukaya algebras constructed in Theorem 5.7 is a homotopy equivalence by an argument using *twice-quilted disks.* A twice-quilted disk (C, Q_1, Q_2) is defined in the same way as once-quilted disks, but with two interior circles $Q_1, Q_2 \subset C$ that are either equal $Q_1 = Q_2$ or with the second $Q_2 \subset \mathrm{int}(Q_1)$ contained inside the first, say with radii $\rho_1 < \rho_2$. The moduli space of twice-quilted treed disks is a cell complex constructed in a similar way to the space of once-quilted treed disks. Denote the moduli space with n leaves and two quiltings by $\overline{\mathcal{M}}_{n,2}$. As before the moduli space admits a stratification by combinatorial type. The combinatorial type of a twice-quilted disk (C, Q_1, Q_2) with boundary markings is a tree

$$\Gamma = (\mathrm{Vert}(\Gamma), \mathrm{Edge}(\Gamma), (h \times t) : \mathrm{Edge}(\Gamma) \to \mathrm{Vert}(\Gamma) \cup \{\infty\})$$

equipped with subsets

$$\mathrm{Vert}^1(\Gamma), \mathrm{Vert}^2(\Gamma) \subset \mathrm{Vert}(\Gamma)$$

corresponding to the quilted components; the set

$$\mathrm{Vert}^{12}(\Gamma) := \mathrm{Vert}^1(\Gamma) \cap \mathrm{Vert}^2(\Gamma)$$

corresponds to the twice-quilted components. The ratios

$$\lambda_S(v) = \rho_2(v)/\rho_1(v), \quad v \in \mathrm{Vert}^{12}(\Gamma)$$

of the radii of the interior circles with radii $\rho_2(v), \rho_1(v), v \in \mathrm{Vert}^{12}(\Gamma)$ are required to be equal for each twice-quilted disk in the configuration, if the configuration has twice-quilted components:

$$\lambda_S(v_1) = \lambda_S(v_2), \quad \forall v_1, v_2 \in \mathrm{Vert}^{12}(\Gamma). \tag{5.13}$$

The stratification of the moduli space of twice-quilted disks $\overline{\mathcal{M}}_{n,2}$ by type is a cell decomposition with cells in bijection with certain expressions involving formal functions f, g and inputs $x_1, \ldots, x_n$. For example, in Figure 5.6 the moduli space of twice-quilted stable disks $\overline{\mathcal{M}}_{2,2}$ is shown without trees; it is a pentagon whose vertices correspond to the expressions $f(g(x_1x_2))$, $f(g(x_1)g(x_2))$, $f(g(x_1))f(g(x_2))$, $((fg)(x_1))((fg)(x_2))$, $(fg)(x_1x_2)$. One can also consider a moduli space of twice-

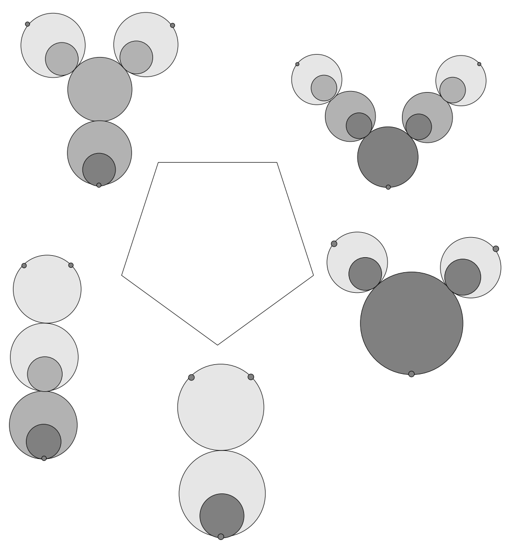

FIGURE 5.6. Twice-quilted disks

quilted disks with interior markings; these are required to lie on components $v \in \mathrm{Vert}(\Gamma)$ which are at least as far away from the root edge as the vertices $v \in \mathrm{Vert}^2(\Gamma)$. That is, if $v \in \mathrm{Vert}(\Gamma)$ lies on a non-self-crossing path from the root edge to an element in $\mathrm{Vert}^2(\Gamma)$ then $v \in \mathrm{Vert}^2(\Gamma)$. Once one allows interior markings, these can bubble off onto *twice-quilted spheres*, which are marked spheres $(C, (z_0, \ldots, z_n))$ equipped with *two* isomorphisms $C - \{z_0\} \to \mathbb{C}$.

As in the quilted case, there is a moduli space of *treed* twice-quilted spheres which assigns lengths to the interior and boundary nodes. The lengths of paths between colored vertices satisfy the balanced condition (5.9) for each color: For any two vertices of the same color $v_1, v_2 \in \mathrm{Vert}^k(\Gamma)$,

$$\sum_{e \in P_+(v_1,v_2)} \ell(e) = \sum_{e \in P_-(v_1,v_2)} \ell(e) \tag{5.14}$$

where $P(v_1,v_2)$ is the (finite length) oriented non-self-crossing path from v_1 to v_2 and $P_+(v_1,v_2)$ resp. $P_-(v_1,v_2)$ is the part of the path pointing towards resp. away from the root edge. In particular this implies that for two vertices of different colors $v_1 \in \mathrm{Vert}^1(\Gamma)$ and $v_2 \in \mathrm{Vert}^2(\Gamma)$ for which there is a (finite length) oriented non-self-crossing path $P(v_1,v_2)$ from v_1 to v_2, let

$$\lambda_T(v_1,v_2) = \sum_{e \in P(v_1,v_2)} \ell(e). \tag{5.15}$$

Then $\lambda_T(v_1,v_2)$ is independent of the choice $v_1 \in \mathrm{Vert}^1(\Gamma)$ and $v_2 \in \mathrm{Vert}^2(\Gamma)$.

Suppose that divisors and perturbations for unquilted and once-quilted disks have already been chosen. That is, there are given compatible almost complex structures J_0, J_1, J_2, metrics G_0, G_1, G_2, divisors D_0, D_1, D_2 and perturbation systems $\underline{P}^0, \underline{P}^1, \underline{P}^2$ for unquilted disks. Furthermore, there are given paths $J_{01}^t, J_{12}^t, J_{02}^t$ of compatible almost complex structures from J_0 to J_1, J_1 to J_2 and J_0 to J_2, paths D_{01}^t from D_0 to D_1, and D_{12}^t from D_1 to D_2, and D_{02}^t from D_0 to D_2. (In our application we are particularly interested in the case $D_0 = D_2$ and the constant path $D_{02}^t = D_0 = D_2$.) Suppose there are given perturbation data $\underline{P}^{ij}$ for once-quilted disks giving rise to morphisms

$$\phi_{ij} : CF(L, \underline{P}^i) \to CF(L, \underline{P}^j), \quad 0 \le i < j \le 2.$$

We have in mind especially the case that $D_0 = D_2$, $D_{01}^t = D_{12}^{1-t}$, and D_{02}^t is the constant path. In this case one may take $\underline{P}^{12,t} = \underline{P}^{01,1-t}$ and $\underline{P}^{02}$ the perturbation system pulled back by the forgetful map forgetting the quilting as in Proposition 5.8.

To construct the moduli spaces of twice-quilted treed disks we extend the families of stabilizing divisors over the universal twice-quilted treed disks. Choose a *domain-dependent parameter* for twice quilted disks: a smooth map

$$\delta^{012} : \triangle \equiv \{(t_1,t_2) \in [-\infty,\infty]^2 \mid t_2 \le t_1\} \to [0,2].$$

For every point $z \in C$ of a twice quilted treed disk C, let $d(z) = (t_1,t_2) \in \triangle$ with t_1 being the signed distance of z to the lowest quilted components $S_v, v \in \mathrm{Vert}^1(\Gamma)$ of C and t_2 being the signed distance of z to the highest quilted components $S_v, v \in \mathrm{Vert}^2(\Gamma)$ of C.

Definition 5.9. A perturbation P_Γ^{012} for twice-quilted treed disks from quilted perturbation systems $\underline{P}^{01} \times \underline{P}^{12}$ to $\underline{P}^{02}$ consists of

(a) a domain-dependent parameter δ_Γ^{012} that agrees
  - with $\delta_{\Gamma_{01}}^{01}$ on $[-\infty,\infty] \times \{-\infty\}$,
  - with $\delta_{\Gamma_{12}}^{12}$ on $\{-\infty\} \times [-\infty,\infty]$ and
  - with $\delta_{\Gamma_{02}}^{02}$ on $\{(t_1,t_2) \in [-\infty,\infty]^2 | t_1 = t_2\}$,

  where $\Gamma_{ij}, 0 \le i \le j \le 2$ is the corresponding type of once-quilted disk;

(b) a smooth family of metrics G_Γ^{012} constant equal to G^0 resp. G^1 resp. G^2 on a neighborhood of the endpoints for which $d = (-\infty,-\infty)$ resp. $d = (\infty,-\infty)$ resp. $d = (\infty,\infty)$ and that agrees

- with $G^{01}_{\Gamma_{01}}$ on $[-\infty,\infty]\times\{-\infty\}$,
- with $G^{12}_{\Gamma_{12}}$ on $\{+\infty\}\times[-\infty,\infty]$ and
- with $G^{02}_{\Gamma_{02}}$ on $\{(t_1,t_2)\in[-\infty,\infty]^2 | t_1=t_2\}$,

where $\Gamma_{ij}, 0\le i\le j\le 2$ is the corresponding type of once-quilted disk;

(c) a domain-dependent Morse function F^{012}_Γ equal to F^0 resp. F^1 resp. F^2 in a neighborhood of the endpoints at $d=(-\infty,-\infty)$ resp. $d=(\infty,-\infty)$ resp. $d=(\infty,\infty)$ and that agrees with $F^{01}_{\Gamma_{01}}$ resp. $F^{12}_{\Gamma_{12}}$ resp. $F^{02}_{\Gamma_{02}}$ on the once quilted disk components of type Γ_{01} resp. Γ_{12} resp. Γ_{02} containing the root resp. the leaves resp. where the quilting radii coincide,

(d) a domain-dependent almost-complex structure J^{012}_Γ such that for every surface component C_i, equal to $J_{\delta^{012}_\Gamma\circ d(z)}$ on $D^{\delta^{012}_\Gamma\circ d(z)}$, in a neighborhood of the spherical nodes, the interior markings and on the boundary of C_i, that is
- equal to $J^0_{\Gamma_0}$, resp. $J^1_{\Gamma_1}$ resp. $J^2_{\Gamma_2}$ on the unquilted components of type Γ_0 resp. Γ_1 resp. Γ_2 at $d=(-\infty,-\infty)$ resp. $d=(\infty,-\infty)$ resp. $d=(\infty,\infty)$ and
- agrees with $J^{01}_{\Gamma_{01}}$ resp. $J^{12}_{\Gamma_{12}}$ resp. $J^{02}_{\Gamma_{02}}$ on the once quilted treed disk components of type Γ_{01} resp. Γ_{12} resp. Γ_{02} containing the root resp. the leaves resp. where the quilting radii coincide.

(e) In cases that the perturbations being connected are identical, one can also require the following invariance property: A perturbation datum is *quilting-independent* if G^{012}_Γ, F^{012}_Γ, and J^{012}_Γ are pull-backs under the forgetful morphism forgetting the quiltings on each once-quilted or twice-quilted disk.

The perturbations above allow the definition of pseudoholomorphic twice-quilted treed disks. Given a treed twice-quilted treed disk C of type Γ, one obtains perturbation data for C by pull-back from the stabilization C^{st} of C, which may be identified with a fiber of the universal twice-quilted disk $\mathcal{U}_\Gamma\to\mathcal{M}_\Gamma$. A *pseudo-holomorphic twice-quilted treed disk* is a twice-quilted disk C together with a map $u: C\to X$ that is J^{012}_Γ-holomorphic on surface components, a F^{012}_Γ-Morse trajectory on boundary tree segments with respect to the metrics $G^{01}_\Gamma, G^{12}_\Gamma, G^{02}_\Gamma$, and constant on the interior part of the tree. Stable and adapted twice-quilted treed disks are defined as in the once-quilted case. In particular, each interior marking z_i maps to the divisor $D^{\delta^{012}_\Gamma(z_i)}$. Suppose that the perturbations $\underline{P}=(P^{012}_\Gamma)$ satisfy coherence and stabilized conditions similar to those for quilted disks. The moduli spaces $\overline{\mathcal{M}}_\Gamma(L,D)$ of adapted stable twice-quilted disks are then compact for each uncrowded combinatorial type Γ of expected dimension $\dim(\mathcal{M}_\Gamma(L,D))$ at most one. The property (e) of Definition 5.9 ensures that the latter zero dimensional spaces $\mathcal{M}_\Gamma(L,D), \dim(\mathcal{M}_\Gamma(L,D))=0$ will not contain non-constant (either once or twice) quilted disks. The one dimensional strata $\mathcal{M}_\Gamma(L,D)$ may involve at most one non-constant once-quilted disk $S_v\subset S$ and if so, $\mathcal{M}_\Gamma(L,D)$ is a constant family over which the latter quilting radius varies freely.

In order to obtain transversality the fiber products involved in the definition of the universal twice-quilted disks in (5.13) must be perturbed, using *delay functions*, as in Seidel [**120**] and Ma'u-Wehrheim-Woodward [**91**]. We define a map incorporating both the condition on ratios and distances between quilted components as follows. Identify $\mathcal{M}_{1,0,2}$ with $[0,\infty]$ as in Figure 5.7. For $n \geq 1, m \geq 0$ denote by

$$\lambda : \mathcal{M}_{n,m,2} \to \mathcal{M}_{1,0,2} \cong [0,\infty]$$

the forgetful morphism forgetting all but the first marking; note that on the interval consisting of only once-quilted disks, λ is essentially equivalent to the map λ_T of (5.15) while on the interval with twice-quilted disks λ is given by the map λ_S of (5.13). Note that λ is also defined in the case $n = 0$, by combining the maps (5.15) and (5.13). Let Γ be a combinatorial type of twice-quilted disks. Define $\overline{\mathcal{M}}_\Gamma^{\text{pre}}$ as

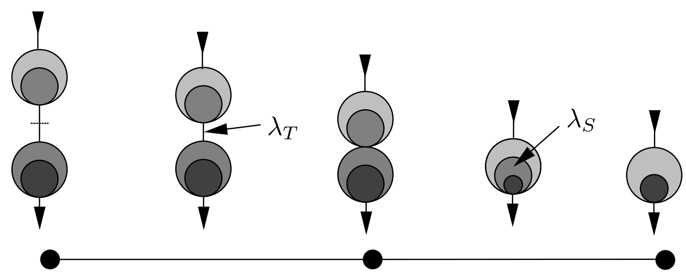

FIGURE 5.7. Moduli of treed twice-quilted disks with one leaf

the product of moduli spaces for the vertices,

$$\overline{\mathcal{M}}_\Gamma^{\text{pre}} = \prod_{v \in \text{Vert}(\Gamma)} \overline{\mathcal{M}}_v.$$

Let k denote the number of twice-quilted vertices and

$$\lambda_\Gamma : \overline{\mathcal{M}}_\Gamma^{\text{pre}} \to \mathbb{R}^k, \quad (r_v) \mapsto \prod_{v \in \text{Vert}^{(12)}(\Gamma)} \lambda(r_v)$$

the map combining the forgetful maps for the twice-quilted components. Then $\overline{\mathcal{M}}_\Gamma = \lambda_\Gamma^{-1}(\Delta)$ where $\Delta \subset \mathbb{R}^k$ is the diagonal. A *delay function* for Γ is a collection of smooth functions depending on $r \in \overline{\mathcal{M}}_\Gamma^{\text{pre}}$

$$\tau_\Gamma = \left(\tau_e \in C^\infty(\overline{\mathcal{M}}_\Gamma^{\text{pre}})\right)_{e \in \text{Edge}(\Gamma)}.$$

Letting $\lambda_i := \lambda(r_{v_i})$ where $\lambda(r_{v_i})$ is the ratio of the radii circles for r_{v_i}, the *delayed evaluation map* is

$$\begin{aligned} \lambda_{\tau_\Gamma} : \prod_{v \in \text{Vert}(\Gamma)} \overline{\mathcal{M}}_v &\to \mathbb{R}^k \\ (r_v, u_v)_{v \in \text{Vert}\,\Gamma} &\mapsto \left(\lambda_i \exp\left(\sum_{e \in p_i} \tau_e(r)\right)\right)_{i=1,\ldots,k} \end{aligned} \tag{5.16}$$

where the sum is the sum of delays along each path p_i to a twice-quilted disk component. Call τ_Γ *regular* if the delayed evaluation map λ_{τ_Γ} is transverse to the diagonal $\Delta \subset \mathbb{R}^k$. Given a regular delay function τ_Γ, we define

$$\overline{\mathcal{M}}_\Gamma := \lambda_{\tau_\Gamma}^{-1}(\Delta). \tag{5.17}$$

For a regular delay function τ_Γ, the delayed fiber product has the structure of a smooth manifold, of local dimension

$$\dim \mathcal{M}_\Gamma = 1 - k + \sum_{v \in \operatorname{Vert} \Gamma} \dim \mathcal{M}_v \tag{5.18}$$

where k is the number of twice-quilted disk components. A collection $\{\tau^d\}_{d \geq 1}$ of delay functions is *compatible* if the following properties hold. Let Γ be a combinatorial type of twice-quilted disk and $v_0, \ldots, v_k$ the vertices corresponding to disk components.

(a) (Subtree property) Suppose that the root component v_0 is not a twice-quilted disk. Let $\Gamma_1, \ldots, \Gamma_{|v_0|-1}$ denote the subtrees of Γ attached to v_0 at its leaves; then $\Gamma_1, \ldots, \Gamma_{|v_0|-1}$ are combinatorial types for nodal twice-quilted disks. Let r_i be the component of $r \in \mathcal{M}_\Gamma^{\mathrm{pre}}$ corresponding to Γ_i. We require that $\tau_\Gamma(r)\big|_{\Gamma_i} = \tau_{\Gamma_i}(r_i)$, i.e., for each edge e of Γ_i, the delay function $\tau_{\Gamma,e}(r)$ is equal to $\tau_{\Gamma_i,e}(r_i)$. See Figure 5.8.

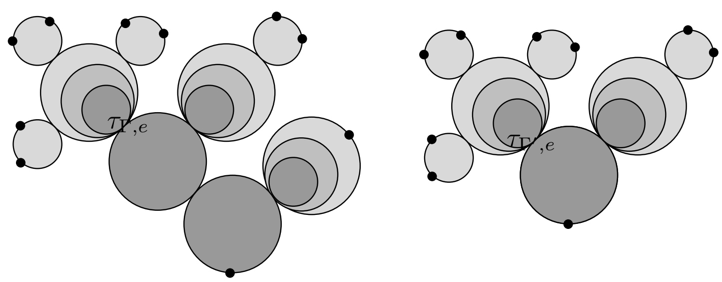

FIGURE 5.8. The (Subtree Property)

(b) (Refinement property) Suppose that the combinatorial type Γ' is a refinement of Γ, so that there is a surjective morphism $f : \Gamma' \to \Gamma$ of trees; let r be the image of r' under gluing. We require that $\tau_\Gamma\big|_U$ is determined by $\tau_{\Gamma'}$ as follows: for each $e \in \operatorname{Edge}(\Gamma)$, and $r \in U$, the delay function is given by the formula

$$\tau_{\Gamma,e}(r) = \tau_{\Gamma',e} + \sum_{e'} \tau_{\Gamma',e'}(r')$$

where the sum is over edges e' in Γ' that are collapsed under gluing and e is the next-furthest-away edge from the root vertex. See Figure 5.9. In the case that the collapsed edges connect twice quilted components with unquilted components, this means that the delay functions are equal for both types, as in the Figure 5.10.

(c) (Core property) Given a combinatorial type Γ, we say that the *core* Γ_0 of Γ is the combinatorial type obtained by removing all vertices above the colored vertices. If two combinatorial types, say Γ and Γ', have the same core Γ_0, let r, r' be disks of type Γ resp. Γ' and r_0, r_0' the disks of type Γ_0 obtained by removing the components except for those corresponding to vertices of Γ_0. If $r_0 = r_0'$ then $\tau_{\Gamma,e}(r) = \tau_{\Gamma',e}(r')$. (That is, the delay functions depend only on the region between the root vertex and the bicolored vertices.)

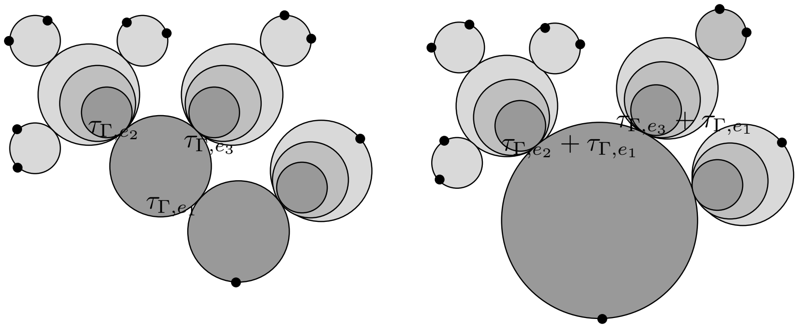

FIGURE 5.9. The (Refinement property), first case

A collection of compatible delay functions is *positive* if, for every vertex $v \in \Gamma_0$ with k leaves labeled in counterclockwise order by $e_1, \ldots, e_k$, the associated delay functions satisfy

$$\tau_{e_1} < \tau_{e_2} < \ldots < \tau_{e_k}. \tag{5.19}$$

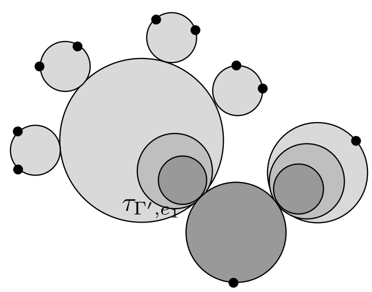

FIGURE 5.10. The (Refinement property), second case

For each type there exist regular, positive delay functions compatible with a collection of given choices on the boundary. For any type Γ the (Subtree property) implies that all the delay functions in $\tau_\Gamma := \tau_\Gamma(L)$ *except* those for the finite edges adjacent to v_0, the root component, are already fixed. We wish to construct regular delay functions for the finite edges $e \in \text{Edge}(\Gamma)$ adjacent to the root vertex $v_0 \in \text{Vert}(\Gamma)$ in a way that is also compatible with the (Refinement property). We ensure compatibility by proceeding inductively on the number of edges connecting to the root vertex v_0 as follows. Suppose that regular delay functions are given for types Γ corresponding to strata of $\mathcal{M}_{n',m,2}$ for $n' < n$, as well as for types Γ' appearing in the (Refinement property) for Γ. We construct inductively on $d \geq 2$ a collection of regular delay functions $\tau_\Gamma := \tau_\Gamma(L)$ for all combinatorial types Γ whose root vertex has $2 \leq d \leq n$ incoming edges. We may assume that Γ has no vertices "beyond the twice-colored vertices" since by the (Core property) the delay functions τ_Γ are independent of the additional vertices. Consider a combinatorial type Γ whose root vertex has d leaves. There is an open neighborhood U of $\partial\mathcal{M}_\Gamma^{\text{pre}}$ in $\overline{\mathcal{M}}_\Gamma^{\text{pre}}$ in which the delay functions τ_Γ for the leaves e adjacent to the root vertex $v \in \text{Vert}(\Gamma)$ are already determined by the compatibility condition (Refinement property) and the inductive hypothesis. We need to show that we may extend the τ_i over the interior of $\overline{\mathcal{M}}_\Gamma^{\mathrm{pre}}$. To set up the relevant function spaces let $l \geq 0$ be an integer and let f be a given C^l function on $\overline{U}$. Let $C^l_f(\mathcal{M}_\Gamma^{\mathrm{pre}})$ denote the Banach manifold of functions with l bounded derivatives on $\mathcal{M}_\Gamma^{\mathrm{pre}}$, equal to f on $\overline{U}$. Let $\Gamma_i, i = 1, \ldots, d$ be the trees attached to the root vertex v_0. Consider the evaluation map

$$\begin{aligned} &\mathrm{ev} : \mathcal{M}_{\Gamma_1} \times \ldots \times \mathcal{M}_{\Gamma_d} \times \mathcal{M}_{v_0} \times \prod_{i=1}^d C^l_{\tau_i}(\mathcal{M}_\Gamma^{\mathrm{pre}}) \to \mathbb{R}^{d-1} \\ &((r_1, u_1), \ldots, (r_n, u_d), (r_0, u_0), \tau_1, \ldots, \tau_d) \mapsto \\ &\left(\lambda_{\Gamma_j}(r_j) \exp(\tau_j(r)) - \lambda_{\Gamma_{j+1}}(r_{j+1}) \exp(\tau_{j+1}(r))\right)_{j=1}^{d-1} \end{aligned}$$

where $r = (r_0, \ldots, r_d)$. Note that 0 is a regular value. The Sard-Smale theorem implies that for l sufficiently large the regular values of the projection

$$\Pi : \mathrm{ev}^{-1}(0) \to \prod_{i=1}^d C^l_{\tau_i}(\mathcal{M}_\Gamma^{\mathrm{pre}})$$

form an open dense set. Taking the intersection over l sufficiently large gives that the set of smooth regular delay functions is comeager. Both the positivity condition (5.19) and the regularity condition are open conditions given an energy bound $E(u) < E$. Therefore the set of smooth, positive, compatible, delay functions that are regular for a given energy bound is non-empty and open. Taking the intersection of these sets over all possible energy bounds $E \in \mathbb{Z}$ we obtain a comeager set of delay functions for which all moduli spaces $\mathcal{M}_\Gamma(L, D)$ are regular. By induction, there exists a smooth, positive, compatible, regular delay function τ_Γ.

Given a collection $\underline{\tau} = (\tau_\Gamma)$ of regular compatible delay functions and a collection of perturbation data $\underline{P}^0, \underline{P}^1, \underline{P}^2, \underline{P}^{01}, \underline{P}^{12}$ and $\underline{P}^{02}$ for unquilted and once-quilted disks, perturbations $\underline{P}^{012} = (P_\Gamma^{012})$ for twice-quilted disks are constructed inductively using the gluing construction and the Sard-Smale theorem. For each stratum $\mathcal{M}_\Gamma(L, D)$, we first use the construction of the previous paragraph to find regular delay functions for the boundary strata $\mathcal{M}_{\Gamma'}(L, D)$, then (after replacing the fiber products in the boundary strata by the *delayed* fiber products of boundary strata using the delay functions chosen) extend the perturbations by the gluing construction. A Sard-Smale argument shows that for perturbations in a comeager subset, the uncrowded moduli spaces of expected dimension at most one are regular of expected dimension.

THEOREM 5.10. (A_∞ homotopies via twice-quilted disks) *Given admissible perturbation systems $\underline{P}^{01}, \underline{P}^{12}, \underline{P}^{02}$ defining morphisms*

$$\phi_{ij} : CF(L, \underline{P}^i) \to CF(L, \underline{P}^j), \quad 0 \leq i < j \leq 2$$

and an admissible perturbation system $\underline{P}^{012}$ for twice-quilted disks, counting treed pseudoholomorphic twice-quilted disks defines a convergent A_∞ homotopy between ϕ_{02} and $\phi_{01} \circ \phi_{12}$.

PROOF. The homotopy is defined by combining homotopies constructed from small variations of ratio. Consider the map $\psi : \overline{\mathcal{M}}_{n,m,2} \to [1, \infty]$ giving the ratio ρ_1/ρ_2 of radii of the inner circles of the twice-quilted disks. For generic values of λ the moduli space $\overline{\mathcal{M}}^\lambda_{n,m,2}(L, D) = \psi^{-1}(\lambda)$ is smooth of expected dimension. We let ϕ_{02}^λ denote the A_∞ morphism obtained by counting twice-quilted disks with ratio of radii $\lambda \in [1, \infty)$. After fixing an energy bound E and a number of leaves n we may divide the interval $[0,1]$ into subintervals $[\lambda_i, \lambda_{i+1}], i = 0, \dots, k-1$ so that each is sufficiently small so that there is a single singular value $\lambda \in [\lambda_i, \lambda_{i+1}]$, contained in the interior of the interval, for which there exist twice-quilted disks of expected dimension zero of energy at most E and n leaves and no such disks with fewer number of leaves. Define $\mathcal{T}_{02}^{\lambda_1, \lambda_{i+1}, \leq E}$ by counting such twice-quilted disks,

$$(5.20) \quad \begin{aligned} &(\mathcal{T}_{02}^{\lambda, \leq E})^n : CF(L; \underline{P}^0)^{\otimes n} \to CF(L; \underline{P}^2) \\ &(x_1, \dots, x_n) \mapsto \sum_{x_0, u \in \overline{\mathcal{M}}_\Gamma^\lambda(L, D; x_0, \dots, x_n)_0} (-1)^{\heartsuit} \epsilon(u)(\sigma(u)!)^{-1} q^{E(u)} y(u) x_0 \end{aligned}$$

where the sum is over combinatorial types Γ of twice-quilted disks. The difference

$$(\phi_{02}^{\lambda_i} - \phi_{02}^{\lambda_{i+1}})^n(x_1, \dots, x_n)$$

is a count of configurations $u : C \to X$ either involving an unquilted disk breaking off $C_1 \subset C$, or a collection of twice-quilted treed disks $C_1, \dots, C_r \subset C$ with $i_1, \dots, i_r$ leaves breaking off from an unquilted treed disk, see [**91**, Section 7]. For degree reasons, because of the fiber product with the diagonal exactly one of these twice-quilted disks lies in the moduli space of expected dimension zero, while the rest have index one. We suppose that the twice-quilted configuration in the expected-dimension-zero moduli space is the $i+1$-st twice-quilted treed disk attached to the unquilted treed disk. Using positivity of the delay functions one obtains that the moduli space of twice quilted disks λ_i and λ_{i+1} and expected dimension zero are cobordant:

$$\begin{array}{lcll} \overline{\mathcal{M}}_{i_j, m_j, 2}^{\lambda + \tau_j}(L, D)_0 & \sim & \overline{\mathcal{M}}_{i_j, m_j, 2}^{\lambda_{i+1} + \tau_j}(L, D)_0 & j > i+2 \\ \overline{\mathcal{M}}_{i_j, m_j, 2}^{\lambda + \tau_j}(L, D)_0 & \sim & \overline{\mathcal{M}}_{i_j, m_j, 2}^{\lambda_i + \tau_j}(L, D)_0 & j \leq i. \end{array}$$

It follows that

$$\phi_{02}^{\lambda_i} - \phi_{02}^{\lambda_{i+1}} = \mu^1_{\phi_{02}^{[\lambda_i, \lambda_{i+1}]}}(\mathcal{T}_{02}^{\lambda_i, \lambda_{i+1}, \leq E}).$$

The facets of $\overline{\mathcal{M}}_{n,m,2}$ with ratio $\lambda = 1$ or $\lambda = \infty$ correspond to either to terms in the definition of composition of A_∞ maps $\phi_{12} \circ \phi_{01} : CF(L, \underline{P}^0) \to CF(L, \underline{P}^2)$, to the components contributing to $\phi_{02} : CF(L, \underline{P}^0) \to CF(L, \underline{P}^2)$ or to terms corresponding to the bubbling off of some markings on the boundary. Composition using (5.6) produces a homotopy

$$\mathcal{T}_{02}^{\leq E} := \mu^2(\mathcal{T}_{02}^{\lambda_{k-1}, \lambda_k, \leq E}, \mu^2(\dots \mu^2(\mathcal{T}_{02}^{\lambda_1, \lambda_2}, \mathcal{T}_{02}^{\lambda_0, \lambda_1, \leq E})))$$

between $\phi_{12} \circ \phi_{01}$ and the identity modulo terms involving powers q^E. Taking the limit $E \to \infty$ defines a homotopy

$$\mathcal{T}_{02} := \lim_{E \to \infty} \mathcal{T}_{02}^{\leq E}$$

between ϕ_{02} and $\phi_{12} \circ \phi_{01}$. Convergence, that is, that $\mathcal{T}_{02}^0(1)$ has coefficients with positive q-valuation, holds since any contributing configuration must contain a non-trivial disk. □

Corollary 5.11. *For any two admissible convergent collections of perturbation data $\underline{P}^0, \underline{P}^1$ the Fukaya algebras $CF(L, \underline{P}^0)$ and $CF(L, \underline{P}^1)$ are homotopy equivalent via strictly unital convergent maps and homotopies in the sense of Lemma 5.3.*

Proof. Using Theorem 5.10 and taking D^t_{02} to be constant and $\underline{P}^{02}$ to be pulled back by the map forgetting the quilting, one obtains a homotopy between the composition of morphisms $\phi_{01} : CF(L, \underline{P}^0) \to CF(L, \underline{P}^1)$, $\phi_{10} : CF(L, \underline{P}^1) \to CF(L, \underline{P}^0)$ and the identity morphism as in Proposition 5.8. Since the perturbation data P_Γ for a type Γ with an infinite weight $\rho(e)$ is a pull-back under the morphism forgetting the first infinite weight, the maps ϕ^k_{01} involving an identity insertion all vanish except ϕ^1_{01}, for which $\phi^1_{01}(e_0) = e_1$. Hence ϕ_{01} is strictly unital. □

5.6. Stabilization

In this section we complete the proof of homotopy invariance of the Fukaya algebras constructed above in the case that the algebras are defined using divisors are not of the same degree or built from homotopic sections. For this we need to recall some results about existence of a Donaldson hypersurface transverse to a given one. Recall from [**28**, Lemma 8.3] that for a constant $\epsilon > 0$, two divisors D, D' intersect *ϵ-transversally* if at each intersection point $x \in D \cap D'$ their tangent spaces T_xD, T_xD' intersect with angle at least ϵ. A result of Cieliebak-Mohnke [**28**, Theorem 8.1] states that there exists an $\epsilon > 0$ such that given a divisor D, there exists a divisor D' of sufficiently high degree ϵ-transverse to D. Moreover, for any $\theta > 0$, ω-tamed almost complex structures θ-close to J making D, D' almost complex exist (provided that the degrees are sufficiently large).

We apply the result of the previous paragraph as follows. Suppose that D^0, D^1 are stabilizing divisors for L, possibly of different degrees. By the previous paragraph, there exists a pair $D^{0,'}$ $D^{1,'}$ of higher degree stabilizing divisors built from homotopic unitary sections over L that are ϵ-transverse to D^0 and D^1, respectively. Let $\underline{P}'_0, \underline{P}'_1$ be perturbation systems for $D^{0,'}, D^{1,'}$. We have already shown that

$$CF(L, \underline{P}'_0) \cong CF(L, \underline{P}'_1) \tag{5.21}$$

are convergent, strictly-unital homotopy equivalent. It remains to show:

Theorem 5.12. *For any admissible perturbation systems $\underline{P}_k, \underline{P}'_k, k = 0, 1$ as above, the Fukaya algebras $CF(L, \underline{P}_k)$ and $CF(L, \underline{P}'_k)$ are convergent, strictly-unital homotopy equivalent.*

Sketch of proof. Denote the subset of almost complex structures close to J preserving $TD^{k,'}$ as well by

$$\mathcal{J}^*(X, D^k \cup D^{k,'}; J_{D^0}, \theta, E) \subset \mathcal{J}^*(X, D^k, J, \theta, E).$$

By [**28**, Corollary 8.20], there exists a path-connected, open, dense set in $\mathcal{J}^*(X, D^k \cup D^{k,'}; J_{D^k}, \theta, E)$ with the property that for any $J \in \mathcal{J}^*(X, D^k \cup D^{k,'}; J_{D^k}, \theta, E)$, neither D^k nor $D^{k,'}$ contain any non-constant pseudoholomorphic spheres of energy at most E, and each pseudoholomorphic sphere meets both D^k and $D^{k,'}$ in at least three points. Fix such an almost complex structure $J_{D^k, D^{k'}}$, and an associated perturbation system $\underline{P}''_k$ using the divisor D^k such that the almost complex structures $J''_{k,\Gamma}$ are equal to $J_{D^k, D^{k,'}}$ on $D^k \cup D^{k,'}$. The argument in the previous section (keeping the divisor constant but changing the almost complex structures) shows that the associated Fukaya algebras $CF(L, \underline{P}_k) \cong CF(L, \underline{P}''_k)$ are homotopy equivalent. Similarly, choose a perturbation system $\underline{P}'_k$ using the divisor $D^{k,'}$. We claim that the Fukaya algebras $CF(L, \underline{P}'_k) \overset{?}{\cong} CF(L, \underline{P}''_k)$ are homotopy equivalent. To see this, we define adapted stable maps adapted to the pair $(D^k, D^{k,'})$: a map is adapted if each interior marking maps to either D^k or $D^{k,'}$, and the first n_k markings map to D^k and the last n'_k markings map to $D^{k,'}$. A perturbation datum morphism $\underline{P} = (P_\Gamma)$ is coherent if it is compatible if with the morphisms of moduli spaces as before: (Cutting edges) axiom, (Collapsing edges/Making an edge or weight finite or non-zero) axiom, and satisfies the (Infinite weights) axiom and (Product) axiom, where now on the unquilted components above resp. below the quilted components the perturbation system is required to depend only on the first n_k resp. last n'_k points mapping to D^k resp. $D^{k,'}$ (that is, pulled back under the forgetful map forgetting the first n_k resp. last n'_k markings). Then the same arguments as before, together with the existence of the homotopy from [**135**], produce the required homotopy equivalence. Putting everything together we have homotopy equivalences $CF(L, \underline{P}_k) \cong CF(L, \underline{P}''_k) \cong CF(L, \underline{P}'_k)$. Applying (5.21) completes the proof. □

Corollary 5.13. *For any stabilizing divisors D^0, D^1 and any convergent admissible perturbation systems $\underline{P}_0, \underline{P}_1$, the Fukaya algebras $CF(L, \underline{P}_0)$ and $CF(L, \underline{P}_1)$ are convergent-homotopy-equivalent.*

Proof. Since homotopy equivalence of A_∞ algebras is an equivalence relation, see Definition 5.1 (g), combining Theorems 5.7 and 5.12 gives the result. □

CHAPTER 6

Fukaya bimodules

Floer cohomology in the Morse model is invariant under Hamiltonian perturbation, as in the various models described in Fukaya-Oh-Ohta-Ono [**46**]. In this section we construct, by counting treed Hamiltonian-perturbed pseudoholomorphic strips, a *Fukaya bimodule* for pairs of Lagrangians equipped with a Hamiltonian perturbation such that the perturbed Lagrangians intersect cleanly. The Fukaya bimodule is isomorphic to the bimodule for the Fukaya algebra if the Lagrangians are identical. The homotopy type of the Fukaya bimodule is independent of the choice of Hamiltonian. In particular, if a Lagrangian brane is displaceable by a Hamiltonian diffeomorphism then the Floer cohomology for any element of the Maurer-Cartan moduli space vanishes. This shows that the version of the Floer cohomology used in this paper is strong enough to deduce standard conclusions about Hamiltonian displaceability.

6.1. A_∞ bimodules

We introduce the following definitions on A_∞ bimodules, with conventions similar to those of Seidel in [**119**]. Let A_0, A_1 be strictly unital A_∞ algebras graded by $\mathbb{Z}_g$ where g is an even integer. An $\mathbb{Z}_g$-graded A_∞ *bimodule* is a $\mathbb{Z}_g$-graded vector space M equipped with operations

$$\mu^{d|e} : A_0^{\otimes d} \otimes M \otimes A_1^{\otimes e} \to M[1-d-e]$$

that satisfy the following relations among homogeneous elements $a_{i,k}, m$:

$$\begin{aligned}(6.1)\quad 0 = &\sum_{i,k} (-1)^{\aleph} \mu^{d-i+1|e}(a_1^0, \dots, \mu_0^i(a_k^0, \dots, a_{k+i-1}^0), a_{k+i}^0, \dots, a_d^0, m, a_1^1, \dots, a_e^1)) \\ &+ \sum_{j,k} (-1)^{\aleph} \mu^{d|e-j+1}(a_1^0, \dots, a_d^0, m, a_1^1, \dots, \mu_1^j(a_k^1, \dots, a_{k+j-1}^1), \dots, a_e^1) \\ &+ \sum_{i,j} (-1)^{\aleph} \mu^{d-i|e-j}(a_1^0, \dots, a_{d-i}^0, \mu^{i|j}(a_{d-i+1}^0, \dots, a_d^0, m, a_1^1, \dots, a_j^1), \dots, a_e^1).\end{aligned}$$

Here we follow Seidel's convention [**119**] of denoting by $(-1)^{\aleph}$ the sum of the reduced degrees to the left of the inner expression, except that m has ordinary (unreduced) degree. A *morphism* ϕ of A_∞ -bimodules M_0 to M_1 of degree $|\phi|$ is a collection of maps

$$\phi^{d|e} : A_0^{\otimes d} \otimes M_0 \otimes A_1^{\otimes e} \to M_1[|\phi| - d - e]$$

satisfying a splitting axiom

$$\sum_{i,j}(-1)^{|\phi|\aleph}\mu_1^{d|e}(a_1^0,\dots,a_{d-i}^0,\phi^{i|j}(a_{d-i+1}^0,\dots,a_d^0,m,a_1^1,\dots,a_j^1),a_{j+1}^1,\dots,a_e^1)$$
$$+\sum_{i,j}(-1)^{|\phi|+1+\aleph}\phi^{d|e}(a_1^0,\dots,a_{d-i}^0,\mu_0^{i|j}(a_{d-i+1}^0,\dots,a_d^0,m,a_1^1,\dots,a_j^1),a_{j+1}^1,\dots,a_e^1)$$
$$+\sum_{i,j}(-1)^{|\phi|+1+\aleph}\phi^{d|e}(a_1^0,\dots,a_{j-1}^0,\mu_0^i(a_j^0,\dots,a_{j+i-1}^0),\dots,a_d^0,m,a_1^1,\dots,a_e^1)$$
$$+\sum_{i,j}(-1)^{|\phi|+1+\aleph}\phi^{d|e}(a_1^0,\dots,a_e^0,m,a_1^1,\dots,\mu_1^j(a_j^1,\dots,a_{j+i-1}),\dots,a_e^1)=0. \tag{6.2}$$

Composition of bimodule morphisms $\phi: M_0 \to M_1, \psi: M_1 \to M_2$ is defined by

$$(\phi\circ\psi)^{d|e}(a_1^0,\dots,a_d^0,m,a_1^1,\dots,a_e^1)$$
$$=\sum_{i,j}(-1)^{|\psi|\aleph}\phi^{i|j}(a_1^0,\dots,a_i^0,\psi_{d-i|e-j}(a_{i+1}^0,\dots,a_d^0,m,a_1^1,\dots,a_{e-j}^1),a_{e-j+1}^1,\dots,a_e^1). \tag{6.3}$$

A *homotopy* of morphisms $\psi_0,\psi_1 : M_0 \to M_1$ of degree zero is a collection of maps $(\phi^{d|e})_{d,e\geq 0}$ such that the difference $\psi_1-\psi_0$ is given by the expression on the left hand side of (6.2). Each of these notions has an extension to the strictly unital case. Suppose that A_0, A_1 are strictly unital with strict units e_0, e_1. A bimodule M is *strictly unital* $\mu^{1,0}(e_0,\cdot)$ and $\mu^{0,1}(\cdot,e_1)$ are the identity, and all other operations involving e_0 and e_1 vanish. A morphism resp. homotopy is *strictly unital* if all operations involving the identities e_0 and e_1 vanish. As usual any A_∞ algebra A is an A_∞ bimodule over itself. The bimodule operations are related to the structure maps by

$$\mu^{d|e}(a_1^0,\dots,a_d^0,m,a_1^1,\dots,a_e^1)$$
$$=(-1)^{1+\diamondsuit}\mu^{d+e+1}(a_1^0,\dots,a_d^0,m,a_1^1,\dots,a_e^1),\quad \diamondsuit=\sum_{j=1}^{d}(|a_j^0|+1), \tag{6.4}$$

see Seidel [**121**, 2.9].

For bimodules satisfying a convergence property the deformed composition maps with infinitely many insertions are well-defined. Given A_∞ algebras A_0, A_1, denote by $MC(A_k), k\in\{0,1\}$ the corresponding Maurer-Cartan solution spaces from (4.51). For each $b\in MC(A_k)$ the cohomology group is denoted $H(\mu_b^{k,1})$. Define for odd elements $b_0\in A_0, b_1\in A_1$ the maps

$$\mu^{d|e,b_0|b_1} : A_0^{\otimes d}\otimes M\otimes A_1^{\otimes e}\to M,$$
$$(a_1^0,\dots,a_d^0,m,a_1^1\dots,a_e^1)\mapsto \sum_{i_0,\dots,i_d,j_0,\dots,j_e}\mu^{d+d'|e+e'}(\underbrace{b_0,\dots,b_0}_{i_0},a_1^0,\underbrace{b_0,\dots,b_0}_{i_1},$$
$$\dots,\underbrace{b_0,\dots,b_0}_{i_d},m,\underbrace{b_1,\dots,b_1}_{j_0},a_1^1,\dots,\underbrace{b_1,\dots,b_1}_{j_e}). \tag{6.5}$$

PROPOSITION 6.1. *Suppose that M is a finitely generated Λ-module. For any odd $b_0\in A_0^+, b_1\in A_1^+$, the operations $\mu^{d|e,b_0|b_1}$ define the structure of an A_∞ bimodule on M. For $b_0\in MC(A_0)$ and $b_1\in MC(A_0)$ we have*

$$(\mu^{0|0,b_0|b_1})^2=0.$$

The cohomology $H(\mu^{0|0,b_0|b_1})$ *is invariant up to homotopy in the sense that if* $M_0 \to M_1$ *is a homotopy equivalence of* A_∞ *bimodules then the induced map* $H(\mu_0^{0|0,b_0|b_1}) \to H(\mu_1^{0|0,b_0|b_1})$ *is an isomorphism.*

Proof. The A_∞ bimodule axiom implies in particular that $(\mu^{0|0,b_0|b_1})^2(a)$ is equal to $(-1)^{|a|}(\mu^{1|0}(\mu_0^{b_0}(1), a) - \mu^{0|1}(a, \mu_0^{b_1}(1))$ for any $a \in A_0$. Since the elements $\mu_0^{b_k}(1)$ are multiples of strict identities, the claim on the square zero follows. It follows from the A_∞ bimodule morphism axiom in (6.2) that for any (convergent) morphism $\phi^{n,m}$ from M_0 to M_1 the sum

$$\phi^{b_0|b_1} = \sum_{n,m\geq 0} \phi^{n|m}(b_0, \ldots, b_0; b_1, \ldots b_1)$$

is a chain map; a similar argument using A_∞ bimodule homotopies implies that homotopic bimodules have isomorphic cohomology groups. □

6.2. Treed strips

The construction of Fukaya bimodules will be similar to the treatment in Fukaya-Oh-Ohta-Ono [**46**], but using the Morse model. Our perturbation systems uses Donaldson hypersurfaces to stabilize the domains as in Cieliebak-Mohnke [**28**]. For this purpose we develop universal curves over moduli spaces of marked strips as follows.

Definition 6.2. (a) (Marked strips) A *marked nodal strip* is a marked nodal disk with two boundary markings. Let S be a connected $(2, n)$-marked nodal disk with markings $\underline{z}$. We write $\underline{z} = (z_-, z_+, z_1, \ldots, z_n)$ where $z_\pm$ are the boundary markings and $z_1, \ldots, z_n$ are the interior markings. We call z_- (resp. z_+) the *incoming* (resp. *outgoing*) marking. Let $S_1, \ldots, S_m$ denote the ordered *strip components* of S connecting z_- to z_+; the remaining components are either *disk components*, if they have boundary, or *sphere components*, otherwise. The nodal strip is stable if it is stable as a marked disk, as in Definition 4.2. Let $\tilde{w}_i := S_i \cap S_{i+1}$ denote the *intermediate node* connecting S_i to S_{i+1} for $i = 1, \ldots, m-1$. Let $\tilde{w}_0 = z_-$ and $\tilde{w}_m = z_+$ denote the incoming and outgoing markings.

(b) (Strip coordinates) Given a nodal strip S denote the curve obtained by removing the nodes connecting strip components and the incoming and outgoing markings

$$S^\times := S - \{\tilde{w}_0, \ldots, \tilde{w}_m\}. \tag{6.6}$$

Each strip component in $S^\times$ may be equipped with coordinates

$$\phi_i : S_i^\times := S_i - \{\tilde{w}_i, \tilde{w}_{i-1}\} \to \mathbb{R} \times [0, 1], \quad i = 1, \ldots, m$$

satisfying the conditions that if j is the standard complex structure on $\mathbb{R} \times [0, 1]$ and j_i is the complex structure on S_i then

$$\phi_i^* j = j_i, \quad \lim_{z \to \tilde{w}_i} \pi_1 \circ \phi_i(z) = -\infty, \quad \lim_{z \to \tilde{w}_{i+1}} \pi_1 \circ \phi_i(z) = \infty$$

where π_i denotes the projection on the i^{th} factor. We denote by

$$f : S^\times \to [0, 1], \quad z \mapsto \pi_2 \circ \phi_i(z) \tag{6.7}$$

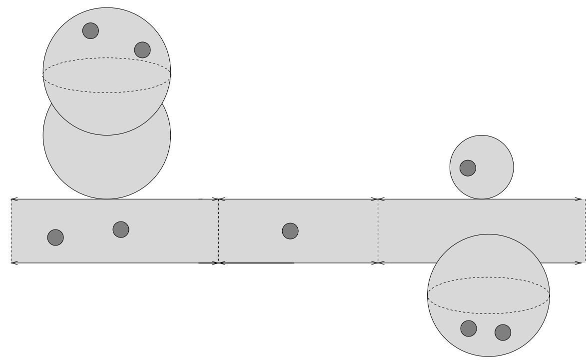

FIGURE 6.1. A stable marked strip

the continuous map induced by the *time* coordinate on the strip components. The time coordinate is extended to nodal marked strips S by requiring constancy on every connected component of $S^\times - \bigcup_i S_i^\times$.

(c) (Partition of the boundary) The boundary of any marked strip S is partitioned as follows. For $b \in \{0,1\}$ denote

$$(\partial S^\times)_b := f^{-1}(b) \cap \partial S \quad \text{so that} \quad \partial S^\times = (\partial S)_0 \cup (\partial S)_1. \tag{6.8}$$

That is, $(\partial S)_b$ is the part of the boundary from z_- to z_+, for $b = 0$, and from z_+ to z_- for $b = 1$. An example of a stable strip is shown in Figure 6.1.

(d) (Treed strips) A *treed strip* is a space $C = S \cup T$ obtained from a marked strip $(S_0, \underline{z})$ (that is, a marked disk with two distinguished boundary markings) by replacing each boundary node and marking of S_0 with a possibly broken edge e of some length $\ell(e)$ and number of breakings $b(e)$; denote by T the union of such edges.

(e) (Combinatorial types) The *combinatorial type* Γ of a treed strip C is defined as in the case of tree disks, but there is a distinguished incoming semi-infinite edge $e_- \in \text{Edge}(\Gamma)$ and a outgoing edge $e_+ \in \text{Edge}(\Gamma)$.

(f) (Isomorphisms) An *isomorphism* of treed marked strips $\phi : C \to C'$ is an isomorphism of the underly treed marked disks preserving the incoming and outgoing edges. A treed strip $C = S \cup T$ is *stable* if the treed pseudoholomorphic marked disk S marked by the intersections $S \cap T$ is stable.

We introduce the following notation for moduli spaces. Let $\overline{\mathcal{M}}_{n_0|n_1,m}$ denote the moduli space of isomorphism classes of stable treed strips with n_0, n_1 boundary markings (using the partition of the boundary) and m interior markings. For Γ a connected type we denote by $\mathcal{M}_\Gamma \subset \overline{\mathcal{M}}_{n_0|n_1,m}$ the moduli space of stable strips of combinatorial type Γ and $\overline{\mathcal{M}}_\Gamma$ its closure. Each moduli space $\overline{\mathcal{M}}_\Gamma$ is naturally a manifold with corners, with local charts obtained by a standard gluing construction. Generally $\overline{\mathcal{M}}_{n_0|n_1,m}$ is the union of several top-dimensional strata. For Γ disconnected, $\overline{\mathcal{M}}_\Gamma$ is the product $\prod_i \overline{\mathcal{M}}_{\Gamma_i}$ of moduli spaces for the connected components Γ_i of Γ.

DEFINITION 6.3. (a) (Universal strip) By the universal strip we mean the complement of the intermediate nodes in the universal treed disk. That is, $\overline{\mathcal{U}}_\Gamma$ is the union of the curves $C^\times$ from (6.6). On the universal strip, the time coordinates (6.7) on the strip components extend to a map

$$f : \overline{\mathcal{U}}_\Gamma \to [0, 1] \tag{6.9}$$

On the subsets $f^{-1}(0), f^{-1}(1)$ with time coordinate equal to zero or one, we have additional maps measuring the distance to the strip components given by summing the lengths of the connecting edges:

$$\ell_b : f^{-1}(b) \to [0, \infty], \quad z \mapsto \sum_{e \in \mathrm{Edge}(z)} \ell(e), \quad b \in \{0, 1\}$$

where $\mathrm{Edge}(z)$ is the set of edges corresponding to nodes between z and the strip components. Thus any point on the universal strip $z \in \overline{\mathcal{U}}_\Gamma$ which lies on a disk component has $f(z) \in \{0, 1\}$.

(b) (Partition by dimension) The universal treed strip can be written as the union of one-dimensional and two-dimensional parts

$$\overline{\mathcal{U}}_\Gamma = \overline{\mathcal{S}}_\Gamma \cup \overline{\mathcal{T}}_\Gamma$$

so that $\overline{\mathcal{S}}_\Gamma \cap \overline{\mathcal{T}}_\Gamma$ is the set of points on the disks or spheres meeting the edges of the tree. Denote by

$$\overline{\mathcal{T}}^b_\Gamma = f^{-1}(b) \cap \overline{\mathcal{T}}_\Gamma$$

the part of the tree corresponding to the minimum resp. maximum values and

$$\overline{\mathcal{T}}^{01}_\Gamma = \overline{\mathcal{T}}_\Gamma - \overline{\mathcal{T}}^0_\Gamma - \overline{\mathcal{T}}^1_\Gamma.$$

6.3. Hamiltonian perturbations

The Fukaya bimodules in this paper will be defined using the Morse model, by a combination of Hamiltonian-perturbed pseudoholomorphic maps on the strip components and Morse trajectories on the edges. We introduce notations for Hamiltonian perturbations and associated perturbed Cauchy-Riemann operators. Let S be a nodal strip. Let

$$K \in \Omega^1(S, \partial S; C^\infty(X))$$

be a one-form with values in smooth functions vanishing on the tangent space to the boundary, that is, a smooth map $TS \times X \to \mathbb{R}$ linear on each fiber of TS, equal to zero on $TS|\partial S$. Denote by

$$\widehat{K} \in \Omega^1(S, \partial S; \mathrm{Vect}(X))$$

the corresponding one form with values in Hamiltonian vector fields. The *curvature* of the perturbation is

$$R_K = \mathrm{d}K + \{K, K\}/2 \in \Omega^2(S, C^\infty(X)) \tag{6.10}$$

where $\{K, K\} \in \Omega^2(S, C^\infty(X))$ is the two-form obtained by combining wedge product and Poisson bracket, see McDuff-Salamon [**95**, Lemma 8.1.6]. Given a map $u : S \to X$, define

$$\overline{\partial}_{J,K} u := (\mathrm{d}u + \widehat{K})^{0,1} \in \Omega^{0,1}(S, u^*TX). \tag{6.11}$$

The map u is *(J,K)-holomorphic* if $\overline{\partial}_{J,K} u = 0$.

For Hamiltonian-perturbed pseudoholomorphic maps the action and energy are related by an equality involving a curvature correction term. Suppose that C is equipped with a compatible metric and X is equipped with a tamed almost complex structure and perturbation K. The *K-energy* of a map $u : C \to X$ is

$$E_K(u) := \frac{1}{2}\int_C |\mathrm{d}u + \widehat{K}(u)|_J^2$$

where the integral is taken with respect to the measure determined by the metric on C and the integrand is defined as in [**95**, Lemma 2.2.1]. If $\overline{\partial}_{J,K} u = 0$, then the K-energy differs from the symplectic area $A(u) := \int_C u^*\omega$ by a term involving the curvature from (6.10):

$$E_K(u) = A(u) + \int_C R_K(u). \tag{6.12}$$

In particular, if the curvature vanishes then the area is non-negative. In the case of a strip

$$C = \mathbb{R} \times [0,1] = \{(s,t) \mid s \in \mathbb{R}, t \in [0,1]\}.$$

and Hamiltonian perturbation $H \in C^\infty([0,1] \times X)$ let K denote the perturbation one-form $K = -H\mathrm{d}t$ and let $E_H := E_K$. If $u : \mathbb{R} \times [0,1] \to X$ has limits $x_\pm : [0,1] \to X$ as $s \to \pm\infty$ then the energy-area relation (6.12) becomes $E_H(u) = A(u) - \int_{[0,1]}(x_+^* H - x_-^* H)\mathrm{d}t$.

Floer trajectories are solutions to the perturbed pseudoholomorphic equation with a translational notion of equivalence. Let $\overline{\partial}_{J,H} = \overline{\partial}_{J,K}$ be the corresponding perturbed Cauchy-Riemann operator from (6.11). A map $u : C \to X$ is *(J,H)-holomorphic* if $\overline{\partial}_{J,H} u = 0$. A *perturbed pseudoholomorphic strip* for Lagrangians L_0, L_1 is a finite energy (J,H)-holomorphic map $u : \mathbb{R} \times [0,1] \to X$ with $\mathbb{R} \times \{b\} \subset L_b, b = 0,1$. An *isomorphism* of Floer trajectories $u_0, u_1 : \mathbb{R} \times [0,1] \to X$ is a translation $\phi : \mathbb{R} \times [0,1] \to \mathbb{R} \times [0,1]$ in the $\mathbb{R}$-direction such that $\phi^* u_1 = u_0$. Denote by

$$\mathcal{M}(L_0, L_1) := \left\{ u : \mathbb{R} \times [0,1] \to X \;\middle|\; \begin{array}{c} \overline{\partial}_{J,H} u = 0 \\ u(\mathbb{R} \times \{b\}) \subset L_b, b \in \{0,1\} \\ E_H(u) < \infty \end{array} \right\} / \mathbb{R}$$

the moduli space of isomorphism classes of Floer trajectories of finite energy, with its quotient topology.

Floer trajectories with Hamiltonian perturbation correspond to unperturbed pseudoholomorphic maps with a perturbed boundary condition. Let $H \in C^\infty([0,1] \times \mathbb{R}, X)$ be a time-dependent Hamiltonian and let $J \in \mathrm{Map}([0,1], \mathcal{J}_\tau(X,\omega))$ be a time-dependent almost complex structure. Suppose that L_0, L_1 are Lagrangians such that $\varphi_1(L_0) \cap L_1$ is transversal. There is a bijection between (J_t, H_t)-holomorphic Floer trajectories $u : \mathbb{R} \times [0,1] \to X$ with boundary conditions L_0, L_1 and $(\varphi_{1-t}^{-1})^* J_t$-holomorphic Floer trajectories with boundary conditions $\varphi_1(L_0), L_1$ obtained by mapping each (L_0, L_1) trajectory $(s,t) \mapsto u(s,t)$ to the $(\varphi_1(L_0), L_1)$-trajectory given by $(s,t) \mapsto \varphi_{1-t}(u(s,t))$ [**42**, Discussion after (7)].

In order to relate bimodules defined using different perturbations we also consider Hamiltonian perturbations on surfaces with strip-like ends. A *surface with strip like ends* consists of a surface with boundary S equipped with a complex structure $j : TS \to TS$, and a collection of embeddings

$$\kappa_e : \pm(0,\infty) \times [0,1] \to S, \quad e = 0, \ldots, n$$

such that $\kappa_e^* j$ is the standard almost complex structure on the strip, and the complement of the union of the images of the maps κ_e is compact. Any such surface has a canonical compactification $\overline{S}$ with the structure of a compact surface with boundary obtained by adding a point at infinity along each strip like end and taking the local coordinate to be the exponential of $\pm 2\pi i \kappa_e$.

Hamiltonian-perturbed pseudoholomorphic maps are in bijection with holomorphic sections of the trivial bundle with respect to a non-product almost complex structure. Given a surface with strip-like ends let $E := S \times X$ denote the product considered as a fiber bundle over S with fiber X. Following [**95**, (8.1.3)], let $\pi_X : E \to X$ denote the projection on the fiber. In local coordinates s, t on S define K_s, K_t by $K = K_s \mathrm{d}s + K_t \mathrm{d}t$. Let

$$\omega_E = \pi_X^* \omega - \pi_X^* \mathrm{d}K_s \wedge \mathrm{d}s - \pi_X^* \mathrm{d}K_t \wedge \mathrm{d}t + (\partial_t K_s - \partial_s K_t) \mathrm{d}s \wedge \mathrm{d}t.$$

The form ω_E is closed, restricts to the two-form ω on any fiber, and defines the structure of a symplectic fiber bundle on E over S. Consider the splitting $TE \cong \pi_X^* TX \oplus (S \times \mathbb{R}^2)$. Let $j_S : TS \to TS$ denote the standard complex structure on S. Define an almost complex structure on E by

$$J_E : TE \to TE, \quad (v, w) \mapsto ((J\hat{K} - \hat{K} j_S) w + Jv, j_S w)$$

where $\hat{K} \in \Omega^1(S, \mathrm{Vect}(X))$ is the Hamiltonian-vector-field-valued one-form associated to K. A smooth map $u : S \to X$ is (J, K)-holomorphic if and only if the associated section $(\mathrm{id} \times u) : S \to E$ is J_E-holomorphic [**95**, Exercise 8.1.5]. Let $\tilde{L}_i = (\partial S)_i \times L_i$ the fiber-wise Lagrangian submanifolds of E defined by L_i. Then $u : S \to X$ has boundary conditions in $(L_i, i = 1, \ldots, m)$ if and only if $\mathrm{id} \times u : S \to E$ has boundary conditions in $(\tilde{L}_i, i = 1, \ldots, m)$.

6.4. Clean intersections

In this section we extend the above results to the case that the union of the cleanly-intersecting Lagrangians is rational using stabilizing divisors. We have in mind especially the case that the two Lagrangians are rational and equal. In the particular case of diagonal boundary conditions, we show that the Floer cohomology is the singular cohomology with Novikov coefficients.

We recall some terminology for clean intersections. A pair $L_0, L_1 \subset X$ of submanifolds intersect *cleanly* if $L_0 \cap L_1$ is a smooth manifold and $T(L_0 \cap L_1) = TL_0 \cap TL_1$. Floer homology for clean intersections was constructed in Pozniak [**109**] and also Schmäschke [**113**, Section 7] under certain monotonicity assumptions, with the Floer differential counting pseudoholomorphic strips. The definition of Floer cohomology in the clean intersection case is a count of configurations of pseudoholomorphic strips, disks, spheres, and Morse trees as in, for example, Biran-Cornea [**15**, Section 4]. Let $L_k, k \in \{0, 1\}$ be compact Lagrangian branes equipped with Morse-Smale pairs (F_k, G_k). In particular, the critical set $\mathrm{crit}(F_k) \subset L_k$ of each Morse function is finite. Let

$$\underline{P}_k = (P_{\Gamma,k} = (J_{\Gamma,k}, F_{\Gamma,k})), k \in \{0, 1\}$$

be regular perturbation data giving rise to moduli spaces of pseudoholomorphic treed disks with boundary in L_k. Consider a a Hamiltonian with flow

$$H_{01} \in C^\infty(X \times [0, 1]), \quad \varphi_t^{01} : X \to X$$

so that $\varphi_1^{01}(L_0) \cap L_1$ is a clean intersection and $\varphi_1^{01}(L_0) \cup L_1$ is rational. That is, some power of the line-bundle-with-connection $\widetilde{X}$ is trivializable over $L_0 \cup L_1$. This includes the special cases of equality $\varphi_1^{01}(L_0) = L_1$ if L_0 is rational. Let

$$F_{01} : \varphi_1^{01}(L_0) \cap L_1 \to \mathbb{R}$$

be a Morse function. By the Morse lemma, the critical set

$$\mathcal{I}(L_0, L_1; H_{01}) := \operatorname{crit}(F_{01}) = \{l \in L_0 \cap L_1 \mid \mathrm{d}F_{01}(l) = 0\}$$

is necessarily finite. Choose a generic metric G_{01} on $\varphi_1^{01}(L_0) \cap L_1$, and let

$$\varphi_t^{01} : L_0 \cap L_1 \to L_0 \to L_1$$

be the time t flow of $-\operatorname{grad}(F_{01}) \in \operatorname{Vect}(L_0 \cap L_1)$. Denote the stable and unstable manifolds of F_{01}:

$$W_x^{\pm} = \left\{ l \in L_0 \cap L_1 \;\middle|\; \lim_{t \to \pm\infty} \varphi_t^{01}(l) = x \right\}.$$

We assume that (F_{01}, G_{01}) is *Morse-Smale*, that is, the stable and unstable manifolds meet transversally

$$T_l W_x^+ + T_l W_y^- = T_l(\varphi_1^{01}(L_0) \cap L_1), \quad \forall l \in W_x^+ \cap W_y^-, \; x, y \in \operatorname{crit}(F).$$

The critical set admits a natural grading map

$$i : \mathcal{I}(L_0, L_1; H_{01}) \to \mathbb{Z}_g$$

obtained by adjusting the Maslov index of paths from $T_x L_0$ to $T_x L_1$ in $T_x X$ for any $x \in \operatorname{crit}(F_{01})$ by the index $i(x) = \dim(W_x^-)$.

The space of Floer cochains is then generated by the finite set of critical points on the given Morse function on the clean intersection:

$$CF(L_0, L_1; H_{01}) = \bigoplus_{x \in \mathcal{I}(L_0, L_1; H_{01})} \Lambda x$$

with $\mathbb{Z}_g$-grading induced by the grading on $\mathcal{I}(L_0, L_1; H_{01})$.

The structure maps for the Fukaya bimodule count configurations containing perturbed pseudoholomorphic strips and gradient segments for the Morse function on the intersection. Given a stable strip C_0 with boundary markings z_-, z_+ let $w_1, \ldots, w_k \in C_0$ denote the nodes appearing in any non-self-crossing path between z_- and z_+. Define a *treed strip*

$$C = C_0 \sqcup \bigsqcup_{i=1}^{k} [0, \ell(w_i)] / \sim$$

by replacing each node w_i by a segment $T_i \cong [0, \ell(w_i)]$ of length $\ell(w_i)$. Denote by

$$T = T_1 \cup \ldots \cup T_k \quad S = \overline{C - T}$$

the *tree resp. surface part* of C. A perturbed pseudoholomorphic strip is then a map from $C = S \cup T$ that is J-holomorphic on the surface part and a F_{01} resp. F_0 resp. F_1 gradient trajectory on each segment in T. See Figure 6.2. Fix *thin parts* of the universal curves: a neighborhood $\overline{\mathcal{T}}_\Gamma^{\text{thin}}$ of the endpoints and a neighborhood $\overline{\mathcal{S}}_\Gamma^{\text{thin}}$ of the markings and nodes. In the regularity construction, these neighborhoods must be small enough so that either a given fiber is in a neighborhood of the boundary, where transversality has already been achieved, or otherwise each segment and each

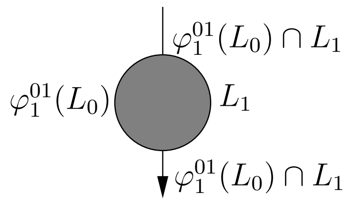

FIGURE 6.2. A treed strip with Lagrangian boundary conditions

disk or sphere component in a fiber meets the complement of the chosen thin parts. For an integer $l \geq 0$ a *domain-dependent perturbation* of F of class C^l is a C^l map

$$F_\Gamma : \overline{\mathcal{T}}_\Gamma \times (\varphi_1^{01}(L_0) \cap L_1) \to \mathbb{R} \tag{6.13}$$

equal to the given function F away from the endpoints:

$$F_{\Gamma,01} | \overline{\mathcal{T}}_\Gamma^{\text{thin},01} = \pi_2^* F_{01}, \quad F_{\Gamma,k} | \overline{\mathcal{T}}_\Gamma^{\text{thin},k} = \pi_2^* F_k, \quad k \in \{0,1\}$$

where π_2 is the projection on the second factor in (6.13). A *domain-dependent almost complex structure* of class C^l for treed disks of type Γ is a map from the two-dimensional part $\overline{\mathcal{S}}_\Gamma$ of the universal curve $\overline{\mathcal{U}}_\Gamma$ to $\mathcal{J}_\tau(X)$ given by a C^l map

$$J_\Gamma : \ \overline{\mathcal{S}}_\Gamma \times X \to \text{End}(TX)$$

gives as the product of pull-backs of maps

$$J_{\Gamma(v)} : \ \overline{\mathcal{S}}_{\Gamma(v)} \times X \to \text{End}(TX)$$

and equal to the given J_D near the nodes and boundary:

$$J_\Gamma | \overline{\mathcal{S}}_\Gamma^{\text{thin}} = \pi_2^* J_D.$$

A *perturbed pseudoholomorphic strip* for the pair (L_0, L_1) consists of a treed disk C and a map $u : C = S \cup T \to X$ such that

(Boundary condition) The Lagrangian boundary condition holds $u((\partial C)_b) \subset L_b$ for $b \in \{0,1\}$.

(Surface equation) On the surface part S of C the map u is J-holomorphic for the given domain-dependent almost complex structure: if j denotes the complex structure on S then

$$J_{\Gamma,u(z),z} \ \mathrm{d}u_S = \mathrm{d}u_S \ j.$$

(Boundary tree equation) On the boundary tree part $T \subset C$ the map u is a collection of gradient trajectories:

$$\frac{d}{ds} u|_{T_b} = -\operatorname{grad}_{F_{\Gamma,b,(s,u(s))}} (u|_{T_b})$$

where s is a local coordinate with unit speed and b is one of the symbols 0, 1 or 01. Thus for each edge $e \in \text{Edge}_-(\Gamma)$ the length of the trajectory is given by the length $u|_{e \subset T}$ is equal to $\ell(e)$.

For each combinatorial type Γ, the moduli space $\mathcal{M}_\Gamma(L_0, L_1)$ of treed strips is locally cut out as the zero set of a Fredholm map as in (4.19).

To obtain regular moduli spaces we introduce a stabilizing divisor. Given a stabilizing divisor $D \subset X - (L_0 \cup L_1)$, a stable strip $u : C \to X$ is *adapted* if and only if (Stable surface axiom) C is a stable marked strip; and (Leaf axiom) each interior leaf T_e lies in $u^{-1}(D)$ and each component of $u^{-1}(D)$ contains an interior leaf T_e. Let $\overline{\mathcal{M}}(L_0, L_1, D)$ denote the set of isomorphism classes of stable adapted Floer trajectories to X, and by $\mathcal{M}_\Gamma(L_0, L_1, D)$ the subspace of combinatorial type Γ. Compactness and transversality properties of the moduli space of Floer trajectories in the case of clean intersection, including exponential decay estimates, can be found in [**134**] and [**113**]. The necessary gluing result can be found in Schmäschke [**113**, Section 7].

In order to define the structure maps in the Fukaya bimodules, we assume that the union of Lagrangians is equipped with the following further structure:

(a) (Local system) Let y_0, y_1 denote the local systems on L_0, L_1 assumed as part of their brane structures. We suppose that $L_0 \cup L_1$ is equipped with a local system $\rho \in \mathcal{R}(L_0 \cup L_1)$ restricting to the given local systems on y_0, y_1. Such a local system is determined by an isomorphism of the corresponding flat line bundles on the intersection $L_0 \cap L_1$, but this isomorphism is not unique.

(b) (Relative spin structure) We suppose that the embedding $L_0 \cup L_1 \to X$ is equipped with a relative spin structure, restricting to the given relative spin structures on L_0, L_1.

Given relative spin structures as above, the construction of orientations proceeds along the lines of Fukaya-Oh-Ohta-Ono [**46**, Chapter 10], see also Wehrheim-Woodward [**133**]. We may ignore the constraints at the interior markings $z_1, \ldots, z_m \in \operatorname{int}(S)$, since the tangent spaces to these markings and the linearized constraints $\mathrm{d}u(z_i) \in T_{u(z_i)}D$ are even dimensional and oriented by the given complex structures. At any regular element $(u : C \to X) \in \mathcal{M}(L_0, L_1, D)$ the tangent space to the moduli space of treed disks is the kernel of the linearized operator

$$T_u\mathcal{M}(L_0, L_1, D) \cong \ker(\tilde{D}_u).$$

The operator $\tilde{D}_u$ is canonically homotopic via family of operators $\tilde{D}_u^t, t \in [0,1]$ to the operator $0 \oplus D_u \oplus \frac{d}{ds}$ given by zero on the variations of complex structure on the domain, D_u on the variations of map $S \subset C$, and $\frac{d}{ds}$ on the variations of map on the tree part $T \subset C$. The deformation $\tilde{D}_u^t, t \in [0,1]$ of operators induces an family of determinant lines over the interval, necessarily trivial, and so (by taking a connection on this family) an identification of determinant lines

$$\det(T_u\mathcal{M}(L_0, L_1, D)) \to \det(T_C\mathcal{M}_\Gamma) \otimes \det(D_u) \tag{6.14}$$

well-defined up to isomorphism. (Here D_u denotes the linearized operator subject to the constraints which require the attaching points of edges mapping to critical points to map to the corresponding unstable manifolds of the Morse function.) In the case of nodes of S mapping to intersection points $x \in L_0 \cap L_1$ the determinant line $\det(D_u)$ is oriented by "bubbling off one-pointed disks", see [**46**, Theorem 44.1] or [**133**, Equation (36)]. That is, for each intersection point $x \in L_0 \cap L_1$ choose a path of Lagrangian subspaces

$$\gamma_x : [0,1] \to \operatorname{Lag}(T_xX), \quad \gamma_x(0) = T_xL_0 \quad \gamma_x(1) = T_xL_1. \tag{6.15}$$

Let S be the unit disk with a single boundary marking $1 \in \partial S$. The path γ_x defines a totally real boundary condition on S on the trivial bundle with fiber T_xX, equipped with a Cauchy-Riemann operator D_x acting on the space of $W^{k,p,-\delta}$ sections of T_xX with totally real boundary conditions, for some Sobolev weighting $\delta > 0$ sufficiently small so that the operator D_x is Fredholm. Let $\det(D_x)$ denote the determinant line for the Cauchy-Riemann operator D_x with boundary conditions γ_x. Denote by

$$i(x) = \dim(\ker(D_x)) - \dim(\operatorname{coker}(D_x)) \in \mathbb{Z}$$

the index of the operator D_x. Let $\mathbb{D}_x^+ = \det(D_x^+)$ and let $\mathbb{D}_x^-$ be the tensor product of the determinant line $\det(D_k^-)$ for the once-marked disk with $\det(T_x L_0) \cong \det(T_x L_1)$. Because the once-marked disks with boundary conditions γ_{x_k} and $\gamma_{\overline{x_k}}$ glue together to a trivial problem on the disk with index $T_{x_k}L$, there is a canonical isomorphisms

$$\mathbb{D}_{x_k}^- \otimes \mathbb{D}_{x_k}^+ \to \mathbb{R}.$$

The orientations for the intersection points are *coherent* if the above isomorphism is orientation preserving with respect to the standard orientation on $\mathbb{R}$. The orientation for a treed strip u is determined by an isomorphism

$$\det(D_u) \cong \mathbb{D}_{x_0}^+ \otimes \mathbb{D}_{x_1}^- \otimes \ldots \otimes \mathbb{D}_{x_d}^-. \tag{6.16}$$

The isomorphism (6.16) is determined by degenerating the surface with strip-like ends to a nodal surface. Thus each end $\epsilon_e, e \in \mathcal{E}(S_i)$ of a component S_i with a node q_k mapping to a intersection point is replaced by a disk $S_{i^\pm(k)}$ with one end attached to the rest of the surface by a node $q_k^\pm$. After combining the orientations on the determinant lines on $S_{i^\pm(k)}$ with coherent orientations on the tangent spaces to the stable manifolds $W_{x_k}^\pm$ in the case of broken edges or semi-infinite edges, one obtains an orientation on the determinant line of the parameterized operator $\det(\tilde{D}_u)$ and so orientations on the regularized moduli spaces $\mathcal{M}_\Gamma(L_0, L_1, D)$. In particular the zero-dimension component of the moduli space inherits an orientation map

$$\epsilon : \mathcal{M}(L_0, L_1, D)_0 \to \{+1, -1\}$$

comparing the constructed orientation to the canonical orientation of a point.

The structure maps in the Fukaya bimodules are defined by weighted counts of perturbed-holomorphic treed strips. The weighting is by powers $q^{A(u)}$ of the formal variable q with exponent given by the vertical symplectic area $A(u)$ for $u \in \mathcal{M}(L_0, L_1)$; while these powers are non-negative for any particular Fukaya bimodule the homotopy invariance maps will require possibly negative powers, so that the homotopy type is only invariant over the Novikov field Λ rather than the Novikov ring. As before let $\sigma(u)$ denote the number of interior markings; $\epsilon(u)$ is the orientation sign while $y(u)$ is the holonomy of the local system. Define

$$CF(L_0, L_1; H_{01}) = \bigoplus_{l \in \mathcal{I}(L_0, L_1; H_{01})} \Lambda l \tag{6.17}$$

Counting Floer trajectories defines operations

$$\mu^{d|e} : CF(L_0)^{\otimes d} \otimes CF(L_0, L_1; H_{01}) \otimes CF(L_1)^{\otimes e} \to CF(L_0, L_1; H_{01})$$

by

$$\begin{aligned} & x_1^0 \otimes \ldots \otimes x_d^0 \otimes x \otimes x_1^1 \otimes \ldots \otimes x_e^1 \\ & \quad \mapsto \sum_{n, u \in \mathcal{M}_{d|e,n}(x_1^0, \ldots, x_d^0, x, x_1^1, \ldots, x_e^1, y)_0} (-1)^{\heartsuit + \diamondsuit} \epsilon(u) y(u) (n!)^{-1} q^{A(u)} y \end{aligned} \tag{6.18}$$

where $\diamondsuit$ is defined in (6.4)

Theorem 6.4. *For admissible collections $\underline{P} = (P_\Gamma)$ of perturbation data the maps $(\mu^{d|e})_{d,e\geq 0}$ induce on $CF(L_0, L_1; H_{01})$ the structure of a strictly unital A_∞ $(CF(L_0), CF(L_1))$-bimodule.*

As in the case of A_∞ algebras, the relation (6.1) follows from a study of the ends of the one-dimensional moduli space. The configurations $u : C \to X$ representing endpoints of the one-dimensional moduli spaces arise when one of the segments e in the configuration becomes length $\ell(e)$ infinity, and so broken. The first term in (6.1) arises when the broken edge e connects a strip $S_1 \subset S$ to a treed disk configuration $S_2 \subset S$ attached at a point where $t = 1$; the second term when a broken edge $e, \ell(e) = \infty$ connects a strip S_2 to a treed disk configuration S_1 attached at a point where $t = 0$; and the third when the broken edge $e, \ell(e) = \infty$ connects two strips S_1, S_2. The sign computation is similar to that for A_∞ algebras, and left to the reader.

6.5. Morphisms

In the remainder of this section we show that up to A_∞ homotopy the Fukaya bimodule is independent of all choices, including the choice of Hamiltonian perturbation. In particular, if a Lagrangian is Hamiltonian displaceable then its Fukaya bimodule is homotopy equivalent to the trivial bimodule. The necessary morphisms of A_∞ bimodules are given by counting *parametrized treed strips*. Let C be a disk with distinguished boundary markings $z_-, z_+ \in \partial C$. A *parametrization* of (C, z_-, z_+) is a holomorphic isomorphism

$$\phi : C - \{z_-, z_+\} \to \mathbb{R} \times [0, 1].$$

A $(d|e, n)$*-marking* of a parametrized strip is a collection of d resp. e resp n markings on $\mathbb{R} \times \{0\}$ resp. $\mathbb{R} \times \{1\}$ resp. in $\mathbb{R} \times (0, 1)$. In the treed version of this moduli space, the $(d|e, n)$-marking becomes a collection of leaves, $d+e$ on the boundary and n in the interior. The moduli space of $(d|e, n)$-leafed treed parametrized strips has a natural compactification $\overline{\mathcal{M}}_{d|e,n,1}$ by stable treed parametrized strips, in which unparametrized strips are allowed to bubble off the ends. This moduli space is equipped with a continuous map $\overline{\mathcal{M}}_{d|e,n,1} \to \overline{\mathcal{M}}_{d|e,n}$ forgetting the parametrization on quilted strip components. The fibers of the forgetful map are canonically oriented so that the positive orientation corresponds to moving the quilting to the left, and so the orientation of $\overline{\mathcal{M}}_{d|e,n}$ induces an orientation on $\overline{\mathcal{M}}_{d|e,n,1}$.

Regularization of the space of pseudoholomorphic parametrized treed strips uses a divisor in the total space of the fibration $\mathbb{R} \times [0, 1] \times X$. Let

$$J_e \in \mathcal{J}(X, \omega), \quad e \in \{0, 1\}$$

be compatible almost complex structures stabilizing for $\varphi_{H_e,1}(L_0) \cup L_1$. Let $\varphi^*_{H_e,1-t} J_e$ denote the corresponding time-dependent almost complex structures, and $\sigma_{k,e} : X \to \widetilde{X}^k$ are asymptotically J_e-holomorphic, uniformly transverse sequences of sections with the property that

$$D'_e = \sigma^{-1}_{k,e}(0), \quad e \in \{0, 1\}$$

are stabilizing for $\varphi_{H_e,1}(L_0) \cup L_1$ for k sufficiently large. The pull-backs $\phi^*_{e,1-t}\sigma_{k,e}$ are then $\phi^*_{e,1-t} J_e$-holomorphic, and any (J_e, H_e)-holomorphic strip with boundary in (L_0, L_1) meets $\phi^{-1}_{e,1-t}(S \times D'_e)$ in at least one point.

These divisors extend over the parametrized strip, so that any Hamiltonian-perturbed pseudoholomorphic strip meets the extended divisor. Denote by $\tilde{E} \to E$ the pull-back of $\widetilde{X} \to X$ to the fibration, equipped with the almost complex structure induced by the given almost complex structure on E. Recall the following from Charest-Woodward [**27**, Theorem 6.1]:

LEMMA 6.5. *Let S be a surface with strip-like ends, let $L_b \subset X$ be rational Lagrangians associated to the boundary components $(\partial S)_b$, and suppose that stabilizing divisors D_e' for the ends $e = 1, \ldots, n$ of S have been chosen as zero sets of asymptotically holomorphic sequences of sections $\sigma_{e,k}$ for k sufficiently large. There exists a asymptotically J_E-holomorphic, uniformly transverse sequence $\sigma_k : E \to \tilde{E}^k$ with the property that for each end e, the pull-back $\kappa_e^* \sigma_k(\cdot + s, \cdot, \cdot)$ converges in C^∞ uniformly on compact subsets to $\phi_{e,1-t}^* \sigma_{e,k}$ as $s \to \pm\infty$. The zero set $D_E = \sigma_k^{-1}(0)$ is approximately holomorphic for k sufficiently large, asymptotic to $(1 \times \phi_{e,1-t})^{-1}(\mathbb{R} \times [0,1] \times D_e')$ for each end $e = 1, \ldots, n$, and stabilizing for holomorphic disks or spheres in the fibers of $E \to S$.*

PROOF. We include the proof for completeness. Let $\mathbb{H} = \{z \in \mathbb{C} \mid \operatorname{Im}(z) \geq 0\}$ denote complex half-space. Let $\overline{E}$ be the fiber bundle over $\overline{S}$ with fiber X defined by gluing together $U_0 = S \times X$ and $U_e = \mathbb{H} \times X, e = 1, \ldots, n$ using the transition maps $\kappa_e \times \phi_{H_e,1\pm t}$ on $\mathbb{H} - \{0\} \cong \mathbb{R} \times [0,1]$ from U_0 to U_e. Denote the projections over U_e to X by $\pi_{X,e}$. The two-forms ω_E on U_0, and $\pi_{X,e}^* \omega$ on U_e glue together to a two-form ω_E on E, making $\overline{E}$ into a symplectic fiber bundle. The fiber-wise symplectic form ω_E above may be adjusted to an honest symplectic form on E by adding a pull-back from the base, and furthermore adjusted so that the boundary conditions have rational union. For the first claim, since ω_E is fiber-wise symplectic, there exists a symplectic form $\nu \in \Omega^2(\overline{S})$ with the property that $\omega_E + \pi^* \nu$ is symplectic, where $\pi : \overline{E} \to \overline{S}$ is the projection. The almost complex structure J_E is compatible with $\omega_E + \pi^* \nu$, and equal to the given almost complex structures $J_B \oplus J_e$ on the ends. For the second claim, let $\tilde{S} \to \overline{S}$ be a line-bundle-with-connection whose curvature is $\nu \in \Omega^2(\overline{S})$. Let $\tilde{E} \to E$ be a line-bundle-with-connection whose curvature is $\omega_E + \pi^* \nu$. Denote by $\tilde{L}_i$ the closure of the image of $(\partial S)_i \times L_i$ for $i = 0, 1$. Fix trivializations of $\tilde{E}$ over $\tilde{L}_{e,0} \cap \tilde{L}_{e,1} \cong L_{e,0} \cap L_{e,1}$ for each end e. By assumption, the line bundle $\tilde{E}$ is trivializable over L_i, hence also $\tilde{L}_i$ by parallel transport along the boundary components. Let $e_k, k = 0, 1$ be ends connected by a connected boundary component labelled by L_i. For any $p_k \in \phi_{H_e,1}(L_0) \cap L_1$ the parallel transport $T(p_0, p_1) \in U(1)$ from p_0 to p_1 is independent of the choice of path. Indeed, any two paths differ up to homotopy by a loop which has trivial holonomy by assumption. After perturbation of the connection and curvature on $\tilde{E}$, we may assume that the parallel transports $T(p_0, p_1)$ are rational for all choices of (p_0, p_1). After taking a tensor power of $\tilde{E}$, we may assume that the parallel transports $T(p_0, p_1)$ are trivial, hence $\tilde{E}$ admits a covariant constant section τ over the union $\tilde{L}_e$.

Donaldson's construction [**35**] implies the existence of a symplectic hypersurface in the total space of the fibration. We show that the hypersurface $D_E \subset E$ may be taken to equal the pullback of one of the given ones $D_e', e \in \{0, 1\}$ on the ends, as follows. Let $\sigma_{m,b} : X \to \widetilde{X}^k$ be an asymptotically holomorphic sequence of sections concentrating on L_b. Let $\sigma_{e,k} : X \to \widetilde{X}^k$ be an asymptotically holomorphic sequence of sections concentrating on $\phi_{H_e,1}(L_{e,0}) \cup L_{e,1}$ and $\sigma_{e,m}$ an asymptotically holomorphic sequence of sections concentrating on $L_{e,m}$, $m \in \{0,1\}$, both asymptotic to the given trivializations on the Lagrangians themselves. For each point $z \in \overline{S}$, let $\sigma_{z,k} : \overline{S} \to \tilde{S}^k$ denote the Gaussian asymptotically holomorphic sequence of sections of $\tilde{S}^k$ concentrated at z as in (6.19). We may assume that the images of the strip-like ends are disjoint. Let V_i be disjoint open neighborhoods $(\partial S)_i - \cup_e \operatorname{Im}(\kappa_e)$ in S. For each $p = (z,x) \in \tilde{L}_i$, let $\sigma_{x,k}$ be either equal to $\sigma_{e,k}$, for $z \in \operatorname{Im}(\phi_e)$ or otherwise equal to $\sigma_{i,k}$ if b lies in V_i. Let P_k be a set of points in $\partial\overline{S}$ such that the balls of g_k-radius 1 cover $\partial\overline{S}$ and any two points of P_k are at least distance $2/3$ from each other, where g_k is the metric defined by $k\nu$. The desired asymptotically-holomorphic sections are obtained by taking products of asymptotically-holomorphic sections on the two factors: Write $\theta(z,x)\tau(z,x) = \sigma_{x,k}(x) \boxtimes \sigma_{z,k}(z)$ so that $\theta(x,z) \in \mathbb{C}$ is the scalar relating the two sections. Define

$$\sigma_k = \sum_{p \in P_k} \theta(z,x)^{-1} \sigma_{x,k} \boxtimes \sigma_{z,k}. \tag{6.19}$$

Then by construction the sections (6.19) are asymptotically holomorphic, since each summand is.

Recall from [**35**] that a sequence $(s_k)_{k\geq 0}$ is *uniformly transverse* to 0 if there exists a constant η independent of k such that for any $x \in X$ with $|s_k(x)| < \eta$, the derivative of s_k is surjective and satisfies $|\nabla s_k(x)| \geq \eta$. The sections σ_k are asymptotically J_E-holomorphic and uniformly transverse to zero over $\partial\overline{S} \times X$, since the sections $\sigma_{i,k}$ and $\sigma_{e,k}$ are uniformly transverse to the zero section. Hence σ_k is also uniformly transverse over a neighborhood of ∂S. Pulling back to E one obtains an asymptotically holomorphic sequence of sections of $\tilde{E}|S$ that is informally transverse in a neighborhood of infinity, that is, except on a compact subset of E. Donaldson's construction [**35**] although stated only for compact manifolds, applies equally well to non-compact manifolds assuming that the section to be perturbed is uniformly transverse on the complement of a compact set. The resulting sequence $\sigma_{E,k}$ is uniformly transverse and consists of asymptotically holomorphic sections asymptotic to the pull-backs of $\sigma_{e,k}$ on the ends. The divisor $D_E = \sigma_{E,k}^{-1}(0)$ is approximately holomorphic for k sufficiently large and equal to the given divisors $\pm(0,\infty) \times D_e$ on the ends, by construction, and concentrated at L_b, over each boundary component $(\partial S)_b$. The stabilizing properties follow. □

A perturbation scheme similar to the one for Floer trajectories makes the moduli spaces transverse. We restrict to the case $S \cong \mathbb{R} \times [0,1]$ Choose a tamed almost complex structure $J_E \in \mathcal{J}_\tau(E, \omega_E + \pi_B^*\nu)$ leaving D_E invariant, so that D_E contains no holomorphic spheres, each holomorphic sphere meets D_E in at least three points, and each disk with boundary in $\tilde{L}_0 \cup \tilde{L}_1$ meets D_E in at least one point. Since D_E is only approximately holomorphic with respect to the product complex structure, the complex structure J_E will not necessarily be of split form, nor will the projection to S necessarily be (J_E, j_S)-holomorphic away from the ends. Furthermore, choose domain-dependent perturbations F_Γ of the Morse functions F_e on $\phi_{H_e,1}(L_0) \cap L_1$, so that F_Γ is a perturbation of F_e on the segments that map to $\phi_{H_e,1}(L_0) \cap L_1$.

Domain-dependent perturbations give a regularized moduli space of adapted treed strips. These are maps $u : C \cong \mathbb{R} \times [0,1] \to E$, homotopic to sections, with the given Lagrangian boundary conditions and mapping the positive resp. negative end of the strip to the positive resp. negative end of E. As before, there is a compactified moduli space $\overline{\mathcal{M}}(L_0, L_1, \underline{D}, \underline{P})$ with a single parametrized strip component, and also including disk, sphere, and unparametrized strip components. For admissible perturbations $\underline{P}$ given as collections $(P_{\Gamma,E} = (F_{\Gamma,E}, J_{\Gamma,E}))$ on E, the moduli space $\overline{\mathcal{M}}(L_0, L_1, \underline{D}, \underline{P}, (x_e))$ of perturbed maps to E with boundary in $\tilde{L}_0 \cup \tilde{L}_1$ and limits $(x_e), e \in \{0,1\}$ has zero and one-dimensional components that are compact and smooth with the expected boundary. In particular the boundary of the one-dimensional moduli spaces $\mathcal{M}_1(L_0, L_1, \underline{D}, \underline{P}, (x_e))$ are 0-dimensional strata $\mathcal{M}_\Gamma(L_0, L_1, \underline{D}, \underline{P}, (x_e))$ corresponding to either a perturbed pseudoholomorphic strip bubbling off on end, or a disk bubbling off the boundary.

Counting rigid elements of the moduli spaces of parametrized treed strips defines morphisms between bimodules. Given two Hamiltonian perturbations H'_{01}, H''_{01} such that the intersections $\varphi_1^{01,'}(L_0) \cap L_1$ and $\varphi_1^{01,''}(L_0) \cap L_1$ are clean, consider a Hamiltonian perturbation $H = H_s \mathrm{d}s + H_t \mathrm{d}t$ equal to $H'_{01}\mathrm{d}t$ for $s \gg 0$ and to $H_{01}\mathrm{d}t''$ for $s \ll 0$. Define operations

$$\begin{aligned}(6.20)\quad \phi^{d|e} : CF(L_0)^{\otimes d} \otimes CF(L_0, L_1; H'_{01}) \otimes CF(L_1)^{\otimes e} \to CF(L_0, L_1; H''_{01}) \\ z_1^0 \otimes \ldots \otimes z_d^0 \otimes z \otimes z_1^1 \otimes \ldots \otimes z_e^1 \mapsto \\ \sum_{u \in \overline{\mathcal{M}}_{d|e,1}(z_1^0,\ldots,z_d^0,z,z_1^1,\ldots,z_e^1,y)_0} (-1)^{\heartsuit + \diamondsuit} \epsilon(u) y(u) (\sigma(u)!)^{-1} q^{A(u)} y\end{aligned}$$

where the cochain groups involved are defined using Λ coefficients; the values $A(u)$ are not necessarily positive because of the additional term in the energy-area relation (6.12).

Proposition 6.6. *For admissible collections of perturbation data, the maps $(\phi^{d|e})_{d,e\geq 0}$ form a strictly unital morphism of A_∞ bimodules from $CF(L_0, L_1; H'_{01})$ to $CF(L_0, L_1; H''_{01})$.*

Proof. The proof is essentially the same as that for A_∞ algebra morphisms in Theorem 5.7. The true boundary components of $\overline{\mathcal{M}}_{d|e,1}(\underline{z}_0, \underline{z}_1, y)_1$ consist of configurations with a broken segment in which either treed strip has broken off or a treed disk has broken off. The former case corresponds to one of the first two terms in (6.2) while the latter corresponds to the last two terms. The first term in (6.2) (in which μ appears before ϕ) has an additional sign coming from the definition of the orientation on $\overline{\mathcal{M}}_{d|e,1}$ as an $\mathbb{R}$-bundle over $\overline{\mathcal{M}}_{d|e}$, so that the orientation of the fiber corresponds to composing parametrization with a translation in the positive direction; this means that the positive orientation on the gluing parameter for the boundary components corresponding to terms of the first type becomes identified with the negative orientation on these fibers, giving rise to the additional sign. The degree of the morphism is zero, which causes those contributions to the sign in (6.2) to vanish. This leaves the contributions from $\aleph$, which are similar to those dealt with before and left to the reader. □

6.6. Homotopies

In this section we show that the morphisms of bimodules introduced above satisfy a gluing law of the type found in topological field theory. The construction uses a moduli space of *twice* parametrized treed strips defined as follows. Let C

be a holomorphic strip, that is, disk with markings z_-, z_+ on the boundary. A *twice-parametrization* is pair $\underline{\phi} = (\phi_0, \phi_1)$ of holomorphic isomorphisms

$$\phi_k : C - \{z_-, z_+\} \to \mathbb{R} \times [0,1], \quad k \in \{0,1\}.$$

An *isomorphism* of a twice-parametrized disks $(C_l, \underline{\phi}_l), l \in \{1,2\}$ is a holomorphic isomorphism C_1 to C_2 that intertwines parametrizations $\phi_{k,l}, k \in \{0,1\}$ for $l \in \{1,2\}$. Each parametrization is equivalent to an affine structure on $C - \{z_+\}$, and in this way a twice-parametrized strip is equivalent to a twice-quilted disk in the language of Chapter 5.5, but with a distinguished incoming boundary marking. In particular, the moduli space of twice-parametrized treed strips with additional boundary markings and interior markings has a compact moduli space with locally toric singularities.

In the compactification, parametrized or unparametrized strips can now bubble off on either end. Suppose that we have chosen Hamiltonian perturbations $H_{01}^a, a = 0,1,2$ so that the intersections $\phi_t^a(L_0) \cap L_1$ are clean where ϕ_t^a denotes the time t flow of H_{01}^a. We extend the Hamiltonian perturbation over the universal twice-parametrized strip and choose generic domain-dependent almost complex structures. The weighted counts of twice-parametrized strips define operations

$$\begin{aligned} (6.21) \quad \tau^{d|e} : CF(L_0)^{\otimes d} \otimes CF(L_0, L_1; H_{01}^0) \otimes CF(L_1)^{\otimes e} \to CF(L_0, L_1; H_{01}^2)[-1] \\ (z_1^0, \dots, z_d^0, z, \dots, z_1^1, \dots, z_e^1, y) \\ \mapsto \sum_{u \in \mathcal{M}_{d|e,2}(z_1^0, \dots, z_d^0, z, z_1^1, \dots, z_e^1, y)_0} (-1)^{\heartsuit + \diamondsuit} \epsilon(u) y(u) (\sigma(u)!)^{-1} q^{A(u)} y. \end{aligned}$$

THEOREM 6.7. *Suppose that admissible perturbation data $\underline{P}^{ab} = (P_\Gamma^{ab})$ for once-parametrized strips have been chosen giving rise to morphisms of A_∞ bimodules*

$$\phi_{ab} : CF(L_0, L_1; H_{01}^a) \to CF(L_0, L_1; H_{01}^b), \quad 0 \le a < b \le 2.$$

For any admissible collection of perturbations $\underline{P} = (P_\Gamma)$ extending the given collections $\underline{P}^{ab} = (P_\Gamma^{ab})$ over the universal curves over moduli spaces of twice-parametrized strips, the resulting operations $(\tau^{d|e})_{d,e \ge 0}$ define a homotopy of morphisms of A_∞ bimodules

$$\phi_{12} \circ \phi_{01} \simeq_\tau \phi_{12} \in \operatorname{Hom}(CF(L_0, L_1; H_{01}^0), CF(L_0, L_1; H_{01}^2)).$$

PROOF. The statement is a consequence of the description of twice-parametrized strips, similar to that of the existence of homotopies between A_∞ functors defined by quilted disks in Theorem 5.10. The components of the boundary of the moduli space of twice-quilted strips $\overline{\mathcal{M}}_{d|e,2}(z_1^0, \dots, z_d^0, z, z_1^1, \dots, z_e^1, y)_1$ that do *not* involve the parametrized strip breaking, or the shaded region disappearing, correspond the terms on the left-hand side of (6.2). The strata where the twice-parametrized strip breaks into once-parametrized strips correspond to the contributions to the composition $\phi_{12} \circ \phi_{01}$, while the components where the parametrized strip vanishes give the identity morphism of A_∞ modules. The sign computation is the same. □

For any $b_0 \in MC(L_0), b_1 \in MC(L_1)$ we denote by $HF(L_0, L_1; H_{01}; b_0, b_1)$ the cohomology of the operator $\mu^1_{b_0,b_1}$. It follows as in Lemma 5.3 that any morphism of A_∞ bimodules corresponding to difference choices of perturbations H_{01}, H_{01}' induces a map

$$HF(L_0, L_1; H_{01}; b_0, b_1) \to HF(L_0, L_1; H_{01}'; b_0, b_1)$$

of cohomology groups, functorially. As a result, $HF(L_0, L_1; H_{01}; b_0, b_1)$ is independent of the choice of the choice of Hamiltonian perturbation. On the other hand, for the case $L_0 = L_1 = L$ the A_∞ bimodule $CF(L_0, L_1; 0)$ is equal to the span of $x \neq x^{\triangledown}, x^{\blacktriangledown}$ in $CF(L)$ itself considered as an A_∞ bimodule over $CF(L)$, so that using Lemma 4.44 we have

$$HF(L, L; b; b) \cong HF(L, b).$$

COROLLARY 6.8. *If a compact rational Lagrangian brane L in a compact rational symplectic manifold X is Hamiltonian displaceable, then $HF(L, b)$ vanishes for every $b \in MC(L)$.*

PROOF. Let $\phi : X \to X$ be a Hamiltonian diffeomorphism with $\phi(L) \cap L = \emptyset$. Then the homotopy equivalences $CF(L) \simeq CF(L, L) \simeq CF(L, \phi(L)) = \{0\}$ imply that $HF(L, b) = \{0\}$ for all $b \in MC(L)$. $\square$

CHAPTER 7

Broken Fukaya algebras

In this section we introduce a version of the Fukaya algebra for Lagrangians in *broken* symplectic manifolds along the lines of *symplectic field theory* as introduced by Eliashberg-Givental-Hofer [**39**]. The symplectic manifold is degenerated, through neck stretching, to a *broken* symplectic manifold. A sequence of pseudoholomorphic curves with respect to the degenerating almost complex structure converges to a *broken pseudoholomorphic map*: a collection of pseudoholomorphic curves in the pieces as well as maps to the neck region. In the version that we consider here introduced by Bourgeois [**23**], these components are connected by gradient trajectories of possibly finite or infinite length.

7.1. Broken curves

First we describe the kind of domains that appear in the particular kind of broken limit we will consider, following Bourgeois-Eliashberg-Hofer-Wysocki-Zehnder [**22**].

Definition 7.1. Let $n, m, s \geq 0$ be integers. A *level of a broken curve* with n boundary markings, m interior markings, and s sublevels consists of

(a) a sequence $C = (C_1, \ldots, C_s)$ of treed nodal curves with boundary, called *sublevels*; in our situation only the first piece C_1 is allowed to have non-empty boundary: $\partial C_2 = \ldots \partial C_s = \emptyset$.
(b) interior markings $\underline{z}_i^{\pm} \subset C_i, \quad i = 1, \ldots, s$.
(c) a collection of *(possibly broken) finite and semi-infinite edges attached to boundary points* that are intervals $I_1, \ldots, I_m$ connecting boundary points in components of C_i for $i = 1, \ldots, s$ with lengths $\ell(I_i)$; since in our situation only C_1 is allowed to have non-empty boundary, these edges only connecting components of C_1; and
(d) a sequence of *intervals* $I_1^i, \ldots, I_{s(i)}^i$ attached to interior points in C_i with finite lengths $\ell(I_k^i)$ independent of k;

Out of the data above one constructs a topological space C by removing the nodes and gluing in the intervals $I_1^i, \ldots, I_{s(i)}^i$. A *broken curve* with k levels is obtained from k curves at a single level $C_1, \ldots, C_k$ by gluing together the endpoints of the semi-infinite interior edges at infinity, as in Figure 7.1. The *combinatorial type* of a broken curve C is the graph Γ obtained by gluing together the types $\Gamma_1, \ldots, \Gamma_k$ of the levels, with the partition into sub-graphs remembered. A broken disk C is defined similarly to a broken curve but the first level C_1 of C is a disjoint union of treed disk and sphere components with connected boundary and segments attached, while the other pieces have surface parts that are spheres; furthermore, the combinatorial type $\Gamma(C)$ is a tree. Weightings are defined as in Definition 4.3: each semi-infinite edge $e \in \text{Edge}_{\to}(\Gamma(C))$ is assigned a weight $\rho(e) \in [0, \infty]$. A broken curve C is *stable*

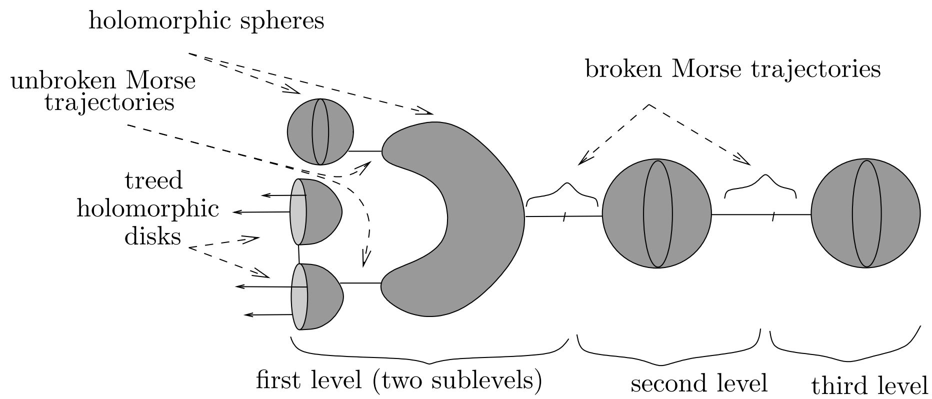

FIGURE 7.1. A broken disk

if it has only finitely many automorphisms $\phi : C \to C$, except for automorphisms of infinite length segments $C_i \cong \mathbb{R}$ with one weighted end $e_- \in \mathcal{E}(C)$ and one unweighted end $e_+ \in \mathcal{E}$.

7.2. Broken maps

A broken map is a map from a broken curve into a broken symplectic manifold, defined as follows.

DEFINITION 7.2. (a) (Broken symplectic manifold) Let X be a compact rational symplectic manifold. Let $Z \subset X$ be a coisotropic hypersurface separating $X \backslash Z$ into components $X^\circ_\subset, X^\circ_\supset$. Suppose that Z has null foliation

$$\ker(\omega|Z) \subset TZ$$

that is a circle fibration over a symplectic manifold Y; that is, there exists a *Reeb vector field* $v \in \text{Vect}(Z)$ taking values in the null foliation whose flow defines an action of S^1 on Z so that $Y := Z/S^1$ is a smooth manifold. By the coisotropic embedding theorem, a neighborhood of Z in X is symplectomorphic to $(-\epsilon, \epsilon) \times Z$ for $\epsilon > 0$ small equipped with a symplectic structure arising from a connection. By a construction of Lerman [**78**] the unions

$$X_\subset := X^\circ_\subset \cup Y, \quad X_\supset := X^\circ_\supset \cup Y$$

have the structure of symplectic submanifolds. Let $N_\pm \to Y$ denote the normal bundle of Y in $X_\subset, X_\supset$, and $N_\pm \oplus \underline{\mathbb{C}}$ the sums with the trivial bundle $\underline{\mathbb{C}} = Y \times \mathbb{C}$. Denote by $\mathbb{P}(N_+ \oplus \underline{\mathbb{C}}) \cong \mathbb{P}(N_- \oplus \underline{\mathbb{C}})$ the projectivized normal bundle, where the isomorphism is induced from

$$\mathbb{P}(N_+ \oplus \underline{\mathbb{C}}) \cong \mathbb{P}((N_+ \oplus \underline{\mathbb{C}}) \otimes N_-)) = \mathbb{P}(N_- \oplus \underline{\mathbb{C}}).$$

The *broken symplectic manifold* arising from the triple $(X_\subset, X_\supset, Y)$ is the topological space

$$\mathbb{X} = X_\subset \cup_Y X_\supset$$

obtained by identifying the copies of Y in $X_\subset$ and $X_\supset$. Thus $\mathbb{X}$ is a stratified space and the link of Y in $\mathbb{X}$ is a disjoint union of two circles. The space Y comes equipped with an isomorphism of normal bundles

$$(TX_\subset)_Y/TY \cong ((TX_\supset)_T/TY)^{-1}. \tag{7.1}$$

(b) (Multiply broken symplectic manifold) For an integer $m \geq 1$ define the $m-1$-*broken symplectic manifold*

$$\mathbb{X}[m] = X_\subset \cup_Y \mathbb{P}(N_\pm \oplus \underline{\mathbb{C}}) \cup_Y \mathbb{P}(N_\pm \oplus \underline{\mathbb{C}}) \cup_Y \ldots \cup_Y X_\supset$$

where there are $m-2$ copies of $\mathbb{P}(N_\pm \oplus \underline{\mathbb{C}})$ called *broken levels*. Define

$$\mathbb{X}[m]_0 = X_\subset, \quad \mathbb{X}[m]_1 = \mathbb{P}(N_\pm \oplus \underline{\mathbb{C}}), \quad , \ldots, \mathbb{X}[m]_m = X_\supset. \tag{7.2}$$

There is a natural action of $\mathbb{C}^\times$ on $\mathbb{P}(N \oplus \mathbb{C})$ given by scalar multiplication on each projectivized normal bundle:

$$\mathbb{C}^\times \times \mathbb{P}(N_\pm \oplus \underline{\mathbb{C}}) \to \mathbb{P}(N_\pm \oplus \underline{\mathbb{C}}), \quad (z, [n, w]) \mapsto z[n, w] := [zn, w].$$

The fixed points of the $\mathbb{C}^\times$ action are the divisors at 0 and ∞:

$$\mathbb{P}(N_\pm \oplus \underline{\mathbb{C}})^{\mathbb{C}^\times} = \{[n, 0]\} \cup \{[0, w]\}$$

where n reps. w ranges over vectors in $N_\pm$ resp. $\underline{\mathbb{C}}$

(c) An almost complex structure $J \in \mathcal{J}(\mathbb{R} \times Z)$ is of *cylindrical form* if there exists an almost complex structure J_Y on Y such that the projection $\pi_Y : \mathbb{R} \times Z \to Y$ is almost complex and J is invariant under the $\mathbb{C}^\times$-action on $\mathbb{R} \times Z$ induced from the embedding in $\mathbb{P}(N_\pm \oplus \underline{\mathbb{C}})$ given by

$$\mathbb{C}^\times \times \mathbb{P}(N_\pm \oplus \underline{\mathbb{C}}) \to \mathbb{P}(N_\pm \oplus \underline{\mathbb{C}}), \quad s\exp(it)(s_0, z) = (s_0 + s, \exp(it)z).$$

That is,

$$D\pi_Y J = J_Y D\pi_Y, \quad J \in \mathcal{J}(\mathbb{R} \times Z)^{\mathbb{C}^\times}.$$

Denote by $\mathcal{J}^{\rm cyl}(\mathbb{R} \times Z)$ and for any $\epsilon > 0$ denote by $\mathcal{J}^{\rm cyl}((0, \infty) \times Z)$ the image of $\mathcal{J}^{\rm cyl}(\mathbb{R} \times Z)$ under restriction. Denote by

$$\mathcal{J}(\mathbb{X}) = \mathcal{J}(X_\subset^\circ) \times_{\mathcal{J}^{\rm cyl}((0,\infty)\times Z)} \mathcal{J}(X_\supset^\circ)$$

the fiber product consisting of tamed almost complex structures of cylindrical form on the neck region. (Note that this definition differs from the one in [**22**, Section 2.1], which does not require the invariance of the almost complex structure under the Reeb flow and so does not suffice for our purposes.)

Definition 7.3. (Broken maps) Let $\mathbb{X} = X_\subset \cup_Y X_\supset$ be a broken symplectic manifold as above, and $L \subset X_\subset$ a Lagrangian disjoint from Y. Let $J \in \mathcal{J}(\mathbb{X})$ be an almost complex structure on of cylindrical form, H be a Morse function on Y and (F, G) a Morse-Smale pair on L. A *broken map* to $\mathbb{X}$ with boundary values in L consists of:

(a) (Broken curve) a broken curve $C = (C_0, \ldots, C_p)$;

(b) (Broken map) a map $u : C \to \mathbb{X}$, that is, a collection of maps (notation from (7.2))

$$u_k : C_k \to \mathbb{X}[p]_k, \quad k = 0, \ldots, p;$$

(c) (Framings at infinity) for each pair of interior nodes $w_\pm \in C$ connected by an interior gradient trajectory $T_e \subset C$, a framing $\tau(w_\pm) : T_{w_\pm} S \to \mathbb{C}$ of the tangent spaces $T_{w_\pm} S$ to the surface S;

satisfying the following conditions:

(a) (Pseudoholomorphicity) On the two-dimensional part $S \subset C$, the map u is J-holomorphic, that is, $\overline{\partial}_J(u|_S) = 0$.
(b) (Gradient flow in the Lagrangian) On the one-dimensional part $T_\circ \subset C$ connecting boundary nodes, u is a segment of a gradient trajectory on each interval component for the Morse function F on L:

$$\left(\frac{d}{dt} + \mathrm{grad}_F\right)(u|T_\circ) = 0.$$

(c) (Intersection multiplicity) If a pseudoholomorphic map $u : C \to X$ has isolated intersections with an almost complex codimension two submanifold $Y \subset X$ then at each point $z \in u^{-1}(Y)$ there is a positive *intersection multiplicity* $\mu(u, z) \in \mathbb{Z}_{>0}$ describing the winding number of a small loop counterclockwise around Y:

$$\mu(u, z) = [u(z + r\exp(i\theta))|_{\theta\in[0,2\pi]}] \in \pi_1(U - (U \cap Y)) \cong \mathbb{Z}$$

where U is a contractible open neighborhood of z and r is sufficiently small so that $u(z + r\exp(i\theta)) \in U$ for all $\theta \in [0, 2\pi]$.
(d) (Gradient flow in the manifold) On the one-dimensional part $T_\bullet \subset C$ connecting interior nodes, u is a segment of a gradient trajectory on each interval component

$$\left(\frac{d}{dt} + \mathrm{grad}_H\right)(u|T_\bullet) = 0;$$

and satisfying the following:

(a) (Matching condition for multiplicities) For any pair of nodes $w_\pm$ of the domain C connected by an interior trajectory, the intersection multiplicities $\mu(w_\pm)$ of the map u with the hypersurface Y are equal:

$$\mu(w_-) = \mu(w_+).$$

(b) (Matching condition for framings) The local framings satisfy the following condition: Choose local coordinates $z_\pm$ on a neighborhood of $w_\pm$ inducing the given framings $\tau(w_\pm)$ on $T_{w_\pm}C$. Let $X_\pm \subset \mathbb{X}[k]$ denote the components receiving the components $S_\pm \subset C$ containing $w_\pm$ and choose local coordinates $(z_1^\pm, \ldots, z_n^\pm)$ so that $Y = \{z_1^\pm = 0\}$ and choose local coordinates on $X_\pm$ compatible with the isomorphism of normal bundles (7.1). Then in local coordinates on $X_\pm$ the map $z_\pm \mapsto u_1(z_\pm)$ has leading order term

$$u_1(z_\pm) \sim z_\pm^{\mu(w_\pm)}. \tag{7.3}$$

We also define the following notions:

(a) (Isomorphisms of broken maps) An isomorphism between broken maps

$$u_i : C_i \to \mathbb{X}[k], \ i \in \{0, 1\}$$

is an isomorphism of domains $\phi : C_0 \to C_1$ together with an element $g \in (\mathbb{C}^\times)^{k-1}$ such that $u_1 \circ \phi = gu_0$, and so that the framings $\tau_1(w_\pm)$ are equivalent up to simultaneous rotation: $\tau_1(w_\pm) = \zeta D_{w_\pm}\phi\tau_0(w_\pm)$ for some $\zeta \in \mathbb{C}^\times$. Note that the leading order condition (7.3) determines the framings $\tau(w_\pm)$ up to an $\mu(w_\pm)$-order root of unity. Since two framings related by simultaneous rotation are considered equivalent, there are $\mu(w_\pm)$ inequivalent framings allowed at each node connected by an internal edge.

(b) (Combinatorial type) The *combinatorial type* Γ of a broken map $u : C \to \mathbb{X}$ is the combinatorial type of the underlying curve C, but with the additional data of the homology class $u_{i,*}[C_i]$ of each component C_i (as a labelling of the vertices) and the intersection multiplicities $m(z_i) \in \mathbb{Z}_{\geq 0}$ with the stabilizing divisor D at each attaching point z_i of an edge or interior node. Let Γ be a type with n leaves (corresponding to trajectories of the Morse function on the Lagrangian) and l broken Morse trajectories on the degenerating divisor. An *admissible labelling* for a Γ is a collection $\underline{l} \in \mathcal{I}(L)^{n+1}$ such that whenever the corresponding label is $x^{\blacktriangledown}$ resp. $x^{\blacktriangledown}$ resp. $x^{\triangledown}$ or the corresponding leaf has weight 0 resp. $[0, \infty]$ resp. ∞.

Denote by $\overline{\mathcal{M}}(L, \mathbb{X}, \mathbb{D})$ the union over types, and by $\mathcal{M}(L, \mathbb{X}, \mathbb{D})$ the locus of types formally of top dimension where there are at most two levels $C_0, C_1 \subset C$ and each edge $T_e, e \in \operatorname{Edge}(\Gamma)$ not connecting two different levels has finite and non-zero length. This ends the Definition.

Remark 7.4. In contrast to the unbroken case, configurations $u : C \to \mathbb{X}$ with two levels are not positive codimension since there is no gluing construction which produces a broken map. On the other hand, configurations u with a neck piece u_i may be glued to broken maps in two different ways, depending on whether that neck piece u_i is glued with the piece u_0 or u_p mapping to $X_\subset$ or $X_\supset$.

Broken maps may be viewed as pseudoholomorphic maps of curves with cylindrical ends, by the removal of singularities argument explained in Tehrani-Zinger [**129**, Lemma 6.6]. This leads to a natural notion of convergence in which the moduli space of broken maps of any given combinatorial type is compact. The compactness statement is essentially a special case of compactness in symplectic field theory [**22**], [**1**], although the particular set-up here has not been considered before. First we recall terminology for the type of cylindrical ends we consider. First we introduce notation for the symplectic manifolds with cylindrical ends: Let $X_\pm^\circ$ denote the manifold obtained by removing the divisor Y, or more generally, for the intermediate pieces $\mathbb{P}(N_\pm \oplus \underline{\mathbb{C}})^\circ \cong \mathbb{R} \times Z$ the manifold obtained by removing the divisors at zero and infinity, isomorphic to Y. We identify a neighborhood of infinity in $\mathbb{P}(N_\pm \oplus C)$ with $\mathbb{R}_{>0} \times Z$ with the almost complex structure induced from a connection on Z and the given almost complex structure on Y.

Recall that the notion of Hofer energy for symplectizations of contact manifolds with fibrating null-foliations, which is a special case of a more general definition for stable Hamiltonian structures in [**22**]. Let $X = \mathbb{R} \times Z$, where Z is equipped with closed two-form $\omega_Z \in \Omega^2(Z)$ with fibrating null-foliation and connection form $\alpha \in \Omega^1(Z)$.

Definition 7.5. (Action and energy)

(a) (Horizontal energy) The *horizontal energy* of a holomorphic map $u = (\phi, v) : (C, j) \to (\mathbb{R} \times Z, J)$ is ([**22**, 5.3])

$$E^h(u) = \int_C v^* \omega_Z.$$

(b) (Vertical energy) The *vertical energy* of a holomorphic map $u = (\phi, v) : (C, j) \to (\mathbb{R} \times Z, J)$ is ([**22**, 5.3])

$$E^v(u) = \sup_{\zeta} \int_C (\zeta \circ \phi) \mathrm{d}\phi \wedge v^* \alpha \tag{7.4}$$

where the supremum is taken over the set of all non-negative C^∞ functions

$$\zeta : \mathbb{R} \to \mathbb{R}, \quad \int_{\mathbb{R}} \zeta(s) \mathrm{d}s = 1$$

with compact support.

(c) (Hofer energy) The *Hofer energy* of a holomorphic map $u = (\phi, v) : (C, j) \to (\mathbb{R} \times Z, J)$ is ([**22**, 5.3]) is the sum

$$E(u) = E^h(u) + E^v(u).$$

(d) (Generalization to manifolds with cylindrical ends) Suppose that X° is a symplectic manifold with cylindrical end modelled on $\mathbb{R}_{>0} \times Z$. The vertical energy $E^v(u)$ is defined as before in (7.4). The Hofer energy $E(u)$ of a map $u : C^\circ \to X^\circ$ from a surface C° with cylindrical ends to X° is defined by dividing X° into a compact piece X^{com} and a cylindrical end diffeomorphic to $\mathbb{R}_{>0} \times Z$. Then we set

$$E(u) = E(u|X^{\mathrm{com}}) + E(u|\mathbb{R}_{\geq 0} \times Z).$$

A compact, Hausdorff moduli space of broken maps may be obtained by imposing an energy bound as well as a stability condition. As in previous cases, the moduli space so obtained will be regularized later using Cieliebak-Mohnke perturbations [**28**], in the cases of low expected dimension.

Definition 7.6. (Stable broken maps) A broken map $u : C \to \mathbb{X}[k]$ is *stable* if it has only finitely many automorphisms ϕ, except for automorphisms of infinite length segments $C_i \cong \mathbb{R}$ with one weighted end and one unweighted end. This means in particular at least one component at each level $u_i : C_i \to \mathbb{X}[p]$ is not a trivial cylinder.

Theorem 7.7. *(c.f. Bourgeois-Eliashberg-Hofer-Wysocki-Zehnder* [**22**]*) Any sequence of finite energy stable broken pseudoholomorphic maps $u_\nu : C_\nu \to \mathbb{X}[k]^\circ$ with bounded Hofer energy $\sup_\nu E(u_\nu) < \infty$ has a Gromov convergent subsequence to a stable limit, and any such convergent sequence has a unique limit.*

Sketch of proof. We will not give a complete proof but rather indicate how the proof can be adapted from the published treatments of sft compactness in Bourgeois-Eliashberg-Hofer-Wysocki-Zehnder [**22**], Abbas [**1**], and Cieliebak-Mohnke [**25**]. In the case that the almost complex structure is domain-independent, and preserves the horizontal subspace, Theorem 7.7 with the Hofer energy bound is essentially a special case of the compactness result in symplectic field theory [**22**, Section 5.4] (with further details and corrections in Abbas [**1**] and alternative approach given in [**25**]) with the additional complication of Lagrangian boundary conditions. Since the Lagrangian L is compact in $X^\circ_\subset$, the Lagrangian boundary conditions do not affect any of the arguments. Thus the question is to extend these results to domain-dependent almost complex structures, and the essential point is that our domain-dependent almost complex structures are domain-independent near the punctures.

Our particular setup corresponds to the case of relative stable maps in Ionel-Parker [**64**] and Li-Ruan [**87**], as explained in Bourgeois et al. [**22**, Remark 5.9]. In particular, asymptotic convergence follows from asymptotic convergence for holomorphic maps to Y; energy quantization for disks in $X_\subset$ implies energy quantization for finite energy holomorphic maps of half-cylinders to $X_\subset^\circ$, where the boundary of the cylinder maps to the Lagrangian L. Energy quantization for holomorphic maps of spheres to Y implies energy quantization for maps of holomorphic spheres to $\mathbb{P}(N_\pm \oplus \underline{\mathbb{C}})$: there exists a constant $\hbar > 0$ such that any holomorphic map $\mathbb{P}(N_\pm \oplus \underline{\mathbb{C}})$ with non-trivial projection to Y has energy at least $\hbar$. Matching of intersection multiplicities is [**22**, Remark 5.9], Tehrani-Zinger [**129**, Lemma 6.6]: By removal of singularities, there is a one-to-one correspondence between finite energy holomorphic curves in $X_\pm^\circ$ resp. $\mathbb{X}[k]^\circ$ and those in $X_\pm$ resp. $\mathbb{X}[k]$ that are not contained in the divisor Y resp. divisors at zero and infinity. Thus the intersection multiplicity is the degree of the cover of the Reeb orbit at infinity. It suffices to show that on each tree or surface part of the domain, a subsequence converges to some limit in the Gromov sense. Since each domain is stable, each surface part has a unique hyperbolic metric so that the boundary is totally geodesic, see Abbas [**1**, I.3.3]. Denote by $r_\nu : C_\nu \to \mathbb{R}_{>0}$ the injectivity radius. The argument of Bourgeois et al. [**22**, Chapter 10], see also Abbas [**1**], shows that after adding finitely many sequences of points to the domain we may assume that the domain C_ν converges to a limit C such that the first derivative $\sup |\mathrm{d}u_\nu|/r_\nu$ is bounded with respect to the hyperbolic metric on the surface part, and with respect to the given metric on the tree part. Thus there exists a limiting map $u : C^\times \to \mathbb{X}$ on the complement $C^\times$ of the nodes so that on compact subsets of the complement of the nodes a subsequence of u_ν converges to u in all derivatives. Removal of singularities and matching conditions then follows from the corresponding results for holomorphic maps: the matching condition for nodes mapping into the cylindrical end is simply the matching condition for the maps to Y, in addition to matching of intersection degrees which is immediate from the description as a winding number. Convergence on the tree part of the domain follows from uniqueness of solutions to ordinary differential equations. The extension to sequences with bounded area is [**22**, Lemma 9.2], or rather, the extension of that Lemma to curves with Lagrangian boundary conditions for Lagrangians not meeting the neck region, for which the proof is the same. □

Remark 7.8. (Exponential decay) Any holomorphic map $u : C^\circ \to X^\circ$ with finite Hofer energy converges *exponentially fast* to a Reeb orbit along each cylindrical end: In coordinates s, t on the cylindrical end diffeomorphic to $\mathbb{R}_{>0} \times Z$, there exists a constant C and constants $s_0, s_1 > 0$ such that for $s > s_1$ the distance $\mathrm{dist}(u(s,t), (\mu(s - s_0), \gamma(t)))$ is bounded by $C \exp(-s)$. This follows from the correspondence with holomorphic maps to the compactification in e.g. Tehrani-Zinger [**129**, Lemma 6.6].

7.3. Broken perturbations

In order to achieve transversality we introduce stabilizing divisors satisfying a compatibility condition with the degeneration and introduce domain-dependent almost complex structures and Morse functions. By a broken divisor we mean a divisor that arises from degeneration of a divisor in the original manifold via neck stretching.

Definition 7.9. (Broken divisors) A *broken divisor* for the broken almost complex manifold $\mathbb{X} := X_\subset \cup_Y X_\supset$ consists of a pair

$$\mathbb{D} = (D_\subset, D_\supset), \quad D_\subset \subset X_\subset,\ D_\supset \subset X_\supset$$

of codimension two almost complex submanifolds $D_\subset, D_\supset$ such that each intersection

$$D_\subset \cap Y = D_\supset \cap Y = D_Y$$

is a codimension two almost complex submanifold D_Y in Y. Given a broken divisor $\mathbb{D} = (D_\subset, D_\supset)$ as above we obtain a divisor

$$D_N := \mathbb{P}(N_\pm | D_Y \oplus \underline{\mathbb{C}}) \subset \mathbb{P}(N_\pm \oplus \underline{\mathbb{C}}).$$

We suppose that each $[D_\subset, D_\supset]$ is dual to a large multiple of the symplectic class on $X_\pm$, that is,

$$[D_\subset] = k[\omega_\subset], \quad [D_\supset] = k[\omega_\supset].$$

Then

$$[D_N] = k\pi_Y^*[\omega_Y]$$

where π_Y is projection onto Y, and as a result does not represent a multiple of any symplectic class on $\mathbb{P}(N_\pm \oplus \underline{\mathbb{C}})$. Thus the divisor D_N can be disjoint from non-constant holomorphic spheres in $\mathbb{P}(N_\pm \oplus \underline{\mathbb{C}})$, namely the fibers. However, holomorphic spheres whose projections to Y are non-constant automatically intersect D_N.

As in the unbroken case, transversality uses almost complex structures equal to a fixed almost complex structure on the stabilizing divisor. We introduce the following notations. For a symplectic manifold X° with cylindrical end, denote by $\mathcal{J}(X^\circ)$ the space of tamed almost complex structures on X° that are of cylindrical form on the end. Given $J_\mathbb{D} = (J_{D_\subset}, J_{D_\supset}) \in \mathcal{J}(\mathbb{X})$, denote by $\mathcal{J}(\mathbb{X}, J_\mathbb{D})$ the space of almost complex structures in $\mathcal{J}(\mathbb{X})$ that agree with $J_\mathbb{D}$ on $D_\subset, D_\supset$:

$$\mathcal{J}(\mathbb{X}, J_\mathbb{D}) = \{\, (J_\subset, J_\supset) \in \mathcal{J}(\mathbb{X}) \ \mid\ J_\subset | D_\subset = J_{D_\subset},\ J_\supset | D_\supset = J_{D_\supset} \,\}. \tag{7.5}$$

Fix a tamed almost complex structure $J_\mathbb{D}$ such that $D_\subset, D_\supset$ contains no non-constant $J_\mathbb{D}$-holomorphic spheres of any energy and any holomorphic sphere meets $D_\subset, D_\supset$ in at least three points, as in [**28**, Proposition 8.14]. By [**27**, Proposition 8.4], for any energy $E > 0$ there exists a contractible open neighborhood $\mathcal{J}^*(\mathbb{X}, J_\mathbb{D}, E)$ of $J_\mathbb{D}$ agreeing with $J_\mathbb{D}$ on $D_\supset, D_\subset$ with the property that $D_\subset, D_\supset$ still contains no non-constant holomorphic spheres and any holomorphic sphere of energy at most E meets $D_\subset, D_\supset$ in at least three points. Denote the fiber product

$$\mathcal{J}^*(\mathbb{X}, J_\mathbb{D}, E) := \mathcal{J}^*(X_\subset, J_{D_\subset}, E) \times_{\mathcal{J}^{\mathrm{cyl}}((0,\infty)\times Z)} \mathcal{J}^*(X_\supset, J_{D_\supset}, E).$$

Given a broken divisor define perturbation data for a broken symplectic manifold as before, but we also perturb the Morse function on the separating hypersurface $Y = X_\subset \cap X_\supset$. For base almost complex structures $J_{D,\pm}$ agreeing on Y, a *perturbation datum* for type Γ of broken maps is a datum

$$P_\Gamma = (J_\Gamma, F_\Gamma, G_\Gamma, H_\Gamma)$$

where

$$J_\Gamma :\ \overline{\mathcal{S}}_\Gamma \to \mathcal{J}(\mathbb{X}, J_\mathbb{D}), \quad F_\Gamma : \overline{\mathcal{T}}_{\Gamma,\circ} \to C^\infty(L)$$
$$G_\Gamma :\ \overline{\mathcal{T}}_{\Gamma,\circ} \to \mathcal{G}(L), \quad H_\Gamma : \overline{\mathcal{T}}_{\Gamma,\bullet} \to C^\infty(Y)$$

and J_Γ is equal to the given almost complex structures $J_{D_\subset}, J_{D_\supset}$ on $D_\subset, D_\supset$ and satisfies the (Locality Axiom).

In order to define perturbation data we note that the domain of a broken curve is not necessarily stable because of Morse trajectories of infinite length. However, given a broken curve C we obtain a stable broken curve $f(C)$ by collapsing unstable components and a perturbation system P_Γ for curves of such type by pulling back $P_{f(\Gamma)}$ under the stabilization map $C \to \mathcal{U}_{f(C)}$. Given a broken map $u : C \to \mathbb{X}[k]$, a *trivial cylinder* is a map to some intermediate piece $\mathbb{P}(N_\pm \oplus \mathbb{C})$ projecting to a constant map to Y. Given a type Γ of broken disk and a perturbation datum P_Γ, an *adapted broken map* is a map $u : C \to \mathbb{X}[k]$ from a broken weighted treed disk C to $\mathbb{X}[k]$ for some k such that each interior leaf $e \subset C$ maps to $\mathbb{D}$ and each component of $u^{-1}(\mathbb{D})$ contains an interior leaf $e \subset T$. An adapted broken map $u : C = S \cup T \to \mathbb{X}$ is *stable* if each level S_i of the surface part S the union of at least one stable curve $S_{i,k} \subset S_i$ and a collection of trivial cylinders $S_{i,l} \cong S^2, D\pi \circ \mathrm{d}u|S_{i,l} = 0$. The coherence, regularity, and stabilizing conditions on perturbation data from the unbroken case in Definition 4.12 generalize naturally. A perturbation system is *admissible* if it satisfies these three conditions.

The following generalizes the compactness and transversality results for Fukaya algebras to the broken case:

THEOREM 7.10. *Let Γ be an uncrowded type of adapted pseudoholomorphic broken treed disk of expected dimension at most one and suppose that admissible perturbation data $P_{\Gamma'}$ have been chosen for all boundary strata $\overline{\mathcal{U}}_{\Gamma'} \subset \overline{\mathcal{U}}_\Gamma$. There exists a comeager subset of the space of admissible perturbation data P_Γ equal to the given perturbation data on lower-dimensional strata such that*

(a) (Transversality) *every element of $\mathcal{M}_\Gamma(\mathbb{X}, L, \mathbb{D})$ is regular;*
(b) (Compactness) *the closure $\overline{\mathcal{M}}_\Gamma(\mathbb{X}, L, \mathbb{D})$ is compact and contained in the adapted uncrowded locus;*
(c) (Tubular neighborhoods) *each uncrowded stratum $\mathcal{M}_\Gamma(\mathbb{X}, L, \mathbb{D})$ of dimension zero has a tubular neighborhood of dimension one in any adjoining uncrowded strata of one higher dimension;*
(d) (Orientations) *the uncrowded strata $\mathcal{M}_\Gamma(\mathbb{X}, L, \mathbb{D})$ of formal dimension at most one are equipped with orientations satisfying the standard gluing signs for inclusions of boundary strata as in the unbroken case; in particular we denote by $\epsilon(u) \in \{\pm 1\}$ the orientation sign associated to elements u of the zero-dimensional moduli spaces $\mathcal{M}(\mathbb{X}, L, \mathbb{D})_0$.*

SKETCH OF PROOF. We describe the differences in the proof from the unbroken case in Theorem 4.28 and Theorem 4.20. Let $u_\nu : C_\nu \to \mathbb{X}$ be a sequence of adapted maps of the given combinatorial type Γ. By Theorem 7.7, there exists a limit $u : C \to \mathbb{X}$ in the Gromov sense that is a broken map. We claim that the limit is adapted and uncrowded. The proof of this claim is the same as that of Theorem 4.28: The stabilization condition on the divisor in (7.5) implies that any sphere bubble resp. disk bubble C_i appearing in the limit C has finitely many but at least three resp. one interior intersection points $u^{-1}(\mathbb{D}) \cap C_i$ with the stabilizing divisor $\mathbb{D}$. Furthermore, by preservation of intersection multiplicity $\#u(C).(\mathbb{D})$ with the divisor, each maximal ghost component $C_i \subset C$ mapping to the divisor $\mathbb{D}$ must contain at least one endpoint $z_j \in C_i$ of an interior leaf. Such a component C_i must be adjacent either to at least two non-ghost components C_j, C_k, a single non-ghost component C_j, or adjacent to two tree segments $T_j, T_j \subset C$. Strata of maps with a component with a point with intersection multiplicity two, or mapping the node to the divisor are codimension at least two, and so these strata do not occur in the limit. Hence the C_i contains at most one marking, so the limit is of uncrowded type.

Transversality as in the unbroken case in Theorem 4.20 is an application of Sard-Smale as in Charest-Woodward [**27**] on the universal space of maps. However, the divisor $\pi^{-1}(D_Y)$ does not intersect every non-trivial map to $\mathbb{P}(N_\pm \oplus \mathbb{C})$ in finitely many points, that is, the divisor on the neck pieces does not represent a multiple of the symplectic class. Any map contained in the divisor must be a multiple of a fiber and these maps are automatically transversal. We introduce a universal moduli space as follows. Let Γ be a type of broken map $u : C \to \mathbb{X}$, and Γ_0 the stabilization of the underlying type of curve C. We begin by covering the universal treed broken disk $\mathcal{U}_{\Gamma_0} \to \mathcal{M}_{\Gamma_0}$ by local trivializations $\mathcal{U}^i_{\Gamma_0} \to \mathcal{M}^i_{\Gamma_0}, i = 1,\ldots,N$. For each local trivialization consider a moduli space defined as follows. Let $\mathrm{Map}^{k,p}_\Gamma(C,\mathbb{X},L,\mathbb{D})$ denote the space of maps of class $k \geq 1, p \geq 1, kp > 2$ (with respect to some connections on $C, X_\subset, X_\supset$) making L totally geodesic) mapping the boundary of C into L, the interior markings into $\mathbb{D}$, and constant (or constant after projection to Y, if the component maps to a neck piece) on each disk with no interior marking. Let $\overline{\mathcal{U}}^{\mathrm{thin}}_{\Gamma_0}$ be a small neighborhood of the nodes and attaching points in the edges $\overline{\mathcal{U}}_\Gamma$, so that the complement in each edge and surface component is open. Let $\mathcal{P}^l_\Gamma(\mathbb{X},L,\mathbb{D})$ denote the space of perturbation data $P_\Gamma = (J_\Gamma,F_\Gamma,G_\Gamma,H_\Gamma)$ of class C^l for Γ_0 equal to the given pair (J,F,G,H) on $\overline{\mathcal{U}}^{\mathrm{thin}}_{\Gamma_0}$, and such that the restriction of P_Γ to $\overline{\mathcal{U}}_{\Gamma'}$ is equal to $P_{\Gamma'}$, for each boundary type Γ'. For any broken curve C of type Γ we obtain perturbation data on C by identifying the stabilization C^{st} of C with a fiber of the universal tree disk $\underline{\mathcal{U}}_{\Gamma_0}$. Let $l \gg k$ be an integer and

$$\mathcal{B}^i_{k,p,l,\Gamma} := \mathcal{M}^i_{\Gamma_0} \times \mathrm{Map}^{k,p}_\Gamma(C,\mathbb{X},L,\mathbb{D}) \times \mathcal{P}^l_\Gamma(\mathbb{X},L,\mathbb{D}).$$

Consider the map given by the local trivialization

$$\mathcal{M}^{\mathrm{univ},i}_{\Gamma_0} \to \mathcal{J}(S),\ m \mapsto j(m).$$

Let $S^{\mathrm{nc}} \subset S$ be the union of disk and sphere components on which the map is non-constant. Let $m(e)$ denote the function giving the intersection multiplicities with the stabilizing divisor, defined on edges e corresponding to intersection points, and consider the fiber bundle $\mathcal{E}^i = \mathcal{E}^i_{k,p,l,\Gamma}$ over $\mathcal{B}^i_{k,p,l,\Gamma}$ given by

$$(7.6)\quad (\mathcal{E}^i_{k,p,l,\Gamma})_{m,u,J} \subset \Omega^{0,1}_{j,J,\Gamma}(S^{\mathrm{nc}},(u|_S)^*T\mathbb{X})_{k-1,p} \oplus \Omega^1(T_\circ,(u|_{T_\circ})^*TL)_{k-1,p} \oplus \Omega^1(T_\bullet,(u|_{T_\bullet})^*TY)_{k-1,p}$$

the space of $0,1$-forms with respect to $j(m), J$ that vanish to order $m(e)-1$ at the node or marking corresponding to each contact edge e. The Cauchy-Riemann and shifted gradient operators applied to the restrictions u_S resp. u_T of u to the two resp. one dimensional parts of $C = S \cup T$ define a C^q section

$$(7.7)\quad \overline{\partial}_\Gamma : \mathcal{B}^i_{k,p,l,\Gamma} \to \mathcal{E}^i_{k,p,l,\Gamma},$$
$$(m,u,J,F_\Gamma,H_\Gamma) \mapsto \left(\overline{\partial}_{j(m),J}u_S, \left(\frac{d}{ds} + \mathrm{grad}_{F_\Gamma}\right)u_{T_\circ}, \left(\frac{d}{ds} + \mathrm{grad}_{H_\Gamma}\right)u_{T_\bullet}\right)$$

where

$$\overline{\partial}_{j(m),J} u := \frac{1}{2}(J \mathrm{d}u_S - \mathrm{d}u_S j(m)), \tag{7.8}$$

and s is a local coordinate with unit speed. The *local universal moduli space* is

$$\mathcal{M}_\Gamma^{\mathrm{univ},i}(\mathbb{X}, L, \mathbb{D}) = \overline{\partial}^{-1} \mathcal{B}^i_{k,p,l,\Gamma}$$

where $\mathcal{B}^i_{k,p,l,\Gamma}$ is embedded as the zero section. This subspace is cut out transversally: by [**28**, Lemma 6.5, Proposition 6.10], the linearized operator is surjective on the two-dimensional part of the domain mapping to $X_\pm$ on which u is non-constant, while at any point z in the interior of an edge in C with $\mathrm{d}u(z) \neq 0$ the linearized operator is surjective by a standard argument. Furthermore, the matching conditions at the nodes are cut out transversally, by an inductive argument given in the unbroken case. Furthermore, for any map to a neck piece $u : \mathbb{P}^1 \to \mathbb{P}(N_\pm \oplus \mathbb{C})$ whose projection to Y is non-constant, the linearized operator is surjective. Let $\eta \in \Omega^{0,1}(u^* T(\mathbb{P}(N_\pm \oplus \underline{\mathbb{C}}))$ be a one-form on one of the intermediate broken pieces S_i such that η lies in the cokernel of the universal linearized operator

$$D_{u,J}(\xi, K) = D_u \xi + \frac{1}{2} K D u j$$

defining the tangent space to the universal moduli space. Variations of tamed almost complex structure of cylindrical type are J-antilinear maps

$$K : T\mathbb{P}(N_\pm \oplus \underline{\mathbb{C}}) \to T\mathbb{P}(N_\pm \oplus \underline{\mathbb{C}})$$

that vanish on the vertical subbundle and are $\mathbb{C}^\times$-invariant. Since the horizontal part of $D_z u$ is non-zero at some $z \in C$, we may find an infinitesimal variation K of almost complex structure *of cylindrical type* by choosing $K(z)$ so that $K(z) D_z u j(z)$ is an arbitrary $(j(z), J(z))$-antilinear map from $T_z C$ to $T_{u(z)} \mathbb{P}(N_\pm \oplus \underline{\mathbb{C}})$. Choose $K(z)$ so that $K(z) D_z u j(z)$ pairs non-trivially with $\eta(u(z))$ and extend $K(z)$ to an infinitesimal almost complex structure K by a cutoff function. Finally, suppose that $u : C_i \to \mathbb{P}(N_\pm \oplus \underline{\mathbb{C}})$ is a component that projects to a point in Y. Such maps are automatically multiple covers of a fiber of $\mathbb{P}(N_\pm \oplus \underline{\mathbb{C}}) \to Y$ and so automatically regular. By the implicit function theorem, $\mathcal{M}_\Gamma^{\mathrm{univ},i}(\mathbb{X}, L, \mathbb{D})$ is a Banach manifold of class C^q, and the forgetful morphism

$$\varphi_i : \mathcal{M}_\Gamma^{\mathrm{univ},i}(\mathbb{X}, L, \mathbb{D})_{k,p,l} \to \mathcal{P}_\Gamma(\mathbb{X}, L, \mathbb{D})_l$$

is a C^q Fredholm map. Let $\mathcal{M}_\Gamma^{\mathrm{univ},i}(\mathbb{X}, L, \mathbb{D})_d \subset \mathcal{M}_\Gamma^{\mathrm{univ},i}(\mathbb{X}, L, \mathbb{D})$ denote the component on which φ_i has Fredholm index d. By the Sard-Smale theorem, for k, l sufficiently large the set of regular values $\mathcal{P}_\Gamma^{i,\mathrm{reg}}(\mathbb{X}, L, D)_l$ of φ_i on $\mathcal{M}_\Gamma^{\mathrm{univ},i}(\mathbb{X}, L, \mathbb{D})_d$ in $\mathcal{P}_\Gamma(\mathbb{X}, L, \mathbb{D})_l$ is comeager. Let

$$\mathcal{P}_\Gamma^{l,\mathrm{reg}}(\mathbb{X}, L, \mathbb{D})_l = \cap_i \mathcal{P}_\Gamma^{i,l,\mathrm{reg}}(\mathbb{X}, L, \mathbb{D})_l.$$

A standard argument shows that the set of smooth domain-dependent $\mathcal{P}_\Gamma^{\mathrm{reg}}(\mathbb{X}, L, \mathbb{D})$ is also comeager. Fix $(J_\Gamma, F_\Gamma) \in \mathcal{P}_\Gamma^{\mathrm{reg}}(\mathbb{X}, L, \mathbb{D})$. By elliptic regularity, every element of $\mathcal{M}_\Gamma^i(\mathbb{X}, L, \mathbb{D})$ is smooth. The transition maps for the local trivializations of the universal bundle define smooth maps $\mathcal{M}_\Gamma^i(\mathbb{X}, L, \mathbb{D})|_{\mathcal{M}_\Gamma^i \cap \mathcal{M}_\Gamma^j} \to \mathcal{M}_\Gamma^j(\mathbb{X}, L, \mathbb{D})_{\mathcal{M}_\Gamma^i \cap \mathcal{M}_\Gamma^j}$. This construction equips the space

$$\mathcal{M}_\Gamma(\mathbb{X}, L, \mathbb{D}) = \cup_i \mathcal{M}_\Gamma^i(\mathbb{X}, L, \mathbb{D})$$

with a smooth atlas. Since $\mathcal{M}_{\Gamma_0}$ is Hausdorff and second-countable, so is $\mathcal{M}_\Gamma(\mathbb{X}, L, \mathbb{D})$ and it follows that $\mathcal{M}_\Gamma(\mathbb{X}, L, \mathbb{D})$ has the structure of a smooth manifold.

Existence of orientations and tubular neighborhoods for codimension one strata involving broken Morse trajectory is similar to that for the unbroken case. However, for strata corresponding to a trajectory of length zero, there is a new gluing result necessary which is proved in Chapter 8.4. □

Remark 7.11. (True and fake boundary components) The formally-codimension-one strata of $\overline{\mathcal{M}}(L, \mathbb{X}, \mathbb{D})$ are of the following types:

(a) Strata of maps

$$u : C = C_0 \cup C_1 \to \mathbb{X}[1]$$

such that one component $C_0' \subset C_0$ has a boundary edge e of length $\ell(e)$ zero. See Figure 7.2.

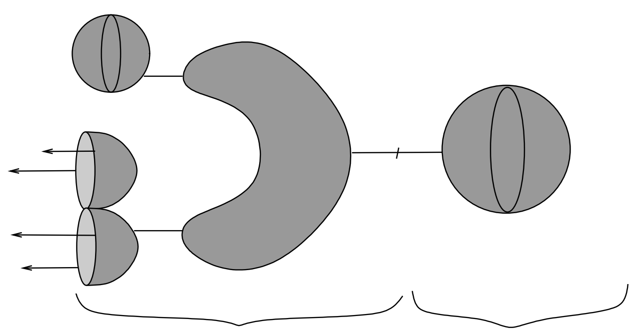

Figure 7.2. Broken disk with a boundary node

(b) Strata of maps

$$u : C = C_0 \cup C_1 \cup C_2 \to \mathbb{X}[2]$$

that is, with three levels C_0, C_1, C_2 connected by broken gradient trajectories of H. See Figure 7.1.

(c) Strata of maps $u : C = C_0 \cup C_1 \to \mathbb{X}[1]$ with two broken segments. See Figure 7.3.

Of these three types, the first two are *fake* boundary types in the sense that the strata $\mathcal{M}_\Gamma(\mathbb{X}, L, \mathbb{D})$ do not represent points in the topological boundary of the union of top dimensional strata $\overline{\mathcal{M}}(\mathbb{X}, L, \mathbb{D})$. In the first type, one can either make the length $\ell(e)$ of the gradient trajectory e of H finite and non-zero or deform the node $w \in C_i \cap C_j$ connecting the two disk components $C_i, C_j \subset C$; this shows that any stratum $\mathcal{M}_\Gamma(\mathbb{X}, L, \mathbb{D})$ of this type Γ is in the closure $\overline{\mathcal{M}}_{\Gamma_k}(\mathbb{X}, L, \mathbb{D}), k \in \{1, 2\}$ of two strata of top dimension. In the second case one can make the length $\ell(e_k), k \in \{1, 2\}$ of either the first gradient trajectory e_1 connecting the first and second levels of u or the second e_2 connecting the second and third levels of u finite, but not both (since the total length $\ell(e_1) + \ell(e_2)$ must be infinite). The last type is a true boundary component since the only deformation is that which deforms the length $\ell(e)$ of the trajectory $u|T_e$ to a finite real number $\ell(e) < \infty$.

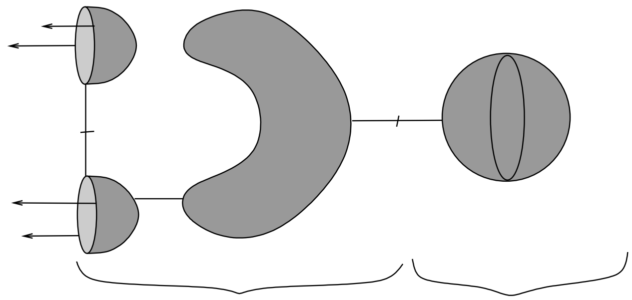

FIGURE 7.3. Broken disk with a broken boundary trajectory

Denote by $\mathcal{M}(\mathbb{X}, L, \mathbb{D}, \underline{l})_d$ the locus of maps expected dimension equal to d. In particular $\mathcal{M}(\mathbb{X}, L, \mathbb{D}, \underline{l})_0$ denotes the space of *rigid maps* that are expected dimension zero in top-dimensional strata. Using the regularized moduli spaces of broken maps we define the composition maps of the broken Fukaya algebra

$$\mu^n : CF(\mathbb{X}, L)^{\otimes n} \to CF(\mathbb{X}, L)$$

on generators by

$$\mu^n(l_1, \ldots, l_n) = \sum_{u \in \mathcal{M}(\mathbb{X}, L, \mathbb{D}, \underline{l})_0} (-1)^{\heartsuit} (\sigma(u)!)^{-1} y(u) q^{E(u)} \epsilon(u) l_0 \tag{7.9}$$

where $\heartsuit = \sum_{i=1}^n i|l_i|$.

THEOREM 7.12. (Broken Fukaya algebra) *For any admissible perturbation system $\underline{P} = (P_\Gamma)$ the maps $(\mu^n)_{n \geq 0}$ satisfy the axioms of a convergent A_∞ algebra $CF(\mathbb{X}, L)$ with strict unit. If the perturbations P_Γ for types Γ with one leaf satisfy the forgetful axiom (4.40) then the maps μ^1 satisfy the weak divisor axiom (4.42). The homotopy type of $CF(\mathbb{X}, L)$ is independent of all choices up to convergent homotopy equivalence.*

The statement of the theorem follows from the transversality and compactness properties of the moduli space of adapted maps of expected dimension at most one in Theorem 7.10. The parts of the map in the pieces of $\mathbb{X}$ other than $X_\subset$ do not affect the sign computation, since all these components are spheres. To prove homotopy invariance one may either repeat the arguments of Chapter 5 replacing quilted treed disks with broken quilted treed disks (which are obtained from quilted disks by replacing each component curve with a broken curve). Alternatively, in Theorem 8.15 below we show that the broken Fukaya algebra is homotopy equivalent to an unbroken one.

7.4. Broken divisors

In the rest of the chapter we show that broken stabilizing divisors exist. The result is an analog of a relative version of Bertini's theorem:

LEMMA 7.13. *Let X be a smooth complex projective variety equipped with an ample line bundle $\mathcal{E}$ and $i : Y \hookrightarrow X$ a smooth subvariety of codimension one. After replacing $\mathcal{E}$ with a tensor power, the following holds: For a given $s_Y \in H^0(i^*\mathcal{E})$*

cutting out a smooth divisor D_Y on Y there exists a section $s \in H^0(\mathcal{E})$ restricting to s_Y and cutting out a smooth divisor on X.

PROOF. Let $\mathcal{E}(Y)$ denote the sheaf of sections vanishing on Y. The exact sequence of sheaves $0 \to \mathcal{E}(Y) \to \mathcal{E} \to i_* i^* \mathcal{E} \to 0$ induces a long exact sequence of cohomology groups including the sequence

$$0 \to H^0(\mathcal{E}(Y)) \to H^0(\mathcal{E}) \to H^0(i_* i^* \mathcal{E}) \to H^1(\mathcal{E}(Y)) \to \ldots.$$

By Kodaira vanishing (see for example Griffiths-Harris [**54**, Theorem 5, p. 159]) $H^1(\mathcal{E}(Y))$ vanishes for sufficiently positive $\mathcal{E}$ and furthermore $\mathcal{E}(Y)$ is generated by its global sections. By the long exact sequence $H^0(\mathcal{E}) \to H^0(i_* i^* \mathcal{E})$ is surjective. Generation by global sections implies that the set of s restricting to s_Y transverse to the zero section is open and dense. Compare with Bertini [**60**, II.8.18]. □

The symplectic version of relative Bertini is obtained by a modification of Donaldson's argument in [**35**]. Let $\widetilde{X} \to X$ be a line-bundle with connection α over X whose curvature two-form $\operatorname{curv}(\alpha)$ satisfies $\operatorname{curv}(\alpha) = (2\pi/i)\omega$. Since our symplectic manifolds are rational we may always assume the existence of such a line bundle after taking a suitable integer multiple $k\omega$ of the symplectic form ω.

DEFINITION 7.14. (Asymptotically holomorphic sequences of sections) Let $(s_k)_{k \geq 0}$ be a sequence of sections of $\widetilde{X}^k \to X$.

(a) The sequence $(s_k)_{k \geq 0}$ is *asymptotically holomorphic* if there exists a constant C and integer k_0 such that for $k \geq k_0$,

$$|s_k| + |\nabla s_k| + |\nabla^2 s_k| \leq C, \quad |\overline{\partial} s_k| + |\nabla \overline{\partial} s_k| \leq C k^{-1/2}. \tag{7.10}$$

(b) The sequence $(s_k)_{k \geq 0}$ is *uniformly transverse* to 0 if there exists a constant η independent of k such that for any $x \in X$ with $|s_k(x)| < \eta$, the derivative of s_k is surjective and satisfies $|\nabla s_k(x)| \geq \eta$.

In both definitions the norms of the derivatives are evaluated using the metric $g_k = k\omega(\cdot, J\cdot)$.

THEOREM 7.15. *Suppose that $X_\subset, X_\supset, Y$ as above are equipped with line bundles $\tilde{X}_\subset, \tilde{X}_\subset, \tilde{Y}$ with connections with curvatures have cohomology classes $[\omega_\subset], [\omega_\supset], [\omega_Y]$ such that*

$$\tilde{X}_\subset|_Y \cong \tilde{Y} \cong \tilde{X}_\supset|_Y.$$

For $k \gg 0$ there exist approximately holomorphic codimension two submanifolds $D_{\subset,k}, D_{\supset,k} \subset X_\pm$ representing $k[\omega_\subset], k[\omega_\supset]$ such that

$$D_{\subset,k} \cap Y = D_{\supset,k} \cap Y = D_{Y,k}$$

for some sequence $D_{Y,k}$ that is also is asymptotically holomorphic represents $k[\omega_Y]$.

The proof will be given after two lemmas below.

LEMMA 7.16. (Extension of asymptotically holomorphic sequences) *Let X be an integral symplectic manifold equipped with a compatible almost complex structure, $\widetilde{X} \to X$ a line bundle with connection whose curvature is the symplectic form and $Y \subset X$ an almost complex (hence symplectic) submanifold. Denote by $\widetilde{Y} \to Y$ the restriction of $\widetilde{X}$ to Y. Given any asymptotically holomorphic sequence $s_{Y,k}$ of sections of $\widetilde{Y}^k \to Y$, there exists an asymptotically holomorphic sequence s_k of sections of $\widetilde{X}^k \to X$ such that $s_{k,\pm}|Y = s_{Y,k}$.*

Proof. We construct an extension by multiplying the given sections over the hypersurface by Gaussians in the normal direction. We may identify X near Y with the normal bundle N of Y in X on a neighborhood U of the zero section. Let $J_N : N \to N$ be the complex structure, $\pi : N \to Y$ denote the projection, P the frame bundle, and $k = \operatorname{rank}(N)$. Identify $N \cong P \times_{U(k)} \mathbb{C}^k$ via the associated bundle construction. The linearization $\widetilde{X}$ on a neighborhood of Y admits an isomorphism $\widetilde{X}^k | U \cong \pi^* \widetilde{Y}^k$, using parallel transport along the normal directions to Y. We may assume that the connection on $\tilde{X}$ in the normal direction is induced via the associated bundle construction from a one-form

$$\alpha_k = \sum_{i=1}^k \frac{k}{4}(z_i \mathrm{d}\overline{z}_i - \overline{z}_i \mathrm{d} z_i) \in \Omega^1(\mathbb{C}^k)$$

where $z_1, \ldots, z_k$ are coordinates on $\mathbb{C}^k$. Let $\phi : N \to \mathbb{R}_{\geq 0}$ denote the norm function, induced from the function $(z_1, \ldots, z_k) \to \frac{1}{2}\sum_{i=1}^k |z_i|^2$ on $\mathbb{C}^k$. Define a Gaussian sequence

$$s_{k,\pm} = (\pi^* s_{Y,k}) \exp(-k\phi^2/4)$$

in a neighborhood of the divisor Y. After multiplication by a cutoff function supported in a neighborhood of size $k^{-1/6}$ of Y, the section extends by zero to all of X. The bound

$$|s_k| + |\nabla s_k| + |\nabla^2 s_k| \leq C$$

follows immediately from the fact that the derivatives of the Gaussian are bounded, and the derivatives are with respect to the Levi-Civita connection for the metric g_k. The bound

$$|\overline{\partial} s_k| + |\nabla \overline{\partial} s_k| \leq C k^{-1/2}$$

follows from the fact that the Gaussian is holomorphic to leading order as in [**35**, (10)]: Consider the splitting TN into the vertical part $T^{\mathrm{ver}}N \cong N$ and its orthogonal complement $T^{\mathrm{hor}}N$. The difference between the almost complex structures $J_0 = \pi^* J_Y \oplus J_N$ and J is the graph of a map as on [**35**, Section 2] so that

$$\mu : \Lambda^{1,0} TN \to \Lambda^{0,1} N, \quad \overline{\partial}_J s = \overline{\partial}_{J_0} s - \mu(\partial_{J_0} s).$$

We may then estimate the failure of s_k to be holomorphic by

$$\begin{aligned} \overline{\partial}_J s_k &= (\overline{\partial}_J - \overline{\partial}_{J_0}) s_k + \overline{\partial}_{J_0} s_k \\ &= -\mu(\partial_{J_0} s_k) + (\overline{\partial}_{\pi^* J_Y} \pi^* s_{Y,k}) e^{-k\phi^2/4} + \pi^* s_{Y,k} \frac{\mu}{2}(k\alpha^{1,0}) e^{-k\phi^2/4}. \end{aligned}$$

Taking norms with respect to the metric induced by $J, k\omega$ we have

$$|\overline{\partial}_J s_k| \leq C k^{-1/2} \phi^2 e^{-k\phi^2/4} + \sup |\overline{\partial}_{J_Y} s_{Y,k}| \leq C k^{-1/2}. \tag{7.11}$$

Similarly

$$\begin{aligned} |\nabla \overline{\partial}_J s_k| &\leq C(|\nabla \mu||\partial_{J_0} s_k| + |\mu||\nabla(k\alpha^{1,0} e^{-k\phi^2/4})||s_{Y,k}| + \sup |\nabla \overline{\partial}_{\pi^* J_Y} \pi^* s_{Y,k}| \\ &\leq C k^{-1/2}(\phi + \phi^3) e^{-k\phi^2/4} + C \sup |\nabla \overline{\partial}_Y s_{Y,k}| \leq C k^{-1/2} \quad \square \end{aligned} \tag{7.12}$$

Lemma 7.17. *Continuing the assumptions of the previous lemma, for any $p \in X - Y$ with $d_k(p, Y) \geq k^{-1/2}$, with* $\operatorname{codim}(Y) = 2$*, there exists an approximately section $s_{p,k}$ satisfying the estimates* (7.11) *and* (7.12) *with the property that $s_{p,k}$ vanishes on Y.*

Proof. We modify the construction of perturbations so that the sections on the hypersurface are unchanged. If $d_k(p,Y) \geq k^{-1/6}$, then the previously chosen locally Gaussian section satisfies the required properties. Indeed in this case $s_{k,p}$ vanishes on Y since Y is outside the support of the cutoff function. So it suffices to assume that p is in the intermediate region

$$d_k(p,Y) \in (k^{-1/2}, k^{-1/6}). \tag{7.13}$$

So fix $p' \in Y$ and choose a local Darboux chart $(z_1, \ldots, z_n)$ near p' so that Y is described locally by $z_1 = 0$ and $\overline{\partial} z_1(p') = 0$. Let p lie in this Darboux chart, satisfying the estimate (7.13). Let $p_1 \neq 0$ denote the first coordinate of the point p. Given an approximately holomorphic sequence $s'_{p,k}$ with sufficiently small support (for example, a Gaussian $s(z) = \exp(-k|z-p|^2)$) the section $s_{p,k}(z) = s'_{p,k}(z)z_1/p_1$ is also approximately holomorphic, uniformly in p as long as $|p_1| > k^{-1/2}$. Indeed the bound

$$|s_{k,p}| + |\nabla s_{k,p}| + |\nabla^2 s_{k,p}| \leq C$$

is immediate, and uniform if $|p_1| > k^{-1/2}$ since $s'_{p,k}$ is Gaussian in $k^{1/2}z_1$. The bound

$$|\overline{\partial} s_{k,p}| + |\nabla \overline{\partial} s_{k,p}| \leq C k^{-1/2}$$

follows from the fact that z_1/p_1 is holomorphic to leading order; see Auroux [**9**, Proof of Proposition 3] where similar approximately holomorphic sections were used to simplify Donaldson's construction [**35**]. □

Proof of Theorem 7.15. We first perturb on the hypersurface, and then use the special perturbations in the previous lemma to perturb away from the hypersurface so that the restriction is unchanged. Let $\mathbb{X} = X_\subset \cup_Y X_\supset$ be a broken symplectic manifold. The sections $s_{\subset,k}, s_{\supset,k}$ from Lemma 7.16 are already asymptotically holomorphic and uniformly transverse in a neighborhood of size $Ck^{-1/2}$ around Y. Recall that in Donaldson's construction [**35**], one has for each k a collection of subsets $V_0 \subset \ldots V_N = X$, where N is independent of k, and one shows that given a combination of the local Gaussians that is approximately holomorphic and transverse section $s_{k,i}$ on V_i, that one can adjust the coefficients of the locally Gaussian functions so that the section is approximately holomorphic and transverse over V_{i+1}. Here we may use sections $s_{p,k}$ in Lemma 7.17 for p of distance at least $k^{-1/2}$ to achieve transversality off of Y. More precisely, in Donaldson's construction [**35**, p. 681] taking V_0 to be, rather than empty, a neighborhood of size $k^{-1/2}$ around Y. Thus only the initial step of Donaldson's construction is different. □

Proposition 7.18. *For any type Γ with $k_\circ \geq 1$ disk components of the surface part resp. $k_\bullet$ sphere components with l levels joined by e cylindrical ends, $m_\circ$ Morse trajectories in L, $m_\bullet$ interior edges, and limits $\underline{l}$ along the n semi-infinite edges mapping to the Lagrangian, the expected dimension of the moduli space $\overline{\mathcal{M}}_\Gamma(\mathbb{X}, L, \mathbb{D}, \underline{l})$ of adapted broken maps of combinatorial type Γ limits $\underline{l}$ with no tangency conditions at the divisor is given by*

(7.14)

$$\dim T_{[u]}\overline{\mathcal{M}}_\Gamma(\mathbb{X}, L, \mathbb{D}, \underline{l}) = (k_\circ - m_\circ)\dim(L) + (k_\bullet - m_\bullet)\dim(X) + I(u) - \dim(W^+_{l_0}) - \sum_{i=1}^{n} \dim(W^-_{l_i}) + n - 3 - \sum_{i=1}^{e}(\dim(Y) + 4s_i) - 2(l-2)$$

where $s_i + 1$ are the multiplicities of u at the intersection points with Y.

PROOF. By construction we have an isomorphism of the tangent space with the kernel of the linearized operator and the cokernel vanishes:

$$T_{[u]}\overline{\mathcal{M}}_\Gamma(\mathbb{X}, L, \mathbb{D}, \underline{l}) \cong \ker(\tilde{D}_u), \quad \{0\} = \operatorname{coker}(\tilde{D}_u).$$

We apply Riemann-Roch for Cauchy-Riemann operators on surfaces with boundary, [**95**, Appendix] to compute the index

$$\operatorname{Ind}(\tilde{D}_u) = \dim T_{[u]}\overline{\mathcal{M}}_\Gamma(\mathbb{X}, L, \mathbb{D}, \underline{l}).$$

The operator $\tilde{D}_u$ is the direct sum of operators $D_{u,S}$ on the surface part $S \subset C$, $D_{u,T}$ on the tree part $T \subset C$, and an operator on the tangent space to the moduli space of treed disks that can be deformed to zero without changing the index. Write S as the union of a disk components $S_{\circ,i}, i = 1, \ldots, k_\circ$ and sphere components $S_{\bullet,j}, j = 1, \ldots, k_\bullet$. By Riemann-Roch

$$\operatorname{Ind}(D_{u,S}) = k_\circ \dim(L) + k_\bullet \dim(X) + I(u).$$

Each boundary resp. interior edge has index $\dim(L)$ resp. $\dim(X)$, since a gradient trajectory is determined by its value at any point. However, there are two matching conditions at the ends of any such edge, so that the semi-infinite edges contribute corrections $\dim(W^-_{l_i})$ appearing from the constraints from the semi-infinite edges, while the boundary resp. interior edges at any fixed level contribute corrections $\dim(L)$ resp. $\dim(X)$. The tangent space to the moduli space of stable disks contributes $n-3$. The edges connecting levels contribute $\sum_{i=1}^e(\dim(Y)+4s_i)$ from the matching and tangency conditions for the Morse trajectories in Y. The factor $2(l-2)$ is the dimension of the group of fiber-wise automorphisms of the neck pieces mapping to $\mathbb{P}(N_\pm \oplus \mathbb{C})$. □

7.5. Reverse flips

Now we specialize to the case that the symplectic manifold is obtained by a small simple reverse flip or blow-up. A symplectic manifold X is obtained from a *smooth reverse flip* if the local model $\tilde{V}$ in (2.7) has positive weights μ_i all equal to 1, that is,

$$(\mu_i > 0) \implies (\mu_i = 1).$$

We say that X is obtained from a small reverse *simple* flip or blow-up if and only if the local model $\tilde{V}$ in 2.7 has all weights equal to ± 1. In this case there exists an embedded projective space $\mathbb{P}^{n_+-1}$ in X and a tubular neighborhood of $\mathbb{P}^{n_+-1}$ in X symplectomorphic to a neighborhood of the zero section in $\mathcal{O}(-1)^{\oplus n_-}$ where $n_+ + n_- = n+1$ and $\mathcal{O}(-1) \to \mathbb{P}^{n_+-1}$ is the tautological bundle.

We apply a degeneration argument so that after degeneration, the mmp transition is given by variation of symplectic quotient. Let $\mathcal{O}(-1)^{n_-}_1$ denote the unit sphere bundle in $\mathcal{O}(-1)^{n_-}$. Then $\mathcal{O}(-1)^{n_-}_1$ is circle-fibered coisotropic fibering over the projectivized bundle $\mathbb{P}(\mathcal{O}(-1)^{n_-}) = \mathbb{P}^{n_+-1} \times \mathbb{P}^{n_--1}$. The variety X degenerates to a broken manifold $(X_\subset, X_\supset)$ where $X_\subset$ is a toric variety

$$X_\subset \cong \operatorname{Bl}_E \mathbb{P}(\mathcal{O}(-1)^{n_-} \oplus \underline{\mathbb{C}})$$

where $E \cong \mathbb{P}^{n_--1}$ is the exceptional locus of the flip. The other piece is a disjoint union

$$X_\supset = X'_\supset \sqcup X''_\supset, \quad X'_\supset = \mathbb{P}(\mathcal{O}(-1)^{n_-} \oplus \underline{\mathbb{C}}), \quad X''_\supset = \operatorname{Bl}_E X.$$

We call $X_\subset$ resp. $X_\supset$ the *exceptional* resp. *remainder* piece of X. The space $X_\subset$ may also be realized as a symplectic quotient

$$X_\subset = (\mathbb{C}^{n_-}_{-1} \oplus \mathbb{C}^{n_+}_{+1} \oplus \mathbb{C}) /\!\!/ (U(1) \times_{\mathbb{Z}_2} U(1)) \tag{7.15}$$

where the action on $\mathbb{C}^{n_-}_{-1}$ is with weight $(-1, 1)$ and on $\mathbb{C}^{n_+}_{+1}$ with weight $(+1, 1)$, and on the last factor of $\mathbb{C}$ with weight $(0, -2)$. The flip is obtained by variation of git quotient in the above local model. The unstable locus changes from $\{0\} \oplus \mathbb{C}^{n_+}$ to $\mathbb{C}^{n_-} \oplus \{0\}$. It follows that under the flip the exceptional locus in X_-

$$(\{0\} \oplus \mathbb{C}^{n_+})) /\!\!/ \mathbb{C}^\times \cong \mathbb{P}^{n_+ - 1}$$

is replaced by the exceptional locus in $X = X_+$

$$(\mathbb{C}^{n_-} \oplus \{0\}) /\!\!/ \mathbb{C}^\times \cong \mathbb{P}^{n_- - 1}.$$

The toric piece is a symplectic quotient as explained in Chapter 2.2. Denote by $T = (S^1)^{n+2}/(S^1 \times S^1)$ the residual torus acting on $X_\subset$. The canonical moment map for the action of $(S^1)^{n+2}$ on $\mathbb{C}^{n+2}$ induces a moment map $\Phi_\subset : X_\subset \to \mathfrak{t}^\vee$. The moment polytope may be written in terms of the normal vectors ν_k as

$$P_\subset := \Phi_\subset(X_\subset) = \{\mu \mid \langle \mu, \nu_k \rangle \geq c_k,\ k = 1, \ldots, n+2\}$$

where the normal vectors ν_k are the projections of minus the standard basis vectors in $\mathbb{R}^{n+2} = \mathrm{Lie}((S^1)^{n+2})$: Using the parametrization

$$(S^1)^n \cong T, \quad (z_1, \ldots, z_n) \mapsto [z_1, \ldots, z_n, 1, 1]$$

and letting ϵ_j be minus the standard basis vectors, we have

$$\nu_1 := \epsilon_1,\ \nu_2 := \epsilon_2, \ldots, \nu_n := \epsilon_n \in \mathfrak{t}^\vee. \tag{7.16}$$

On the other hand, from the description of the weights in (7.15) we have

$$\nu_{n+1} := \epsilon_1 + \ldots + \epsilon_{n_-} - \epsilon_{n_-+1} - \ldots - \epsilon_n,\ \nu_{n+2} := -\epsilon_1 + \ldots - \epsilon_{n_-}. \tag{7.17}$$

We assume

$$c_1 = c_2 = \ldots c_n = 0,\ c_{n+1} = \epsilon, c_{n+2} \gg 0$$

where the constant $\epsilon > 0$ represents the size of the exceptional divisor.

The regular Lagrangian described in Definition 3.1 and referred to in Theorem 1.1 is a toric moment fiber: Define

$$\lambda = \epsilon(1, \ldots, 1)/(n_+ - n_-), \quad L = \Phi_\subset^{-1}(\lambda). \tag{7.18}$$

By the Cho-Oh classification in Proposition 3.3, L is regular in the sense of 3.1. For example, if $n_+ = 2, n_- = 1$ then the corresponding transition is a blow-down of curve in a surface. The moment polytope has normal vectors $\epsilon_1, \epsilon_2, \epsilon_1 - \epsilon_2, -\epsilon_1$. The moment polytope is

$$\{(\lambda_1, \lambda_2) \mid \lambda_1 \geq 0, \lambda_2 \geq 0, \lambda_1 - \lambda_2 \geq \epsilon, \lambda_1 \leq c_4\}$$

where $c_4 \gg 0$. Then L is the fiber over (ϵ, ϵ). See Figure 7.4 where the point λ is shown as a shaded dot inside the moment polytope, which is a trapezoid.

In order to study the broken Fukaya algebra we introduce a Morse function on the separating hypersurface of standard form. The pieces $X_\subset, X_\supset$ are joined by the disconnected hypersurface

$$Y = Y_0 \cup Y_\infty \quad Y_0 \cong Y_\infty \cong \mathbb{P}^{n_+ - 1} \times \mathbb{P}^{n_- - 1}$$

namely the divisors in $X_\subset$ at 0 and ∞. Let

$$H : Y_0 \cong Y_\infty \to \mathbb{R}, \quad y \mapsto \langle \Phi_Y(y), \xi \rangle$$

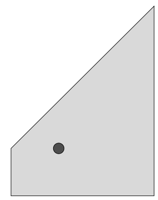

FIGURE 7.4. The polytope for the blow-up of the projective plane

be a Morse function obtained as the pairing of the moment map $\Phi_Y : Y \to \mathbb{R}^{n_- + n_+}$ with a generic vector ξ, which we take to be $((1, \ldots, n_-), (1, \ldots, n_+))$. For $1 \leq i \leq k$ let

$$[\epsilon_i] := [0, \ldots, 1, 0, \ldots, 0] \in \mathbb{P}^{k-1}$$

denote the point whose homogeneous coordinates are all zero except for the i-th coordinate. The critical points of H are the fixed points for the torus action

$$\operatorname{crit}(H) = \left\{ ([\epsilon_{i_-}], [\epsilon_{i_+}]) \in \mathbb{P}^{n_- -1} \times \mathbb{P}^{n_+ -1}, \quad i_\pm \leq n_\pm \right\}. \tag{7.19}$$

Consider the one-parameter subgroup $\mathbb{C}^\times \to (\mathbb{C}^\times)^{n_- + n_+}$ generated by ξ inside the standard torus of dimension $n_- \times n_+$ acting on $\mathbb{P}^{n_- -1} \times \mathbb{P}^{n_+ -1}$. The stable manifolds consist of those points that flow to $([\epsilon_{i_-}], [\epsilon_{i_+}])$ under the $\mathbb{C}^\times$-action:

$$W^-_{([\epsilon_i],[\epsilon_j])} = \left\{ (z_1, z_2) \in \mathbb{P}^{n_- -1} \times \mathbb{P}^{n_+ -1} \;\middle|\; \lim_{z \to 0} z(z_1, z_2) = ([\epsilon_{i_-}], [\epsilon_{i_+}]) \right\}.$$

The Morse cycles are those points (z_-, z_+) such that the homogeneous coordinates of $z_\pm$ above index $i_\pm$ vanish. That is,

$$W^-_{([\epsilon_{i_-}],[\epsilon_{i_+}])} \cong \mathbb{P}^{i_- -1} \times \mathbb{P}^{i_+ -1}, \quad \text{for } i_- \leq n_-, i_+ \leq n_+.$$

For example, when $n_- = 2, n_+ = 1$ the transition corresponds to the blow-down of a curve in a surface, $Y_0 \cong Y_\infty \cong \mathbb{P}^1$ and any Morse cycle is either all of $\mathbb{P}^1$ or a point.

In order to study the broken Fukaya algebra, we choose the standard complex structure on the toric piece. This suffices to achieve transversality:

PROPOSITION 7.19. *There exist admissible perturbation data $\underline{P} = (P_\Gamma)$ for the broken manifold $\mathbb{X} = (X_\subset, X_\supset)$ with the properties that*

(a) *the almost complex structure $J_{\subset,\Gamma}$ on $X_\subset \subset \mathbb{X}$ is domain-independent and equal to the standard $T_\mathbb{C}$-invariant complex structure from (2.9); and*
(b) *the Morse function H_Γ on $Y_0 \cup Y_\infty$ is domain-independent and equal to a component of the moment map on $Y_0 \cong Y_\infty \cong \mathbb{P}^{n_+ -1} \times \mathbb{P}^{n_- -1}$ as in (7.19).*

We first prove a result on holomorphic spheres in toric varieties:

LEMMA 7.20. *Let X be a smooth compact toric variety with torus-invariant prime boundary divisors $D_i, i = 1, \ldots, N$. Let $D_i, i \in I$ be a collection of prime boundary divisors with the following property:*

> *For each $D_i, i \in I$ there exists a collection of linearly equivalent divisors $D_j, j \in I(i)$ with the property that $\cap D_j, j \in I(i) = 0$.*

Let $u : \mathbb{P}^1 \to X$ be a holomorphic sphere not contained in the union $\cup_{i \notin I} D_i$. Then $H^1(\mathbb{P}^1, u^*TX) = \{0\}$ and the evaluation map at any point is a submersion.

PROOF. We write X as a git quotient of $\mathbb{C}^N$, so that the i-th factor of $T\mathbb{C}^N$ descends to $\mathcal{O}(D_i) \subset TX$. Thus up to the addition of a trivial vector bundle we have

$$TX \cong \bigoplus_{i=1}^{N} \mathcal{O}(D_i)$$

and it suffices to show that the higher cohomology of each $H^1(\mathbb{P}^1, u^*\mathcal{O}(D_i))$ vanishes. If $i \notin I$, then $u(\mathbb{P}^1)$ is not contained in D_i and so the intersection number $u(\mathbb{P}^1).D_i = \deg(u^*\mathcal{O}(D_i))$ is positive. On the other hand, if $i \in I$ then D_i is linearly equivalent to some $D_j, j \in I(i)$ not containing $u(\mathbb{P}^1)$, so $\deg(u^*\mathcal{O}(D_i))$ is positive in this case as well. Hence $H^1(\mathbb{P}^1, u^*\mathcal{O}(D_i))$ vanishes, as claimed. $\square$

REMARK 7.21. The toric piece $X_\subset$ is Fano. Indeed by Kleiman's criterion [**69**] it suffices to check that the anticanonical degree of holomorphic curves $v : S \to X_\subset$ is positive, $v.K^{-1} > 0$. Since any such holomorphic curve degenerates to a union of rational torus-invariant holomorphic curves, it suffices to check the condition on torus-invariant rational curves. By Remark 3.1 $X_\subset$ is a $\mathbb{P}^1$-bundle over $\mathbb{P}^{n_- -1} \times \mathbb{P}^{n_+ -1}$. The irreducible invariant holomorphic curves lie in either a fiber or in the divisors D_0, D_∞ isomorphic to $\mathbb{P}^{n_- -1} \times \mathbb{P}^{n_+ -1}$. The restriction of the canonical bundle to D_0, D_∞ is $\mathcal{O}(n_- \pm 1) \otimes \mathcal{O}(n_+ \mp 1)$, and the claim follows.

PROOF OF PROPOSITION 7.19. By the git presentation (7.15) the exceptional piece $X_\subset$ is a toric variety. Any holomorphic sphere contained in a broken configuration mapping to $X_\subset$ is not contained in $Y_0 \cup Y_\infty$. The remaining prime boundary divisors in $X_\subset$ are those which fiber over a prime boundary divisor of $Y_0 \cong Y_\infty \cong \mathbb{P}^{n_+ -1} \times \mathbb{P}^{n_- -1}$. These are of two types: the prime boundary divisors in $\mathbb{P}^{n_+ -1}$ and those in $\mathbb{P}^{n_- -1}$. In each case, each prime boundary divisor D_i is linearly equivalent to all the remaining prime boundary divisors of its type, and the intersection of all such boundary divisors is empty. By Lemma 7.20, any holomorphic sphere in $X_\subset$ not contained in $Y = Y_0 \cup Y_\infty$ is regular and the evaluation map at any point is a submersion.

We claim that perturbation data with almost complex structure equal to the standard almost complex structure on the toric piece and standard Morse function on the separating hypersurface are admissible. As in Cho-Oh [**31**], all holomorphic disks (necessarily given by the Blaschke products in (3.2)) are regular, and by the previous paragraph, all the spheres in the middle piece are regular as well. The same holds after imposing constraints at the interior leaves, that is, requiring the markings to map to stable manifolds of H as long as the interior markings lie on distinct points on the disk component, since the Blaschke description (3.2) shows that the moduli space of such disks is transversally cut out by varying the positions of the roots $a_{i,j}$. Note that the only rigid configurations are those with a single interior special point on the disk component, since the roots $a_{i,j}$ may always be chosen arbitrarily. Configurations $u : C \to \mathbb{X}$ where two interior nodes w_1, w_2 lie on a ghost bubble $S_v \subset S \subset C$ are *not* transversally cut out, but automatically deform to configurations $u' : C \to \mathbb{X}$ with the two nodes w_1', w_2' at distinct points of the disk component $S_{v'} \subset S$, and so never lie on a component $\mathcal{M}_\Gamma(\mathbb{X}, L)$ of the moduli space of broken treed disks of expected dimension $\dim(\mathcal{M}_\Gamma(\mathbb{X}, L)) \leq 1$ at most one. Thus the moduli spaces of expected dimension at most one are regular on the piece mapping to $X_\subset$. A repeat of the previous arguments shows that generic admissible such perturbation data on $X_\supset$ make the moduli spaces $\mathcal{M}_\Gamma(\mathbb{X}, L)$ of expected dimension at most one regular. $\square$

The following is the main result of this section; it states that the broken Floer theory of the Lagrangian torus is unobstructed and non-trivial.

THEOREM 7.22. *Let $L \subset X_\subset$ be a regular Lagrangian brane that is a Lagrangian torus orbit in the toric piece $X_\subset$. If $\underline{P} = (P_\Gamma)$ is a collection of admissible perturbations such that each almost complex structure is constant equal to the standard complex structure on $X_\subset$ and the Morse function on $Y_0 \cong Y_\infty$ for the nodes attaching to the disk components is the standard one on $Y_0 \cong Y_\infty \cong \mathbb{P}^{n_- - 1} \times \mathbb{P}^{n_+ - 1}$ then*

(a) *the broken Fukaya algebra $CF(\mathbb{X}, L)$ is a projectively flat for any local system y and some $b(y)$ depending on y, that is, $\mu_y^{0,b(y)}(1)$ is a multiple of the strict identity and so $(m_{1,y}^{b(y)})^2 = 0$; and*

(b) *there exist local systems $y_i \in \mathcal{R}(L), i = 1, \ldots, n_+ - n_-$ such that the Floer cohomology is non-vanishing: $H(m_{1,Y}^{b(y)}) = H(L) \neq 0$.*

PROOF. The argument for weak unobstructedness of the Floer theory is a dimension count using the classification of disks in (3.2). Let

$$u = (u_\subset, u_1, \ldots, u_k, u_\supset) : C_\subset \cup C_1 \cup \ldots \cup C_k \cup C_\supset \to X_\subset \cup \mathbb{P}(N_\pm \oplus \mathbb{C}) \oplus \ldots \mathbb{P}(N_\pm \oplus \mathbb{C}) \cup X_\supset \tag{7.20}$$

be a broken disk contributing to $\mu_0(1)$, that is, with no boundary leaves, with intersection multiplicities $\underline{s}$ with the hypersurface Y and mapping to the unstable manifolds of $x_i, i = 1, \ldots, m$ at the intersection points with Y. Each such constraint raises the required Maslov index of $u_\subset$ by 2, so that

$$I(u_\subset) + \sum_{i=1}^{m} (2 - \deg(x_i) - 2s_i) + \deg(l) - 2 = 0 \tag{7.21}$$

where m is the number of points mapping to Y. On the other hand, the requirement that $u_\subset(z_k)$ meet $\mathbb{P}^i \times \mathbb{P}^j$ at a given point $z_k \in S_\subset$ implies that for any $l > i$, the component u_l has a zero at z_k; the number of such constraints is the degree of x_i. Furthermore, if u_0 is the component whose vanishing corresponds to intersections with Y then the intersection multiplicity condition also requires that u_0 have a zero of order s_i at z_k. Each of these zeroes $u_\subset^{-1}(Y)$ contributes two to the Maslov index $I(u_\subset)$ of $u_\subset$. For the moduli space to contain some element $[u]$ it must hold that

$$I(u_\subset) \geq \sum_{i=1}^{m} (\deg(x_i) + 2s_i). \tag{7.22}$$

Combining (7.22) with (7.21) it follows that

$$\deg(l) \leq 2 - 2m$$

with $\deg(l) = 2$ only if $m = 0$ and $I(u_\subset) = 0$. From the Blaschke classification (3.2) the requirement $I(u_\subset) = 0$ implies that $u_\subset | S_\subset$ is constant. No such disks contribute to $\mu_y^0(1)$ since any configuration contributing to $\mu_y^0(1)$ must have at least one non-constant component. Hence, $\deg(l) < 2 - 2m \leq 0$. Since $\deg(l) \geq 0$, we have $\deg(l) = 0$. By assumption, there is a unique element $l \in \mathcal{I}^{\text{geom}}(L)$ of degree zero, the geometric unit $l = x^\blacktriangledown$. It follows that $CF(\mathbb{X}, L)$ is projectively flat, that is,

$\mu_y^0(1)$ is a multiple of the geometric unit $x^{\blacktriangledown}$ for any local system y. As in 4.43, for similar reasons only configurations with no disks contribute to the sum (4.36) which becomes $\mu_y^1(x^{\blacktriangledown}) = x^{\triangledown} - x^{\blacktriangledown}$. Hence if $\mu_y^1(0) = W(y)x^{\blacktriangledown}$ then $W(y)x^{\blacktriangledown} \in \tilde{MC}(L)$.

Local systems for which the Floer theory is non-trivial are found by computing the critical points of the potential. The leading order terms in the potential $\mathcal{W}(y)$ are as in the toric case and can be read off from (7.16), (7.17): If ϵ denotes the size of the symplectic class in the exceptional locus $H^2(\mathbb{P}^{n_- -1}) \cong \mathbb{R}$ then

$$\mathcal{W}(y_1, \ldots, y_{n-1}) = q^{\epsilon} \left(y_1 + \ldots + y_{n-1} + \frac{y_1 \cdots y_{n_+}}{y_{n_+ +1} \cdots y_{n-1}} \right) + \text{higher order}$$

where the higher order terms have q-exponent at least ϵ. Indeed, each of these terms corresponds to a Maslov index two disk in the toric piece $X_{\subset}$ with a single exception: In the case of a blow-up $n_- = 1, n_+ = n$ there is also a contribution from the disk meeting the exceptional divisor $\mathbb{P}^{n_+ -1}$ which, in the broken limit, consists of a disk in $X_{\subset}$ meeting Y_0 and a holomorphic sphere mapping to a fiber of $X'_{\supset} \to \mathbb{P}^{n_+ -1}$; the corresponding contribution to the potential is $q^{\epsilon} y_1 \ldots y_{n-1}$. The leading order potential

$$\mathcal{W}_0(y_1, \ldots, y_{n-1}) = q^{\epsilon} \left(y_1 + \ldots + y_{n-1} + \frac{y_1 \cdots y_{n_+}}{y_{n_+ +1} \cdots y_{n-1}} \right)$$

has partial derivatives for $i \leq n_+$ resp. $I > n_+$

$$q^{-\epsilon} y_i \partial_{y_i} \mathcal{W}_0(y_1, \ldots, y_{n-1}) = y_i + \quad \text{resp. -} \quad \frac{y_1 \cdots y_{n_+}}{y_{n_+ +1} \cdots y_{n-1}}.$$

Setting all partial derivatives equal to zero we obtain

$$y_1 = \ldots = y_{n_+} = -y_{n_+ +1} = \ldots = -y_{n-1}$$

and $y_1^{n_+ - n_-} = (-1)^{n_-}$. Hence $\mathcal{W}_0(y)$ has a non-degenerate critical point at certain roots of unity. One can solve for a critical point of $\mathcal{W} = \mathcal{W}_0 +$ higher order using the Theorem 4.37 by varying the local system. For a comeager subset of perturbations, the one-leaf moduli spaces admit forgetful maps as in Lemma 4.33. Thus the critical points give local systems $y \in \mathcal{R}(L)$ for the broken Fukaya algebra $CF(\mathbb{X}, L)$ for which there exists Maurer-Cartan solutions $b \in MC(L)$ for which the Floer cohomology $HF(L, b)$ is non-vanishing by Proposition 4.35. □

CHAPTER 8

The break-up process

In this Chapter we show that the Fukaya algebra is homotopy equivalent to the broken Fukaya algebra obtained in the sft limit, combined with a family of broken Fukaya algebras in which the sum of the lengths of the Morse trajectories connecting the pieces in the building varies.

8.1. Varying the length

We first introduce a version of the broken Fukaya algebra where the gradient segments in the separating hypersurface sum to a finite, prescribed length parameter. The broken Fukaya algebras $CF(\mathbb{X}, L, \varsigma)$ for various choices of length parameter ς will be A_∞ homotopy equivalent. For $\varsigma = 0$, we will relate the broken Fukaya algebra with the unbroken one, while the case $\varsigma = \infty$ is computationally more tractable since any gradient trajectory of infinite length must pass through a critical point.

Definition 8.1. Let $n, m, s \geq 0$ be integers. A ς-*broken disk* with s sublevels consists of a treed disk $C = S \cup T$ and a decomposition of the surface part $S = S_0 \cup \ldots \cup S_s$ into *sublevels* of possibly disconnected surfaces $S_0, \ldots, S_s$ where only the first piece S_0 is allowed to have non-empty boundary, that is, $\partial S_1 = \ldots = \partial S_s = \emptyset$, and satisfying the following conditions:

(a) (Same Length Axiom) the edges $T_e \subset T$ connecting adjacent components S_i to S_{i+1} have the same length ℓ_i;
(b) (Non-adjacent sublevel axioms) there are no edges connecting non-adjacent components $S_i, S_j, |i - j| \geq 2$;
(c) (Total distance axiom) for any component S_i define the *distance* to S_0 by summing the lengths of the segments connecting S_0 and S_i:

$$\operatorname{dist}(S_0, S_i) = \sum_{j=1}^{i-1} \ell_i.$$

Then the distance between S_0 and the last component S_s is ς.

To achieve regular perturbations of the moduli spaces we introduce the notation of broken disks adapted to a broken Donaldson hypersurface. Let $\mathcal{M}_\Gamma^\varsigma(\mathbb{X}, L, \mathbb{D})$ denote the moduli space of stable adapted ς-broken weighted treed disks with boundary in L of type Γ. Counting adapted broken maps with lengths ς as in (7.9) defines a ς-broken Fukaya algebra $CF(\mathbb{X}, L, \varsigma)$, by the same arguments as in the case $\varsigma = \infty$ treated in the previous Chapter. To prove that the broken Fukaya algebra is independent of the length, we construct morphisms $CF(\mathbb{X}, L, \varsigma_0) \to CF(\mathbb{X}, L, \varsigma_1)$ using treed quilted disks where the total distance from the first to last level varies over the quilting. We also choose a function which specifies the length of the gradient trajectories connecting the pieces of the broken map. Choose a continuous function

$$\ell : \mathcal{U}_{n,1} \to [\varsigma_0, \varsigma_1]$$

on the moduli space of quilted disks as follows. Require that

$$(d(z) = \infty) \implies (\ell(z) = \varsigma_0)$$
$$(d(z) = -\infty) \implies (\ell(z) = \varsigma_1)$$

and furthermore ℓ is constant on any disk or sphere component in a fiber of $\mathcal{U}_{n,1}$. A *quilted* $[\varsigma_0, \varsigma_1]$*-broken disk* is obtained from a quilted disk by replacing each disk or sphere with a broken disk or sphere with the property that on any broken disk or sphere component C_i, the distance from the first to last sublevel is $\ell(C_i)$. Given perturbation data $\underline{P}^0$ and $\underline{P}^1$ with respect to metrics $G^0, G^1 \in \mathcal{G}(L)$ over unquilted treed disks for D^0 resp. D^1, a *perturbation morphism* $\underline{P}^{01}$ is defined as in Chapter 5.3. A *pseudoholomorphic quilted* $[\varsigma_0, \varsigma_1]$*-broken treed disk* $u : C \to X$ of combinatorial type Γ is a continuous map from a quilted treed $[\varsigma_0, \varsigma_1]$-broken disk C that is smooth on each component, J_Γ^{01}-holomorphic on the surface components, F_Γ^{01}-Morse trajectory with respect to the metric G_Γ^{01} on each boundary tree segment of disk type $e \in \mathrm{Edge}_\circ(T)$, and an H_Γ^{01}-gradient trajectory on the tree segments of sphere type $e \in \mathrm{Edge}_\bullet(\Gamma)$. The moduli space of quilted broken disks $\overline{\mathcal{M}}_{n,1}(\mathbb{X}, L)$ then has the same transversality and compactness property as in the unquilted case.

Counting broken quilted holomorphic disks defines homotopy equivalences of the broken Fukaya algebras for various choices. Given an admissible perturbation morphism $\underline{P}^{01}$ from $\underline{P}^0$ to $\underline{P}^1$, define

$$\begin{aligned} (8.1) \quad \phi^n &: CF(L; \underline{P}^0, \varsigma_0)^{\otimes n} \to CF(L; \underline{P}^1, \varsigma_1) \\ (x_1, \ldots, x_n) &\mapsto \sum_{x_0, u \in \mathcal{M}_\Gamma^{[\varsigma_0,\varsigma_1]}(L, D, x_0, \ldots, x_n)_0} (-1)^{\heartsuit} \epsilon(u) (\sigma(u)!)^{-1} q^{E(u)} y(u) x_0 \end{aligned}$$

where the sum is over quilted disks in strata of dimension zero with $x_1, \ldots, x_n$ incoming labels. Similarly counting twice-quilted disks gives homotopy equivalences between the various morphisms. Thus the homotopy type of $CF(\mathbb{X}, L, \varsigma)$ and non-vanishing of the broken Floer cohomology is independent of all choices (including the choice of ς) up to homotopy equivalence.

8.2. Breaking a symplectic manifold

Next we study the relationship between the broken Fukaya algebra and the unbroken Fukaya algebra. We recall some terminology from Bourgeois-Eliashberg-Hofer-Wysocki-Zehnder [**22**].

Definition 8.2. (a) (Neck-stretched manifold) Let X be a closed almost complex manifold, and $Z \subset X$ a separating real hypersurface of the form in Definition 7.2 (a). Let X° denote the manifold with boundary obtained by cutting open X along Z. Let Z', Z'' denote the resulting copies of Z. For any $\tau > 0$ let

$$X^\tau = X^\circ \cup \bigcup_{Z'' = \{-\tau\} \times Z, \{\tau\} \times Z = Z'} [-\tau, \tau] \times Z$$

obtained by gluing together the ends Z', Z'' of X° using a neck $[-\tau, \tau] \times Z$ of length 2τ.

To achieve compactness and gluing, we restrict to almost complex structures that are of standard form on the neck region as in the following:

DEFINITION 8.3. (Cylindrical almost complex structures) Let $\pi_Y : Z \to Y$ be a circle bundle over a symplectic manifold Y and $X = \mathbb{R} \times Z$. Let

$$\pi_{\mathbb{R}} : X \to \mathbb{R}, \quad \pi_Z : X \to Z, \quad \pi : X \to Y$$

the projections onto factors resp. onto Y and $\ker(D\pi) \subset TX$ the vertical subspace. Let $\omega_Z = \pi_Y^* \omega_Z \in \Omega^2(Z)$ denote the pullback of the symplectic form ω_Y to Z. Then

$$V = \ker(D\pi_Y) \oplus \ker(D\pi_Z) \subset TX \tag{8.2}$$

is a rank two subbundle complementary to

$$H = \ker(\alpha) \subset \pi_Z^* TZ \subset TX.$$

The $\mathbb{R}$ action by translation on $\mathbb{R}$ and $U(1)$ action on Z combine to a smooth $\mathbb{C}^\times \cong \mathbb{R} \times U(1)$ action on X. An almost complex structure J on X is called *cylindrical* if and only if

(a) $J\frac{\partial}{\partial t} = v$, with t the coordinate on $\mathbb{R}$ and
(b) J is invariant under the $\mathbb{C}^\times$-action.

REMARK 8.4. Any cylindrical almost complex structure induces an almost complex structure J_Y on Y by projection, so that $D\pi(Jw) = J_Y D\pi w$ for any $w \in TX$. Since J preserves the vertical component $\ker(D\pi)$ this formula defines $J_Y D\pi w$ independent of the choice of w. There are isomorphisms of complex vector bundles

$$TX \cong H \oplus V, \quad H \cong \pi_Y^* TY, \quad V \cong X \times \mathbb{C}. \tag{8.3}$$

REMARK 8.5. The neck-stretched manifold X^τ is diffeomorphic to X by a family of diffeomorphisms given on the neck region by a map $(-\tau,\tau)\times Z \to (-\epsilon,\epsilon)\times Z$ equal to the identity on Z and a translation in a neighborhood of $\pm\tau$. Given an almost complex structure J on X that is of cylindrical form on $(-\epsilon,\epsilon)\times Z$, we obtain an almost complex structure J^τ on X^τ by using the same cylindrical almost complex structure on the neck region. Via the diffeomorphism $X^\tau \to X$ described above, we obtain an almost complex structure on X also denoted J^τ.

8.3. Breaking perturbation data

The next lemmas allow us to develop a perturbation scheme for broken pseudoholomorphic maps compatible with degeneration.

LEMMA 8.6. (Symplectic sums for pairs) *Suppose that $X_\subset, X_\supset$ are symplectic manifolds both containing a symplectic hypersurface $Y \subset X_\subset, X_\supset$ such that the normal bundles $N_\subset, N_\supset$ of Y in $X_\subset, X_\supset$ are inverse, that is, there exists an isomorphism $N_\subset \cong N_\supset^{-1}$. Suppose furthermore that $D_\subset, D_\supset \subset X_\subset, X_\supset$ are codimension two symplectic submanifolds such that $D_\subset, D_\supset \cap Y = D_Y$ for some $D_Y \subset Y$. Then the symplectic sum*

$$D = D_\subset \#_{D_Y} D_\supset$$

is naturally a symplectic submanifold of $X = X_\subset \#_Y X_\supset$ preserved by a compatible almost complex structure of cylindrical form on the neck region.

PROOF. The statement is an application of symplectic local models as in Lerman [78] and Gompf [52]. Choose a metric on $X_\subset, X_\supset$ near Y so that $D_\subset, D_\supset$ is totally geodesic in $X_\subset, X_\supset$, as in [86, Lemma 6.8]; this can be done in stages so that the metrics on Y (considered as submanifolds of $X_\subset, X_\supset$) agree. The geodesic exponential map $N_\subset \to X_\subset, N_\supset \to X_\supset$ identifies a neighborhood of the zero section in $N_\subset, N_\supset$ restricted to D_Y with a neighborhood of D_Y in $D_\subset, D_\supset$. Choose a unitary connection α_+ on $N_\subset$, let α_- denote the dual connection on $N_\supset$ and let $\rho : N \to \mathbb{R}_{\geq 0}$ denote the norm function. Let $\pi : N \to Y$ denote the projection. The two forms

$$\pi^*\omega_Y + \mathrm{d}(\alpha_+, \rho) \in \Omega^2(N_\subset), \quad \pi^*\omega_Y - \mathrm{d}(\alpha_-, \rho) \in \Omega^2(N_\supset)$$

are symplectic in a neighborhood of the zero section and the punctured normal bundles $N_\subset^\circ, N_\supset^\circ$ may be glued together to form the symplectic sum.

To globalize this procedure we show that the symplectic normal forms may be chosen so that the identification of the divisors is preserved. The constant rank embedding theorem in Marle [83] identifies a neighborhood of D_Y in D with $N|D_Y$ symplectically. Consider the identification of N with a neighborhood of Y on $X_\subset, X_\supset$ which maps $N|D_Y$ to $D = D_\subset$ or $D = D_\supset$. Let ω_1 denote the pull-back of the symplectic form on $X_\subset, X_\supset$ and ω_0 the symplectic form induced from a connection on N. We have $\omega_1 - \omega_0|(D \cup Y) = 0$. The intersection $D \cap Y$ is transverse, so there exists a deformation retract of a neighborhood U of Y to $(D \cup Y) \cap U$ in $X_\subset$ or $X_\supset$. The standard homotopy formula produces a one-form $\beta \in \Omega^1(U)$ such that

$$\mathrm{d}\beta = \omega_1 - \omega_0, \quad \beta|((D \cup Y) \cap U) = 0.$$

Then $\omega_t = t\omega_1 + (1-t)\omega_0$ satisfies $\frac{d}{dt}\omega_t = \mathrm{d}\beta$. Define a vector field

$$v_t \in \mathrm{Vect}(U), \quad \iota(v_t)\omega_t = \beta.$$

Then v_t vanishes on $(D \cup Y) \cap U$ and defines a symplectomorphism from ω_0 to ω_1 in a neighborhood of Y equal to the identity on D. Thus the gluing of local models induces an identification of the divisors as well. The almost complex structure on N preserves $N|D_Y$ and induces a cylindrical almost complex structure on the symplectic sum preserving D. □

LEMMA 8.7. *Suppose that $L \subset X_\subset$ is a rational Lagrangian. There exists a family of divisors D^τ on X so that the image of L in X is exact in the complement of D^τ, and D^τ degenerates as $t \to 0$ to a broken symplectic manifold $\mathbb{D}$ where $D_\subset, D_\supset$ are divisors in $X_\subset, X_\supset$ so that L is exact in the complement of $D_\subset$.*

PROOF. We first construct a suitable Donaldson hypersurface in the broken symplectic manifold. Let $D_\subset \subset X_\subset$ be a divisor making L exact in the complement, and such that the intersection $D_\subset \cap Y = D_Y$ is a Donaldson hypersurface for Y. We may assume that $D_\subset, D_\supset$ are the zero set of some element in an asymptotically holomorphic sequence of sections covariant constant on L. By Theorem 7.15, there exists a Donaldson hypersurface $D_\supset \subset X_\supset$ with $Y \cap D_\supset = D_Y$. We now apply the gluing procedure of Lemma 8.6. Since the normal bundles $N_\subset, N_\supset$ of Y in $X_\subset, X_\supset$ are inverses, the restrictions $N_\subset, N_\supset|D_Y$ are also inverses. Then (Y, D_Y) has a symplectic tubular neighborhood of the form $(N_\subset, N_\subset|D_Y)$ resp $(N_\supset$ resp. $N_\supset|D_Y)$, where the latter has symplectic structure induced by the choice of connection as in the Lemma 8.6. Furthermore, D is almost complex with respect to every element of the family J^τ by construction, and is the zero set of some element in an asymptotically holomorphic sequence of sections of $\tilde{X}^k$. The connections on the bundles $\tilde{X}_\subset, \tilde{X}^k_\supset$ glue together to a connection on $\tilde{X}^k$ making ω exact in the complement of D, and for which the section defining D is covariant constant on L. Then L is exact in the complement $X - D$ by a standard argument using Stokes' theorem, c.f. [**27**, Section 3]. □

We now construct a system of perturbations for the breaking process. A *breaking disk* is a treed holomorphic disk $u : C \to X^\tau$ with a *breaking parameter* $\tau \in [0, \infty]$ such that if $\tau < \infty$ resp. $\tau = \infty$ then the disk is unbroken resp. broken. The compactified moduli space of breaking disks of type Γ is denoted $\overline{\mathcal{M}}_\Gamma$ with a universal curve $\overline{\mathcal{U}}_\Gamma$ that has surface part $\overline{\mathcal{S}}_\Gamma$ and tree part $\overline{\mathcal{T}}_\Gamma$. Breaking quilted disks and twice-quilted disks can be defined similarly.

Definition 8.8. (Perturbation data) A *perturbation datum* for breaking curves of type Γ is a datum $P_\Gamma = (J_\Gamma, F_\Gamma, G_\Gamma, H_\Gamma)$ consisting of maps

$$J_\Gamma : \overline{\mathcal{S}}_\Gamma \to \mathcal{J}^\tau(X), \quad \begin{pmatrix} F_\Gamma : \overline{\mathcal{T}}_{\Gamma,\circ} \to C^\infty(L) \\ G_\Gamma : \overline{\mathcal{T}}_{\Gamma,\circ} \to \mathcal{G}(L) \end{pmatrix}, \quad H_\Gamma : \overline{\mathcal{T}}_{\Gamma,\bullet} \to C^\infty(Y).$$

In particular, suppose that J_Γ is a collection of domain-dependent almost complex structures for disks equal to a fixed cylindrical structure $J_{\mathbb{R}\times Z}$ in a tubular neighborhood of the separating hypersurface $Z \subset X$. Then by gluing in lengths of neck τ we obtain a family J^τ_Γ of domain-dependent almost complex structures for breaking disks.

Associated to any morphism of combinatorial types is a morphism of perturbation data, as before. However in this case there is a new kind of morphism $\Gamma \to \Gamma'$ of combinatorial types corresponding to *gluing of cylindrical end symplectic manifolds*, that is, making the neck length τ in 7.2 finite. In this case a collection of perturbations $\underline{P} = (P_\Gamma)$ is *coherent* if, in addition to the usual conditions in the coherence axioms, we require that the restriction of $P_{\Gamma'}$ to $\mathcal{U}_\Gamma$ is equal to P_Γ whenever a neck length τ is set to infinity. Given a map $u : C = S \cup T \to X$ we obtain an element

(8.4)
$$\overline{\partial} u = \left(J_\Gamma \circ \mathrm{d}(u|S) \circ j + \mathrm{d}(u|S), \frac{d}{ds} u|T_\bullet + \mathrm{grad}_{F_\Gamma} u|T_\bullet, \frac{d}{ds} u|T_\circ + \mathrm{grad}_{H_\Gamma} u|T_\circ \right)$$
$$\in \Omega^{0,1}(S, (u|S)^* TX) \times \Omega^1(T_\bullet, (u|T_\bullet)^* TX) \times \Omega^1(T_\circ, (u|T_\circ)^* TL).$$

We say that u is P_Γ-holomorphic if $\overline{\partial} u = 0$.

Definition 8.9. Given a perturbation datum an *adapted breaking map* $u : C = S \cup T \to X$ of type Γ is a breaking disk C with breaking parameter $\tau \in [0, \infty)$ together with an adapted map $C \to X$ that is P_Γ-holomorphic for $\tau < \infty$, or an adapted map $C \to \mathbb{X}$ for $\tau = \infty$ that is a broken P_Γ-holomorphic map.

For generic breaking perturbations, the moduli space of adapted breaking maps is cut out transversally, by a repeat of the Sard-Smale arguments described in the proof of Theorem 4.20.

A theorem analogous to Theorem 7.7 holds in the case of a breaking symplectic manifold. Compare with, for example, [**25**, Theorem 1.1]. The proof is similar, but allowing for domain-dependent almost complex structures away from the nodes.

THEOREM 8.10. *Let $\underline{P} = (P_\Gamma)$ be a coherent collection of perturbation data for breaking disks. Let $\tau_\nu \to \infty$ be a sequence of neck lengths. Any sequence of adapted stable maps $u_\nu : C_\nu \to X^{\tau_\nu}$ holomorphic with respect to J_Γ with bounded energy converges, after passing to a subsequence, to a stable adapted broken map $u : C \to \mathbb{X}[k]$, for some k, with the same index.*

8.4. Getting back together

We have tubular neighborhoods of the strata as in Theorem 4.20, in particular, a tubular neighborhood of the boundary corresponding to the breaking. This gluing result is similar but slightly different from that in Bourgeois-Oancea [**21**]. Much more complicated gluing theorems in symplectic field theory have been proved in Hutchings-Taubes [**63**] and Hofer-Wysocki-Zehnder (see e.g. [**61**]) both of which involve obstructions arising from multiple branched covers of Reeb orbits. Here any such cover corresponds to a fiber of a bundle with projective line fibers, and so one has transversality automatically (although one also has to achieve transversality with the diagonal at the nodes). Recall from Definition 8.2 that $C^{\delta_1,\ldots,\delta_k}$ resp. $X_\delta := X^{|\ln(\delta)|}$ are obtained from $C, \mathbb{X}$ by gluing in necks of length $|\ln(\delta_i)|$ resp. $|\ln(\delta)|$ at each node of C resp. the divisor Y.

THEOREM 8.11. *Suppose that $u : C \to \mathbb{X}$ is a 0-regular broken map with multiplicities $\mu_1, \ldots, \mu_k$ at the separating hypersurface $Y \subset \mathbb{X}$. Then there exists $\delta_0 > 0$ such that for each gluing parameter $\delta \in (0, \delta_0)$ there exists an unbroken map $u_\delta : C^{\delta/\mu_1,\ldots,\delta/\mu_k} \to X_\delta$, with the property that u_δ depends smoothly on δ and $\lim_{\delta\to 0}[u_\delta] = [u]$. For any energy bound $E > 0$ there exists δ_0 such that for $\delta < \delta_0$ the correspondence $[u] \mapsto [u_\delta]$ defines a bijection between the moduli spaces $\mathcal{M}_\Gamma^{<E}(X_\delta, L)$ and $\mathcal{M}_\Gamma^{<E}(\mathbb{X}, L)$ for any combinatorial type Γ.*

PROOF. As for other gluing theorems in pseudoholomorphic curves the proof is an application of a quantitative version of the implicit function theorem for Banach manifolds. The steps are: construction of an approximation solution; construction of an approximate inverse to the linearized operator; quadratic estimates; application of the contraction mapping principle, and surjectivity of the gluing construction. We describe the proof of this gluing result in the following simple case: Let C be a broken curve with two sublevels C_+, C_-. A standard gluing procedure creates, for any small *gluing parameter* $\delta \in \mathbb{C}$, a curve C^δ obtained by removing small disks $U_\pm$ around the node in the surface part $w \in S \subset C$ and gluing using a map given in local coordinates by $z \mapsto \delta/z$. That is, C^δ is obtained by replacing S with $S^\delta - (U_+ - U_-)/(z \sim \delta/z)$ and leaving the tree part the same.

Step 0: Fix local trivializations of the universal treed disk and the associated families of complex structures and metrics on the domains and targets. Let $u_- : C_- \to X_- := X_\subset$ and $u_+ : C_+ \to X_+ := X_\supset$ with $C_\pm = S_\pm \cup T_\pm$ be treed holomorphic disks agreeing at a point $u_+(w_{+-}) = u_-(w_{-+}) \in Y$. Let $\Gamma_\pm$ denote the combinatorial types of $u_\pm$ and let

$$\mathcal{U}^i_{\Gamma_\pm} \to \mathcal{M}^i_{\Gamma_\pm} \times S_\pm, , i = 1, \ldots, l$$

be local trivializations of the universal treed disk, identifying each nearby fiber with $(C^\circ_\pm, \underline{z}, \underline{w})$ such that each point in the universal treed disk is contained in one of these local trivializations. We may assume that $\mathcal{M}^i_{\Gamma_\pm}$ is identified with an open ball in Euclidean space so that the fiber $C^\circ_\pm$ corresponds to 0. Similarly, we assume we have a local trivialization of the universal bundle near the glued curve giving rise to a family of complex structures

$$\mathcal{M}_\Gamma^i \to \mathcal{J}(S^\delta) \tag{8.5}$$

of complex structures on the two-dimensional locus $S^\delta \subset C^\delta$ that are constant on the neck region. We consider metrics on the punctured curves $C_\pm^\circ$ that are cylindrical on the neck region given as the image of the end coordinates

$$\epsilon_\pm^C : \pm[0,\infty) \times S^1 \to C_\pm^\circ.$$

Introduce cylindrical ends for $X_- := X_\subset, X_+ := X_\supset$ so that the embeddings

$$\epsilon_\pm^X : \mp[0,\infty) \times Z$$

are isometric. Both the glued target X_{δ^μ} and glued domain C^δ are defined by removing the part of the end with $|s| > |\ln(\delta)|$ and identifying $(s,t) \sim (s - |\ln(\delta)|, t)$ for $(s,t) \in (0, |\ln(\delta)|) \times S^1$ resp. $(s,t) \sim (s - |\mu \ln(\delta)|, t)$ for $(s,t) \in (0, |\ln(\delta)| \times Z)$.

Step 1: Define an approximate solution by gluing together the two solutions using a cutoff function. Choose a cutoff function

$$\beta \in C^\infty(\mathbb{R}, [0,1]), \quad \begin{cases} \beta(s) = 0 & s \le 0 \\ \beta(s) = 1 & s \ge 1 \end{cases}. \tag{8.6}$$

We may suppose by Remark 7.8 and a shift in coordinates that the maps $u_\pm$ are asymptotic to $(\mu s, t^\mu z)$ for some $z \in Z$. The maps $u_\pm^\pm$ considered locally as maps to X° are asymptotic to the trivial cylinder $(\mu s, t^\mu z)$:

$$\lim_{s\to\infty} d(u_+(s,t), (\mu s, t^\mu z)) = \lim_{s\to -\infty} d(u_-(s,t), (\mu s, t^\mu z)) = 0.$$

Denote by $\exp_x : T_x X \to X$ geodesic exponentiation with respect to the given cylindrical metric on the neck region. Let $\zeta_\pm$ be the section related by geodesic exponentiation in cylindrical coordinates to the maps $u_\pm$

$$u_\pm(s,t) = \exp_{(\mp\mu s, t^\mu z)}(\zeta_\pm(s,t)).$$

Define u_δ^{pre} to be equal to $u_\pm$ away from the neck region, while on the neck region of C^δ with coordinates s,t define

$$\begin{aligned} u_\delta^{\text{pre}}(s,t) &= \exp_{(\mu s, t^\mu z)}(\zeta^\delta(s,t)), \\ \zeta^\delta(s,t) &= \beta(-s)\zeta_-(-s + |\ln(\delta)|/2, t) + \beta(s)\zeta_+(s - |\ln(\delta)|/2, t)). \end{aligned} \tag{8.7}$$

In other words, one translates u_+, u_- by some amount $|\ln(\delta)|$, and then patches them together using the cutoff function and geodesic exponentiation.

Step 2: Define a map between suitable Banach spaces whose zeroes describe pseudoholomorphic curves near to the approximate solution. Denote by

$$(s,t) \in [-|\ln(\delta)|/2, |\ln(\delta)|/2] \times S^1$$

the coordinates on the neck region in C^δ created by the gluing. Let $\lambda \in (0,1)$ be a *Sobolev weight*. Define a *Sobolev weight function*

$$\kappa_\lambda^\delta : C^\delta \to [0,\infty), \quad (s,t) \mapsto \beta(|\ln(\delta)|/2 - |s|)p\lambda(|\ln(\delta)|/2 - |s|)$$

where $\beta(|s| - |\ln(\delta)|/2)p\lambda(|s| - |\ln(\delta)|/2))$ is by definition zero on the complement of the neck region. We will also use similar weight functions on the punctured curves

$$\kappa_\lambda^\pm : C_\pm^\circ \to [0,\infty), \quad (s,t) \mapsto \beta(|s|)p\lambda|s|$$

Pseudoholomorphic maps near the pre-glued solution are cut out locally by a smooth map of Banach spaces. Given a smooth map $u : C^\delta \to X$, let

$$\Omega^0(C^\delta, u^*TX, T_{w_\pm}Y) = \Omega^0(S^\delta, (u|S)^*TX, T_{w_\pm}Y) \oplus \Omega^0(T, (u|T)^*TL)$$

the space of infinitesimal deformations of the map, preserving the condition $u(w_\pm) \in Y$. Given an element and sections

$$m \in \mathcal{M}_\Gamma^i, \quad \xi_S : S^\delta \to u_S^*TX, \quad \xi_T : T \to u_T^*TL, \quad \xi = (\xi_S, \xi_T)$$

define as in Abouzaid [**3**, 5.38] a norm based on the decomposition of the section into a part constant on the neck and the difference:

$$\begin{aligned} (8.8) \quad \|(m,\xi)\|_{1,p,\lambda}^p &:= \|m\|^p + \|\xi_S\|_{1,p,\lambda}^p + \|\xi_T\|_{1,p,\lambda}^p \\ \|\xi_S\|_{1,p,\lambda}^p &:= \|(\xi_S(0,0))\|^p + \int_{C^\delta} (\|\nabla \xi_S\|^p \\ &\quad + \|\xi_S - \beta(|\ln(\delta)|/2 - |s|)\mathcal{T}^u(\xi_S(0,0))\|^p) \exp(\kappa_\lambda^\delta) \mathrm{d}\,\mathrm{Vol}_{C^\delta} \end{aligned}$$

where $\mathcal{T}^u$ is parallel transport from $u^{\mathrm{pre}}(0,t)$ to $u^{\mathrm{pre}}(s,t)$ along $u^{\mathrm{pre}}(s',t)$. Denote by $\Omega^0(C^\delta, u^*TX, T_wY)_{1,p,\delta}$ the space of $W^{1,p}_{\mathrm{loc}}$ sections with finite norm (8.8); these are sections whose difference from a covariant-constant TY-valued section on the neck has an exponential decay behavior governed by the Sobolev constant λ. Pointwise geodesic exponentiation defines a map (using Sobolev multiplication estimates)

$$(8.9) \qquad \exp_{u_\delta^{\mathrm{pre}}} : \Omega^0(C^\delta, (u_\delta^{\mathrm{pre}})^*TX_{\delta^\mu})_{1,p,\lambda} \to \mathrm{Map}_{1,p}(C^\delta, X_{\delta^\mu})$$

where $\mathrm{Map}_{1,p}(C^\delta, X_{\delta^\mu})$ denotes maps of class $W_{1,p}^{\mathrm{loc}}$ from C^δ to X_{δ^μ}. Similarly for the punctured surfaces we have Sobolev norms

$$\begin{aligned} (8.10) \quad \|(m,\xi)\|_{1,p,\lambda} &:= \|m\|^p + \|\xi_S\|_{1,p,\lambda} + \|\xi_T\|_{1,p,\lambda}, \\ \|\xi_S\|_{1,p,\lambda} &:= \|\xi(0,0)\|^p + \int_{C^\delta} (\|\nabla \xi\|^p + \\ &\quad \|\xi_S - \beta(|s|)\mathcal{T}^u \xi(0,0)\|^p) \exp(\kappa_\lambda^\pm) \mathrm{d}\,\mathrm{Vol}_{C_\pm^\circ} \Big)^{1/p}. \end{aligned}$$

Geodesic exponentiation defines maps

$$(8.11) \qquad \exp_{u_\delta^{\mathrm{pre}}} : \Omega^0(C_\pm^\circ, (u_\delta^{\mathrm{pre}})^*TX)_{1,p,\lambda} \to \mathrm{Map}_{1,p,\lambda}(C_\pm^\circ, X_\pm^\circ)$$

where, by definition, $\mathrm{Map}_{1,p,\lambda}(C_\pm^\circ, X_\pm^\circ)$ is the space of $W_{1,p}^{\mathrm{loc}}$ maps from $C_\pm^\circ$ to $X_\pm$ that differ from a Reeb orbit at infinity by an element of $\Omega^0(C_\pm^\circ, X_\pm^\circ)_{1,p,\lambda}$ (which may vary at infinity because of the inclusion of constant maps on the end in the Banach space). Since the surface part S^δ satisfies a uniform cone condition and the metrics on X_{δ^μ} are uniformly bounded, one has uniform Sobolev embedding estimates and multiplication estimates.

Results on elliptic operators on manifolds with cylindrical ends in Lockhart-McOwen [**82**] imply that the linearized Cauchy-Riemann operators are Fredholm. For the closed manifolds $C_\pm$, we have linearized operators

$$D_{u_\pm} : \Omega^0(C_\pm, u_\pm^*TX)_{1,p} \to \Omega^{0,1}(C_\pm, u_\pm^*TX)_{0,p}$$

where by definition $\Omega^0(C_\pm, u_\pm^* TX)_{1,p}$ consists of those sections ξ satisfying various constraints such as tangency to the divisor at infinity $\xi(w_\pm) \in TY$, see (7.6). In the case of the cylindrical end manifolds $u_\pm^\circ : C_\pm^\circ \to X$, the assumption $\lambda \in (0,1)$ on the Sobolev decay constant implies that the linearized operators

$$D_{u_\pm}^\circ : \Omega^0(C_\pm^\circ, u_\pm^* TX_\pm^\circ)_{1,p,\lambda} \to \Omega^{0,1}(C_\pm^\circ, u_\pm^* TX_\pm^\circ)_{1,p,\lambda}$$

are Fredholm. As in [**3**, 4.18] there is an inclusion

$$\ker(D_{u_\pm}^\circ) \to \ker(D_{u_\pm}) \tag{8.12}$$

defined by removal of singularities at infinity: The space $\Omega^0(C_\pm^\circ, u_\pm^* TX_\pm^\circ)_{1,p,\lambda}$ includes into $\Omega^0(C_\pm, u_\pm^* TX_\pm)_{1,p,\lambda}$ because of the exponential decay estimates while conversely elliptic regularity implies any element of $\ker(D_{u_\pm})$ is smooth near $w_\pm$ and so decays exponentially in logarithmic coordinates; note that a change to the constant term on the neck corresponds to a change to the derivative of the map at the point at infinity. Since any translation on the neck extends to a diffeomorphism of the domain, any element $\xi_\pm \in \ker(\tilde{D}_{u_\pm})$ is equivalent mod $\ker(\tilde{D}_{u_\pm})$ to a section taking values in $T_y Y$ at infinity. By the regularity assumption the fiber products

$$\ker(\tilde{D}_{u_-^\circ}) \times_{T_y Y} \ker(\tilde{D}_{u_+^\circ}) \cong \ker(\tilde{D}_{u_-}) \times_{T_y Y} \ker(\tilde{D}_{u_+}) \tag{8.13}$$

are transversally cut out.

The space of pseudoholomorphic maps near the pre-glued solution is cut out locally by a smooth map of Banach spaces. For a $0,1$-form $\eta \in \Omega^{0,1}(C^\delta, u^* TX)$ define

$$\|\eta\|_{0,p,\lambda} = \left(\int_{C^\delta} \|\eta\|^p \exp(\kappa_\lambda^\delta) \mathrm{d}\,\mathrm{Vol}_{C^\delta} \right)^{1/p}.$$

Parallel transport using an almost-complex connection defines a map

$$\mathcal{T}_{u_\delta^{\mathrm{pre}}}(\xi) : \ \Omega^{0,1}(C^\delta, (u_\delta^{\mathrm{pre}})^* TX)_{0,p,\lambda} \to \Omega^{0,1}(C, (\exp_{u_\delta^{\mathrm{pre}}}(\xi))^* TX)_{0,p,\lambda}.$$

Define

$$\mathcal{F}_\delta : \mathcal{M}_\Gamma^i \times \Omega^0(C^\delta, (u_\delta^{\mathrm{pre}})^* TX)_{1,p} \to \Omega^{0,1}(C^\delta, (u_\delta^{\mathrm{pre}})^* TX)_{0,p} \tag{8.14}$$

$$(m_S, m_T, \xi_S, \xi_T) \mapsto \left(\mathcal{T}_{u_\delta^{\mathrm{pre}}}(\xi_S)^{-1} \overline{\partial}_{J_\Gamma, j(m_S)} \exp_{u_\delta^{\mathrm{pre}}}(\xi_S), \ (\frac{d}{dt} + \mathrm{grad}_{m_T}(H_\Gamma))(\xi_T) \right).$$

Treed pseudoholomorphic maps close to u_δ^{pre} correspond to zeroes of $\mathcal{F}_\delta$. In addition, because we are working in the adapted setting, our curves C^δ have a collection of interior leaves $e_1, \ldots, e_n$. We require that the constant maps on the interior edges have values

$$(\exp_{u_\delta^{\mathrm{pre}}}(\xi))(T_{e_i}) \subset D, \quad i = 1, \ldots, n.$$

By choosing local coordinates near the attaching points $z_i = e_i \cap S$, these constraints may be incorporated into the map $\mathcal{F}_\delta$ to produce a map

$$\begin{aligned} \mathcal{F}_\delta : \mathcal{M}_\Gamma^i \times \Omega^0(C_\delta, (u_\delta^{\mathrm{pre}})^* TX)_{1,p} & \\ \to \Omega^{0,1}(C^\delta, (u_\delta^{\mathrm{pre}})^* TX)_{0,p} \oplus \bigoplus_{i=1}^n T_{u(z_i)} X / T_{u(z_i)} D & \end{aligned} \tag{8.15}$$

whose zeroes correspond to *adapted* pseudoholomorphic maps near the preglued map u_δ^{pre}.

Step 3: Estimate the failure of the approximate solution to be an exact solution. The one-form $\mathcal{F}_\delta(0)$ has contributions created by the cutoff function as well as the difference between $J_{u_\pm}$ and $J_{u_\delta^{\mathrm{pre}}}$:

$$\begin{aligned}\|\mathcal{F}_\delta(0)\|_{0,p,\lambda} &= \|\overline{\partial}\exp_{(\mu s,t^\mu z)}(\beta(-s)\zeta_-(-s+|\ln(\delta)|/2,t)\\ &\quad +\beta(s)\zeta_+(s-|\ln(\delta)|/2,t))\|_{0,p,\lambda}\\ &= \|(D\exp_{(\mu s,t^\mu z)}(\mathrm{d}\beta(-s)\zeta_-(-s+|\ln(\delta)|/2,z)\\ &\quad +\mathrm{d}\beta(s)\zeta_+(s-|\ln(\delta)|/2,t))+\\ &\quad (\beta(-s)\mathrm{d}\zeta_-(-s+|\ln(\delta)|/2,z)\\ &\quad +\beta(s)\mathrm{d}\zeta_+(s-|\ln(\delta)|/2,t)))^{0,1}\|_{0,p,\lambda}.\end{aligned}$$

Holomorphicity of $u_\pm$ implies an estimate

$$\|((\beta(-s)\mathrm{d}\zeta_-(-s+|\ln(\delta)|/2,z)+\beta(s)\mathrm{d}\zeta_+(s-|\ln(\delta)|/2,t)))^{0,1}\|_{0,p,\lambda} \le Ce^{-|\ln(\delta)|(1-\lambda)} = C\delta^{1-\lambda},$$

c.f. Abouzaid [**3**, 5.10]. Similarly from the terms involving the derivatives of the cutoff function and exponential convergence of $\zeta_\pm$ to 0 we obtain an estimate

$$\|\mathcal{F}_\delta(0)\|_{0,p,\lambda} < C\exp(-|\ln(\delta)|(1-\lambda)) = C\delta^{1-\lambda} \tag{8.16}$$

with C independent of δ.

Step 4: Construct a uniformly bounded right inverse for the linearized operator of the approximate solution from the given right inverses of the pieces. Given an element $\eta\in\Omega^{0,1}(C^\delta,(u^{\mathrm{pre}})^*T(\mathbb{R}\times Z))_{0,p}$, one obtains elements

$$\underline{\eta} = (\eta_-,\eta_+)\in\Omega^{0,1}(C_\pm^\circ,u_\pm^*TX_\pm^\circ)$$

by multiplication with the cutoff function and parallel transport $\mathcal{T}^{u_\pm}$ to $u_\pm$ along the path $\exp_{(\mu s,t^\mu z)}(\rho(\zeta^\delta(s,t)+(1-\rho)\zeta_\pm(s,t))),\rho\in[0,1]$. Define

$$\eta_+ = \mathcal{T}^{u_+}\beta(s-1/2)\eta,\quad \eta_- = \mathcal{T}^{u_-}\beta(1/2-s)\eta.$$

Since the fiber product (8.13) is transversally cut out, for any $\eta_\pm\in\Omega^{0,1}(C,u_\pm^*TX)_{0,p,\lambda}$ there exists

$$(\xi_+,\xi_-)\in\Omega^0(C_\pm^\circ,u^*TX_\pm^\circ)_{1,p,\lambda},\quad \tilde{D}_{u_\pm^\circ}\xi_\pm=\eta_\pm,\quad \xi_+(w_{+-})=\xi_-(w_{-+})$$

where $w_{\pm\mp}\in C_\pm$ are the nodal points considered as the points at infinity in $C_\pm^\circ$ and furthermore equal at infinity to an element

$$\xi_\infty\in T_{u_\pm(\infty)}Y$$

and at the markings constrained to lie at the divisor satisfying the constraints

$$\xi_\pm(z_i)\in T_{u_\pm(z_i)}D_i.$$

Define $Q^\delta\eta$ equal to (ξ_-,ξ_+) away from $[-|\ln(\delta)|/2,|\ln(\delta)|/2]\times Z$ and on the neck region by patching these solutions together using a cutoff function

$$\begin{aligned}Q^\delta\eta := {}&\beta\left(-s-\frac{1}{4}|\ln(\delta)|\right)((\mathcal{T}^{u_-})^{-1}\xi_- - \mathcal{T}^u\xi_\infty)\\ &+\beta\left(s+\frac{1}{4}|\ln(\delta)|\right)((\mathcal{T}^{u_+})^{-1}\xi_+ - \mathcal{T}^u\xi_\infty)\\ &+\mathcal{T}^u\xi_\infty\in\Omega^{0,1}(C^\delta,(u_\delta^{\mathrm{pre}})^*TX)_{1,p,\lambda}\end{aligned} \tag{8.17}$$

where $\mathcal{T}^\delta_\pm$ denotes parallel transport from $u_\pm$ to u^{pre} along the path

$$\exp_{(\mu s, t^\mu z)}(\rho(\zeta^\delta(s,t) + (1-\rho)\zeta_\pm(s,t))), \rho \in [0,1].$$

Since

$$\eta = (\mathcal{T}^{u_-})^{-1}\eta_- + (\mathcal{T}^{u_+})^{-1}\eta_+$$

we have

$$\begin{aligned} \|\tilde{D}_{u^\delta_{\mathrm{pre}}} Q^\delta \eta - \eta\|_{1,p,\lambda} &= \|\tilde{D}_{u^{\mathrm{pre}}_\delta} Q^\delta \eta - (\mathcal{T}^{u_-})^{-1}\tilde{D}_{u^\delta_-}\xi_- \\ &\quad -(\mathcal{T}^{u_+})^{-1}\tilde{D}_{u^\delta_+}\xi_+\|_{1,p,\lambda} \\ &\leq C\exp((1-\lambda)|\ln(\delta)/4|)\|\eta\|_{0,p,\lambda} \\ &\quad + C\|\mathrm{d}\beta(s - |\ln(\delta)|/4)Q^\delta_- \underline{\eta}\|_{0,p,\lambda} \\ &\quad + C\|\mathrm{d}\beta(-s - |\ln(\delta)|/4)Q^\delta_+ \underline{\eta}\|_{0,p,\lambda}. \end{aligned}$$

Here the first term arises from the difference between $\tilde{D}_{u^{\mathrm{pre}}_\delta}$ and $(\mathcal{T}^{u_\pm})^{-1}\tilde{D}_{u_\pm}\mathcal{T}^{u_\pm}$ and the second from the derivative $\mathrm{d}\beta$ of the cutoff function β. The difference in the exponential factors

$$\kappa^\pm_\lambda = \kappa^\delta_\lambda \exp(\pm 2s\lambda), \quad \mp s \geq |\ln(\delta)|/2$$

in the definition of the Sobolev weight functions implies that possibly after changing the constant C, we have

$$\|\mathrm{d}\beta(s - |\ln(\delta)|/4)Q^\delta_\pm \eta\|_{1,p,\lambda} < Ce^{-\lambda|\ln(\delta)|/2} = C\delta^{\lambda/2}.$$

Hence one obtains an estimate as in Fukaya-Oh-Ohta-Ono [**46**, 7.1.32], Abouzaid [**3**, Lemma 5.13]: for some constant $C > 0$, for any $\delta > 0$

$$\|\tilde{D}_{u^{\mathrm{pre}}_\delta} Q^\delta - \mathrm{Id}\| < C\min(\delta^{\lambda/2}, \delta^{(1-\lambda)/4}). \tag{8.18}$$

It follows that for δ sufficiently large an actual inverse may be obtained from the Taylor series formula

$$\tilde{D}^{-1}_{u^{\mathrm{pre}}_\delta} = (Q^\delta \tilde{D}_{u^{\mathrm{pre}}_\delta})^{-1} Q^\delta = \sum_{k \geq 0} (I - Q^\delta \tilde{D}_{u^{\mathrm{pre}}_\delta})^k Q^\delta.$$

Step 5: Obtain a uniform quadratic estimate for the non-linear map. After redefining $C > 0$ we have for all m, ξ, m_1, ξ_1

$$\|D_{m,\xi}\mathcal{F}_\delta(m_1,\xi_1) - \tilde{D}_{u^{\mathrm{pre}}_\delta}(m_1,\xi_1)\| \leq C\|m,\xi\|_{1,p,\lambda}\|m_1,\xi_1\|_{1,p,\lambda}. \tag{8.19}$$

To prove this we require some estimates on parallel transport. Let

$$\mathcal{T}^{\delta,x}_z(m,\xi) : \Lambda^{0,1}T^*_z C_\delta \otimes T_x X \to \Lambda^{0,1}_{j^\delta(m)} T^*_z C_\delta \otimes T_{\exp_x(\xi)} X$$

denote pointwise parallel transport. Consider its derivative

$$D\mathcal{T}^{\delta,x}_z(m,\xi,m_1,\xi_1;\eta) = \nabla_t|_{t=0}\mathcal{T}^{\delta,x}_z m + tm_1, \xi + t\xi_1)\eta.$$

For a map $u : C \to x$ we denote by $D\mathcal{T}^{\mathrm{pre}}_u$ the corresponding map on sections at u^δ_{pre}. By Sobolev multiplication (for which the constants are uniform because of the uniform cone condition on the metric on C^δ and uniform bounds on the metric on X_{δ^μ}) there exists a constant c such that

$$\|D\mathcal{T}^\delta_u(m,\xi,m_1,\xi_1;\eta)\|_{0,p,\lambda} \leq c\|(m,\xi)\|_{1,p,\lambda}\|(m_1,\xi_1)\|_{1,p,\lambda}\|\eta\|_{0,p,\lambda}. \tag{8.20}$$

Differentiate the equation

$$\mathcal{T}^\delta_u(m,\xi)\mathcal{F}_\delta(m,\xi) = \overline{\partial}_{j^\delta(m)}(\exp_{u^\delta_{\mathrm{pre}}}(\xi)))$$

with respect to (m_1, ξ_1) to obtain

$$(8.21)\quad D\mathcal{T}_u^\delta(m,\xi,m_1,\xi_1,\mathcal{F}_\delta(m,\xi)) + \mathcal{T}_u^\delta(m,\xi)(D\mathcal{F}_\delta(m,\xi,m_1,\xi_1)) = (D\overline{\partial})_{j^\delta(m),\exp_{u^\delta}(\xi)}(Dj^\delta(m,m_1), D\exp_{u^\delta}(\xi,\xi_1)).$$

The pointwise inequality

$$|\mathcal{F}_\delta(m,\xi)| < c|\mathrm{dexp}_{u_\delta^{\mathrm{pre}}(z)}(\xi)| < c(|\mathrm{d}u_\delta^{\mathrm{pre}}| + |\nabla\xi|)$$

holds for m, ξ sufficiently small. Together with the estimate (8.20) we obtain a pointwise estimate

$$|\mathcal{T}_u^\delta(\xi)^{-1} D\mathcal{T}_{u^\delta_{\mathrm{pre}}}(m,\xi,m_1,\xi_1,\mathcal{F}_\delta(m,\xi))| \le c(|\mathrm{d}u^\delta_{\mathrm{pre}}| + |\nabla\xi|)\,|(m,\xi)|\,|(\xi_1,m_1)|.$$

Hence

$$(8.22)\quad \|\mathcal{T}_u^\delta(\xi)^{-1} D\mathcal{T}_u^\delta(m,\xi,m_1,\xi_1,\mathcal{F}_\delta(m,\xi))\|_{0,p,\lambda} \le c(1 + \|\mathrm{d}u^\delta\|_{0,p,\lambda} + \|\nabla\xi\|_{0,p,\lambda})\|(m,\xi)\|_{L^\infty}\|(\xi_1,m_1)\|_{L^\infty}.$$

It follows that

$$(8.23)\quad \|\mathcal{T}_u^\delta(\xi)^{-1} D\mathcal{T}_u^\delta(m,\xi,m_1,\xi_1,\mathcal{F}_\delta(m,\xi))\|_{0,p,\lambda} \le c\|(m,\xi)\|_{1,p,\lambda}\|(m_1,\xi_1)\|_{1,p,\lambda}$$

since the $W^{1,p}$ norm controls the L^∞ norm by the uniform Sobolev estimates. Then, as in McDuff-Salamon [**95**, Chapter 10], Abouzaid [**3**] there exists a constant $c > 0$ such that for all δ sufficiently small, after another redefinition of C we have

$$(8.24)\quad \|\mathcal{T}_u^\delta(\xi)^{-1}\tilde{D}_{\exp_{u^\delta_{\mathrm{pre}}}(\xi)}(D_m j^\delta(m_1), D_{\exp_{u^\delta_{\mathrm{pre}}}(\xi)}\xi_1)) - \tilde{D}_{u^\delta_{\mathrm{pre}}}(m_1,\xi_1)\|_{0,p,\lambda} \le C\|m,\xi\|_{1,p,\lambda}\|m_1,\xi_1\|_{1,p,\lambda}.$$

Combining these estimates completes the proof of claim (8.19).

Step 6: Apply the implicit function theorem to obtain an exact solution. Recall Floer's version of the Picard Lemma, [**41**, Proposition 24]). The set-up is as follows. Let $f : V_1 \to V_2$ be a smooth map between Banach spaces that admits a Taylor expansion $f(v) = f(0) + df(0)v + N(v)$ where $df(0) : V_1 \to V_2$ has a right inverse $G : V_2 \to V_1$ satisfying the uniform bound

$$\|GN(u) - GN(v)\| \le C(\|u\| + \|v\|)\|u - v\|$$

for some constant C. Let $B(0,\epsilon)$ denote the ϵ-ball centered at $0 \in V_1$ and assume that

$$\|Gf(0)\| \le 1/8C.$$

The conclusion of the lemma is that for $\epsilon < 1/4C$, the zero-set of $f^{-1}(0) \cap B(0,\epsilon)$ is a smooth submanifold of dimension $\dim(\mathrm{Ker}(df(0)))$ diffeomorphic to the ϵ-ball in $\mathrm{Ker}(df(0))$.

Floer's Picard lemma together with the estimates (8.16), (8.18), (8.19) produce a unique solution $m(\delta), \xi(\delta)$ to the equation $\mathcal{F}_\delta(m(\delta),\xi(\delta)) = 0$ for each δ, such that the maps $u(\delta) := \exp_{u_\delta^{\mathrm{pre}}}(\xi(\delta))$ depend smoothly on δ. Note that Picard lemma itself does not give that the maps u_δ are distinct, since each u_δ is the result of applying the contraction mapping principle in a different Sobolev space.

Step 7: Surjectivity of the gluing construction. We show that the gluing construction gives a bijection. Consider a family of adapted maps

$$[u'_\delta : C(\delta) \to X_\delta], \quad |\ln(\delta)| \to \infty.$$

By the compactness result Theorem 7.7, there exists a subsequence that converges to a broken map $u : C \to \mathbb{X}$. Note that we do not require the domain $C(\delta)$ to be given by the same gluing parameter as the gluing parameter δ for X_δ. By definition of Gromov convergence the curve $C(\delta)$ is obtained from C using a gluing parameter δ_C, which is a function of the gluing parameter δ for the breaking of target to $\mathbb{X}$ and converges to zero as $\delta \to 0$.

To prove the bijection we must show that any such family of maps is in the image of the gluing construction. Since the implicit function theorem used to construct the gluing gives a unique solution in a neighborhood, it suffices to show that the maps $[u'_\delta]$ are close, in the Sobolev norm used for the gluing construction, to the approximate solution u_δ^{pre} defined by (8.7).

We think of the map on the neck region is being composed of a horizontal and vertical part. For the horizontal part of the map $\pi \circ u'_\delta : C(\delta) \to Y$, the necessary estimate is a consequence of the standard result on pseudoholomorphic cylinders of small energy, see for example Frauenfelder-Zemisch [**44**, Lemma 3.1]. Denote by $A(l)$ the annulus

$$A(l) = [-l/2, l/2| \times S^1.$$

Since there is no area loss in the limit, for any $\epsilon > 0$ there exists $\delta' > \delta_C$ such that the restriction of $\pi \circ u'_\delta$ to the annulus $A(|\ln(\delta')|/2)$ satisfies the energy estimate of [**44**, Lemma 3.1]. Thus

$$\pi u'_\delta(s,t) = \exp_{\pi u^{\mathrm{pre}}(s,t)} \xi^h(s,t), \quad \|\xi^h(s,t)\| \leq \epsilon(e^{s-|\ln(\delta')|/2} + e^{|\ln(\delta')|/2 - s}) \qquad s \in [-|\ln(\delta')|/2, |\ln(\delta')|/2]. \tag{8.25}$$

A similar estimate holds for the higher derivatives $D^k\xi^h(s,t)$ by elliptic regularity, for any $k \geq 0$.

For the vertical component we wish to compare the given family of maps with the trivial cylinder, c.f. B. Parker [**108**, Lemma 5.13]. For $l < |\ln(\delta')|$, but still very large, consider the $\mathbb{C}^\times$-bundle $P \to A(l)$ obtained from $\mathbb{R} \times Z \to Y$ by pull-back under $\pi \circ u'_\delta | A(l)$. The connection on Z induces the structure of a holomorphic $\mathbb{C}^\times$-bundle on P, which is necessarily holomorphically trivializable. That is, there exists a $\mathbb{R} \times S^1$-equivariant diffeomorphism

$$(\pi \times \varpi) : P \to A(l) \times \mathbb{R} \times S^1$$

mapping the complex structure $(u_\delta^{\mathrm{pre}})^* J_\Gamma$ on P to the standard complex structure $J_{(\mathbb{R}\times S^1)^2}$ on the right-hand-side. We claim that the holomorphic trivialization may be chosen to differ from the one given by parallel transport from the trivial cylinder by an estimate similar to (8.25). To see this, note that the bundle $Z \to Y$ is an S^1 bundle and the almost complex structure on $\mathbb{R} \times Z$ is induced from the almost complex structures on the base and fiber and a connection, given as a one-form in $\Omega^1(Z)^{S^1}$. Over any subset $U \subset Y$ we may trivialize $P|U \cong U \times S^1$ using geodesic exponentiation from the fiber. Since u takes values in $\pi^{-1}(U)$, the pull-back connection in the given trivialization is a one-form $\alpha \in \Omega^1(U)$. Any other trivialization of $u^*(\mathbb{R} \times Z)_U$ is then given by a $\mathbb{C}^\times$-valued gauge-transformation

$$\exp(\zeta) : U \to \mathbb{C}^\times, \zeta = (\zeta_s, \zeta_t).$$

The trivialization is holomorphic if the complex gauge transform of the connection is trivial. Thus we wish to solve an inhomogeneous Cauchy-Riemann equation of the form

$$\alpha = \alpha_s \mathrm{d}s + \alpha_t \mathrm{d}t = \exp(\zeta)^{-1} \mathrm{d} \exp(\zeta) = \mathrm{d}\zeta_s + *\mathrm{d}\zeta_t.$$

Write the connection and infinitesimal gauge transformation in terms of its Fourier expansion

$$\alpha(s,t) = \sum_{n\in\mathbb{Z}} \alpha_n(s)\exp(int), \quad \zeta(s,t) = \sum_{n\in\mathbb{Z}} \zeta_n(s)\exp(int)$$

where we identify $S^1 \cong [0, 2\pi]/(0 \sim 2\pi)$. The Fourier coefficients $\zeta_n, n \in \mathbb{Z}$ of ζ satisfy an equation

$$\left(\frac{d}{ds} - n\right)\zeta_n(s) = \alpha_n(s). \tag{8.26}$$

An explicit solution of (8.26) is given by integration

$$\zeta_n(s)\exp(-n(s - s_0(n))) = \int_{s_0(n)}^{s} \alpha_n(s')\mathrm{d}s'$$

so that the solution $\zeta_n(s)$ vanishes on $s_0(n)$. We make a careful choice of the Dirichlet condition $\zeta_n(s_0(n)) = 0$ for the n-th Fourier coefficient ζ_n so that the solution $\zeta(s,t)$ satisfies the same exponential decay condition (8.25) as the connection α. Define

$$s_0(n) = \begin{cases} l/2 & n > 0 \\ 0 & n = 0 \\ -l/2 & n < 0 \end{cases}.$$

Now the estimate on the neck region (8.25) implies by integration

$$\begin{aligned} \|\zeta_n(s)\| &= \left\| \exp(-n(s - s_0(n))) \int_{s_0(n)}^{s} \int_{t\in S^1} \alpha(s')\exp(-int)\mathrm{d}t\mathrm{d}s'/2\pi \right\| \\ &\le (1/2\pi) \begin{cases} \epsilon(l/2 + s)\exp(-(|\ln(\delta')|/2 + s)) & n < 0 \\ \epsilon\exp(-(|\ln(\delta')|/2 - |s|) & n = 0 \\ \epsilon(l/2 - s)\exp(-(|\ln(\delta')|/2 - s)) & n > 0 \end{cases}. \end{aligned}$$

For l sufficiently large absorb the prefactor $(l/2 - |s|)$ at the cost of weakening the exponential decay constant to some $\rho \in (\lambda, 1)$:

$$(|\ln(l(\delta))|/2 - |s|)\exp(-(|\ln(\delta')|/2 - |s|)) \le \exp\left(-\rho(|\ln(\delta')|/2 - |s|)\right).$$

Thus for $k = 0$ and any $\epsilon > 0$ we have for l sufficiently large the exponential decay holds:

$$\|\zeta_{A(l)}\|_{k,2} \le \epsilon\exp(-\rho(|\ln(\delta')|/2 - l))).$$

The same arguments applied to the uniform bound on the k-th derivative proves the same estimate for the Sobolev $k,2$-norm for any $k \ge 0$. By Sobolev embedding one obtains a C^{k-2}-estimate for $\zeta(s,t)$ of the form: For any $\epsilon > 0$ there exists $l = l(\delta)$ sufficiently large so for $(s,t) \in A(l-1)$,

$$\sup_{m\le k-2} |D^m\zeta(s,t)| \le C\epsilon\exp(-\rho(|\ln(\delta')|/2 - |s|))$$

where C is a uniform-in-δ Sobolev embedding constant. Thus the holomorphic trivialization of the $\mathbb{C}^\times$-bundle P is exponentially small over the middle of the cylinder as claimed.

Having constructed a holomorphic trivialization, we may now compare the given holomorphic cylinder with the trivial cylinder. Write

$$\varpi(p) = (\varpi_s(p), \varpi_t(p)) \in \mathbb{R} \times S^1, \quad \forall p \in P.$$

Since the complex structure is constant in the local trivialization the difference between the given map and the trivial cylinder

$$(s,t) \mapsto (\mu s, t^\mu)^{-1} \varpi(u'_\delta(s,t)) = (\varpi_s(u'(s,t)) - \mu s, t^{-\mu} \varpi_t(u'_\delta(s,t))) \tag{8.27}$$

is also holomorphic. By uniform convergence of u'_δ to u on compact sets, we have

$$(\pm|\ln(\delta)|/2 + \mu s, t^\mu)^{-1} u'_\delta(\pm|\ln(\delta_C)|/2 + s, t) \to (0,1)$$

as $s \to \mp\infty$ in cylindrical coordinates on $X^\circ_\pm$. Thus the difference

$$(\mu s, t^\mu)^{-1} \varpi(u'_\delta(s,t))$$

is holomorphic and converges uniformly in all derivatives to the constant map $\pm(|\ln(\delta)|/2 - \mu(|\ln(\delta_C)|))$ on the components of $A(l) - A(l-1) \cong [0,1] \times S^1$ as $\delta \to 0$ and $l \to \infty$. Writing

$$\partial(\mu s, t^\mu)^{-1} \varpi(u'_\delta(s,t)) = \xi''_\delta(s,t)$$

the map $\xi''_\delta(s,t)$ is also holomorphic in s,t and converges to zero uniformly on the ends of the cylinder. It follows from the annulus lemma [**44**, 3.1] that for any ϵ, there exists l sufficiently large so that

$$\|\xi''_\delta(s,t)\| \le \epsilon(e^{s-l/2} + e^{-l/2-s}). \tag{8.28}$$

In particular

$$u'_\delta(s,t) = (\mu(s-s_0), t^\mu t_0^{-1}) \xi'_\delta(s,t).$$

for some $(s_0(\delta), t_0(\delta)) \in \mathbb{R} \times S^1$ converging to $(0,1)$ as $\delta \to 0$. In particular the difference of lengths

$$\mu|\ln(\delta_C)| - |\ln(\delta)| \to 0$$

converges to zero. That is, the gluing parameters δ_C for the domains of u_δ satisfy $\delta\delta_C^{-\mu} \to 1$ as $\delta \to 0$.

We now complete the proof that the given family of solutions is close to the pre-glued solution. Choose $\epsilon > 0$. We write

$$C(\delta) = \exp_{C^{\delta_C}(m'_\delta)}, \quad u'_\delta(s,t) = \exp_{u^{\text{pre}}_\delta(s,t)} \xi'_\delta(s,t)$$

and claim that

$$\|(m'_\delta, \xi'_\delta)\|^p_{1,p,\lambda} < \epsilon$$

for δ sufficiently small. By assumption m'_δ converges to zero so for δ sufficiently small

$$\|m'_\delta\|^p < \epsilon/2. \tag{8.29}$$

Abusing notation we write $\|\xi'_\delta|_{A(l(\delta))}\|_{1,p,\lambda}$ for the expression obtained by replacing the integral over C^δ in (8.8) with $A(l(\delta))$ so that

$$\|\xi'_\delta\|^p_{1,p,\lambda} = \|\xi'_\delta|_{A(l(\delta))}\|^p_{1,p,\lambda} + \|\xi'_\delta|_{C^\delta - A(l(\delta))}\|^p_{1,p,\lambda}.$$

By uniform convergence of u'_δ on compact sets, there exists $l(\delta)$ with $|\ln(\delta)| - |\ln(l(\delta))| \to \infty$ such that

$$\|\xi'_\delta|_{C^\delta - A(l(\delta))}\|_{1,p,\lambda} < \epsilon/4. \tag{8.30}$$

Since each holomorphic trivialization ϖ_i differs from the trivialization of $P_i|U$ by an exponentially small factor on the middle of the neck, we have

$$\|\xi'_\delta(0,0)\| < \epsilon/8.$$

Write the trivial cylinder as a geodesic exponentiation from the preglued solution

$$(\underline{\mu}s, t\underline{\mu}) = \exp_{u^{\mathrm{pre}}_\delta}(\xi^{\mathrm{triv}}_\delta(s,t)).$$

The restriction of ξ'_δ to the neck region $A(l)$ has $1,p,\lambda$-norm given by integrating the product of (8.28) with the exponential weight function κ^δ_λ

$$\begin{aligned}
\|\xi'_\delta|_{A(l(\delta))}\|^p_{1,p,\lambda} &\leq 2^{p-1}\left(\|\xi'_\delta - \xi^{\mathrm{triv}}_\delta|_{A(l(\delta))}\|^p_{1,p,\lambda} + \|\xi^{\mathrm{triv}}_\delta|_{A(l(\delta))}\|^p_{1,p,\lambda}\right)\\
&\leq 2^p\|\xi''_\delta\|^p_{1,p,\lambda} + 2^{p-1}\|\xi^{\mathrm{triv}}_\delta|_{A(l(\delta))}\|^p_{1,p,\lambda}\\
&\leq 2^p\epsilon\left(e^{-p\rho(|\ln(\delta')|-l)+p\lambda(|\ln(\delta)|-l)}/(\rho-\lambda)\right.\\
&\qquad \left.+e^{p(\lambda-1)(|\ln(\delta')|-l)/2}/(1-\lambda)\right).
\end{aligned}$$

For $|\ln(\delta') - l|$ sufficiently large, the last expression is bounded by $\epsilon/4$, so

$$\|\xi'_\delta|_{A(l)}\|^p_{1,p,\lambda} \leq \epsilon/4. \tag{8.31}$$

Combining (8.29), (8.30) and (8.31) completes the proof for the case of two levels joined by a single node.

The case of multiple levels joined by multiple nodes is similar. We choose gluing parameters $\delta'_1,\ldots,\delta'_l$ assigned to the hypersurfaces separating levels and gluing in necks $Z\times[0,|\ln(\delta'_i)|]$ to an obtain an unbroken symplectic manifold X_δ with length $\sum_{i=1^l}|\ln(\delta'_i)|$. For each node pair of nodes $w_\pm$ mapping to the i-th copy of Y and multiplicity $\mu(w_\pm)$ we assign a gluing parameter $\delta(w_\pm) = (\delta'_i)^{1/\mu(w_\pm)}$. Then gluing in trivial cylinders into the neck regions creates an approximate solution as before. Similar arguments to those for a single node show that the approximate solution is nearby an actual family of solutions, and that any family of solutions arises in this way. □

COROLLARY 8.12. *Let $\underline{P}$ be an admissible perturbation system for broken disks. For each $E > 0$, there exists τ_0 such that if $\tau > \tau_0$ then $\mathcal{M}^{<E}(X,L,\underline{P}_\tau)_0$ is independent of τ and every element is regular.*

PROOF. By Theorem 8.10, any sequence $[u_\nu]$ in $\mathcal{M}(X_{\tau_\nu},L)$ with bounded energy and $\tau_\nu\to\infty$ has a subsequence that converges to an element of $\mathcal{M}(\mathbb{X},L)$. If ξ_ν is a sequence of elements in the cokernel of $\tilde{D}_{u_\nu}$ with norm one then after passing to a subsequence one obtains an element in the cokernel of the limiting linearized operator $\tilde{D}_u$. Since $\tilde{D}_u$ is surjective by assumption, $\tilde{D}_{u_\nu}$ is surjective for sufficiently large ν as well. □

REMARK 8.13. After a further small perturbation we may assume that the moduli spaces $\mathcal{M}^{<E}(X,L,\underline{P}^\tau)_{\leq 1}$ of expected dimension at most one are also regular. This perturbation may be chosen sufficiently small so that $\mathcal{M}^{<E}(X,L,\underline{P}^\tau)_0$ is unchanged, up to a C^0-small bijection, by the perturbation. As a results, the composition maps defined by counts of elements of $\mathcal{M}^{<E}(X,L,\underline{P}^\tau)_0$ satisfy the A_∞ axiom.

8.5. The infinite length limit

Using the gluing result of the previous chapter, we identify the broken theory with the infinite length limit of the unbroken theory.

PROPOSITION 8.14. *The composition maps $\mu^{n,\tau}$ of the A_∞ algebra $CF(L, \underline{P}_\tau)$ have a limit as $\tau \to \infty$:*

$$\mu^{n,\infty} := \lim_{\tau\to\infty} \mu^{n,\tau}. \tag{8.32}$$

The limit $CF(X, L, \underline{P}_\infty)$ is convergent-A_∞ -homotopy equivalent to $CF(L, \underline{P}_\tau)$ for any finite τ.

PROOF. First we show that the limit in (8.32) exists. For any energy bound E, the terms in $\mu^{n,\tau}$ of order at most q^E are independent of t for $\tau > \tau_0$ by Corollary 8.12. The A_∞ axiom mod q^E follows from the A_∞ axiom for $\mu^{n,\tau}$ modulo q^E; since this holds for any E, the A_∞ axiom holds on the nose.

Second we construct a strictly-unital, convergent-homotopy-equivalence from the limit to the Fukaya algebra for any finite neck length. Recall that a count of quilted disks defines a homotopy equivalences

$$\phi_\tau : CF(L, \underline{P}_\tau) \to CF(L, \underline{P}_{\tau+1}), \quad \psi_\tau : CF(L, \underline{P}_{\tau+1}) \to CF(L, \underline{P}_\tau).$$

We claim that for any energy bound E, the terms in ϕ_τ with coefficient $q^{E(u)}, E(u) < E$ vanish for sufficiently large τ except for constant disks. Indeed, otherwise there would exist a sequence of breaking quilted disks with arbitrarily large τ in a component of the moduli space with expected dimension zero and bounded energy. By Theorem 8.10, the limit would be a broken quilted disk in a component of the moduli space of expected dimension -1, a contradiction.

The claim implies that there exist limits of the successive compositions of the homotopy equivalences. Consider the composition

$$\phi_{(n)} := \phi_\tau \circ \phi_{\tau+1} \circ \ldots \circ \phi_{\tau+n} : CF(L, \underline{P}_\tau) \to CF(L, \underline{P}_{\tau+n+1}).$$

The bijection in Corollary 8.12 implies that the limit

$$\phi = \lim_{n\to\infty} \phi_{(n)} : CF(L, \underline{P}_\tau) \to \lim_{n\to\infty} CF(L, \underline{P}_{\tau+n})$$

exists. Similarly the limit

$$\psi = \lim_{n\to\infty} \psi_{(n)}, \quad \psi_{(n)} := \psi_\tau \circ \psi_{\tau+1} \circ \ldots \circ \psi_{\tau+n}$$

exists. Since the composition of strictly unital morphisms is strictly unital, the composition ψ is strictly unital mod terms divisible by q^E for any E, hence strictly unital.

The limiting morphisms are also homotopy equivalences. Let h_n, g_n denote the homotopies satisfying

$$\phi_{(n)} \circ \psi_{(n)} - \mathrm{Id} = \mu^1(h_{(n)}), \quad \psi_{(n)} \circ \phi_{(n)} - \mathrm{Id} = \mu^1(g_{(n)}),$$

obtained as in Definition 5.5 from the homotopies relating $\phi_\tau \circ \psi_\tau$ and $\psi_\tau \circ \phi_\tau$. In particular, $h_{(n+1)}, g_{(n+1)}$ differ from $h_{(n)}, g_{(n)}$ by expressions involving counting *twice-quilted* breaking disks. For any $E > 0$, for τ sufficiently large all terms in $h_{(n+1)} - h_{(n)}$ are divisible by q^E. It follows that the infinite composition

$$h = \lim_{n\to\infty} h_{(n)}, \quad g = \lim_{n\to\infty} g_{(n)}$$

exists and gives a homotopy equivalence $\phi \circ \psi$ resp. $\psi \circ \phi$ and the identities. □

THEOREM 8.15. *Suppose that* $\underline{P}_\tau$ *converges to an admissible perturbation system* $\underline{P}$ *for broken treed disks as above. Then the limit* $\lim_{\tau\to\infty} CF(X, L, \underline{P}_\tau)$ *is equal to the broken Fukaya algebra* $CF(\mathbb{X}, L)$*, and in particular* $CF(X, L, \underline{P}_\tau)$ *is homotopy equivalent to* $CF(\mathbb{X}, L)$.

PROOF. By Theorem 8.11, for any given energy bound $E > 0$ there exists a bijection between the moduli spaces $\mathcal{M}^{<E}(X, L, \underline{P}_\tau)_0$ and $\mathcal{M}^{<E}(\mathbb{X}, L, \underline{P}_\infty)_0$. These moduli spaces define the structure coefficients of the Fukaya algebras $CF(L, \underline{P}_\tau)$ for τ sufficiently large resp. the structure coefficients of $CF(\mathbb{X}, L)$. The bijection preserves the area $A(u)$ of each map $u : C \to X$ as well as the homology class $[\partial u] \in H_1(L)$ of the restriction ∂u of the map u to the boundary of S. Hence the bijection preserves the holonomies of the local system $y(u)$. It follows from the construction of coherence property (4.30) on the orientation of the stable and unstable manifolds $W_x^\pm$ of the gradient flow $-\operatorname{grad}(F)$ on the separating hypersurface Y that the bijection $\mathcal{M}(X, L, \underline{P}_\tau)_0 \to \mathcal{M}(\mathbb{X}, L, \underline{P}_\infty)_0$ is orientation preserving. The last statement follows from Corollary 8.14. □

PROOF OF THEOREM 1.1. We combine the computation in the sft limit with the homotopy equivalence above. Let X be obtained by a small reverse flip or blowup, $L \subset X$ a regular Lagrangian near the exceptional locus. Let $\underline{P}_\tau$ be as in Theorem 8.15. The limit $CF(\mathbb{X}, L) = \lim_{\tau\to\infty} CF(X, L, \underline{P}_\tau)$ has non-empty Maurer-Cartan space $MC(\mathbb{X}, L)$ and Floer cohomology $HF(\mathbb{X}, L, y_i, b(y_i))$ non-trivial for $n_+ - n_-$ local systems $y_i, i = 1, \ldots, n_+ - n_-$ and weakly bounding cochains $b(y_i)$, by Theorems 7.22 and 8.15. The same holds for any finite τ, by Theorem 5.2. □

8.6. Examples

In this chapter we describe Lagrangians associated to mmp transitions in each of the examples of Chapter 3. We remark that the results of Palmer-Woodward [**107**] show that the non-triviality of the Floer cohomology in Propositions 8.17, 8.20, 8.21 below hold for times far away from the critical time, but before the next transition in the mmp.

8.6.1. A standard Lagrangian in a Darboux chart. Standard Lagrangians in Darboux charts in compact symplectic manifolds are unobstructed, although their Floer cohomology is often trivial. Let $x \in X$ and let $q_1, p_1, \ldots, q_n, p_n \in C^\infty(U)$ be Darboux coordinates on an open neighborhood U of $x \in X$ centered at 0.

PROPOSITION 8.16. *The standard Lagrangian tori* $L = \{(p_j^2 + q_j^2) = c_j,\ j = 1, \ldots, n\}$ *for some small constants* $c_1, \ldots, c_n > 0$ *have* $MC(L)$ *non-empty and trivial Floer cohomology* $HF(L, b) \cong \{0\}$ *for any* $b \in MC(L)$.

PROOF. Choose a sphere around x consider the degeneration of X by neck-stretching along the sphere to a broken symplectic manifold $(X_\subset, X_\supset)$. The Lagrangian L becomes a toric moment fiber in $X_\subset \cong \mathbb{P}^{n-1}$ and so has non-empty Maurer-Cartan space $MC(L)$. On the other hand, L is displaceable for c_j sufficiently small and so has trivial Floer cohomology $HF(L, b)$ for $b \in MC(L)$. □

8.6.2. Toric symplectic manifolds. Continuing Chapter 2.2 the following describe Floer-non-trivial torus orbits in toric varieties, which give special cases of the results in [**47**].

PROPOSITION 8.17. *Suppose that X is a compact symplectic toric manifold, X_t an mmp running with t_0 a singular time, $x \in X_{t_0}$ a singular point, and $\lambda_0 = \Phi(t_0)$ the moment image of the singular point. Then for $t = t_0 - \epsilon$ for ϵ small, the Lagrangian $L = \Phi^{-1}_{t_0-\epsilon}(\lambda)$ is regular and so has non-vanishing Floer cohomology for some local system $y \in \mathcal{R}(L)$ and weakly bounding cochain $b(y)$.*

EXAMPLE 8.18. (Blow-up of a product of projective lines) The disks in the case of blow-up of $\mathbb{P}^1 \times \mathbb{P}^1$ are shown in Figure 3.1. The image of each disk is one-dimensional, since the angular direction in each disk is tangent to the level sets of the moment map. On the other hand, the image of each disk (shown roughly as dotted lines in the figure) is non-linear since the map from V to $X = V /\!/ G$ is non-linear. In Figure 3.1, the areas of the three Maslov-index-two disks of smallest area are all equal.

EXAMPLE 8.19. Continuing Example 2.4 of del Pezzo surfaces, the points giving Floer non-trivial tori in Theorem 1.1 are darkly shaded in Figure 2.1. The fiber over the medium-shaded point is also Floer-non-trivial.

8.6.3. Polygon spaces. Continuing Chapter 2.3, regular labellings of triangulated polygons give rise to Floer-non-trivial torus orbits in polygon space; see Nishinou-Nohara-Ueda [**101**] and Nohara-Ueda [**102**] for another approach to Floer-non-trivial Lagrangian tori in these spaces.

PROPOSITION 8.20. *Let $\mu \in \mathbb{R}^{n-3}_{\geq 0}$ be a regular labelling, and $\mu(t)$ the family of labellings obtained by replacing each μ_i with $\mu_i - t$ which becomes singular at first time $t = t_i$. Then for $\epsilon > 0$ sufficiently small and $t \in (t_i - \epsilon, t_i)$, any labelling $\mu(t)$ has the property that $\Psi^{-1}(\mu)$ has non-trivial Floer cohomology $HF(\Psi^{-1}(\mu), y, b) \neq 0$ for some local system y and weakly bounding cochain b.*

8.6.4. Moduli spaces of flat bundles on punctured spheres. By similar arguments, Theorem 1.1 implies the existence of Floer non-trivial tori in representation varieties:

PROPOSITION 8.21. *Suppose that $\mu_1, \ldots, \mu_n$ are generic labels and $\mathcal{R}(\mu_1, \ldots, \mu_n)$ is the corresponding moduli space of flat $SU(2)$-bundles on the n-holed two-sphere Σ. Let $\mathcal{P} = \{P\}$ be a pants decomposition of Σ with ordered boundary circles $C_1, \ldots, C_{n-3} \subset \Sigma$ and suppose that $\lambda = (\lambda_1, \ldots, \lambda_{n-3})$ is a regular labelling with looseness t_1 given by the first transition time t_1 in the mmp. Then for t_1 sufficiently small the Goldman Lagrangian $\Psi^{-1}(\lambda)$ has non-trivial Floer cohomology, $HF(\Psi^{-1}(\lambda)) \cong H(\Psi^{-1}(\lambda)) \neq 0$.*

An example of a labelling giving a Floer non-trivial torus is shown in Figure 8.1.

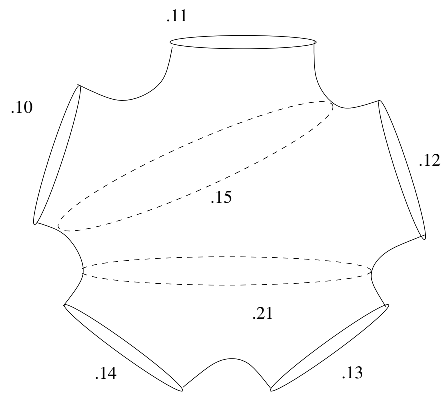

FIGURE 8.1. A labelling giving a Floer-non-trivial torus

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