DEFORMATIONS OF SYMPLECTIC VORTICES

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Abstract. We prove a gluing theorem for a symplectic vortex on a compact complex curve and a collection of holomorphic sphere bubbles. Using the theorem we show that the moduli space of regular strongly stable symplectic vortices on a fixed curve with varying markings has the structure of a stratified-smooth topological orbifold. In addition, we show that the moduli space has a non-canonical C^1-orbifold structure.

Contents

1. Introduction

In this paper we generalize the following result on existence of universal deformations for stable (pseudo-)holomorphic maps. Let (X, ω) be a compact symplectic manifold equipped with a compatible almost complex structure J, and (Σ, j) a compact nodal complex curve. A map $u : \Sigma \to X$ is *holomorphic* if

$$\overline{\partial} u := J_u \circ \mathrm{d}u - \mathrm{d}u \circ j = 0$$

on each component of Σ. One naturally has the notion of a stratified-smooth *family* of holomorphic maps, and hence the notion of a *deformation*, namely the germ of a family around the central fiber together with an isomorphism of the central fiber with the given map. Recall that a deformation is *universal* if any other deformation is obtained from it by pullback, in a unique way, by a map of parameter spaces. A holomorphic map $u : \Sigma \to X$ is *regular* if the linearized Cauchy-Riemann operator is surjective. The following theorem is the result of the well-known gluing construction for holomorphic maps, c.f. Ruan-Tian [21] or the text McDuff-Salamon [15, Chapter 10] in the case of genus zero:

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Theorem 1.0.1. *A regular holomorphic map $u : \Sigma \to X$ admits a stratified-smooth universal deformation iff it is stable.*

The construction of the universal deformation proceeds via the implicit function theorem. For each element in the infinitesimal deformation space of the stable map one first produces an approximate solution and then applies the implicit function theorem to find an exact solution. Unfortunately one uses a different Sobolev space for each "gluing parameter" controlling the domain, which means that it is rather tricky to show that each nearby stable holomorphic map occurs only once in the resulting family. A slightly jazzed up version of the above theorem implies that the gluing construction gives rise to orbifold charts on the regular locus of the moduli space of stable holomorphic maps. Uniqueness of the universal deformations implies that the smooth structures on each stratum are independent of the Sobolev spaces used in the implicit function theorem. One can make these charts C^1-compatible by suitable choices of *gluing profiles*, that is, coordinates on the local deformation spaces; however the C^1-structure on the moduli space is not canonical. The first part of the paper contains an exposition of the above theorem, which is rather scattered in the literature.

The main result of the paper is a generalization of the theorem above to certain *gauged (pseudo)holomorphic maps*, namely *symplectic vortices* as introduced by Mundet [16] and Cieliebak, Gaio and Salamon, see [5]. Let G be a compact Lie group and X a Hamiltonian G-manifold equipped with a moment map $\Phi : X \to \mathfrak{g}^*$ and an invariant almost complex structure J. Let Σ be a compact smooth holomorphic curve with complex structure j and equipped with an area form Vol_Σ. A *gauged holomorphic map* with values in X consists of a smooth principal G-bundle $P \to \Sigma$, a connection A on P, and a smooth section $u : \Sigma \to P(X) := P \times_G X$ such that $\overline{\partial}_A u = 0$ where $\overline{\partial}_A$ is defined using the splitting given by the connection A and the complex structures J, j. Let $F_A \in \Omega^2(\Sigma, P(\mathfrak{g}))$ denote the curvature of A and $P(\Phi) : P(X) \to P(\mathfrak{g})$ the map induced by Φ. The space of gauged holomorphic maps admits a formal symplectic structure depending on a choice of invariant metric on $\mathfrak{g}$ so that the action of the group of gauge transformations is formally Hamiltonian. A *symplectic vortex* is a pair in the zero level set of the moment map: a pair (A, u) such that

$$\overline{\partial}_A u = 0, \quad F_A + u^* P(\Phi)\,\mathrm{Vol}_\Sigma = 0.$$

Thus the moduli space $M(\Sigma, X)$ of symplectic vortices is the symplectic quotient of the space of gauged maps by the group of gauge transformations. In certain cases where the moduli spaces are compact Cieliebak, Gaio, Mundet, and Salamon [4] and Mundet [16] constructed invariants that we will call *gauged Gromov-Witten invariants* by integration over these moduli spaces. In general $M(\Sigma, X)$ admits a compactification $\overline{M}(\Sigma, X)$ consisting of *polystable symplectic vortices* given by allowing u to develop holomorphic sphere bubbles in the fibers of $P(X)$. A polystable vortex is *strongly stable* if the principal component has finite automorphism group, and *regular* if a certain linearized operator is surjective, that is, the moduli space is formally smooth. Our main result is the following:

Theorem 1.0.2. *Let Σ, X be as above. A regular strongly stable symplectic vortex from Σ to X admits a universal stratified-smooth deformation.*

Using the deformations constructed in Theorem 1.0.2 we prove that the moduli space $\overline{M}^{\text{reg}}(\Sigma, X)$ of regular strongly stable symplectic vortices admits the structure of an oriented stratified-smooth topological orbifold, and (non-canonically) the structure of a C^1-orbifold. The first statement implies that if $\overline{M}^{\text{reg}}(\Sigma, X)$ is compact then it carries a rational fundamental class. The second statement implies for example, that if the target carries a group action then the usual equivariant localization theorems hold for the induced group action on the moduli space. In the case that X is a smooth projective variety, algebraic methods explain in [11] give similar results and provide virtual fundamental classes on the moduli space. However, the symplectic gluing construction is interesting in its own right, not in the least because it potentially extends to the case of Lagrangian boundary conditions. We understand that a forthcoming paper of Mundet i Riera and Tian gives a gluing construction for two symplectic vortices, when the structure group is the circle group.

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2. Deformations of holomorphic curves

The following section is essentially a review of the material that can be found at the beginning of Siebert [22], with a few additional comments incorporating terminology of Hofer, Wysocki, and Zehnder [13, Appendix]. In the first part we review the holomorphic construction of universal deformations of stable curves. In the second part, we study smooth deformations of curves.

2.1. **Holomorphic families of stable curves.** A compact, complex *nodal curve* Σ is obtained from a collection $(\Sigma_1, \ldots, \Sigma_k)$ of smooth, compact, complex curves by identifying a collection of distinct *nodal points*

$$w = \{\{w_1^-, w_1^+\}, \ldots, \{w_m^-, w_m^+\}\}.$$

For $l = 1, \ldots, m$, we denote by $\Sigma_{i^\pm(l)}$ the components such that $w_l^\pm \in \Sigma_{i^\pm(l)}$. A point $z \in \Sigma$ is *smooth* if it is not equal to any of the nodal points. A *marked nodal curve* is a nodal curve together with a collection $z = (z_1, \ldots, z_n)$ of distinct, smooth points. An *isomorphism* of marked nodal curves (Σ_0, z_0) to (Σ_1, z_1) is an isomorphism $\phi : \Sigma_0 \to \Sigma_1$ of nodal curves such that $\phi(z_{0,i}) = z_{1,i}$ for $i = 1, \ldots, n$. A marked nodal curve is *stable* if it has finite automorphism group, that is, each component contains at least three marked or nodal points if genus zero, or one special point if genus one.

The *combinatorial type* $\Gamma(\Sigma)$ of Σ is the graph whose vertices are the components and edges are the nodes and markings of Σ. The map $\Sigma \mapsto \Gamma(\Sigma)$ extends to a functor from the category of marked nodal curves to the category of graphs. In particular, there is a canonical homomorphism $\mathrm{Aut}(\Sigma) \to \mathrm{Aut}(\Gamma(\Sigma))$, whose kernel is the product of the automorphism groups of the components of Σ (with nodes replaced by markings.)

Let S be a complex variety (or scheme). A *family of nodal curves* over S is a complex variety (or scheme) Σ_S equipped with a proper flat morphism $\pi : \Sigma_S \to S$, such that each fiber $\Sigma_s, s \in S$ is a nodal curve. A *deformation* of a marked nodal curve Σ is a germ of a family of marked nodal curves Σ_S over a pointed space $(S, 0)$ together with an isomorphism $\varphi : \Sigma_0 \to \Sigma$ of the *central fiber* Σ_0 with Σ. A deformation (Σ_S, φ) of Σ is *versal* iff any other deformation $(\Sigma'_{S'} \to S', \varphi')$ is induced from a map $\psi : S' \to S$ in the sense that there exists an isomorphism ϕ of Σ' with the fiber product $\Sigma_S \times_S S'$ in a neighborhood of the central fiber Σ_0. A versal deformation is *universal* if the map ϕ is the unique such map inducing the identity on Σ_0. A deformation has *fixed type* if the combinatorial type of the fiber is constant. A *universal deformation of fixed type* is a deformation of fixed type, which is universal in the above sense for deformations of fixed type. The space $\mathrm{Def}(\Sigma)$ of *infinitesimal deformations* of Σ is the tangent space $T_0 S$ of the base S of a universal deformation, well-defined up to isomorphism. We write $\mathrm{Def}_\Gamma(\Sigma)$ for the space of infinitesimal deformations of fixed type. Let $\tilde{\Sigma}$ be the normalization of Σ, so that $\mathrm{Def}_\Gamma(\Sigma)$ is isomorphic to the space of deformations of $\tilde{\Sigma}$ equipped with the additional markings $w_1^\pm, \ldots, w_m^\pm$ obtained by lifting the nodes. The general theory of deformations, see for example [7] in the analytic setting, shows that any marked nodal curve Σ admits a versal deformation with smooth parameter space S. Σ admits a universal deformation $\Sigma_S \to S$ if and only if Σ is stable. Furthermore, the space $\mathrm{Def}(\Sigma)$ of the space of infinitesimal deformations admits a canonical isomorphism with $H^{0,1}(\Sigma, T\Sigma[-z_1 - \ldots - z_n])$, where $T\Sigma[-z_1 - \ldots, -z_n]$ is the sheaf of vector fields vanishing at $z_1, \ldots, z_n$.

The relationship between the various deformation spaces (in the case with markings, fixed type, etc.) is given as follows. The space of *infinitesimal automorphisms* $\mathrm{aut}(\Sigma, z)$ of (Σ, z) is the space $\mathrm{Vect}(\Sigma, z) = H^0(\Sigma, T\Sigma[-z_1 - \ldots - z_n])$ of holomorphic vector fields vanishing at the marked points. The short exact sequence of sheaves

$$0 \to \oplus_{i=1}^n T_{z_i}\Sigma \to T\Sigma \to T\Sigma[-z_1 - \ldots - z_n] \to 0$$

gives a long exact sequence in cohomology [12, p. 94]

$$0 \to \mathrm{Vect}(\Sigma, z) \to \mathrm{Vect}(\Sigma) \to \bigoplus_{i=1}^n T_{z_i}\Sigma \to \mathrm{Def}(\Sigma, z) \to \mathrm{Def}(\Sigma) \to 0.$$

From now on, we omit the markings from the notation, and study deformations of a nodal marked curve $\Sigma = (\Sigma, z)$. By $T_{w_i^\pm}\Sigma$, we mean the tangent space in the component of Σ containing $w_i^\pm$. A *gluing parameter* for the i-th node is an element $\delta_i \in T_{w_i^+}\Sigma \otimes T_{w_i^-}\Sigma$. The canonical conormal sequence [12, p. 100] gives rise to an exact sequence

$$0 \to \mathrm{Def}_\Gamma(\Sigma) \to \mathrm{Def}(\Sigma) \to \bigoplus_{i=1}^m T_{w_i^+}\Sigma \otimes T_{w_i^-}\Sigma \to 0. \tag{1}$$

After trivialization of the tangent spaces the gluing parameters are identified with complex numbers.

Universal deformations of a smooth marked curve can be constructed for example using Teichmüller theory [8] or by Hilbert scheme methods [12, p. 102]. Later we will need an explicit gluing construction of a holomorphic universal deformation of a stable marked curve. This construction seems to be well-known, but the only proof we could find in the literature is Siebert [22]. The idea is to remove small neighborhoods of the nodes, and glue the remaining components together. A *local coordinate near a smooth point* $z \in \Sigma$ is a neighborhood U of z and a holomorphic isomorphism κ of U with a neighborhood of 0 in the tangent line $T_z\Sigma$, whose differential $T_zU \to T_z\Sigma$ is the identity.

Remark 2.1.1. The space of local coordinates near z is convex, since if κ_0, κ_1 are local coordinates then any combination $t\kappa_0 + (1-t)\kappa_1$ is still holomorphic and has the same differential at z, and so by the inverse function theorem is a holomorphic isomorphism in a neighborhood of z.

Any gluing parameter δ_i induces an identification

$$T_{w_i^+}\Sigma - \{0\} \to T_{w_i^-}\Sigma - \{0\}, \quad \lambda_i^+ \mapsto \delta_i/\lambda_i^-.$$

Given local coordinates for the nodes of Σ and a set of gluing parameters $\delta = (\delta_1, \ldots, \delta_m)$, define a (possibly nodal) curve Σ^δ by gluing together small disks around the node w_i by $z \mapsto \delta_i/z$, for every gluing parameter δ_i that is non-zero, where z is the local coordinate given by κ_i. That is,

$$\Sigma^\delta = \bigcup_{i=1}^k \Sigma_i - \{w_1^\pm, \ldots, w_m^\pm\}/(z \sim (\kappa_i^+)^{-1}(\delta_i/\kappa_i^-(z)), i = 1, \ldots, m) \tag{2}$$

for pairs of points in the two components such that both coordinates are defined. In particular, the choice of local coordinates near the nodes defines a splitting of the sequence (1).

The gluing construction works in families as follows. Let $I_\Gamma^{i,\pm} \to S_\Gamma$ resp. $I_\Gamma \to S_\Gamma$ denote the vector bundle whose fiber at $s \in S_\Gamma$ is the tangent line at the j-node resp. tensor product of tangent lines at the nodes,

$$I_{\Gamma,s}^{i,\pm} = T_{w_{i,s}^\pm}\Sigma_s, \quad I_{\Gamma,s} = \bigoplus_{j=1}^m T_{w_{i,s}^-}\Sigma_s \otimes T_{w_{i,s}^+}\Sigma_s. \tag{3}$$

Let $\Sigma_{S_\Gamma} \to S_\Gamma$ be a family of nodal curves of the same combinatorial type Γ, with nodal points $(w_{S_\Gamma,j}^\pm)_{i=1}^m$. A *holomorphic system of local coordinates for the i-th node* is a holomorphic map κ_i from a neighborhood $U_{i,\pm}$ of the zero section in $I_S^{i,\pm}$ to

Σ_S which is an isomorphism onto its image and induces the identity at any point in the zero section. Given a holomorphic system of coordinates for each node $\kappa = (\kappa_1^+, \kappa_1^-, \ldots, \kappa_m^+, \kappa_m^-)$ the gluing construction (2) produces a family $\Sigma_S \to S$ over an open neighborhood S of the zero section in the bundle $I \to S_\Gamma$ of gluing parameters.

Theorem 2.1.2. [22, Proposition 2.4] *If $\Sigma_{\Gamma,S}$ is a family giving a universal deformation of fixed type, then Σ_S is a universal deformation of any of its fibers, and in particular is independent up to isomorphism of deformations of the choice of local coordinates κ.*

The following properties of universal deformations of stable curves will be used later:

Theorem 2.1.3. [22, Lemma 2.7] *For any universal deformation (Σ_S, φ), the action of automorphisms* $\mathrm{Aut}(\Sigma)$ *of* Σ *extends to an action of* $\mathrm{Aut}(\Sigma)$ *on* Σ_S*, possibly after shrinking S. For any universal deformation, there exists a neighborhood of the central fiber such that any two fibers Σ_S contained in the neighborhood are isomorphic, if and only if they are related by an automorphism of* Σ.

If Σ is not stable, then the above construction produces a *minimal versal deformation* of Σ. That is, $\Sigma_S \to S$ is versal, and any other versal deformation given by a family $\Sigma'_{S'} \to S'$ is obtained by pull-back by a map $S' \to S$.

Algebraic families of connected stable nodal curves with genus g and n markings form the objects of a smooth *Deligne-Mumford stack* $\overline{M}_{g,n}$ [6] which admits a coarse moduli space with the structure of a normal projective variety. The maps $\mathrm{Def}(\Sigma) \to \overline{M}_{g,n}, \ s \mapsto [\Sigma_s]$ (restricted to a neighborhood of 0) provide $\overline{M}_{g,n}$ with an atlas of holomorphic orbifold charts.

2.2. **Stratified-smooth families of stable curves.** We extend the definition of families and deformations to smooth and stratified-smooth settings. Given a family $\Sigma_S \to S$ of compact complex nodal curves, let

$$S = \bigcup S_\Gamma, \quad S_\Gamma = \{ s \in S, \quad \Gamma(\Sigma_s) = \Gamma \}$$

denote the stratification by combinatorial type of the fiber. It follows from the gluing construction of the previous section that if $\Sigma_S \to S$ is a family giving a universal deformation, then each S_Γ is a smooth manifold, and the restriction Σ_{Γ,S_Γ} of Σ_{S_Γ} to S_Γ gives a universal deformation of fixed type Γ. By a *smooth family* of curves of fixed type Γ we mean a fiber bundle $\Sigma_{\Gamma,S_\Gamma} \to S_\Gamma$ with fibers of type Γ and smoothly varying complex structure. In the nodal case, it is obtained from a smooth family of smooth holomorphic curves, identified using a collection of pairs of smooth sections (nodes).

Lemma 2.2.1. *Holomorphic universal deformations of fixed type are also universal in the category of smooth deformations of Σ. That is, let $\Sigma_S \to S, \varphi$ be a universal holomorphic deformation of fixed type of a nodal curve Σ. Any smooth deformation $\Sigma'_{S'} \to S', \varphi'$ of nodal curves of fixed type is obtained by pull-back $\Sigma_S \to S$ by a smooth map $S' \to S$.*

Proof. By the construction of local slices for the action of diffeomorphisms in [8], [20, Chapter 9]. □

Similarly we can define *continuous* families of holomorphic curves, which correspond to continuous maps $S' \to S$ to the parameter space S for a universal holomorphic deformation. The following spells out the definition without reference to the universal holomorphic deformation.

Definition 2.2.2. Let Γ_0, Γ_1 be graphs. A *simple contraction* τ is a pair of maps $\mathrm{Vert}(\tau) : \mathrm{Vert}(\Gamma_0) \to \mathrm{Vert}(\Gamma_1)$ and a bijection $\mathrm{Edge}(\tau) : \mathrm{Edge}(\Gamma_0) \to \mathrm{Edge}(\Gamma_1) \cup \{\emptyset\}$ such that Γ_1 is obtained from Γ_0 identifying the head and tail of the *contracting edge* e such that $\mathrm{Edge}(\tau)(e) = \emptyset$. A *contraction* is a sequence of simple contractions.

Definition 2.2.3. A *continuous family* of nodal holomorphic curves consists of topological spaces Σ_S, a surjection $\Sigma_S \to S$, and a collection of (possibly nodal) holomorphic structures j_{Σ_s} on the fibers $\Sigma_s, s \in S$, which vary continuously in s in the following sense: for every $s_0 \in S$ there exists for s in a neighborhood of s_0 of some combinatorial type Γ,

(a) contractions $\tau_s : \Gamma(\Sigma_{s_0}) \to \Gamma$, constant in $s \in S_\Gamma$;
(b) for every node $\{w_i^\pm\}$ collapsed under τ_s, a pair of local coordinates $\kappa_i^\pm : W_i^\pm \to \mathbb{C}$
(c) for every component $\Sigma_{s_0,i}$ of Σ_{s_0}, a real number $\epsilon_s > 0$ and maps

$$\phi_{i,s} : \Sigma_{s_0,i} - \cup_{w_k^\pm \in \Sigma_{s_0,i}, \tau_s(w_k^\pm) = \emptyset} (\kappa_k^\pm)^{-1}(B_{\epsilon_s}) \to \Sigma_{s,\tau_s(i)}$$

such that

(a) for any s, the images of the maps $\phi_{i,s}$ cover Σ_s;
(b) for any nodal point $w_k^\pm$ of Σ_s joining components $\Sigma_{s,i^\pm(k)}$, there exists a constant $\lambda_s \in \mathbb{C}^*$ such that $(\kappa_k^+ \circ \phi_{s,i^+(k)}^{-1} \circ \phi_{s,i^-(k)} \circ (\kappa_k^-)^{-1})(z) = \lambda_s z$, if the former is defined, and $\lambda_s \to 0$ as $s \to s_0$.
(c) for any smooth $z \in \Sigma_{s_0,i}$, $\lim_{s \to s_0}(\phi_{i,s}(z)) = z$;
(d) $\phi_{i,s}^* j_{\Sigma_{s,\tau_s(i)}}$ converges to $j_{\Sigma_{s_0,i}}$ uniformly in all derivatives on compact sets;
(e) if z_i is contained in $\Sigma_{s_0,k}$, then $z_i = \lim_{s \to s_0} \phi_{s,k}^{-1}(z_{i,s})$.

A *stratified-smooth* family of curves is a continuous family $\Sigma_S \to S$ over a stratified base $S = \bigcup_\Gamma S_\Gamma$ such that the restriction Σ_{S_Γ} of Σ_S to S_Γ is a smooth family of fixed type Γ. A *stratified-smooth deformation* of a nodal curve Σ is a germ of a stratified-smooth family of nodal curves Σ_S equipped with an isomorphism of the central fiber Σ_0 with Σ. A *universal stratified-smooth deformation* of Σ is a deformation with the property that any other stratified-smooth deformation $\Sigma'_{S'} \to S'$ is obtained by pull-back by maps $\psi : S' \to S$, $\phi : \Sigma \times_S S' \to \Sigma'_{S'}$, and any two isomorphisms ϕ, ϕ' inducing the identity on Σ are equal.

Any universal holomorphic deformation is also a universal stratified-smooth deformation, essentially by Lemma 2.2.1. In the stratified-smooth setting, the analog of Theorem 2.1.3 fails and we need an additional definition:

Definition 2.2.4. A universal stratified-smooth deformation $(\pi : \Sigma_S \to S, \phi)$ is *strongly universal* if π is a universal deformation of any of its fibers, and two fibers of π are isomorphic, if and only if they are related by the action of $\mathrm{Aut}(\Sigma)$.

The construction of universal deformations extends to the smooth setting as follows. Let $\Sigma_{S_\Gamma} \to S_\Gamma$ be a smooth family of curves of fixed type Γ. A *smooth system of local coordinates for the i-th node of* Σ_{S_Γ} is a smooth map κ_i from a neighborhood $U_{i,\pm}$ of the zero section in $I^{i,\pm}$ to Σ_{S_Γ} which is an isomorphism onto its image and induces the identity at zero. Given a universal deformation $(\Sigma_{S_\Gamma} \to S_\Gamma, \varphi)$ of fixed type Γ and a smooth system of local coordinates, applying the gluing construction (2) gives a smooth family $\Sigma_S \to S$ over an open neighborhood S of 0 in $\mathrm{Def}(\Sigma)$.

Theorem 2.2.5. *Let Σ be a stable curve. The family $\Sigma_S \to S \subset \mathrm{Def}(\Sigma)$ constructed by gluing from a family $\Sigma_{\Gamma,S} \to S \subset \mathrm{Def}_\Gamma(\Sigma)$ of fixed type, using any smooth family of local coordinates κ near the nodes, gives a strongly universal stratified-smooth deformation of Σ.*

Proof. Let $\Sigma^\kappa_{S^\kappa} \to S^\kappa$ be a family constructed via gluing using a smooth family of local coordinates κ as in (2), and $\Sigma_S \to S$ a universal deformation using a holomorphic family of local coordinates by the same construction (2). By universality, there exists a map $\psi : S^\kappa \to S$ so that $\Sigma_{\psi(s)} \cong \Sigma^\kappa_s$. It suffices to show that ψ is a diffeomorphism on each stratum. Consider the canonical map from $T_\delta S^\kappa$ to $\mathrm{Def}(\Sigma^\delta)$, which maps an infinitesimal change in the parameter space S^κ to the corresponding infinitesimal deformation of Σ^δ, which we identify with an element of $\Omega^0(\Sigma^\delta, \mathrm{End}(T\Sigma^\delta))$. Let $U \subset \Sigma^\delta$ denote the gluing region, that is, the image of the union of domains of the local coordinates. The deformations generated by the gluing parameters are supported in the gluing region U. On the other hand, linearly independent deformations of fixed type $\mathrm{Def}_\Gamma(\Sigma)$ generate deformations of the glued curve that are linearly independent on $\Sigma^\delta - U$, for sufficiently small U. (The generated deformations will not vanish on U, because of the varying local coordinates.) Thus the map $\mathrm{Def}_\Gamma(\Sigma) \to \Omega^0(\Sigma - U, \mathrm{End}(T\Sigma)|\Sigma - U)$ is injective; it follows that $TS^\kappa \to \mathrm{Def}(\Sigma^\delta)$ is injective, hence an isomorphism by a dimension count. This shows that the map $S^\kappa \to S$ is a covering. Let κ_t be a family of local coordinates interpolating between κ and a holomorphic family. The corresponding family ψ_t interpolates between the identity and ψ. Since each ψ_t is a covering and ψ_0 is the identity, each ψ_t is a diffeomorphism. □

The strongly universal deformations above defined using smooth families of local coordinates provide smooth orbifold charts on $\overline{M}_{g,n}$. Since the space of local coordinates is convex, one can construct the local coordinates for each stratum compatibly. Namely, let Γ' be a combinatorial type degenerating to Γ. Local coordinates for the nodes of $M_{g,n,\Gamma}$ induce local coordinates for $M_{g,n,\Gamma'}$, in a neighborhood of $M_{g,n,\Gamma}$, via the gluing construction (2).

Definition 2.2.6. A *compatible system of local coordinates* for $\overline{M}_{g,n}$ is a system of local coordinates for the nodes of each stratum $M_{g,n,\Gamma}$, so that the local coordinates on any stratum $M_{g,n,\Gamma'}$ are induced from those on $M_{g,n,\Gamma}$, in a neighborhood of $M_{g,n,\Gamma}$.

Compatible systems of local coordinates can be constructed by induction on the dimension of $M_{g,n,\Gamma}$, using convexity on the space of local coordinates in Remark 2.1.1.

One can modify the gluing construction above by choosing a different smooth structure on the space of gluing parameters. In the language of Hofer, Wysocki and Zehnder [13, Appendix],

Definition 2.2.7. A *gluing profile* is a diffeomorphism $\varphi : (0,1] \to [0,\infty)$. The diffeomorphism given by $\varphi(\delta) = -1 + 1/\delta$ will be called the *standard gluing profile*; $\varphi(\delta) = e^{1/\delta} - e$ will be called the *exponential gluing profile*, and $\varphi(\delta) = -\ln(\delta)$ the *logarithmic gluing profile.*

The set of gluing profiles naturally forms a partially ordered set: Write $\varphi_1 \geq \varphi_0$ and say φ_1 is *softer than* φ_0 if $\varphi_1^{-1}\varphi_0$ extends to a diffeomorphism of $[0,1]$. Write $\varphi_1 > \varphi_0$ and say that φ_1 is *strictly softer* than φ_0 if the derivatives of $\varphi_1^{-1}\varphi_0 : [0,1] \to [0,1]$ vanish at 0. The exponential gluing profile, standard gluing profile, and logarithmic gluing profile form a decreasingly soft sequence in this partial order.

Fix a gluing profile φ, and consider once again the gluing construction.

Definition 2.2.8. Given a nodal curve Σ with local coordinates κ near the nodes, and a collection of gluing parameters $\delta = (\delta_1, \ldots, \delta_m)$, the *glued curve* $\Sigma(\delta, \varphi)$ is defined by gluing together small disks:

$$\Sigma^{\delta,\varphi,\kappa} := \left(\bigcup_{i=1}^{k} \Sigma_i - \{w_1^{\pm}, \ldots, w_m^{\pm}\} \right) / \sim \tag{4}$$

where the equivalence relation $\sim$ is given by

$$z \sim (\kappa_i^+)^{-1}(\exp(-\varphi(|\delta_i|) - \sqrt{-1}\arg(\delta_i))/\kappa_i^-(z), \quad ,z \in U_i^-, \quad i = 1, \ldots, m.$$

More generally, given a family $\Sigma_{S_\Gamma} \to S_\Gamma$ of curves of constant combinatorial type Γ and a system of local coordinates near the nodes κ, the construction (4) produces a family of curves $\Sigma^{\kappa,\varphi}_{S^{\kappa,\varphi}} \to S^{\kappa,\varphi}$ where $S^{\kappa,\varphi}$ is the product of S with the space of gluing parameters.

Let Σ be a compact, complex nodal curve. For any gluing profile φ and any collection κ of local coordinates near the nodes, the family $\Sigma^{\kappa,\varphi}_{S^{\kappa,\varphi}} \to S^{\kappa,\varphi}$ is a stratified-smooth strongly universal deformation, since it is so for the standard gluing profile. Let $\kappa = (\kappa_\Gamma)$ be a compatible system of local coordinates near the nodes, for each combinatorial type Γ. Each stratified-smooth universal deformation above defines a *classifying map*

$$S^{\kappa,\varphi}/\operatorname{Aut}(\Sigma) \to \overline{M}_{g,n}, \quad s \mapsto [\Sigma_s] \tag{5}$$

which is a homeomorphism onto its image, possibly after shrinking the parameter space $S^{\kappa,\varphi}$. (To obtain a precise meaning for "classifying map" it is necessary to pass to the stack-theoretic viewpoint, which we do not discuss here.) The maps (5) provide $\overline{M}_{g,n}$ with a compatible set of stratified-smooth orbifold charts, since the transition maps are the identity on the space of gluing parameters by construction, and smooth on each stratum. We denote by $\overline{M}_{g,n}^{\kappa,\varphi}$ the smooth structure on $\overline{M}_{g,n}$ defined by the system of local coordinates κ near the nodes and the gluing profile φ; the use of this smooth structure seems to have been suggested by Hofer. It seems that these smooth structures might depend on the choice of κ, except in the case of the logarithmic gluing profile, in which case one has a canonical smooth structure.

The forgetful maps with respect to these non-standard smooth structures have regularity properties that are worse than those with respect to the standard smooth structure. For $2g + n > 3$ we have forgetful morphisms $f_i : \overline{M}_{g,n} \to \overline{M}_{g,n-1}$ by forgetting the i-th marking and collapsing unstable components. There are two possibilities: a genus zero component with one marking and two nodes is replaced by a point; a genus zero component with two markings and one node is replaced by a single marking. For any gluing profile, the maps f_i are smooth away from the locus where collapsing occurs. We say a local coordinate on a genus zero curve is *standard* if it extends to an isomorphism with the projective line. The forgetful morphism f_i is smooth near the locus of one node, two marking components if the local coordinates are standard and $\delta \mapsto \exp(\varphi(\delta))^{-1}$ is smooth, that is, φ is at least as hard as the logarithmic gluing profile. The forgetful morphism f_i is smooth near the locus of curves containing components with two nodes and one marking if the map $\delta_1, \delta_2 \mapsto \varphi^{-1}(\varphi(\delta_1) + \varphi(\delta_2))$ is smooth. For example, in the logarithmic gluing profile we have $(\delta_1, \delta_2) \mapsto \delta_1\delta_2$, which is smooth, while for the standard gluing profile collapsing a component gives the map $(\delta_1, \delta_2) \mapsto \delta_1\delta_2/(\delta_1 + \delta_2)$ in the local gluing parameters, which is not smooth.

3. Deformations of holomorphic maps from curves

This section reviews the construction of stratified-smooth universal deformations for stable (pseudo)holomorphic maps. The proof relies on a gluing theorem, of the sort given by Ruan-Tian [21]; our approach follows that of McDuff-Salamon [15] who treat the genus zero case. A different set-up for gluing is described in Fukaya-Oh-Ohta-Ono [9], and explained in more detail in Abouzaid [1]. The gluing construction gives rise to charts for the moduli space of regular stable maps.

3.1. **Stable maps.** Let (X, ω) be a compact symplectic manifold and $\mathcal{J}(X)$ the space of compatible almost complex structures on X. Let $J \in \mathcal{J}(X)$.

Definition 3.1.1. A *marked nodal J-holomorphic map* to X consists of a nodal curve Σ, a collection $z = (z_1, \ldots, z_n)$ of distinct, smooth points on Σ, and a J-holomorphic map $u : \Sigma \to X$. An *isomorphism* of marked nodal maps from (Σ_0, z_0, u_0) to (Σ_1, z_1, u_1) is an isomorphism of nodal curves $\psi : \Sigma_0 \to \Sigma_1$ such that $\psi(z_{0,i}) = z_{1,i}$ for $i = 1, \ldots, n$ and $u_1 \circ \psi = u_0$. A marked nodal map (Σ, u, z) is *stable* if it has finite automorphism group or equivalently each component Σ_i of genus zero resp. one for which u_i is constant has at least three resp. one special (nodal or marked) point. The *homology class* of stable map $u : \Sigma \to X$ is $u_*[\Sigma] \in H_2(X, \mathbb{Z})$.

A *continuous family* of J-holomorphic maps over a topological space S is a continuous family of nodal curves $\Sigma_S \to S$ (see Definition 2.2.3) and a continuous map $u_S : \Sigma_S \to X$ which is fiberwise holomorphic. That is, for each $s_0 \in S$ and each nearby combinatorial type Γ we have

(a) a sequence of contractions $\tau_s : \Gamma(\Sigma_{s_0}) \to \Gamma$, constant in $s \in S_\Gamma$;
(b) for every node $\{w_i^\pm\}$ collapsed under τ_s, a pair of local coordinates $\kappa_i^\pm : W_i^\pm \to \mathbb{C}$
(c) for every component $\Sigma_{s_0,i}$ of Σ_{s_0}, a real number $\epsilon_s > 0$ converging to 0 as $s \to s_0$ and maps

$$\phi_{i,s} : \Sigma_{s_0,i} - \cup_{w_k^\pm \in \Sigma_{s_0,i}, \tau_s(w_k^\pm) = \emptyset} (\kappa_k^\pm)^{-1}(B_{\epsilon_s}) \to \Sigma_{s,\tau_s(i)}$$

such that

(a) for any s, the images of the maps $\phi_{i,s}$ cover Σ_s;
(b) for any nodal point $w_k^\pm$ of Σ_s joining components $\Sigma_{s,i^\pm(k)}$, there exists a constant $\lambda_s \in \mathbb{C}^*$ such that $(\kappa_k^+ \circ \phi_{s,i^+(k)}^{-1} \circ \phi_{s,i^-(k)} \circ (\kappa_k^-)^{-1})(z) = \lambda_s z$ where defined, and $\lambda_s \to 0$ as $s \to s_0$;
(c) for any smooth $z \in \Sigma_{s_0,i}$, $\lim_{s \to s_0}(\phi_{i,s}(z)) = z$;
(d) $\phi_{i,s}^* j_{\Sigma_{s,\tau_s(i)}}$ converges to $j_{\Sigma_{s_0,i}}$ uniformly in all derivatives on compact sets;
(e) if z_i is contained in $\Sigma_{s_0,k}$, then $z_i = \lim_{s \to s_0} \phi_{s,k}^{-1}(z_{i,s})$;
(f) $\phi_{i,s}^* u_s$ converges to u_{s_0} uniformly in all derivatives on compact sets.

Remark 3.1.2. It follows from the assumption that $u_S : \Sigma_S \to X$ is continuous that the homology class $u_{s,*}[\Sigma_s]$ is locally constant in $s \in S$. Indeed continuity implies that for s sufficiently close to s_0, u_S is homotopic to a map of the form $v_S \circ \gamma_S$ where $\gamma_s : \Sigma_S \to \Sigma_{s_0}$ is a map to the central fiber Σ_{s_0} which collapses the gluing regions to the node. Since each $\gamma_s = \gamma_S | \Sigma_s$ maps $[\Sigma_s]$ to $[\Sigma_{s_0}]$, the claim follows.

In particular, taking S to be the topological space given as the closure of the set S^* of rational numbers of the form $1/i, i \in \mathbb{Z}_{>0}$, we say that a sequence of holomorphic maps $u_i : \Sigma_i \to X$ *Gromov converges* if it extends to a continuous family over S. To state the Gromov compactness theorem, recall that the *energy* of a map $u : \Sigma \to X$ is

$$E(u) = \frac{1}{2} \int |du|^2.$$

Theorem 3.1.3 (Gromov compactness)**.** *Let X, ω, J be as above. Any sequence $u_i : \Sigma_i \to X$ of stable holomorphic maps with bounded energy has a Gromov convergent subsequence. Furthermore, the limit is unique.*

For references and discussion, see for example [14, Theorem 1.8]. The definition of Gromov convergence passes naturally to equivalence classes of stable maps. A subset C of $\overline{M}_{g,n}(X,d)$ is *Gromov closed* if any sequence in C has a limit point in C, and *Gromov open* if its complement is closed. The Gromov open sets form a topology for which any convergent sequence is Gromov convergent, by an argument using [15, Lemma 5.6.5]. Furthermore, any convergent sequence has a unique limit. Gromov compactness implies that for any $E > 0$, the union of $\overline{M}_{g,n}(X,d)$ over $d \in H_2(X,\mathbb{Z})$ with $(d,[\omega]) < E$ is a compact, Hausdorff space.

Definition 3.1.4. Let X, ω, J be as above. A *stratified-smooth* family of nodal J-holomorphic maps over a space S is a pair (Σ_S, u_S) of a stratified-smooth family of nodal curves $\Sigma_S \to S$ together with a continuous map $u_S : \Sigma_S \to X$ such that the restriction u_s of u to any fiber Σ_s is holomorphic, and the restriction of u_S to any stratum Σ_{S_Γ} is smooth. A *stratified-smooth deformation* of a stable J-holomorphic map (Σ, u) is a germ of a stratified-smooth family (Σ_S, u_S) together with an isomorphism of nodal curves $\iota : \Sigma_0 \to \Sigma$ such that $\iota^* u = u_0$. A deformation (Σ_S, u_S, ι) of (Σ, u) is *versal* if any other (germ of) family of marked, nodal curves $(\Sigma', \Sigma_0) \to (S', 0)$ is induced from a map $\psi : S' \to S$ in the sense that there exists an isomorphism $\phi : \Sigma' \to \Sigma \times_S S'$ in a neighborhood of the central fiber Σ_0, and u' is obtained by composing projection on the first factor with u. A versal deformation is *universal* if the map ϕ above is the unique map inducing the identity on Σ_0.

3.2. Smooth universal deformations of regular stable maps of fixed combinatorial type.

Let $u : \Sigma \to X$ be a stable map. For $p > 2$ define a fiber bundle $\mathcal{E} \to \mathcal{B}$ by

$$\mathcal{B} = \mathcal{J}(\Sigma) \times \operatorname{Map}(\Sigma, X)_{1,p}, \quad \mathcal{E}_{j,u} = \Omega^{0,1}(\Sigma, u^*TX)_{0,p},$$

where the latter is the space of $(0,1)$-forms with respect to the pair (j, J). Consider the Cauchy-Riemann section,

$$\overline{\partial} : \mathcal{B} \to \mathcal{E}, \quad (j,u) \mapsto \overline{\partial}_j u, \quad \overline{\partial}_j u = \frac{1}{2}(\mathrm{d}u \circ j - J_u \circ \mathrm{d}u).$$

Let

$$\operatorname{ev} : \mathcal{B} \to X^{2m}, \quad u \mapsto (u(w_1^-), u(w_1^+), \ldots, u(w_m^-), u(w_m^+))$$

denote the map evaluating at the nodal points. The space of stable maps of type Γ is given as $(\overline{\partial}, \operatorname{ev})^{-1}(0 \times \Delta^m)$ where $\Delta \subset X \times X$ is the diagonal. To obtain a Fredholm map, we quotient by diffeomorphisms of Σ, or equivalently, restrict to a minimal versal deformation $\Sigma_S \to S$ of Σ of fixed type. This means that for each $\zeta \in \operatorname{Def}_\Gamma(\Sigma)$ near 0 we have a complex structure $j(\zeta)$ on Σ, which we may assume agrees with $j = j(0)$ near the nodes. Then the Cauchy-Riemann section induces a map

$$\operatorname{Def}_\Gamma(\Sigma) \times \Omega^0(\Sigma, u^*TX) \to \mathcal{E}.$$

Linearizing the Cauchy-Riemann section, together with the differences at the nodes, gives rise to a Fredholm operator

$$\tilde{D}_u : \mathrm{Def}_\Gamma(\Sigma) \times \Omega^0(\Sigma, u^*TX) \to \Omega^{0,1}(\Sigma, u^*TX) \oplus \bigoplus_{i=1}^m u(w_i^\pm)^*TX$$

(6)
$$\tilde{D}_u(\zeta, \xi) := \left(\pi_\Sigma^{0,1}(\nabla\xi - \frac{1}{2}J(u)\mathrm{d}uDj(\zeta) - \frac{1}{2}J_u(\nabla_\xi J)_u \partial u)), (\xi(w_i^+) - \xi(w_i^-))_{i=1}^m \right)$$

given by the linearized Cauchy-Riemann operator on each component, and the difference of the values of the section at the nodes $w_1^\pm, \ldots, w_m^\pm$. The map $u = (\Sigma, u, z)$ is *regular* if $\tilde{D}_u$ is surjective. This is independent of the choice of representatives $j(\zeta)$: any two such choices $j'(\zeta), j(\zeta)$ are related by a diffeomorphism of Σ. The *space of infinitesimal deformations of* u *of fixed type* is

$$\mathrm{Def}_\Gamma(u) = \ker(\tilde{D}_u)/\operatorname{aut}(\Sigma).$$

The *space of infinitesimal deformations of* u is

$$\mathrm{Def}(u) = \mathrm{Def}_\Gamma(u) \oplus \bigoplus_{i=1}^m T_{w_j^+}\Sigma \otimes T_{w_j^-}\Sigma$$

where Γ is the type of u.

Theorem 3.2.1. *Let* X, ω, J *be as above. A regular marked nodal* J*-holomorphic map* $u = (\Sigma, u, z)$ *admits a strongly universal deformation* (Σ_S, u_S, z_S) *with parameter space* $S \subset \mathrm{Def}_\Gamma(u)$ *of fixed type if and only if* u *is stable.*

Proof. Let (Σ, u) be a stable map to X and $\Sigma_S \to S \subset \mathrm{Def}_\Gamma(\Sigma)$ a minimal versal deformation of Σ of fixed type constructed in (2). We may write any map C^0-close to u as $\exp_u(\xi)$ for some $\xi \in \Omega^0(\Sigma, u^*TX)$. Let $\Psi_u(\xi) : u^*TX \to \exp_u(\xi)^*TX$ denote parallel transport along geodesics with respect to the Hermitian connection $\tilde{\nabla} = \nabla - \frac{1}{2}J(\nabla J)$; here ∇ is the Levi-Civita connection, see [15, Chapter 2]. This defines an isomorphism

$$\Psi_u(\xi)^{-1} : \Omega_j^{0,1}(\Sigma, \exp_u(\xi)^*TX) \to \Omega_j^{0,1}(\Sigma, u^*TX). \tag{7}$$

where subscript j denotes the space of $0,1$-forms taken with respect to the complex structure j on Σ. There is an isomorphism of $\Omega_{j(\zeta)}^{0,1}(\Sigma, u^*TX)$ with $\Omega_j^{0,1}(\Sigma, u^*TX)$ given by composing the inclusion

$$\Omega_{j(\zeta)}^{0,1}(\Sigma, u^*TX) \to \Omega^1(\Sigma, u^*TX)_\mathbb{C} = \Omega^1(\Sigma; u^*TX) \otimes_\mathbb{R} \mathbb{C}$$

with the projection $\Omega^1(\Sigma; u^*TX) \otimes_\mathbb{R} \mathbb{C} \to \Omega_j^{0,1}(\Sigma, u^*TX)$. We denote by

$$\Psi_j(\zeta) : \Omega_j^{0,1}(\Sigma, u^*TX) \to \Omega_{j(\zeta)}^{0,1}(\Sigma, u^*TX)$$

the resulting map; one can think of this as a connection over the space of complex structures on Σ on the bundle whose fiber is the space of $0,1$-forms with respect to

$j(\zeta)$. By composing $\Psi_u(\xi)^{-1}$ and $\Psi_j(\zeta)^{-1}$ we obtain an identification

$$\Psi_{j,u}(\zeta,\xi)^{-1} : \Omega^{0,1}_{j(\zeta)}(\Sigma, \exp_u(\xi)^* TX) \to \Omega^{0,1}_j(\Sigma, u^*TX). \tag{8}$$

Define

$$\mathcal{F}_u : \mathrm{Def}_\Gamma(\Sigma) \times \Omega^0(\Sigma, u^*TX) \to \Omega^{0,1}_j(\Sigma, u^*TX)$$
$$(\zeta,\xi) \mapsto \Psi_{j,u}(\zeta,\xi)^{-1}(\overline{\partial}_{j(\zeta)}(\exp_u(\xi))).$$

The operator $\tilde{D}_u$ is the linearization of $\mathcal{F}_u$. The implicit function theorem implies that if u is regular then the zero set of $\mathcal{F}_u$ is modelled locally on a neighborhood of 0 in $\ker(\tilde{D}_u)$. Furthermore, by elliptic regularity the zero set consists entirely of smooth J-holomorphic maps [15, Section B.4]. Thus we obtain a smooth family of stable maps in a neighborhood of 0 in $\ker(\tilde{D}_u)$. The action of $\mathrm{Aut}(u)$ on the space of stable maps with domain Σ induces an inclusion of the Lie algebra $\mathrm{aut}(u)$ into $\ker(\tilde{D}_u)$. Restricting to $\mathrm{Def}_\Gamma(u)$, identified with a complement of $\mathrm{aut}(\Sigma)$ (that is, a slice for the $\mathrm{Aut}(u)$ action) gives a family $(\Sigma_S, u_S) \to S \subset \mathrm{Def}_\Gamma(u)$ of fixed type. The family (Σ_S, u_S), together with the canonical identification ι of the central fiber with Σ, is a universal smooth deformation of fixed type. Indeed, another smooth family $(\Sigma_{S'}, u_{S'})$ over a base S' is in particular a deformation of the underlying curve. After shrinking S', each fiber of $(\Sigma_{s'}, u'_{s'})$ corresponds to a zero of $\mathcal{F}_u$, and so lies in the image of the map given by the implicit function theorem. The uniqueness part of the implicit function theorem gives a smooth map $\psi : S' \to \mathrm{Def}_\Gamma(u)$ and an identification $\Sigma_{S'} \to \psi^*\Sigma_S$. Any two such maps inducing the same map on the central fiber are close in a neighborhood of the central fiber. Since the automorphism groups of the central fibre are discrete, any automorphism group is discrete. Thus any two such automorphisms defined in a neighborhood of the central fiber, and equal on the central fiber must be equal in a neighborhood of the central fiber. This shows that the identification is unique, so that the deformation given by the gluing construction is universal.

If u is not stable, then it has no universal deformation since the identification with the central fiber is unique only up to a continuous family of automorphisms. □

Let $M^{\mathrm{reg}}_{g,n,\Gamma}(X,d)$ denote the moduli space of regular stable maps of combinatorial type Γ. A family u_S over $S \subset \mathrm{Def}_\Gamma(u)$ induces a map

$$S \to M^{\mathrm{reg}}_{g,n,\Gamma}(X,d), \;\; s \mapsto [u_s] \tag{9}$$

where $[u_s]$ denotes the isomorphism class of $u_s : \Sigma_s \to X$.

Theorem 3.2.2. *For any g,n,d and combinatorial type Γ with m nodes, $M^{\mathrm{reg}}_{g,n,\Gamma}(X,d)$ has the structure of a smooth orbifold of dimension $(1-g)(\dim(X)-6)+2(c_1(TX),d)-2m+2n$, with tangent space at $[u]$ isomorphic to $\mathrm{Def}_\Gamma(u)$.*

Proof. By Theorem 3.2.1, the maps (9) for families giving universal deformations are homeomorphisms onto their image and provide compatible charts. The dimension formula follows from Riemann-Roch: The index of $\tilde{D}_u$ may be deformed to a complex linear operator by homotoping the zero-th order terms (which define a compact operator) to zero. □

3.3. **Constructing stratified-smooth deformations of varying type.** The main result of this section is Theorem 1.0.1, which is probably well-known, cf. [19], [21], but for which we could not find an explicit reference. The theorem itself will not be used, but the estimates involved in the proof will be needed later for the corresponding result for vortices. The proof uses a gluing construction for holomorphic maps, which produces from a smooth family of holomorphic maps of fixed type, a stratified-smooth family of maps of varying type.

Step 1: Approximate Solution

Definition 3.3.1. Let Σ be a compact, complex nodal curve. A *gluing datum* for Σ consists of

(a) a collection of gluing parameters $\delta = (\delta_1, \ldots, \delta_m)$ in the bundle I of (3);
(b) local coordinates $\kappa_j^\pm$ near the nodes $w_j^\pm$ for $j = 1, \ldots, m$;
(c) a parameter ρ which describes the width of the annulus on which the gluing of maps is performed;
(d) a gluing profile φ, see Definition 2.2.7;
(e) a smooth cutoff function

$$\alpha : \mathbb{C} \to [0,1], \quad \alpha(z) = \left\{ \begin{array}{ll} 0 & |z| \le 1 \\ 1 & |z| \ge 2 \end{array} \right\}. \tag{10}$$

We first treat the case that φ is the standard gluing profile. Let a gluing datum be given, and Σ^δ denote the glued curve from (2). Let $u : \Sigma \to X$ be a holomorphic map. Near each node w_k let $i^\pm(k)$ denote the components on either side of w_k. In the neighborhoods $U_k^\pm$ (assuming they have been chosen sufficiently small) define maps

$$\xi_k^\pm : U_k^\pm \to T_{x_k} X, \quad u_{i^\pm(k)}(z) = \exp_{x_k}(\xi_k^\pm(z))$$

where $x_k = u(w_k)$ and $\exp_{x_k} : T_{x_k} X \to X$ denotes geodesic exponentiation. Given a holomorphic map $u : \Sigma \to X$, and a gluing datum $(\delta, \kappa, \rho, \varphi, \alpha)$ define the *pre-glued map* by interpolating between the maps on the various components using the given cutoff function and local coordinates: $u^\delta = u(z)$ for $z \notin \cup_k U_k^\pm$ and otherwise

$$u^\delta(z) = \exp_{x_k}(\alpha(\kappa_k^\pm(z)/\rho|\delta_k|^{1/2})\xi_k^\pm(z)) \quad z \in U_k^\pm. \tag{11}$$

Remark 3.3.2. The same formula but with domain Σ (not the glued curve) defines an *intermediate map* $u_0^\delta : \Sigma \to X$ which is constant near the nodes. The right inverse of $\tilde{D}_{u_0^\delta}$ will be used in the gluing construction.

First we estimate the failure of u^δ to satisfy the Cauchy-Riemann equation. Define on Σ^δ the C^0-metric g by the identification

$$\Sigma^\delta = \Sigma - \bigcup_{k,\pm} \kappa_k^\pm(B_{|\delta_k|^{1/2}}(0)) / \left(\kappa_k^+(\partial B_{|\delta_k|^{1/2}}(0)) \sim \kappa_k^-(\partial B_{|\delta_k|^{1/2}}(0)) \right) \tag{12}$$

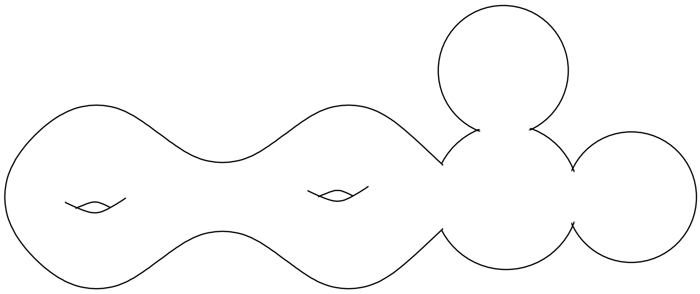

Figure 1. Continuous metric on a glued curve

using a Kähler metric on Σ, see Figure 1. The generalized Sobolev spaces $W^{l,p}$ with respect to this metric are defined for $p \geq 1$ and integers $l \in \{0, 1\}$, see [2] or [3]. For any vector bundle E we denote by $\Omega(\Sigma^\delta, E)_{l,p,\delta}$ the space of $W^{l,p}$ forms with values in E. If $p = \infty$ the norm is independent of δ and we drop it from the notation. Let $\| \cdot \|_{k,p,\delta}$ denote the Sobolev $W^{k,p}$-norm on $\Omega^0(\Sigma^\delta, u^{\delta,*}TX)$ defined using the δ-dependent metric (12).

Proposition 3.3.3. *Suppose that $u : \Sigma \to X$ is a stable map, and $u^\delta : \Sigma^\delta \to X$ is the pre-glued map defined in* (11)*, defined for δ sufficiently small. There is a constant c and an $\epsilon > 0$ such that if $\|\delta\| < \epsilon, \rho > 1/\epsilon$, and $|\delta_k|^2 \rho < \epsilon, \quad k = 1, \ldots, m$ then*

$$\|\overline{\partial} u^\delta\|^p_{0,p,\delta} \leq c \sum_{k=1}^m (|\delta_k|^{1/2}\rho)^2.$$

Proof. Compare with McDuff-Salamon [15, Chapter 10]. The error term $\overline{\partial} u(\delta)$ can be estimated by terms of two types; those involving derivatives of the cutoff functions and those involving derivatives of the map ξ_k. The derivative of $\exp_{x_k}$ is approximately the identity near the node. The derivatives of α grow like $1/\rho|\delta_k|^{1/2}$, while the norm of $\xi_k^\pm$ is bounded by a constant times $\rho|\delta_k|^{1/2}$ on the gluing region. The term involving the derivatives of α is bounded and supported on a region of area less than $\pi\rho^2|\delta_k|$ for each node. The derivatives of $\xi_k^\pm$ are also uniformly bounded, and the area bound gives the required estimate. □

Let $\Sigma_S \to S$ with $S \subset \mathrm{Def}_\Gamma(\Sigma)$ be a family giving a minimal versal deformation of Σ of fixed type, and $\Sigma_{S_\delta(\delta)} \to S_\delta \subset \mathrm{Def}(\Sigma^\delta)$ a family giving a minimal versal deformation of Σ^δ. The gluing construction (2) applied to the family Σ_S produces a map

$$\mathrm{Def}_\Gamma(\Sigma) \to \mathcal{J}(\Sigma^\delta), \quad \zeta \mapsto j^\delta(\zeta) \tag{13}$$

which maps any deformation of the original curve to the corresponding deformation of the glued curve. In other words, any variation of complex structure on Σ of fixed type induces a variation of complex structure on Σ^δ. Similarly, for any $\xi \in$

$\Omega^0(\Sigma, u^*TX)$ we obtain an element $\xi^\delta \in \Omega^0(\Sigma^\delta, u^*TX)$ by interpolating between the components as in (**??**).

Proposition 3.3.4. *Suppose that u, u^δ are as above, and $(\zeta,\xi) \in \mathrm{Def}_\Gamma(u)$. There is a constant c and an $\epsilon > 0$ such that if $\|\delta\| < \epsilon, \rho > 1/\epsilon$, $\|\zeta\| + \|\xi\|_{1,p} \le \epsilon$, and $|\delta_k|^2\rho < \epsilon$ for $k = 1,\dots,m$ then*

$$\|\overline{\partial}_{j^\delta(\zeta)} \exp_{u^\delta}(\xi^\delta)\|^p_{0,p,\delta} \le c \sum_{k=1}^m (|\delta_k|^{1/2}\rho)^2.$$

Step 2: Uniformly bounded right inverse

We wish to show that the map in Proposition 3.3.4 can be corrected to obtain a holomorphic map. Define

$$\mathcal{F}^\delta_u : \mathrm{Def}_\Gamma(\Sigma) \times \Omega^0(\Sigma^\delta, u^{\delta,*}TX) \to \Omega^{0,1}(\Sigma^\delta, u^{\delta,*}TX) \tag{14}$$
$$(\zeta,\xi) \mapsto \Psi_{j,u^\delta}(\zeta,\xi)^{-1}(\overline{\partial}_{j^\delta(\zeta)}(\exp_{u^\delta}(\xi))).$$

Here the operator Ψ_{j,u^δ} is as in (8). Let $\tilde{D}^\delta_u(\xi)$ be the associated linear operator, that is, the linearization of (14) at ξ. This operator naturally extends to a map from Sobolev $1,p$-completion of the second factor of the domain to the $0,p$-completion of the codomain. We denote by $\tilde{D}^\delta_u := \tilde{D}^\delta_u(0)$. We will construct an approximate inverse

$$T_\delta : \Omega^{0,1}(\Sigma^\delta, u^{\delta,*}TX) \to \mathrm{Def}_\Gamma(\Sigma) \oplus \Omega^0(\Sigma^\delta, u^{\delta,*}TX) \tag{15}$$

to $\tilde{D}^\delta_u$. The construction depends on a carefully chosen cutoff function:

Lemma 3.3.5. [15, Section 10.3] *For any $\delta > 0, \rho > 1$ there exists a function $\beta_{\rho,\delta} : \mathbb{R}^2 \to [0,1]$ that satisfies*

$$\beta_{\rho,\delta}(z) = \begin{cases} 0 & |z| \le \sqrt{\delta/\rho} \\ 1 & |z| \ge \sqrt{\delta\rho} \end{cases}$$

and for all $\xi \in W^{1,p}(B_{\rho|\delta_k|})$ satisfying $\xi(0) = 0$

$$\|(\nabla\beta_{\rho,\delta})\xi\|_{0,p} \le c\log(\rho^2)^{-1+1/p}\|\xi\|_{1,p}, \quad \|\beta_{\rho,\delta}\|_{1,2} \le C\log(\rho^2)^{-1/2}. \tag{16}$$

Recall the map u_0 from Remark 3.3.2.

Lemma 3.3.6. *For sufficiently small δ there exists a right inverse $Q_{u_0^\delta}$ of $\tilde{D}_{u_0^\delta}$ with image the L^2-perpendicular of the kernel of $\tilde{D}_{u_0^\delta}$.*

Proof. Consider the maps defined by parallel transport using the modified Levi-Civita connection, $\Pi^u_{u_0^\delta} : \Omega^0(\Sigma, (u_0^\delta)^*TX) \to \Omega^0(\Sigma, u^*TX)$ and $\Psi^u_{u_0^\delta} : \Omega^{0,1}(\Sigma, (u_0^\delta)^*TX) \to \Omega^{0,1}(\Sigma, u^*TX)$. The operator $\Psi^u_{u_0^\delta}\tilde{D}_{u_0^\delta}(\Pi^u_{u_0^\delta})^{-1}$ approaches the operator $\tilde{D}_u$ as $\delta \to 0$, c.f. [15, Remark 10.2.2]. The statement of the lemma follows. □

Define the approximate right inverse for $\tilde{D}_u^\delta$ by composing the right inverse $Q_{u_0^\delta}$ with a cutoff and extension operator: $T_\delta := P_\delta Q_{u_0^\delta} K_\delta$, defined as follows. The *cutoff operator*

$$K_\delta : \Omega^{0,1}(\Sigma^\delta, u^{\delta,*}TX)_{0,p,\delta} \to \Omega^{0,1}(\Sigma, u_0^{\delta,*}TX)_{0,p}$$

is defined by

$$(K_\delta(\eta))(z) = \begin{cases} \eta(z) & z \notin \bigcup_{k,\pm} B_{|\delta_k|^{1/2}}(w_k^\pm) \\ 0 & \text{otherwise} \end{cases}.$$

We have $\|K_\delta \eta\|_{0,p} \le \|\eta\|_{0,p,\delta}$ by definition of the $0,p,\delta$ norm. The *extension operator*

$$P_\delta : \mathrm{Def}_\Gamma(\Sigma) \oplus \Omega^0(\Sigma, u_0^\delta * TX)_{1,p,\delta} \to \mathrm{Def}(\Sigma^\delta) \oplus \Omega^0(\Sigma^\delta, u^{\delta,*}TX)_{1,p,\delta}$$

is defined as follows. For each component Σ_i let Σ_i^* denote the complements of small balls around the nodes

$$\Sigma_i^* = \Sigma_i - \bigcup_{l, w_l^\pm \in \Sigma_i} B_{|\delta_l|^{1/2}/\rho}(w_l^\pm)$$

and the inclusion $\pi_i : \Sigma_i^* \to \Sigma^\delta$ induces a map $\pi_{i,*} : \Omega^0(\Sigma_i^*, u_i^* TX) \to \Omega^0(\Sigma^\delta, u^{\delta,*}TX)_{0,p}$. Define

$$P_\delta(\zeta, \xi) = (\zeta^\delta, \xi^\delta)$$

where ζ^δ is the image of ζ under the gluing map (13) and ξ^δ is obtained by patching together the sections ξ; on the gluing region arising from gluing the k-th node w_k the section ξ^δ is given by the sum

$$\pi_{i^+(k),*}\beta_{\rho,\delta_k}(\xi_{i^+(k)} - \xi(w_k)) + \pi_{i^-(k),*}\beta_{\rho,\delta_k}(\xi_{i^-(k)} - \xi(w_k)) + \xi(w_k).$$

Fix a metric $\|\cdot\|$ on the finite-dimensional space $\mathrm{Def}_\Gamma(\Sigma)$ and define

$$\|(\zeta, \xi)\|_{1,p,\delta} = \left(\|\zeta\|^p + \|\xi\|^p_{1,p,\delta} \right)^{1/p}.$$

Proposition 3.3.7. *Let $u : \Sigma \to X$ be a stable map. There exist constants $c, C > 0$ such that if $\|\delta\| < c$ then the approximate inverse T_δ of* (15) *satisfies*

$$\|(\tilde{D}_u^\delta T_\delta - I)\eta\|_{0,p,\delta} \le \frac{1}{2}\|\eta\|_{0,p,\delta}, \quad \|T_\delta\| < C.$$

Proof. By construction T_δ is an exact right inverse for $\tilde{D}_u^\delta$ away from gluing region. In the gluing region the variation of complex structure on the curve vanishes and $D_u^\delta = D_{x_k}$, the standard Cauchy-Riemann operator with values in $T_{x_k}X$. So

$$\begin{aligned} \tilde{D}_u^\delta T_\delta \eta - \eta &= \sum D_{x_k} \beta_{\rho,\delta}(z)(\xi_{i^\pm(k)}(z) - \xi_{i^\pm(k)}(w_k)) \\ &= \sum (D_{x_k} \beta_{2;\rho,\delta}(z)) \xi_{i^\pm(k)}(z) \end{aligned}$$

since $K_\delta \eta = 0$ on $B_{|\delta|^{1/2}}(0)$ in the components adjacent to the node. Since $p > 2$, the $0,p,\delta$-norm of the right hand side is controlled by the ordinary L^p norm. By (16) we have

$$\|\tilde{D}_u^\delta T_\delta \eta - \eta\|_{0,p,\delta} \le \sum_k c |\log(\rho)|^{2/p-2} \|\xi_{i^\pm(k)} - \xi(w_k)\|_{1,p}.$$

The last factor is bounded by $\|K_\delta \eta\|_{0,p}$, by the uniform bound on $Q_{u_0^\delta}$, and hence $\|\eta\|_{0,p,\delta}$, by the uniform bound on K_δ. □

Define a right inverse Q_δ to $\tilde{D}_u^\delta$ by the formula

$$Q_\delta = T_\delta(\tilde{D}_u^\delta T_\delta)^{-1} = \sum_{k\geq 0} T_\delta(\tilde{D}_u^\delta T_\delta - I)^k.$$

The uniform bound on T_δ from Lemma 3.3.7 implies a uniform bound on Q_δ.

Step 3: Uniform quadratic estimate

Proposition 3.3.8. *Let $u : \Sigma \to X$ be a stable map. For every constant $c > 0$ there exist constants $c_0, \delta_0 > 0$ such that if $u \in \mathrm{Map}(\Sigma, X)_{1,p}, \xi \in \Omega^0(\Sigma^\delta, u^{\delta,*}TX)_{1,p}$*

$$\|\mathrm{d}u\|_{0,p} \leq c_0, \quad \|\xi\|_{L_\infty} \leq c_0 \quad \|\zeta\| \leq c_0, \quad \|\delta\| < \delta_0$$

then

$$\|(D\mathcal{F}_u^\delta(\zeta,\xi,\zeta_1,\xi_1) - \tilde{D}_u^\delta)(\zeta_1,\xi_1)\|_{0,p,\delta} \leq c\|\zeta,\xi\|_{1,p,\delta}(\|\zeta_1,\xi_1\|_{1,p,\delta}.$$

Here $D\mathcal{F}_u^\delta(\zeta,\xi,\zeta_1,\xi_1)$ denotes the derivative evaluated at ζ,ξ, applied to ζ_1,ξ_1. We use similar notation throughout the discussion. The proof uses uniform estimates on Sobolev embedding:

Lemma 3.3.9. *There exists a constant $c > 0$ independent of δ such that the embedding*

$$\Omega^0(\Sigma^\delta, u^{\delta,*}TX)_{1,p,\delta} \to \Omega^0(\Sigma^\delta, u^{\delta,*}TX)_{0,\infty}$$

has norm less than c.

Proof. One writes the Sobolev norms as a contribution from each component of the curve Σ. Then on each piece, the metric near the boundary is uniformly comparable with the flat metric. The claim then follows from [2, Chapter 4] which shows that the constants in the Sobolev embeddings depend only on the dimensions of the cone in the cone condition. □

Proof of Proposition. For simplicity we assume a single gluing parameter δ. Let $\Psi_u^{\delta,x}(\zeta,\xi) : \Lambda^{0,1}T_z^*\Sigma^\delta \otimes T_xX \to \Lambda^{0,1}_{j^\delta(\zeta)}T_z^*\Sigma \otimes T_{\exp_x(\xi)}X$ denote pointwise parallel transport as in (7) using the parallel transport using the modified Levi-Civita connection, and projecting onto the $0,1$-part of the form defined using the complex structure $j^\delta(\zeta)$ obtained from gluing $j(\zeta)$, see (13). Let

$$\Theta_u^{\delta,x}(\zeta,\xi,\zeta_1,\xi_1;\eta) = \tilde{\nabla}_t \Psi_u^{\delta,x}(\zeta + t\zeta_1, \xi + t\xi_1)\eta.$$

For ξ,η sufficiently small there exists a constant c such that

$$|\Theta_u^{\delta,x}(\zeta,\xi,\zeta_1,\xi_1;\eta)| \leq c\|\xi,\zeta\|\|\xi_1,\zeta_1\|\|\eta\| \tag{17}$$

where the norms on the right-hand side are any norms on the finite dimensional vector spaces $T_\Sigma M_{g,n,\Gamma}$ and T_xX. This estimate is uniform in δ, for the variation in complex structure vanishes in a neighborhood of the nodes. Differentiate the equation $\Psi_u^\delta(\zeta,\xi)\mathcal{F}_u^\delta(\zeta,\xi) = \overline{\partial}_{j^\delta(\zeta)}(\exp_{u^\delta}(\xi)))$ with respect to (ζ_1,ξ_1) to obtain

$$\Theta_{u^\delta}(\zeta,\xi,\zeta_1,\xi_1,\mathcal{F}_u^\delta(\zeta,\xi)) + \Psi_u^\delta(\zeta,\xi)(D\mathcal{F}_u^\delta(\zeta,\xi,\zeta_1,\xi_1)) = \tilde{D}_u^\delta(\xi, Dj^\delta(\zeta,\zeta_1), D\exp_{u^\delta}(\xi,\xi_1)). \tag{18}$$

Using the pointwise inequality

$$|\mathcal{F}_u^\delta(\zeta,\xi)| < c|\mathrm{d}\exp_{u^\delta}(\xi)| < c(|\mathrm{d}u^\delta| + |\nabla\xi|)$$

for ζ,ξ sufficiently small, the estimate (17) on Φ produces a pointwise estimate

$$|(\Psi_u^\delta)^{-1}(\xi)\Theta_u^\delta(\zeta,\xi,\zeta_1,\xi_1,\mathcal{F}_u^\delta(\zeta,\xi))| \le c(|\mathrm{d}u^\delta| + |\nabla\xi|)\,|(\xi,\zeta)|\,|(\xi_1,\zeta_1)|.$$

Hence

$$\|\Psi_{u^\delta}^{-1}(\xi)\Theta_u^\delta(\zeta,\xi,\zeta_1,\xi_1,\mathcal{F}_u^\delta(\zeta,\xi))\|_{0,p} \le c(1 + \|\mathrm{d}u^\delta\|_{0,p} + \|\nabla\xi\|_{0,p})\|(\xi,\zeta)\|_{0,\infty}\|(\xi_1,\zeta_1)\|_{0,\infty}. \tag{19}$$

It follows that

$$\|\Psi_u^\delta(\xi)^{-1}\Theta_u^\delta(\zeta,\xi,\zeta_1,\xi_1,\mathcal{F}_u^\delta(\zeta,\xi))\|_{0,p} \le c\|(\xi,\zeta)\|_{1,p}\|(\xi_1,\zeta_1)\|_{1,p} \tag{20}$$

since the $W^{1,p}$ norm controls the L^∞ norm by Lemma 3.3.9.

We next show that there exists a constant $c > 0$ such that uniformly in δ,

$$\|\Psi_{u^\delta}(\xi)^{-1}\tilde{D}_u^\delta(\xi, Dj^\delta(\zeta,\zeta_1), D\exp_{u^\delta}(\xi,\xi_1)) - \tilde{D}_u^\delta(\zeta_1,\xi_1)\|_{0,p} \le c\|\zeta,\xi\|_{1,p}\|\zeta_1,\xi_1\|_{1,p}. \tag{21}$$

Indeed differentiate (14) to obtain

$$\tilde{D}_u^\delta(\xi, Dj^\delta(\zeta,\zeta_1), D\exp_{u^\delta}(\xi,\xi_1)) = \nabla^{0,1}_{j^\delta(\zeta)} D\exp_{u^\delta}(\xi,\xi_1) - \frac{1}{2}\pi^{0,1}_{j^\delta(\zeta)} J_{\exp_{u^\delta}(\xi)}\mathrm{d}\exp_{u^\delta}(\xi) Dj^\delta(\zeta,\zeta_1) - J_{\exp_{u^\delta}(\xi)}(\nabla_{D\exp_u(\xi,\xi_1)} J_{\exp_{u^\delta}(\xi)})\partial\exp_{u^\delta}(\xi). \tag{22}$$

Hence

$$\tilde{D}_u^\delta(\xi, Dj^\delta(\zeta,\zeta_1), D\exp_{u^\delta}(\xi,\xi_1)) - \Psi_{u^\delta}(\xi)\tilde{D}_u^\delta(\zeta_1,\xi_1) = \Pi_1 + \Pi_2 + \Pi_3$$

where the three terms Π_1,Π_2,Π_3 are

$$\Pi_1 = \nabla^{0,1}_{j^\delta(\zeta)} D\exp_{u^\delta}(\xi,\xi_1) - \Psi_{u^\delta}(\xi)\nabla^{0,1}_{j^\delta(0)}\xi_1$$

$$\Pi_2 = -\frac{1}{2}\pi^{0,1}_{j^\delta(\zeta)} J_{\exp_u(\xi)}\mathrm{d}\exp_{u^\delta}(\xi)Dj^\delta(\zeta,\zeta_1) + \Psi_{u^\delta}(\xi)\pi^{0,1}_{j^\delta(0)}\frac{1}{2}J_{u^\delta}\mathrm{d}u^\delta Dj^\delta(0,\zeta_1)$$

$$\Pi_3 = -\frac{1}{2}J_{\exp_u(\xi)}(\nabla^{0,1}_{D\exp_{u^\delta}(\xi,\xi_1)} J_{\exp_{u^\delta}(\xi)})\partial_{j^\delta(\zeta)}\exp_{u^\delta}(\xi) + \frac{1}{2}\Psi_{u^\delta}(\xi)J_{u^\delta}(\nabla_{\xi_1}J_u)\partial_{j^\delta(0)}u^\delta.$$

The first difference has norm given by

$$\begin{aligned} &|\pi^{0,1}_{j^\delta(\zeta)}(\nabla(D\exp_{u^\delta}(\xi,\xi_1)) - \Psi_{u^\delta}(\xi)\nabla\xi_1)| \\ &\le |\pi^{0,1}_{j^\delta(\zeta)}(\nabla(D\exp_{u^\delta}(\xi,\xi_1)) - D\exp_{u^\delta}(\xi,\nabla\xi_1))| + |\pi^{0,1}_{j^\delta(\zeta)}(D\exp_{u^\delta}(\xi,\nabla\xi_1) - \Psi_{u^\delta}(\xi)\nabla\xi_1)| \\ &\le c|\nabla\xi||\xi_1| + c(|\zeta| + |\xi|)|\nabla\xi_1| + c|\mathrm{d}u^\delta||\xi||\xi_1|. \end{aligned} \tag{23}$$

We write for the second difference

$$
\begin{aligned}
(24)\quad & |\pi^{0,1}_{j^\delta(\zeta)}(J_{\exp_{u^\delta}(\xi)} \mathrm{d}\exp_{u^\delta}(\xi) Dj^\delta(\zeta,\zeta_1) - \Psi_{u^\delta}(\xi) J_{u^\delta} \mathrm{d}u^\delta Dj^\delta(0,\zeta_1))| \\
& \leq |\pi^{0,1}_{j^\delta(\zeta)}(J_{\exp_{u^\delta}(\xi)} \mathrm{d}\exp_{u^\delta}(\xi) Dj^\delta(\zeta,\zeta_1) - J_{\exp_{u^\delta}(\xi)} \Psi_{u^\delta}(\xi) \mathrm{d}u^\delta Dj^\delta(0,\zeta_1))| + \\
& \quad |\pi^{0,1}_{j^\delta(\zeta)}(J_{\exp_{u^\delta}(\xi)} \Psi_{u^\delta}(\xi) \mathrm{d}u^\delta Dj^\delta(0,\zeta_1) - \Psi_{u^\delta}(\xi) J_u \mathrm{d}u^\delta Dj^\delta(0,\zeta_1))| \\
& \leq c(|\zeta| + |\xi| + |\mathrm{d}u^\delta| + |\nabla\xi|)|\zeta_1|.
\end{aligned}
$$

The third term can be estimated pointwise by

$$
\begin{aligned}
& |J_{\exp_{u^\delta}(\xi)}(\nabla_{D\exp_{u^\delta}(\xi,\xi_1)} J_{\exp_{u^\delta}(\xi)})\partial_{j^\delta(\zeta)} \exp_{u^\delta}(\xi) - \Psi_{u^\delta}(\xi) J_{u^\delta}(\nabla_{\xi_1} J_{u^\delta})\partial_{j^\delta(0)} u^\delta| \\
\leq\ & |J_{\exp_{u^\delta}(\xi)}(\nabla_{D\exp_u(\xi,\xi_1)} J_{\exp_{u^\delta}(\xi)})\partial_{j^\delta(\zeta)} \exp_{u^\delta}(\xi) - J_{\exp_{u^\delta}(\xi)}(\nabla_{D\exp_{u^\delta}(\xi,\xi_1)} J_{\exp_{u^\delta}(\xi)})\partial_{j^\delta(0)} \exp_{u^\delta}(\xi)| \\
& + |J_{\exp_{u^\delta}(\xi)}(\nabla_{D\exp_u(\xi,\xi_1)} J_{\exp_{u^\delta}(\xi)})\partial_{j^\delta(0)} \exp_{u^\delta}(\xi) - \Psi_{u^\delta}(\xi) J_{u^\delta}(\nabla_{\xi_1} J_{u^\delta})\partial_{j^\delta(0)} u^\delta| \\
& \leq c|\zeta|(|\mathrm{d}u^\delta| + |\nabla\xi|)|\xi_1| + c(|\mathrm{d}u^\delta| + |\nabla\xi|)(|\xi_1|)
\end{aligned}
$$

for ξ sufficiently small. Combining these estimates and integrating, using the $0, p, \delta$-norms on $\mathrm{d}u$, $\nabla\xi, \nabla\xi_1$ and the L^∞ norms on the other factors and Lemma 3.3.9, completes the proof. □

Step 4: Implicit Function Theorem

For any $(\zeta_0, \xi_0) \in \mathrm{Def}_\Gamma(u)$ we denote by ζ_0^δ the deformation of Σ^δ defined in (13) and by ξ_0^δ the section of $u^{\delta,*}TX$ defined as in (11).

Theorem 3.3.10. *Let $u : \Sigma \to X$ be a stable map. There exist constants $\epsilon_0, \epsilon_1 > 0$ such that for any $(\zeta_0, \xi_0, \delta) \in \ker \tilde{D}_u \times \mathbb{R}^m$ of norm at most ϵ_0, there is a unique $(\zeta_1, \xi_1) = (\tilde{D}_u^\delta)^* \eta_1$ of norm at most ϵ_1 such that the map $\exp_{u^\delta}(\xi_0^\delta + \xi)$ is $j^\delta(\zeta_0 + \zeta_1)$-holomorphic, and depends smoothly on ζ_0, ξ_0.*

Proof. The first claim is an application of the quantitative version of the implicit function theorem (see for example [15, Appendix A.3]) using the uniform error bound from Proposition 3.3.3, uniformly bounded right inverse from Proposition 3.3.7, and uniform quadratic estimate from Proposition 3.3.8. □

Step 5: Rigidification

In the previous step we have constructed a family of stable maps which we will show eventually gives rise to a parametrization of all nearby stable maps. A more natural way of parametrizing nearby stable maps involves examining the intersections with a family of codimension two submanifolds. For example, this construction of charts is that given in the algebraic geometry approach of Fulton-Pandharipande [10]. In order to carry this out in the symplectic approach, we study the differentiability of the evaluation maps. Let $u_S : \Sigma_S \to X$ over a parameter space $S \subset \mathrm{Def}(u)$ be the family of maps defined in the previous step. The following is similar to [9, Lemma A1.59].

Theorem 3.3.11. *If u_S is constructed using the exponential gluing profile φ and $U \subset \Sigma$ is an open neighborhood of the nodes then the map $(z,s) \mapsto u_s(z)$ is differentiable on a neighborhood of $(\Sigma - U) \times \{0\}$.*

Proof. For simplicity, we assume that there is a single gluing parameter δ. Differentiability for δ is studied in McDuff-Salamon [15, Section 10.6]. The discussion in our case is somewhat easier, because we use a fixed right inverse in the gluing construction. Given $(\zeta,\xi) \in \mathrm{Def}_\Gamma(\Sigma) \times \Omega^0(\Sigma^\delta, (u^\delta)^*TX)$, we constructed a unique correction (ζ_1, ξ_1) in the image of the right inverse such that $\overline{\partial}_{j^\delta(\zeta_0+\zeta_1)} \exp_{u^\delta}(\xi_0 + \xi_1) = 0$. For δ fixed, (ζ_1,ξ_1) depends smoothly on (ζ_0,ξ_0), by the implicit function theorem. Hence the evaluation at $z \in \Sigma - U$ also depends smoothly on (ζ_0,ξ_0).

The computation of the derivative with respect to the gluing parameter is complicated by the fact that for each δ a different implicit function theorem is applied to obtain the correction. Let $\tilde{D}_\delta = D\mathcal{F}_{u^\delta}$. Differentiating the equation $\mathcal{F}_{u^\delta}(\zeta_0^\delta + \zeta_1, \xi_0^\delta + \xi_1) = 0$ with respect to δ gives

$$\tilde{D}_\delta \left(\frac{d}{d\delta}\zeta_1, D\exp_{u^\delta}(\xi_0^\delta + \xi_1, 0, \frac{d}{d\delta}\xi_1) \right) = -\tilde{D}_\delta \left(0, D\exp_{u^\delta}(\xi_0^\delta + \xi_1, \frac{d}{d\delta}u^\delta, 0) \right).$$

From (11) we have in the gluing region,

$$\begin{aligned} \frac{d}{d\delta}\overline{\partial} u^\delta &= \frac{d}{d\delta}\overline{\partial}\exp_x(\left(\alpha(\rho\varphi^{-1/2}|z|)\xi(z))\right) \\ &= D\exp_x\left(\alpha(\rho\varphi^{-1/2}|z|)\xi(z), \alpha'(\rho\varphi^{-1/2}|z|)|z|\xi(z)\frac{d}{d\delta}\varphi^{-1/2}\rho\right)^{0,1}. \end{aligned}$$

Hence there exists a constant C depending on ρ,α but not on δ such that

$$\left|\frac{d}{d\delta}\overline{\partial} u^\delta\right| \le C\left|\frac{d}{d\delta}\varphi^{-1/2}\right|. \tag{25}$$

Now $\mathrm{d}\varphi$ is given by

$$\mathrm{d}(e^{1/\delta} - e)^{-1/2} = (1/2)(e^{1/\delta} - e)^{-3/2} e^{1/\delta}\delta^{-2}\mathrm{d}\delta = (1/2)(e^{1/3\delta} - e^{-2/3\delta+1})^{-3/2}\delta^{-2}\mathrm{d}\delta.$$

For δ small, this is less than $\frac{1}{2}e^{-1/2\delta}\delta^{-2}$. Integrating and using the pointwise estimate (25) we obtain for some constant $C > 0$,

$$\left\|\frac{d}{d\delta}\overline{\partial} u_s^\delta\right\|_{0,p} \le Ce^{-1/2\delta}\delta^{-2} \le Ce^{-1/\delta}$$

for sufficiently small δ. Now the uniform quadratic estimates imply that $\tilde{D}_\delta = D\mathcal{F}_{u^\delta}(\zeta,\xi)$ is uniformly bounded from below on the right inverse of $\tilde{D}_u^\delta = D\mathcal{F}_u^\delta(0,0)$, for (ζ_0,ξ_0) sufficiently small. It follows that

$$\left\|(\frac{d}{d\delta}\zeta_1, \frac{d}{d\delta}\xi_1)\right\|_{1,p} \le Ce^{-1/\delta}$$

for ζ_0, ξ_0, δ sufficiently small as well. Hence the same is true for the evaluation $\frac{d}{d\delta}\xi_1(z)$ for $z \in \Sigma - U$. In particular, $\lim_{\delta\to 0}(\partial_\delta \exp_{u^\delta}(\xi_0^\delta + \xi_1))(z) = 0$. It follows that the differential of the evaluation map has a continuous limit as $\delta \to 0$, which completes the proof of the Theorem. □

Using the evaluation maps in the previous step, we construct embeddings of the families constructed above into suitable moduli spaces of stable marked curves, given by adding additional marked points which map to fixed submanifolds in X. A codimension two submanifold $Y \subset X$ is *transverse* to $u : \Sigma \to X$ if u meets Y transversally in a single point $u(z)$.

Definition 3.3.12. Let $u : \Sigma \to X$ be a stable map. Given any family $Y = (Y_1, \ldots, Y_\ell)$ of codimension two submanifolds transverse to u and a family Σ_S, u_S, z_S with parameter space S of an n-marked stable map (Σ, u, z), the *rigidified family* of $n + \ell$-marked nodal surfaces is defined by

$$\Sigma_S^{Y,u} := (\Sigma_S, (z_{1,S}, \ldots, z_{n+\ell,S})) \to S, \qquad u_s(z_{n+i,s}) \in Y_i. \tag{26}$$

Proposition 3.3.13. *Let u_S be a family of stable maps over a parameter space $S \subset \operatorname{Def}(u)$ given by the gluing construction using a gluing profile φ and system of coordinates κ. Suppose that the evaluation map $\operatorname{ev} : (\Sigma - U) \times S \to X$ is C^1, and that the rigidified family has stable underlying curves. Then the rigidified family of curves $\Sigma_S^{Y,u}$ is C^1 with respect to the gluing profile and local coordinates, that is, the map $S \mapsto \overline{M}_{g,n+l}, \quad s \mapsto \Sigma_s^{Y,u}$ is C^1 with respect to the smooth structure defined by φ, κ.*

Proof. By the implicit function theorem for C^1 maps and differentiability of evaluation maps from the previous subsection. □

Definition 3.3.14. Let Y, u be as above. The pair (Y, u) is *compatible* if

(a) each Y_j intersects u transversally in a single point $z_j \in \Sigma$;
(b) if $\xi \in \ker(\tilde{D}_u)$ satisfies $\xi(z_{n+j}) \in T_{u(z_{n+j})} Y_j$ for $j = 1, \ldots, l$ then $\xi = 0$;
(c) the curve Σ marked with the additional points $z_{n+1}, \ldots, z_{n+\ell}$ is stable;
(d) if some automorphism of (Σ, u) maps z_i to z_j then Y_i is equal to Y_j.

The second condition says that there are no infinitesimal deformations which do not change the positions of the extra markings.

Proposition 3.3.15. *Let u be a parametrized regular stable map, and u_S the stratified-smooth universal deformation constructed in Theorem 3.3.10 with base $S \subset \operatorname{Def}(u)$. There exists a collection Y compatible with u. Furthermore if the evaluation map is C^1 as in Proposition 3.3.13 then $\Sigma_S^{Y,u}$ defines an C^1-immersion of S into $\operatorname{Def}(\Sigma^{Y,u})$.*

Proof. First we show the existence of a compatible collection. Given a regular stable map $(\Sigma, z = (z_1, \ldots, z_n), u : \Sigma \to X)$, choose $Y_1, \ldots, Y_k$ transverse u on the unstable components of Σ, so that $\Sigma_1 = (\Sigma, (z_1, \ldots, z_{n+k}))$ is a stable curve. Let $\Sigma_{S_1,1} \to S_1$ denote a universal deformation of Σ_1. By universality, the family $\Sigma_S^{Y,u}$ is induced by a map $\psi : S \to S_1$. We successively add marked points until ψ is an immersion: Suppose that ψ is not an immersion. Then we may choose an additional marked point $z_{n+k+1} \in \Sigma$ such that $\mathrm{d}\,\mathrm{ev}_{n+k+1}$ is non-trivial on $\ker D\psi$. Since u is holomorphic, $\mathrm{d}u(z_{n+k+1})$ is rank two at z_{n+k+1}. Let $Y_{n+k+1} \subset X$ be a codimension two submanifold containing $u(z_{n+k+1})$ such that u is transverse to Y_{n+k+1} at z_{n+k+1}, and Y_{n+k+1} is transversal to ev_{n+k+1} at Σ, u. Suppose $z_{n+n'+1}$ has orbit $z_{n+k+1}, z_{n+k+2}, \ldots, z_{n+l}$ under the group $\mathrm{Aut}(u)$. Repeating the same submanifold for each marking related by automorphisms gives a collection invariant under the action of automorphisms. The map ψ_1 for the new family has property that the dimension of $\ker(D\psi_1)$ has dimension at least two less than that of $\ker(D\psi)$. It follows that the procedure terminates after adding a finite number of markings. The last claim follows from the second condition in Definition 3.3.14. □

Step 6: Surjectivity

In this section, we show that the family constructed above contains a Gromov neighborhood of the central fiber. First we show:

Proposition 3.3.16. *There exists a constant $\epsilon > 0$ such that any stable map (Σ_1, u_1) with complex structure on Σ_1 given by $j^\delta(\zeta)$ for some $\zeta \in \mathrm{Def}(\Sigma^\delta)$, and $u_1 := \exp_{u^\delta}(\xi)$ with $\|\zeta\|^2 + \|\xi\|^2_{1,p,\delta} < \epsilon$ is of the form in Theorem 3.3.10 for some $(\xi_1, \zeta_1) \in \mathrm{Im}(\tilde{D}^\delta_u)^*$.*

Proof. Compare with [15, Section 10.7.3]. Let (ζ, ξ) be as in the statement of the Proposition. We claim that $(\zeta, \xi) = (\zeta_0^\delta, \xi_0^\delta) + (\zeta_1, \xi_1)$ for some $(\zeta_0, \xi_0) \in \ker(\tilde{D}_u)$ and $(\zeta_1, \xi_1) \in \mathrm{Im}((D^\delta_u)^*)$ with small norm. It then follows by the implicit function theorem that (ζ_1, ξ_1) is the solution given in Theorem 3.3.10. Now $\zeta = \zeta_0^\delta$ for some $\zeta_0 \in \mathrm{Def}_\Gamma(\Sigma)$ and gluing parameters δ, because $\mathrm{Def}(\Sigma^\delta)$ is the direct sum of the image of $\mathrm{Def}_\Gamma(\Sigma)$ and $\mathbb{C}^m$. By the gluing theorem for indices (see e.g. [23, Theorem 2.4.1]), the image of $\mathrm{Def}_\Gamma(u)$ under the gluing map projects isomorphically onto $\ker(\tilde{D}^\delta_u)$ for δ sufficiently small, and so $\mathrm{Def}_\Gamma(u)$ is transverse to $\mathrm{Im}(\tilde{D}^\delta_u)^*$, for δ sufficiently small. The claim then follows from the inverse function theorem. □

Given a regular stable u with stable domain, consider the family of J-holomorphic maps u_S produced by Theorem 3.3.10 with parameter space a neighborhood S of 0 in $\mathrm{Def}(u)$, equipped with a canonical identification ι of the central fiber with the original map u. In the case that the domain Σ is not a stable (marked) curve, we choose codimension two submanifolds $Y = (Y_1, \ldots, Y_l)$ meeting u transversally so that Σ with the additional marked points is stable. Applying this to the family u_S gives a family of marked stable maps u_S^Y with $n + l$ marked points over a parameter space $S \subset \mathrm{Def}(u^Y)$ in the deformation space of the map with the additional marked points. Now $\mathrm{Def}(u^Y) \cong \mathrm{Def}(u) \oplus \bigoplus_{i=1}^l T_{z_i}\Sigma$ includes the deformations of the markings, but these are fixed by requiring that the additional marked points map to the given collection Y. Forgetting the additional marked points gives a family u_S of stable maps with n marked points over a neighborhood of 0 in $\mathrm{Def}(u)$.

Proposition 3.3.17. *(u_S, ι) is a versal stratified-smooth deformation of u, and in fact u_S gives a versal stratified-smooth deformation of any of its fibers.*

Proof. First suppose that Σ is stable. Let $(u^1_{S^1}, \iota^1)$ be another stratified-smooth deformation of u with parameter space S^1. Let $\Sigma_S \to S \subset \mathrm{Def}(\Sigma)$ be a minimal versal deformation of Σ. The family $\Sigma^1_{S^1}$ is obtained by pull-back of Σ_S by a stratified-smooth map $\psi : S^1 \to S$. By definition the map u^1_s converges to the central fiber in the Gromov topology as s converges to the base point $0 \in S^1$. The exponential decay estimate of [15, Lemma 4.7.3] for holomorphic cylinders of small energy imply that for s sufficiently close to 0, Σ^1_s, u^1_s is given by exponentiation, $u^1_s = \exp_{u^\delta}(\xi)$ for some $\xi \in \Omega^0(u^{\delta,*}TX)$ with $\|\xi\|_{1,p} < \epsilon_1$, for s sufficiently close to 0. Proposition 3.3.16 produces a stratified-smooth map $\psi : S^1 \to \mathrm{Def}(u)$ such that $u^1_{S^1}$ is the pull-back of ψ. To show that the deformation (u_S, ι) is universal, let $\phi_j : \Sigma^1_{S^1} \to \psi_j^* \Sigma_S, j = 0, 1$ be isomorphisms of families inducing the identity on the central fiber. The difference between the two automorphisms is an automorphism of the family $\Sigma^1_{S^1}$ inducing the identity on the central fiber; since the automorphism group of the central fiber is discrete, the automorphism must be the identity. In the case that Σ is not stable, after adding marked points passing through $Y_1, \ldots, Y_l$, we obtain a family $u^{1,Y}_{S^1}$ of stable maps with $n + l$ marked points. By the case with stable domain, this family is obtained by pull-back of u^Y_S by some map $S^1 \to S$. Hence $u^1_{S^1}$ is obtained by pull-back by the same map. The argument for an arbitrary fiber is similar and left to the reader. □

Remark 3.3.18. In the case that Σ is unstable, it seems likely that restricting the family of Theorem 3.3.10 to $\mathrm{Def}(u)$ (that is, the perpendicular of $\mathrm{aut}(\Sigma)$) also gives a universal deformation, but we do not know how to prove this. The problem is that in this case, several different gluing parameters give the same curve, and we do not have an implicit function theorem for varying gluing parameter.

Step 7: Injectivity

By injectivity, we mean that the family constructed above contains each nearby stable map exactly once, up to the action of $\mathrm{Aut}(u)$. This is part of what we called "strongly universal" in Definition 2.2.4.

Theorem 3.3.19. *The versal deformations constructed in Step 6 above are strongly universal.*

Proof. Let u_S be a deformation constructed as in Step 6, using the exponential gluing profile. Let $\Sigma_{1,S^1} \to S^1$ be a family giving a universal deformation of the curve $\Sigma^{Y,u}$ obtained by adding the additional markings mapping to the given submanifolds. By Definition 3.3.14, the family $\Sigma^{Y,u}_S$ induces a map $\phi : S \to S_1$ whose differential is injective in a neighborhood of 0. By the inverse function theorem for C^1 maps, ϕ induces a homeomorphism onto its image. In particular, any two distinct fibers of $\Sigma^{Y,u}_S$ are non-isomorphic, and so two fibers of Σ_S are isomorphic if and only if they are related by a permutation of the markings. After shrinking S, this happens only if the permutation is induced by an automorphism of u. Given another family $u'_{S'} : \Sigma_{S'} \to S'$ corresponding to a deformation of a fiber of $u_S \to S$, by the uniqueness part of the implicit function theorem, a map $\phi' : S' \to S_1$ so that $u'_{S'}$ is obtained by pull-back from u_S, and this map is unique by the injectivity just proved. This shows that u_S gives a stratified-smooth universal deformation of any of its fibers, and so is strongly universal. □

The Theorem implies that the families in the universal deformations constructed above define stratified-smooth-compatible charts for the moduli space $\overline{M}_{g,n}(X,d)$. That is, for any stratum $M_{g,n,\Gamma}(X,d)$, the restriction of the charts given by the universal deformation of some map of type Γ to $M_{g,n,\Gamma}(X,d)$ are smoothly compatible.

Corollary 3.3.20. *Let X, J be as above. For any $g \geq 0, n \geq 0$, the strongly universal stratified-smooth deformations of parametrized regular stable maps provide $\overline{M}^{\mathrm{reg}}_{g,n}(X)$ with the structure of a stratified-smooth topological orbifold.*

In order to apply localization one needs to know that the fixed point sets admit tubular neighborhoods. For this it is helpful to know that $\overline{M}^{\mathrm{reg}}_{g,n}(X,d)$ admits a C^1 structure. In order to obtain compatible charts, we construct the local coordinates inductively as in Definition 2.2.6, starting with the strata of highest codimension.

Proposition 3.3.21. *Let X, J be as above. For any compatible system of local coordinates near the nodes, the strongly universal deformations constructed using the exponential gluing profile equip $\overline{M}^{\mathrm{reg}}_{g,n}(X)$ with the structure of a C^1-orbifold.*

Proof. We claim that the charts induced by the universal deformations are C^1-compatible, assuming they are constructed from the same system of local coordinates near the nodes. Given two sets of submanifolds Y_1, Y_2, define $Y = Y_1 \cup Y_2$. The family $\Sigma^{Y,u}$ admits proper étale forgetful maps $\Sigma^{Y,u}_S \to \Sigma^{Y_j,u}_S, \quad j = 1,2$. The fiber consists of reorderings of the additional marked points induced by the action of $\mathrm{Aut}(\Sigma, u)$, and the diagram provided by $\Sigma^{Y,u}$ expresses the composition as a smooth C^1-morphism of orbifolds. □

Remark 3.3.22. Any compact C^1 orbifold admits a compatible C^∞ structure, in analogy with the situation with manifolds. Indeed, as is well known any orbifold admits a presentation as the quotient of a manifold (namely its orthogonal frame bundle) by a locally free group action, and so the orbifold case follows from the equivariant case proved in Palais [18]. Hence $\overline{M}^{\mathrm{reg}}_{g,n}(X,d)$ if compact admits a (non-canonical) smooth structure. Presumably the compactness assumption may be removed but we have not proved that this is so. See however the construction of smoothly compatible Kuranishi charts in [9, Appendix].

4. Deformations of symplectic vortices

We begin by reviewing the theory of symplectic vortices introduced by Mundet i Riera [16] and Salamon and collaborators [5]. Let Σ be a compact complex curve,

G a compact Lie group, and $\pi : P \to \Sigma$ a smooth principal G-bundle. Given any left G-manifold F we have a left action of G on $P \times F$ given by $g(p, f) = (pg^{-1}, gf)$ and we denote by $P(F) = (P \times F)/G$ the quotient, that is, the associated fiber bundle with fiber F. Let X be a compact Hamiltonian G-manifold with symplectic form ω and moment map $\Phi : X \to \mathfrak{g}^*$. The action of G on X induces an action on $\mathcal{J}(X)$; and we denote by $\mathcal{J}(X)^G$ the invariant subspace. Let $\psi : \Sigma \to BG$ be a classifying map for P, so that $P \cong \psi^* EG$ and $P(X) \cong \psi^* EG \times_G X \cong \psi^* X_G$ where $X_G = EG \times_G X$. Continuous sections $u : \Sigma \to P(X)$ are in one-to-one correspondence with lifts of ψ to X_G. The homology class $\deg(u)$ of the section u is defined to be the homology class $\deg(u) \in H_2^G(X, \mathbb{Z})$ of the corresponding lift. Let $\mathcal{A}(P)$ be the space of smooth connections on P, and $P(\mathfrak{g})$ the adjoint bundle. For any $A \in \mathcal{A}(P)$, let $F_A \in \Omega^2(\Sigma, P(\mathfrak{g}))$ the curvature of A. Any connection $A \in \mathcal{A}(P)$ induces a map of spaces of almost complex structures

$$\mathcal{J}(X)^G \to \mathcal{J}(P(X)), \quad J \mapsto J_A$$

by combining the almost complex structures on X and Σ using the splitting defined by the connection. Let $\Gamma(\Sigma, P(X))$ denote the space of smooth sections of $P(X)$. Consider the vector bundle

$$\bigcup_{u \in \Gamma(\Sigma, P(X))} \Omega^{0,1}(\Sigma, u^* T^{\mathrm{vert}} P(X)) \to \Gamma(\Sigma, P(X)). \tag{27}$$

We denote by $\overline{\partial}_A$ the section given by the Cauchy-Riemann operator defined by J_A. A *gauged map* from Σ to X is a datum (P, A, u) where $A \in \mathcal{A}(P)$ and $u : \Sigma \to P(X)$ is a section. A *gauged holomorphic map* is a gauged map (P, A, u) such that $\overline{\partial}_A u = 0$. Let $\mathcal{H}(P, X)$ be the space of gauged holomorphic maps with underlying bundle P. Let $\mathcal{G}(P)$ denote the group of gauge transformations

$$\mathcal{G}(P) = \{a : P \to P, a(pg) = a(p)g, \quad \pi \circ a = \pi\}.$$

The Lie algebra of $\mathcal{G}(P)$ is the space of sections $\Omega^0(\Sigma, P(\mathfrak{g}))$ of the adjoint bundle $P(\mathfrak{g}) = P \times_G \mathfrak{g}$. We identify $\mathfrak{g} \to \mathfrak{g}^*$, and hence $P(\mathfrak{g}) \to P(\mathfrak{g}^*)$, using an invariant metric on $\mathfrak{g}$. Let $P(\Phi) : P(X) \to P(\mathfrak{g})$ denote the map induced by the equivariant map $\Phi : X \to \mathfrak{g}$.

Definition 4.0.23. A gauged holomorphic map $(A, u) \in \mathcal{H}(P, X)$ is a *symplectic vortex* (or vortex for short) if it satisfies

$$F_A + \mathrm{Vol}_\Sigma \, u^* P(\Phi) = 0.$$

An *n-marked* symplectic vortex is a vortex (A, u) together with n-tuple $z = (z_1, \ldots, z_n)$ of distinct points on Σ. A marked vortex (A, u, z) is *stable* if it has finite automorphism group.

The equation in the definition can be interpreted as the zero level set condition for a formal moment map for the action of the group of gauge transformations, see [16], [5]. The *energy* of a gauged holomorphic map (A, u) is given by

$$E(A, u) = \frac{1}{2} \int_\Sigma \left(|\mathrm{d}_A u|^2 + |F_A|^2 + |u^* P(\Phi)|^2 \right) \mathrm{Vol}_\Sigma \, .$$

The *equivariant symplectic area* of a pair (A, u) is the pairing of the homology class $\deg(u)$ with the class $[\omega_G = \omega + \Phi] \in H^2(X_G)$,

$$D(A, u) = (\deg(u), [\omega_G]) = ([\Sigma], u^*[\omega_G]).$$

Lemma 4.0.24. *Suppose* Vol_Σ *is the Kähler form for the metric on* Σ*. The energy and equivariant area are related by*

$$E(A, u) = D(A, u) + \int_\Sigma \left(|\overline{\partial}_A u|^2 + \frac{1}{2} |F_A + \mathrm{Vol}_\Sigma \, u^* P(\Phi)|^2 \right) \mathrm{Vol}_\Sigma \,. \tag{28}$$

Proof. See [4, Proposition 2.2]. □

In particular, for any symplectic vortex the energy and action are equal. Let $M(P, X)$ denote the moduli space of vortices

$$M(P, X) := \mathcal{H}(P, X) /\!\!/ \mathcal{G}(P) = \{F_A + \mathrm{Vol}_\Sigma \, u^* P(\Phi) = 0\} / \mathcal{G}(P).$$

Let $M_n(P, X)$ denote the moduli space of n-marked vortices, up to gauge transformation, and $M_n(\Sigma, X) = \bigcup_{P \to \Sigma} M_n(P, X)$ the union over types of bundles P. Clearly, $M_n(\Sigma, X)$ is homeomorphic to the product $M(\Sigma, X) \times M_n(\Sigma)$ where $M(\Sigma, X) := M_0(\Sigma)$ and $M_n(\Sigma)$ denotes the configuration space of n-tuples of distinct points on Σ.

We wish to study families and deformations of symplectic vortices. For families with smooth domain, the definitions are straightforward:

Definition 4.0.25. A *smooth family of vortices* on a principal G-bundle P on Σ over a parameter space S consists of a family of connections depending smoothly on $s \in S$, that is, a smooth map $A_S : S \times P \to T^*P \otimes \mathfrak{g}$ on P such that the restriction A_s of A_S to any $\{s\} \times P$ is a connection, together with a smooth family of (pseudo)holomorphic sections $u_S = (u_s)_{s \in S}$, such that each pair $(A_s, u_s), s \in S$ is a symplectic vortex. A *deformation* of (A, u) is a germ of a smooth family (A_S, u_S) together with an isomorphism (gauge transformation) relating (A_0, u_0) with (A, u). A deformation is *universal* if it satisfies the condition as in Definition 3.1.4, and *strongly universal* if it satisfies the conditions in Definition 2.2.4.

We define a *linearized operator* associated to a vortex as follows. Define

$$\mathrm{d}_{A,u} : \Omega^1(\Sigma, P(\mathfrak{g})) \oplus \Omega^0(\Sigma, u^*TP(X)) \to \Omega^2(\Sigma, P(\mathfrak{g})) \tag{29}$$
$$\mathrm{d}_{A,u}(a, \xi) := \mathrm{d}_A a + \mathrm{Vol}_\Sigma \, u^* L_\xi P(\Phi).$$

Here $L_\xi P(\Phi)$ denotes the derivative of $P(\Phi)$ with respect to the vector field generated by ξ, and $u^* L_\xi P(\Phi)$ its evaluation at u. Define an operator

$$\mathrm{d}^*_{A,u} : \Omega^1(\Sigma, P(\mathfrak{g})) \oplus \Omega^0(\Sigma, u^*TP(X)) \to \Omega^0(\Sigma, P(\mathfrak{g})) \tag{30}$$
$$\mathrm{d}^*_{A,u}(a, \xi) = \mathrm{d}^*_A a + u^* L_{J\xi} P(\Phi).$$

(This is not the adjoint of operator in (32), but rather defined by analogy with the case X trivial.) It is shown in [5, Section 4] that if (A, u) is stable then the set

$$W_{A,u} = \{(A + a, \exp_u(\xi)), (a, \xi) \in \ker \mathrm{d}^*_{A,u}\}$$

is a slice for the gauge group action near (A, u). Define

$$\begin{aligned} (31) \quad \mathcal{F}_{A,u} : \Omega^1(\Sigma, P(\mathfrak{g})) \oplus \Omega^0(\Sigma, u^* T^{\mathrm{vert}} P(X)) \\ \to (\Omega^0 \oplus \Omega^2)(\Sigma, P(\mathfrak{g})) \oplus \Omega^{0,1}(\Sigma, u^* T^{\mathrm{vert}} P(X)) \\ (a, \xi) \mapsto \left(F_{A+a} + \mathrm{Vol}_\Sigma \exp_u(\xi)^* P(\Phi), \mathrm{d}^*_{A,u}(a, \xi), \Psi_u(\xi)^{-1} \overline{\partial}_{A+a} \exp_u(\xi) \right). \end{aligned}$$

Let

$$\Omega^1(\Sigma, P(\mathfrak{g})) \to \Omega^1(\Sigma, u^* T^{\mathrm{vert}} P(X)), \quad a \mapsto a_X$$

denote the map induced by the infinitesimal action. The linearization of the last component (31) is

$$D_{A,u}(a, \xi) = (\nabla_A \xi)^{0,1} + \frac{1}{2} J_u (\nabla_\xi J)_u \partial_A u + a_X^{0,1}.$$

Here $0,1$ denotes projection on the $0,1$-component. The linearized operator for a vortex (A, u) is the operator

$$\begin{aligned} (32) \quad \tilde{D}_{A,u} = (\mathrm{d}_{A,u}, \mathrm{d}^*_{A,u}, D_{A,u}) : \Omega^1(\Sigma, P(\mathfrak{g})) \oplus \Omega^0(\Sigma, u^* T^{\mathrm{vert}} P(X)) \\ \to (\Omega^0 \oplus \Omega^2)(\Sigma, P(\mathfrak{g})) \oplus \Omega^{0,1}(\Sigma, u^* T^{\mathrm{vert}} P(X)) \end{aligned}$$

A vortex (A, u) is *regular* if the operator $\tilde{D}_{A,u}$ is surjective. A marked vortex (A, u, z) is regular if the underlying unmarked vortex is regular. The *space of infinitesimal deformations* of (A, u) is $\mathrm{Def}(A, u) := \ker(\tilde{D}_{A,u})$.

Theorem 4.0.26. *Any regular vortex with smooth domain (A, u) has a strongly universal smooth deformation if and only if it is stable.*

Proof. Give the spaces of connections and sections the structure of Banach manifolds by taking completions with respect to Sobolev norms $1, p$ for 1-forms, and $0, p$ for 0 and 2-forms. For $p > 2$, the map $\mathcal{F}_{A,u}$ is a smooth map of Banach spaces.

$$\begin{aligned} (33) \quad \mathcal{F}_{A,u} : \Omega^1(\Sigma, P(\mathfrak{g}))_{1,p} \oplus \Omega^0(\Sigma, u^* T^{\mathrm{vert}} P(X))_{1,p} \\ \to (\Omega^0 \oplus \Omega^2)(\Sigma, P(\mathfrak{g}))_{0,p} \oplus \Omega^{0,1}(\Sigma, u^* T^{\mathrm{vert}} P(X))_{0,p} \end{aligned}$$

equivariant for the action of the group $\mathcal{G}(P)_{2,p}$ of gauge transformations of class $2, p$. Suppose that (A, u) is regular and stable. By the implicit function theorem, there is a local homeomorphism

$$\ker(\tilde{D}_{A,u}) \to \left\{ \begin{array}{c} F_{A+a} + \mathrm{Vol}_\Sigma (\exp_u(\xi))^* P(\Phi) = 0 \\ \overline{\partial}_{A+a}(\exp_u(\xi)) = 0 \\ \mathrm{d}^*_{A,u}(a, \xi) = 0 \end{array} \right\}.$$

This gives rise to a family $(A_S, u_S) \to S$ over a neighborhood S of 0 in $\ker(\tilde{D}_{A,u})$. By [5, Theorem 3.1], (A_S, u_S) is a smooth family, assuming (A, u) is smooth. Given any other family $(A'_{S'}, u'_{S'}) \to S'$ of stable vortices with $(A'_0, u'_0) = (A, u)$, the implicit function theorem provides a smooth map $S' \to S$ so that $(A'_{S'}, u'_{S'})$ is obtained from (A, u) by pull-back. The first property of the universal deformation is a consequence of the slice condition; the second property follows from the fact that the projection $\ker(\tilde{D}_{A,u}) \to \ker(\tilde{D}_{A_s, u_s})$ is an isomorphism for sufficiently small s. $\square$

Let $M_n^{\mathrm{reg}}(\Sigma, X)$ denote the moduli space of regular, stable n-marked symplectic vortices from Σ to X. We denote by $(c_1^G(TX), d)$ the pairing of d with the first Chern class $c_1^G(P(TX) \to P(X))$

Theorem 4.0.27. *Let Σ, X, J be as above. $M_n^{\mathrm{reg}}(\Sigma, X)$ has the structure of a smooth orbifold with tangent space at $[A, u]$ isomorphic to* $\mathrm{Def}(A, u)$*, and dimension of the component of homology class $d \in H_2^G(X)$ is given by*

$$\dim(M_n^{\mathrm{reg}}(\Sigma, X, d)) = (1 - g)(\dim(X) - 2\dim(G)) + 2((c_1^G(TX), d) + n).$$

Proof. Charts for $M_n^{\mathrm{reg}}(\Sigma, X)$ are provided by the strongly universal deformations. The dimension of the tangent space at $[A, u]$ is given by the index of the linearized operator $\tilde{D}_{A,u}$, which deforms via Fredholm operators to the sum of the operator $\mathrm{d}_A \oplus \mathrm{d}_A^*$ for the connection, which has index $2\dim(G)(g - 1)$, and the linearized Cauchy-Riemann operator on the nodal curve, which has index $(1 - g)\dim(X) + 2n + 2(c_1^G(TX), d)$ by Riemann-Roch, if (A, u) has equivariant homology class d (which determines the first Chern class of P by projection.) □

4.1. **Polystable vortices.** The moduli space of symplectic vortices admits a compactification which allows bubbling of the section in the fibers.

Definition 4.1.1. A *nodal gauged marked holomorphic map* from Σ to X consists of a datum $(\hat{\Sigma}, P, A, u, z)$ where $P \to \Sigma$ is a principal G-bundle, $A \in A(P)$ is a connection, $\hat{\Sigma}$ is a marked nodal curve, $v : \hat{\Sigma} \to \Sigma$ is a holomorphic map of degree $[\Sigma]$, and $u : \hat{\Sigma} \to P(X)$ is a J_A-holomorphic map from a nodal curve $\hat{\Sigma}$ such that $\pi \circ u$ has class $[\Sigma]$. In other words,

(a) $\hat{\Sigma}$ is a connected nodal complex curve consisting of a *principal component* Σ_0 equipped with an isomorphism with Σ together with a number of projective lines $\Sigma_1, \ldots, \Sigma_k$. We denote by $w_1^{\pm}, \ldots, w_m^{\pm}$ the nodes. For each $i = 1, \ldots, m$, we denote by $w_i^0 \in \Sigma_0$ the attaching point to the principal component.
(b) $(A, u) \in \mathcal{H}(P, X)$ is a gauged holomorphic map from Σ to X;
(c) for each non-principal component Σ_i, a holomorphic map $u_i : \Sigma_i \to P(X)_{w_i^0}$;
(d) $z = (z_1, \ldots, z_n) \in \hat{\Sigma}$ are distinct, smooth points of $\hat{\Sigma}$.

Let $\mathcal{H}(\hat{\Sigma}, P(X))$ denote the space of nodal gauged holomorphic sections with domain $\hat{\Sigma}$ and bundle P. The group of gauge transformations $\mathcal{G}(P)$ acts on $\mathcal{H}(\hat{\Sigma}, P(X))$ by $g(A, u) = (g^*A, g \circ u)$. The generating vector field for $\zeta \in \Omega^0(\Sigma, P(\mathfrak{g}))$ acting on $\mathcal{H}(\hat{\Sigma}, P(X))$ at $(\hat{\Sigma}, A, u)$ is the tuple given by

$$\zeta_{\mathcal{H}(\hat{\Sigma}, P(X))}(\hat{\Sigma}, A, u) = (\mathrm{d}_A\zeta, u_0^* P(\zeta_X), (u_i^* P(\zeta_X(w_i^0)))_{i=1}^k) \tag{34}$$

in $\Omega^1(\Sigma, P(\mathfrak{g})) \oplus \Omega^0(\hat{\Sigma}, u^* T^{\mathrm{vert}} P(X))$. Here $P(\zeta_X) \in \Omega^0(\Sigma, P(\mathrm{Vect}(X)))$ is the fiber-wise vector field generated by ζ and $u_0^* P(\zeta) \in T^{\mathrm{vert}} P(X)$ is the evaluation at u_0. Similarly for the bubble components $u_1, \ldots, u_k$ in the fibers $P(X)_{w_1^0}, \ldots, P(X)_{w_k^0}$. A slice is given by taking the perpendicular to the tangent spaces to the $\mathcal{G}(P)$-orbits.

We will assume for simplicity that the stabilizer of the $\mathcal{G}(P)$ action on the principal component is finite, so that a slice is given locally by the kernel of d^*_{A,u_0}, that is, the Coulomb gauge condition on the principal component. The implicit function theorem shows that any nearby pair (A_1, u_1) is gauge equivalent to a pair of the form $(A + a, \exp_u(v))$ with $(a, v) \in \ker \mathrm{d}^*_{A,u_0}$.

Definition 4.1.2. A *nodal vortex* is a stable nodal gauged holomorphic map such that the principal component is an vortex. A nodal vortex $(\hat{\Sigma}, A, u, z)$ is *polystable* if each sphere bubble Σ_i on which u_i is constant has at least three marked or singular points, and *stable* if it has finite automorphism group. An *isomorphism* of nodal vortices $(\hat{\Sigma}, A, u, z), (\hat{\Sigma}', A', u', z')$ consists of an automorphism of the domain, acting trivially on the principal component, and a corresponding automorphism of the principal bundle mapping (A, u) to (A', u') and mapping the markings z to z'. For any nodal section $u : \hat{\Sigma} \to P(X)$, the *homology class* of u is defined as the sum of the homology class $d_0 \in H_2^G(X, \mathbb{Z})$ of the principal component u_0 and the homology classes $d_i \in H_2(X, \mathbb{Z}), i = 1, \ldots, k$ of the sphere bubbles, using the inclusion $H_2(X, \mathbb{Z}) \to H_2^G(X, \mathbb{Z})$ given by equivariant formality. The *combinatorial type* $\Gamma(\hat{\Sigma}, A, u, z)$ of a gauged nodal map is a rooted graph whose vertices represent the components of $\hat{\Sigma}$, whose finite edges represent the nodes, semi-infinite edges represent the markings, and whose root vertex represents the principal component.

Note that there is no condition for points on the principal component. In particular, nodal gauged holomorphic maps with no markings can be polystable. The term *polystable* is borrowed from the vector bundle case. In that situation, a bundle is *stable* if it is flat and has only central automorphisms and polystable if it is a direct sum of stable bundles of the same slope. Any flat bundle is automatically polystable; a bundle is *semistable* if it is grade equivalent to a polystable bundle. In particular, the moduli space of stable bundles is definitely *not* compact, and we feel that the vortex terminology should include this fact as a special case.

From now on, we fix the bundle P.

Definition 4.1.3. Let X as above. A *smooth family of fixed type* of nodal vortices to X consists of a smooth family $\hat{\Sigma}_S \to S$ of nodal curves of fixed type, a smooth family of holomorphic maps $v_S : \hat{\Sigma}_S \to \Sigma$ of class $[\Sigma]$, a smooth family $u_S : \hat{\Sigma}_S \to P(X)$ of maps, and a smooth family $A_S : S \times P \to T^*P$ of connections over S. A *smooth deformation* of a nodal vortex $(A, \hat{\Sigma}, u, z)$ of fixed type consists of a germ of a smooth family $(A_S, \hat{\Sigma}_S, u_S, z_S)$ of nodal vortices of fixed type together with an identification ι of of the central fiber with $(A, \hat{\Sigma}, u, z)$. A *stratified-smooth family of marked nodal symplectic vortices* is a datum $(\hat{\Sigma}_S, A_S, u_S, z_S)$ consisting of a stratified-smooth family $\hat{\Sigma}_S \to S$ of nodal curves, a stratified-smooth family of holomorphic maps $v : \hat{\Sigma}_S \to \Sigma$ of class $[\Sigma]$, a stratified-smooth family A_S of connections on P, a stratified-smooth family of maps $u_S : \hat{\Sigma}_S \to P(X)$; such that each triple $(\hat{\Sigma}_s, A_s, u_s, z_s)$ is a marked nodal symplectic vortex. A *family of polystable* *symplectic vortices* is a family of marked nodal symplectic vortices such that any fiber is polystable.

A smooth vector bundle $E \to \hat{\Sigma}$ is a collection of smooth vector bundles E_i over the components Σ_i of $\hat{\Sigma}$, equipped with identifications of the fibers at nodal points $E_{i^+(w_k^+)} \to E_{i^-(w_k^-)}, k = 1, \ldots, m$. We denote by $\Omega(\hat{\Sigma}, E)$ the sum over components, $\Omega(\hat{\Sigma}, E) = \bigoplus_{i=1}^k \Omega(\Sigma_i, E_i)$ where $E_i = E|\Sigma_i$.

Definition 4.1.4. For a polystable vortex $(\hat{\Sigma}, A, u)$, let $\tilde{D}_{A,u}$ denote the *linearized operator*

$$\begin{aligned} (35) \quad & \Omega^1(\Sigma, P(\mathfrak{g})) \oplus \Omega^0(\hat{\Sigma}, u^* T^{\mathrm{vert}} P(X)) \\ & \to (\Omega^0 \oplus \Omega^2)(\Sigma, P(\mathfrak{g})) \oplus \Omega^{0,1}(\hat{\Sigma}, u^* T^{\mathrm{vert}} P(X)) \oplus \bigoplus_{k=1}^m T^{\mathrm{vert}}_{u(w_k^\pm)} P(X) \end{aligned}$$

given by the linearized vortex operator $(\mathrm{d}_{A,u_0}, D_{A,u_0})$ on the principal component, the linearized Cauchy-Riemann operator $\tilde{D}_{u_i}$ on the bubbles, the slice operator d_{A,u_0}^*, and the difference operator on the fibers over the nodes

$$\Omega^0(\hat{\Sigma}, u^* T^{\mathrm{vert}} P(X)) \to \bigoplus_{i=1}^m T^{\mathrm{vert}}_{u(w_i^\pm)} P(X), \quad \xi \mapsto (\xi(w_i^+) - \xi(w_i^-))_{i=1}^m.$$

(A, u) is *regular* if $\tilde{D}_{A,u}$ is surjective. The space of infinitesimal deformations of A, u of fixed type is $\mathrm{Def}_\Gamma(A, u) := \ker \tilde{D}_{A,u} / \mathrm{aut}(\hat{\Sigma})$ where $\mathrm{aut}(\hat{\Sigma})$ denotes the group of infinitesimal automorphisms acting trivially on the principal component. The space of infinitesimal deformations of A, u is

$$\mathrm{Def}(A, u) := \mathrm{Def}_\Gamma(A, u) \oplus \bigoplus_{i=1}^m T_{w_i^+} \hat{\Sigma} \otimes T_{w_i^-} \hat{\Sigma}$$

consisting of a deformation of fixed type together with a collection of gluing parameters.

4.2. **Constructing deformations of symplectic vortices.**

First we construct deformations of fixed type.

Theorem 4.2.1. *A regular polystable vortex has a strongly universal smooth deformation of fixed type if and only if it is stable.*

The proof is by the implicit function theorem applied to the map

$$\begin{aligned} (36) \quad \mathcal{F}_{A,u}(a, \xi) = & (F_{A+a} + \mathrm{Vol}_\Sigma (\exp_{u_0}(\xi_0))^* P(\Phi), \mathrm{d}_{A,u_0}^*(a, \xi), \\ & \Psi_{u_0}(\xi_0)^{-1} \overline{\partial}_A u_0, (\Psi_{u_i}(\xi_i)^{-1} \overline{\partial} u_i)_{i=1}^k, (\xi(w_i^+) - \xi(w_i^-))_{i=1}^m) \end{aligned}$$

whose linearization is $\tilde{D}_{A,u}$. The proof is left to the reader. We denote by $M_{n,\Gamma}(\Sigma, X, d)$ of the moduli space of isomorphism classes of polystable vortices of combinatorial type Γ of homology class $d \in H_2^G(X, \mathbb{Z})$, and $M_{n,\Gamma}^{\mathrm{reg}}(\Sigma, X, d)$ the regular locus.

Corollary 4.2.2. *$M^{\mathrm{reg}}_{n,\Gamma}(\Sigma, X, d)$ has the structure of a smooth orbifold of dimension given, if Σ is connected, by $(1-g)(\dim(X) - 2\dim(G)) + 2(n + (c_1^G(TX), d) - m)$ where m is the number of nodes.*

We now prove that a regular stable symplectic vortex from Σ to X admits a strongly universal stratified-smooth deformation if it is strongly stable, that is, Theorem 1.0.2. We explain the construction for a single bubble only, so that $\hat{\Sigma}$ is the union of a principal component $\Sigma_+ = \Sigma$ and a holomorphic sphere Σ_-, attached by a single pair $w_\pm$ of nodes. We denote by (A, u_+) the restriction to the principal component and by u_- the bubble, so that $x := u_+(w_+) = u_-(w_-)$ and $u = (u_+, u_-)$. We choose a local coordinate near w, equivariant for the action of the automorphism group $\mathrm{Aut}(A, u)$ in the sense that $\mathrm{Aut}(A, u)$ acts on the local coordinate by multiplication by roots of unity. The construction depends on the following choices:

Definition 4.2.3. A *gluing datum* for $(\hat{\Sigma}, A, u)$ consists of

(a) neighborhoods $U_\pm$ of the nodes $w_\pm$;
(b) *local coordinates* $\kappa = (\kappa_+, \kappa_-)$ on $U_\pm$;
(c) a trivialization $P|_{U_0} \to G \times U_0$ on U_0;
(d) a *gluing parameter* δ;
(e) an *annulus parameter* ρ ;
(f) a *cutoff function* α as in (10).

Step 1: Approximate Solution

Given a nodal vortex (A, u) as above and a gluing datum we wish to define an approximate solution to the vortex equations (A, u^δ). Let $\exp_x : T_x X \to X$ denote the exponential map defined by the metric on X. Define sections

$$\xi_\pm : U_\pm \to T_x X, \quad u(z) = \exp_x(\xi_\pm(z)).$$

Let $\hat{\Sigma}^\delta$ denote the surface obtained by gluing; since the bubble is genus zero, this surface is isomorphic to Σ but not canonically. Define the *pre-glued section* u^δ : $\hat{\Sigma}^\delta \to P(X)$,

$$(37)\quad u^\delta(z) = \exp_x \Big(\alpha(|\kappa_+(z)|/\rho|\delta|^{1/2})(\xi_+(z) - \xi_\pm(w^\pm)) + \\ \alpha(|\kappa_-(z)|/\rho|\delta|^{1/2})(\xi_-(z) - \xi_\pm(w^\pm)) + \xi_\pm(w^\pm) \Big)$$

for $|\kappa_\pm(z)| \le 2|\delta|^{1/2}\rho^2$; elsewhere let $u^\delta(z) = u(z)$, using the identification of Σ with $\hat{\Sigma}$ away from the gluing region. We do not modify A in the bubble region; this is because after re-scaling the connection on the bubble is already close to the trivial connection.

The pair (A, u^δ) is an approximate solution to the vortex equations in the following Sobolev norms. Let $\Omega(\Sigma, P(\mathfrak{g}))_{k,p}$ denote the $W^{k,p}$-completion using the standard metric. Let g^δ denote the C^0 metric on the glued surface in (12). Let

$\Omega^{0,1}(\Sigma^\delta, u^{\delta,*}T^{\mathrm{vert}}P(X))_{k,p,\delta}$. Let

$$\mathcal{H}_\delta := \Omega^2(\Sigma, P(\mathfrak{g}))_{0,3} \oplus \Omega^0(\Sigma, P(\mathfrak{g}))_{0,3} \oplus \Omega^{0,1}(\Sigma^\delta, u^{\delta,*}T^{\mathrm{vert}}P(X))_{0,3,\delta}$$

with norm

$$\|(\phi, \psi, \eta)\|_\delta^2 = \|\phi\|_{0,3}^2 + \|\psi\|_{0,3}^2 + \|\eta\|_{0,3,\delta}^2. \tag{38}$$

Let

$$\mathcal{I}_\delta := \Omega^1(\Sigma, P(\mathfrak{g}))_{1,3} \oplus \Omega^0(\Sigma^\delta, u^{\delta,*}T^{\mathrm{vert}}P(X))_{1,3,\delta}$$

with norm

$$\|(a, \xi)\|_\delta^2 = \|a\|_{1,3}^2 + \|\xi\|_{1,3,\delta}^2.$$

Locally the moduli space of polystable vortices is in bijection with the zero set of the map

$$\begin{aligned} (39) \quad & \mathcal{F}^\delta_{A,u} : \mathcal{I}_\delta \to \mathcal{H}_\delta \\ & (a, \xi) \mapsto \Big(F_{A+a} + \mathrm{Vol}_\Sigma \exp_{u^\delta}(\xi)^* P(\Phi), \mathrm{d}^*_{A,u_+}(a, \xi), \Psi_{u^\delta}(\xi)^{-1}\overline{\partial}_{A+a} \exp_{u^\delta}(\xi) \Big) . \end{aligned}$$

Here the second component enforces a slice condition. That $\mathcal{F}^\delta_{A,u}$ is well-defined follows from Sobolev embedding: In particular, there is a δ-uniform embedding

$$\Omega^0(\Sigma^\delta, u^{\delta,*}T^{\mathrm{vert}}P(X))_{1,3,\delta} \to \Omega^0(\Sigma^\delta, u^{\delta,*}T^{\mathrm{vert}}P(X))_{0,\infty}, \tag{40}$$

since the dimensions of the cone in the cone condition [2, Chapter 4] are uniformly bounded in δ, and the metric uniformly comparable to the flat metric.

Lemma 4.2.4. *Let (A, u) be a symplectic vortex on a nodal curve with a single node $w = (w^+, w^-)$. There exist constants $c_0, c_1 > 0$ such that if $|\delta| < c_1, \rho > 1/c_1$ and$|\delta|\rho^4 < c_1$ then the pair $(A, u^\delta) \in \mathcal{A}(P) \times \Gamma(P(X))$ satisfies*

$$\|\mathcal{F}^\delta_{A,u}(0, 0)\| \le c_0 |\delta|^{1/3} \rho^{2/3}. \tag{41}$$

Proof. The expression $\overline{\partial}_A u^\delta$ can be expressed as a sum of terms involving derivatives of the cutoff function α, terms involving derivatives of $\xi_\pm$, and terms involving the connection A on the bubble region. The derivative of α is bounded by $C/\rho|\delta|^{1/2}$, while the norm of ξ_j is bounded by $C\rho|\delta|^{1/2}$ on the gluing region. Hence the term involving the derivative of α is bounded and supported on a region of area less than $C|\delta|\rho^2$. In the given trivialization we have

$$\overline{\partial}_A u = \overline{\partial} u + A^{0,1}_X(u)$$

where $A^{0,1}$ is the $0,1$-form defined by $A \in \Omega^1(B_R, \mathfrak{g})$ and $A^{0,1}_X(u)$ is the corresponding form with values in $T^{\mathrm{vert}}P(X) \otimes_\mathbb{R} \mathbb{C}$. We have

$$\|A^{0,1}_X(u)\|_{0,3,\delta} \le \|A^{0,1}_X(u)\|_{0,3}$$

since $p \ge 2$; for $p = 2$ the $W^{0,3,\delta}$ and $W^{0,3}$ norms are the same, by conformal invariance; for $p > 2$ the $0,3$-norm is strictly greater. Hence

$$\|\overline{\partial}_A u^\delta\|_{0,3,\delta} \le C \max(|\delta|^{1/3}\rho^{2/3}, |\delta|).$$

The moment map term $F_A + (u^\delta)^* P(\Phi) \operatorname{Vol}_\Sigma$ vanishes except on $|\kappa_+| \le \rho|\delta^{1/2}|$, where it is uniformly bounded. Hence for δ small

$$\| F_A + (u^\delta)^* P(\Phi) \operatorname{Vol}_\Sigma \|_{0,3} \le C \rho^{2/3} |\delta|^{1/3}.$$

The statement of the lemma follows. □

We also wish to perform the gluing construction in families, that is, for each nearby vortex and gluing parameter we wish to find a solution to the vortex equations on Σ^δ. Define

$$(42)\quad \mathcal{F}^{D,\delta}_{A,u} : \operatorname{Def}_\Gamma(A,u) \times \mathcal{I}_\delta \to \mathcal{H}_\delta, \quad (a, \xi, a_1, \xi_1) \mapsto \Big(F_A + a_0 + a_1 + \operatorname{Vol}_\Sigma \exp_{u^\delta}(\xi_0^\delta + \xi_1)^* P(\Phi), \mathrm{d}^*_{A,u^\delta}(a_0 + a_1, \xi_0^\delta + \xi), \Psi_{u^\delta}(\xi_0^\delta + \xi_1)^{-1} \overline{\partial}_{A + a_0 + a_1} \exp_{u^\delta}(\xi_0^\delta + \xi_1) \Big)$$

The following is proved in the same way as Lemma 4.2.4 and left to the reader:

Lemma 4.2.5. *Let (A,u) be as above. There exist constants $c_0, c_1 > 0$ such that if $|\delta| < c_1, \rho > 1/c_1$, $\|(a,\xi)\| \le c_1$ and $|\delta|\rho^4 < c_1$ then*

$$(43)\qquad \| \mathcal{F}^{D,\delta}_{A,u}(a, \xi, 0, 0) \| \le c_0 |\delta|^{1/3} \rho^{2/3}.$$

Step 2: Uniformly bounded right inverse

In preparation for the construction of the uniformly bounded right inverse of $\tilde{D}_\delta$ we define the *intermediate family* (A, u_0^δ) of gauged holomorphic maps on the nodal curve $\hat{\Sigma}$ is the family defined by the equations (4.2.4), using the identification of $\hat{\Sigma}$ and $\hat{\Sigma}^\delta$ away from the gluing region. Thus u_0^δ is constant in a neighborhood of the node $w^\pm$. We identify $(u_0^\delta)^* T^{\mathrm{vert}} P(X)$ with $u^* T^{\mathrm{vert}} P(X)$ by geodesic parallel transport.

Lemma 4.2.6. *The operator $\tilde{D}_{A,u_0^\delta}$ converges in the operator norm to $\tilde{D}_{A,u}$ as $\delta \to 0, \rho \to \infty, \delta\rho^4 \to 0$.*

Proof. The section u_0^δ converges in the $W^{1,3}$ norm to u as $\rho^2|\delta|^{1/2} \to 0$. It follows that the operator $\xi \mapsto \operatorname{Vol}_\Sigma (u_0^\delta)^* L_\xi P(\Phi)$ converges to $\xi \mapsto \operatorname{Vol}_\Sigma u^* L_\xi P(\Phi)$. Hence $\mathrm{d}_{A,u_0^\delta,\epsilon}$ converges to $\mathrm{d}_{A,u,\epsilon}$, and similarly for $\mathrm{d}^*_{A,u_0,\epsilon}$. The operator D_{A,u_0^δ} converges to D_{A,u_0}, as in Lemma 3.3.6. □

Proposition 4.2.7. *Let (A,u) be a nodal vortex, and (A,u^δ) the approximate solution constructed above. There exist constants $c, C > 0$ such that if $|\delta| < c$ then there exists an approximate right inverse T_δ of the parametrized linear operator $\tilde{D}_\delta := \tilde{D}_{A,u^\delta}$ that is, a map $T_\delta : \mathcal{H}_\delta \to \mathcal{I}_\delta$ such that*

$$\|(\tilde{D}_\delta T_\delta - I)\eta\|_\delta \le \frac{1}{2}\|\eta\|_\delta, \quad \|T_\delta \eta\|_\delta \le C\|\eta\|_\delta.$$

Given such an approximate inverse, we obtain a uniformly bounded right inverse Q_δ to $\tilde{D}_\delta$ by the formula

$$Q_\delta = T_\delta(\tilde{D}_\delta T_\delta)^{-1} = \sum_{k\geq 0} T_\delta(\tilde{D}_\delta T_\delta - I)^k.$$

Proof of 4.2.7. By the regularity assumption, $\tilde{D}_{A,u}$ is surjective when restricted to the space of vectors (a,ξ) such that $\xi_0(0) = \xi_1(\infty)$. By Lemma 4.2.6, $\tilde{D}^0_\delta := \tilde{D}_{A,u_0^\delta}$ is surjective for sufficiently small ρ, δ, when restricted to the same space. The approximate right inverse is constructed by composing a cutoff operator K_δ, right inverse Q_δ, and gluing operator R_δ, as follows. (In other words, the right inverse is defined by truncating the given functions, applying the right inverse for the linearized operator on the nodal curve, and then gluing back together using cutoff functions again.) For simplicity we assume that there is a single node. Define the *cutoff operator*

$$K_\delta : \Omega^{0,1}(\hat{\Sigma}^\delta, u^{\delta,*}T^{\mathrm{vert}}P(X))_{0,3,\delta} \to \Omega^{0,1}(\hat{\Sigma}, (u_0^\delta)^*T^{\mathrm{vert}}P(X))_{0,3} \tag{44}$$

$$K_\delta(\eta) = \begin{cases} 0 & \kappa_\pm(z) \in B_{|\delta|^{\frac{1}{2}}}(0) \\ \eta & \text{otherwise} \end{cases}.$$

Then $\|K_\delta(\eta)\|_{0,3} \leq \|\eta\|_{0,3,\delta}$. Define the *gluing operator*

$$R_\delta : \Omega^0(\hat{\Sigma}, (u_0^\delta)^*TX)_{1,3} \to \Omega^0(\hat{\Sigma}^\delta, u^{\delta,*}T^{\mathrm{vert}}P(X))_{1,3,\delta}$$

as follows. Let $\hat{\Sigma}^*_\pm$ denote the complements of small balls around the nodes $\hat{\Sigma}^*_\pm = \Sigma_\pm - B_{\rho^2|\delta|^{1/2}}(w^\pm)$. Let $\pi_\pm : \hat{\Sigma}^*_\pm \to \hat{\Sigma}^\delta$ denote the inclusions. These induce maps of sections with compact support in $\hat{\Sigma}^*_\pm$, $\pi_{\pm,*} : \Omega^0_c(\hat{\Sigma}^*_\pm, u^*_\pm TX) \to \Omega^0(\hat{\Sigma}^\delta, u^{\delta,*}TX)$. Define $R_\delta(\xi) = \xi^\delta$ where

$$\xi^\delta = \pi_{+,*}\beta_{\rho,\delta}(\xi_+ - \xi_+(w_+)) + \pi_{-,*}\beta_{\rho,\delta}(\xi_- - \xi_-(w_-)) + \xi(w)$$

for $\kappa_\pm(z) \in B_{|\delta|^{1/2}\rho^2}(0)$ and $\xi^\delta = \xi$ otherwise. Here $\xi^\pm(w^\pm)$ is the value of ξ at the node. Define $T_\delta := (I \times R_\delta)Q_\delta(I \times K_\delta)$. That is, if $(a,\xi) = Q_\delta K_\delta(\phi,\psi,\eta)$ then $T_\delta(\phi,\psi,\eta) = (a,\xi^\delta)$. The map T_δ is the required approximate right inverse. The difference $(\tilde{D}_\delta T_\delta - I)(\phi,\psi,\eta)$ is the sum of terms

$$\mathrm{d}_{A,u^\delta}(a,\xi^\delta) - \phi, \quad \mathrm{d}^*_{A,u_0}(a,\xi^\delta) - \psi, \quad D_{A,u^\delta}(a,\xi^\delta) - \eta.$$

By definition $\mathrm{d}_{A,u_0^\delta}(a,\xi) = \phi$, so the first difference has contributions involving the difference between u_0^δ and u^δ, and between ξ^δ and ξ. But since $\mathrm{d}_{A,u}(a,\xi) = \mathrm{d}_A a + u^* L_{\xi_X} P(\Phi)$, these terms are zeroth order in u, ξ,

$$\|(u^\delta)^* L_{\xi_X^\delta} P(\Phi) - u_0^\delta L_{\xi_X} P(\Phi)\|_{0,3} \leq C|\delta|^{1/3}\|\xi\|_{C^0};$$

that is, a constant times the area of $\pi_{-,*}(\hat{\Sigma}^*_-)$, which goes to zero as δ does. A similar discussion holds for the second difference. The third difference has terms arising from the cutoff function and the term $a_X^{0,1}(u)$ on the bubble region $|\kappa_+(z)| \leq |\delta|^{1/2}$. We have

$$\|a_X^{0,1}(u)\|_{0,3,\delta} \leq \|a_X^{0,1}(u)\|_{0,3} \leq C|\delta|^{1/3}.$$

since $p \geq 2$; for $p = 2$ the $W^{0,3,\delta}$ and $W^{0,3}$ norms are the same, by conformal invariance; for $p > 2$ the $0,3$-norm is strictly greater. The term involving the derivative of the cutoff function satisfies

$$\|\mathrm{d}\beta_{\rho,\delta}(\xi - \xi(w))\|_{0,3,\delta} \leq c \log(\rho^2)^{-2/3} \|(\xi - \xi(w))\|_{1,3}$$

by Lemma 3.3.5, and so vanishes in the limit $\rho \to \infty$. Using the uniform bound on Q_δ, the total difference is bounded by $C(\log(\rho^2)^{-2/3} + |\delta|^{1/3})\|(\phi, \psi, \eta)\|$, and so vanishes in the limit $\delta \to 0, \rho \to \infty, |\delta|\rho^2 \to 0$. The uniform bound on T_δ follows from the uniform bound on Q_δ and the cutoff and extension operators. □

Step 3: Uniform quadratic estimate

Proposition 4.2.8. *There exist constants $c, C > 0$ such that if $\|\xi\|_{1,3,\delta} < c$, $|\delta| < c, \rho > 1/c$ and $|\delta|\rho^4 < c$ then the map $\mathcal{F}^\delta_{A,u}$ satisfies a quadratic bound*

$$\|D\mathcal{F}^\delta_{A,u}(a_1, \xi_1) - \tilde{D}_{A,u^\delta}(a_1, \xi_1)\|_\delta \leq C\|a, \xi\|_\delta \|a_1, \xi_1\|_\delta.$$

Proof. The norm of the non-linear part of the curvature $\|[a, a_1]\|_{0,3}$ is bounded by Sobolev multiplication. The other term appearing in the first vortex equation satisfies

$$\| \exp_{u^\delta}(\xi_0)^* L_{\xi_1,X} P(\Phi) - (u^\delta)^* L_{\xi_1,X} P(\Phi)\|_{0,3} \leq C\|\xi_0\|_{1,3,\delta}\|\xi_1\|_{1,3,\delta}$$

for some constant C independent of δ, using that $W^{1,3,\delta}$ norm controls the $W^{0,3}$ norm uniformly. The non-linear terms in the Cauchy-Riemann equation are estimated as in Theorem 4.2.8 and [15, Section 3.5, Lemma 10.3.1]; note that we are fixing the complex structure on Σ, which avoids the more complicated analysis we gave in the previous section. The second vortex equation also involves a term of mixed type $\Psi_u(\xi + \xi_1)^{-1}(a_1)^{0,1}_X(\exp_u(\xi + \xi_1)) - \Psi_u(\xi)^{-1}(a_1)^{0,1}_X(\exp_u(\xi))$. It follows from uniform Sobolev embedding that this difference has $0, 3, \delta$-norm bounded by $C\|a_1\|_{1,3}\|\xi_1\|_{1,3,\delta}$ for some constant C independent of δ. □

Step 4: Implicit Function Theorem

Theorem 4.2.9. *Let (A, u) be a regular stable nodal vortex of combinatorial type Γ. There exist constants $\epsilon_0, \epsilon_1 > 0$ such that for every $(a, \xi, \delta) \in \mathrm{Def}_\Gamma(A, u)$ with norm less than ϵ_0, there exists a unique (ϕ, ψ, η) of norm less than ϵ_1 such that if $(a_1, \xi_1) = Q_\delta(\phi, \psi, \eta)$ then $(A + a_0 + a_1, \exp_{u^\delta}(\xi_0^\delta + \xi_1))$ is a symplectic vortex in Coulomb gauge with respect to (A, u^δ). The solution depends smoothly on a_0, ξ_0, and transforms equivariantly the action of $\mathcal{G}(P)_{A,u}$ on $\mathrm{Def}_\Gamma(A, u)$.*

Proof. Uniform error and quadratic estimates are those for $\mathcal{F}^\delta_{A,u}$ in Lemmas 4.2.4, 4.2.7, and 4.2.8, in a uniformly bounded neighborhood of 0 in $\mathrm{Def}_\Gamma(A, u)$. Then the first claim is an application of the quantitative version of the implicit function theorem (see for example [15, Appendix A.3]). Equivariance follows from uniqueness of the solution given by the implicit function theorem, since the map $\mathcal{F}^{D,\delta}_{A,u}$ is equivariant for the action of $\mathcal{G}(P)_{A,u}$. □

Step 5: Rigidification

As in the case of holomorphic maps in the previous section, there is a more natural way of parametrizing nearby symplectic vortices which involves examining the intersections of the sections with submanifolds of $P(X)$, and framings induced by parallel transport. First we study the differentiability of the evaluation maps. The gluing construction of the previous step gives rise to a deformation (A_S, u_S) of (A, u) with parameter space a neighborhood S of 0 in $\operatorname{Def}(A, u)$, and so a map $S \to \overline{M}_n(\Sigma, X), \ s \mapsto (\hat{\Sigma}_s, A_s, u_s)$. Consider the map

$$\operatorname{ev} : (\hat{\Sigma} - U) \times S \to P(X), \quad (z, s) \mapsto u_s(z). \tag{45}$$

Proposition 4.2.10. *The map* ev *of* (45) *is* C^1 *for the family constructed by gluing in Theorem 4.2.9 using the exponential gluing profile.*

Proof. We denote by $u_S^{\mathrm{pre}} : \hat{\Sigma}_S \to X$ the family obtained by pre-gluing only, that is, omitting the step which solves for an exact solution. We denote by $\operatorname{ev}^{\mathrm{pre}}$ the map

$$\operatorname{ev}^{\mathrm{pre}} : (\hat{\Sigma} - U) \times S \to P(X), \quad (z, s) \mapsto u_s^{\mathrm{pre}}(z).$$

This map is independent of the gluing parameters, and is therefore C^1. We write $s = (a_0, \xi_0)$ and $A_s = A + a_0 + a_1, u_s = \exp_{u_s^{\mathrm{pre}}}(\xi_0^\delta + \xi_1)$. The corrections a_1, ξ_1 depend smoothly on a_0, ξ_0, by the implicit function theorem, and so $\xi_1(z)$ depends smoothly on a_0, ξ_0. Next we take the derivative with respect to the gluing parameter. Let (A, u) be a nodal symplectic vortex, (A, u^δ) the pre-glued pair (we omit the parameter ρ controlling the diameter of the gluing region from the notation) and consider the equation $\mathcal{F}_{A,u^\delta}(a_0 + a_1, \xi_0^\delta + \xi_1) = 0$. Let $\tilde{D}_\delta$ denote the derivative of $\mathcal{F}_{A,u^\delta}$. Differentiating with respect to δ gives

$$\tilde{D}_\delta \left(\frac{d}{d\delta} a_1, D \exp_{u^\delta}(\xi_0^\delta; 0, \frac{d}{d\delta}\xi_1) \right) = -\tilde{D}_\delta \left(0, D\exp_{u^\delta}(\xi_0^\delta; \frac{d}{d\delta} u^\delta, \frac{d}{d\delta}\xi_0^\delta) \right).$$

The same arguments as in the proof of Theorem 3.3.11 show that there exists a constant $C > 0$ such that the right hand side is bounded in norm by $Ce^{-1/\delta}$. On the other hand, the norm of the left-hand side $\tilde{D}_\delta$ is uniformly bounded from below in terms of the norm of $\frac{d}{d\delta}a_1, \frac{d}{d\delta}\xi_1$, by the quadratic estimates. It follows that $(\frac{d}{d\delta}a_1, \frac{d}{d\delta}\xi_1)$ is also bounded in norm by $Ce^{-1/\delta}$. Hence $\lim_{\delta \to 0} \partial_\delta \operatorname{ev} = 0$. It follows that $D\operatorname{ev}$ has a continuous limit as $\delta \to 0$. □

Choose a path $\gamma : [0, 1] \to \Sigma$ in the principal component and an element $\phi_0 \in P_{\gamma(0)}$. Let $\tau_\gamma(A) : P_{\gamma(0)} \to P_{\gamma(1)}$ denote parallel transport. By an *m-framed* family of marked curves, we mean a family of curves together with an m-tuple of points in P. Given a family $(\hat{\Sigma}_S, A_S, u_S)$ of gauged holomorphic maps over a parameter space S, a collection of codimension two submanifolds $Y = (Y_1, \ldots, Y_k)$ in $P(X)$, and a collection of paths $\gamma = (\gamma_1, \ldots, \gamma_l)$ with the same initial point y_0 to $y_j, j = 1, \ldots, l$, define a family of marked, framed curves $\hat{\Sigma}_S^{Y,u,\gamma,A} \to S$ by requiring that the additional marked points z_{n+i} map to Y_i, and the framings are given by parallel transport along the paths γ_i.

Definition 4.2.11. The data Y, γ, A, u are *compatible* if

(a) each Y_j intersects u_j transversally in a single point $z_j \in \hat{\Sigma}$;
(b) if $(a, \xi) \in \ker \tilde{D}_{A,u}$ satisfies $\xi(z_{n+j}) \in T_{u(z_{n+j})} P(X)$ for $j = 1, \dots, k$ and $D_A \tau_{\gamma_i}(a) = 0$ for $i = 1, \dots, l$ then $(a, \xi) = 0$.
(c) the curve $\hat{\Sigma}$ marked with the additional points $z_{n+1}, \dots, z_{n+k}$ is stable.
(d) if some automorphism of $(\hat{\Sigma}, u)$ maps z_i to z_j then Y_i is equal to Y_j.

The second condition says that there are no infinitesimal deformations which do not change the positions of the extra markings or framings.

Proposition 4.2.12. *Let (A, u) be a parametrized regular stable nodal vortex, and $(A_S, u_S) \to S$ the stratified-smooth universal deformation constructed by the gluing construction. There exists a collection (Y, γ) compatible with (u, A). Furthermore, if (Y, γ) is compatible with (A, u), then the rigidified family $\hat{\Sigma}_S^{Y,u,\gamma,A} \to S$ of* (26) *is a stratified-smooth deformation of the marked-curve-with-framings $\hat{\Sigma}^{Y,u,\gamma,A}$ which defines an immersion of S into the parameter space for the universal deformation of the central fiber.*

Proof. First we show the existence of a compatible collection. Suppose that the second condition is not satisfied for some (a, ξ). Suppose first that $\xi \neq 0$. Let z_{n+1} be a point with $\xi(z_{n+1}) \neq 0$, and choose a codimension two submanifold Y_{n+1} transverse to u near $u(z_{n+1})$, and such that TY_{n+1} does not contain $\xi(z_{n+1})$. Adding Y_{n+1} to the list of submanifolds decreases the dimension of the space of (a, ξ) satisfying the condition in (b) by at least one. Repeating this process, we may assume that the only elements satisfying the condition in (b) have $\xi = 0$. Suppose that ξ is zero, so that a is necessarily non-zero. Choose an additional marked point y_{l+1} and a path γ_{l+1} from the base point y_0 to y_{l+1} such that the derivative of the parallel transport over γ with respect to a over is non-zero. Appending γ_{l+1} to the list of path decreases the dimension of (a, ξ) satisfying the condition in (b) by at least one. Hence the process stops after finitely many steps, after which the kernel is trivial. The proof of the second claim is similar to Proposition 3.3.15 and will be omitted. □

Step 6: Surjectivity

We show that any nearby vortex appears in the family constructed above. First, we show:

Proposition 4.2.13. *Let (A, u) be a regular strongly stable symplectic vortex. There exists a constant $\epsilon > 0$ such that if $(A_1, u_1) = (A + a, \exp_{u^\delta}(\xi))$ with $\|a\|_{1,3} + \|\xi\|_{1,3,\delta} \leq \epsilon$ then after gauge transformation we have $(A_1, u_1) = (A + a_0 + a_1, \exp_{u^\delta}(\xi_0^\delta + \xi_1))$ for some $(a_0, \xi_0) \in \ker(\tilde{D}_{A,u})$ and (a_1, ξ_1) in the image of Q_δ.*

Proof. We claim that for some constant $C > 0$, we have $(a, \xi) = (a_0, \xi_0^\delta) + (a_1, \xi_1)$ for some $(a_0, \xi_0) \in \ker \tilde{D}_{A,u}$ and $(a_1, \xi_1) \in \operatorname{Im} \tilde{D}^*_{A,u^\delta}$ with norm $\|(a_1, \xi_1)\| \leq C \|(a_1, \xi_1)\|$.

Given the claim, the proposition follows by the uniqueness statement of the implicit function theorem. For any $c > 0$ there exists δ_0 such that for $\delta < \delta_0$,

$$\|\tilde{D}_{A,u^\delta}(a_0^\delta, \xi_0^\delta)\| \leq c\|(a_0^\delta, \xi_0^\delta)\|$$

by estimates similar to those of Lemma 4.2.4. Thus the image of $\ker \tilde{D}_{A,u}$ is transverse to $\operatorname{Im} \tilde{D}^*_{A,u^\delta}$, for δ sufficiently small, since it meets $\operatorname{Im} \tilde{D}^*_{A,u^\delta}$ trivially and projects isomorphically onto $\ker \tilde{D}_{A,u^\delta}$, by gluing for indices, as in [23, Theorem 2.4.1]. The claim then follows from the inverse function theorem. □

Given a strongly stable symplectic vortex (A, u) with stable domain $\hat{\Sigma}$, let (A_S, u_S) be the family given by the gluing construction above. Otherwise, if $\hat{\Sigma}$ is not stable, let $Y = (Y_1, \ldots, Y_l)$ be a collection of codimension two submanifolds of $P(X)$, and consider the family (A, u^Y) with additional marked points given by requiring that the additional marked points z_{n+i} map to Y_i. Let (A_S, u_S) denote the family obtained by applying the gluing construction for (A_S, u_S^Y), and then forgetting the additional marked points.

Lemma 4.2.14. *Suppose that (A_i, u_i) Gromov converges to (A, u). After a sequence of gauge transformations, for any ϵ, there exists i_0 such that if $i > i_0$ then there exists $\delta, (a_i, \xi_i)$ satisfying $(A_i, u_i) = (A + a_i, \exp_{u^\delta}(\xi_i))$ with $\|a_i\|_{1,3} + \|\xi_i\|_{1,3,\delta} \leq \epsilon$.*

Proof. By definition of Gromov convergence, after gauge transformation A_i C^0-converges to A and converges uniformly in all derivatives on the complement of the bubbling set [17]. The exponential decay estimate [17, Lemma A.2.2] show that u_i converges to u on the complement of the nodes, uniformly in all derivatives on compact sets, and whose derivative on the gluing region is uniformly bounded in the δ-dependent metric. It follows that $u_i = \exp_{u^\delta}(\xi_i)$ for some δ and $\xi_i \in \Omega^0(\Sigma^\delta, (u^\delta)T^{\mathrm{vert}}P(X))$ with $\|\xi_i\|_{1,3,\delta} < \epsilon$. To obtain the improved convergence for the connection, note that $F_{A_i} + (u_i)^*P(\Phi) = 0$ and the corresponding equations for the limit (A, u) imply that (omitting Vol_Σ from the notation)

$$F_{A_i} - F_A = \mathrm{d}_A(A_i - A) - (A - A_i) \wedge (A - A_i) = (u_i)^*P(\Phi) - u^*P(\Phi).$$

Since $u_i^*P(\Phi)$ is bounded and converges to $u^*P(\Phi)$ on the complement of the bubbling set, and A_i converges to A in C^0 hence $W^{0,3}$, the right hand side converges to 0 in $W^{0,3}$ as $i \to \infty$. After gauge transformation we may assume that $\mathrm{d}_A^*(A - A_i) = 0$. Then the elliptic estimate for the operator $\mathrm{d}_A \oplus \mathrm{d}_A^*$ implies that $A - A_i$ converges to zero in $W^{1,3}$. □

Corollary 4.2.15. *(A_S, u_S) is a stratified-smooth versal deformation of $(\hat{\Sigma}, A, u)$.*

Proof. Proposition 4.2.13 implies that any family $(A^1_{S^1}, u^1_{S^1})$ is obtained by pull-back from (A_S, u_S), in case $\hat{\Sigma}$ is stable, or obtained from the family obtained by adding the marked points mapping to submanifolds, in general. □

Step 7: Injectivity

We show that any nearby vortex appears once in our family, up to the action of $\mathrm{Aut}(A,u)$; this is part of the following:

Theorem 4.2.16. *Any family (A_S,u_S) constructed by gluing using the exponential gluing profile is a strongly universal stratified-smooth deformation of (A,u).*

Proof. Let $\overline{Z}_n(P,X)$ denote the moduli space of marked symplectic vortices up to equivalences that involve only the identity gauge transformation, so that $\overline{M}_n(P,X) = \overline{Z}_n(P,X)/\mathcal{G}(P)$. Let (A,u) be a stable marked vortex, and $W_{A,u}$ a slice for the gauge group action on $\overline{Z}(P,X)$, so that

$$W_{A,u}/\mathcal{G}(P)_{A,u} \to \overline{M}_n(P,X)$$

is a homeomorphism onto its image. Let $\mathrm{Aut}_0(A,u)$ denote the subgroup of $\mathrm{Aut}(A,u)$ acting trivially on P, so that $\mathcal{G}(P)_{A,u} = \mathrm{Aut}(A,u)/\mathrm{Aut}_0(A,u)$ is the stabilizer of (A,u) under the gauge action. Let (A_S,u_S) denote a universal deformation of (A,u) constructed by gluing using the exponential gluing profile. We claim that the map

$$S/\mathrm{Aut}_0(A,u) \to W_{A,u}, \quad [s] \mapsto [A_s,u_s] \tag{46}$$

is an injection. Indeed, rigidification produces an injection

$$S/\mathrm{Aut}_0(A,u) \to \overline{M}_{n+k,l}(\Sigma)/\mathrm{Aut}_0(A,u), \quad [s] \mapsto [\Sigma^{A_s,u_s,Y,\gamma}] \tag{47}$$

where $\mathrm{Aut}_0(A,u)$ acts by re-ordering the marked points. Since this map factors through (46), the claim follows. If $(A_{S^1},u^1_{S^1})$ is a family of symplectic vortices giving a deformation of any fiber of (A_S,u_S) then Corollary 4.2.15 together with injectivity shows that this family is obtained by pull-back by some map $S^1 \to S$. Hence (A_S,u_S) is a stratified-smooth strongly universal deformation of (A,u). □

Theorem 4.2.17. *Let X be a Hamiltonian G-manifold equipped with a compatible invariant almost complex structure $J \in \mathcal{J}(X)^G$. The maps*

$$S \to \overline{M}_n(\Sigma,X), \quad s \mapsto [A_s,u_s] \tag{48}$$

associated to the universal deformations constructed above equip the locus $\overline{M}_n^{\mathrm{reg}}(\Sigma,X)$ of regular strongly stable symplectic vortices with the structure of a stratified-smooth orbifold. If the local coordinates near the nodes are chosen compatibly and the gluing profile is the exponential gluing profile, then the deformations provide $\overline{M}_n^{\mathrm{reg}}(\Sigma,X)$ with the structure of a C^1-orbifold.

Proof. It suffices to show that the charts given by two sets Y_j,γ_j are compatible. Define $Y = Y_1 \cup Y_2$ and $m = m_1 + m_2$ the total number of extra points. Similarly let γ be the union of γ_1 and γ_2 of total number $l = l_1 + l_2$. The family $\hat{\Sigma}_S^{Y,u,\gamma,A}$ admits a proper étale forgetful map $\hat{\Sigma}_S^{Y,u,\gamma,A} \to \hat{\Sigma}_S^{Y_j,u,\gamma_j,A}, \quad j = 1,2$ whose fiber consists of the re-orderings of the points for Y induced by automorphisms of $\mathrm{Aut}(A,u)$ that fix the ordering for Y_j. It follows that the corresponding charts are C^1-compatible. □

Remark 4.2.18. As discussed in Remark 3.3.22, the Theorem implies that if $\overline{M}_n^{\mathrm{reg}}(\Sigma,X)$ is compact then it admits a (non-canonical) smooth structure.

Let $\overline{M}_n(\Sigma)$ denote the moduli space of stable maps to Σ with homology class $[\Sigma]$, n markings and genus that of Σ, or in other words, *parametrized* stable curves with principal component isomorphic to Σ. Forgetting the pair (A, u) gives a forgetful morphism $\overline{M}_n^{\mathrm{reg}}(\Sigma, X) \to \overline{M}_n(\Sigma)$. Using the differentiable structure defined above, the evaluation maps are differentiable but unfortunately the forgetful morphisms are not, unless one uses a different gluing profile for the moduli space of vortices with one less marking. More precisely, the forgetful morphism $\overline{M}_n^{\mathrm{reg}}(\Sigma, X) \to \overline{M}_n(\Sigma)$ is continuous and C^1 near any pair (A, u) whose domain is stable as an element of $\overline{M}_n(\Sigma)$, and a submersion near the boundary of $\overline{M}_n(\Sigma)$. For the standard smooth structure on $\overline{M}_n(\Sigma)$, the forgetful morphism $\overline{M}_n^{\mathrm{reg}}(\Sigma, X) \to \overline{M}_n(\Sigma)$ is smooth.

The gluing construction has various parametrized versions. For example, in [11] we consider a moduli space of *polystable polarized vortices*, which consist of a vortex together with a lift of the connection to the Chern-Simons line bundle. In each of these cases one applies the implicit function theorem using the linearized operator for the parametrized problem to prove that any *parametrized regular* polystable vortex has a strongly universal deformation in the parametrized sense. In particular, any regular polystable polarized vortex has a strongly universal deformation etc.

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