DUISTERMAAT-HECKMAN DISTRIBUTIONS FOR GROUP VALUED MOMENT MAPS

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Abstract. We introduce equivariant Liouville forms and Duistermaat-Heckman distributions for Hamiltonian group actions with group valued moment maps. The theory is illustrated by applications to moduli spaces of flat connections on 2-manifolds.

1. Introduction

One of the fundamental invariants of a Hamiltonian G-space M in symplectic geometry is the Duistermaat-Heckman measure $\mathfrak{m}$ on the dual of the Lie algebra $\mathfrak{g}^*$, defined as the push-forward of the canonical volume form under the moment map. The measure $\mathfrak{m}$ encodes volumes of reduced spaces, and by the Duistermaat-Heckman theorem its derivatives describe mixed Chern numbers for the corresponding level set.

More generally there are "twisted" DH-distributions which contain information on more complicated intersection pairings on reduced spaces. DH-distributions have a number of interesting properties [13] and they can be computed using localization techniques [13, 8, 6].

In [1] a definition was given of a Hamiltonian G-space (M, ω, Φ) for which the moment map Φ takes values not in $\mathfrak{g}^*$ but in the group G itself. Basic examples are G-conjugacy classes, with moment map the inclusion into G, and moduli spaces of flat connections on surfaces with boundary, with moment map the holonomies around the boundary components. Many concepts from the theory of $\mathfrak{g}^*$-valued moment maps carry over to this setting. In particular, there is a notion of reduction and the reduced spaces are symplectic.

One of the differences to Hamiltonian spaces in the usual sense is that non-degeneracy of the 2-form ω is replaced by a more complicated condition involving the moment map. Hence the top exterior power of ω does not in general define a volume form on M.

We will show in this paper that there exist, nevertheless, canonical volume forms on group valued Hamiltonian G-spaces whose push-forward $\mathfrak{m}$ under Φ plays the role of DH-measures. Our construction uses an exotic version of equivariant de Rham theory, the "group valued" equivariant de Rham theory developed in [2]. Its definition is similar to that of the usual equivariant de Rham cohomology, however the defining complex of equivariant differential forms is non-commutative.

Just as in the $\mathfrak{g}^*$-valued theory, the DH-distributions $\mathfrak{m}$ for group valued Hamiltonian G-spaces describe volumes of reduced spaces, and by an extension of the DH theorem its derivatives give formulas for mixed characteristic numbers of the level sets. Again, there are more general "twisted" DH-distributions which encode more complicated intersection pairings on reduced spaces. In [3] we prove a localization theorem for group valued equivariant cohomology, which computes the DH-distributions in terms of fixed point contributions.

The Duistermaat-Heckman measure and the volume form appear in the following way in the theory of moduli spaces of flat connections on 2-manifolds: Suppose G is a compact, connected and simply connected Lie group, together with an invariant inner product on its Lie algebra. For all $h \geq 0$ the space G^{2h} carries the structure of a group valued Hamiltonian G-space, where the action is by conjugation on each factor and the moment map $\Phi:\, G^{2h} \to G$ reads

$$\Phi(a_1, b_1, \ldots, a_h, b_h) = \prod_{i=1}^{h} [a_i, b_i].$$

The reduced space at the group unit, $M(\Sigma) := \Phi^{-1}(e)/G$, is the moduli space of all flat G-connections on a compact, oriented 2-manifold Σ of genus h, and it is shown in [1] that the symplectic structure obtained by reduction is equal to that coming from the gauge theory construction in Atiyah-Bott [4, 5]. The equivariant Liouville form on G^{2h} can be explicitly computed in this case, and the associated volume form is found to coincide with Haar measure $\mathrm{d}\,\mathrm{Vol}_{G^{2h}}$. Hence, the push-forward $\mathfrak{m} = \Phi_*(\mathrm{d}\,\mathrm{Vol}_{G^{2h}})$ plays the role of the Duistermaat-Heckman measure, and its value at the group unit is (up to a normalizing constant) the symplectic volume of the moduli space $M(\Sigma)$, as computed by Witten [24, 25]. The argument extends to surfaces with boundary, with prescribed holonomies around the boundary circles.

The plan of this article is as follows. In Section 2 we provide background material on group valued Hamiltonian G-spaces. In particular, we recall how to obtain the symplectic structure on moduli spaces of flat connection from this point of view. Section 3 is a review of the construction of equivariant Liouville forms for Hamiltonian G-spaces. In Section 4 we describe the "group valued" equivariant cohomology which we use in Section 5 to construct Liouville forms, volume forms and DH-distributions for group valued moment maps. Section 6 describes their behavior under direct products and under "exponentiating". In Section 7 we show that the DH-distributions encode intersection pairings on reduced spaces. In Section 8 we extend our constructions (most of which use an assumption that G be a product of a simply connected group and a torus) to arbitrary compact Lie groups. In Section 9 we apply our theory to moduli spaces of flat connections.

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2. Group valued Hamiltonian G-spaces

In this section we give a brief introduction to the theory of Hamiltonian G-spaces with group valued moment map. Throughout this paper G denotes a compact, connected Lie group with Lie algebra $\mathfrak{g}$. For any G-manifold M and all $\xi \in \mathfrak{g}$, we denote by ξ_M the corresponding fundamental vector field: $\xi_M = \frac{d}{dt}|_{t=0} \exp(-t\xi)^*$. Given a basis e_a of $\mathfrak{g}$ we will denote by ι_a the contractions and by L_a the Lie derivatives with respect to $(e_a)_M$. The structure constants f^c_{ab} of $\mathfrak{g}$ in such a basis are defined by $[e_a, e_b] = f^c_{ab} e_c$ (using summation convention).

2.1. **$\mathfrak{g}^*$-valued Hamiltonian G-spaces.** A Hamiltonian G-space in the usual sense (that is, with $\mathfrak{g}^*$-valued moment map) is a triple (M, ω_0, Φ_0) consisting of a G-space M, a 2-form ω_0, and an equivariant map $\Phi_0 \in C^\infty(M, \mathfrak{g}^*)^G$ satisfying

$$\begin{aligned} \mathrm{d}\omega_0 &= 0 && \text{(Cocycle condition)} \\ \iota(\xi_M)\omega_0 &= \mathrm{d}\langle \Phi_0, \xi \rangle && \text{(Moment map condition)} \\ \ker((\omega_0)_m) &= \{0\} && \text{(Non-degeneracy).} \end{aligned}$$

The first two conditions imply that ω_0 is G-invariant. Sometimes we drop the last condition, in which case we call (M, ω_0, Φ_0) a *degenerate* Hamiltonian G-space.

Fundamental examples for $\mathfrak{g}^*$-valued Hamiltonian G-spaces are coadjoint orbits $\mathcal{O} \subset \mathfrak{g}^*$, with moment map the embedding $\Phi_0 : \mathcal{O} \hookrightarrow \mathfrak{g}^*$. The 2-form is uniquely determined by the moment map condition, and is explicitly given by the Kirillov-Kostant-Souriau (KKS) formula

$$\omega_0\big((\xi_1)_{\mathcal{O}}(\mu), (\xi_2)_{\mathcal{O}}(\mu)\big) = \langle \mu, [\xi_1, \xi_2] \rangle \tag{1}$$

for all $\mu \in \mathcal{O}, \quad \xi_1, \xi_2 \in \mathfrak{g}$.

Given any Hamiltonian G-space (M, ω_0, Φ_0), with 0 a regular value of Φ_0, the reduced space (symplectic quotient) $M_{red} := \Phi_0^{-1}(0)/G$ naturally carries the structure of a symplectic orbifold. More generally, if $\mu \in \mathfrak{g}^*$ is a regular value one defines $M_\mu = \Phi_0^{-1}(\mu)/G_\mu$ where G_μ is the stabilizer of μ. Letting $\mathcal{O} = G \cdot \mu$ be the coadjoint orbit through μ, and $\mathcal{O}^- = G \cdot (-\mu)$, there is a natural isomorphism ("shifting-trick")

$$M_\mu \cong M_{\mathcal{O}} := (M \times \mathcal{O}^-)_{red}.$$

2.2. **Group valued Hamiltonian G-spaces.** In the definition of a group valued Hamiltonian G-space the target of the moment map is the group G itself, with G acting by conjugation. Let the Lie algebra $\mathfrak{g}$ be equipped with an invariant inner product $\cdot$, used to identify $\mathfrak{g}^* \cong \mathfrak{g}$. The group G carries a bi-invariant closed 3-form

$$\eta = \frac{1}{12}\theta \cdot [\theta, \theta] = \frac{1}{12}\overline{\theta} \cdot [\overline{\theta}, \overline{\theta}]$$

where $\theta, \overline{\theta} \in \Omega^1(G, \mathfrak{g})$ are the left/right invariant Maurer-Cartan forms, respectively.

Definition 2.1. A G-valued Hamiltonian G-space is a triple (M, ω, Φ) consisting of a G-manifold M, a 2-form ω, and an equivariant map $\Phi \in C^\infty(M, G)^G$ satisfying the following three conditions:

$$\begin{array}{rcll} \mathrm{d}\omega & = & \Phi^*\eta & (\Phi\text{-relative cocycle condition}) \\ \iota(\xi_M)\omega & = & \frac{1}{2}\Phi^*(\theta + \overline{\theta}) \cdot \xi & (\text{Moment map condition}) \\ \ker \omega_m & = & \{\xi_M(m) |\ \mathrm{Ad}_{\Phi(m)}\,\xi = -\xi\} & (\text{Minimal degeneracy condition}). \end{array}$$

We will sometimes omit the minimal degeneracy condition, in which case we call (M, ω, Φ) a *degenerate* G-valued Hamiltonian G-space. The definition can be motivated as follows. The second condition is the natural G-analogue of the moment map condition for $\mathfrak{g}^*$-valued Hamiltonian G-spaces. Since

$$\iota(\xi_G)\eta = -\frac{1}{2}\mathrm{d}(\theta + \overline{\theta}) \cdot \xi,$$

the first two conditions imply that $\mathcal{L}_{\xi_M}\omega = 0$ so that ω is G-invariant. Since $\Phi^*\overline{\theta} = \mathrm{Ad}_\Phi\, \Phi^*\theta$, the moment map condition forces

$$\ker \omega_m \supseteq \{\xi_M(m) |\ \mathrm{Ad}_{\Phi(m)}\,\xi = -\xi\}.$$

so that the third condition is a minimal degeneracy condition for ω. See [1] for more details, and further motivation in terms of the Cartan model of equivariant cohomology. One of the results of this paper we will be a much more attractive formulation of the minimal degeneracy condition.

Notice that if the group G is abelian, the form η vanishes and the first and third condition say that the 2-form ω is symplectic. In this case, our definition reduces to the usual definition of a torus-valued moment map (cf. [19, 14]).

2.3. **Examples of G-valued Hamiltonian spaces.**

2.3.1. *Conjugacy classes.* The most fundamental examples for group valued Hamiltonian G-spaces are conjugacy classes $\mathcal{C} \subset G$, with moment map the embedding $\Phi : \mathcal{C} \hookrightarrow G$. Again the 2-form is uniquely determined by the moment map condition and is given by

$$\omega((\xi_1)_{\mathcal{C}}(g), (\xi_2)_{\mathcal{C}}(g)) = \frac{1}{2}\,(\mathrm{Ad}_g - \mathrm{Ad}_{g^{-1}})\xi_1 \cdot \xi_2 \quad (\xi_1, \xi_2 \in \mathfrak{g}). \tag{2}$$

for all $g \in \mathcal{C}$. Equation (2) is the group analogue to the KKS formula (1), with the skew-adjoint operator ad_μ on $\mathfrak{g}$ replaced by the skew-adjoint operator $\frac{1}{2}(\mathrm{Ad}_g - \mathrm{Ad}_{g^{-1}})$.

Notice that the 2-form ω is degenerate if and only if $\mathrm{Ad}_g : \mathfrak{g} \to \mathfrak{g}$ $(g \in \mathcal{C})$ has an eigenvalue equal to -1. In fact it can happen that ω vanishes identically; this is the case if and only if g squares to an element in the center $Z(G)$. Conjugacy classes also illustrate the following fact:

> *G-valued Hamiltonian G-spaces for non-simply connected Lie groups G need not be orientable.*

A counter-example is the conjugacy class of the rotation group $G = \mathrm{SO}(3)$ corresponding to rotations by an angle 180^o, which is isomorphic to $\mathbb{R}P(2)$.

2.3.2. *The double.* The double $D(G)$ is the analogue for group valued Hamiltonian spaces to the cotangent bundle T^*G. Let $D(G) = G \times G$, with the group G^2 acting by

$$(g_1, g_2) \cdot (a, b) = (g_1\, a g_2^{-1}, g_2\, b g_1^{-1}).$$

Define a 2-form

$$\omega = \frac{1}{2}(\theta^1 \cdot \overline{\theta}^2 + \overline{\theta}^1 \cdot \theta^2)$$

where the superscripts denote the projections to the respective factor in $D(G) = G \times G$, and a moment map

$$\Phi:\ D(G) \to G^2,\ (a, b) \mapsto (ab, a^{-1}b^{-1}).$$

Then $(D(G), \omega, \Phi)$ is a group valued Hamiltonian G^2-space.

2.3.3. *The spinning 4-sphere.* Let S^4 be the unit sphere in $\mathbb{R}^5$, equipped with the rotation action of SU(2) induced from the defining action on $\mathbb{C}^2$ and the identification $\mathbb{R}^5 = \mathbb{C}^2 \oplus \mathbb{R}$. We will show that S^4 with this action admits the structure of a group valued Hamiltonian SU(2) space.

Choose the basis $e_a = \sigma_a$ of $\mathfrak{su}(2)$ given by Pauli matrices

$$\sigma_1 = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix},\quad \sigma_2 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix},\quad \sigma_3 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix},$$

and define an invariant metric on $\mathfrak{su}(2)$ by declaring this basis to be orthonormal. A matrix $A = tI + n_a\sigma_a$ is contained in SU(2) if and only if $t^2 + n_a n_a = 1$. Using the metric to pull down indices, the structure constants are $f_{abc} = -2\epsilon_{abc}$. Let $u : \mathbb{C}^2\backslash\{0\} \to \mathfrak{su}(2)$ be the equivariant function

$$u = u_a\sigma_a = \frac{1}{||z||^2}\big((|z_1|^2 - |z_2|^2)\sigma_1 + 2\,\mathrm{Im}(z_1\overline{z_2})\sigma_2 + 2\,\mathrm{Re}(z_1\overline{z_2})\sigma_3\big)$$

taking values in the unit sphere. Since the restriction of the function $(\mathbb{C}^2\backslash\{0\}) \times \mathbb{R} \to \mathbb{R}$, $(z_1, z_2, s) \mapsto \cos(\frac{\pi s}{2})/||z||^2$ to S^4 extends smoothly over the north pole $\mathcal{N} = (0, 0, 1)$ and the south pole $\mathcal{S} = (0, 0, -1)$, it follows that the map

$$\Phi(z_1, z_2, s) = \sin(\frac{\pi s}{2}) + \cos(\frac{\pi s}{2}) u_a\sigma_a$$

restricts to a smooth, equivariant map $\Phi : S^4 \to \mathrm{SU}(2)$. Now let $\Theta \in \Omega^1(\mathbb{C}^2\backslash\{0\}, \mathbb{R}^3)$ be the unique connection 1-form such that Θ vanishes on radial vectors, and let $\omega \in \Omega^2(S^4\backslash\{\mathcal{S}, \mathcal{N}\})$ be the 2-form

$$\omega = -\frac{\pi}{2} u_a\Theta_a\, \mathrm{d}s - \frac{1}{8}\sin(\pi s)\epsilon_{abc}u_a \mathrm{d}u_b \mathrm{d}u_c.$$

We claim that ω pulls back to a smooth 2-form on all of S^4, and that (S^4, ω, Φ) is a group valued Hamiltonian SU(2)-space.

a. *ω extends to a smooth 2-form on S^4.* Near the north pole $\mathcal{N}$, we may take (z_1, z_2) as coordinates on S^4. To show that ω is smooth near $\mathcal{N}$ we compare to the 2-form on $\mathbb{C}^2$,
$$\omega_0 = \frac{i\pi}{2} \mathrm{d}z_a \, \mathrm{d}\overline{z}_a.$$
The moment map for the SU(2)-action on $\mathbb{C}^2$ is given by $\Phi_0(z_1, z_2) = \frac{\pi}{2}||z||^2 \, u$. Away from the origin, ω_0 can be re-written in terms of Θ:
$$\omega_0 = \frac{\pi}{2} u_a \Theta_a \mathrm{d}||z||^2 - \frac{\pi}{8} \, ||z||^2 \, \epsilon_{abc} u_a \mathrm{d}u_b \mathrm{d}u_c.$$
(Indeed, since ω_0 is completely determined by all 1-forms $\iota_a \omega_0$ it suffices to verify the moment map condition; this is easily done using $\iota_a \mathrm{d}u_s = L_a u_s = 2\epsilon_{ast} u_t$.) The difference between ω and ω_0 is
$$\omega - \omega_0 = \frac{1}{8}(\pi \, ||z||^2 - \sin(\pi ||z||^2))\epsilon_{abc} u_a \mathrm{d}u_b \mathrm{d}u_c.$$
which is smooth at the origin. A similar argument applies to the south pole $\mathcal{S} \in S^4$.

b. ω satisfies the moment map condition. Using $\iota_a \mathrm{d}u_s = L_a u_s = 2\epsilon_{ast} u_t$ we have
$$\iota_a \omega = -\frac{\pi}{2} u_a \mathrm{d}s + \frac{1}{2} \sin(\pi s) \mathrm{d}u_a.$$
On the other hand, for the Maurer-Cartan form one finds
$$\begin{aligned} \Phi^* \overline{\theta}_a &= -\frac{\pi}{2} u_a \mathrm{d}s + \frac{1}{2} \sin(\pi s) \mathrm{d}u_a - \cos^2(\frac{\pi s}{2}) \epsilon_{abc} u_b \mathrm{d}u_c \\ \Phi^* \theta_a &= -\frac{\pi}{2} u_a \mathrm{d}s + \frac{1}{2} \sin(\pi s) \mathrm{d}u_a + \cos^2(\frac{\pi s}{2}) \epsilon_{abc} u_b \mathrm{d}u_c \end{aligned}$$
This shows $\iota_a \omega = \frac{1}{2} \Phi^*(\theta_a + \overline{\theta}_a)$.

c. *The condition $d\omega = \Phi^* \eta$.* Since almost all orbits have codimension one, it suffices to verify $\iota_a(\mathrm{d}\omega - \Phi^* \eta) = 0$. Using $\iota_a \eta = -\frac{1}{2} \mathrm{d}(\overline{\theta}_a + \theta_a)$ this follows from the moment map condition together with invariance of ω.

2.4. Constructions with group valued Hamiltonian G-spaces.

2.4.1. *Fusion.* Given a Hamiltonian $G \times G$-space (in the usual sense), the same space with diagonal G-action and sum of the two moment map components is a Hamiltonian G-space. Similarly, for any group valued Hamiltonian $G \times G$-space the same space with diagonal action and *product* of the moment map components is a group valued Hamiltonian G-space. However, it is necessary to modify the 2-form as well:

Theorem 2.2. [1] *Let H, G be compact connected Lie groups and $(M, \omega, (\Phi_1, \Phi_2, \Psi))$ a group valued Hamiltonian $G \times G \times H$-space. Let $\tilde{M} = M$ with diagonal $G \times H$-action, and 2-form*
$$\tilde{\omega} = \omega + \frac{1}{2} \Phi_1^* \theta \cdot \Phi_2^* \overline{\theta}.$$

Then $(\tilde{M}, \tilde{\omega}, (\Phi_1\Phi_2, \Psi))$ *is a group valued Hamiltonian* $G \times H$*-space called the (internal) fusion of* M*.*

In the particular case $M = M_1 \times M_2$ where M_j are group valued Hamiltonian $G \times H_j$-spaces, we write $\tilde{M} =: M_1 \circledast M_2$ and call this the fusion product of M_1 and M_2. It is shown in [1] that the fusion product is associative and commutative: That is, the group valued Hamiltonian spaces obtained by fusions with different orderings of the G-factors are isomorphic.

2.4.2. *Reduction.* Just as for Hamiltonian spaces one has the notion of reduction: Let $(M, \omega, (\Phi, \Psi))$ be a group valued Hamiltonian $G \times H$-space and suppose that the identity element $e \in G$ is a regular value of Φ. As shown in [1], this implies that the G-action on $\Phi^{-1}(e)$ is locally free. Moreover, the 2-form ω and the map Ψ descend to the reduced space

$$M_{red} := \Phi^{-1}(e)/G.$$

giving M_{red} the structure of a group valued Hamiltonian H-space. (Strictly speaking M_{red} may have orbifold singularities unless the G-action is free; however the definition of a group valued Hamiltonian space extends directly to orbifolds). In particular, if $H = \{e\}$ this quotient is symplectic. If one drops the regularity assumption, the reduced space M_{red} is a stratified symplectic space in the sense of Sjamaar-Lerman [22]. That is, it is a singular space stratified by smooth symplectic manifolds, the singularities are conical and are given by certain normal forms described in [22].

More generally, if $g \in G$ is a regular value of Φ one can define the reduction $M_g = \Phi^{-1}(g)/G_g$ which is a group valued Hamiltonian H-space. Letting $\mathcal{C} = G \cdot g$ be the corresponding conjugacy class, and $\mathcal{C}^- = G \cdot (g^{-1})$, there is a natural isomorphism ("shifting trick")

$$M_g \cong M_{\mathcal{C}} := (M \circledast \mathcal{C}^-)_{red}.$$

Example 2.3. For any group valued Hamiltonian $G \times H$-space M, there is a canonical isomorphism $(D(G) \circledast M)_{red} = M$ (where we are taking the fusion product with respect to the $\{e\} \times G \subset G^2$-action on $D(G)$). In particular, $D(G)_{\mathcal{C}} = (D(G) \circledast \mathcal{C}^-)_{red} = \mathcal{C}^-$.

Example 2.4. The moment map for the spinning 4-sphere is a surjection onto SU(2), and defines a circle fibration except over the two elements of the center (where the fiber is a point). It follows that S^4 is a multiplicity-free space, that is, all reduced spaces $(S^4)_g$ are points.

2.4.3. *Exponentials.* There is a way of "exponentiating" Hamiltonian G-spaces in the usual sense to group valued Hamiltonian spaces. We use the inner product on $\mathfrak{g}$ to identify $\mathfrak{g}^* \cong \mathfrak{g}$. Let $J : \mathfrak{g} \to \mathbb{R}$ be the determinant of the Jacobian of the exponential map $\exp : \mathfrak{g} \to G$. Let $\varpi \in \Omega^2(\mathfrak{g})$ be the image of the closed form $\exp^* \eta \in \Omega^3(\mathfrak{g})$ under the de Rham homotopy operator $\Omega^\star(\mathfrak{g}) \to \Omega^{\star-1}(\mathfrak{g})$.

Theorem 2.5. [1] *Let (M, ω_0, Φ_0) is a Hamiltonian space in the usual sense (possibly degenerate). Then (M, ω, Φ) with moment map $\Phi = \exp(\Phi_0)$ and 2-form $\omega = \omega_0 + \Phi_0^* \varpi$ is a (degenerate) group valued Hamiltonian G-space. The form ω satisfies the minimal degeneracy condition at $m \in M$ if and only if $(\omega_0)_m$ is non-degenerate and $J(\Phi_0(m)) \neq 0$.*

Conversely, if (M, ω, Φ) is a group valued Hamiltonian G-space such that the exponential map has a well-defined inverse $\log : U \to \mathfrak{g}$ over an open subset $U \subset G$ containing $\Phi(M)$, then $(M, \omega - \log(\Phi)^* \varpi, \log(\Phi))$ is a Hamiltonian G-space in the usual sense.

2.5. **Example: Moduli spaces of flat connections.** We now explain the construction of the symplectic structure on moduli spaces of flat connections using group valued moments maps (cf [1]). For simplicity, we make the assumption that the Lie group G is connected and simply connected. Let Σ_h^r be the compact, connected, oriented 2-manifold with of genus h with r boundary components, $B_1, \ldots, B_r$. Since G is simply connected, every principal G-bundle over Σ_h^r is trivial.

Identify the space of Lie-algebra valued 1-forms $\Omega^1(\Sigma_h^r, \mathfrak{g})$ with the space of connections on the trivial bundle $\Sigma_h^r \times G$. For any $A \in \Omega^1(\Sigma_h^r, \mathfrak{g})$ let F_A be its curvature. For every boundary circle B_j we choose a base point $p_j \in B_j$. Let $\mathrm{Hol}_{B_j} : \Omega^1(\Sigma_h^r, \mathfrak{g}) \to G$ be the map which takes a connection to the holonomy around the loop based at p_j and winding once around B_j in positive direction. Let $C^\infty(\Sigma_h^r, G)$ be the gauge group, with gauge action $g \cdot A = \mathrm{Ad}_g(A) - g^* \overline{\theta}$. Given a collection of conjugacy classes $\mathcal{C}_1, \ldots, \mathcal{C}_r$ let

$$M(\Sigma_h^r, \underline{\mathcal{C}}) = \frac{\{A \in \Omega^1(\Sigma_h^r, \mathfrak{g}) |\ F_A = 0,\ \mathrm{Hol}_{B_j}(A) \in \mathcal{C}_j \text{ for all } j\}}{C^\infty(\Sigma_h^r, G)}$$

be the moduli space of flat connections with specified holonomies. For $r \geq 1$ and generic conjugacy classes $\mathcal{C}_j$ it is a finite dimensional compact orbifold, and according to Atiyah-Bott [5, 4] it carries a natural symplectic structure. (For non-generic conjugacy classes or in the case without boundary it is a stratified symplectic space in the sense of Sjamaar-Lerman [22].)

In [1] it was shown that the symplectic form can be obtained by reduction from a space with group valued moment map: First, let $M(\Sigma_1^1) = \widetilde{D}(G) = G \times G$ be the fusion of the double. The G-action is the action by conjugation on each factor, and the moment map is a Lie group commutator $(a, b) \mapsto [a, b] = aba^{-1}b^{-1}$. The space $M(\Sigma_1^1)$ can be interpreted as a moduli space of all flat connections on a 1-punctured torus Σ_1^1, up to gauge transformations that are trivial at the base point; the moment map is the holonomy around the boundary loop. Similarly the fusion product

$$M(\Sigma_1^1) \circledast \ldots \circledast M(\Sigma_1^1) \circledast \mathcal{C}_1 \circledast \ldots \circledast \mathcal{C}_r$$

can be interpreted as a moduli space of flat connections on Σ_h^{r+1}: More precisely it is the space of flat connections such that the holonomies around the first r boundary components are contained in the specified conjugacy classes, divided by gauge transformations $g \in C^\infty(\Sigma_h^{r+1})$ for which $g(p_{r+1}) = e$. The moment map is interpreted as the holonomy around remaining boundary component B_{r+1}. Reduction corresponds to setting this holonomy equal to e and dividing out the residual gauge action. Therefore,

$$M(\Sigma_h^r, \mathcal{C}) = \big(M(\Sigma_1^1) \circledast \ldots \circledast M(\Sigma_1^1) \circledast \mathcal{C}_1 \circledast \ldots \circledast \mathcal{C}_r\big)_{red}.$$

It is proved in [1] that the symplectic structure obtained by this procedure agrees with that from the gauge-theory construction in Atiyah-Bott. The case without boundary is included since $M(\Sigma_h^0) = M(\Sigma_h^1, \{e\})$. The symplectic volume of the spaces $M(\Sigma_h^r, \mathcal{C})$ was computed by Witten [25]. Using the result of this paper, we will give in Section 9 an elementary proof of Witten's formulas.

3. Equivariant Liouville form for $\mathfrak{g}^*$-valued moment maps

For any manifold M together with a closed 2-form ω_0, we define the Liouville form as the exponential $\exp \omega_0$. Its top form degree part $(\exp \omega_0)_{[top]}$ is a volume form if and only if ω_0 is non-degenerate. If (M, ω_0, Φ_0) is a (possibly degenerate) Hamiltonian G-space the 2-form, and therefore the Liouville form, have equivariant extensions to cocycles in equivariant cohomology. In this section we summarize the definition and basic facts about equivariant Liouville forms, which we will then extend to group valued moment maps.

3.1. **The Weil algebra.** Let G be a compact Lie group with Lie algebra $\mathfrak{g}$, and let $e_a \in \mathfrak{g}$ be a basis.

The Weil algebra $W_G^\star = \oplus_r W_G^r$ is the tensor product of the symmetric algebra and the exterior algebra over $\mathfrak{g}^*$,

$$W_G^r = \bigoplus_{2j+k=r} S^j\mathfrak{g}^* \otimes \wedge^k \mathfrak{g}^*.$$

Let $y^a \in \wedge^1\mathfrak{g}^*$ and $v^a \in S^1\mathfrak{g}^*$ be the generators corresponding to the dual basis $e^a \in \mathfrak{g}^*$.

The coadjoint action of G on $\mathfrak{g}^*$ induces actions on $S\mathfrak{g}^*$ and $\wedge\mathfrak{g}^*$, hence also on W_G; let $L_a = L_a \otimes 1 + 1 \otimes L_a$ be the Lie derivatives. Let $\iota_a = 1 \otimes \iota_a : W_G^\star \to W_G^{\star-1}$ be the extension of the natural contraction mapping on $\wedge\mathfrak{g}^*$. Finally let the differential d be given by the formula

$$\mathrm{d} = (L_a \otimes 1)y^a + (v^a - \frac{1}{2} f^a_{bc} y^b y^c)\iota_a. \tag{3}$$

The derivations ι_a, L_a and d satisfy the same (super-) bracket relations as contractions, Lie derivatives, and exterior differential for a G-manifold M:

$$\begin{aligned} &[\iota_a, \iota_b] = 0, \quad [\iota_a, \mathrm{d}] = L_a, \quad [L_a, \iota_b] = f^c_{ab}\iota_c, \\ &[L_a, \mathrm{d}] = 0, \quad [L_a, L_b] = f^c_{ab} L_c, \quad [\mathrm{d}, \mathrm{d}] = 0. \end{aligned} \tag{4}$$

The elements y^a are analogues to connection 1-forms, since $\iota_a y^b = \delta^b_a$. By a result of Cartan [11] the differential complex (W_G, d) is acyclic. Hence W_G is morally the de Rham complex of the classifying bundle EG.

We will also need a "Weil algebra with generalized coefficients", defined as follows. Let $\mathcal{E}'(\mathfrak{g}^*)$ be the convolution algebra of compactly supported distributions. The space of distributions supported at 0 is a subalgebra, and is naturally identified with $S\mathfrak{g}^*$ by the isomorphism sending the generator $v^a \in S\mathfrak{g}^*$ to the distribution $\frac{\partial}{\partial t}|_{t=0}\delta_{te_a}$. Define $\widehat{W}_G := \mathcal{E}'(\mathfrak{g}^*) \otimes \wedge\mathfrak{g}^*$. The derivations ι_a, L_a, d extend to $\widehat{W}_G$, still satisfying (4).

Let $\tau_0 \in C^\infty(g^*) \otimes \wedge\mathfrak{g}^*$ be the function

$$\tau_0(\mu) = \exp(-\frac{1}{2} f_{ab}^c y^a y^b \mu_c). \tag{5}$$

It is invertible and acts on $\widehat{W}_G$ by multiplication, preserving the subspace W_G. Conjugation by τ_0^{-1} simplifies the differential (cf. [2], Section 5.1)

$$\mathrm{Ad}(\tau_0^{-1})\,\mathrm{d} = v^a \iota_a, \quad \mathrm{Ad}(\tau_0^{-1})\,\iota_a = \iota_a - f_{ab}^c \mu_c y^b.$$

Under the natural pairing of $\widehat{W}_G$ with $\Omega(\mathfrak{g}^*) = C^\infty(\mathfrak{g}^*) \otimes \wedge\mathfrak{g}$, this says that $\mathrm{Ad}(\tau_0^{-1})\,\mathrm{d}$ and $\mathrm{Ad}(\tau_0^{-1})\,\iota_a$ are dual to operators on $\Omega(\mathfrak{g}^*)$,

$$\mathrm{Ad}(\tau_0^{-1})\,\mathrm{d} = -\mathrm{d}^*, \quad \mathrm{Ad}(\tau_0^{-1})\,\iota_a = -(\mathrm{d}\mu_a + \iota_a)^*. \tag{6}$$

3.2. **Equivariant cohomology.** Given a G-manifold M, one defines the equivariant cohomology of M to be the cohomology of the basic subcomplex, consisting of elements which are both invariant and horizontal:

$$H_G^\star(M) := H^\star((W_G \otimes \Omega(M))_{basic}).$$

Under our assumption that G is compact, it can be shown that this definition agrees with the topological definition of equivariant cohomology (over $\mathbb{R}$). The space $H_G^\star(M)$ is a module over the ring $H_G^\star(\mathrm{pt}) = (S^\star\mathfrak{g}^*)^G$ of invariant polynomials. Replacing W_G with $\widehat{W}_G$ we also define a $\mathbb{Z}_2$-graded space

$$\widehat{H}_G(M) = H((\widehat{W}_G \otimes \Omega(M))_{basic})$$

which is a module over the ring $\widehat{H}_G(\mathrm{pt}) = \mathcal{E}'(\mathfrak{g}^*)^G$ of compactly supported invariant distributions. If M is compact and oriented integration over M defines natural maps

$$\int_M : H_G(M) \to (S\mathfrak{g}^*)^G, \quad \int_M : \widehat{H}_G(M) \to \mathcal{E}'(\mathfrak{g}^*)^G.$$

Let $\Pi : \widehat{W}_G \to \mathbb{R}$ be the natural projection, induced by the inclusion $\{e\} \hookrightarrow G$. It is a product $\Pi = \int_{\mathfrak{g}^*} \otimes P_{hor}$ of horizontal projection $P_{hor} : \wedge^\star\mathfrak{g}^* \to \wedge^0\mathfrak{g}^* = \mathbb{R}$ and the integration map $\int_{\mathfrak{g}^*} : \mathcal{E}'(\mathfrak{g}^*) \to \mathbb{R}$. Since Π is a chain map and a ring map, it induces for any G-manifold M a ring homomorphism in cohomology, $\widehat{H}_G(M) \to H(M)$.

The operator $P_{hor} \otimes 1$ on $\widehat{W}_G \otimes \Omega(M)$ restricts to an algebra isomorphism

$$P_{hor} \otimes 1 : (\widehat{W}_G \otimes \Omega(M))_{basic} \cong (\mathcal{E}'(\mathfrak{g}^*) \otimes \Omega(M))^G.$$

The space $\widehat{C}_G(M) = (\mathcal{E}'(\mathfrak{g}^*) \otimes \Omega(M))^G$ equipped with the differential d_G induced by this isomorphism is called the *Cartan model* for $\widehat{H}_G(M)$. One finds that $\mathrm{d}_G = 1 \otimes \mathrm{d} - v^a \otimes \iota_a$.

3.3. **Equivariant Liouville forms for Hamiltonian G-spaces.** We first describe a form $\Lambda_0 \in \widehat{W}_G \otimes \Omega(\mathfrak{g}^*)$ that appears in the definition of the Liouville form. View the function $\mu \mapsto \tau_0(\mu)\delta_\mu$ as an element of $\widehat{W}_G \otimes \Omega^0(\mathfrak{g}^*)$ and let

$$\Lambda_0 := e^{-y^a d\mu_a}\tau_0(\mu)\delta_\mu \in (\widehat{W}_G \otimes \Omega^*(\mathfrak{g}^*))^G.$$

Notice that $\tau_0^{-1}\Lambda_0$ is is the kernel of the identity map of $\widehat{W}_G$, using the pairing with $\Omega(\mathfrak{g}^*)$ introduced at the end of Section 3.1. Equation (6) hence shows that

$$\mathrm{d}\Lambda_0 = 0, \qquad \iota_a\Lambda_0 = -\mathrm{d}\mu_a\ \Lambda_0. \tag{7}$$

Suppose M is a G-manifold, together with an equivariant map $\Phi_0 \in C^\infty(M, \mathfrak{g}^*)^G$ and a 2-form $\omega_0 \in \Omega^2(M)$. We define the equivariant Liouville form of the triple (M, ω_0, Φ_0) to be the form

$$\mathcal{L}_0 = e^{\omega_0}\Phi_0^*\Lambda_0 \in \widehat{W}_G \otimes \Omega(M).$$

From (7) it follows that the equivariant Liouville form $\mathcal{L}_0$ is closed if and only if ω_0 is closed, and that its horizontality is equivalent to the moment map condition for Φ_0. Recall that the combination of these two conditions implies that ω_0 and hence also $\mathcal{L}_0$ are invariant. In this case $\mathcal{L}_0$ defines a cohomology class

$$[\mathcal{L}_0] \in \widehat{H}_G(M).$$

The 2-form ω_0 is non-degenerate if and only if the top degree part of the form

$$\Gamma_0 = (\Pi \otimes 1)\mathcal{L}_0 = e^{\omega_0}$$

is a volume form; in this case $\mathrm{Vol}(M) = \int_M e^{\omega_0}$ is called the Liouville volume of M.

3.4. **Example: Coadjoint orbits.** Let $\Phi_0 : \mathcal{O} \hookrightarrow \mathfrak{g}^*$ be a coadjoint orbit. Use an invariant inner product on $\mathfrak{g}$ to identify $\mathfrak{g} \cong \mathfrak{g}^*$. Let $\kappa = \kappa^a e_a \in \Omega^1(\mathcal{O}, \mathfrak{g})$ be the unique Lie-algebra valued 1-form on $\mathcal{O}$ defined by

$$\iota(\xi_{\mathcal{O}}(\mu))\kappa_\mu = \mathrm{pr}_{\mathfrak{g}_\mu^\perp}\,\xi$$

where $\mathrm{pr}_{\mathfrak{g}_\mu^\perp} : \mathfrak{g} \to \mathfrak{g}$ is the orthogonal projection onto $\mathfrak{g}_\mu^\perp$. The symplectic form (1) on $\mathcal{O}$ can be written

$$\omega_0 = -\frac{1}{2}(\mathrm{ad}_\mu)_{ab}\kappa^a\kappa^b.$$

Since $(\Phi_0)^*\mathrm{d}\mu_a = (\mathrm{ad}_\mu)_{ab}\kappa^b$ and $(\mathrm{ad}_\mu)_{ab} = f_{abc}\mu_c$, it follows that the equivariant Liouville form equals

$$\mathcal{L}_0 = \exp(-\frac{1}{2}(\mathrm{ad}_\mu)_{ab}(\kappa^a + y^a)(\kappa^b + y^b))\delta_{\Phi_0},$$

which is just $\tau_0\delta_{\Phi_0}$ with y^a replaced by $y^a + \kappa^a$. Using the operator $\exp(\kappa^a\iota_a)$ generating this shift we have shown that

$$\mathcal{L}_0 = \exp(\kappa^a\iota_a)\ \Phi_0^*(\tau_0\delta).$$

The volume form on $\mathcal{O}$ can be computed as the top degree part of $\Gamma_0 = \exp\omega_0$: Given $\mu \in \mathcal{O}$, let $\det_{\mathfrak{g}_\mu^\perp}(\mathrm{ad}_\mu) > 0$ be the determinant of ad_μ, viewed as an automorphism of

$\mathfrak{g}_\mu^\perp$. There is a unique orientation on $\mathfrak{g}_\mu^\perp \cong T_\mu\mathcal{O}$ for which the Pfaffian $\det^{\frac{1}{2}}_{\mathfrak{g}_\mu^\perp}(-\operatorname{ad}_\mu)$ is positive. Assume with no loss of generality that the first $l = \dim \mathfrak{g}_\mu^\perp$ elements of the basis e_a are an oriented basis of $\mathfrak{g}_\mu^\perp$. Then $\kappa^1 \ldots \kappa^l$ is the Riemannian volume form $\mathrm{d}\,\mathrm{Vol}_{Riem}$ form on $\mathcal{C}$, and the definition of the Pfaffian shows that

$$(\Gamma_0)_{[top]} = \det{}^{\frac{1}{2}}_{\mathfrak{g}_\mu^\perp}(-\operatorname{ad}_\mu)\ \mathrm{d}\,\mathrm{Vol}_{Riem}\,.$$

This gives the well-known formula

$$\mathrm{Vol}(\mathcal{O}) = |\det{}_{\mathfrak{g}_\mu^\perp}(\operatorname{ad}_\mu)|^{\frac{1}{2}}\ \frac{\mathrm{Vol}\, G}{\mathrm{Vol}\, G_\mu}$$

where $\mathrm{Vol}(G)$ and $\mathrm{Vol}(G_\mu)$ are the Riemannian volumes of G and G_μ.

3.5. **Twisted DH-distributions.** Let (M, ω_0, Φ_0) be a compact, oriented (possibly degenerate) Hamiltonian G-space, with Liouville form $\mathcal{L}_0$. The integral

$$\mathfrak{m}_0 = \int_M \mathcal{L}_0 \in \mathcal{E}'(\mathfrak{g}^*)^G,$$

is called the *Duistermaat-Heckman measure*. It is equal to the push-forward $\mathfrak{m}_0 = (\Phi_0)_*(\Gamma_0)_{[top]}$ where the orientation is used to view $(\Gamma_0)_{[top]}$ as a (signed) measure. More generally, given an equivariant differential form $\beta_0 \in (W_G \otimes \Omega(M))_{basic}$, the *twisted* Duistermaat-Heckman distribution $\mathfrak{m}_0^{\beta_0}$ is defined by

$$\mathfrak{m}_0^{\beta_0} = \int_M \beta_0 \mathcal{L}_0 \in \mathcal{E}'(\mathfrak{g}^*)^G.$$

The map $\beta_0 \mapsto \mathfrak{m}_0^{\beta_0}$ vanishes on coboundaries and therefore descends to a map in cohomology $H_G(M) \to \mathcal{E}'(\mathfrak{g}^*)^G$. The distributions $\mathfrak{m}_0^{\beta_0}$ appear for example in work of Jeffrey-Kirwan [16], Duistermaat [12], Vergne [23] and Paradan [21]. They are supported on the image of Φ and their singular support is contained in the set of singular values of Φ_0.

At regular values μ of Φ_0 they encode intersection pairings of reduced spaces:

$$\frac{\mathfrak{m}_0^{\beta_0}}{\mathrm{d}\,\mathrm{Vol}_{\mathfrak{g}}^*}\Big|_{\mathcal{O}} = \frac{\mathrm{Vol}\, G}{k\, \mathrm{Vol}\, \mathcal{O}} \int_{M_\mathcal{O}} \kappa_\mathcal{O}(\beta_0) e^{\omega_\mathcal{O}}. \tag{8}$$

Here $\mathcal{O}$ is the coadjoint orbit through μ, $(M_\mathcal{O}, \omega_\mathcal{O})$ the reduced space, k the number of elements in a generic stabilizer for the G-action, and $\kappa_\mathcal{O} : H_G(M) \to H(M_\mathcal{O})$ the "Kirwan map" defined by pull-back to $\Phi_0^{-1}(\mathcal{O})$ followed by the isomorphism $H_G(\Phi_0^{-1}(\mathcal{O})) \cong H(M_\mathcal{O})$. For a proof of (8) see e.g. [12].

4. Group valued equivariant cohomology

Before we extend the definition of equivariant Liouville forms to group valued moment maps, we define a complex of group-valued equivariant forms. The Liouville form will be a closed form in this complex.

This section summarizes material from the paper [2] to which we refer for proofs and further details.

4.1. **The non-commutative Weil algebra.** Suppose G is a compact connected Lie group, and let its Lie algebra $\mathfrak{g}$ be equipped with an invariant inner product. Let $e_a \in \mathfrak{g}$ be an orthonormal basis and f_{abc} the structure constants for this basis.

Let $\mathrm{Cl}(\mathfrak{g})$ be the Clifford algebra of $\mathfrak{g}$, defined as the quotient of the tensor algebra by the ideal generated by all $\xi \otimes \xi - \frac{1}{2}\xi \cdot \xi$. Thus if $x_a \in \mathrm{Cl}(\mathfrak{g})$ are the generators with respect to the basis $\{e_a\}$, we have $[x_a, x_b] = \delta_{ab}$ (using graded commutators). We let $\sigma : \mathrm{Cl}(\mathfrak{g}) \to \wedge\mathfrak{g}$ be the symbol map defined by

$$\sigma(x_{j_1} \dots x_{j_s}) = y_{j_1} \dots y_{j_s}, \quad j_1 < \dots < j_s$$

Let $\mathcal{E}'(G)$ be the convolution algebra of (compactly supported) distributions on G. The universal enveloping algebra $U(\mathfrak{g})$ embeds into $\mathcal{E}'(G)$ as the subalgebra of distributions with support at the group unit, by the map given on generators as $u_a \mapsto \frac{d}{dt}|_{t=0} \exp(te_a)$.

The non-commutative Weil algebra $\mathcal{W}_G$ and the non-commutative Weil algebra with generalized coefficients are the $\mathbb{Z}_2$-graded algebras

$$\mathcal{W}_G := U(\mathfrak{g}) \otimes \mathrm{Cl}(\mathfrak{g}), \quad \widehat{\mathcal{W}}_G := \mathcal{E}'(G) \otimes \mathrm{Cl}(\mathfrak{g}).$$

The Lie derivatives for the diagonal G-action are inner derivations $L_a = \mathrm{ad}(u_a + g_a)$ where

$$g_a = -\frac{1}{2} f_{abc}\, x_b\, x_c. \tag{9}$$

Define a derivation $\mathrm{d} = \mathrm{ad}(\mathcal{D})$ where

$$\mathcal{D} = u_a x_a - \frac{1}{6} f_{abc} x_a x_b x_c.$$

It is shown in [2] that

$$\mathcal{D}^2 = \frac{1}{2} u_a u_a - \frac{1}{48} f_{abc} f_{abc}. \tag{10}$$

Since (10) is contained in the center $U(\mathfrak{g})^G$ of the universal enveloping algebra, it follows that $\mathrm{d}^2 = \mathrm{ad}(\mathcal{D}^2) = 0$ showing that d is a differential. Putting $\iota_a = \mathrm{ad}(x_a)$, the derivations ι_a, L_a and d satisfy the bracket relations (4), for example Cartan's formula $[\iota_a, \mathrm{d}] = L_a$ follows from $[x_a, \mathcal{D}] = g_a$.

Let $\mathrm{Spin}(\mathfrak{g})$ be the spin group, defined as the image of $\mathrm{so}(\mathfrak{g}) \subset \mathrm{Cl}^{(2)}_{even}(\mathfrak{g})$ under the (Clifford) exponential map. The G-action on $\mathrm{Cl}(\mathfrak{g})$ defines a homomorphism

$$\mathfrak{g} \to \mathrm{so}(\mathfrak{g}) \subset \mathrm{Cl}(\mathfrak{g}),\ \mu \mapsto \mu_a g_a$$

where g_a was defined in (9). If we make the assumption that G is a product of a simply connected group and a torus, it exponentiates to a map $\tau:\ G\to \mathrm{Spin}(\mathfrak{g})\hookrightarrow \mathrm{Cl}(\mathfrak{g})$. Thus

$$\tau(\exp\,\mu)=\exp(-\frac{1}{2}f_{abc}\,\mu_a\,x_b\,x_c) \tag{11}$$

which shows that formally, τ is a group analogue to the function τ_0 introduced in (5). By definition, $\tau(g_1)\tau(g_2)=\tau(g_1g_2)$ and $g\cdot x=\mathrm{Ad}(\tau(g))x$ for all $x\in \mathrm{Cl}(g)$.

The function τ acts on $\widehat{\mathcal{W}}_G$ by multiplication from the left. The conjugates of d and ι_a under τ are dual to operators on $\Omega(G)$, as follows. Use the left-invariant Maurer-Cartan forms θ_a to trivialize T^*G to identify $\Omega(G)\cong C^\infty(G)\otimes\wedge\mathfrak{g}^*$, and use the symbol map to identify $\widehat{\mathcal{W}}_G=\mathcal{E}'(G)\otimes\wedge\mathfrak{g}$. Under the pairing given by these identifications, one finds that the τ-conjugates of the operators d,ι_a are dual to operators on $\Omega(G)$:

$$\mathrm{Ad}(\tau^{-1})(\mathrm{d})=-(\mathrm{d}+\eta)^*,\quad \mathrm{Ad}(\tau^{-1})(\iota_a)=-(\iota_a+\frac{1}{2}(\theta_a+\overline{\theta}_a))^*. \tag{12}$$

Moreover, the map τ takes the multiplication map $\mathrm{Mult}_{\mathcal{W}}$ for $\widehat{\mathcal{W}}_G$ into a map dual to the comultiplication $\exp(-\frac{1}{2}\theta^1_a\overline{\theta}^2_a)\circ \mathrm{Mult}^*_G:\Omega^*(G)\to\Omega^*(G\times G)$

$$\tau^{-1}\circ \mathrm{Mult}_{\mathcal{W}}\circ(\tau\otimes\tau)=(e^{-\frac{1}{2}\theta^1_a\overline{\theta}^2_a}\circ \mathrm{Mult}^*_G)^* \tag{13}$$

where the superscripts denote the pull-backs to the respective G-factor.

One of the main results in [2] is the construction of a linear map (called quantization map) $\mathcal{Q}:\ \widehat{W}_G\to\widehat{\mathcal{W}}_G$ which restricts to an isomorphism $W_G\to\mathcal{W}_G$ and which satisfies

$$[\mathcal{Q},\iota_a]=0,\quad [\mathcal{Q},L_a]=0,\quad [\mathcal{Q},\mathrm{d}]=0.$$

The map $\mathcal{Q}$ is most easily described by duality to a map $\Omega(G)\to\Omega(\mathfrak{g})$,

$$\tau^{-1}\circ\mathcal{Q}\circ\tau_0:=(e^{\varpi}\circ\exp^*)^*, \tag{14}$$

where $\varpi\in\Omega^2(\mathfrak{g})^G$ was introduced in Section 2.4.3. In [2] we give an explicit formula for $\mathcal{Q}$ and show in particular that on the subalgebra $\mathcal{E}'(\mathfrak{g})\subset\widehat{\mathcal{W}}_G$, it restricts to the Duflo map $\exp_*\circ J^{\frac{1}{2}}:\ \mathcal{E}'(\mathfrak{g})\to\mathcal{E}'(G)$. Here $J^{\frac{1}{2}}$ is the square root of the Jacobian of the exponential map $\exp:\ \mathfrak{g}\to G$.

4.2. **Group valued equivariant cohomology.** For any G-manifold M, the equivariant cohomology algebras $\mathcal{H}_G(M)$ and $\widehat{\mathcal{H}}_G(M)$ are defined as the cohomology of the basic subcomplexes

$$(\mathcal{W}_G\otimes\Omega(M))_{basic}=(\mathcal{W}_G\otimes\Omega(M))^G_{hor},\quad (\widehat{\mathcal{W}}_G\otimes\Omega(M))_{basic}=(\widehat{\mathcal{W}}_G\otimes\Omega(M))^G_{hor}.$$

They are modules over the ring of Casimir elements $\mathcal{H}_G(\mathrm{pt})=U(\mathfrak{g})^G$ resp. of invariant distributions $\widehat{\mathcal{H}}_G(\mathrm{pt})=\mathcal{E}'(G)^G$. If M is compact and oriented, integration over M defines maps

$$\int_M:\ \mathcal{H}_G(M)\to U(\mathfrak{g})^G,\qquad \int_M:\ \widehat{\mathcal{H}}_G(M)\to\mathcal{E}'(G)^G.$$

The quantization map $\mathcal{Q}: \widehat{W}_G \to \widehat{\mathcal{W}}_G$ induces linear maps

$$\mathcal{Q}: \widehat{H}_G(M) \to \widehat{\mathcal{H}}_G(M), \quad H_G(M) \to \mathcal{H}_G(M),$$

the second of which is an isomorphism. By Theorem 8.1 in [2] these two maps are actually *ring* maps. For the case $M = \{pt\}$, this becomes the fact that the Duflo map restricts to ring homomorphisms

$$\exp_* \circ J^{\frac{1}{2}}: \mathcal{E}'(\mathfrak{g})^G \to \mathcal{E}'(G)^G, \quad S(\mathfrak{g})^G \to U(\mathfrak{g})^G.$$

The quantization map is clearly functorial with respect to pull-backs and integrations. For example, if M is compact and oriented, we have the identity

$$\int_M \mathcal{Q}(\beta) = \mathcal{Q}(\int_M \beta) = \exp_*(J^{\frac{1}{2}} \int_M \beta) \tag{15}$$

for all $\beta \in \widehat{H}_G(M)$.

4.3. **Cartan model.** Consider the horizontal projection map $P_{hor} = \prod_\beta \iota_\beta x_\beta : \mathrm{Cl}(\mathfrak{g}) \to \mathbb{R}$. Using the symbol map to identify $\mathrm{Cl}(\mathfrak{g}) \cong \wedge\mathfrak{g}$ it is equal to projection to $\wedge^0\mathfrak{g}$, however it is not a ring map for the Clifford multiplication. We extend P_{hor} to projection maps $\widehat{\mathcal{W}}_G \to \mathcal{E}'(G)$ and $\mathcal{W}_G \to U(\mathfrak{g})^G$. For any G-manifold M, horizontal projection $(P_{hor} \otimes 1)$ in the Weil algebra part induces vector space isomorphisms

$$(\widehat{\mathcal{W}}_G \otimes \Omega(M))_{basic} \cong (\mathcal{E}'(G) \otimes \Omega(M))^G, \quad (\mathcal{W}_G \otimes \Omega(M))_{basic} \cong (U(\mathfrak{g}) \otimes \Omega(M))^G \tag{16}$$

The spaces

$$\widehat{\mathcal{C}}_G(M) = (\mathcal{E}'(G) \otimes \Omega(M))^G, \quad \mathcal{C}_G(M) = (U(\mathfrak{g}) \otimes \Omega(M))^G,$$

equipped with the differential d_G and algebra structure induced by this isomorphism, is called the *Cartan model* for $\widehat{\mathcal{H}}_G(M)$. The ring structure on the Cartan model induced by (16) has an explicit description. First, define a new ring structure on $\Omega(M)$ by setting

$$\beta_1 \odot \beta_2 = \mathrm{diag}_M^* \, e^{\frac{1}{2}\iota_a^1 \iota_a^2}(\beta_1 \otimes \beta_2)$$

where $\mathrm{diag}_M : M \to M \times M$ is the diagonal embedding, and ι_a^1, ι_a^2 are the contractions on $\Omega(M \times M) = \Omega(M) \otimes \Omega(M)$ with respect to the first respectively second G-factor. (Notice that the space of invariant forms is a subring for the new ring structure). Let $\odot$ be the ring structure on $\widehat{C}_G(M)$ induced by the given ring structure (convolution) on $\mathcal{E}'(G)$ and the new ring structure $\odot$ on $\Omega(M)$.

Proposition 4.1. [1] *The map (16) is a ring map for the product $\odot$ on $\widehat{\mathcal{C}}_G(M)$, and a chain map provided $\widehat{\mathcal{C}}_G(M)$ is equipped with the differential*

$$d_G = 1 \otimes d - \frac{1}{2}(u_a^L + u_a^R) \otimes \iota_a + \frac{f_{abc}}{24} 1 \otimes \iota_a \iota_b \iota_c.$$

Here u_a^L and u_a^R are the operators of left/right convolution by u_a.

As a direct consequence, the map

$$\Pi \otimes 1 : (\widehat{\mathcal{W}}_G \times \Omega(M))_{basic} \to \Omega(M)^G$$

given by composition of $P_{hor} \otimes 1$ and $\int_G$ is a ring map for the product $\odot$ on $\Omega(M)^G$, and also a chain map if the space $\Omega(M)^G$ is equipped with the differential

$$\mathrm{d} + \frac{1}{24} f_{abc} \iota_a \iota_b \iota_c.$$

We will also need the following result, describing "restriction to the diagonal" in terms of Cartan models:

Proposition 4.2. [2] *Let M be a $G \times G$-manifold, and $\tilde{M}$ the same manifold with diagonal G-action. Let $\alpha \in (\widehat{\mathcal{W}}_{G\times G} \otimes \Omega(M))_{basic}$, and let $\tilde{\alpha}$ be its image under the map* $\mathrm{Mult}_{\mathcal{W}} : (\widehat{\mathcal{W}}_{G\times G} \otimes \Omega(M))_{basic} \to (\widehat{\mathcal{W}}_G \otimes \Omega(\tilde{M}))_{basic}$. *Then*

$$(P_{hor} \otimes 1)(\tilde{\alpha}) = \big((\mathrm{Mult}_G)_* \otimes e^{\frac{1}{2}\iota_a^1 \iota_a^2}\,\big) \circ (P_{hor} \otimes 1)(\alpha).$$

5. Liouville form for group valued Hamiltonian spaces

For the next few Sections we will make the following assumption on the group G:

Assumption (S): *G is a direct product of a compact, connected, simply connected Lie group and a torus.*

This restriction will be lifted in Section 8. Assumption (S) is required for the function $\tau : G \to \mathrm{Cl}(\mathfrak{g})$ to be well-defined. It also implies that the half-sum ρ of positive roots (for any given choice of maximal torus and positive Weyl chamber) is a weight for G, parametrizing an irreducible representation V_ρ. The character $\chi_\rho \in C^\infty(G)$ of this representation is given by

$$\frac{\chi_\rho(g)}{\dim V_\rho} = \det{}^{\frac{1}{2}}\big(\frac{1+\mathrm{Ad}_g}{2}\big) \tag{17}$$

using the unique smooth square root which equal 1 at $g = e$.

5.1. Definition of the equivariant Liouville form. In this section we construct the equivariant Liouville form for any group valued Hamiltonian G-space (M, ω, Φ) as a cocycle for group valued equivariant cohomology. We first give the description in the Weil model $(\widehat{\mathcal{W}}_G \otimes \Omega(M))_{basic}$. The two maps $G \to \mathrm{Cl}(\mathfrak{g})$, $g \mapsto \tau(g)$ and $G \mapsto \mathcal{E}'(G)$, $g \mapsto \delta_g$ combine into a map $G \to \widehat{\mathcal{W}}_G$, $g \mapsto \tau(g)\delta_g$, which can be viewed as an element of $\widehat{\mathcal{W}}_G \otimes \Omega^0(G)$. The group analogue to Λ_0 is the form

$$\Lambda = e^{-x_a \overline{\theta}_a}\,\tau(g)\delta_g = \tau(g) e^{-x_a \theta_a} \delta_g \in \widehat{\mathcal{W}}_G \otimes \Omega(G).$$

Similar to $\tau_0^{-1}\Lambda_0$, the form $\tau^{-1}\Lambda$ is the kernel of the identity map of $\widehat{\mathcal{W}}_G$, using the pairing with $\Omega(G)$ introduced in Section 4.1. Hence (12) implies

$$\mathrm{d}\Lambda = -\eta\,\Lambda, \quad \iota_a \Lambda = -\frac{1}{2}(\theta_a + \overline{\theta}_a)\Lambda. \tag{18}$$

The definition of a Liouville form for G-valued Hamiltonian spaces is parallel to that for $\mathfrak{g}^*$-valued Hamiltonian spaces, the form Λ replacing the form Λ_0:

Definition 5.1. Suppose that M is a G-manifold together with a 2-form ω and an equivariant map $\Phi : M \to G$. We define the *equivariant Liouville form* of (M, ω, Φ) by

$$\mathcal{L} := e^{\omega}\Phi^*\Lambda \in \widehat{\mathcal{W}}_G \otimes \Omega(M)$$

It is immediate from the properties of Λ that $\mathcal{L}$ is closed if and only if $\mathrm{d}\omega = \Phi^*\eta$, and that horizontality of $\mathcal{L}$ is equivalent to the moment map condition in Definition 2.1. (Recall from Section 2.2 that the combination of these two conditions implies that ω, hence also $\mathcal{L}$, is invariant.) That is, (M, ω, Φ) is a (degenerate) Hamiltonian G-space if and only if $\mathcal{L}$ represents a cohomology class $[\mathcal{L}] \in \widehat{\mathcal{H}}_G(M)$. As for Hamiltonian spaces we define a differential form

$$\Gamma = (\Pi \otimes 1)\mathcal{L} \in \Omega(M)^G,$$

and if M is compact and oriented we refer to $\mathrm{Vol}(M) = \int_M \Gamma$ as the (signed) Liouville volume.

Theorem 5.2. *Let (M, ω, Φ) be a possibly degenerate group valued Hamiltonian G-space. Then:*

a. *The space (M, ω, Φ) is non-degenerate if and only if the top degree part $\Gamma_{[top]}$ is a volume form.*
b. *$\Gamma_{[top]}$ is related to the top degree part of $\exp\omega$ by*

$$\frac{\Phi^*\chi_\rho}{\dim V_\rho}\Gamma_{[top]} = (e^{\omega})_{[top]}.$$

The proof of Theorem 5.2 will be given in Section 5.3. It will require the following explicit description of Γ. By definition

$$\Gamma = (\Pi \otimes 1)(e^{\omega}\Phi^*\Lambda) = e^{\omega}\,\Phi^*(\Pi \otimes 1)(\Lambda) = e^{\omega}\Phi^*\tau'$$

where

$$\tau' = (P_{hor} \otimes 1)(e^{-x_a\overline{\theta}_a}\tau) \in \Omega^*(G).$$

Proposition 5.3. *The differential form τ' on G is given by the formula*

$$\tau'_g = \frac{\chi_\rho(g)}{\dim V_\rho}\exp\Big(-\frac{1}{4}\,(\frac{\mathrm{Ad}_g - 1}{\mathrm{Ad}_g + 1})_{ab}\overline{\theta}_a\overline{\theta}_b\Big). \tag{19}$$

The right hand side of (19) *is well-defined in the sense that the zeroes of χ_ρ compensate the singularities of the exponential.*

Proof. The operator P_{hor} combines with the factor $\exp(-x_a\overline{\theta}_a)$ to produce the symbol map: Indeed σ can be defined as the composition of two maps $\exp(-2x_by_b) : \mathrm{Cl}(\mathfrak{g}) \to \mathrm{Cl}(\mathfrak{g}) \otimes \wedge\mathfrak{g}$ and $P_{hor} \otimes 1 : \mathrm{Cl}(\mathfrak{g}) \otimes \wedge\mathfrak{g} \to \wedge\mathfrak{g}$. It follows that the form τ' is obtained from the symbol $\sigma(\tau)$ by replacing $2y_a$ with $\overline{\theta}_a$.

As a special case of [7], Proposition 3.13, together with the formula (17) for χ_ρ, the symbol of τ is given by

$$\sigma(\tau(g)) = \frac{\chi_\rho(g)}{\dim V_\rho} \exp\Big(- \big(\frac{\mathrm{Ad}_g -1}{\mathrm{Ad}_g +1}\big)_{ab} y_a y_b \Big), \tag{20}$$

which proves Proposition 5.3. ☐

Using Equation (19) we have found the following expression for the differential form Γ,

$$\Gamma = e^\omega \frac{\chi_\rho(\Phi)}{\dim V_\rho} \exp\Big(- \frac{1}{4} \big(\frac{\mathrm{Ad}_\Phi -1}{\mathrm{Ad}_\Phi +1}\big)_{ab} \Phi^*\overline{\theta}_a \Phi^*\overline{\theta}_b \Big). \tag{21}$$

Since $\mathcal{L}$ is basic and closed, the discussion in previous Section shows that Γ is closed under the differential $\mathrm{d} + \frac{1}{24} f_{abc}\iota_a\iota_b\iota_c$ on $\Omega(M)^G$.

Remark 5.4. Originally, we discovered the volume form Γ_{top} directly in terms of the above differential form expression. The similarity of the form τ' with the symbol of the map $\tau : G \to \mathrm{Cl}(\mathfrak{g})$ was pointed out to us by N. Berline and B. Kostant.

5.2. **The Liouville form for a conjugacy class.** Before we continue the discussion of the equivariant Liouville form in the general case, let us examine the example of a conjugacy class in more detail.

Let $\Phi : \mathcal{C} \hookrightarrow G$ be a conjugacy class. By equivariance the Liouville form is determined by its value at any point $g \in \mathcal{C}$. Let $\mathfrak{g}_g$ be the Lie algebra of the centralizer G_g of g and let $\mathrm{pr}_{\mathfrak{g}_g^\perp} : \mathfrak{g} \to \mathfrak{g}_g^\perp$ be projection onto its orthogonal complement. Let $\kappa = \kappa_a e_a \in \Omega^1(\mathcal{C}, \mathfrak{g})$ be the Lie algebra valued 1-form defined by

$$\iota(\xi_\mathcal{C}(g))\kappa_g = \mathrm{pr}_{\mathfrak{g}_g^\perp}(\xi) \tag{22}$$

for all $\xi \in \mathfrak{g}$. The 2-form ω on $\mathcal{C}$ (cf. (2)) can be written

$$\omega = -\frac{1}{4}(\mathrm{Ad}_g - \mathrm{Ad}_{g^{-1}})_{ab}\kappa_a\kappa_b.$$

Proposition 5.5. *The equivariant Liouville form for the conjugacy class $\mathcal{C} = G \cdot g$ is given by the formula*

$$\mathcal{L} = \exp(-\iota_a \otimes \kappa_a)\Phi^*(\tau\, \delta.). \tag{23}$$

The top form degree part of the form Γ is a volume form on $\mathcal{C}$, and in particular defines an orientation. The corresponding Liouville volume of $\mathcal{C}$ reads

$$\mathrm{Vol}(\mathcal{C}) = |\det_{\mathfrak{g}_g^\perp}(\mathrm{Ad}_g -1)|^{\frac{1}{2}} \frac{\mathrm{Vol}\, G}{\mathrm{Vol}\, G_g}. \tag{24}$$

where $\mathrm{Vol}\, G$ and $\mathrm{Vol}\, G_g$ are the Riemannian volumes. The volume form $\Gamma_{[top]}$ is related to the top exterior power of the 2-form ω on $\mathcal{C}$ by

$$(e^\omega)_{[top]} = \frac{\Phi^*\chi_\rho}{\dim V_\rho} \Gamma_{[top]} \tag{25}$$

Proof. Our proof uses two formulas from Clifford calculus. Suppose V is an oriented Euclidean vector space, and $S \in \mathrm{so}(V)$. Let $x_a \in \mathrm{Cl}(V)$ be the generators corresponding to some choice of oriented, orthonormal basis of V, and let κ_a be odd elements in some commutative super-algebra $\mathcal{A}$. Then

$$(26)\qquad \big(\prod_\beta \kappa_\beta\big)\,\iota_\beta \exp(\frac{1}{2}S_{ab}x_a x_b) = \det{}^{\frac{1}{2}}(2\sinh(\frac{S}{2}))\,\kappa_1 \dots \kappa_n$$

where the square root is defined as a Pfaffian. Moreover,

$$(27)\qquad \exp(-\iota_a\kappa_a)\exp\big(\frac{1}{2}S_{ab}x_a x_b\big) = \exp(\varpi_2)\exp(-x_r\gamma_r)\exp\big(\frac{1}{2}S_{ab}x_a x_b\big)$$

where $\gamma_r = (1-e^S)_{rs}\kappa_s$ and $\varpi_2 = \frac{1}{2}(\sinh(S))_{ab}\kappa_a\kappa_b$. Both equations are proved by block-diagonalizing the matrix S. For (26) this is carried out in Mathai-Quillen [18], Proposition 2.10 and for (27) in [2], Lemma 7.5.

Choose $\mu \in \mathfrak{g}$ such that $\exp\mu = g$. We will apply Equations (26) and (27) to $V = \mathfrak{g}_g^\perp$ and $S = -\mathrm{ad}_\mu$, with κ as defined in (22). Notice that $\tau(g) = \exp(\frac{1}{2}S_{ab}x_a x_b)$ takes its values in the Clifford algebra $\mathrm{Cl}(V) \subset \mathrm{Cl}(\mathfrak{g})$.

We may assume with no loss of generality that the basis of $\mathfrak{g}$ is chosen in such a way that the first l elements $e_1, \dots, e_l$ are a basis for $\mathfrak{g}_g^\perp$.

Then ϖ_2 is just the 2-form ω on $\mathcal{C}$ and $\gamma_r = \Phi^*\overline{\theta}_r$. Equation (27) therefore becomes

$$\exp(-\iota_a\kappa_a)\tau(g) = \exp(\omega)\,\exp(-x_a\overline{\theta}_a)\,\tau(g)$$

which proves (23). Since taking the top degree part commutes with the projection $\Pi\otimes 1 = \big(\int_G \otimes P_{hor}\big)\otimes 1$,

$$\Gamma_{[top]} = (P_{hor}\otimes 1)(\int_G \mathcal{L})_{[top]}.$$

Using (23) and (26),

$$(\int_G \mathcal{L})_{[top]} = \prod_{\beta=1}^{l}(-\iota_\beta\otimes\kappa_\beta)\tau(g) = \det{}^{\frac{1}{2}}_{\mathfrak{g}_g^\perp}\big(2\sinh(\frac{\mathrm{ad}_\mu}{2})\big)\kappa_1\dots\kappa_l \neq 0.$$

Since this expression does not contain Clifford variables x_a, it is not affected by the horizontal projection P_{hor}, showing that $\Gamma_{[top]}$ is a volume form. Since $\pm\kappa_1\dots\kappa_l$ is the Riemannian volume form on $T_\mu\mathcal{C}$, and since $|\det\big(2\sinh(\frac{\mathrm{ad}_\mu}{2})\big)| = |\det_{\mathfrak{g}_g^\perp}(\mathrm{Ad}_g - 1)|$, this proves (24). Finally, since

$$\exp(\omega)_{[top]} = \det{}^{\frac{1}{2}}_{\mathfrak{g}_g^\perp}(\sinh(\mathrm{ad}_\mu))\mathrm{d}\,\mathrm{Vol}_{Riem}$$

by definition of the Pfaffian, and since

$$\frac{\det^{\frac{1}{2}}_{\mathfrak{g}_g^\perp}(\sinh(\mathrm{ad}_\mu))}{\det^{\frac{1}{2}}_{\mathfrak{g}_g^\perp}(2\sinh(\frac{\mathrm{ad}_\mu}{2}))} = \det{}^{\frac{1}{2}}(\cosh(\frac{\mathrm{ad}_\mu}{2})) = \frac{\chi_\rho(\exp\mu)}{\dim V_\rho}$$

we obtain the formula (25). □

5.3. **Proof of Theorem 5.2.** We are now in position to prove Theorem 5.2, saying that the top degree part of Γ is a volume form if and only if ω is minimally degenerate.

Proof of Theorem 5.2. Let (M, ω, Φ) be a group valued Hamiltonian G-space, possibly degenerate. Let $m \in M$, and $g = \Phi(m)$. There is a natural splitting of the tangent space $T_m M$ given by

$$T_m M = \mathfrak{g}_g^\perp \oplus E$$

where $\mathfrak{g}_g^\perp$ is embedded by means of the fundamental vector fields and $E = (\mathrm{d}_m \Phi)^{-1}(T_g(G_g))$. The 2-form on $\mathfrak{g}_g^\perp$ is just the 2-form corresponding to its identification with the tangent space to the conjugacy class $G \cdot g$. It is verified in Section 7 of [1] that this splitting is ω-orthogonal, that is, $\omega(v_1, v_2) = 0$ for $v_1 \in \mathfrak{g}_g^\perp$ and $v_2 \in E$ and that the restriction $\omega_E = \omega|_E$ is non-degenerate if and only if ω satisfies the minimal degeneracy condition at m, that is if and only if $\ker \omega_m = \{\xi_M(m) | \xi \in \ker(\mathrm{Ad}_g + 1)\}$. Hence

$$\Gamma_m = \Gamma_{\mathfrak{g}_g^\perp} \wedge \exp(\omega_E)$$

where $\Gamma_{g_g^\perp}$ is the form corresponding to the identification $\mathfrak{g}_g^\perp \cong T_g(G \cdot g)$, and therefore

$$(\Gamma_m)_{[top]} = (\Gamma_{\mathfrak{g}_g^\perp})_{[top]} \wedge \exp(\omega_E)_{[top]}.$$

Since $(\Gamma_{\mathfrak{g}_g^\perp})_{[top]}$ is non-vanishing, it follows that $(\Gamma_m)_{[top]} \neq 0$ if and only if ω_E is non-degenerate. This proves the first assertion, and the second assertion follows from

$$(\exp(\omega_{\mathfrak{g}_g^\perp}))_{[top]} = \frac{\chi_\rho(g)}{\dim V_\rho} (\Gamma_{\mathfrak{g}_g^\perp})_{[top]}$$

by (25). □

Remark 5.6. In contrast to Hamiltonian G-spaces, the volume form for a group valued Hamiltonian G-space depends not only on the 2-form but also on the moment map. For example, if (M, ω, Φ) is a group valued Hamiltonian G-space and $c \in Z(G)$ a central element, then $\Phi' = c\Phi$ is a moment map for the same group action and 2-form ω. The new Liouville form is $\mathcal{L}' = \tau(c)\delta_c \mathcal{L}$. Since ρ is a weight for G, it defines a homomorphism $T \to S^1$, $t \mapsto t^\rho$. We have

$$\tau(c) = c^\rho, \tag{28}$$

as one can see for example from the formula (20) for the symbol, $\sigma(\tau)(c) = \frac{\chi_\rho(c)}{\dim V_\rho} = c^\rho$. Therefore, $\mathcal{L}' = c^\rho \, \delta_c \, \mathcal{L}$ and

$$\Gamma' = c^\rho \Gamma.$$

One has $c^\rho = \pm 1$: indeed, $c^{2\rho} = 1$ since 2ρ is a weight for $G/Z(G)$. In general $c^\rho \neq 1$ so that Γ' may differ from Γ by sign.

5.4. **DH-distributions.** The definition of DH-distributions for group valued Hamiltonian spaces (M, ω, Φ) is analogous to that for $\mathfrak{g}^*$-valued moment maps (Section 3.5). Let (M, ω, Φ) be a compact, oriented, possibly degenerate G-valued Hamiltonian G-space, with Liouville form $\mathcal{L}$. The integral

$$\mathfrak{m} := \int_M \mathcal{L} \in \mathcal{E}'(G)^G$$

is called the Duistermaat-Heckman measure. It is equivalently given as a push-forward, $\mathfrak{m} = \Phi_*(\Gamma)_{[top]}$. More generally, given any equivariant differential form $\beta \in (\mathcal{W}_G \otimes \Omega(M))_{basic}$ we define

$$\mathfrak{m}^\beta = \int_M \beta\mathcal{L} \in \mathcal{E}'(G)^G.$$

The map

$$(\mathcal{W}_G \otimes \Omega(M))_{basic} \to \mathcal{E}'(G)^G,\ \beta \mapsto \mathfrak{m}^\beta$$

descends to a linear map in cohomology, $\mathcal{H}_G(M) \to \mathcal{E}'(G)^G$. One can define a similar map $\widehat{\mathcal{H}}_G(M) \to \mathcal{E}'(G)^G$ – only on $\mathcal{H}_G(M)$ however one has the following locality property of the distributions $\mathfrak{m}^\beta$:

Proposition 5.7. *For any $\beta \in (\mathcal{W}_G \otimes \Omega(M))_{basic}$, the support of the distribution $\mathfrak{m}^\beta$ is contained in the image of Φ and its singular support in the set of singular values of Φ.*

Proof. By Proposition 4.1, and since $(P_{hor} \otimes 1)\mathcal{L} = \Gamma\delta_\Phi$,

$$\mathfrak{m}^\beta = \int_M (\beta\mathcal{L}) = \int_M (P_{hor} \otimes 1)(\beta\mathcal{L}) = \int_M (P_{hor} \otimes 1)(\beta) \odot \Gamma\delta_\Phi.$$

Writing $(P_{hor} \otimes 1)(\beta) = \sum_J u^J \otimes \beta_J \in (U(\mathfrak{g}) \otimes \Omega(M))^G$ we find

$$\mathfrak{m}^\beta = \sum_J u^J \int_M (\beta_J \odot \Gamma)\,\delta_\Phi = \sum_J u^J\ \Phi_*(\beta_J \odot \Gamma)_{[top]}.$$

This proves the proposition since the support of $\Phi_*(\beta_J \odot \Gamma)_{[top]}$ is contained in the image of Φ and the singular support in the set of singular values, and since left convolution by u^J is given by application of a differential operator. $\square$

In what follows, we will often find it convenient to express DH-distributions in terms of their Fourier coefficients: Given an irreducible representation V_λ labeled by a dominant weight λ, with character χ_λ, we have

$$\mathfrak{m}^\beta(g) = \sum_\lambda \langle \mathfrak{m}^\beta, \chi_\lambda \rangle \chi_\lambda(g^{-1})\ \frac{\mathrm{d}\,\mathrm{Vol}_G}{\mathrm{Vol}\,G}. \tag{29}$$

For a conjugacy class $\mathcal{C} \subset G$, the Duistermaat-Heckman measure $\mathfrak{m}_\mathcal{C}$ is given by

$$\mathfrak{m}_\mathcal{C} = \mathrm{Vol}(\mathcal{C})\,\delta_\mathcal{C}$$

where $\delta_{\mathcal{C}}$ is the unique invariant measure supported on $\mathcal{C}$ and with integral 1, and where $\mathrm{Vol}(\mathcal{C})$ is given by (24). Its Fourier coefficients are, therefore,

$$\langle \mathfrak{m}_{\mathcal{C}}, \chi_\lambda \rangle = \mathrm{Vol}(\mathcal{C}) \chi_\lambda(\mathcal{C}). \tag{30}$$

5.5. **Example: The double $D(G)$.** As an example, we calculate the volume form and the DH-distribution for the double $D(G)$. We claim:

Proposition 5.8. *Let G be a compact, connected, simply connected Lie group and let $D(G) = G \times G$ be the double. The volume form $\Gamma_{[top]}$ on $D(G)$ is equal to the Riemannian volume form $d\mathrm{Vol}_{G\times G}$ on $D(G)$ (for the canonical orientation). In particular, the Liouville volume $\mathrm{Vol}(D(G))$ is equal to the Riemannian volume, $\mathrm{Vol}(G)^2$. The Fourier coefficients of the DH measure are*

$$\langle \mathfrak{m}, \chi_{\lambda_1} \otimes \chi_{\lambda_2} \rangle = \mathrm{Vol}(G)^2 \ \delta_{\lambda_1, \lambda_2}. \tag{31}$$

Proof. We use the trivialization of the tangent bundle $T(D(G)) = D(G) \times (\mathfrak{g} \oplus \mathfrak{g})$ by means of left invariant vector fields. The skew-symmetric matrix describing ω at a point (a, b) has block form,

$$B = \frac{1}{2} \begin{pmatrix} 0 & -\mathrm{Ad}_b - \mathrm{Ad}_{a^{-1}} \\ \mathrm{Ad}_a + \mathrm{Ad}_{b^{-1}} & 0 \end{pmatrix}$$

which has determinant

$$\det(B) = \det\Big(\frac{\mathrm{Ad}_b + \mathrm{Ad}_{a^{-1}}}{2}\Big) \det\Big(\frac{\mathrm{Ad}_a + \mathrm{Ad}_{b^{-1}}}{2}\Big) = \det\Big(\frac{\mathrm{Ad}_{ab} + 1}{2} \ \frac{\mathrm{Ad}_{a^{-1}b^{-1}} + 1}{2}\Big).$$

This proves

$$(\exp \omega)_{[top]} = \det{}^{\frac{1}{2}}(B) \mathrm{d}\,\mathrm{Vol}_{G\times G} = \det{}^{\frac{1}{2}}\Big(\frac{\mathrm{Ad}_{\Phi_1} + 1}{2} \ \frac{\mathrm{Ad}_{\Phi_2} + 1}{2}\Big) \,\mathrm{Vol}_{G\times G}$$

(Here we take the unique smooth square root which is equal 1 at $(a, b) = (e, e)$; the sign is verified by evaluating both sides at (e, e)). By the second part of Theorem 5.2 this shows that $\Gamma_{[top]} = \mathrm{d}\,\mathrm{Vol}_{G\times G}$. The measure $\mathfrak{m} \in \mathcal{E}'(G \times G)$ is the push-forward of $\mathrm{d}\,\mathrm{Vol}_{G\times G}$ under the map $(a, b) \mapsto (ab, a^{-1}b^{-1})$. We calculate

$$\langle \mathfrak{m}, \chi_{\lambda_1} \otimes \chi_{\lambda_2} \rangle = \int_{D(G)} \chi_{\lambda_1}(ab) \chi_{\lambda_2}(a^{-1}b^{-1}) \ \mathrm{d}a\, \mathrm{d}b = \mathrm{Vol}(G)^2 \delta_{\lambda_1, \lambda_2}.$$

∎

6. Fusion products and exponentials

In this Section we discuss the behavior of the equivariant Liouville form under the two basic operations with group valued Hamiltonian spaces: Fusion and Exponentials.

6.1. **Multiplicative properties of the Liouville form.** Suppose both G and H are a product of a connected, simply connected Lie group and a torus.

Theorem 6.1 (Diagonal G-action). *Let (M,ω,Φ) be a group valued Hamiltonian $G\times G\times H$-space (possibly degenerate), with moment map $\Phi=(\Phi_1,\Phi_2,\Psi)$, and let $(\tilde{M},\tilde{\omega},(\Phi_1\Phi_2,\Psi))$ be its internal fusion. Denote by $\mathcal{L},\tilde{\mathcal{L}}$ the Liouville forms and let $\Gamma=(\Pi\otimes 1)\mathcal{L}$ and $\tilde{\Gamma}=(\Pi\otimes 1)\tilde{\mathcal{L}}$. Then*

$$\tilde{\mathcal{L}}=(\mathrm{Mult}_{\mathcal{W}}\otimes 1)\mathcal{L} \tag{32}$$

and

$$\tilde{\Gamma}=\exp(\frac{1}{2}\iota^1_a\iota^2_a)\,\Gamma. \tag{33}$$

In particular, the top degree part does not change: $\tilde{\Gamma}_{[top]}=\Gamma_{[top]}$. If $\beta\in(\mathcal{W}_{G\times G\times H}\otimes\Omega(M))_{basic}$, and $\tilde{\beta}$ its image in $(\mathcal{W}_{G\times H}\otimes\Omega(\tilde{M}))_{basic}$, the corresponding DH-distributions $\mathfrak{m}^{\beta}$ and $\tilde{\mathfrak{m}}^{\tilde{\beta}}$ are related by push-forward under the G-multiplication Mult_G,

$$\tilde{\mathfrak{m}}^{\tilde{\beta}}=(\mathrm{Mult}_G)_*\mathfrak{m}^{\beta}. \tag{34}$$

Proof. Recall that $\tau^{-1}\Lambda$ is the kernel of the identity map of $\widehat{\mathcal{W}}_G$. Therefore, the kernel for the multiplication map is $\tau^{-1}\Lambda^1\Lambda^2$, where $\Lambda^1,\Lambda^2\in\widehat{\mathcal{W}}_G\otimes\Omega^*(G\times G)$ be the pull-backs of Λ to the first/second factor. Equation (13) is equivalent to

$$\Lambda^1\Lambda^2=e^{-\frac{1}{2}\theta^1_a\overline{\theta}^2_a}(1\otimes\mathrm{Mult}^*_G)\Lambda. \tag{35}$$

Since $\tilde{\omega}=\omega+\frac{1}{2}\theta^1_a\overline{\theta}^2_a$, Equation (32) follows directly from (35) and the definition of the Liouville form. Equation (33) is a consequence of (32) and Proposition 4.2. Equation (34) follows since $\tilde{\beta}\tilde{\mathcal{L}}$ is the image of $\beta\mathcal{L}$ under the map $(\widehat{\mathcal{W}}_{G\times G\times H}\otimes\Omega(M))_{basic}\to(\widehat{\mathcal{W}}_{G\times H}\otimes\Omega(\tilde{M}))_{basic}$. $\square$

In the special case that $M=M_1\times M_2$ is the direct product of two group valued Hamiltonian G-spaces, Theorem 6.1 says:

Corollary 6.2. *If (M_j,ω_j,Φ_j) (j=1,2) are two group valued Hamiltonian G-spaces, with Liouville forms $\mathcal{L}_1,\mathcal{L}_2$, then the Liouville form for the fusion product $M_1\circledast M_2$ is the product of the Liouville forms, $\mathcal{L}_1\mathcal{L}_2$, and the volume form is the direct product of the volume forms, $(\Gamma_1)_{[top]}\times(\Gamma_2)_{[top]}$.*

Given equivariant forms $\beta_j\in(\widehat{\mathcal{W}}_G\otimes\Omega(M_j))_{basic}$, the twisted DH distribution for their product $\beta_1\beta_2\in(\widehat{\mathcal{W}}_G\otimes\Omega(M_1\times M_2))_{basic}$ is the convolution product $\mathfrak{m}_1^{\beta_1}*\mathfrak{m}_2^{\beta_2}$ of the twisted DH distribution $\mathfrak{m}_j^{\beta_j}$ for the factors.

Note that this result looks quite complicated if it is spelled out in terms of the differential form expression (21) for $\Lambda=\Gamma\,\delta_\Phi$!

Equation (34) can be re-phrased in terms of Fourier coefficients as follows. Suppose $H = \{e\}$ for simplicity and let λ be a dominant weight for G. Then (34) says that

$$\langle \mathfrak{m}^{\tilde{\beta}}, \chi_\lambda \rangle = \frac{\langle \mathfrak{m}^{\beta}, \chi_\lambda \otimes \chi_\lambda \rangle}{\dim V_\lambda}. \tag{36}$$

This follows from the formula for convolution of characters (cf. [10], Proposition (4.16)):

$$\Big(\chi_\lambda \frac{\mathrm{d}\,\mathrm{Vol}_G}{\mathrm{Vol}\,G}\Big) * \Big(\chi_\nu \frac{\mathrm{d}\,\mathrm{Vol}_G}{\mathrm{Vol}\,G}\Big) = \delta_{\lambda,\nu} \frac{1}{\dim V_\lambda} \Big(\chi_\lambda \frac{\mathrm{d}\,\mathrm{Vol}_G}{\mathrm{Vol}\,G}\Big).$$

Example 6.3. Recall that the volume form on the double $D(G)$ was simply the Riemannian volume form and that the Fourier coefficients were calculated in (31). The Theorem tells us that the volume form for its internal fusion $\tilde{D}(G)$ is still the Riemannian volume form, and that the Fourier coefficients for the DH measure are

$$\langle \tilde{\mathfrak{m}}, \chi_\lambda \rangle = \frac{(\mathrm{Vol}\,G)^2}{\dim V_\lambda}. \tag{37}$$

6.2. **The Liouville form for exponentials.** In this section we compare the equivariant Liouville forms of a Hamiltonian space (M, ω_0, Φ_0) and of its exponential.

Since $\mathcal{Q}(\Lambda_0)$ is the kernel of the quantization map $\widehat{W}_G \to \widehat{\mathcal{W}}_G$, the description (12) of the quantization map is equivalent to the equation

$$\mathcal{Q}(\Lambda_0) = e^{\varpi}(1 \otimes \exp^*)\Lambda. \tag{38}$$

This implies the following Theorem.

Theorem 6.4 (Exponentials). *Let (M, ω_0, Φ_0) be a (possibly degenerate) Hamiltonian G-space, with equivariant Liouville form $\mathcal{L}_0$. Then $(M, \omega_0 + \Phi_0^*\varpi, \exp(\Phi_0))$ is a group valued Hamiltonian G-space, with Liouville form $\mathcal{L}$ the quantization*

$$\mathcal{L} = \mathcal{Q}(\mathcal{L}_0). \tag{39}$$

The top degree parts of the forms $\Gamma_0 = (\Pi \otimes 1)\mathcal{L}_0$ and $\Gamma = (\Pi \otimes 1)\mathcal{L}$ are related by

$$\Gamma_{[top]} = (\Phi_0^* J^{\frac{1}{2}})\ (\Gamma_0)_{[top]}. \tag{40}$$

Let $\beta_0 \in (W_G \otimes \Omega(M))_{basic}$ be a cocycle and $\beta = \mathcal{Q}(\beta_0)$ its quantization. If M is compact and oriented,

$$\mathfrak{m}^{\beta} = \exp_*(J^{\frac{1}{2}} \mathfrak{m}_0^{\beta_0}). \tag{41}$$

Proof. The first two statements are immediate from (38). Equation (40) follows from (38) and the definitions of $\mathcal{L}_0$ and $\mathcal{Q}$. Since $\mathcal{Q}$ is a ring homomorphism in cohomology, $\mathcal{Q}([\beta_0\mathcal{L}_0]) = \mathcal{Q}([\beta_0])\mathcal{Q}([\mathcal{L}_0]) = [\beta\mathcal{L}]$. Therefore, Equation (41) follows from (15). $\square$

Given two Hamiltonian G-spaces $(M_j, \omega_0^j, \Phi_0^j)$ ($j = 1, 2$), the direct product $M_1 \times M_2$ with diagonal G-action becomes a (possibly degenerate) group valued Hamiltonian G-space in two different ways, according whether one takes the product before or after fusion. The two resulting moment maps are

$$\exp(\Phi_0^1 + \Phi_0^2) \quad \text{resp.} \quad \exp(\Phi_0^1)\exp(\Phi_0^2).$$

If we denote the equivariant Liouville forms for M_j by $\mathcal{L}_0^j$ (viewed as forms on the product $M_1 \times M_2$), the Liouville forms for the first method is $\mathcal{Q}(\mathcal{L}_0^1\mathcal{L}_0^2)$ and for the second method it reads $\mathcal{Q}(\mathcal{L}_0^1)\mathcal{Q}(\mathcal{L}_0^2)$.

Combining Theorem 6.4 with the fact [2] that $\mathcal{Q}$ is a ring homomorphism in cohomology, we have:

Proposition 6.5. *Given two Hamiltonian G-spaces $(M_j, \omega_0^j, \Phi_0^j)$ ($j = 1, 2$) with equivariant Liouville forms $\mathcal{L}_0^j$ (viewed as forms on the product $M_1 \times M_2$), the forms $\mathcal{Q}(\mathcal{L}_0^1\mathcal{L}_0^2)$ and $\mathcal{Q}(\mathcal{L}_0^1)\mathcal{Q}(\mathcal{L}_0^2)$ are cohomologous.*

7. Intersection pairings on reduced spaces

At regular values of Φ the twisted DH distributions $\mathfrak{m}^\beta$ have a simple interpretation. Recall (see e.g. [15]) that for any G-manifold X with a locally free G-action, the pull-back map $\Omega(X/G) \to \Omega(X)_{basic} \subset (\mathcal{W}_G \otimes \Omega(X))_{basic}$ induces an isomorphism in cohomology, $H_G(X) \to H(X/G)$. In particular, if (M, ω, Φ) is a group valued Hamiltonian G-space and $\mathcal{C}$ some conjugacy class which is contained in the set of regular values of Φ, we obtain a map

$$\kappa_{\mathcal{C}} : \mathcal{H}_G(M) \to H(M_{\mathcal{C}})$$

by composition of the isomorphism $\mathcal{Q}^{-1} : \mathcal{H}_G(M) \cong H_G(M)$, pull-back to the level set $H_G(M) \to H_G(\Phi^{-1}(\mathcal{C}))$, and isomorphism $H_G(\Phi^{-1}(\mathcal{C})) \cong H(\Phi^{-1}(\mathcal{C})/G)$.

Theorem 7.1. *Let (M, ω, Φ) be a compact, oriented, group valued Hamiltonian G-space (possibly degenerate), and $\beta \in \mathcal{H}_G(M)$. If the conjugacy class $\mathcal{C}$ is contained in the set of regular values of Φ, the value of the function $\frac{\mathfrak{m}^\beta}{d\,\mathrm{Vol}_G}$ on $\mathcal{C}$ is given by the formula*

$$\frac{\mathfrak{m}^\beta}{d\,\mathrm{Vol}_G}\Big|_{\mathcal{C}} = \frac{\mathrm{Vol}\, G}{k\,\mathrm{Vol}\,\mathcal{C}} \int_{M_{\mathcal{C}}} \kappa_{\mathcal{C}}(\beta) e^{\omega_{\mathcal{C}}}$$

where $\mathrm{Vol}\mathcal{C}$ is the Liouville volume of the conjugacy class $\mathcal{C}$ and k is the number of elements in a generic stabilizer for the G-action on $\Phi^{-1}(\mathcal{C})$.

Proof. We begin by proving the theorem for the simplest case $\mathcal{C} = \{e\}$. Passing to a neighborhood $\Phi^{-1}(U)$ where $U \subset G$ is a sufficiently small invariant open neighborhood of e, we can assume that (M, ω, Φ) is the exponential of some Hamiltonian G-space (M, ω_0, Φ_0). Put $\beta_0 = \mathcal{Q}^{-1}(\beta)$. Since $\mathfrak{m}^\beta = \exp_*(J^{\frac{1}{2}}\mathfrak{m}_0^{\beta_0})$ we have

$$\frac{\mathfrak{m}^\beta}{\mathrm{d}\,\mathrm{Vol}_G}\Big|_e = \frac{\mathfrak{m}_0^{\beta_0}}{\mathrm{d}\,\mathrm{Vol}_{\mathfrak{g}^*}}\Big|_0 = \frac{\mathrm{Vol}\, G}{k} \int_{M_{red}} \kappa_0(\beta_0),$$

where we have used (8). The result follows since $\kappa_0(\beta_0) = \kappa_e(\beta)$. We now reduce the case of arbitrary conjugacy classes $\mathcal{C}$ to the case of the trivial conjugacy class by means of the shifting-trick,

$$M_\mathcal{C} = (M \circledast \mathcal{C}^-)_{red}.$$

Let $\delta_{\mathcal{C}^-} \in \mathcal{E}'(G)$ be the invariant measure with support $\mathcal{C}^- = G \cdot g^{-1}$ and integral 1. The Duistermaat-Heckman measure for the conjugacy class $\mathcal{C}^-$ is $\mathrm{Vol}(\mathcal{C})\,\delta_{\mathcal{C}^-}$. Letting $\mathfrak{m}^\beta$ be the twisted DH-distribution for M, the corresponding distribution for the product $M \circledast \mathcal{C}^-$ is the convolution $\mathrm{Vol}(\mathcal{C})\mathfrak{m}^\beta * \delta_{\mathcal{C}^-}$ (see Corollary 6.2). By the above we have

$$\mathrm{Vol}(\mathcal{C}) \frac{\mathfrak{m}^\beta * \delta_{\mathcal{C}^-}}{\mathrm{d}\,\mathrm{Vol}_G}\Big|_e = \frac{1}{k}\int_{M_\mathcal{C}} \kappa_\mathcal{C}(\beta) e^{\omega_\mathcal{C}}.$$

On the other hand, the value of the convolution product with $\delta_\mathcal{C}$ at the group unit reads

$$\frac{\mathfrak{m}^\beta * \delta_\mathcal{C}}{\mathrm{d}\,\mathrm{Vol}_G}\Big|_e = \frac{\mathfrak{m}^\beta}{\mathrm{d}\,\mathrm{Vol}_G}\Big|_\mathcal{C}.$$

Comparing the two expressions completes the proof. □

The above result can be reformulated in terms of the Fourier coefficients of $\mathfrak{m}^\beta$. Evaluating (29) on $\mathcal{C}$ we find

$$\int_{M_\mathcal{C}} \kappa_\mathcal{C}(\beta) e^{\omega_\mathcal{C}} = \frac{k \,\mathrm{Vol}\,\mathcal{C}}{(\mathrm{Vol}\, G)^2} \sum_\lambda \langle \mathfrak{m}^\beta, \chi_\lambda\rangle \chi_\lambda(\mathcal{C}^-) \tag{42}$$

provided the sum on the right hand side converges. In case of convergence problems apply (as in K. Liu [17]) the smoothing operator $e^{t\Delta}$, $t > 0$ to $\mathfrak{m}^\beta$, where Δ is the Laplace-Beltrami operator on G. Recall that for all dominant weights λ,

$$\Delta \chi_\lambda = -(||\lambda + \rho||^2 - ||\rho||^2)\chi_\lambda.$$

This gives

$$\int_{M_\mathcal{C}} \kappa_\mathcal{C}(\beta) e^{\omega_\mathcal{C}} = \frac{k \,\mathrm{Vol}\,\mathcal{C}}{(\mathrm{Vol}\, G)^2} \lim_{t\to 0^+} \sum_\lambda e^{-t||\lambda+\rho||^2}\, \langle \mathfrak{m}^\beta, \chi_\lambda\rangle \chi_\lambda(\mathcal{C}^-)$$

where the sum is absolutely convergent for all $t > 0$ and the limit $t \mapsto 0^+$ exists.

As a special case, suppose $P \in U(\mathfrak{g})^G$ is in the center of the universal enveloping algebra and $\beta = P \otimes 1$ is its image under the map $U(\mathfrak{g})^G = (\mathcal{W}_G)_{basic} \to (\mathcal{W}_G \otimes \Omega(M))_{basic}$. In this case $\mathfrak{m}^\beta$ is obtained from $\mathfrak{m}$ by application of the bi-invariant differential operator corresponding to P,

$$\mathfrak{m}^\beta = P(\mathfrak{m}).$$

The pre-image $\beta_0 = \mathcal{Q}^{-1}\beta$ is just the invariant polynomial on $\mathfrak{g}$ corresponding to P under the Duflo isomorphism $S(\mathfrak{g})^G \cong U(\mathfrak{g})^G$, and the class $\kappa_\mathcal{C}(\beta) = \kappa(\mathcal{Q}(\beta_0))$ is the characteristic class of the bundle $\Phi^{-1}(\mathcal{C}) \to M_\mathcal{C}$ attached to the invariant polynomial $p = \beta_0$ by the Chern-Weil homomorphism $S(\mathfrak{g})^G \to H(M_\mathcal{C})$.

8. Generalization to arbitrary compact Lie groups

Our construction of an equivariant Liouville form was made under the assumption (S) that the group G is a product of a simply connected group and a torus. Recall from Section 2.3.1 that without this assumption, group valued Hamiltonian G-spaces are not necessarily orientable.

We will show that dropping Assumption (S) naturally leads to Liouville volumes with values in the orientation bundle. Recall that the orientation bundle o_M for any manifold M is the associated bundle $\mathcal{P} \times_{\mathbb{Z}_2} \mathbb{R}$, where the principal $\mathbb{Z}_2$-bundle $\mathcal{P}$ is the oriented double cover of M.

One defines the space of twisted differential forms

$$\Omega_t(M) := \Omega(M, o_M)$$

as sections of $\wedge T^*M \otimes o_M$. It is a module over the space $\Omega(M)$ of differential forms on M. The real line bundle $\wedge^n T^*M \otimes o_M$ is isomorphic to the density bundle of M; hence the space $\Omega_t^n(M)$ is just the space of smooth densities of M.

For any compact, connected Lie group G there is a finite, connected covering $\widehat{G} \to G$, where $\widehat{G}$ is a product of a simply connected group and a torus. Its kernel is a subgroup $R \subseteq \pi_1(G)$.

Suppose (M, ω, Φ) is a group valued Hamiltonian G-space. Let $\widehat{M} \to M$ be the covering space obtained by pulling back the covering $\widehat{G} \to G$ under the map Φ, and let $\widehat{\Phi} : \widehat{M} \to \widehat{G}$ be the corresponding map. Let $\widehat{\omega} \in \Omega^2(\widehat{M})$ be the pull-back of ω. The G-action on M and the conjugation action on $\widehat{G}$ induce a G-action on $\widehat{M}$; letting $\widehat{G}$ act by the covering map to G, the triple $(\widehat{M}, \widehat{\omega}, \widehat{\Phi})$ is a group valued Hamiltonian $\widehat{G}$-space. For all $c \in R$ let $S_c : \widehat{M} \to \widehat{M}$ denote the corresponding deck transformation. By construction all deck transformations commute with the $\widehat{G}$-action, preserve the 2-form, and satisfy $S_c^*\widehat{\Phi} = c\widehat{\Phi}$. If $\hat{g} \in \widehat{G}$ maps to $g \in G$ there is a natural isomorphism of reduced spaces

$$\widehat{M}_{\hat{g}} \cong M_g.$$

On $\widehat{M}$ our previous considerations apply, and we obtain a Liouville form

$$\widehat{\mathcal{L}} = \exp(\widehat{\omega})\ \exp(-x_a \widehat{\Phi}^* \overline{\theta}_a)\, \tau(\widehat{\Phi})\, \delta_{\widehat{\Phi}} \in \mathcal{E}'(\widehat{G}) \otimes \mathrm{Cl}(\mathfrak{g}) \otimes \Omega(\widehat{M}).$$

We have

$$S_c^* \widehat{\mathcal{L}} = \tau(c) \delta_c \widehat{\mathcal{L}}.$$

Recall from (28) that $\tau(c) = c^\rho = \pm 1$. It follows that S_c changes the orientation by c^ρ, and the form $\widehat{\mathcal{L}}$ satisfies $S_c^* \widehat{\mathcal{L}} = c^\rho \delta_c\, \widehat{\mathcal{L}}$. Pushing forward to $\mathcal{E}'(G)$, multiplication by δ_c becomes an identity, and $\widehat{\mathcal{L}}$ becomes a closed form $\mathcal{L}' \in \widehat{\mathcal{W}}_G \otimes \Omega(\widehat{M})$ with property $S_c^* \mathcal{L}' = c^\rho \mathcal{L}'$, or equivalently a closed form

$$\mathcal{L} \in (\widehat{\mathcal{W}}_G \otimes \Omega_t(M))_{basic}.$$

Similarly $\widehat{\Gamma}$ descends to a form $\Gamma \in \Omega_t(M)$ with top degree part $\Gamma_{[top]}$ a nowhere vanishing density, so that the volume $\mathrm{Vol}(M) = \int_M \Gamma_{[top]}$ is defined. The Duistermaat-Heckman measure $\mathfrak{m}$ is as before defined as $\mathfrak{m} = \int_M \mathcal{L} = \Phi_* \Gamma_{[top]}$, and given $\beta \in (\mathcal{W}_G \otimes \omega(M))_{basic}$ one can once again define twisted DH-distributions

$$\mathfrak{m}^\beta = \int_M \beta \mathcal{L}.$$

Theorem 7.1 interpreting these distributions in terms of intersection pairings on reduced spaces goes through verbatim, as does its proof.

Example 8.1. Let (M, ω, Φ) be a group valued Hamiltonian G-space. Let $\widehat{G}$ act on M by the covering $\widehat{G} \to G$. Suppose Φ admits a lift to an equivariant map $\widehat{\Phi} : M \to \widehat{G}$. (This happens for instance for the fusion of the double $\widetilde{D}(G)$.) Then $(M, \omega, \widehat{\Phi})$ is a group valued Hamiltonian $\widehat{G}$-space. The space $\widehat{M}$ considered above is just $\widehat{M} = M \times R$, with $\widehat{\Phi}(m, c) = c\widehat{\Phi}(m)$. In particular, if G is semi-simple and $\widehat{G}$ its universal cover (so that $R = \pi_1(G)$, the space M_{red} has $\#\pi_1(G)$ connected components, given as reduced spaces $\widehat{M}_c$ for $c \in \pi_1(G) \subset Z(\widehat{G})$.

Example 8.2. Suppose that $\mathcal{C} \subset G$ is a conjugacy class. Then $\widehat{\mathcal{C}}$ is just the pre-image of $\mathcal{C}$ in $\widehat{G}$, and therefore is a finite union of conjugacy classes in $\widehat{G}$.

One easily verifies that the formulas for the DH-measures of the double $D(G)$ (Proposition 5.8), its fusion $\widetilde{D}(G)$, or for a conjugacy class $\mathcal{C} \subset G$ (Proposition 5.5) continue to hold.

9. Application to moduli spaces of flat connections

In this Section we explain how Witten's formulas for volumes of moduli spaces of flat connections follow from the results of this paper. For other proofs of the volume formulas, see K. Liu [17], Bismut-Labourie [9], or Meinrenken-Woodward [20].

Let G be a compact, connected Lie group. Using the notation from Section 2.5, Theorem 6.1 says that the volume form on the space $M(\Sigma_1^1) = \widetilde{D}(G) = G \times G$ is just the Riemannian volume form, and the volume form on the fusion product

$$M(\Sigma_1^1) \circledast \ldots \circledast M(\Sigma_1^1) \circledast \mathcal{C}_1 \circledast \ldots \circledast \mathcal{C}_r$$

is just the direct product of the volume forms on the factors. Consequently, letting $\mathfrak{m}_1^1$ and $\mathfrak{m}_{\mathcal{C}}$ be the DH measures for $M(\Sigma_1^1)$ (cf. (37)) resp. for a conjugacy class $\mathcal{C}$ (cf. (30)) the Fourier coefficients of the DH-measure $\mathfrak{m}$ are

$$\langle \mathfrak{m}, \chi_\lambda \rangle = \frac{(\langle \mathfrak{m}_1^1, \chi_\lambda \rangle)^h \prod_{j=1}^r \langle \mathfrak{m}_{\mathcal{C}_j}, \chi_\lambda \rangle}{(\dim V_\lambda)^{r+h-1}} = (\mathrm{Vol}\, G)^{2h} \, \frac{\prod_{j=1}^r \mathrm{Vol}(\mathcal{C}_j)\, \chi_\lambda(\mathcal{C}_j)}{(\dim V_\lambda)^{2h+r-1}}$$

For $h \geq 2$ and $r > 1$, and generic conjugacy classes $\mathcal{C}_j$ the action of G on the level set $\Phi^{-1}(e)$ is locally free, with generic stabilizer the center $Z(G)$. In this case the symplectic volume of the reduced space is given by $\#Z(G)$ times the value of $\mathfrak{m}$ at the group unit.

It follows that that the symplectic volume of the moduli space $M(\Sigma_h^r, \underline{\mathcal{C}})$ is given by the formula

$$\mathrm{Vol}(M(\Sigma_h^r, \underline{\mathcal{C}})) = \#Z(G)\ \mathrm{Vol}(G)^{2h-2} \prod_{j=1}^{r} \mathrm{Vol}(\mathcal{C}_j) \sum_{\lambda} \frac{\prod_{j=1}^{r} \chi_\lambda(\mathcal{C}_j)}{(\dim V_\lambda)^{2h+r-2}}$$

which is Witten's formula [25].

If G is simply connected, one obtains from this formula the volume of the moduli space of flat connection on a surface Σ_h^0 without boundary since this moduli spae is just $M(\Sigma_h^1, \{e\})$. (This is as long as the generic stabilizer is discrete, by the continuity properties of the DH-measure.)

A mild complication arises for non-simply connected groups. Indeed, if G is semi-simple but not simply connected not every G-bundle $\mathcal{P}$ over Σ_0^h is trivial. The bundle $\mathcal{P}$ becomes trivial if restricted to the surface with boundary Σ_h^1 obtained from Σ_0^h by removing a small disk, and also over the disk itself. The gluing function along the boundary is a map $S^1 \to G$, whose homotopy class defines the topological type of $\mathcal{P}$. Therefore the topological types of G-bundles over Σ_0^h are classified by elements of $\pi_1(G)$.

Let $M(\Sigma_h^1) := \widetilde{D}(G) \circledast \ldots \circledast \widetilde{D}(G) \cong G^{2h}$, let $\Phi : M(\Sigma_h^1) \to G$ be the moment map and $\widehat{\Phi} : M(\Sigma_h^1) \to \widehat{G}$ the unique lift such that $\widehat{\Phi}(e, \ldots, e) = e$. Given $c \in \pi_1(G) \subset Z(G)$ the quotient $M^{(c)}(\Sigma_0^h) = \widehat{\Phi}^{-1}(c)/\widehat{G}$ is the moduli space of flat connections on the bundle parametrized by c. The reduced space $M(\Sigma_h^0) := \Phi^{-1}(e)/G = M(\Sigma_h^1)_{red}$ is the space of isomorphism classes of flat G-bundles over Σ_h^0, and is a disjoint union

$$M(\Sigma_h^0) = \coprod_{c \in \pi_1(G)} M^{(c)}(\Sigma_h^0).$$

The DH-measure $\mathfrak{m}$ for the space $(M(\Sigma_h^1), \omega, \hat{\Phi})$ has Fourier coefficients

$$\langle \mathfrak{m}, \chi_\lambda \rangle = \mathrm{Vol}(G)^{2h} (\dim V_\lambda)^{2h-1}$$

for any dominant weight λ of $\hat{G}$. Since the generic stabilizer for the $\hat{G}$-action has $\#Z(\hat{G}) = \#Z(G)\, \#\pi_1(G)$ elements, and $\mathrm{Vol}(\hat{G}) = \#\pi_1(G)\ \mathrm{Vol}(G)$, Equation (42) yields

$$\mathrm{Vol}(M^{(c)}(\Sigma_h^0)) = \mathrm{Vol}(G)^{2h-2} \frac{\#Z(G)}{\#\pi_1(G)} \sum_{\lambda} \frac{\chi_\lambda(c^{-1})}{(\dim V_\lambda)^{2h-1}} \tag{43}$$

in accordance with Witten's result ([25], Section 4.1). Summing over all $c \in \pi_1(G)$ we recover the fact that the volume of the moduli space of flat bundles $\mathrm{Vol}(M(\Sigma_h^0))$ is given by the same formula as for the simply connected case (as a sum over irreducible characters for $G = \hat{G}/\pi_1(G)$).

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