CLASSIFICATION OF AFFINE VORTICES

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Abstract. We prove a Hitchin-Kobayashi correspondence for affine vortices generalizing a result of Jaffe-Taubes [19] for the action of the circle on the complex line. Namely, suppose a compact connected Lie group K acts on a Kähler manifold X with proper moment map so that stable=semistable for the action of the complexified Lie group G and X is equivariantly convex at infinity. Then, for some sufficiently divisible integer n, there is a bijection between gauge equivalence classes of K-vortices with target X modulo gauge and isomorphism classes of maps from the weighted projective line $\mathbb{P}(1,n)$ to X/G that map the stacky point at infinity $\mathbb{P}(n)$ to semistable locus of X^{ss}. The results allow the construction and partial computation of the quantum Kirwan map in Woodward [40], and play a role in the conjectures of Dimofte, Gukov, and Hollands [10] relating vortex counts to knot invariants.

Contents

1. Introduction

A *vortex* in geometric analysis is a pair consisting of a connection and bundle section which satisfies an equation involving the curvature with a non-linear (usual quadratic) term depending on the section, as well as a first order equation for the section. Vortices arise naturally in several areas of mathematical physics and geometry; our interest in them arises from their interpretation as equivariant generalizations of pseudoholomorphic maps. Our goal in this paper is to classify vortices, in the special case that the domain is the complex line. In this setting, Jaffe-Taubes [19] provided a classification in the first non-trivial case of finite energy vortices with circle group $K = U(1)$ and target space $X = \mathbb{C}$. In this paper we generalize their classification to arbitrary compact connected Lie groups K and fiber bundles whose

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fiber is a Kähler manifold X with a Hamiltonian K-action. Here X is either compact or is equivariantly convex at infinity with a proper moment map. More precisely we classify *affine vortices* with target X, consisting of pairs (A, u) where A is the connection on the principal bundle $P = \mathbb{C} \times K$, and $u : \mathbb{C} \to P(X) := (P \times X)/K$ is a holomorphic section and (A, u) satisfies the vortex equation:

$$*F_A + \Phi(u) = 0. \tag{1}$$

Here F_A is the curvature of A, $\Phi : X \to \mathfrak{k}^\vee \simeq \mathfrak{k}$ is the moment map on X for the K-action, and $*$ is the Hodge star for the standard metric on $\mathbb{C}$.

In the special case of the action of the circle group on the complex line, Jaffe-Taubes [19] show that the gauge-equivalence class of a finite energy vortex is completely determined by the zeros of the section u, so that the moduli space of gauge equivalence classes of finite energy vortices is a symmetric product. This gives a description of the space of vortices with target $\mathbb{C}$ in terms of holomorphic data which from a more modern perspective may be viewed as a *Hitchin-Kobayashi correspondence.* The first example of such a correspondence by Narasimhan and Seshadri [25] shows that stable holomorphic bundles over a Riemann surface correspond to irreducible unitary representations of the fundamental group. Donaldson [11] reproves the Narasimhan-Seshadri theorem in a differential-geometric setting, replacing irreducible unitary representations by an equivalent object - the minima of the Yang-Mills functional. The extension of this result to higher dimensional base manifolds involved replacing a Yang-Mills connection by a Hermitian-Einstein connection. A holomorphic vector bundle admits a Hermitian-Einstein connection if and only if it is stable - this was proved for Kähler surfaces in [12] and for general compact Kähler manifolds by [35].

In the case of vortices the corresponding holomorphic objects are holomorphic bundles over compact Kähler manifolds with additional data, for example a prescribed holomorphic section. Vortices are the zeros of the *vortex functional* given by the norm-square of the left hand side of (1). The zeros of the vortex functional correspond to minima of the Yang-Mills-Higgs functional - also called the energy functional for gauged holomorphic pairs. Bradlow's paper [4] defines a stability condition for such objects and relates it to the existence of zeros of the vortex functional. Results in Bradlow [4] are used to investigate the moduli space of finite energy vortices in [5], [2] and [3]. Garcia-Prada [27], [28], [29] provides a different approach to similar results via dimensional reduction, achieving in some cases more general results. Mundet [24] generalizes these results by allowing the fiber to be a Kähler Hamiltonian manifold. All these Hitchin-Kobayashi results are infinite-dimensional versions of the abstract setting laid out in Kempf-Ness [20] and Kirwan [21], in which the main idea is that the symplectic quotient coincides with the geometric invariant theory quotient. We call the pair of a holomorphic bundle with a section a *gauged holomorphic map.*

The stability condition for gauged holomorphic map looks surprisingly simple when the base manifold is a non-compact Riemann surface with cylindrical or Euclidean ends. The finite energy condition for vortices ensures that the singularity at infinity can be removed, and the gauged holomorphic map extends to some principal bundle over a compactified Riemann surface obtained by adding 'points at infinity'. The stability condition is that under the gauged map, the point at infinity is mapped to the semistable locus $X^{\rm ss}$. In the case of $\Sigma = \mathbb{C}$, if in addition we assume that G acts freely on the semistable locus, the gauged holomorphic map extends to a principal bundle over $\mathbb{P}^1$. If on the other hand, the G action on $X^{\rm ss}$ has finite stabilizers, $\mathbb{P}^1$ is replaced by the weighted projective line $\mathbb{P}(1,n)$. Here n is an integer that is divisible by the order of the stabilizer groups $|G_x|$ for all $x \in X^{\rm ss}$. The weighted projective line $\mathbb{P}(1,n)$ is the quotient of $\mathbb{C}^2 - \{0\}$ by the action of $\mathbb{C}^\times$ with weights $1, n$.

Our main result is a bijection between finite energy affine vortices and a moduli space of gauged holomorphic maps: the space of gauged holomorphic maps on the weighted projective line $\mathbb{P}(1,n)$ (the principal K-bundle is allowed to vary) to X that map the point at infinity $\mathbb{P}(n)$ to $X^{\rm ss}$ modulo the group of complex gauge transformations. Alternatively this space can be defined as the isomorphism classes of the space of pairs $(P_{\mathbb{C}}, u)$ where $P_{\mathbb{C}} \to \mathbb{P}(1,n)$ is a principal G-bundle and $u : \mathbb{P}(1,n) \to P_{\mathbb{C}} \times_G X$ is a holomorphic section that maps $\mathbb{P}(n)$ to $P(X^{\rm ss})$.

Theorem 1.1. (Classification of affine vortices) *Let X, G, K be as above. Let n be a positive integer such that for any $x \in X^{\rm ss}$, the order of the stabilizer group $|G_x|$ divides n. Then there exists a bijection between gauge equivalence classes of affine K-vortices with target X, and gauged holomorphic maps $(P_{\mathbb{C}}, u)$ from $\mathbb{P}(1,n)$ to X such that over $\mathbb{P}(n)$ the section u maps to $P_{\mathbb{C}} \times_G X^{\rm ss}$ modulo complex gauge transformations.*

The moduli space of gauged holomorphic maps in the theorem has a convenient stack-theoretic interpretation. The weighted projective line is a Deligne-Mumford stack or an orbifold. Recall that if C is a complex curve, then by a definition of Deligne-Mumford the category of morphisms from C to the *quotient stack* X/G is the category of pairs of a principal G-bundle over C together with a section of the associated fiber bundle $P \times_G X$. The quotient stack X/G contains as a proper open substack the geometric invariant theory quotient $X/\!\!/G$, here defined as the stack-theoretic quotient of the semistable locus by the action of G. The same is true when C is replaced by $\mathbb{P}(1,n)$ (see [22]). $M^G(\mathbb{P}(1,n), X)$ is the coarse moduli space of the open substack of $\mathrm{Hom}(\mathbb{P}(1,n), X/G)$ that satisfies $u(\infty) \in X/\!\!/G$. In the case that G is a torus acting on a finite dimensional complex vector space X, bundles on $\mathbb{P}(1,n)$ and sections of the associated vector bundle can be classified explicitly.

A recent paper of Xu [41] complements our work. It proves similar results as this paper for $U(1)$-vortices with fiber $X = \mathbb{C}^m$ using results of [4]. It also shows a correspondence between compactifications of the space of affine vortices modulo gauge on the one side and the space of gauged holomorphic maps over $\mathbb{P}^1$, that are semistable at ∞.

The Hitchin-Kobayashi correspondence for affine vortices in Theorem 1.1 is partly motivated by a certain quantization of the Kirwan map that arises in the study of Gromov-Witten invariants of geometric invariant theory quotients. Namely Kirwan

[21] constructs a map from the equivariant cohomology $H_G(X,\mathbb{Q})$ to the cohomology of the quotient $H(X/\!\!/G,\mathbb{Q})$. A Gromov-Witten generalization of [21] called the quantum Kirwan map, suggested by Gaio-Salamon [15], maps the equivariant quantum cohomology of X to the quantum cohomology of the quotient $X/\!\!/G$. Salamon and Ziltener [42] suggested to defined the quantum Kirwan map by a count of affine vortices. the quantum Kirwan map is defined by counting affine vortices on X. The work of the second author [26], [40] generalizes the manifolds for which this map is defined, by removing monotonicity and asphericity assumptions on X. Also, the group action can have finite stabilizers on the zero-level set of the moment map, i.e. the symplectic quotient can be an orbifold. This paper is part of that project. It provides an algebraic description for the moduli space of vortices, and from there, the quantum Kirwan map can be defined using Behrend-Fantechi machinery. A very important result in this context is a compactification for the space of affine vortices modulo gauge proved by Ziltener ([42], [43]).

An additional recent motivation arises from *knot-invariants via vortex counting* conjectures of Dimofte, Gukov, and Hollands [10]. In these conjectures, the equivariant index (defined via localization) of the moduli space of affine vortices is conjectured to be a certain knot invariant. Our results allow the identification of the moduli space of affine vortices with the *quasimap* spaces discussed in, for example, Bertram, Ciocan-Fontanine, and Kim [1]. The space of matrix-valued vortices in Example 1.3 appears as the relevant space of vortices for a torus knot in the vortex counting conjectures of [10].

The proof uses the gradient flow result (see Theorem 3.2) of the functional

$$(A,u) \mapsto \|{*F_A} + \Phi(u)\|_{L^2(\Sigma)}$$

where Σ is a compact Riemann surface. This is proved by the first author in [37]. Using this result, the vortex equation can be solved on a succession of annuli of increasing sizes that exhaust $\mathbb{C}$. This sequence of vortices converges to a solution on the whole plane. This process involves a number of complications, such as that the sequence of solutions on the balls converges without bubbling.

We give some examples of the main result in the case of torus actions on vector spaces. Given a finite energy affine vortex (A,u), it extends to a gauged holomorphic map over a principal K-bundle $P \to \mathbb{P}(1,n)$. The characteristic class $[P]$ of P in $H_2(BK,\mathbb{Q})$ is a topological invariant of (A,u) (see remark 2.5). If K is a torus then $H_2(BK,\mathbb{Q})$ is isomorphic to the subset $\{\lambda \in \mathfrak{k} \,|\, \exists n \in \mathbb{Z},\ e^{2\pi n\lambda} = \mathrm{Id}\}$ of rational elements $\mathfrak{k}$ and we call the image of $[P]$ in $\mathfrak{k}$ the *degree* of the vortex. In the following proposition, we classify affine vortices of fixed degree.

Proposition 1.2. *Suppose that G is a complex torus acting on a finite dimensional complex vector space X with weights $\mu_1,\dots,\mu_k \in \mathfrak{g}^\vee$ contained in an open half space, and spanning $\mathfrak{g}^\vee$. Then there is a bijection between affine vortices of degree $d \in \mathfrak{k} \subset \mathfrak{g}$ and isomorphism classes of tuples of polynomial maps $u = (u_1,\dots,u_k) : \mathbb{C} \to X$ satisfying*

(a) *the degree of u_j is at most $\langle \mu_j, d\rangle$ for each $j = 1,\dots,k$; and*

(b) *if*

$$u(\infty) := \left(u_j(\infty) := \begin{cases} u_j^{(\langle \mu_j, d\rangle)}/\langle \mu_j, d\rangle! & \langle \mu_j, d\rangle \in \mathbb{Z} \\ 0 & \textit{otherwise} \end{cases} \right)_{j=1}^{k}$$

denotes the vector of leading order coefficients ($\langle \mu_j, d\rangle$-th derivatives) with integer exponents, then $u(\infty) \in X^{\mathrm{ss}}$.

Two such tuples are isomorphic if they are related by the action of G.

We defer the proof to Section 4.

Example 1.3. (a) (Jaffe-Taubes classification) If $X = \mathbb{C}$ with $\mu_1 = 1$, then affine vortices of class d correspond to polynomials of degree exactly d up to the action of scalar multiplication, hence classified by their zeroes. This recovers the Jaffe-Taubes [19] result.

(b) (Matrix-valued vortices and quot schemes) If $X = M_n(\mathbb{C})$, the space of $n \times n$ matrices, and $G = GL_n$ acts by left multiplication, then the semistable locus consists of invertible matrices and the action on the semistable locus is free. Theorem 3.1 gives a classification of vortices according to the following data: By Grothendieck's theorem [18], any holomorphic vector bundle on $\mathbb{P}^1$ splits as a sum of line bundles

$$P \times_{GL_n(\mathbb{C})} \mathbb{C}^n \cong \mathcal{O}_{\mathbb{P}^1}(\lambda_1) \oplus \ldots \oplus \mathcal{O}_{\mathbb{P}^1}(\lambda_n), \quad \lambda_1 \geq \ldots \geq \lambda_n$$

where $\mathcal{O}_{\mathbb{P}^1}(\lambda)$ is the λ-th tensor power of the hyperplane bundle. The associated X bundle is then $P(X) = \mathcal{O}_{\mathbb{P}^1}(\lambda_1)^{\oplus n} \oplus \ldots \oplus \mathcal{O}_{\mathbb{P}^1}(\lambda_n)^{\oplus n}$. A section u of $P(X)$ may be viewed as a matrix-valued function on $\mathbb{C}$. The semistability condition at infinity is then the condition that the leading order terms of u form an invertible matrix. Thus u defines a morphism of sheaves which is generically an isomorphism, providing a connection to the quot schemes considered in [1].

We briefly sketch the contents of the paper. Section 2 defines gauged holomorphic maps and extends the definitions to the case when the base manifold Σ is an orbifold $\mathbb{P}(1, n)$. In section 3, we give an analytic version of the main theorem and prove the surjectivity part of the main theorem 3.1. Section 4 proves injectivity. The proof relies on removal of singularities at infinity for finite energy vortices which is proved in section 5.

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2. Background

In this section we introduce basic notation for gauged holomorphic maps and vortices, especially gauged holomorphic maps on weighted projective lines.

2.1. **Kähler Hamiltonian manifolds.** We introduce the following notation for Hamiltonian group actions. Let G be a complex reductive Lie group with maximal compact subgroup K, so that G is the complexification of K. Let (X, ω, J) be a Kähler manifold on which G acts holomorphically and K acts symplectically. A *moment map* is a K-equivariant map $\Phi : X \to \mathfrak{k}^\vee$ such that

$$\iota(\xi_X)\omega = \mathrm{d}\langle \Phi, \xi \rangle, \ \forall \xi \in \mathfrak{k},$$

where $\xi_X \in \mathrm{Vect}(X)$ given by the infinitesimal action of ξ on X. The action of K is *Hamiltonian* if there exists a moment map $\Phi : X \to \mathfrak{k}^\vee$. Since K is compact, $\mathfrak{k}$ has an Ad-invariant metric. We fix such a metric and $\mathfrak{k}^\vee$ and so the moment map becomes a map $\Phi : X \to \mathfrak{k}$. We assume X is equipped with a Hamiltonian action and fix the moment map. If X is a polarized projective G-variety, the geometric invariant theory quotient is defined by $X/\!\!/G := X^{\mathrm{ss}}/\sim$ where X^{ss} is the semi-stable locus and $\sim$ is the orbit closure relation. In the rest of this paper, we assume

Assumption 2.1. *The G-action on X^{ss} has finite stabilizers.*

It follows that $X/\!\!/G$ is the orbit space of the action of G on X^{ss}. From Kempf-Ness [20] and Kirwan [21], the geometric invariant theory quotient $X/\!\!/G$ is homeomorphic to the symplectic quotient $\Phi^{-1}(0)/K$. Assumption 2.1 guarantees that the K action on $\Phi^{-1}(0)$ has finite stabilizers, and that $X^{\mathrm{ss}} = G\Phi^{-1}(0)$. Under these assumptions we have $X^{\mathrm{ss}} = G\Phi^{-1}(0)$.

2.2. **Connections and curvature.** We begin with basic notions of connections, curvature, and their behavior under gauge transformations. Let Σ be a manifold and $P \to \Sigma$ a principal K-bundle. A *connection* is a K-equivariant one-form $A \in \Omega^1(P, \mathfrak{k})^K$, that satisfies $A(\xi_P) = \xi$ for $\xi \in \mathfrak{k}$. The *space of connections* $\mathcal{A}(P)$ is an affine space modelled on $\Omega^1(\Sigma, P(\mathfrak{k}))$, where $P(\mathfrak{k}) = P \times_K \mathfrak{k}$ is the *adjoint bundle*. In case P is the trivial bundle $\Sigma \times K$, there is a trivial connection D, and the adjoint bundle has a trivialization $P(\mathfrak{k}) \simeq \Sigma \times \mathfrak{k}$. Then, the space of connections is

$$\mathcal{A}(P) = \mathrm{D} + \Omega^1(\Sigma, \mathfrak{k}).$$

On any bundle over Σ, where the fiber has a K-action, a connection A defines a covariant derivative d_A. For example on the bundle $\Sigma \times X$, writing $A = \mathrm{D} + a$,

$$\mathrm{d}_A := u \mapsto \mathrm{d}u + a_u \in \Omega^1(\Sigma, u^*TX).$$

At a point $x \in \Sigma$, $a_u(x)$ is the infinitesimal action of $a(x)$ at $u(x)$. The *curvature* of a connection A is a two-form $F_A \in \Omega^2(\Sigma, P(\mathfrak{k}))$. In particular, on a trivial bundle, for a connection $A = \mathrm{D} + a$,

$$F_A := \mathrm{d}a + [a \wedge a]/2 \in \Omega^2(\Sigma, \mathfrak{k}).$$

The curvature varies with the connection as

$$F_{A+ta} = F_A + t\mathrm{d}_A a + \frac{t^2}{2}[a \wedge a].$$

A *gauge transformation* is an automorphism of P - it is an equivariant bundle map $P \to P$. Alternatively, it is a section of the bundle $P \times_K K \to \Sigma$, where K acts on itself by conjugation. The group of gauge transformations on P is denoted $\mathcal{K}(P)$. On the trivial bundle $\Sigma \times K$, $k \in \mathcal{K}(P)$ is a map $k : \Sigma \to K$. It acts on a connection $A = \mathrm{D} + a$ as

$$k(A) = \mathrm{D} + (\mathrm{d}k k^{-1} + \mathrm{Ad}_k\, a).$$

Differentiating, we see that the infinitesimal action of $\xi : \Sigma \to \mathfrak{k}$ on A is $-\mathrm{d}_A \xi$.

2.3. **Complex gauge transformations and holomorphic structures.** We introduce notation for complex gauge transformations. Associated to the principal K-bundle $P \to \Sigma$, we have a principal G-bundle $P_{\mathbb{C}} := P \times_K G$ on Σ. A *complex gauge transformation* is a G-equivariant bundle automorphism $P_{\mathbb{C}} \to P_{\mathbb{C}}$. The group of complex gauge transformations is denoted by $\mathcal{G}(P)$. It is the space of sections $g : \Sigma \to P \times_K G$, where K acts on G by conjugation. Recall that

$$K \times \mathfrak{k} \to G, \quad (k, s) \mapsto k e^{is} \tag{2}$$

is an isomorphism. So, a complex gauge transformation g can be written as $g = k e^{i\xi}$, where $k \in \mathcal{K}(P)$ and $\xi \in \mathrm{Lie}(\mathcal{K}(P)) = \Gamma(P(\mathfrak{k}))$. We next explain the action of $\mathcal{G}(P)$ on the space of connections $\mathcal{A}(P)$.

Unitary connections determine holomorphic structures on fiber bundles via the associated bundle construction. Let $P \to \Sigma$ be a principal K-bundle and X a complex manifold with a K-action. A connection $A \in \mathcal{A}(P)$ determines a holomorphic structure on the associated bundle $P(X) := P \times_K X$ by the $\overline{\partial}_A$ operator:

$$\overline{\partial}_A := \Gamma(\Sigma, P(X)) \ni u \mapsto (d_A u)^{0,1} \in \Omega^{0,1}(\Sigma, u^* T^{vert} P(X)).$$

In particular, taking $X = G$ produces a holomorphic G-bundle $P_{\mathbb{C}}$. Conversely, given a holomorphic bundle $P_{\mathbb{C}}$, a choice of a section $\sigma : \Sigma \to P_{\mathbb{C}}/K$ gives a principal K-bundle P by pullback of the bundle $P_{\mathbb{C}} \to P_{\mathbb{C}}/K$, which is naturally a submanifold of $P_{\mathbb{C}}$. The intersection $TP \cap J(TP)$ defines a connection in TP [32].

The group of complex gauge transformations $\mathcal{G}(P)$ acts on the space of holomorphic structures $\mathcal{C}$ on $P_{\mathbb{C}}$, this action pulls back to an action of $\mathcal{G}(P)$ on the space of connections $\mathcal{A}(P)$. The correspondence between connections and holomorphic structures gives an infinitesimal isomorphism

$$T_A \mathcal{A} = \Omega^1(\Sigma, P(\mathfrak{k})) \to T_C \mathcal{C} = \Omega^{0,1}(\Sigma, P(\mathfrak{g})), \quad a \mapsto a^{0,1}.$$

The complex structure on $\mathcal{C}$ pulls back to a complex structure on $\mathcal{A}$ given by $J_{\mathcal{A}} a = a \circ j_\Sigma = *a$. For any $\xi \in P(\mathfrak{k})$, the infinitesimal action of $i\xi$ on A is $- * \mathrm{d}_A \xi$.

2.4. **Gauged holomorphic maps.** Roughly speaking "gauging" an object means introducing a connection into the picture. By gauged holomorphic maps we mean pairs of a connection and a section holomorphic with respect to the corresponding complex structure on the fiber bundle. In this section we assume that Σ is a Riemann surface, equipped with a conformal structure $j_\Sigma : T\Sigma \to T\Sigma$ and a volume form $\omega_\Sigma \in \Omega^2(\Sigma)$ that induces the metric. A *gauged holomorphic map* (A, u) from P to X consists of a connection A and a section u of $P(X)$ that is holomorphic with respect to $\overline{\partial}_A$. The space of gauged holomorphic maps from P to X is called $\mathcal{H}(P, X)$. A *symplectic vortex* is a gauged holomorphic map that satisfies

$$*F_A + \Phi(u) = 0.$$

The *energy* of a gauged holomorphic map (A, u) is

$$E(A, u) := \int_\Sigma (|F_A|^2 + |\mathrm{d}_A u|^2 + |\Phi \circ u|^2)\omega_\Sigma.$$

In the literature, the energy map E is also called the Yang-Mills-Higgs functional. For $\Sigma = \mathbb{C}$, finite energy symplectic vortices have good asymptotic properties (see [15] section 11, [43]). The complexified gauge group acts on $\mathcal{H}(P, X)$ pairwise:

$$g : (A, u) \mapsto (g(A), gu) = ((g^{-1})^* A, gu).$$

This action preserves holomorphicity (see [37]) but not the vortex equation unless the gauge transformation is unitary. From the holomorphic viewpoint, a gauged holomorphic map (A, u) on a Riemann surface Σ can be described by the following data :

(a) A G-equivariant holomorphic map $u : P_\mathbb{C} \to X$, where $P_\mathbb{C} = P \times_K G$ and its holomorphic structure is given by A
(b) and a section $\sigma : \Sigma \to P_\mathbb{C}/K$.

The complex gauge equivalence class of (A, u) is specified by the map u, that is, it does not depend on σ.

In this paper we are mostly interested in the case of a trivial bundle. When P is the trivial bundle $\Sigma \times K$, the section u is a map $u : \Sigma \to X$. The formula for the covariant derivative on $P(X)$ becomes

$$\mathrm{d}_A u = \mathrm{d}u + a_u \in \Omega^1(\Sigma, u^*TX).$$

Here $A = \mathrm{D} + a$ and a_u is the infinitesimal action of a on u - i.e. for any $x \in \Sigma$, $a_u(x) = a(x)_{u(x)}$. Then,

$$\overline{\partial}_A u = \overline{\partial} u + a_u^{0,1}.$$

2.5. **Gauge theory on weighted projective lines.** In the Hitchin-Kobayashi correspondence we wish to prove, gauged holomorphic maps on orbifolds play an important role. For orbifolds, we follow classical definitions [31]. Gauged holomorphic maps on orbifolds are similar to J-holomorphic curves on orbifolds described in [6]. We will not repeat full definitions here, but just describe what the definitions give in the specific case that the orbifold is the weighted projective line $\mathbb{P}(1, n)$.

We begin with some notation for weighted projective lines. For n a positive integer, $\mathbb{P}(1, n)$ is the quotient of $\mathbb{C}^2 - \{0\}$ by the action of $\mathbb{C}^\times$ with weights $1, n$. It is covered by two orbifold charts, $\tilde{U}_1$ and U_2 where $\tilde{U}_1 = U_2 = \mathbb{C}$ and the equivalences are :

$$\begin{aligned} z &\sim e^{2\pi i/n} z \qquad && z \in \tilde{U}_1 \\ z^{-n} &\sim w && 0 \neq w \in U_2, 0 \neq z \in \tilde{U}_1 \end{aligned} \tag{3}$$

We refer to $\tilde{U}_1/\sim$ as U_1.

Principal bundles on the weighted projective line can be described in terms of these charts via the clutching construction as follows. Let $\sigma_n : \tilde{U}_1 \to \tilde{U}_1,\ z \mapsto e^{2\pi i/n} z$ denote the diffeomorphism giving the action of the generator of $\mathbb{Z}_n$. A principal K-bundle $P \to \mathbb{P}(1,n)$ is described as follows. Let $\tilde{P}|_{\tilde{U}_1}$ denote the lift of $P|_{U_1}$ to $\tilde{U}_1$. P is given by gluing the trivial bundles $\tilde{P}|_{\tilde{U}_1} \simeq \tilde{U}_1 \times K$ and $P|_{U_2} \simeq U_2 \times K$ under the equivalences

$$
\begin{aligned}
&(z,h) \sim (\sigma_n z, \mu(z)h) \qquad (z,h) \in \tilde{U}_1 \times K \\
&(z,h) \sim (w, \tau(z)h) \qquad 0 \neq z \in \tilde{U}_1, w \in U_2, w = \frac{1}{z^n}, h \in K
\end{aligned}
\tag{4}
$$

where $\mu : \tilde{U}_1 \to K$ defines a $\mathbb{Z}_n$-action on $\tilde{U}_1 \times K$ and and $\tau : \tilde{U}_1 \backslash \{0\} \to K$ satisfies $\tau(e^{2\pi i/n} z) = \tau(z)\mu(z)^{-1}$. Note that the fiber over the singular point $0 \in \tilde{U}_1$ may not be K, it could just be $K/\mathbb{Z}_n$. We remark in the case $n = 1$, $\mu = \mathrm{Id}$ and $\tau : \mathbb{C}^\times \to K$ is the standard transition function. Two orbifold bundles over $\mathbb{P}(1,n)$ given by transition functions (μ_0, τ_0) and (μ_1, τ_1) are isomorphic if there exist smooth functions $\phi_1 : \tilde{U}_1 \to K$ and $\phi_2 : U_2 \to K$ satisfying

$$
\begin{aligned}
\phi^{-1}(\sigma_n(z))\mu_0(z)\phi_1(z) &= \mu_1(z) \quad \forall z \in \tilde{U}_1 \\
\phi_2^{-1}(w)\tau_0(z)\phi_1(z) &= \tau_1(z) \quad \forall z \in \tilde{U}_1, w \in U_2, w = z^{-n}.
\end{aligned}
$$

The clutching construction for bundles above is related to the classification of principal bundles up to isomorphism. In the case without singularities, the set of principal K-bundles $P \to \mathbb{P}^1$ is in bijection with $\pi_1(K)$. This can be seen as follows: the deformation retract of the transition map $\mathbb{C}^* \to K$ is a loop $S^1 \to K$, whose homotopy type determines the bundle. The loop $S^1 \to K$ can be deformed to a geodesic loop $\theta \mapsto e^{\lambda\theta}$, where $\lambda \in \mathfrak{k}$ satisfies $e^{2\pi\lambda} = \mathrm{Id}$. In the case of orbifold singularities, given a bundle $P \to \mathbb{P}(1,n)$, the transition maps would now produce a geodesic path $\theta \in [0, 2\pi] \mapsto e^{\lambda\theta}$, $\lambda \in \mathfrak{k}$ satisfies $e^{2\pi n\lambda} = \mathrm{Id}$. The isomorphism type of the bundle P is determined by the homotopy class of the *classifying path* $\theta \mapsto e^{\lambda\theta}$, that is, deformations keeping the endpoints Id and $e^{2\pi\lambda}$ fixed or by applying Ad_k to the path for some $k \in K$. In this way one obtains a bijection between isomorphism classes of K-bundles over $\mathbb{P}(1,n)$ and elements of $\exp(2\pi i n\cdot)^{-1}(1)$, up to conjugacy.

Connections on principal bundles can be described via their restriction to trivialization associated to the clutching construction. A connection on $P \to \mathbb{P}(1,n)$ is given by connections on trivializations $\tilde{U}_1 \times K$ and $U_2 \times K$ that satisfy the equivalences (4):

(a) The connection $A|_{\tilde{U}_1}$ satisfies

$$
\sigma_n^* A = \mu(A), \tag{5}
$$

where μ as a gauge transformation.

(b) By the above condition, $\sigma_n^*(\tau(A)) = \tau(A)$ on $\tilde{U}_1 \backslash \{0\}$, so $\tau(A)$ descends to a connection on $\tilde{U}_1 \backslash \{0\}/\mathbb{Z}_n$. We require that this descended connection is $A|_{U_2 \backslash \{0\}}$.

Finally, we describe gauge transformations in terms of the canonical atlas. A gauge transformation k on P consists of $\tilde{k}_1 = k|_{\tilde{U}_1} : \tilde{U}_1 \to K$ and $k_2 = k|_{U_2} : U_2 \to K$ satisfying the equivalences (4) -

(a) $\sigma_n^* k_1 = \mu k_1 \mu^{-1}$ and
(b) $\tau k_1|_{\tilde{U}_1 \backslash \{0\}} \tau^{-1}$ descends to k_2.

The orbifold $\mathbb{P}(1,n)$ has smooth locus given by a copy of $\mathbb{C}$, and if $n \neq 1$ has an "orbifold singularity" at ∞. More precisely, the orbifold point is the quotient $\mathbb{P}(n)$ of $\mathbb{C}^*$ by $\mathbb{C}^*$ acting with weight n. As a groupoid this is equivalent to the quotient of a point by $\mathbb{Z}_n$, that is, $B\mathbb{Z}_n$.

Finally we describe gauged holomorphic maps on weighted projective lines. Let $P \to \mathbb{P}(1,n)$ be a principal K-bundle. A *gauged holomorphic map* (A,u) from $\mathbb{P}(1,n)$ to X consists of gauged holomorphic maps on the bundles $\tilde{U}_1 \times K$ and $U_2 \times K$ that satisfy the equivalence conditions (4)

(a) $(A,u)|_{\tilde{U}_1}$ satisfies $\sigma_n^*(A,u) = \mu(A,u)$ (viewing μ as a gauge transformation).
(b) By the above condition, $\sigma_n^*(\tau(A,u)) = \tau(A,u)$ on $\tilde{U}_1 \backslash \{0\}$, so it descends to a gauged holomorphic map on $\tilde{U}_1 \backslash \{0\}/\mathbb{Z}_n$. We require that this descended map is $(A,u)|_{U_2 \backslash \{0\}}$.

Holomorphic bundles on $\mathbb{P}(1,n)$ can be described in a similar way to the unitary bundles, the only difference is that the transition functions $\mu : \tilde{U}_1 \to G$ and $\tau : \tilde{U}_1 \backslash \{0\} \to G$ will be holomorphic maps. Now any holomorphic principal bundle over $\mathbb{C}$ is trivial (see remark 19.6 in [9]). So, a gauged holomorphic map can just be specified by $u : \tilde{U}_1, U_2 \to X$ and the transition functions τ and μ.

The next two technical propositions show that we can complex gauge-transform a gauged holomorphic map on a weighted projective line to a *standard form* near infinity. We identify the complement of the orbifold point with $\mathbb{C}$, and let B_R denote an open ball of radius R around 0.

Proposition 2.2. *Suppose A is a connection on $P \to \mathbb{P}(1,n)$. There is a complex gauge transformation g on P, and a trivialization of P over $\mathbb{C}$ so that $gA|_{\mathbb{C}} = \mathrm{D} + \lambda d\theta$ on $\mathbb{C} \backslash B_R$ for some $R > 0$, $\lambda \in \mathfrak{k}$ satisfying $e^{2\pi n \lambda} = \mathrm{Id}$. Conversely, if A is a connection on $\mathbb{C}$ satisfying the above condition, then it extends to a connection on a principal bundle over $\mathbb{P}(1,n)$.*

Proof. We can transform A to a flat connection in a neighborhood of infinity by complex gauge transformation as follows. Let A be a connection on a principal bundle $P \to \mathbb{P}(1,n)$. Choose $R_1 > 0$ and consider $B_{R_1} \subseteq \tilde{U}_1$. By Theorem A.3, there is a unique $s : B_{R_1} \to \mathfrak{k}$ with $s|_{\partial B_{R_1}} = 0$, so that $e^{is}A$ is a flat connection. By the uniqueness of s and the symmetry of A (see (5)), the complex gauge transformation is symmetric under the $\mathbb{Z}_n$ action, i.e. $s \circ \sigma_n = \mathrm{Ad}_\mu s$. Let $\eta : \tilde{U}_1 \to [0,1]$ be a radially symmetric cut-off function that is 1 on B_R and vanishes on $\tilde{U}_1 \backslash B_{2R_1}$. It is easy to see that $e^{i\eta s}$ defines a complex gauge transformation g on all of $\mathbb{P}(1,n)$. The connection gA is flat near infinity.

Next we do a further unitary gauge transformation so that the connection is in standard form, working over U_2. Let $R = (2R_1)^{-n}$. Choose a trivialization $P|_{U_2} \to U_2 \times G$ so that gA is in radial gauge outside B_R, and since gA is flat, $gA = \mathrm{D} + a(\theta)\mathrm{d}\theta$, for some $a : S^1 \to \mathfrak{k}$. We now produce a gauge transformation $k : U_2 \to K$ such that $kgA = \mathrm{D} + \lambda \mathrm{d}\theta$ for some $\lambda \in \mathfrak{k}$. Let $k_1 : [0, 2\pi] \to K$ be the solution of

$$\frac{k_1^{-1} dk_1}{\mathrm{d}\theta} = a(\theta), \qquad\qquad k_1(0) = \mathrm{Id}\,.$$

The path $\theta \mapsto k_1(\theta)$ can be homotoped to a geodesic $\theta \mapsto e^{\lambda\theta}$, $\lambda \in \mathfrak{k}$. Then, $e^{\lambda\theta} k_1^{-1}$ is a gauge transformation on $U_2 \backslash B_R$ that is homotopic to the identity and it transforms gA to $\mathrm{D} + \lambda \mathrm{d}\theta$ on $U_2 \backslash B_R$. By using a cut-off function, $e^{\lambda\theta} k^{-1}$ can be extended to a gauge transformation k on all of U_2. The holonomy of kgA about infinity is $e^{2\pi\lambda}$. Since $g(A)$ has trivial holonomy for loops close to 0 in $\tilde{U}_1$, we have $e^{2\pi n\lambda} = \mathrm{Id}$.

For the converse, we construct a bundle $P \to \mathbb{P}(1, n)$. Set $\mu = e^{-2\pi\lambda}$ and $\tau = e^{n\lambda\theta}$. We are given $A|_{U_2}$. A connection $A|_{\tilde{U}_1}$ can be constructed using the transition function τ. $A|_{\tilde{U}_1}$ will be trivial on $B_{R^{-n}} \subseteq \tilde{U}_1$. □

Remark 2.3. (a) (Choice of orbifold singularity) Suppose A is a connection on the trivial bundle $\mathbb{C} \times K$ of the form mentioned in the above proposition, i.e. $A = \mathrm{D} + \lambda \mathrm{d}\theta$ on $\mathbb{C} \backslash B_R$ with $e^{2\pi\lambda n} = \mathrm{Id}$. We can extend A to a connection on principal bundles not just over $\mathbb{P}(1, n)$, but also $\mathbb{P}(1, mn)$ for any positive integer m.

(b) (Choice of standard form) In the lemma above, the infinitesimal holonomy λ produces the the classifying path for the bundle as $\theta \mapsto e^{\lambda\theta}$. The choice of λ is unique up to the action of Ad_K. In our description of bundles over $\mathbb{P}(1, n)$, this path can be recovered from the transition functions μ, τ as $\lim_{r\to 0}\{\theta \in [0, \frac{2\pi}{n}] \mapsto \tau^{-1}(r, 0)\tau(r, \theta)\}$. The limit path may not exist, but this determines a homotopy class of paths from Id to $\mu(0)^{-1}$ in K. (Recall $\tau^{-1}(r, 0)\tau(r, \frac{2\pi}{n}) = \mu(r, 0)^{-1}$.)

The next result follows easily from Proposition 2.2.

Proposition 2.4. (Standard form near infinity for gauged holomorphic maps) *Let $P \to \mathbb{P}(1, n)$ be a principal K-bundle and let (A, u) be a gauged holomorphic map from P to X. There is a complex gauge transformation g on P and a trivialization of P over $\mathbb{C}$ so that $g(A, u)$ satisfies the following:*

(a) *There is a $\lambda \in \mathfrak{k}$ so that $gA = \mathrm{D} + \lambda \mathrm{d}\theta$ on $\mathbb{C} \backslash B_R$ for some $R > 0$. The element λ satisfies $e^{2\pi n\lambda} = \mathrm{Id}$.*

(b) *For any $\theta \in [0, 2\pi)$, $\lim_{r\to\infty} e^{-\lambda\theta} u(r, \theta) = x$ and $e^{2\pi\lambda} x = x$.*

Conversely, any gauged holomorphic map from $\mathbb{C} \times K$ to X that satisfies the above conditions extends to a map on $\mathbb{P}(1, n)$ for some principal bundle $P \to \mathbb{P}(1, n)$. The isotropy n of the point at infinity can be taken to be any positive integer so that $e^{2\pi n\lambda} = \mathrm{Id}$.

Remark 2.5 (Topological invariants of affine vortices). The second equivariant homology class of a gauged holomorphic map on $\mathbb{P}(1,n)$, $[(P,A,u)] \in H_2^K(X,\mathbb{Q})$ is obtained by pushing forward the rational fundamental class of the domain $[\mathbb{P}(1,n)]$ under a map $P \times_K X \to EK \times_K X$ given by a classifying map for P. It is a deformation invariant of (P,A,u). By Theorem 3.1, one can also associate to any finite energy affine vortex (A,u) a homology class $[(P,A,u)]$. The class $[(P,A,u)] \in H_2^K(X,\mathbb{Q})$ is integral if the G action on $X^{\rm ss}$ is free. Forgetting u yields a map

$$f : H_2^K(X,\mathbb{Q}) \to H_2^K(\text{point},\mathbb{Q}) = H_2(BK,\mathbb{Q}).$$

The class $[P] := f_*([(P,A,u)])$ determines the topology of the principal bundle $P \to \mathbb{P}(1,n)$. This topological information is precisely a choice of $\lambda \in \mathfrak{k}$ satisfying $e^{2\pi n\lambda} = \mathrm{Id}$ modulo Ad_K (see remark 2.3).

3. From a holomorphic map to a vortex

We prove a slightly refined version of the main result Theorem 1.1 relating affine vortices to gauged holomorphic maps from orbifold lines. This requires the following analytic definition: Fix $p > 2$. We call a gauged holomorphic map (A,u) on $P \to \mathbb{P}(1,n)$ *p-weak* if it is smooth on $\mathbb{C}$ and on $B_{\tilde{R}} \subset \tilde{U}_1$, which is a neighbourhood of ∞, $(A,u)|_{B_{\tilde{R}}} \in L^p \times W^{1,p}$. A *(complex) gauge transformation* on P is *p-weakly extendable* if it is smooth on $\mathbb{C}$ and in $W^{1,p}$ in a neighbourhood of ∞. Denote by $\mathcal{G}(P)_{\rm we}$ the group of p-weakly extendable gauge transformations. By the Sobolev embedding theorem, any p-weakly extendable (A,u) resp. g is continuous and so $u(\infty)$ resp. $g(\infty)$ is well-defined. The following is the analytic version of the main result of the paper.

Theorem 3.1. *Suppose X is Kähler manifold with Hamiltonian action of a compact Lie group K, which is either compact or equivariantly convex at infinity with a proper moment map. Let G be the complexification of K, and suppose G acts locally freely on $X^{\rm ss}$. Let n be an integer such that for any $x \in X^{\rm ss}$, $|G_x|$ divides n. Fix $2 < p < 2(1+\frac{1}{n})$.*

(a) *Let (A,u) be a gauged holomorphic map from $\mathbb{C}$ to X that extends smoothly to a map over some principal bundle $P \to \mathbb{P}(1,n)$, and suppose $u(\infty) \in X^{\rm ss}$. There is a p-weakly extendable complex gauge transformation $g \in \mathcal{G}(P)_{\rm we}$ such that $g(A,u)|_{\mathbb{C}}$ is a smooth finite energy symplectic vortex, which is unique up to left multiplication by a unitary gauge transformation.*

(b) *Conversely, given any finite energy symplectic vortex, there is a K-bundle $P \to \mathbb{P}(1,n)$ so that (A,u) extends to a p-weak gauged holomorphic map on P. There is a p-weakly extendable complex gauge transformation $g \in \mathcal{G}(P)_{\rm we}$ so that $g(A,u)$ is smooth over $\mathbb{P}(1,n)$. The gauged holomorphic map $g(A,u)$ is unique up to complex gauge transformations in $\mathcal{G}(P)$.*

The proof depends on the following result about heat flow of the first author [37]. Let

$$c_0 = \inf\{|\Phi(x)| : K\text{-action on } x \text{ does not have infinitesimal stabilizers.}\}. \tag{6}$$

By the assumption 2.1, $c_0 > 0$.

Theorem 3.2. *([37] Theorem 4.4.1) Let Σ be a compact Riemann surface without boundary and (A_0, u_0) is a gauged holomorphic map on a principal bundle $P \to \Sigma$. In addition, assume that (A_0, u_0) satisfies $E(A_0, u_0) \leq c_0^2 \operatorname{vol}(\Sigma)$. Then, there is a complex gauge transformation $g = e^{i\xi}$, $\xi \in W^{2,p}(\Sigma, P(\mathfrak{k}))$ (for any $p > 2$) such that $g(A, u)$ is a vortex. Up to unitary gauge equivalence, $g(A, u)$ is the unique vortex in the complex gauge orbit of (A, u).*

This result is obtained by studying the gradient flow of (A_0, u_0) in the space of gauged pairs on Σ under the vortex functional

$$(A, u) \mapsto \|F_{(A,u)}\|^2_{L^2(\Sigma)}, \quad F_{(A,u)} := *F_A + \Phi(u).$$

The trajectory of the flow $t \mapsto (A_t, u_t)$ lies in the complex gauge orbit of (A_0, u_0) and it converges to a critical point of the functional. The bound on the energy of (A_0, u_0) ensures that the critical point corresponds to $F = 0$, i.e. it is a vortex and that the limit is also in the complex gauge orbit of (A_0, u_0). The flow has the property of decreasing energy. This is seen by the following energy identity.

Lemma 3.3. (Cieliebak et al. [7]) *Let Σ be a closed compact Riemann surface and P a principal G-bundle on it. A pair $(A, u) \in \mathcal{A}(P) \times \Gamma(\Sigma, P(X))$ satisfies*

$$\begin{aligned} &\frac{1}{2}\int_\Sigma |F(A)|^2 + |\Phi \circ u|^2 + |d_A u|^2 \operatorname{dvol}_\Sigma \\ &\qquad = \int_\Sigma |\overline{\partial}_A u|^2 + \frac{1}{2}| * F_A + \Phi(u)|^2 \operatorname{dvol}_\Sigma + \langle \omega_X - \Phi, u\rangle, \end{aligned} \tag{7}$$

where $\langle \omega_X - \Phi, u\rangle = \int_\Sigma u^*\omega - d\langle \Phi(u), A\rangle$

The pairing $\langle \omega_X - \Phi, u\rangle$ is a topological invariant of (A, u) and is preserved by the flow, since $\|F_{(A_t,u_t)}\|^2_{L^2}$ decreases with time t, the same is the case with the energy $E(A_t, u_t)$.

We recall several results need for the proof of Theorem 3.1 (i). The first one says that any gauged holomorphic map which is an approximate solution to the vortex equations may be complex gauge transformed into one. It is proved by combining proposition 4.3.1 and proposition 4.4.2 in [37].

Proposition 3.4. *Let $p > 2$. Suppose Σ is a compact Riemann surface, possibly with boundary. Let $(A_i, u_i) \in \mathcal{H}(P, X)_{L^p \times W^{1,p}}$ be a sequence and $(A_\infty, u_\infty) \in \mathcal{H}(P, X)_{L^p \times W^{1,p}}$ be such that $A_i \to A_\infty$ in L^p and there is a finite set $Z \subseteq \Sigma$ so that $u_i \to u_\infty$ in C^0 on compact subsets of $\Sigma \backslash (Z \cup \partial\Sigma)$. Also, $F_i := *F(A_i) + u_i^*\Phi \to 0$ in $W^{-1,p}$. In case $\partial\Sigma = \emptyset$, we further assume that $u_\infty(\Sigma\backslash Z) \cap X^{\mathrm{ss}} \neq \emptyset$. Then, there exist constants C and i_0 so that for $i > i_0$, there is a complex gauge transformation $\exp i\xi_i$, $\xi_i \in W^{2,p}_\delta(\Sigma, P(\mathfrak{k}))$ so that $(\exp i\xi_i)(A_i, u_i)$ is a vortex and satisfies $\|\xi_i\|_{W^{2,p}} < 8C\|F_i\|_{L^p}$.*

The next result we will need is proposition 4.3.2 in [37]. Roughly it says that in a complex gauge orbit, there is at most one vortex up to gauge. The proof is reproduced, because it will be useful in understanding the corresponding result for affine vortices.

Proposition 3.5. *Let Σ be a compact Riemann surface, possibly with boundary. Let (A_0, u_0), $(A_1, u_1) \in \mathcal{H}(P, X)$ be vortices on a principal bundle $P \to \Sigma$ that are related by a complex gauge transformation g, i.e. $(A_1, u_1) = g(A_0, u_0)$ and assume $g(\partial\Sigma) \subseteq K$. Then, (A_0, u_0) and (A_1, u_1) are gauge-equivalent, i.e. $g \in \mathcal{K}$.*

Proof. After a gauge transformation, we can assume $(A_1, u_1) = e^{i\xi}(A_0, u_0)$, where $\xi \in \Gamma(\Sigma, P(\mathfrak{k}))$ and $\xi|_{\partial\Sigma} = 0$. Let $(A_t, u_t) := e^{it\xi}(A_0, u_0)$. We know $F_{A_0,u_0} = F_{A_1,u_1} = 0$. For $\xi|_{\partial\Sigma} = 0$,

$$\begin{aligned}\frac{d}{dt}\int_\Sigma \langle *F_{A_t,u_t}, \xi\rangle &= \int_\Sigma \langle \mathrm{d}_{A_t}^*\mathrm{d}_{A_t}\xi + u_t^* d\Phi(J(\xi)_{u_t}), \xi\rangle_{\mathfrak{k}} \\ &= \|\mathrm{d}_{A_t}\xi\|_{L^2}^2 + \int_\Sigma \omega_{u_t}((\xi)_{u_t}, J(\xi)_{u_t}) \geq 0.\end{aligned}$$

The inequality is strict for non-zero ξ. So, $\xi = 0$ and (A_0, u_0) and (A_1, u_1) are gauge-equivalent. □

We prove Theorem 3.1 (i) i.e. given a gauged holomorphic map on a principal bundle $P \to \mathbb{P}(1, n)$ with target X, it can be converted to a finite energy affine vortex via a weak complex gauge transformation. We first give an outline of the proof of Theorem 3.1 (i) in case $X/\!\!/G$ is a manifold, i.e. we assume $n = 1$ and G acts freely on X^{ss}. We are given a gauged holomorphic map (A, u) on a principal bundle $P \to \mathbb{P}^1$. We equip $\mathbb{P}^1$ with a family of metrics dvol_R depending on a real parameter R that interpolate between the Euclidean metric on $\mathbb{C} \subset \mathbb{P}^1$ and the Fubini-Study metric on $\mathbb{P}^1$. The metric dvol_R is defined so that it is equal to the Euclidean metric on the ball B_R and is the Fubini Study metric near infinity. By Proposition 3.6, we complex gauge transform (A, u) such that its energy is bounded under all these metrics. Then, we perform heat flow (Theorem 3.2) on $\mathbb{P}^1$ under the metric dvol_{R_i} where R_i is a sequence of numbers increasing to infinity. $\mathrm{vol}_R(\mathbb{P}^1)$ increases with R, so for large enough R, the limit of the heat flow is a vortex. This produces a sequence of vortices (A_i, u_i) on B_{R_i} that are complex gauge equivalent to (A, u) and whose energies are bounded. The sequence (A_i, u_i) converges to a limit vortex (A_∞, u_∞) on $\mathbb{C}$ modulo bubbling at a finite number of points. We prove that there is actually no bubbling and the limit vortex is in fact complex gauge equivalent to (A, u).

Next, we outline the differences in the proof of Theorem 3.1 (i) in the case where the action of G on X^{ss} has non-trivial finite stabilizers, i.e. $X/\!\!/G$ is an orbifold and $n > 1$. Since this case involves an orbifold base, Theorem 3.2 does not apply. So, we work on a cover $\tilde{\mathbb{P}}$, which is bi-holomorphic to $\mathbb{P}^1$. The cover is equipped with a ramified covering map

$$\pi : \tilde{\mathbb{P}} \to \mathbb{P}(1, n) \quad [x : y] \mapsto [x^n : y].$$

To see that this map is well-defined, recall that $\mathbb{P}(1, n)$ can be defined as

$$\mathbb{P}(1, n) := (\mathbb{C}^2 \backslash \{0\})/\sim, \quad (x, y) \sim (\lambda^n x, \lambda y)\ \forall \lambda \in \mathbb{C}^\times.$$

We consider $\pi^*(A, u)$ and perform all the following steps maintaining symmetry under the $\mathbb{Z}_n$-action. The Euclidean area form on $\mathbb{C} \subset \mathbb{P}(1, n)$ pulls back to an area form

$$\mathrm{dvol}_{\mathrm{orb}} = n^2(|x|^2 + |y|^2)^{n-1} dx \wedge dy$$

on $\mathbb{C} \subset \tilde{P}$. In order to mimic the procedure used in the $n = 1$ case, we would like to use the metric $\mathrm{dvol}_{\mathrm{orb}}$ on increasing balls in $\mathbb{C} \subset \tilde{\mathbb{P}}$. But, unfortunately this metric degenerates at the origin. We define a family of metrics dvol_R (see (8)) that is equal to $\mathrm{dvol}_{\mathrm{orb}}$ on the annulus $A(1/R, R)$ of radii $1/R$ and R, it is equal to the Fubini-Study metric in a neighborhood of ∞ and a constantly scaled Euclidean metric in a neighborhood of 0. The modification near the origin is to remove the degeneracy of the metric. The first step in the proof is to modify the gauged holomorphic map $\pi^*(A, u)$ defined on $\tilde{\mathbb{P}}$ so that its energy is bounded under the metric dvol_R for all R. This is handled in proposition 3.6. By applying heat flow, we produce a sequence of gauged holomorphic maps (A_i, u_i). By symmetry arguments they descend to $\mathbb{C} \subset \mathbb{P}(1, n)$. The pair (A_i, u_i) satisfies the vortex equation under the Euclidean metric on a sequence of increasing annuli $A(1/R_i, R_i)$. To get better behavior near the origin, we apply a further complex transformation supported in $B(2/R_i)$ that ensures that under the Euclidean metric on $\mathbb{C} \subset \mathbb{P}(1, n)$ the L^2 norms F_{A_i,u_i} go to 0 as $i \to \infty$. This step is carried out in proposition 3.13. The maps (A_i, u_i) converge to a limit vortex (A_∞, u_∞) on $\mathbb{C}\backslash\{0\}$ modulo bubbling. Because of the improved behavior of (A_i, u_i) in B_{1/R_i}, we are able to remove the singularity of (A_∞, u_∞) at 0. The rest of the proof is broadly similar to the $n = 1$ case.

We now give the details of the construction. Let $\eta : \mathbb{C} \to [0, 1]$ be a radially symmetric cut-off function that is 1 in the unit ball B_1 and 0 on $\mathbb{C}\backslash B_2$. For any $R > 0$, define a cut-off function $\eta_R : \mathbb{C} \to [0, 1]$ as $\eta_R(x) := \eta(x/R)$. For any $R > 2$, define a metric on dvol_R on $\tilde{\mathbb{P}}$ as

$$\begin{aligned}\mathrm{dvol}_R := \Big(&\frac{1 - \eta_R}{(1 + x^2 + y^2)^2} + \eta_R(1 - \eta_{1/2R})n^2(x^2 + y^2)^{n-1} \\ &+ \eta_{1/2R} n^2 R^{-2n+2}\Big) dx \wedge dy,\end{aligned} \tag{8}$$

where (x, y) are Euclidean coordinates on $\mathbb{C} \subset \tilde{\mathbb{P}}$. In case $n = 1$, it simplifies to

$$\mathrm{dvol}_R := \left(\frac{1 - \eta_R}{(1 + x^2 + y^2)^2} + \eta_R\right) dx \wedge dy.$$

We give the chart-based description of $\tilde{\mathbb{P}}$. Recall $\mathbb{P}(1, n)$ is constructed using two charts $\tilde{U}_1$ and U_2. $\mathbb{P}$ is made up of charts $\tilde{U}_1$, $\tilde{U}_2$. $\pi : \tilde{U}_2 \to U_2 := w \mapsto w^n$ is an n-cover ramified at 0.

We now prove a key technical result, which say that any gauged holomorphic map may be gauge-transformed so that the energy with respect to the R-independent metrics defined above is uniformly bounded in R.

Proposition 3.6. *Let (A, u) be a gauged holomorphic map defined on a principal bundle $P \to \mathbb{P}(1, n)$, and suppose $u(\infty) \in X^{\mathrm{ss}}$. Then there exists a smooth complex* *gauge transformation $g \in \mathcal{G}(P)$ such that*

$$\sup_R E_R(\pi^*(g.(A,u))) < \infty. \tag{9}$$

Here $\pi : \tilde{\mathbb{P}} \to \mathbb{P}(1,n)$ is the projection and E_R is the energy of the gauged holomorphic map under the metric dvol_R on $\tilde{\mathbb{P}}$.

Proof of Proposition 3.6. We first point out that conformally rescaling the metric by a scalar-valued function greater than one has the effect of decreasing the L^2 norm of the two-form F_A and increasing the L^2 norm of the zero-form $\Phi(u)$. The L^2 norm of the one-form $d_A u$ is independent of the metric. We wish to find a smooth g such that

$$\begin{aligned} F_{gA} &\equiv 0 \quad \text{in a neighborhood of } 0, \\ \Phi(gu) &\equiv 0 \quad \text{in a neighborhood of } \mathbb{P}(n). \end{aligned} \tag{10}$$

For the first condition, we use Theorem A.3 which produces a unique element $s : \Omega \to P(\mathfrak{k})$ satisfying $s|_{\partial\Omega} \equiv 0$ such that $g(A)$ is a flat connection on Ω. By taking Ω to be a neighborhood of 0, and using a cut-off function, we can produce $g_1 \in \mathcal{G}(P)$ that makes the connection flat in a neighborhood of 0 and g_1 is identity away from this neighborhood.

For the second condition $\Phi(gu) = 0$, observe that there is a neighborhood B of $\mathbb{P}(n)$ in $\mathbb{P}(1,n)$ that is mapped by u to X^{ss}. For any $x \in X^{\mathrm{ss}}$, there is a unique $s \in \mathfrak{k}$ such that $\Phi(e^{is}x) = 0$, and s varies smoothly with x. Consider a cover of B: $\tilde{B} \subset \tilde{U}_1$, and we work with the trivialization of P over $\tilde{U}_1$. There exists $\xi : \tilde{B} \to \mathfrak{k}$ such that $\Phi(e^{i\xi}u) \equiv 0$ on $\tilde{B}$. Recall u satisfies $u \circ \sigma_n = \mu u$. The uniqueness of ξ implies

$$\xi \circ \sigma_n = \mathrm{Ad}_\mu \xi. \tag{11}$$

By using a cut-off function, ξ can be extended to all of $\tilde{U}_1$ such that it vanishes away from $\tilde{B}$ and still satisfies the symmetry relation (11). The element $g := g_1 e^{i\xi}$ satisfies the conditions in (10), by construction. This is enough to prove the bound (9) because the metric dvol_R increases with R in B, a neighborhood of $\mathbb{P}(n)$, and decreases with R in Ω, a neighborhood of 0. For the rest of the space $\mathbb{P}(1,n) \backslash (B \cup \Omega)$, there is a uniform upper and lower bound on the metric. □

To prove Theorem 3.1 (i), we need the following lemma on convergence of vortices:

Lemma 3.7 ([30], [43])**.** *Let $p > 2$ and Σ_i be a sequence of precompact sets exhausting a Riemann surface Σ*

$$\Sigma_1 \Subset \Sigma_2 \Subset \cdots \Subset \Sigma \qquad\qquad \bigcup_i \Sigma_i = \Sigma.$$

Suppose for each i, (A_i, u_i) is a smooth vortex on Σ_i,

$$\sup_i E(\Sigma_i, (A_i, u_i)) < \infty$$

and there is a compact set $S \subset X$ containing the images of all the u_i. Then, passing to a subsequence, there are gauge transformations $k_i \in H^2(\Sigma_i, K)$, a finite set $Z \subset \Sigma$ and a finite energy vortex $(A_\infty, u_\infty) \in H^1_{\text{loc}} \times H^2_{\text{loc}}$ over $\mathbb{C}$ such that

(a) *$k_i A_i \rightharpoonup A_\infty$ in H^1 on compact subsets of $\mathbb{C}$, and strongly in L^p.*
(b) *$u_i \rightharpoonup u_\infty$ in H^2 on compact subsets of $\mathbb{C}\backslash Z$ and strongly in $W^{1,p}$, C^0.*

Proof. This lemma is a combination of results in [43] and [30]. We provide an outline of the proof. The bounded energy condition implies a curvature bound

$$\|F(A_i)\|_{L^2(\Sigma_i)} < c$$

for all i. By Uhlenbeck's theorem for non-compact domains (Theorem A' in [39]), after passing to a subsequence, there are gauge transformations $k_i \in H^2(\Sigma_i)$ and a limit connection A_∞ on the trivial bundle $\Sigma \times K$ such that $k_i A_i \rightharpoonup A_\infty$ in H^1 on compact subsets of Σ.

We first prove convergence on compact sets for bounded first derivatives. On a compact set $Q \subset \Sigma$, if there is a bound on $\|d_{A_i} u_i\|_{L^p(Q)}$, then a subsequence of $k_i u_i$ converges weakly in $H^2(Q)$. To see this, write $k_i A_i = \mathrm{D} + a_i$. Since all the u_i map to a compact set and a_i is bounded in L^p, $(a_i)_{u_i}$ is also bounded in L^p, hence there is an $L^p(Q)$ bound on $d(k_i u_i)$. This implies $k_i u_i$ converges weakly in $W^{1,p}(Q)$ and strongly in C^0 to a limit u_∞ (see Theorem B.4.2 in [23]). By Lemma 3.8, after passing to a subsequence $k_i u_i \rightharpoonup u_\infty$ in $H^2(Q)$.

In the absence of a first derivative bound, one has bubbling. As in Ott [30], there is a finite set $Z \subset \Sigma$ where bubbling occurs. That is, after passing to a subsequence, $\|d_{A_i} u_i\|_{L^p(Q)}$ is bounded on compact subsets $Q \subset \Sigma\backslash Z$. The map u_∞ extends to a continuous map on Σ. Since we are in the Kähler case, this has a much easier proof than that in Ott [30]: Choose $z_0 \in Z$ and a small neighborhood $U \subset \Sigma$ so that $U \cap Z = \{z_0\}$. By Lemma 4.3, there is a complex gauge transformation $g \in W^{1,p}(U) \hookrightarrow C^0(U)$ such that gA_∞ is the trivial connection. Complex gauge transformations preserve holomorphicity, so $\overline{\partial}(gu_\infty) = 0$ on $U\backslash\{z_0\}$ and the image $gu_\infty(B_R\backslash\{0\})$ is contained in a compact set. By the removable singularity theorem of complex analysis, we have gu_∞, and hence u_∞ extends continuously over z_0. Finally $u_\infty \in H^2_{\text{loc}}$ as in the proof of Lemma 3.8.

□

The following elliptic regularity result (Theorem 3.1 in Cieliebak et al.[8]) is standard, but we provide a self-contained proof.

Lemma 3.8. (Elliptic regularity for gauged holomorphic maps) *Let $s \geq 1$ be an integer, $\Sigma \subseteq \mathbb{C}$ be pre-compact and (A_i, u_i) be a sequence of gauged holomorphic maps such that $A_i \rightharpoonup A_\infty$ in $H^s(\overline{\Sigma})$ and $u_i \to u_\infty$ in $C^0(\overline{\Sigma})$, then $u_i \rightharpoonup u_\infty$ in $H^{s+1}(\overline{\Sigma}')$, for any Σ', whose closure is contained in* $\mathrm{int}(\Sigma)$.

Proof. The proof is by induction on s. The base case is proved in the same way as the induction step. First note that it suffices to work locally in X: Choose an atlas $X = \cup_\alpha \mathcal{V}_\alpha$ such that $\mathcal{V}_\alpha$ is bi-holomorphic to an open subset of $\mathbb{C}^n$. Since $u_i \to u_\infty$

in C^0, we can find a finite cover $\Sigma = \cup_\beta \mathcal{U}_\beta$ such that for $u_i(\mathcal{U}_\beta)$ is contained in a single $\mathcal{V}_\beta$ for large i. So, now we can think of u_i as mapping to $\mathbb{C}^n$.

In each chart we apply a combination of Sobolev multiplication and regularity theorems. As in [8], write $A_i = \mathrm{D} + \Phi_i dx + \Psi_i dy$, where Ψ_i, $\Phi_i \in H^s(\Sigma, \mathfrak{k})$ and the holomorphicity equation for (A_i, u_i) is

$$-\overline{\partial} u_i = \Phi_i(u_i) + J_X \Psi_i(u_i) \tag{12}$$

We know that both A_i and u_i weakly converge in H^s, so Ψ_i, Φ_i and u_i are uniformly bounded in H^s. We next show that $\Phi_i(u_i)$ and $\Psi_i(u_i)$ are uniformly bounded in H^s. For this, define an operator L_x for every $x \in X$,

$$L_x : \mathfrak{k} \to T_x X, \quad \xi \mapsto \xi_X(x). \tag{13}$$

So, given $u : \Sigma \to X$, we obtain a section

$$L_u \in \Gamma(\Sigma, \mathrm{Hom}(\mathfrak{k}, T_u X))$$

of the vector bundle $\mathrm{Hom}(\mathfrak{k}, T_u X)$ on Σ. $\Phi_i(u_i)$ can be seen as a product $\Phi_i(u_i) = L(u_i)\Phi_i$. Since L is smooth, $\|L(u_i)\|_{H^s(\mathcal{U}_\alpha)} < c$ for all i, α. By Sobolev multiplication (Proposition A.2), for $s \geq 1$,

$$\|L(u_i)\Phi_i\|_{H^s(\mathcal{U}_\alpha)} \leq c\|L(u_i)\|_{H^s(\mathcal{U}_\alpha)}\|\Phi_i\|_{H^s(\mathcal{U}_\alpha)}.$$

By holomorphicity, $\|\overline{\partial} u_i\|_{H^s(\mathcal{U}_\alpha)} < c$ for all i, α. By elliptic regularity for curves in $\mathbb{C}^n$,

$$\|u_i\|_{H^{s+1}(\mathcal{U}'_\alpha)} \leq c(\|\overline{\partial} u_i\|_{H^s(\mathcal{U}_\alpha)} + \|u_i\|_{L^2(\mathcal{U}_\alpha)}).$$

where $\overline{\mathcal{U}}'_\alpha \subseteq \mathcal{U}_\alpha$ and c depends on $\mathcal{U}_\alpha$, $\mathcal{U}'_\alpha$. By picking $\mathcal{U}'_\alpha$ such that they cover Σ', we obtain a uniform bound on $\|u_i\|_{H^{s+1}(\Sigma')}$ and so $u_i \rightharpoonup u_\infty$ in $H^{s+1}(\Sigma')$. □

We define a weak version of vortices as follows. Let (A, u) lie in $L^p_{\mathrm{loc}} \times W^{1,p}_{\mathrm{loc}}$ for some $p > 2$. For such a pair, the vortex equation holds in a weak sense if $F_{A,u}$ vanishes in $W^{-1,p}$, i.e. for all $\xi \in W^{1,p}_0(\Sigma, P(\mathfrak{k}))$, we have $\int_\Sigma \langle F_{A,u}, \xi \rangle = 0$. On a compact subset of the domain the energy is well-defined, because the curvature term $F_A = -\Phi(u) \in C^0$.

Lemma 3.9 (Regularity for vortices)**.** *Suppose Σ be a Riemann surface which can be exhausted by a sequence of increasing compact sets $\Sigma_i \subset \Sigma$ that are deformation retracts of Σ. Given a finite energy vortex $(A, u) \in L^p_{\mathrm{loc}} \times C^0$ on the trivial bundle $\Sigma \times K$. There is a gauge transformation $k \in W^{1,p}_{\mathrm{loc}}$ such that $k(A, u)$ is smooth on Σ.*

Proof. This result is a variation of Proposition 77 in [43], which works with vortices on $\mathbb{C}$, but the same procedure works when the base space is as in this lemma. In our proof, we work with a compact Σ , possibly with boundary and find $k \in H^2$ that makes (A, u) smooth on $\Sigma \backslash \partial\Sigma$. If Σ is non-compact, the same procedure can be applied on all Σ_i in a controlled way, the resulting gauge transformations can be patched as in Proposition 77 in [43] to produce one on all of Σ.

Suppose Σ is compact and $(A, u) \in W^{m,p}_{\mathrm{loc}} \times W^{m+1,p}_{\mathrm{loc}}$, where $mp > 2$. The idea of the proof is to find a smooth reference connection A_0 and put A is Coulomb gauge with respect to A_0 by a gauge transformation $k \in W^{m+1,p}$. Writing $kA = A_0 + \alpha$, we have $d_{A_0}^* \alpha = 0$. Since $u \in W^{m+1,p}(\Sigma)$, $\Phi(u)$ is also in the same class and by the vortex equation $F_A \in W^{m+1,p}(\Sigma) \hookrightarrow W^{m,p}(\Sigma)$. Writing

$$d_{A_0}\alpha = F_{kA} - \frac{1}{2}[\alpha \wedge \alpha] \implies d_{A_0}\alpha \in W^{m,p}. \tag{14}$$

So, we have $kA \in W^{m+1,p}_{\mathrm{loc}}(\Sigma \backslash \partial\Sigma)$. By elliptic regularity (similar to proof of Lemma 3.8), we have $W^{m+1,p}_{\mathrm{loc}}$ control over ku on $\Sigma \backslash \partial\Sigma$.

This procedure can be applied inductively to gauge transform (A, u) to a smooth vortex on $\Sigma \backslash \partial\Sigma$. When $A \in L^p$, it is possible to gauge transform A to a connection in Coulomb gauge (see Theorem 8.3 in [39]). However, the step (14) requires that $mp > 2$, which fails when $(A, u) \in L^p_{\mathrm{loc}} \times C^0$. In that case, we just have to apply (14) many times and inductively improve the regularity of the connection.

Given a $p > 2$, there exists m and a sequence $2 < q_0 < \cdots < q_{m-1} \leq 4$, $q_m > 4$ such that

$$q_0 = p, \quad q_i < \frac{2q_{i-1}}{4 - q_{i-1}} \quad i \geq 1.$$

Suppose $\alpha \in L^{q_i}$, by Hölder's inequality $[\alpha \wedge \alpha] \in L^{q_i/2}$. F_{kA} is in C^0 and hence also in $L^{q_i/2}$. This implies $\alpha \in W^{1,q_i/2} \hookrightarrow L^{q_{i+1}}$. By repeating this procedure, we end up with $\alpha \in L^{q_m}$. Using (14) one more time, we get $\alpha \in W^{1,q_m/2}$. By elliptic regularity, $u \in W^{2,q_m/2}_{\mathrm{loc}}(\Sigma \backslash \partial\Sigma)$. Now, the inductive step of the previous paragraph is applicable.

□

We now prove Theorem 3.1 (i) for the case $n = 1$, i.e. $X /\!\!/ G$ is a manifold.

Proof of Theorem 3.1 (i) for $n = 1$. Given a gauged holomorphic map (A_0, u_0) on a principal bundle $P \to \mathbb{P}^1$, by Proposition 3.6 we can assume

$$E_R(A_0, u_0) < E_0 \quad \forall R > 0,$$

where E_R is the energy under the metric dvol_R. So, there exists R_0 such that $E_R(A_0, u_0) \leq c_0^2 \operatorname{vol}_R(\Sigma)$ for all $R \geq R_0$, here c_0 is as defined in (6). Then for $R \geq R_0$, Theorem 3.2 is applicable for the gauged holomorphic map (A_0, u_0) on $\mathbb{P}^1$ under the metric dvol_R. Choose an increasing sequence $R_i \to \infty$, with $R_1 \geq R_0$. By Theorem 3.2, (A_0, u_0) is complex gauge equivalent to a vortex (A_i, u_i) under the metric dvol_{R_i}. By Lemma 2.7 in [8], there is a compact set $S \subset X$ that contains the images of all the u_i. The pair (A_i, u_i), when restricted to B_{R_i}, is a vortex under the Euclidean metric. It satisfies $E(A_i, u_i, B_{R_i}) \leq E(A_i, u_i, \mathbb{P}^1) \leq E_0$. By modifying the g_is by gauge transformations on B_{R_i}, if necessary, we may assume that (A_i, u_i) is smooth (by Theorem 3.1 in [8]).

First we introduce the bubbling set. The sequence $\{(A_i, u_i)\}$ is a sequence of vortices on B_{R_i}, which exhaust $\mathbb{C}$. They satisfy an energy bound, so Lemma 3.7 is applicable. So, there is a subsequence of (A_i, u_i) (still denoted by the same subscripts), a sequence of gauge transformations $k_i \in H^2(B_{R_i})$, a finite set $Z \subseteq \mathbb{C}$

and a finite energy vortex (A_∞, u_∞) on $\mathbb{C}$ so that $k_iA_i \rightharpoonup A_\infty$ in H^1 on compact subsets of $\mathbb{C}$ and $k_iu_i \rightharpoonup u_\infty$ in H^2 on compact subsets of $\mathbb{C}\backslash Z$. Since (A_∞, u_∞) has finite energy, $\Phi(u_\infty(z)) \to 0$ as $z \to \infty$. So, we may assume, by increasing R if necessary, that $u_\infty(\mathbb{C}\backslash B_R) \subseteq X^{\rm ss}$. Also, we can assume $Z \subseteq B_R$ and so for large i, $u_i(\mathbb{C}\backslash B_R) \subseteq X^{\rm ss}$.

First we use the fact that by construction there is no bubbling on the annulus region to construct a complex gauge transformation on this region. The image $u_i(\overline{B}_{2R}\backslash B_R) \subseteq X^{\rm ss}$ for all i including $i = \infty$. Since G acts freely on $X^{\rm ss}$, there exist complex gauge transformations $g_i' \in H^2(\overline{B}_{2R}\backslash B_R)$ such that $g_i'k_i(A_i, u_i) = (A_\infty, u_\infty)$ on $\overline{B}_{2R}\backslash B_R$. Further $g_i' \rightharpoonup \mathrm{Id}$ in $H^2(B_{2R}\backslash B_R)$. Write $g_i' = k_i'e^{i\xi_i'}$, where $k_i' : \overline{B}_{2R}\backslash B_R \to K$ and $\xi_i' : \overline{B}_{2R}\backslash B_R \to \mathfrak{k}$. Since (2) is a smooth function, $\xi_i' \rightharpoonup 0$ in $H^2(\overline{B}_{2R}\backslash B_R)$.

We find a complex gauge transformation around a large ball around infinity that makes the pair into a vortex. Consider a cut-off function $\eta : \overline{B}_{2R} \to [0,1]$ that is 1 in a neighbourhood of ∂B_{2R} and 0 in a neighbourhood of $\overline{B}_R$. The product $\eta\xi_i' : B_{2R} \to \mathfrak{k}$ is well-defined. Furthermore $e^{i\eta\xi_i'}k_i(A_i, u_i)$ is a gauged holomorphic map on $\overline{B}_{2R}$ that is gauge-equivalent to (A_∞, u_∞) in a neighborhood of the boundary. It satisfies $F_{e^{i\eta\xi_i'}k_i(A_i,u_i)} \to 0$ in $W^{-1,p}$. This is because $F_{e^{i\eta\xi_i'}k_i(A_i,u_i)}$ is non-zero only in the annulus $B_{2R}\backslash B_R$, and in this region we have convergence of A_i, u_i and ξ_i' in L^p, $W^{1,p}$ and $W^{1,p}$ respectively. By proposition 3.4, for large i, there exist $\xi_i'' \in W_0^{1,p}(B_{2R}, \mathfrak{k})$, such that $e^{i\xi_i''}e^{i\eta\xi_i'}k_i(A_i, u_i)$ is a vortex on B_{2R}.

We show that there is no bubbling in the sequence. Proposition 3.4 shows that $\xi_i'' \to 0$ in $W^{1,p}(B_{2R})$. Set $(A_i'', u_i'') := e^{i\xi_i''}e^{i\eta\xi_i'}k_i(A_i, u_i)$. We recall the following:

- (A_i'', u_i'') are vortices on B_{2R},
- $\eta\xi_i', \xi_i'' \to 0$ in $W^{1,p}(B_{2R})$,
- $k_iA_i \to A_\infty$ in $L^p(\overline{B}_{2R})$ and
- $k_iu_i \to u_\infty$ in $W^{1,p}$ on compact subsets of $\overline{B}_{2R}\backslash Z$.

The gauged maps (A_i'', u_i'') are equal to (A_∞, u_∞) on ∂B_{2R} up to gauge equivalence. So, by Proposition 3.5, for all i, (A_i'', u_i'') are in the same gauge orbit as each other. We can write $(A_i'', u_i'') = h_i(A_0'', u_0'')$, where $h_i \in W^{1,p}(B_{2R}, K)$. Since $A_i'' \to A_\infty$ in $L^p(\overline{B}_{2R})$, by standard elliptic regularity arguments (see [37, Lemma 4.3.4]), after passing to a subsequence $h_i \rightharpoonup h_\infty$ in $W^{1,p}(B_{2R})$ (and strongly in C^0), $A_\infty = h_\infty A_0''$. This implies that the sequence u_i'', which is same as h_iu_0'' C^0-converges to $h_\infty u_0''$. Since u_i'' has a C^0-limit, it must be same as u_∞ and so the bubbling set Z is empty.

The argument above also shows that (A_∞, u_∞) is in the complex gauge orbit of (A_i, u_i). In particular,

$$(A_\infty, u_\infty) = g_\infty(A_0, u_0), \quad \text{on } \overline{B}_{2R}, \tag{15}$$

where $g_\infty = h_\infty e^{i\xi''}e^{i\eta\xi} : \overline{B}_{2R} \to G$.

Next, we claim that g_∞ can be defined on all of $\mathbb{C}$ while still satisfying (15). Since G acts freely on $X^{\rm ss}$ and $u_i(\mathbb{C}\backslash B_{2R}) \subset X^{\rm ss}$ for all i including $i = \infty$, the definition of g_∞ can be extended to all of $\mathbb{C}$ in a way that $u_\infty = g_\infty u_0$ on $\mathbb{C}$. We now explain why $g_\infty A_0 = A_\infty$. For $x \in X^{\mathrm{ss}}$ close to $\Phi^{-1}(0)$, we have

$$\mathfrak{k}_x \cap J\mathfrak{k}_x = \phi. \tag{16}$$

Since $u_\infty(\infty) \in \Phi^{-1}(0)$, by increasing R, if necessary, we can assume that this condition is satisfied for all $u_\infty(c)$ for $c \in \mathbb{C}\backslash B_R$. Let $g_\infty A_0 - A_\infty = a$. Since both (gA_0, u_∞) and (A_∞, u_∞) are holomorphic, we have $a^{0,1}_{u_\infty} = 0$. Write $a = a_x dx + a_y dy$, a_x, $a_y : \mathbb{C}\backslash B_R \to \mathfrak{k}$. Then, $a_x(u) + Ja_y(u) = 0$ and by (16), $a = 0$ on $\mathbb{C}\backslash B_R$.

We next show that the complex gauge transformation g_∞ needed to make the pair into a vortex has the claimed regularity. Using Lemma 3.9, we can modify g_∞ by a gauge transformation in $W^{1,p}_{\mathrm{loc}}$, so that $g_\infty(A_0, u_0)$ is smooth. By applying Lemma 3.12 in neighbourhoods in $\mathbb{C}$, we conclude that $g_\infty : \mathbb{C} \to G$ is smooth. Write $g_\infty = k_\infty e^{i\xi_\infty}$. We assumed $\Phi(u_0) = 0$ outside a compact set in $\mathbb{C}$. We also have $\lim_{r\to\infty} u_\infty(r, \theta) \in \Phi^{-1}(0)$ for any θ. This implies that $\xi_\infty(r, \theta) \to 0$ as $r \to \infty$. We now have a complex gauge transformation $e^{i\xi_\infty}$ on P that is smooth on $\mathbb{C}$ and continuous at ∞, we show that $e^{i\xi_\infty} \in \mathcal{G}(P)_{\mathrm{we}}$. Given the finite energy vortex $e^{i\xi_\infty}(A, u)$, Theorem 3.1(ii) gives $g \in \mathcal{G}(P')_{\mathrm{we}}$ so that $ge^{i\xi_\infty}(A, u)$ is a smooth gauged holomorphic map on $P' \to \mathbb{P}(1, n)$. (The proof of part (ii) of the theorem is independent of part (i)). By using the arguments in the uniqueness part of the proof of Theorem 3.1(ii), P' and P are isomorphic, so we assume $P' = P$. The proof of uniqueness in (ii) still applies if we weaken the assumption g_1, $g_2 \in \mathcal{G}(P)_{\mathrm{we}}$, and instead just assume that g_1, g_2 are smooth on $\mathbb{C}$ and extend continuously over ∞. So, $ge^{i\xi_\infty} \in \mathcal{G}(P)$ is smooth. Since $g \in \mathcal{G}(P)_{\mathrm{we}}$, the same is true for $e^{i\xi_\infty}$ also. That the vortex $e^{i\xi_\infty}(A, u)$ is unique up to gauge transformations, is proved separately in proposition 4.6 using Theorem 3.1 (ii). □

The proof of Theorem 3.1 (i) in the case of orbifold singularities follows the same pattern. We set down the additional details in each step of the proof.

Proof of Theorem 3.1 (i) for $n > 1$. We are given a gauged holomorphic map (A_0, u_0) on a principal bundle $P \to \mathbb{P}(1, n)$. We first lift (A_0, u_0) to a gauged holomorphic map $(\tilde{A}_0, \tilde{u}_0)$ on $\pi^*P \to \tilde{\mathbb{P}}$. On $\mathbb{C} \subset \tilde{\mathbb{P}}$, $(\tilde{A}_0, \tilde{u}_0)$ is symmetric under the $\mathbb{Z}_n$-action. This means that on the trivialization over $\mathbb{C} \subset \tilde{\mathbb{P}}$, $(\tilde{A}_0, \tilde{u}_0)$ is preserved by $\frac{2\pi}{n}$-rotations of the domain. On the trivialization over $\tilde{\mathbb{P}}\backslash\{0\}$, we have

$$\sigma_n^*(\tilde{A}_0, \tilde{u}_0) = \mu(\tilde{A}_0, \tilde{u}_0).$$

We proceed as in the $n = 1$-case. By proposition 3.6, we can assume

$$E_R(\tilde{A}_0, \tilde{u}_0, \tilde{\mathbb{P}}) < E_0 \quad \forall R > 0,$$

where E_R denotes the energy under the dvol_R metric. Let $R_i \to \infty$ be an increasing sequence and $\tilde{R}_i := R_i^{1/n}$. Using Theorem 3.2, we produce a sequence $(\tilde{A}_i, \tilde{u}_i)$ of $\mathrm{dvol}_{\tilde{R}_i}$-vortices on $\tilde{\mathbb{P}}$. They have bounded energy and there is a compact set $S \subset X$ containing the images of all the u_i. Further, on the trivialization on $\mathbb{C} \subset \tilde{\mathbb{P}}$, $(\tilde{A}_i, \tilde{u}_i)$ is symmetric under the $\mathbb{Z}_n$-action. So, $(\tilde{A}_i, \tilde{u}_i)$ descend to gauged holomorphic maps on $\mathbb{P}(1, n)$. These are vortices on $B_{R_i}\backslash B_{1/R_i} \subset \mathbb{C} \subset \mathbb{P}(1, n)$. In order to ensure better behavior in B_{1/R_i}, we apply Proposition 3.13 to the sequence $(\tilde{A}_i, \tilde{u}_i)$ on $\tilde{\mathbb{P}}$. This gives a sequence of complex gauge transformations $\tilde{g}_i$, which are identity on $\tilde{\mathbb{P}} \backslash B_{2/\tilde{R}_i}$ and are symmetric under the $\mathbb{Z}_n$-action. So, the descended maps $(A_i, u_i) := (\pi^*)^{-1}(\tilde{g}_i(\tilde{A}_i, \tilde{u}_i)$ are well-defined. (A_i, u_i) is a vortex on $B_{R_i} \backslash B_{2^n/R_i}$ and satisfies

$$\lim_{i \to \infty} \|F_{(A_i,u_i)}\|_{L^2(B_{R_i})} = 0. \tag{17}$$

The energy bound on $(\tilde{A}_i, \tilde{u}_i)$ under $\mathrm{dvol}_{\tilde{R}_i}$ metric implies a bound

$$E(A_i, u_i, A(2^n/R_i, R_i)) < c \tag{18}$$

under the Euclidean metric.

First we introduce the bubbling set. The energy of the connection $\|F(A_i)\|^2_{L^2(B_{R_i})}$ is bounded for all i using (17), (18) and because all the maps u_i have image contained in a compact set. So there is a bound on $\|\Phi(u_i)\|_{L^2(B_{2^n/R_i})}$. By Uhlenbeck compactness for non-compact domains (Theorem A' in [39]), we obtain a sequence of gauge transformations $k_i \in H^2(B_{R_i})$ and a connection A_∞ on the trivial bundle $\mathbb{C} \times K$ such that $k_i A_i \rightharpoonup A_\infty$ in H^1 in compact subsets of $\mathbb{C}$. We apply Lemma 3.7 to the sequence of vortices on domains $B_{R_i} \backslash B_{2^n/R_i}$. The Lemma implies that there is a finite set $Z \subset \mathbb{C} \backslash \{0\}$ and a finite energy limit vortex (A_∞, u_∞), where A_∞ is the limit connection found earlier and u_∞ is defined on $\mathbb{C} \backslash \{0\}$ such that $k_i u_i \rightharpoonup u^0_\infty$ in H^2 on compact subsets of $\mathbb{C} \backslash (Z \cup \{0\})$. Choose R such that $Z \subseteq B_R \backslash B_{1/R}$.

The singularity of u_∞ at 0 can be removed by the following argument. The pair (A_∞, u_∞) is a vortex on $B_R \backslash \{0\}$. Since the image of $k_i u_i$ is contained in a compact set $S \subset X$ for all i, the image of u_∞ is also contained in S. By Lemma 4.3, there is a complex gauge transformation $g \in W^{1,p}(B_R) \hookrightarrow C^0(B_R)$ such that gA_∞ is the trivial connection. Complex gauge transformations preserve holomorphicity, so $\overline{\partial}(gu_\infty) = 0$ on $B_R \backslash \{0\}$ and the image $gu_\infty(B_R \backslash \{0\})$ is contained in a compact set. By the removable singularity theorem of complex analysis, we have gu_∞, and hence u_∞ extends continuously over 0.

Secondly we find a complex gauge transformation on a large annulus making the pair into a vortex. We have $k_i u_i \rightharpoonup u_\infty$ in H^2 on compact subsets of $\mathbb{C} \backslash (Z \cup \{0\})$. As in the $n = 1$ case, we can assume, possibly by increasing R, that $u_i(\mathbb{C} \backslash B_R) \subseteq X^{\mathrm{ss}}$ for all i including $i = \infty$. For the existence of complex gauge transformations $g'_i \in H^2(\overline{B}_{2R} \backslash B_R)$ such that $g'_i k_i(A_i, u_i) = (A_\infty, u_\infty)$, we use Lemma 3.10. After passing to a subsequence, $g'_i \to \mathrm{Id}$ in $H^2(B_{2R} \backslash B_R)$.

Thirdly we find a complex gauge transformation so that the vortex equations are satisfied on a large ball. As before, write $g'_i = k'_i e^{i\xi'_i}$, where $k'_i : \overline{B}_{2R} \backslash B_R \to K$ and $\xi'_i : \overline{B}_{2R} \backslash B_R \to \mathfrak{k}$. We still have $F_{e^{i\eta\xi'_i} k_i(A_i,u_i)} \to 0$ in $L^2(B_{2R})$, using the arguments in the $n = 1$ case and (17). Further, the sequence $\xi''_i \in W^{1,p}_0(B_{2R})$ can be chosen such that $e^{i\xi''_i} e^{i\eta\xi'_i} k_i(A_i, u_i)$ is a vortex on B_{2R} and $\xi''_i \to \mathrm{Id}$ in $W^{1,p}(B_{2R})$.

The argument that there is no bubbling, and that the limit is in the complex-gauge-orbit of the pair is the same as in the case without orbifold singularity. Using

Lemma 3.11, we can define g_∞ on all of $\mathbb{C}$ such that it satisfies $g_\infty u_0 = u_\infty$. The rest of the proof is the same as in the case without orbifold singularity. □

Lemmas 3.10, 3.11 and 3.12 were used in the proof of Theorem 3.1 (i).

Lemma 3.10. *Suppose $\Sigma \subseteq \mathbb{C}$ is compact and that the maps $u_i : \Sigma \to X^{\mathrm{ss}}$ converge in $W^{m,p}$ to $u_\infty : \Sigma \to X^{\mathrm{ss}}$ for some $m \geq 1$ and $p \geq 2$. Assume further that the maps are related by complex gauge transformations, then it is possible to choose $g_i : \Sigma \to G$ such that $u_i = g_i u_\infty$, and a subsequence of g_i converges in $W^{m,p}$ to* Id.

Proof. Since the maps u_i and u_∞ are related by complex gauge transformations, we can write $u_i = g_i u_\infty$ for some $g_i : \Sigma \to G$. Let $x \in X/\!\!/G$ and $y \in \pi_G^{-1}(x)$. We can find a slice at y, i.e. a locally closed G_y-invariant submanifold $V \subset X^{\mathrm{ss}}$ containing y and so that there is an isomorphism $V \times_{G_y} G \to GV$. So, $V \times G \to GV$ is a $|G_y|$-cover. Suppose $U \subset \Sigma$ is such that $u_\infty(U) \subset V$, $u_i(U) \subset V$ for large i. Σ can be covered by a finite number of sets of the form of U. Choose lifts $\tilde{u}_i = (v_i, G_i)$ so that $G_i^{-1} G_j = g_i^{-1} g_j$. Since $u_i \to u_\infty$, a subsequence of $\tilde{u}_i$ converges in $W^{m,p}$. (This is because: $G \times V \to GV$ is a local diffeomorphism and has inverses locally. $\cup_i u_i(U) \subset V$ is pre-compact, and in this set the derivatives of the inverse of $G \times V \to GV$ are bounded.) So, G_i converges and consequently $g_i \to g_\infty$. Working with the chosen cover of Σ, by successively passing to subsequences, we obtain a limit g_∞ on all of Σ. The element g_∞ automatically stabilizes u_∞. So by replacing g_i by $g_\infty^{-1} g_i$, we can obtain a sequence of complex gauge transformations that converge to Id in $W^{m,p}$. □

Lemma 3.11. *Let $R > 0$. Suppose u_0, $u_1 : \mathbb{C} \backslash B_R \to X^{\mathrm{ss}}$ be holomorphic functions such that $u_0(x)$ and $u_1(x)$ are in the same G-orbit for every $x \in \mathbb{C} \backslash B_R$. Further, assume there exists a continuous function $g : B_{2R} \backslash B_R \to G$ such that $g u_0 = u_1$. Then, the definition of g can be extended to all of $\mathbb{C} \backslash B_R$ so that it satisfies $g u_0 = u_1$.*

Proof. The size of the stabilizer subgroup $|G_{u_0(x)}|$ is upper-semicontinuous on $\mathbb{C} \backslash B_R$. For any positive integer m, the set $\{x \in X^{\mathrm{ss}} : |G_x| = m\}$ is a complex submanifold of X^{ss}. Since the maps u_i are holomorphic, for a generic $x \in \mathbb{C} \backslash B_R$, the size $|G_{u_0(x)}|$ is a constant, say m_0. For a discrete set of points, $Y \subseteq \mathbb{C} \backslash B_R$, the size of the stabilizer would be a multiple of m_0.

We first assume $Y = \emptyset$. For a neighborhood of $x \in \mathbb{C} \backslash B_R$, there are m_0 continuous functions $g_1, \ldots, g_{m_0}$ to G that satisfy $g_i u_0 = u_1$. They are disjoint branches : i.e. for $i \neq j$, $g_i \neq g_j$ at all points. These functions are well-defined over any contractible neighborhood. They patch up globally on $\mathbb{C} \backslash B_R$ because one of the branches agrees with $g : B_{2R} \backslash B_R \to G$ given in the hypothesis of the lemma.

Suppose Y is non-empty and pick $z_0 \in Y$. As in the proof of Lemma 3.10, we use a G_{z_0}-invariant slice V in the neighborhood U of z_0, so that there is an isomorphism $V \times_{G_{z_0}} G \to GV$. $V \times G \to GV$ is a $|G_{z_0}|$-cover. The neighborhood U can be chosen small enough that the stabilizers G_z are of size m_0 for all $z \in U \backslash \{z_0\}$. Choose lifts $\tilde{u}_i = (v_i, G_i)$ of u_i, $i = 0, 1$ in a way that $v_0 = v_1$. We can ensure $v_0 = v_1$: it can be done by choosing the lifts such that $v_0(z) = v_1(z)$ for a single point $z \neq z_0$. Now,

$g := G_1 G_0^{-1}$ satisfies $gu_0 = u_1$. By making different choices of lifts $(\tilde{u}_0, \tilde{u}_1)$, we can define m_0 branches $g_1, \dots, g_{m_0} : U \to G$, this is because in the neighborhood of any point $z \neq z_0$, there are m_0 branches. Now, the proof proceeds in the same way as the case when $Y = \emptyset$. □

Lemma 3.12. (Regularity of complex gauge transformations) *Let Σ be a compact Riemann surface and $P \to \Sigma$ be a principal K-bundle, and $P_{\mathbb{C}} := P \times_K G$ a G-bundle. Suppose $A \in \mathcal{A}(P)$ is a smooth connection and $g \in \mathcal{G}(P)_{2,p}$ a complex transformation such that gA is also smooth. Then g is smooth.*

Proof. We use the relation on $P_{\mathbb{C}}$: $\overline{\partial}_{gA} = g \circ \overline{\partial}_A \circ g^{-1}$. $a := gA - A$ is smooth and

$$\overline{\partial}_{gA} - \overline{\partial}_A = a^{0,1} = g\overline{\partial}_A(g^{-1}) = -\overline{\partial}_A g g^{-1}.$$

Therefore $a^{0,1} g = -\overline{\partial}_A g$. The smoothness of g follows by elliptic bootstrapping. □

Proposition 3.13. *Let $R_i \to \infty$ be an increasing sequence and $\tilde{R}_i := R_i^{1/n}$. Let $(\tilde{A}_i, \tilde{u}_i)$ a sequence of $\mathrm{dvol}_{\tilde{R}_i}$-vortices on $\tilde{\mathbb{P}}$ that are symmetric under the $\mathbb{Z}_n$-action and whose energies are bounded. There exist complex gauge transformations $e^{i\tilde{s}_i} \in W^{2,p}_{\mathrm{loc}}(\mathcal{G}(P))$ that are symmetric under the $\mathbb{Z}_n$-action, are identity on $\tilde{\mathbb{P}} \backslash B_{2/\tilde{R}_i}$ and satisfy the following: Let $(A_i, u_i) := (\pi^{-1})^*(e^{i\tilde{s}_i}(\tilde{A}_i, \tilde{u}_i))$ and $F := *F_{A_i} + \Phi(u_i)$, where the Hodge star is taken with respect to the Euclidean norm. Then, $\|F\|_{L^2(B_{R_i})} \to 0$ as $i \to \infty$.*

Proof. Let $R > 0$, and $\tilde{B}_R \subset \tilde{U}_2$ be the unit ball centered at 0 in $\tilde{\mathbb{P}}$. Let

$$\sigma_R : \tilde{B}_1 \to \tilde{B}_{1/R} \quad x \mapsto x/R$$

be the dilation map. We have

$$\begin{aligned} \|\sigma^*_{\tilde{R}_i} F(\tilde{A}_i)\|_{L^2(\tilde{B}_2)} &= \frac{1}{\tilde{R}_i} \|F(\tilde{A}_i)\|^{Euc}_{L^2(\tilde{B}_{2/\tilde{R}_i})} \leq \tilde{R}_i^{-2n+1} \|F(\tilde{A}_i)\|^{\mathrm{dvol}_{\tilde{R}_i}}_{L^2(\tilde{B}_{2/\tilde{R}_i})} \\ &= \tilde{R}_i^{-2n+1} \|\Phi(u_i)\|^{\mathrm{dvol}_{\tilde{R}_i}}_{L^2(\tilde{B}_{2/\tilde{R}_i})} \leq c\tilde{R}_i^{-3n+1}. \end{aligned}$$

By Uhlenbeck's local theorem (Theorem 2.1 in [36]), for large i, we can choose a trivialization corresponding to each connection $\sigma^*_{\tilde{R}_i} \tilde{A}_i$, under which the connection matrices are uniformly bounded and are invariant under the $\mathbb{Z}_n$-action. We remark that this requires a symmetric version of the local theorem that given a symmetric connection, there exists a symmetric gauge transformation that puts the connection in Coulomb gauge. The symmetric version can be proved in a straightforward way by going through the steps of the proof of the local theorem, we omit the proof in this paper. By Lemma 4.3 and Remark 4.5, there is a constant c such that for large enough i, there exists $\tilde{s}_i \in H^2_\delta(\tilde{B}_{2/\tilde{R}_i}, \mathfrak{k})$ such that $e^{i\tilde{s}_i}(\tilde{A}_i)$ is a flat connection and

$$\|\sigma^*_{\tilde{R}_i} \tilde{s}_i\|_{H^2(\tilde{B}_2)} \leq c\|\sigma^*_{\tilde{R}_i} F(\tilde{A}_i)\|_{L^2(\tilde{B}_2)} \leq c\tilde{R}_i^{-3n+1}. \tag{19}$$

By the uniqueness of s_i, it is symmetric under $\mathbb{Z}_n$-action. The bound (19) implies a C^0 bound $\|\tilde{s}_i\|_{C^0} \leq c\tilde{R}_i^{-3n+1}$, and this bound is independent of metric. We now move to work on a neighborhood of 0 in $\mathbb{P}(1,n)$. Let

$$s_i := (\pi^{-1})^*\tilde{s}_i,\ A_i := (\pi^{-1})^*\tilde{A}_i\ u_i := (\pi^{-1})^*\tilde{u}_i,\ R_i' := R_i^n,\ \eta_i := \eta_{2/R_i'}.$$

We claim the required complex gauge transformations are $e^{i\eta_i s_i}$. We first consider the restriction to $B_{1/R_i'} \subset \mathbb{P}(1,n)$. By construction $F(e^{i\eta_i s_i}A_i) = 0$ on $B_{1/R_i'}$. By the C^0 bound on s_i, the image $e^{i\eta_i s_i}u_i$ is still contained in a compact set in X, and hence there is a C^0 bound on $\Phi(e^{i\eta_i s_i}u_i)$. This implies

$$\begin{aligned}\|F_{e^{i\eta_i s_i}(A_i,u_i)}\|_{L^2(B_{1/R_i'})} &= \|\Phi(e^{i\eta_i s_i}u_i)\|_{L^2(B_{1/R_i'})}\\ &\leq cR_i^{-3n+1}(R_i')^{-1} = c(R_i')^{-4+\frac{1}{n}}.\end{aligned} \tag{20}$$

Next, we obtain a bound in the annulus $A(\frac{1}{R_i'},\frac{2}{R_i'})$. For $\Phi(u_i)$, as above, using the C^0 bound on s_i, we have

$$\|\Phi(e^{i\eta_i s_i}u_i) - \Phi(u_i)\|_{L^2(A(1/R_i',2/R_i'))} \leq \|s_i\|_{C^0} \leq c(R_i')^{-4+\frac{1}{n}}. \tag{21}$$

To obtain an L^2 bound on the two-form

$$\Omega_i := F(e^{i\eta_i s_i}A_i) - F(A_i)$$

in the annulus $A(\frac{1}{R_i'},\frac{2^n}{R_i'})$, we work with its pull back to $\tilde{B}_2\backslash\tilde{B}_1$ via $\pi\circ\sigma_{R_i}$. Since we have a H^1-bound on the connection matrices $(\pi^{-1})^*\tilde{A}_i$, we can apply Lemma A.4 to say

$$\|\sigma_{R_i}^*(e^{i\eta\tilde{s}_i}\tilde{A}_i - \tilde{A}_i)\|_{H^1(A(1,2))} \leq c\|\sigma_{R_i}^*\tilde{s}_i\|_{H^2(\tilde{B}_2)} \leq cR_i^{-3n+1}.$$

The curvature map $A \mapsto F(A) : H^1 \to L^2$ is bounded, so we have

$$\|\sigma_{R_i}^*(F(e^{i\eta\tilde{s}_i}\tilde{A}_i) - F(\tilde{A}_i))\|_{L^2(A(1,2))} \leq \|\sigma_{R_i}^*(e^{i\eta\tilde{s}_i}\tilde{A}_i - \tilde{A}_i)\|_{H^1(A(1,2))} \leq cR_i^{-3n+1}.$$

The bound on the two-form Ω_i now requires to be transported to the annulus $A(1/R_i', 2^n/R_i')$ equipped with the Euclidean norm.

$$\begin{aligned}\|\Omega_i\|^{Euc}_{L^2(A(\frac{1}{R_i'},\frac{2^n}{R_i'}))} &= c\|\Omega_i\|^{dvol_i}_{L^2(A(\frac{1}{R_i},\frac{2}{R_i}))}\\ &= cR_i\|\sigma_{R_i}^*\Omega_i)\|^{dvol_i}_{L^2(A(1,2))} \leq cR_i^{-3n+2} = c(R_i')^{-3+\frac{2}{n}}.\end{aligned} \tag{22}$$

Using (21), (22) and recalling $F_{A_i,u_i} = 0$ on the annulus $A(\frac{1}{R_i'},\frac{2^n}{R_i'})$, we have

$$\begin{aligned}\|F_{e^{i\eta s_i}(A_i,u_i)}\|_{L^2(A(\frac{1}{R_i'},\frac{2^n}{R_i'}))} &\leq \|\Phi(e^{i\eta_i s_i}u_i) - \Phi(u_i)\|_{L^2(A(1/R_i',2/R_i'))}\\ &\quad + \|\Omega_i\|_{L^2(A(\frac{1}{R_i'},\frac{2^n}{R_i'}))} \leq c(R_i')^{-3+\frac{1}{n}}.\end{aligned} \tag{23}$$

The product ηs_i vanishes on $\mathbb{P}(1,n)\backslash B_{2^n/R_i'}$, so $F_{e^{i\eta s_i}(A_i,u_i)} = 0$ on $\mathbb{C}\backslash B_{2^n/R_i'}$, so (20) and (23) together finish the proof of Proposition 3.13. □

4. From vortices to holomorphic maps

In this section, we prove Theorem 3.1 (ii). The main tool used is the following result on the asymptotic behavior of a finite energy vortex (A,u) on $\mathbb{C}$, which is a slight generalization of a result of Ziltener [44].

Proposition 4.1 (Exponential Decay for Vortices). *Suppose G acts locally freely on X^{ss}. Let n be a positive integer such that for any $x \in X^{\mathrm{ss}}$, the order of the stabilizer group $|G_x|$ divides n. Let (A,u) be a finite energy vortex on $\mathbb{C}$ with target X. Then, for every $\epsilon > 0$, there is a constant C such that*

$$|F_A(z)|^2 + |\mathrm{d}_A u(z)|^2 + |\Phi(u(z))|^2 \le C|z|^{-2-\frac{2}{n}+\epsilon}, \quad \forall z \in \mathbb{C}\backslash B_1. \tag{24}$$

The norms are taken with respect to the standard Euclidean metric on $\mathbb{C}$.

Ziltener [44] proves this result for $n = 1$, in section 5, we explain how it generalizes to the case when $X /\!\!/ G$ is an orbifold. The following are conclusions of Proposition 4.1. The proofs appear in section 5.

Proposition 4.2 (Removal of singularity for vortices at infinity). *Assume the setting of Proposition 4.1. For any $2 < p < \frac{2}{1-\frac{1}{n}}$ and $0 < \gamma < \frac{1}{n} - 1 + \frac{2}{p}$, there exist a constant c, $x_0 \in \Phi^{-1}(0)$ and $k_0 \in W^{1,p}([0,2\pi], K)$ such that*

$$\lim_{r\to\infty} \max_{\theta\in[0,2\pi]} d(x_0, k_0(\theta)u(re^{i\theta})) = 0. \tag{25}$$

Further, if the restriction of A in radial gauge to the circle $\{|z| = r\} \simeq S^1$ is $\mathrm{D} + a\mathrm{d}\theta$, for $r \ge 1$,

$$\|k_0^{-1}\partial_\theta k_0 + a(r,\cdot)\|_{L^p([0,2\pi],K)} < cr^{-\gamma}. \tag{26}$$

Necessarily, x_0 is fixed by $k_0(2\pi)$, which is of finite order.

Proof of Theorem 3.1 (ii). We work on the chart $\tilde{U}_1 \simeq \mathbb{C}$ defined in (3). In this chart the pair (A,u) lifts to $(\tilde{A},\tilde{u})(z) := (A,u)(z^{-n})$, defined over $\mathbb{C}\backslash\{0\}$.

We first find a gauge transformation $\tilde{k}$ on $\tilde{U}_1\backslash\{0\}$ so that $\tilde{k}\tilde{A} \in L^p_{\mathrm{loc}}(\tilde{U}_1)$ and $\tilde{k}\tilde{u}$ extends continuously over 0. Define $\tilde{k}(r,\theta) := k_0(n\theta)$ where k_0 is given by Proposition 4.2. Recall that k_0 was only defined on $[0,2\pi]$, which can be extended as $k_0(\theta+2\pi) = k_0(2\pi)k_0(\theta)$. This way, $\tilde{k}$ is a well-defined gauge transformation on $\mathbb{C}\backslash\{0\}$. By Proposition 4.1, $\tilde{k}\tilde{u}$ extends continuously over ∞, with $\tilde{k}\tilde{u}(\infty) = x_0$. To show $\tilde{k}\tilde{A}$ is in $L^p_{\mathrm{loc}}(\tilde{U}_1)$, we assume $\tilde{A}$ is in radial gauge, so $\tilde{A} = \mathrm{D} + a\mathrm{d}\theta$. Then, using (26)

$$\|\tilde{k}\tilde{A}\|^p_{L^p(B_1)} = \int_0^1 \int_0^{2\pi} |\frac{1}{r}(k_0^{-1}\partial_\theta k_0 + a(r^{-n},\theta))|^p r\mathrm{d}\theta dr \le c\int_0^1 r^{n\gamma p+1-p} dr.$$

This expression is finite because $2 < p < 2(1+\frac{1}{n})$, so we can choose

$$\frac{1}{n}(1-\frac{2}{p}) < \gamma < \frac{1}{n} - 1 + \frac{2}{p}.$$

Proposition 4.2 is still applicable with these values.

Next, we show that for some small $\tilde{R}$, $\tilde{k}\tilde{A}$ can be complex gauge-transformed to the trivial connection on $B_{\tilde{R}} \subset \tilde{U}_1$. This also ensures that $\tilde{u}$ is holomorphic around $0 \in \tilde{U}_1$. For $0 < \tilde{R} < 1$, let $\sigma_{\tilde{R}} : B_1 \to B_{\tilde{R}}$ denote the dilation function $x \mapsto \tilde{R}x$. Letting $\tilde{k}\tilde{A} = \mathrm{D} + \tilde{a}$, we have $\sigma_{\tilde{R}}^*(\tilde{k}\tilde{A}) = \mathrm{D} + R\tilde{a}(R\cdot)$. Now

$$\|\sigma_R^*(\tilde{k}\tilde{A})\|_{L^p(B_1)} = \|\tilde{R}\tilde{a}(\tilde{R}\cdot)\|_{L^p(B_1)} = \tilde{R}^{1-\frac{2}{p}}\|\tilde{a}\|_{L^p(B_{\tilde{R}})}.$$

For some $\tilde{R} \in (0,1]$, $\|\sigma_{\tilde{R}}^*\tilde{A}\|_{L^p(B_1)}$ is small enough that Lemma 4.3 is applicable, which means that there is a unique $\tilde{\xi} \in W_0^{1,p}(B_{\tilde{R}})$ and $\tilde{k}_1 \in W^{1,p}(B_{\tilde{R}}, K)$ so that $F_{e^{i\tilde{\xi}}\tilde{k}\tilde{A}} = 0$ and $\tilde{k}_1 e^{i\tilde{\xi}}\tilde{k}\tilde{A} = d$. Choose $\tilde{k}_1$ so that $\tilde{k}_1(0) = \mathrm{Id}$. We observe that $\sigma_n^*(\tilde{k}A) = k_0(2\pi)A$. By the uniqueness of $\tilde{\xi}$ and $\tilde{k}_1$, we see that $\tilde{\xi} \circ \sigma_n = \mathrm{Ad}_{k_0(2\pi)}\tilde{\xi}$ and $\tilde{k}_1 \circ \sigma_n = k_0(2\pi)\tilde{k}_1 k_0(2\pi)^{-1}$. Putting everything together, we have

$$\sigma_n^*(\tilde{k}_1 e^{i\tilde{\xi}}\tilde{k}) = k_0(2\pi)\tilde{k}_1 e^{i\tilde{\xi}}\tilde{k}.$$

Looking at $\tilde{u}$, we have $\overline{\partial}(\tilde{k}_1 e^{i\tilde{\xi}}\tilde{k}\tilde{u}) = 0$ on $B_{\tilde{R}}\backslash\{0\}$ and $\tilde{k}_1 e^{i\tilde{\xi}}\tilde{k}\tilde{u}$ is continuous at 0. So, we can work on a chart of X and use the removal of singularities for holomorphic functions to show that $\tilde{k}_1 e^{i\tilde{\xi}}\tilde{k}\tilde{u}$ extends holomorphically over 0.

Next, we find a complex gauge transformation g over $\mathbb{C}$ (U_2 in the notation of (3)) corresponding to $\tilde{k}_1 e^{i\tilde{\xi}}\tilde{k}$ on $B_{\tilde{R}}\backslash\{0\}$ and show that $g(A,u)$ extends to a smooth gauged holomorphic map on a bundle $P \to \mathbb{P}(1,n)$. Let $R = \tilde{R}^{-1/n}$. Suppose $k_0 : [0,2\pi] \to K$ is homotopic to the geodesic $\theta \mapsto e^{-\lambda\theta}$ with $k_0(2\pi) = e^{-2\pi\lambda}$. The complex gauge transformation $e^{n\lambda\theta}\tilde{k}_1 e^{i\tilde{\xi}}\tilde{k}$ is symmetric under the $\mathbb{Z}_n$-action, so it descends to a complex gauge transformation g on $\mathbb{C}\backslash B_R$ that satisfies $e^{n\lambda\theta}\tilde{k}_1 e^{i\tilde{\xi}}\tilde{k}(z) = g(\frac{1}{z^n})$. The map $g : \mathbb{C}\backslash B_R \to G$ is homotopic to identity. This is because on $B_{\tilde{R}}\backslash\{0\} \subset \tilde{U}_1$, both $\tilde{k}^{-1}\tilde{k}_1 e^{i\tilde{\xi}}\tilde{k}$ and $e^{n\lambda\theta}\tilde{k}$ are homotopic to identity. Therefore, g extends to a complex gauge transformation defined on all of $\mathbb{C}$. The gauged holomorphic map $g(A,u)$ satisfies the conditions in Proposition 2.4 - i.e. on $\mathbb{C}\backslash B_{2R}$, $gA = \mathrm{D} + \lambda \mathrm{d}\theta$ and $\lim_{r\to\infty} e^{-\lambda\theta}gu(r,\theta) = x_0$. So, $g(A,u)$ extends to a gauged holomorphic curve over $\mathbb{P}(1,n)$. The complex gauge transformation $g : \mathbb{C}\backslash B_R \to G$ is smooth because both u and gu are smooth and $u(\mathbb{C}\backslash B_R) \subset X^{\mathrm{ss}}$, where G acts locally freely. So, the extension $g : \mathbb{C} \to G$ can be chosen so that it is smooth. It follows from the construction of $\tilde{k}_1 e^{i\tilde{\xi}}\tilde{k}$ that, $g(A,u)$ extends to a smooth gauged holomorphic map over $P \to \mathbb{P}(1,n)$ described by transition functions $\mu = e^{-2\pi\lambda}$ and $\tau = e^{n\lambda\theta}$.

This indeed proves the existence statement of the theorem: the complex gauge transformation g can be written as $g = e^{i\zeta}k$, where $k \in \mathcal{K}(\mathbb{C})$ and $\zeta : \mathbb{C} \to \mathfrak{k}$ are smooth. We claim that $e^{-i\zeta} \in \mathcal{G}(P)_{\mathrm{we}}$, so that we can say that $k(A,u)$ extends to a weak gauged holomorphic map over $P \to \mathbb{P}(1,n)$. Let $\tilde{\zeta} : \tilde{U}_1 \to \mathfrak{k}$ be defined as $\tilde{\zeta}(z) = \zeta(\frac{1}{z^n})$. Following the calculation of ζ is continuous at ∞, hence $\tilde{\zeta}$ is well-defined over $\tilde{U}_1$. The complex gauge transformation $e^{-i\zeta}$ on U_2 corresponds to $e^{-n\lambda\theta}e^{-i\zeta}e^{n\lambda\theta}$ on $B_{\tilde{R}/2^n} \subset \tilde{U}_1$. By straightforward calculations, this is equal to $\tilde{k}_1 e^{-i\xi}\tilde{k}_1^{-1}$, which is in $W^{1,p}(B_{\tilde{R}/2^n})$. Since $k(A,u)$ extends to a weak gauged holomorphic map over P, the same can be said about (A, u), since k just contributes to the choice of trivialization of $P|_{\mathbb{C}}$.

Finally we show uniqueness. Given a finite energy vortex (A, u), the bundle $P \to \mathbb{P}(1, n)$ is determined uniquely by the path $\{\theta \in [0, \frac{2\pi}{n}] \mapsto k_0(\theta)\}$ up to homotopy and the action of Ad_k for $k \in K$ (see remark 2.3). The path k_0 is determined using Proposition 4.1, the choice of this equivalence class is unique because it has to satisfy the condition

$$\lim_{r\to\infty} \max_{\theta\in[0,2\pi]} d(x_0, k_0(\theta)u(re^{i\theta})) = 0, \quad \text{where } x_0 = \lim_{r\to\infty} u(r, 0) \in \Phi^{-1}(0).$$

Now suppose $g_1, g_2 \in \mathcal{G}(P)_{\mathrm{we}}$ so that $g_i(A, u)$ is a smooth gauged holomorphic map on P. On $B_{\tilde{R}} \subset \tilde{U}_1$, $g_1 u$, $g_2 u$ are smooth maps to X^{ss}, where G acts locally freely. So, $g_1^{-1} g_2$ is smooth in this region and hence $g_1^{-1} g_2 \in \mathcal{G}(P)$. □

The following lemma is used in the proof of Theorem 3.1 (ii).

Lemma 4.3. *Let Σ be a Riemann surface with boundary. Let $P := \Sigma \times K$ be the trivial principal K-bundle on Σ. Let D be the trivial connection on P. There are constants c_1, c_2 so that the following holds. Let $A = \mathrm{D} + a$ be a connection on P so that $a \in \Omega^1(\Sigma, \mathfrak{k})_{L^p}$. If $\|a\|_{L^p(\Sigma)} < c_1$, there is $\xi \in W_0^{1,p}(B_1, \mathfrak{k})$ satisfying $F_{e^{i\xi}A} = 0$ and $\|\xi\|_{W^{1,p}} \leq c_2 \|a\|_{L^p}$.*

Further, on any contractible closed set $\Sigma' \subset \mathrm{int}\,\Sigma$, there is a gauge transformation $k \in W^{1,p}(\Sigma', K)$ so that $ke^{i\xi}A = d$ on Σ'. The gauge transformation k is unique up to left multiplication by a constant element in K.

On a Riemann surface Σ with boundary, if the Lie group K is connected, then any principal K-bundle $P \to \Sigma$ is trivializable. Hence it suffices to consider only trivial bundles in Lemma 4.3.

Remark 4.4 (The space $W^{-1,p}$). We recall that $W^{-1,p} = (W_0^{1,p^*})^*$, where W_0^{1,p^*} is the subspace of W^{1,p^*} sections whose boundary trace vanishes, $\frac{1}{p} + \frac{1}{p^*} = 1$ and the dual is taken under the L^2 pairing. The operator

$$d^* : L^p(\Omega^1(\Sigma, \mathfrak{k})) \to W^{-1,p}(\Sigma, \mathfrak{k})$$

has norm bound ≤ 1, as this is the dual of the operator $-d : W_0^{1,p^*} \to L^{p^*}$. The same is true of the operator d mapping L^p sections to $W^{-1,p}$.

Proof of Lemma 4.3. The proof is by an application of the implicit function theorem on the function

$$\mathcal{F}^A : W_0^{1,p}(B_1, \mathfrak{k}) \to W^{-1,p}(B_1, \mathfrak{k}), \quad \xi \mapsto *F_{e^{i\xi}A}.$$

The linearization at ξ is

$$D\mathcal{F}^A(\xi) = d^*_{e^{i\xi}A} d_{e^{i\xi}A} : W_0^{1,p}(B_1, \mathfrak{k}) \to W^{-1,p}(B_1, \mathfrak{k}). \tag{27}$$

Step 1: *There is a constant c_1 such that if a connection $A = \mathrm{D} + a$ satisfies $\|a\|_{L^p} < c_1$, then $D\mathcal{F}^A(0)$ is invertible and the inverse has norm independent of a.*

The operator $\mathrm{d}^*\mathrm{d} : W_0^{1,p}(\Sigma, \mathfrak{k}) \to W^{-1,p}(\Sigma, \mathfrak{k})$ is an isomorphism (see appendix D in [39]). Further,

$$(\mathrm{d}_A^*\mathrm{d}_A - \mathrm{d}^*\mathrm{d})\xi = *[a \wedge *d\xi] + d^*[a, \xi] + *[a \wedge *[a \wedge \xi]]. \tag{28}$$

By Sobolev multiplication (Proposition A.2),

$$\|(\mathrm{d}_A^*\mathrm{d}_A - \mathrm{d}^*\mathrm{d})\xi\|_{W^{-1,p}} \leq c\|a\|_{L^p}\|\xi\|_{W^{1,p}},$$

so the operator norm satisfies $\|(\mathrm{d}_A^*\mathrm{d}_A - \mathrm{d}^*\mathrm{d})\| \leq c\|a\|_{L^p}$. If this quantity is small enough, by a Neumann series argument, $d_A^*d_A$ is invertible with bounded norm. Let $\Theta := (\mathrm{d}_A^*\mathrm{d}_A - \mathrm{d}^*\mathrm{d})(d^*d)^{-1}$. If $\|\Theta\| < 1$, then the operator $\mathrm{Id} + \Theta$ is invertible and the inverse has a bound

$$\|(\mathrm{Id} + \Theta)^{-1}\| \leq \sum_{i=0}^{\infty} \|\Theta\|^i.$$

Further,

$$(d_A^*d_A)^{-1} = (d^*d)^{-1}(\mathrm{Id} + \Theta)^{-1}.$$

We can pick c_1 such that if $\|a\|_{L^p} < c_1$, then,

$$\|(\mathrm{d}_A^*\mathrm{d}_A)^{-1}\| \leq 2\|(\mathrm{d}^*\mathrm{d})^{-1}\|.$$

In the statement of the implicit function theorem (Proposition A.1), we take $C := 2\|(\mathrm{d}^*\mathrm{d})^{-1}\|$.

Step 2: *Given a constant c_1, there is a constant c such that for any connection $A = \mathrm{D} + a$ satisfying $\|a\|_{L^p} < c_1$ and $\|\xi\|_{W^{1,p}} < 1$,*

$$\|D\mathcal{F}_\xi^A - D\mathcal{F}_0^A\| < c\|\xi\|_{W^{1,p}}.$$

Write

$$D\mathcal{F}_\xi^A - D\mathcal{F}_0^A = (d_{(\exp i\xi)A}^* d_{(\exp i\xi)A} - \mathrm{d}_A^*\mathrm{d}_A)$$

Let $(\exp i\xi)A = A + \alpha$. Then,

$$(d_{(\exp i\xi)A}^* d_{(\exp i\xi)A} - \mathrm{d}_A^*\mathrm{d}_A)\xi_1 = *[\alpha \wedge *d\xi_1] + d^*[\alpha \wedge \xi_1] + *[\alpha \wedge *[\alpha \wedge \xi_1]].$$

As in Step 1, the operator norm of $D\mathcal{F}_\xi - D\mathcal{F}_0$ is bounded as

$$\|d_{(\exp i\xi)A}^* d_{(\exp i\xi)A} - \mathrm{d}_A^*\mathrm{d}_A\| \leq c\|\alpha\|_{L^p} \leq c\|\xi\|_{W^{1,p}},$$

where the last inequality is by Lemma A.4. If $\|a\|_{L^p} < c_1$, the constant c is independent of a and ξ.

Step 3: *Finishing the proof.*

By the result in Step 2, for a small enough $\delta > 0$,

$$\|\xi\|_{W^{1,p}} < \delta \implies \|D\mathcal{F}_\xi^R - D\mathcal{F}_0^R\| \leq \frac{1}{2C}.$$

The function

$$(\xi, \Delta\xi) \mapsto D\mathcal{F}^A(\xi)\Delta\xi : W_0^{1,p}(B_1, \mathfrak{k}) \times W_0^{1,p}(B_1, \mathfrak{k}) \to W^{-1,p}(B_1, \mathfrak{k})$$

is continuous, this can be seen using the identity (28), Lemma A.4 and Sobolev multiplication (proposition A.2). So $\mathcal{F}^A$ is differentiable and the implicit function theorem is applicable. By Proposition A.1, if $\|F_A\|_{-1,p} < \frac{\delta}{4C}$, we get the result. This can be ensured if $\|a\|_{L^p}$ is small enough because by the multiplication theorem,

$$\|F_A\|_{-1,p} \leq c(\|a\|_{L^p} + \|a\|^2_{L^p}).$$

We next prove the second statement of the lemma. First note that a flat connection is gauge-equivalent to a connection that is smooth away from the boundary $\partial\Sigma$. Indeed, this is proved in [38] (Theorem 3.1) for a base manifold without boundary, but since we only want interior regularity, the proof is identical. We briefly outline this proof: For an L^p connection A, if we pick a smooth connection $\tilde{A}_0$ close enough to A (in the L^p norm), there is a gauge transformation $k \in W^{1,p}$ that puts A in coulomb gauge with respect to $\tilde{A}_0$, that is,

$$\mathrm{d}^*_{\tilde{A}_0}(kA - \tilde{A}_0) = 0.$$

This is the content of the local slice theorem (Theorem 8.3) in [39]. Now, one has control over da and d^*a, using elliptic bootstrapping as in [38], it can be shown that kA is smooth away from $\partial\Sigma$. Similar to the first statement in the lemma, here we rely on the ellipticity of $\Delta : W^{1,p}_0 \to W^{-1,p}$.

Finally note that a smooth flat connection on a contractible set is gauge equivalent to the trivial connection. The uniqueness statement in the lemma is obvious because the trivial connection is preserved only by constant gauge transformations. □

Remark 4.5. Lemma 4.3 is true for higher regularity connections. Suppose $k \in \mathbb{Z}_{\geq 0}$, $p > 1$ such that $(k+1)p > 2$. If $a \in W^{k,p}$, then $\xi \in W^{k+1,p}$ can be produced satisfying the conditions of lemma 4.3. The proof of the higher regularity statement is analogous and is more standard.

The following is the last component in the proof of the main Theorem 3.1. To state it, we assume the result of part (ii) of this theorem.

Proposition 4.6. (At most one vortex in a complex gauge orbit) *Suppose* (A_0, u_0) *and* (A_1, u_1) *are finite energy vortices on* $\mathbb{C}$ *that extend to a weak gauged holomorphic map over a bundle* $P \to \mathbb{P}(1,n)$*, that are related by a complex gauge transformation* $g \in \mathcal{G}(P)_{\mathrm{we}}$*. Then* g *is a unitary gauge transformation, i.e.* $g \in \mathcal{K}(P)_{\mathrm{we}}$*.*

Proof. The complex gauge transformation g can be written as $g = ke^{i\xi}$, where $k \in \mathcal{K}(P)_{\mathrm{we}}$ and $\xi \in \Gamma(\mathbb{P}(1,n), P(\mathfrak{k}))_{\mathrm{we}}$. We can assume $k = \mathrm{Id}$ and $(A_1, u_1) = e^{i\xi}(A_0, u_0)$. The proof proceeds in the same way as the corresponding result for vortices on a compact Riemann surface (Proposition 3.5). Let $(A_t, u_t) := e^{it\xi}(A, u)$ for $t \in [0,1]$. We write

$$\begin{aligned}\frac{d}{dt}\int_{\mathbb{C}} \langle *F_{A_t,u_t}, \xi\rangle &= \int_{\mathbb{C}} \langle \mathrm{d}^*_{A_t}\mathrm{d}_{A_t}\xi + u_t^* d\Phi(J(\xi)_{u_t}), \xi\rangle_{\mathfrak{k}} \\ &= \|\mathrm{d}_{A_t}\xi\|^2_{L^2} + \int_{\mathbb{C}} \omega_{u_t}((\xi)_{u_t}, J(\xi)_{u_t}) + \lim_{r\to\infty}\int_{\partial B_r} \langle \nabla_{A_t,\nu}\xi, \xi\rangle_{\mathfrak{k}}\end{aligned} \tag{29}$$

where $\nabla_{A_t,\nu}\xi$, is the covariant derivative of ξ along ν, the outward normal unit vector field to ∂B_r.

However, for the above computation to make sense, we need to show that the terms $\|\mathrm{d}_{A_t}\xi\|_{L^2(\mathbb{C})}$ and $\int_{\mathbb{C}} \omega_{u_t}((\xi)_{u_t}, J(\xi)_{u_t})$ are finite and the boundary term vanishes.

Step 1: $\|d_{A_t}\xi\|_{L^2} < \infty$ *for* $t \in [0,1]$.
We work on the chart containing infinity - this is $\tilde{U}_1$ from (3). Consider $B_{\tilde{R}} \subset \tilde{U}_1$, for some $\tilde{R} > 0$. We have $A_0 \in L^p(B_{\tilde{R}})$ and $\xi \in W^{1,p}(B_{\tilde{R}})$. Using Lemma A.4 for the action of the complex gauge transformation $e^{it\xi}$ on the connection A_0, we can say $A_t - A_0 \in L^p(B_{\tilde{R}})$. Since $p > 2$, $\|d_{A_t}\xi\|_{L^2(B_{\tilde{R}})}$ is finite. Let $R = (\tilde{R})^{-1/n}$. The norms $\|d_{A_t}\xi\|_{L^2(\mathbb{C}\backslash B_R)}$ and $\|d_{A_t}\xi\|_{L^2(B_{\tilde{R}})}$ are equal, so both are finite, and hence $\|d_{A_t}\xi\|_{L^2} < \infty$ for $0 \leq t \leq 1$.

Step 2: $\int_{\mathbb{C}} \omega_{u_t}((\xi)_{u_t}, J(\xi)_{u_t}) < \infty$ *for* $t \in [0,1]$.
We use asymptotic decay of vortices (Proposition 4.1) to obtain an asymptotic bound on $|\xi|$: fix an $0 < \epsilon < \frac{2}{n}$, let $\delta = \frac{2}{n} - \epsilon$, then for an affine vortex (A,u) there is a constant c so that for $z \in \mathbb{C}\backslash B_1$,

$$e_{A,u}(z) \leq c|z|^{-2-\delta}.$$

In particular, $\Phi(u_0(z))$, $\Phi(u_1(z)) \leq c|z|^{-1-\frac{\delta}{2}}$.

To shorten notation, for any $c > 0$, define $Z_c := \{|\Phi| < c\} \subset X$. For any $x \in X$, define

$$\Psi_x : \mathfrak{k} \to \mathfrak{k}, \quad s \mapsto \Phi(e^{is}x) - \Phi(x).$$

For all $x \in \Phi^{-1}(0)$, the derivative at 0, $d\Psi_x(0)$ is invertible. Since Φ is proper, we can pick $\epsilon_1 > 0$ such that $d\Psi_x(0)$ is invertible for all $x \in Z_{\epsilon_1}$. Define

$$\Psi : Z_{\epsilon_1} \times \mathfrak{k} \to Z_{\epsilon_1} \times \mathfrak{k} \quad (z,s) \mapsto \Psi_z(s) = (z, \Phi(e^{is}z) - \Phi(z)).$$

For all $z \in Z_{\epsilon_1}$, the map $d\Psi(z,0)$ is an isomorphism, so Ψ is locally an isomorphism. i.e. there is an open neighborhood of U of $(z,0)$, such that $\Psi : U \to f(U)$ is a diffeomorphism. By the compactness of Z_{ϵ_1}, we can say: there is $\epsilon_2 > 0$ such that Ψ has a smooth inverse on $Z_{\epsilon_1} \times B_{\epsilon_2}(\mathfrak{k})$. Set $\epsilon := \min\{\epsilon_1, \epsilon_2\}$. We can conclude that there is a constant c such that for x, $y \in Z_\epsilon$ satisfying $y = e^{is}x$ for some $s \in \mathfrak{k}$,

$$|s| < c|\Phi(x) - \Phi(y)|. \tag{30}$$

We can pick $0 < \epsilon_2' < \epsilon_2$ such that for all $z \in Z_{\epsilon_1}$, the straight line connecting 0 and any point in $\Psi_z^{-1}(B_{\epsilon_2'})$ is contained in $\Psi_z^{-1}(B_{\epsilon_2})$. Set $\epsilon' := \min\{\epsilon_1, \epsilon_2'\}$. For any x, $y \in Z_{\epsilon'}$ satisfying $y = e^{is}x$ for some $s \in \mathfrak{k}$, the curve $\{e^{its}x : t \in [0,1]\}$ is contained in Z_ϵ.

Let $R > 0$ be large enough that both u_0, u_1 map $\mathbb{C}\backslash B_R$ to $Z_{\epsilon'}$. Recall $u_1 = e^{i\xi}u_0$. By the preceding discussion, $u_t(\mathbb{C}\backslash B_R) \subset Z_\epsilon$ for all $t \in [0,1]$ and by (30),

$$|\xi(x)| < c|\Phi(u_1(x)) - \Phi(u_0(x))| < c|z|^{-1-\frac{\delta}{2}} \tag{31}$$

for $x \in \mathbb{C}\backslash B_R$. For $z \in Z_\epsilon$, the operator L_z (see (13)) is uniformly bounded. So,

$$|\omega_{u_t}((\xi)_{u_t}, J(\xi)_{u_t})(z)| < c|z|^{-2-\delta}$$

for $z \in \mathbb{C}\backslash B_R$ and hence,

$$\int_{\mathbb{C}} \omega_{u_t}((\xi)_{u_t}, J(\xi)_{u_t}) < \infty, \quad 0 \leq t \leq 1.$$

STEP 3: $\lim_{r\to\infty} \int_{\partial B_r} \langle \nabla_{A_t,\nu}\xi, \xi\rangle_{\mathfrak{k}} = 0$ *for* $t \in [0,1]$.
In step 1, we showed that A_t is in $L^2(\mathbb{C}\backslash B_R)$, but this isn't enough to prove anything about $A_t|_{\partial B_r}$. We now obtain a C^0-bound under suitable local trivializations of P. For this we cover $\mathbb{C}\backslash B_R$ by identical open sets : let $S \subseteq \mathbb{C}$ be an open set with smooth boundary such that

$$[-\frac{3}{4}, \frac{3}{4}] \times [-\frac{3}{4}, \frac{3}{4}] \subseteq S \subseteq [-1,1] \times [-1,1].$$

Then, $\{S + (x,y) : |x|, |y| \geq R-2\}$ is a cover of $\mathbb{C}\backslash B_R$. Let $S'' \Subset S' \Subset S$ be such that their translates (by integers) also cover $\mathbb{C}\backslash B_R$. The quantity $\langle \mathrm{d}_{A_t}\xi, \xi\rangle_{\mathfrak{k}} \in \Omega^1(\Sigma)$ is gauge-invariant, so on each $S + (x,y)$, we can choose a different trivialization to study it. We focus on a single set $S_{xy} := S + (x,y)$ and let $r := \sqrt{x^2+y^2}$. In the following discussion, the constant c is independent of (x,y) and r. Fix a trivialization of $P|_{S_{xy}}$ so that $A_0 = \mathrm{D} + a_0$ is in Uhlenbeck gauge i.e.

$$\|a_0\|_{W^{1,p}(S_{xy})} < c\|F(A_0)\|_{L^p(S_{xy})} < c. \tag{32}$$

Under this trivialization, we write $A_t = \mathrm{D} + a_t$. By applying a gauge transformation $k : S_{xy} \to K$, we can put A_1 in Uhlenbeck gauge - i.e. if $kA_1 = \mathrm{D} + \tilde{a}_1$ then,

$$\|\tilde{a}_1\|_{W^{1,p}(S_{xy})} < c\|F(A_1)\|_{L^p(S_{xy})} < c. \tag{33}$$

Denote $g = ke^{i\xi}$, so $gA_0 = \mathrm{D} + \tilde{a}_1$. As in the proof of Lemma 3.12, we can write $a^{0,1}g = -\overline{\partial}_{A_0}g$, where $a = \tilde{a}_1 - a_0$. By elliptic regularity,

$$\begin{aligned}\|g\|_{W^{1,p}(S'_{xy})} &\leq c(\|\overline{\partial}g\|_{L^p(S_{xy})} + \|g\|_{L^p(S_{xy})})\\ &\leq c(\|a^{0,1}g\|_{L^p(S_{xy})} + \|a_0^{0,1}g\|_{L^p(S_{xy})}) + c\|g\|_{L^\infty(S_{xy})})\end{aligned}$$

There is a L^∞ bound on g from (31) and the fact that K is compact. Together with the bounds on a and a_0 (from (32) and (33)), this shows that $\|g\|_{W^{1,p}(S'_{xy})} \leq c$. By applying elliptic regularity again, we can show $\|g\|_{W^{2,p}(S''_{xy})} \leq c$. Since (2) is an isomorphism,

$$\|\xi\|_{W^{2,p}(S''_{xy})} \leq c. \tag{34}$$

Repeating the calculations in Lemma A.4 for different Sobolev spaces, we get

$$\|a_t - a_0\|_{W^{1,p}(S''_{xy})} < c.$$

Together with the bound on a_0, this uniformly bounds $\|a_t\|_{W^{1,p}(S_{xy})}$. By Sobolev embedding

$$\|a_t\|_{C^0(S_{xy})} < c \tag{35}$$

for $0 \leq t \leq 1$.

Consider the integral $\int_{\partial B_r} \langle \nabla_{A_t,\nu}\xi, \xi\rangle_{\mathfrak{k}}$. We partition the curve into segments, so that each segment lies in a single set S''_{xy}. Write $\nabla_{A_t,\nu}\xi = \frac{\partial}{\partial\nu}\xi + [(a_t)_\nu, \xi]$. We have

C^0 bounds on $\frac{\partial}{\partial\nu}\xi$ and $(a_t)_\nu$ using (34) and (35). Together with the asymptotic bound on ξ (31), the result of Step 3 is proved.

STEP 4: *Finishing the proof.*
We have shown that the equation (29) holds and the boundary term vanishes as $r \to \infty$. So,

$$\frac{d}{dt}\int_{\mathbb{C}} \langle *F_{A_t,u_t}, \xi\rangle > 0$$

if $\xi \neq 0$. Since $F_{A_0,u_0} = F_{A_1,u_1} = 0$, this implies that $\xi = 0$. □

Proof of Proposition 1.2. It is enough to consider the case of a line bundle $X = \mathbb{C}$ on which G acts with weight $w \in \mathfrak{g}^\vee$. P is a principal bundle with $[P] = d$. Then the associated orbifold line bundle $P(X)$ has degree (pairing of the first Chern class with the rational fundamental class) $\langle w, d\rangle \in \mathbb{Q}$. By our set-up $\langle w, d\rangle = \frac{m}{n}$ for some $m \in \mathbb{Z}_{\geq 0}$. Up to isomorphism, there is a unique holomorphic line bundle on $\mathbb{P}(1, n)$ of this degree [33, Theorem 5.1], this line bundle is described as: $L = (\tilde{U}_1 \times \mathbb{C}) \sqcup (U_2 \times \mathbb{C})/\sim$ where $\sim$ is as follows

$$\begin{aligned} &(z, l) \sim (\sigma_n z, e^{-2\pi i m/n} l) \qquad (z, l) \in \tilde{U}_1 \times \mathbb{C} \\ &(z, l) \sim (w, z^m l) \qquad 0 \neq z \in \tilde{U}_1, w \in U_2, w = \frac{1}{z^n}, \in \mathbb{C} \end{aligned}$$

In the framework of section 2.5, the transition functions $\mu : \tilde{U}_1 \to \mathbb{C}^\times$ and $\tau : \tilde{U}_1 \backslash \{0\} \to \mathbb{C}^\times$ are holomorphic and are $\mu(z) := e^{-2\pi i m/n}$ and $\tau(z) = z^m$. Holomorphic sections on $u : \mathbb{P}(1, n) \to P(X)$ are of the following form:

$$U_2 \ni w \mapsto u_0 + u_1 w + \cdots + u_k w^k, \quad \tilde{U}_1 \ni z \mapsto z^m(u_0 + u_1 z^{-n} + \cdots + u_k z^{-kn}),$$

where $k = \lfloor m/n \rfloor$. It is clear that $u(\infty) = 0$ if m/n is not an integer, and otherwise $u(\infty) = u_k$. □

5. Asymptotic decay for vortices

Proposition 4.1 is a consequence of a decay result for vortices on a cylinder (Proposition 5.3) and a result about the limit behavior of u as $z \to \infty$ (proposition 5.4).

Definition 5.1 (Energy density)**.** Suppose $\lambda : B_1 \subset \mathbb{C} \to \mathbb{R}_{\geq 0}$ is a smooth function. Let (A, u) be a vortex from the principal bundle $P \to B_1$ to X with respect to the metric $\omega = \lambda^2 ds \wedge dt$. The *energy density* of (A, u) is

$$e_{(A,u)}(z) := \frac{1}{2}\left(|F_A(z)|^2 + |\mathrm{d}_A u(z)|^2 + |\Phi(u(z))|^2\right)$$

for any $z \in \Sigma$. The norms are defined as : for any form $\alpha \in \Omega^*(B_1, \mathfrak{k})$, if $\alpha \wedge_\lambda *\alpha = f ds \wedge dt$, then $|\alpha(z)|^2 = |f|^2$. Here, $*_\lambda$ denotes the Hodge star under the metric $\lambda^2 ds \wedge dt$.

Definition 5.2 (Admissible metric)**.** On a half cylinder $\Sigma := \{(s,t) : s \geq 0, t \in \mathbb{R}/a\mathbb{Z}\}$, the metric $\omega_\Sigma = \lambda^2 ds \wedge dt$ is *admissible* if

$$\lambda \geq \frac{2\pi}{a m_{\Phi^{-1}(0)}} \qquad \sup_\Sigma(\Delta(\lambda^{-2}) + |d(\lambda^{-1})|^2) < 2m^2_{\Phi^{-1}(0)} \tag{36}$$

where $m_{\Phi^{-1}(0)} = \inf_{x \in \Phi^{-1}(0)} |L_x|$. L_x is defined by (13).

Proposition 5.3 (Decay for vortices on the half cylinder, [44], Theorem 1.3)**.** *Let Σ be a half cylinder*

$$\Sigma := \{(s,t) : s \geq 0, t \in \mathbb{R}/a\mathbb{Z}\}$$

with an admissible metric $\omega_\Sigma = \lambda^2 ds \wedge dt$ where $\lambda = \lambda(s,t)$ is a positive function. Suppose G acts freely on X^{ss} and (A,u) is a finite energy vortex from the trivial bundle $\Sigma \times K$ to X such that $\overline{u(\Sigma)}$ is compact. Then, for every $\epsilon > 0$, there is a constant C such that

$$e_{(A,u)}(s+it) \leq C\lambda^{-2} e^{(-\frac{4\pi}{a}+\epsilon)s}.$$

Proposition 5.4. *Suppose (A,u) is a finite energy vortex on the half cylinder Σ (described in proposition 5.3). Let $\pi_G : X^{\mathrm{ss}} \to X/\!\!/G$ denote the projection. Then, $\lim_{s\to\infty} \pi_G \circ u$ exists.*

Proof. The proof of Proposition 5.4 uses a combination of results from [44] and [23]. The main difficulty is that the action of G on X^{ss} is not free. [44] considers vortices where this action is free. The composition $\pi_G \circ u$ is a J-holomorphic curve on $X/\!\!/G$, but we can't use the results of [23] directly because $X/\!\!/G$ has orbifold singularities.

For the mean value inequality, we use Lemma 3.3 in [44]. The proof of this result works when the action of K on $\Phi^{-1}(0)$ is locally free. In the setting of Proposition 5.3, this gives : there exists a number $s_0 \geq \frac{1}{2}$ such that for $z = (s,t) \in \{s \geq s_0\}$,

$$e_{(A,u)}(z) \leq \frac{32}{\pi} E((A,u), B_{\frac{1}{2}}(z)).$$

Since $E((A,u),\Sigma)$ is finite, the right side goes to 0 as $s \to \infty$. So, for large enough s_0, $\Phi(u(s,t))$ is close enough to 0 and so $u(s,t) \in X^{\mathrm{ss}}$, and $u_G := \pi_G \circ u$ is well-defined and is a holomorphic curve on $X/\!\!/G$. Also,

$$\begin{aligned} \ell(u_G(\{s = s_1, t \in \mathbb{R}/a\mathbb{Z}\})) &\leq \int_0^a |du_G(s_1,t)| dt \\ &\leq \int_0^a |\mathrm{d}_A u(s_1,t)| dt \to 0 \quad \text{as } s_1 \to \infty. \end{aligned} \tag{37}$$

Now, we switch to working on $X/\!\!/G$ to prove the result. For every $p \in X/\!\!/G$, there is a neighbourhood U_p and a uniformizing chart (V_p, G_p, π) such that $V_p \subset \mathbb{C}^m$, G_p is a finite group acting on U_p and $\pi : U_p \to V_p$ induces a homeomorphism from U_p/G_p to V_p (see [6]). Each U_p has a G_p-invariant symplectic form that descends to the symplectic form on $X/\!\!/G$. $X/\!\!/G$ is compact, so it can be covered by a finite number of such neighbourhoods $U_1, \dots, U_k$. Fix a constant $\delta_0 > 0$ such that for any $p \in X/\!\!/G$, $B_p(\delta_0) \subset U_i$ for some $i \in \{1, \dots, k\}$. If the length of the loop $\gamma : S^1 \to X/\!\!/G$ is less than δ_0, it can be lifted to the cover in some uniformizing chart and the isoperimetric inequality (Theorem 4.4.1, [23]) can be applied. The rest of the proof can be completed in the same way as the proof of the removal of singularities result for J-holomorphic curves in [23] (theorem 4.1.2). We need the second paragraph of the proof of Lemma 4.5.1 (this requires Stokes' theorem for orbifolds, which can be proved by passing to a cover locally using regularizing charts), followed by the proof of Theorem 4.1.2. Note that we don't require holomorphic extension of u_G over the singularity. □

Proposition 4.1 now follows in a straightforward way.

Proof of Proposition 4.1. Map $\mathbb{C}\backslash B_1$ to the half cylinder Σ setting $a = 2\pi$ and the change of coordinates $\Sigma \ni z \mapsto e^z \in \mathbb{C}\backslash B_1$. Equip Σ with the pullback metric. By Proposition 5.4, $u_G(\infty) := \lim_{s\to\infty} u_G(s,t)$ exists. Let $x \in \pi_G^{-1}(u_G(\infty))$, and let S be a slice for the G-action at x. This means $G \times_{G_x} S \to X^{\mathrm{ss}}$ is a diffeomorphism onto its image. $\pi : G \times S \to X^{\mathrm{ss}}$ is a $|G_x|$-cover, equip $G \times S$ with the metric $\pi^*\omega_X$. The left K-action is free and has moment map $\pi^*\Phi$. n divides G_x, so for some large s_0, $u(\Sigma_{s>s_0}) \subseteq GS$ and it lifts to $\tilde{u} : \tilde{\Sigma}_{s>s_0} \to G \times S$. Here $\tilde{\Sigma}_{s>s_0} = \{(s,t) : s \geq s_0, t \in \mathbb{R}/2\pi n\mathbb{Z}\}$ is an n-cover of $\Sigma_{s>s_0}$ equipped with the pullback metric. Now, Proposition 5.3 can be applied to the lift $(\tilde{A}, \tilde{u}) : \tilde{\Sigma}_{s\geq s_0} \to G \times S$, and this proves Proposition 4.1. □

Proof of Proposition 4.2. We work in cylindrical co-ordinates. Map $\mathbb{C}\backslash B_1$ to the half cylinder $\Sigma = \{(s,t) : s \geq 0, t \in [0, 2\pi)\}$, with change of coordinates $\Sigma \ni z \mapsto e^z \in \mathbb{C}\backslash B_1$. The Euclidean metric on $\mathbb{C}\backslash B_1$ pulls back to $\omega_\Sigma = e^{2s}\mathrm{d}s \wedge \mathrm{d}t$ on Σ. The connection A can be put in radial gauge, so that on Σ, $A = \mathrm{D} + a(s,t)dt$. The proof now is exactly same as the proof of proposition D.7 (B) in [42]. The only asymptotic result used in that proof is: for some $\delta > 0$,

$$\sup_t(|\frac{\partial}{\partial s}u| + e^{\delta s}|\Phi(u)|) \leq e^{-\delta s} \quad \forall s \geq 0.$$

In our case, by Proposition 4.1 we have this result for $\delta = \frac{2}{n} - \epsilon$. The conclusion of proposition D.7 (B) consists of (25) and (26) in the form:

$$\|k_0^{-1}\partial_\theta k_0 + a(r,\cdot)\|_{W^{1,p}([0,2\pi],K)} < cr^{(-1+\frac{2}{p}+\frac{\delta}{2})}.$$

To obtain our result, we choose $\epsilon = 2(\frac{1}{n} - 1 + \frac{2}{p} - \gamma)$. □

Appendix A. Some analytic results

In this section, we collect some analytic results used in the proof. The first two results are standard. The following is proposition A.3.4 in [23].

Proposition A.1. (Implicit function theorem) *Let $F : X \to Y$ be a differentiable map between Banach spaces. Suppose that $DF(0)$ is surjective and has a right inverse Q, with $\|Q\| \leq c$. For all $x \in B_\delta$, $\|DF(x) - DF(0)\| < \frac{1}{2c}$. If $\|F(0)\| < \frac{\delta}{4c}$, then $F(x) = 0$ has a solution in B_δ. This solution is the unique in B_δ satisfying $x \in \operatorname{Im} Q$.*

Proposition A.2 (Sobolev multiplication)**.** *Let $\Omega \subseteq \mathbb{R}^n$ be a closed domain with smooth boundary, not necessarily compact. The multiplication operator*

$$\begin{aligned} W^{s_1,p_1}(\Omega) \times W^{s_2,p_2}(\Omega) &\to W^{s_3,p_3} \\ (f,g) &\mapsto fg \end{aligned}$$

is bounded if $s_1 + s_2 \geq 0$, $s_3 \leq \min(s_1, s_2)$ and $s_3 - \frac{n}{p_3} < s_1 - \frac{n}{p_1} + s_2 - \frac{n}{p_2}$.

If Ω is compact, $s_1 + s_2 > 0$ and $s_3 < \min(s_1, s_2)$, the operator is compact.

The following result is Theorem 1 in [13] applied to the case where the base manifold is a Riemann surface.

Theorem A.3 (Donaldson [13], Theorem 1)**.** *Let Σ be a compact Riemann surface with boundary, and let A be a H^1 connection on the trivial bundle $\Sigma \times K$. There is a unique $s \in H^2(\Sigma, \mathfrak{k})$ satisfying $s|_{\partial\Sigma} \equiv 0$ and $e^{is}A$ is a flat connection.*

Lemma A.4 (Norm bound for $\mathcal{G}_{\mathbb{C}}$ action on $\mathcal{A}$)**.** *Let Σ be a compact Riemann surface, possibly with boundary. Let k, $p > 0$ be such that $(k+1)p > 0$. For any $\epsilon > 0$, there is a constant C so that the following is satisfied.*

For any $W^{k,p}$ connection $A = \mathrm{D} + a$ on the trivial bundle $\Sigma \times K$ which satisfies $\|a\|_{W^{k,p}(\Sigma)} < \epsilon$ and any $\xi \in W^{k+1,p}(\Sigma, \mathfrak{k})$ that satisfies $\|\xi\|_{W^{k+1,p}} < 1$, $(\exp i\xi)A \in W^{k,p}$ and

$$\|(\exp i\xi)A - A\|_{W^{k,p}(\Sigma)} \leq C\|\xi\|_{W^{k+1,p}(\Sigma)}.$$

Proof. The infinitesimal action of $i\xi$ on a connection A is $*\mathrm{d}_A\xi$. Suppose $a_t : [0,1] \to H^1_A(\mathfrak{k})$ is the solution of the equation

$$\frac{da_t}{dt} = *d_{A+a_t}\xi = *\mathrm{d}\xi + *[a_t \wedge \xi], \quad a_0 = a. \tag{38}$$

Then, $(\exp i\xi)A = \mathrm{D} + a_1$. The equation (38) has a solution on the time interval $[0,1]$ because

$$W^{k,p} \ni a \mapsto *(d\xi + [a,\xi]) \in W^{k,p}$$

is Lipschitz, so $t \mapsto a_t$ is continuously differentiable.

Suppose $a_t \neq 0$ for all $t \in [0,1]$. Then,

$$\frac{d}{dt}\|a_t\|_{W^{k,p}} \leq \|\frac{da_t}{dt}\|_{W^{k,p}} \leq c(\|\xi\|_{W^{k+1,p}} + \|a_t\|_{W^{k,p}}\|\xi\|_{W^{k+1,p}}). \tag{39}$$

Since $\|\xi\|_{W^{k+1,p}} < 1$, we have $\frac{d}{dt}\|a_t\|_{W^{k,p}} \leq c(1 + \|a\|_{W^{k,p}})$. Then,

$$\|a_t\|_{W^{k,p}} \leq e^{ct}(1 + \|a_0\|_{W^{k,p}}) - 1 \leq C(\epsilon).$$

Now, we use (39) again and write $\frac{d}{dt}\|a_t\|_{W^{k,p}} \leq C\|\xi\|_{W^{k+1,p}}$. This proves the result if $a_t \neq 0$ for all $t \in [0,1]$. If that is not the case, we apply the procedure on the intervals $[0,t_0]$ and $[t_1,1]$, where t_0, t_1 are the smallest and largest time values for which $\|a_t\| \leq \epsilon$. This would produce the same bound. ∎

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