# The classification of transversal multiplicity-free group actions

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**Abstract**

Multiplicity-free Hamiltonian group actions are the symplectic analogs of multiplicity-free representations, that is, representations in which each irreducible appears at most once. The most well-known examples are toric varieties. The purpose of this paper is to show that under certain assumptions multiplicity-free actions whose moment maps are transversal to a Cartan subalgebra are in one-to-one correspondence with a certain collection of convex polytopes. This result generalizes a theorem of Delzant concerning torus actions.

## Contents



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## 1 Introduction

A Hamiltonian action is called *multiplicity-free* if all the symplectic reduced spaces are zero dimensional, or equivalently if the invariant functions form an abelian Poisson algebra [15]. The name comes from the fact that the geometric quantization of any multiplicity-free action is a multiplicity-free representation, that is, contains each irreducible representation at most once (at least if the action admits a Kähler quantization [4]; see also [13] and [26].) Recall that by a theorem of Kirwan [21], if a compact Lie group acts on a compact connected symplectic manifold in a Hamiltonian fashion, then the intersection of the image of the moment map with a positive Weyl chamber is a convex polytope, called the *moment* (or *Kirwan) polytope*. The moment polytope is the classical limit of the set of highest weights of irreducibles appearing the geometric quantization of the action, in a sense that can be made precise. The *multiplicity-free conjecture* suggests that any two multiplicity-free actions of compact connected Lie groups on compact connected symplectic manifolds with the same moment polytope and same principal isotropy subgroup are isomorphic. In the case of torus actions, this conjecture was proved by Delzant [7]. Delzant also characterized exactly which polytopes occur and showed that any multiplicity-free torus action (under the assumptions above) is the Hamiltonian action associated to a toric variety.

In this paper we prove the multiplicity-free conjecture for actions of non-abelian groups whose moment maps are transversal to a Cartan subalgebra, and that satisfy some further assumptions which will be described later. We also characterize the convex polytopes that occur as moment polytopes of these actions by a simple symmetry condition involving local Weyl groups. This generalizes the first half of Delzant's theorem.[1] The project was suggested by V. Guillemin.

The motivating examples of the multiplicity-free actions studied in this paper are Hamiltonian actions underlying certain multiplicity-free branching laws in the representation theory of compact groups. By a multiplicity-free branching law, we mean an irreducible representation of a compact connected group $H$ that is multiplicity-free under the action of $G$. The corresponding Hamiltonian group action is by the Borel-Weil theorem a coadjoint orbit of $H$, considered as a Hamiltonian $G$-space. The results in this paper do not apply to all of the actions corresponding to multiplicity-free branching laws, but do apply to the most famous of these, the action of $U(n)$ on a generic coadjoint orbit of $U(n+1)$.

To state the main result precisely, let $G$ be a compact connected Lie group, $T \subset G$ a maximal torus, and $W$ the Weyl group of $T \subset G$, which acts on $\mathfrak{t}^* \subset \mathfrak{g}^*$. Let $\mathfrak{t}^*_+$ be a closed positive Weyl chamber.

**Definition 1.1** We say that a convex polytope $\Delta \subset \mathfrak{t}^*_+$ of maximal dimension is *reflective* at $x \in \Delta$ if[2]

[1] Iglesias [19] and Delzant [8] have proved generalizations of Delzant's theorem for groups of low rank. Sjamaar and Guillemin [27] have proved a generalization for groups of arbitrary rank, which depends on an assumption on the stabilizers.

[2] The term reflective has no connection with the term *reflexive* in e.g. [3].

(a) the set of hyperplanes that intersect $\Delta$ in facets (i.e., codimension 1 faces) that contain $x$ is invariant under the stabilizer $W_x$ of $x$, and

(b) any open facet of $\Delta$ containing $x$ in its closure is contained in the open positive Weyl chamber int $\mathfrak{t}^*_+$.

$\Delta$ is called reflective if $\Delta$ is reflective at all points $x \in \Delta$. Condition (a) is equivalent to requiring that if $H$ is a hyperplane such that $H \cap \Delta$ is a facet containing $x$, and $w \in W_x$ then $H \cap w\Delta$ is a facet of $w\Delta$. That is, the facets of $\Delta$ "continue through the walls".

Given a convex polytope $\Delta \subset \mathfrak{t}^*_+$ of maximal dimension and a point $x \in \Delta$, let $V(x) \subset \mathfrak{g}$ be a set of inward-pointing normal vectors to facets (i.e. codimension 1 faces) of $\Delta$ meeting $x$. The elements of $V(x)$ are defined up to multiplication by positive scalars. Note that condition (a) in Definition 1.1 is equivalent to requiring that for each $v \in V(x)$ and $w \in W_x$ we have $wv \in \mathbb{R}V(x)$.

**Example 1.2** Consider the polytopes shown in Figure 1, in the case $G = U(2)$.

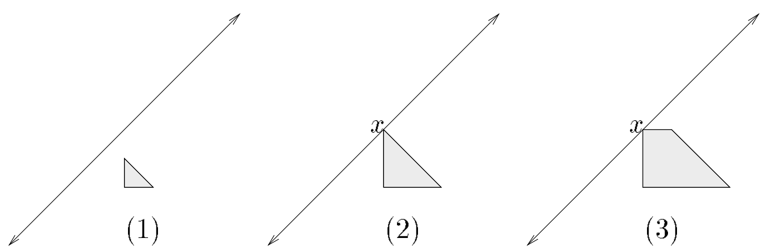


Figure 1: Reflective and non-reflective polytopes

(1) is reflective, since $W_x$ is trivial for all $x \in \Delta$.

(2) is not reflective. Let $x$ be the vertex lying in the Weyl wall. Then $V(x) = \{(1,0), (-1,-1)\}$ and the nontrivial element $w$ of $W_x$ acts by $w(x,y) = (y,x)$. The vector $w(1,0) = (0,1)$ does not lie in $\mathbb{R}V(x)$.

(3) is reflective, since $V(x) = \{(1,0), (0,-1)\}$ is invariant under $w$, up to sign.

$\Delta$ is called *simple* at $x$ if $V(x)$ is linearly independent. If $V(x)$ consists of rational vectors with respect to the lattice $L := \exp^{-1}(\mathrm{Id})$, then we will always assume that $V(x)$ is in minimal form, that is, for any $v \in V(x)$ and $a \in \mathbb{R}$ we have $av \in V(x)$ if and only if $a \in \mathbb{Z}$. We say $\Delta$ is *Delzant* at $x$ if $V(x)$ is a sub-basis of the lattice $L$, that is, a basis of the lattice $L \cap \mathrm{span}(V(x))$. $\Delta$ is simple (resp. Delzant) if $\Delta$ is simple (resp. Delzant) at all points $x \in \Delta$.

If $M$ is a compact connected Hamiltonian $G$-space with moment map $\Phi : M \to \mathfrak{g}^*$, then $\Phi(M) \cap \mathfrak{t}^*_+$ is called the moment, or Kirwan polytope. We say that $M$ is transversal if $\Phi$ is transversal to $\mathfrak{t}^*$, that is, for all $p \in \Phi^{-1}(\mathfrak{t}^*)$, we have

$$\mathrm{Im}\ \mathrm{d}\Phi_p + \mathfrak{t}^* = \mathfrak{g}^*. \qquad (1)$$

The main result of this paper is that

**Theorem 1.3** *The map assigning to each action its moment polytope is a bijection between "torsion-free", transversal, multiplicity-free actions of G on compact, connected symplectic manifolds and reflective, Delzant convex polytopes of maximal dimension in* $\mathfrak{t}^*_+$.

"Torsion-free" means that certain isotropy groups (including the principal isotropy subgroup) are trivial. The exact definition is given in Definition 6.1.

There are two results which provide simple criteria in terms of the moment polytope for determining when an action lies within the scope of the classification Theorem 1.3. In a separate paper [30] we prove

**Theorem 1.4** *A multiplicity-free action of a compact group on a compact connected symplectic manifold that is locally free on a dense subset is transversal if and only if the moment polytope is simple and reflective.*

Proposition 6.2 of this paper states that that under certain assumptions a transversal, multiplicity-free space is "torsion-free" if and only if the moment polytope is Delzant.

Our approach is completely symplectic. That is, we nowhere assume the existence of complex structures, although of course the motivating examples possess them. The basic technique is to study the inverse image $Y_+$ of the positive Weyl chamber under the moment map. $Y_+$ is a multiplicity-free $T$-space. and except for compactness satisfies the assumptions of Delzant's theorem. In the Kähler situation, $Y_+$ is not necessarily a complex submanifold, since the moment map is not algebraic. The approach taken here is therefore very different from the one taken in algebraic geometry, where the analogous situation is known as "spherical", and there is a classification theorem involving "colored fans". (See [5],[23].) At present it seems unclear how the two theories are related. An important outstanding question is whether all of the symplectic manifolds studied here are Kähler, that is, whether the second half of Delzant's theorem generalizes.[3]

It was originally hoped that the study of transversal multiplicity-free actions would lead to a (weighted) lattice-point counting formula similar to those coming from the study of toric varieties (see e.g. [22]) but lack of a description of the Todd class of these manifolds has so far prevented any progress towards this goal.

This paper is organized into the proofs of the different parts of Theorem 1.3. After section 2, which discusses examples, and section 3, which reviews Delzant's work, we show in section 4 that the moment polytope of any transversal, multiplicity-free space is reflective. In section 5, we discuss the consequences of the definition of reflective 1.1, and give combinatorial proofs of some properties of the moment polytopes of these actions first discovered by Guillemin and Souza. In section 6, we show that the question of whether or not the moment polytope is Delzant is related to the triviality of certain discrete stabilizers. In section 7, we prove a relationship between the orbit-type decomposition of the manifold and the decomposition of the moment polytope into open

[3] Added in proof: This question has been answered negatively by the author, [31], using work of S. Tolman, and independently by F. Knop.

faces. In section 8 we make a preliminary attempt to show that given a reflective Delzant polytope $\Delta$, we can construct a torsion-free, transversal, multiplicity-free space with moment polytope $\Delta$. In section 9, we show that two actions satisfying these assumptions and having the same polytope are isomorphic. In section 10, we develop a gluing technique which completes the construction begun in section 8.

I would like to thank Victor Guillemin and Regina Souza for sharing with me their unpublished work on multiplicity-free transversal actions, which was the starting point for this project. It's also a pleasure to thank Victor Guillemin for constant support and encouragement, and Yael Karshon, Allen Knutson, Eugene Lerman, Eckhard Meinrenken, Reyer Sjamaar, and Sue Tolman for many suggestions and ideas.

## 2 Examples

The motivating examples of transversal multiplicity-free spaces are certain coadjoint orbits of compact Lie groups, under the actions of subgroups. For simplicity, in each example the Lie algebra is identified with its dual using an invariant inner product.

### 2.1 Generic $U(3)$-orbits under the action of $U(2)$.

This example is taken from [16, page 364]. Let $\lambda = (\lambda_1, \lambda_2, \lambda_3)$ with $\lambda_1 > \lambda_2 > \lambda_3$ and let $M_\lambda$ be the coadjoint orbit of $U(3)$ containing

$$i \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix} \in \mathfrak{u}(3)^*.$$

To simplify notation, we multiply $M_\lambda$ by $-i$, so that $M_\lambda$ becomes the set of hermitian matrices with eigenvalues $\lambda_1, \lambda_2$, and $\lambda_3$. By embedding $G = U(2)$ in $U(3)$ by $U \to \mathrm{diag}(1, U)$ we induce on $M_\lambda$ the structure of a Hamiltonian $U(2)$-space. The moment map $\Phi$ for the $U(2)$ action is the projection of $M_\lambda$ onto $\mathfrak{u}(2)^*$.

**Proposition 2.1** *The moment polytope $\Delta_\lambda = \Phi(M_\lambda) \cap \mathfrak{t}^*_+$ is equal to $[\lambda_2, \lambda_1] \times [\lambda_3, \lambda_2]$.*

Proof - An element $A \in M_\lambda$ lies in $\Phi^{-1}(\mathfrak{t}^*_+)$ if and only if $A$ is of the form

$$A = \begin{pmatrix} a_0 & a_1 & a_2 \\ a_1^* & \mu_1 & 0 \\ a_2^* & 0 & \mu_2 \end{pmatrix} \tag{2}$$

where $\mu_1 \geq \mu_2$, $a_0 \in \mathbb{R}$, and $a_1, a_2 \in \mathbb{C}$. Thus $\Phi^{-1}(\mathfrak{t}^*_+)$ is the set of matrices of the form (2) such that

$$\begin{aligned} (\lambda_1 - \lambda)(\lambda_2 - \lambda)(\lambda_3 - \lambda) &= \det(A - \lambda I) \\ &= (\mu_1 - \lambda)(\mu_2 - \lambda)(a_0 - \lambda - \frac{a_1 a_1^*}{\mu_1 - \lambda} - \frac{a_2 a_2^*}{\mu_2 - \lambda}), \end{aligned}$$

that is,

$$P(\lambda) = \frac{(\lambda_1-\lambda)(\lambda_2-\lambda)(\lambda_3-\lambda)}{(\mu_1-\lambda)(\mu_2-\lambda)} = (a_0-\lambda) - \frac{a_1a_1^*}{\mu_1-\lambda} - \frac{a_2a_2^*}{\mu_2-\lambda}. \tag{3}$$

To solve (3) for $\mu_1$ and $\mu_2$ we separate into two cases.

**Case** $\mu_1 \neq \mu_2$. (3) holds if and only if

$$-a_1a_1^* = \mathrm{Res}_{\mu_1} P(\lambda) = \frac{(\lambda_1-\mu_1)(\lambda_2-\mu_1)(\lambda_3-\mu_1)}{\mu_2-\mu_1} \tag{4}$$

$$-a_2a_2^* = \mathrm{Res}_{\mu_2} P(\lambda) = \frac{(\lambda_1-\mu_2)(\lambda_2-\mu_2)(\lambda_3-\mu_2)}{\mu_1-\mu_2} \tag{5}$$

and finally

$$a_0 = P(\lambda_0) + \lambda_0 + \frac{a_1a_1^*}{\mu_1-\lambda_0} + \frac{a_2a_2^*}{\mu_2-\lambda_0}. \tag{6}$$

for some $\lambda_0 \neq \mu_1, \mu_2$. Since $-a_1a_1^*$ and $\mu_2-\mu_1$ are negative, (4) has a solution if and only if the numerator $(\lambda_1-\mu_1)(\lambda_2-\mu_1)(\lambda_3-\mu_1)$ is positive, that is, $\mu_1 \in [\lambda_2, \lambda_1]$ or $\mu_1 < \lambda_3$. Similarly, (5) has a solution if $\mu_2 \in [\lambda_3, \lambda_2]$ or $\mu_2 > \lambda_1$. Since $\mu_1 \geq \mu_2$, the possibilities $\mu_1 < \lambda_3$ and $\mu_2 > \lambda_1$ are impossible.

**Case** $\mu_1 = \mu_2$. (3) holds if and only if the pole in $P(\lambda)$ at $\lambda = \mu_1$ is order 1, that is, $\lambda_i = \mu_1$ for some $i \in \{1,2,3\}$, $a_0$ satisfies (6) and

$$-(a_1a_1^* + a_2a_2^*) = \prod_{j\neq i}(\lambda_j - \mu_1)$$

which has a solution exactly when $\mu_1 = \mu_2 = \lambda_2$. □

These calculations show that the level sets $\Phi^{-1}(x)$ for $x \in \Delta_\lambda$ are as follows. (e.g. for $x$ lying in the left open facet of $\Delta_\lambda$, $\Phi^{-1}(x) \cong S^1$.)

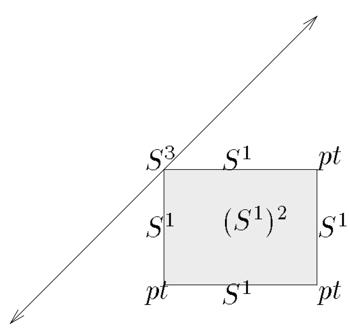


Figure 2: The sets $\Phi^{-1}(x)$ for $x \in \Delta_\lambda$

The action of $T$ on a generic fiber $\Phi^{-1}(x) \cong T$ is free, so $T$ acts effectively on $Y_+ = \Phi^{-1}(\mathrm{int}\ \mathfrak{t}_+^*)$, and $G$ acts effectively on $M_\lambda$. Since dim $M_\lambda = 6 = (\dim + \mathrm{rank})U(2)$, $M_\lambda$ is multiplicity-free. The following lemma, which will be proved in section 4, shows that $M_\lambda$ is transversal.

**Lemma 2.2 (Guillemin)** *If $M$ is any Hamiltonian $G$-space with moment map $\Phi : M \to \mathfrak{g}^*$, then $M$ is transversal if and only if for all $x \in \Delta$, the semisimple part $K_x$ of $G_x$ acts locally freely on $\Phi^{-1}(x)$.*

Let $x$ be the unique point of $\Delta_\lambda$ contained in the Weyl wall, so that $G_x = U(2)$. It follows from the above computation that the semisimple part, $SU(2)$, of $U(2)$ acts freely on $\Phi^{-1}(x) \cong S^3 \cong SU(2)$, so $M_\lambda$ is transversal.

## 2.2 Symplectic cuts of $M_\lambda$

We begin by defining symplectic cutting, following E. Lerman [24]. We then apply the technique to $M_\lambda$ to produce examples of both transversal and non-transversal actions.

Let $M$ be a Hamiltonian $G$-space with moment map $\Phi : M \to \mathfrak{g}^*$, and let $f : M \to \mathbb{R}$ be the moment map for a circle action which commutes with the action of $G$. Define $g : M \times \mathbb{C} \to \mathbb{R}$ by

$$g(m, z) = f(m) + |z|^2/2.$$

With respect to the standard product symplectic structure on $M \times \mathbb{C}$, $g$ is the moment map for the diagonal circle action, which is equivariant with respect to the action of $U(2)$ on the left factor. For $a \in \mathbb{R}$, the level set $g^{-1}(a)$ has the form

$$g^{-1}(a) \cong f^{-1}(a) \cup (f^{-1}(-\infty, a) \times S^1).$$

If $S^1$ acts freely on $f^{-1}(a)$, then the reduced space

$$g^{-1}(a)/S^1 \cong f^{-1}(a)/S^1 \cup f^{-1}(-\infty, a)$$

is by the Marsden-Weinstein-Meyer theorem a Hamiltonian $G$-space. Following Lerman, we call $M_{\leq a} := g^{-1}(a)/S^1$ the *symplectic cut* of $M$ at $a$.

Now let $M = M_\lambda$ and let $f = \mathrm{Tr}\ \Phi$ be the Hamiltonian of the action of the center, $Z$, of $U(2)$ on $M_\lambda$. By our previous description of the level sets $\Phi^{-1}(x)$ we see that $Z$ acts freely on $\Phi^{-1}(x)$ except for the vertices in int $\mathfrak{t}_+^*$

$$x = (\lambda_2, \lambda_3), \ x = (\lambda_1, \lambda_3), \ \text{ or } \ x = (\lambda_1, \lambda_2).$$

The key point is that $Z$ does act freely on $\Phi^{-1}(\lambda_2, \lambda_2)$. Hence $Z$ acts freely on $f^{-1}(a) \cap \Phi^{-1}(\mathfrak{t}_+^*)$ except for

$$a = \lambda_2 + \lambda_3, \ a = \lambda_1 + \lambda_3, \ \text{ or } a = \lambda_1 + \lambda_3.$$

Since $Z$ acts $U(2)$-equivariantly, $Z$ acts freely on $f^{-1}(a)$ except for these values of $a$. If $(M_\lambda)_{\leq a}$ is smooth, its moment polytope is

$$(\Delta_\lambda)_{\leq a} = \{x \in \Delta_\lambda |\ \mathrm{Tr}(x) \leq a\}.$$

In particular, if $2\lambda_2 < \lambda_1 + \lambda_3$, then $(M_\lambda)_{\leq 2\lambda_2}$ is smooth and has the moment polytope pictured in Figure 1, (2). The polytopes in (1) and (3) are the moment polytopes of the spaces $(M_\lambda)_{\leq 2\lambda_2 \pm \epsilon}$ for $\epsilon$ small. Note that by the Duistermaat-Heckman theorem, these spaces are equivariantly homeomorphic! That is, the polytopes in Figure 1, (1), (2), and (3) correspond to a variation of the symplectic structure on $(M_\lambda)_{\leq 2\lambda_2}$. As we noted before, only (1) and (3) are reflective.

We conclude by verifying the prediction of Theorem 1.4. Let $\Phi_\epsilon$ denote the moment map of the action of $U(2)$ on $(M_\lambda)_{\leq 2\lambda_2+\epsilon}$. If $\epsilon < 0$, then $\Phi_\epsilon^{-1}(\lambda_2, \lambda_2) = \emptyset$. If $\epsilon > 0$, then $\Phi_\epsilon^{-1}(\lambda_2, \lambda_2) \cong \Phi^{-1}(\lambda_2, \lambda_2) \cong S^3$, on which $SU(2)$ acts freely. If $\epsilon = 0$, then $\Phi_\epsilon^{-1}(\lambda_2, \lambda_2) \cong \Phi^{-1}(\lambda_2, \lambda_2)/Z \cong S^2$ on which $SU(2)$ cannot act freely. By Guillemin's Lemma 2.2 $(M_\lambda)_{\leq 2\lambda_2+\epsilon}$ is transversal if and only if $\epsilon \neq 0$.

## 2.3 Generic coadjoint orbits of $SO(6)$ under the action of $SO(5)$

In this section we study a family of Hamiltonian $SO(5)$-structures on $SO(6)/SO(2)^3$. As with the previous example, this family includes both transversal and non-transversal actions. Some members of the family have the same moment polytopes, and we identify the equivariant symplectomorphism guaranteed by Theorem 1.3.

Let $\lambda = (\lambda_1, \lambda_2, \lambda_3)$ with $\lambda_1 > \lambda_2 > |\lambda_3|$ and let $M_\lambda$ be the coadjoint orbit of

$$\begin{pmatrix} 0 & \lambda_1 & 0 & 0 & 0 & 0 \\ -\lambda_1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \lambda_2 & 0 & 0 \\ 0 & 0 & -\lambda_2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \lambda_3 \\ 0 & 0 & 0 & 0 & -\lambda_3 & 0 \end{pmatrix} \in \mathfrak{so}(6)^*.$$

Let $G = SO(5)$ which embeds in $SO(6)$ by $A \to \mathrm{diag}(A, 1)$.

**Case** $\lambda_3 \neq 0$. An argument similar to that for the action of $U(2)$ on $U(3)/U(1)^3$ shows that the moment polytope is $\Delta_\lambda = [\lambda_2, \lambda_1] \times [|\lambda_3|, \lambda_2]$, the level sets $\Phi^{-1}(x)$, $x \in \Delta_\lambda$ are as shown in Figure 3, and $M_\lambda$ is a transversal, multiplicity-free $SO(5)$-space.

**Case** $\lambda_3 = 0$**.** The moment polytope $\Delta_\lambda = [\lambda_2, \lambda_1] \times [0, \lambda_2]$ and the level sets $\Phi^{-1}(x)$ are as shown in Figure 4.

As before, $M_\lambda$ is multiplicity-free. However, if for example $x = (\lambda_2, 0)$ then $K_x = SO(3)$ which cannot act locally freely on $\Phi^{-1}(x) \cong S^2$. By Guillemin's Lemma 2.2, $M_\lambda$ is not transversal. The moment polytope $\Delta_\lambda$ violates part (2) of the definition of reflective 1.1.

Now let $\lambda_3 \neq 0$ and let $\lambda' = (\lambda_1, \lambda_2, -\lambda_3)$ so that $\Delta_\lambda = \Delta_{\lambda'}$. By Theorem 1.3, there must exist an $SO(5)$-equivariant symplectomorphism $\varphi : M_\lambda \cong M_{\lambda'}$. One such map is defined by $\varphi(p) = \mathrm{Ad}^*(g)p$ where $g \in O(6)$ is the element $g = \mathrm{diag}(1, 1, 1, 1, 1, -1)$. The union $M_\lambda \cup M_{\lambda'}$ is a coadjoint orbit of $O(6)$, whose components are $SO(6)$-orbits. The components are $SO(5)$-symplectomorphic because

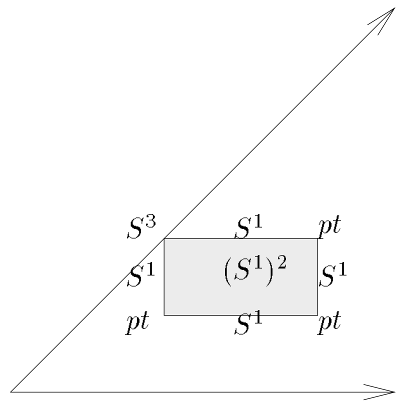


Figure 3: The level sets $\Phi^{-1}(x)$ for $x \in \Delta_\lambda$ for $\lambda_3 \neq 0$.

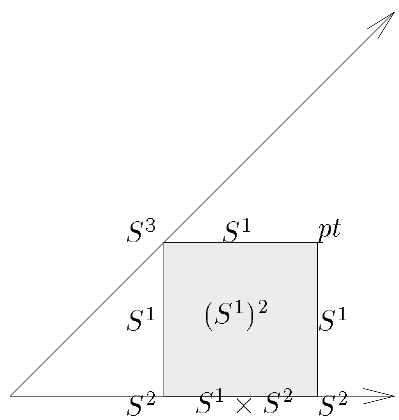


Figure 4: The level sets $\Phi^{-1}(x)$ for $x \in \Delta_\lambda$ for $\lambda_3 = 0$.

the outer automorphism of $SO(6)$ has $SO(5)$ as its fixed point set. One can see this easily from the Dynkin diagram.

## 3 The abelian case

In this section we review the proof of Delzant's theorem. The proof here is almost the same as Delzant's proof in [7]. The differences are that our construction uses symplectic cutting, and the uniqueness argument uses Čech cohomology.

Let $G$ be a torus of dimension $n$. Since $G$ is abelian, the points in the principal stratum of the orbit-type decomposition of $M$ have the same stabilizer, so that $G$ acts generically freely (that is, freely on a dense subset) if and only if the action is effective (that is, $G$ injects into $\mathrm{Diff}(M)$.)

**Theorem 3.1 (Delzant)** *Let $G$ be a torus. There is a one-to-one correspondence between effective, multiplicity-free actions of the torus $G$ on compact, connected symplectic manifolds, and*

*Delzant polytopes of maximal dimension in $\mathfrak{g}^*$. Furthermore, any such action is the Hamiltonian action associated to a toric variety.*

### 3.1 Multiple symplectic cuts

Lerman [24] defines symplectic cutting for torus actions as follows. Suppose that $(M, \omega)$ is a Hamiltonian $G$-space with moment map $\Phi : M \to \mathfrak{g}^*$, and that

$$f : M \to \mathbb{R}^d$$

is a moment map for a $G$-equivariant action of a $d$-dimensional torus $T^d$. Define

$$g : M \times \mathbb{C}^d \to \mathbb{R}^d$$

by

$$g(m, z) = f(m) - |z|^2/2$$

where $|z|^2/2$ denotes $(|z_1|^2/2, \dots, |z_d|^2/2) \in \mathbb{R}^d$. $g$ is a moment map for the diagonal action of $T^d$ on $M \times \mathbb{C}^d$. For $a \in \mathbb{R}^d$, define the symplectic cut $M_{\leq a}$ of $M$ at $a$ as the symplectic reduction $M_{\leq a} := g^{-1}(a)/T^d$. $M_{\leq a}$ can be written as a disjoint union

$$M_{\leq a} \cong \bigcup_{I \subset \{1, \dots, d\}} \left( \bigcap_{i \in I} f_i^{-1}(a_i) \cap \bigcap_{i \notin I} f_i^{-1}(-\infty, a_i) \right) /T^I. \tag{7}$$

Here, $T^I$ denotes the torus with Lie algebra span$(\{e_i, i \in I\})$ and $e_1, \dots, e_d$ is the standard basis for $\mathbb{R}^d$. By the Marsden-Weinstein-Meyer theorem, if $T^d$ acts freely on $g^{-1}(a)$ then $M_{\leq a}$ is a smooth Hamiltonian $G$-space. The space $M_{\geq a}$ is defined in an analogous way, as the reduction of $M \times \mathbb{C}^d$ at $a$ by the anti-diagonal action of $T^d$.

**Examples**

(1) Let $M = \mathbb{C}^n$, $G = T^n$, and $\Phi(z_1, \dots, z_n) = (|z_1|^2/2, \dots, |z_n|^2/2)$. Define $f = \sum_{i=1}^n |z_i|^2/2$. Then

$$g(z_1, \dots, z_n, w) = \sum |z_i|^2/2 + |w|^2/2$$

so $g^{-1}(a)/S^1 = \mathbb{C}P^n$. The moment polytope of $M_{\leq a}$ is

$$\Delta_{\leq a} = \{x \in (\mathbb{R}_+)^n | \sum x_i/2 \leq a\},$$

i.e., the standard n-simplex.

(2) The following example illustrates that small symplectic cuts are "blow-ups" of fixed point manifolds. (See [24].) Let $M = \mathbb{C}P^2 = \{(z_1, z_2, z_3) | \sum |z_i|^2/2 = 1\}/S^1$, $\Delta$ be the moment polytope of $M$, and $f = |z_2|^2/2$. For $a \in (0, 1)$, the symplectic cut $M_{\leq a}$ is smooth and has moment polytope

$$\Delta_{\leq a} = \{(x_1, x_2) \in \Delta |\; x_2 \leq a\}.$$

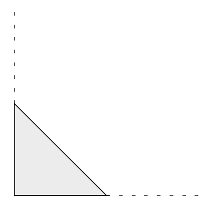

Figure 5: Constructing $\mathbb{C}P^2$ by symplectic cutting

An element of $M_{\leq a}$ is an equivalence class

$$[(z_1, z_2, z_3, w)]$$

where

$$\sum |z_i|^2/2 = 1, \qquad |z_2|^2 + |w|^2 = 2a$$

and

$$(z_1, z_2, z_3, w) \equiv (gz_1, gz_2, gz_3, w) \equiv (z_1, g'z_2, z_3, g'w)$$

for elements $g, g' \in U(1)$. The map

$$(z_1, z_2, z_3, w) \to (wz_1, z_2, wz_3)$$

induces a map

$$M_{\leq a} \to (\mathbb{C}^3 - \{0\})/\mathbb{C}^* \cong \mathbb{C}P^2$$

which is onto and one-to-one, except at $[(0, 1, 0)]$, which has fiber $\mathbb{C}P^1$. That is, $M_{\leq a}$ is the "blow-up" of $\mathbb{C}P^2$ at $[(0, 1, 0)]$.

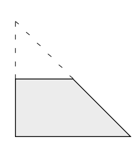

Figure 6: Blowing up $\mathbb{C}P^2$.

### 3.2 Construction

Let $G$ be an n-torus. Choose a basis for $\mathfrak{g}$ and an inner product so that $\mathfrak{g} \cong \mathfrak{g}^* \cong \mathbb{R}^n$, and let $\Delta \subset \mathbb{R}^n$ be a Delzant polytope of maximal dimension. In this subsection we present Lerman's version of the construction of an effective multiplicity-free $G$-action with moment polytope $\Delta$ by symplectic cutting on $\mathbb{C}^n$.

Let $F_1, \ldots, F_d$ be the facets of $\Delta$, $v_1, \ldots, v_d \in \mathfrak{g}$ their inward-pointing normal vectors, and $\lambda_1, \ldots, \lambda_d \in \mathbb{R}$ constants such that

$$F_i = \Delta \cap \{x \in \mathfrak{g}^* | \langle x, v_i \rangle = \lambda_i\}.$$

Let $M = \mathbb{C}^n$ and $\Phi : M \to \mathbb{R}^n$ the moment map

$$\Phi(z_1, \ldots, z_n) = const + (|z_1|^2/2, \ldots, |z_n|^2/2)$$

where $const \in \mathbb{R}^n$ is chosen so that $\Delta$ lies in the interior of $\Phi(M)$. Define $f : M \to \mathbb{R}^d$ by

$$f_i = \langle \Phi, v_i \rangle$$

and let $M_{\geq\lambda}$ be the symplectic cut of $M$ at $\lambda \in \mathbb{R}^d$ with respect to $f$. If $M_{\geq\lambda}$ is smooth, its moment polytope is

$$\{x \in \Phi(M) | \langle x, v_i \rangle \geq \lambda_i\} = \Delta.$$

By equation (7) without loss of generality it suffices to show that $T^k$ acts freely on

$$\bigcap_{i=1}^{k} f_i^{-1}(a_i) \cap \bigcap_{i=k+1}^{d} f_i^{-1}(-\infty, a_i) \subset \Phi^{-1}(\overline{F_1} \cap \ldots \cap \overline{F_k}).$$

If $\overline{F_1} \cap \ldots \cap \overline{F_k}$ is non-empty, by the definition of a Delzant polytope, the map $T^k \to G$ given by

$$\exp(\sum t_i e_i) \to \exp(\sum t_i v_i)$$

is an injection. Here $e_1, \ldots, e_k$ is the standard basis for the Lie algebra $\mathbb{R}^k$ of $T^k$. Since $G$ acts freely on $\Phi^{-1}(\overline{F_1} \cap \ldots \cap \overline{F_k})$, $T^k$ acts freely, and so $M_{\geq\lambda}$ is smooth.

Since symplectic reductions of symplectic manifolds with invariant Kähler structures are Kähler (see [13]) $M_{\geq\lambda}$ is Kähler. The action of $G$ extends to an action of the complexified group $G_{\mathbb{C}}$. Since $G$ is a torus, the dimension of a $G_{\mathbb{C}}$-orbit containing $p$ is twice the dimension of the $G$-orbit containing $p$. $M_{\geq\lambda}$ has an open $G_{\mathbb{C}}$ orbit, and is therefore a toric variety.

### 3.3 Consequences of multiplicity-free

Let $M$ be a compact, connected symplectic manifold with a Hamiltonian action of a torus $G$. There are two natural decompositions of $M$:

$$M = \bigcup_{F \subset \Delta} \Phi^{-1}(F)$$

where the union is over faces $F$ of $\Delta$, and the orbit-type decomposition. The following lemma shows that if the action is effective and multiplicity-free, these two decompositions are related.

**Lemma 3.2 (Delzant)** *Let $G$ be a torus, and let $M$ be a compact connected symplectic manifold with an effective, multiplicity-free action of $G$ and moment polytope $\Delta$. If $F$ is a face of $\Delta$, and $p$ is any point in $\Phi^{-1}(F)$, then the stabilizer, $G_p$, of $p$ is connected and its Lie algebra $\mathfrak{g}_p$ is the annihilator $F^0$ of $F$ in $\mathfrak{g}^*$.*

Proof - Let $N$ be the component of the fixed point set of $G_p$ containing $p$, which is a symplectic submanifold of $M$ containing the orbit $Gp$ of $p$. Since $G$ is abelian, $Gp$ is isotropic, so

$$\dim N \geq 2\dim Gp = 2\dim G/G_p. \tag{8}$$

On the other hand, let $V = T_p(M)$. Under the action of $G_p$, $V$ breaks up into the sum of irreducibles. Let $V_{triv}$ (resp. $V_{non-triv}$) be the sum of the trivial (resp. non-trivial) irreducibles, which are symplectic sub-representations.

By the equivariant Darboux theorem, $V$ is a local model for the action of $G_p$ on $M$ near $p$. That is, there exists a neighborhood of 0 in $V$ that is $G_p$-symplectomorphic to a neighborhood of $p$ in $M$. Clearly, $V_{triv}$ is a local model for $N$. Since $G_p$ acts effectively, $G_p$ injects into $\mathrm{Sp}(V_{non-triv})$. Since $G_p$ is abelian,

$$\dim\ V_{non-triv} \geq 2\dim G_p$$

and

$$\begin{aligned} \dim N &= \dim V - \dim V_{non-triv} && (9)\\ &\leq 2\dim G - 2\dim G_p && (10)\\ &= 2\dim G/G_p. && (11)\end{aligned}$$

Equations (8) and (11) imply that $\dim N = 2\dim G/G_p$ and $\dim V_{non-triv} = 2\dim G_p$. Let

$$\rho : G_p \to \mathrm{Sp}(V_{non-triv})$$

be the representation of $G_p$ on $V_{non-triv}$. Then $\rho(G_p)$ is a maximal torus in $\mathrm{Sp}(V_{non-triv})$, and since $\rho$ is an injection, $G_p$ is connected. Let

$$V_{non-triv} \cong \mathbb{C}^k$$

be the decomposition of $V_{non-triv}$ into $G_p$ irreducibles, so that $\rho(G_p) \cong U(1)^k$ and has Lie algebra $\mathfrak{u}(1)^k \cong \mathbb{R}^k$. Let $e_1, \ldots, e_k \in \mathbb{R}^k$ be the standard basis vectors, and define $v_i \in \mathfrak{g}_p$ by

$$v_i = (d\rho)^{-1}(e_i)$$

so that $\{v_1, \ldots, v_k\}$ is a basis for $L \cap \mathfrak{g}_p$. The moment map $\langle \Phi, v_i\rangle$ for the action of $\exp(\mathbb{R}v_i)$ is locally

$$\langle \Phi, v_i\rangle = |z_i|^2/2$$

and therefore has a local minimum at $p$. By a Morse-theoretic lemma of Atiyah [1] and Guillemin and Sternberg [12] $\langle \Phi, v_i\rangle$ has a global minimum at $p$. $\Phi(p)$ lies in the intersection of $k$ supporting hyperplanes with normal vectors $v_1, \ldots, v_k$. Since $v_1, \ldots, v_k$ are linearly independent, the face $F$ containing $\Phi(p)$ is at most dimension $n - k$.

On the other hand, $G/G_p$ acts generically freely on the symplectic manifold $N$, so that $\Phi(N)$ is a convex polytope of dimension $n - k$, and since $G/G_p$ acts freely at $p$, $\Phi(p)$ lies in the interior of $\Phi(N)$. Therefore, the dimension of $F$ is exactly $n - k$, and its annihilator is $\mathfrak{g}_p$. $\square$

**Corollary 3.3** *Let $M$ be a compact connected symplectic manifold with an effective multiplicity-free action of $G$ and moment polytope $\Delta$. Then*

1. *The polytope $\Delta$ is a Delzant polytope.*

2. *A vector $v \in \mathfrak{g}$ is normal to a facet $F$ of $\Delta$ if and only if the fixed point set of $\exp(\mathbb{R}v) \subset G$ acting on $M$ has a component of codimension 2.*

3. *Let $F$ be any face, and define $G_F := G_p$ for any element $p \in \Phi^{-1}(F)$. Let $H_F \subset G$ be a subgroup such that $G = G_F \times H_F$. ($H_F$ exists since $G_F$ is connected.) Then $\Phi^{-1}(F)$ is a principal $H_F$-bundle over $F$.*

Proof - The first assertion is contained in the proof of the lemma above. To prove the second assertion, note the tangent space to $\Phi^{-1}(F)$ at $p$ is $V_{triv}$, so that $\Phi^{-1}(F)$ is dimension $2 \dim F$, By the lemma, the fixed point set of $\exp(\mathbb{R}v)$ is the union of sets $\Phi^{-1}(F)$ such that $v$ is normal to $F$, and the assertion follows. To prove the third assertion, note that since $H_F$ acts freely at $p \in \Phi^{-1}(F)$, $\operatorname{Im} d\Phi_p$ must be at least dimension $\dim F$, and so $\Phi$ restricts to a submersion of $\Phi^{-1}(F)$ onto $F$. Therefore, the fibers are smooth, of dimension $2\dim F - \dim F = \dim F = \dim H_F$. Because the fibers of $\Phi$ over $F$ are connected by a lemma of Atiyah [1] and Guillemin and Sternberg [12], $H_F$ acts transitively and freely on the fibers of $\Phi$, and the claim follows. □

### 3.4 Uniqueness

Let $M_1$ and $M_2$ be compact, connected symplectic manifolds with effective, multiplicity-free actions of a torus $G$ and suppose that $\Phi_1(M_1) = \Phi_2(M_2) =: \Delta$. The purpose of this subsection is to prove that in this situation $M_1$ and $M_2$ are equivariantly symplectomorphic.

Let $F$ be a face of $\Delta$, $x$ a point in $F$, and for $i = 1, 2$ let $p_i$ be a point in $\Phi_i^{-1}(x)$. By Lemma 3.2, $G_{p_1} = G_{p_2} = G_F$. By the local normal form theorem of Marle and Guillemin and Sternberg (see e.g. [28, Proposition 2.5], there exist neighborhoods of $p_i$ in $M_i$ that are $G$-equivariantly symplectomorphic to neighborhoods of the zero section in

$$G \times_{G_F} ((\mathfrak{g}/\mathfrak{g}_F)^* \oplus W_i)$$

where $W_i := (T(Gp_i))^{\omega_i}/T(Gp_i)$ are the symplectic slices to the orbits $Gp_i \subset M_i$. In our case, one checks that $W_i \cong V_{non-triv,i}$, and since $G_F$ is isomorphic to a maximal torus in $Sp(W_i)$, the vector spaces $W_1$ and $W_2$ are isomorphic symplectic representations of $G_F$. It follows that there exists neighborhoods, $U_i$ of $p_i$ in $M_i$ and an equivariant symplectomorphism $U_1 \cong U_2$. Since $M_1$ and $M_2$ are multiplicity-free, the sets $U_i$ are of the form $\Phi_i^{-1}(V)$, for some neighborhood $V$ of $x$ in $\Delta$.

The construction of a global equivariant symplectomorphism follows from a Čech cohomology argument,[4] which is a modification of one given by Haefliger and Salem [17] for actions of tori

[4]This argument is joint work with E. Lerman and S. Tolman

on orbifolds. Let $\{V_j\}$ be a good cover of $\Delta$ such that for each $j$ there exists an equivariant symplectomorphism $\varphi_j : \Phi_1^{-1}(V_j) \to \Phi_2^{-1}(V_j)$. Consider the sheaf

$$\mathrm{Diff}(\omega, \Phi, G)$$

which assigns to any $V \subset \Delta$ the group of $G$-equivariant symplectomorphisms of $\Phi^{-1}(V)$ which preserve the moment map $\Phi$. For each $i$ and $j$, define

$$\psi_{ij} = \varphi_i^{-1}\varphi_j.$$

The collection $\{\psi_{ij}\}$ defines a cocycle

$$\psi \in \check{C}^1(\Delta, \mathrm{Diff}(\omega, \Phi, G)).$$

It suffices to show that $\psi$ is the boundary of some element $\{\tau_i\} \in \check{C}^0(\Delta, \mathrm{Diff}(\omega, M/G, G))$. Indeed, the collection of maps $\varphi_i\tau_i$ satisfy

$$\varphi_i\tau_i = \varphi_j\tau_j$$

on $\Phi_1^{-1}(V_i \cap V_j)$, and therefore define a global equivariant symplectomorphism $M_1 \to M_2$.

To show that $\psi$ is cohomologically trivial, let $C_G^\infty$ be the sheaf which assigns to each open subset $V \subset \Delta$ the $G$-invariant smooth functions on $\Phi^{-1}(V)$. There is a morphism of sheaves

$$C_G^\infty \to \mathrm{Diff}(\omega, \Phi, G)$$

given by assigning to some $f \in C_G^\infty(V)$ the time one exponential of the corresponding Hamiltonian vector field. Let $\mathcal{R}$ be the constant sheaf which assigns to each open set $V$ a copy of the real numbers $\mathbb{R}$. Let $\mathcal{L}$ be the constant sheaf which assigns to each open subset of $\Delta$ the abelian group $L = \ker \exp : \mathfrak{g} \to G$. There is a morphism of sheaves

$$\mathcal{L} \oplus \mathcal{R} \to C_G^\infty$$

given by assigning to some $(l, r) \in L \oplus \mathbb{R}$ the $G$-invariant function $r + \langle \Phi, l \rangle$.

**Lemma 3.4** *Let $(M, \omega)$ be a compact connected symplectic manifold with a multiplicity-free, effective action of a torus $G$ with moment map $\Phi : M \to \mathfrak{g}^*$. Then the sequence of sheaves $0 \to \mathcal{L} \oplus \mathcal{R} \to C_G^\infty \to \mathrm{Diff}(\omega, \Phi, G) \to 0$ is exact.*

Proof - First, we show that the third arrow is surjective. For any $x \in \Delta$, let $V$ be a contractible neighborhood of $x \in \Delta$, and let $U = \Phi^{-1}(V)$. We can assume that $V$ is small enough so that $U$ is an invariant tubular neighborhood of the orbit $\Phi^{-1}(x)$. Let $\varphi : U \to U$ be an equivariant symplectomorphism, which preserves the moment map. Since $\Delta \cong M/G$, $\varphi$ acts trivially on the set of orbits $M/G$. A lemma of Haefliger, Salem, and Schwarz [17, Theorem 3.1] implies that there exists a smooth invariant map $\theta : M \to G$ such that $\varphi(p) = \theta(p)p$. Since $V$ is simply connected, $\theta$ lifts to a map $\overline{\theta} : M \to \mathfrak{g}$. Define $X \in \mathrm{Vect}(U)$ by

$$X_p = \overline{\theta(p)}_p^{\#}.$$

Since $\overline{\theta}$ is $G$-invariant, $X$ is equal to $\theta(p)^{\#}$ on the $G$-orbit through $p$. Therefore, the time one exponential of $X$ is $\varphi$. It remains to show that $X$ is Hamiltonian. Recall the following facts about the basic subcomplex $(\Omega^*_{basic}(U), d)$.

(1) If $\varphi$ is an equivariant diffeomorphism of $U$ that induces the identity on $U/G$, then $\varphi$ acts trivially on $\Omega_{basic}$.

(2) If $U$ is a tubular neighborhood of an orbit, then $H^*_{basic}(U) = 0$.

(1) is proved by looking at the top-dimensional stratum of the orbit-type decomposition. (2) follows since $U$ is equivariantly contractible to the orbit, on which any basic form vanishes.

To show that $X$ is Hamiltonian, let $Y \in \mathfrak{g}$. By definition of the moment map

$$\omega(X, Y^{\#}) = X\langle \Phi, Y\rangle = 0.$$

Since $X$ and $\omega$ are $G$-invariant, $\imath_X\omega$ is also $G$-invariant, and so $\imath_X\omega$ is basic. Because $d$ of any basic form is also basic, $\mathcal{L}_X\omega = d\imath_X\omega$ is basic as well. Since $\varphi = \exp_1 X$ is symplectic,

$$\begin{aligned} 0 &= \int_0^1 \frac{d}{dt}(\exp_t X)^*\omega \; dt \\ &= \int_0^1 (\exp_t X)^*\mathcal{L}_X\omega \; dt \\ &= \int_0^1 \mathcal{L}_X\omega \; dt \qquad \text{(by (1))} \\ &= \mathcal{L}_X\omega. \end{aligned}$$

By (2), $\imath_X\omega = df$ for some $G$-invariant function $f$ defined on $U$.

Exactness at $\mathcal{L} \oplus \mathcal{R}$ follows from the assumption that $G$ acts effectively. It remains to show exactness at $C^\infty_G$. Let $V \subset \Delta$ be a connected open set, and let $(l, r)$ be any element of $L \oplus \mathbb{R}$. The Hamiltonian vector field of $f_l = r + \langle \Phi, l\rangle$ is the vector field $l^{\#}$, which has time one exponential the identity map. Conversely, suppose that $f \in C^\infty_G(\Phi^{-1}(V))$ has Hamiltonian vector field $X_f$, and the time one exponential of $X_f$ is the identity. Since $X_f\Phi = 0$, $X_f$ is tangent to the orbits of $G$. Since $X_f$ is $G$-invariant, on any orbit $Gp$ of $G$, $X_f$ must equal $Y^{\#}$, for some $Y \in \mathfrak{g}$. We can therefore define a map $\bar{\theta} : \Delta \to \mathfrak{g}$ by requiring that $\bar{\theta}(x)^{\#} = X_f$ on $\Phi^{-1}(x)$. (The map $\bar{\theta}$ is not uniquely defined on faces of $\Delta$.) Since $G$ acts generically freely on the set $\Phi^{-1}(\text{int }\Delta)$, the image of $\bar{\theta}$ lies generically in $L$, and since $L$ is discrete, $\bar{\theta}$ is generically constant, equal to some fixed element $l \in L$. It follows that $X_f = X_{f_l}$ on $\Phi^{-1}(\text{int }\Delta)$, and by continuity on $M$. The one-forms $df$ and $df_l$ are therefore equal, and the difference $f - f_l$ is a constant $r \in \mathbb{R}$. $\square$

Consider the long exact sequence of Čech cohomology groups associated to the short exact sequence above. Because of the existence of invariant partitions of unity, $C^\infty_G$ is a fine sheaf, and therefore $\check{H}^i(\Delta, \text{Diff}(\omega, G, \Phi)) \cong \check{H}^{i+1}(\Delta, \mathcal{L})$ for all $i > 0$. Since $\Delta$ is contractible, the latter groups are zero. It follows that $[\psi] \in \check{H}^1(\Delta, \text{Diff}(\omega, \Phi, G))$ is zero, as required. $\square$

Finally we note for future reference the following lemma.[5]

**Lemma 3.5** *Let $G$ be a torus, let $M$ be a compact manifold with an action of $G$, and suppose that the fixed point set $M_G$ is discrete. Let $\omega_1$ and $\omega_2$ be $G$-invariant symplectic forms on $M$, and suppose that $\Phi_1$ and $\Phi_2$ are moment maps for the action of $G$ on $(M, \omega_1)$ and $(M, \omega_2)$ respectively. If $\Phi_1|_{M_G} = \Phi_2|_{M_G}$, then $[\omega_1] = [\omega_2]$.*

Proof - Let $i : M_G \to M$ denote the inclusion of $M_G$ in $M$. By [2, page 23] the map $i^* : H^*_G(M) \to H^*_G(M_G)$ in equivariant cohomology is injective. Let $\tilde{\omega_1}$ and $\tilde{\omega_2}$ be the equivariant extensions of $\omega_1$ and $\omega_2$ given by $\Phi_1$ and $\Phi_2$. The assumption guarantees that $i^*[\tilde{\omega_1}] = i^*[\tilde{\omega_2}]$ which implies that $[\tilde{\omega_1}] = [\tilde{\omega_2}]$ and hence $[\omega_1] = [\omega_2]$. $\square$

## 4 Consequences of transversality

Since Im $d\Phi_p$ is the annihilator $\mathfrak{g}_p^0$ of $\mathfrak{g}_p$ (see [12]) the transversality condition

$$\text{Im } d\Phi_p + \mathfrak{t}^* = \mathfrak{g}^*. \tag{12}$$

is equivalent to

$$\mathfrak{g}_p^0 + \mathfrak{t}^* = \mathfrak{g}^*$$

or

$$\mathfrak{g}_p \cap \mathfrak{t}^\perp = \{0\}. \tag{13}$$

By the equivariance of $\Phi$, the subalgebra $\mathfrak{g}_p$ is contained in $\mathfrak{g}_{\Phi(p)}$, so if $\mathfrak{g}_{\Phi(p)} = \mathfrak{t}$ then (13) is automatic.

We now prove Lemma 2.2, which has already been used several times. For each $x \in \mathfrak{t}^*$, let $K_x$ be the semisimple part of the stabilizer, $G_x$, of $x$, and $\mathfrak{k}_x$ its Lie algebra.

Proof of Lemma 2.2 - By definition, $K_x$ acts locally freely at $p \in \Phi^{-1}(x)$ if and only if $\mathfrak{k}_x \cap \mathfrak{g}_p = \{0\}$. Suppose that $K_x$ acts locally freely at $p$. Since $\mathfrak{g}_x \cap \mathfrak{t}^\perp \subset \mathfrak{k}_x \cap \mathfrak{t}^\perp$ and $\mathfrak{g}_p \subset \mathfrak{g}_x$ we have

$$\begin{aligned} \mathfrak{g}_p \cap \mathfrak{t}^\perp &= (\mathfrak{g}_p \cap \mathfrak{g}_x) \cap \mathfrak{t}^\perp \\ &= (\mathfrak{g}_p \cap \mathfrak{k}_x) \cap \mathfrak{t}^\perp = \{0\} \end{aligned}$$

as required.

On the other hand, suppose $M$ is transversal and $X \in \mathfrak{k}_x \cap \mathfrak{g}_p$. We must show that $X = 0$. We claim that for some $k \in K_x$, the vector $\text{Ad}(k)X$ lies in $\mathfrak{t}^\perp$. It follows that $X \in \mathfrak{g}_{kp} \cap \mathfrak{t}^\perp$, which if $X \neq 0$ violates (13). An invariant inner product gives an identification of the orbit $K_x X$ with the coadjoint orbit $K_x X^* \subset \mathfrak{k}_x^*$. The orbit $K_x X^*$ has a Hamiltonian action of the maximal torus, $T_x$, of

[5] I would like to thank the referee for pointing out the right form of this lemma.

$K_x$, whose moment map, $\pi$, is the projection onto $\mathfrak{t}_x^*$. It suffices to show that $\pi^{-1}(0)$ is non-empty. This is a consequence of Kostant convexity: If $W_x$ is the Weyl group of $T_x \subset K_x$, the vector

$$\frac{1}{|W_x|} \sum_{w \in W_x} wX^* \tag{14}$$

lies in $\pi(K_x X^*)$ by convexity. Since (14) is Weyl invariant, and $K_x$ is semisimple, (14) is the zero vector. □

By the transversality assumption, $\Phi^{-1}(\mathfrak{t}^*)$ is a smooth submanifold, but in general $\Phi^{-1}(\mathfrak{t}^*)$ is not symplectic. Guillemin observed that Lemma 2.2 produces local models for $\Phi^{-1}(\mathfrak{t}^*)$ where it is not a symplectic submanifold. Let $\sigma$ be a Weyl wall, that is, a connected component of the orbit-type decomposition of $\mathfrak{t}_+^*$, and define $G_\sigma = G_x$ for any $x \in \sigma$. By the cross-section theorem of Guillemin and Sternberg [16, Theorem 26.2], there exists a neighborhood, $U$, of $\Phi^{-1}(\sigma)$ in $\Phi^{-1}(\mathfrak{g}_\sigma^*)$ that is symplectic. Let $K_\sigma$ be the semi-simple part of $G_\sigma$ with Lie algebra $\mathfrak{k}_\sigma$. Let $P$ denote the projection

$$P : \mathfrak{g}_\sigma^* \to \mathfrak{k}_\sigma^*.$$

Then $P \circ \Phi$ is a moment map for the action of $K_\sigma$ on $U$. By definition, $\Phi^{-1}(\sigma)$ is an open subset of

$$(P \circ \Phi)^{-1}(0) = \Phi^{-1}(\mathfrak{z}_\sigma^*),$$

where $\mathfrak{z}_\sigma$ is the Lie algebra of the center $Z_\sigma$ of $G_\sigma$. Since $K_\sigma$ acts locally freely on $\Phi^{-1}(\sigma)$, $\Phi^{-1}(\sigma)$ is a coisotropic submanifold, and by the equivariant coisotropic embedding theorem (see [16, page 315]) there exists an equivariant symplectomorphism $\psi$ of a neighborhood of $\Phi^{-1}(\sigma)$ in $U$ with a neighborhood of $\Phi^{-1}(\sigma)$ in

$$\Phi^{-1}(\sigma) \times \mathfrak{k}_\sigma^*. \tag{15}$$

The Hamiltonian structure on (15) is given as follows. Let

$$\alpha \in \Omega^1(\Phi^{-1}(\sigma), \mathfrak{k}_\sigma)$$

be a $G_\sigma$-invariant connection form for the action of $K_\sigma$. That is, if $X \in \mathfrak{k}_\sigma$, then $\alpha(X^\#) = X$. Let $\omega_\sigma$ be the pullback of $\omega$ to $\Phi^{-1}(\sigma)$. Then the symplectic form on $\Phi^{-1}(\sigma) \times \mathfrak{k}_\sigma^*$ is

$$\omega_\sigma + d\langle \alpha, \pi \rangle$$

where $\pi$ denotes the projection $\Phi^{-1}(\sigma) \times \mathfrak{k}_\sigma^*$ onto $\mathfrak{k}_\sigma^*$. The action of $G_\sigma$ is given by $g(p,k) = (gp, \mathrm{Ad}^*(g)k)$, and the moment map for the action of $K_\sigma$ is the projection $(p,k) \mapsto k$. Denote by $i$ the inclusion of $\Phi^{-1}(\mathfrak{t}^*)$ in $M$, and by $j$ the inclusion of $\Phi^{-1}(\sigma) \times \mathfrak{t}_\sigma^*$ in $\Phi^{-1}(\sigma) \times \mathfrak{k}_\sigma^*$. Restricting $\psi$ to $\Phi^{-1}(\mathfrak{t}^*)$, we have:

**Lemma 4.1 (Guillemin)** *Let $(M, \omega)$ be a compact Hamiltonian $G$-space with moment map $\Phi : M \to \mathfrak{g}^*$ such that $\Phi$ is transversal to $\mathfrak{t}^*$. Then for any Weyl wall $\sigma$, there exists near a $T$-equivariant diffeomorphism $\psi$ of a neighborhood of $\Phi^{-1}(\sigma)$ in $\Phi^{-1}(\mathfrak{t}^*)$ with a neighborhood of $\Phi^{-1}(\sigma) \times \{0\}$ in $\Phi^{-1}(\sigma) \times \mathfrak{t}_\sigma^*$ such that $\psi_*(i^*\omega) = j^*(\omega_\sigma + d\langle \alpha, \pi \rangle)$.*

That is, we have a local model for $\Phi^{-1}(\mathfrak{t}^*)$ despite the fact that $i^*\omega$ is degenerate at $\Phi^{-1}(\sigma)$.

## 4.1 Consequences of transversal and multiplicity-free

The main result of this section is

**Proposition 4.2** *Let $M$ be a compact, connected, transversal, multiplicity-free $G$-space containing a dense subset on which $G$ acts locally freely. Then the moment polytope of $M$ is reflective.*

We begin by giving an alternative formulation of the definition of a reflective polytope in the case that the elements of $V(x)$ are rational vectors.

**Lemma 4.3** *Let $\Delta \subset \mathfrak{t}^*_+$ be a convex polytope, $x \in \Delta$, and suppose that $V(x) \subset \mathfrak{t}$ consists of rational vectors in minimal form. Then $\Delta$ is reflective at $x$ if and only if (1) the set $V(x) \cup -V(x)$ is invariant under $W_x$, and (2) all open facets of $\Delta$ meeting $x$ are contained in* $\mathrm{int}\ \mathfrak{t}^*_+$. *Furthermore, $\Delta$ is reflective if and only if $\Delta$ is reflective at vertices.*

Proof - Assume that $\Delta$ is reflective at $x$. If $v \in V(x)$ is in minimal form, then since $W$ leaves $L$ invariant, $wv$ is also in minimal form. Therefore $c(wv) \in V(x)$ for some constant $c \in \mathbb{R}$ if and only if $\pm wv \in V(x)$. The proof of the last assertion is left to the reader. □

Proof of Proposition 4.2 - We have to show that any open facet $F$ of $\Delta$ is *interior*, that is, lies in the open chamber int $\mathfrak{t}^*_+$. For any simple root $\alpha$, the hyperplane $\ker\alpha$ is supporting, so $F \cap \ker\alpha$ is either empty or $F$. Therefore, $F$ is contained in some Weyl wall, $\sigma$. Since $K_\sigma$ acts locally freely on $\Phi^{-1}(\sigma)$, by the Marsden-Weinstein-Meyer theorem, $Y_\sigma := \Phi^{-1}(\sigma)/K_\sigma$ is Hamiltonian $Z_\sigma$-space with moment image $\Delta \cap \sigma$, which has at worst orbifold singularities. Since any orbit is isotropic,

$$\dim \Delta \cap \sigma \leq \frac{1}{2} \dim Y_\sigma. \tag{16}$$

But we know that

$$\begin{aligned} \dim Y_\sigma &= \dim M - (\dim \mathfrak{g}^* - \dim \sigma) - \dim K_\sigma & (17)\\ &= (\dim + \mathrm{rank})(G) - \dim G + \dim Z_\sigma - \dim K_\sigma & (18)\\ &= 2\dim Z_\sigma + \mathrm{rank}\ K_\sigma - \dim K_\sigma. & (19) \end{aligned}$$

Since $K_\sigma$ is semisimple, $\dim K_\sigma \geq 3$ rank $K_\sigma$, so

$$\dim \Delta \cap \sigma \leq \frac{1}{2} \dim Y_\sigma \leq \dim Z_\sigma - \mathrm{rank}\ K_\sigma = n - 2\mathrm{codim}\ \sigma \tag{20}$$

which at most $n-2$, if $\sigma \neq \mathrm{int}\ \mathfrak{t}^*_+$. This observation is due to R. Souza [29]. Therefore, $\sigma$ cannot contain $F$.

To prove that $V(x)$ is rational and $V(x) \cup -V(x)$ is $W_x$-invariant we need the following extension of Delzant's Lemma 3.2, noted in [8]. For any Hamiltonian $G$-space $M$ with moment map $\Phi : M \to \mathfrak{g}^*$, we denote by $Y_+$ the inverse image $Y_+ := \Phi^{-1}(\mathrm{int}\ \mathfrak{t}^*_+)$, which is a connected Hamiltonian $T$-space (see e.g. [25].)

**Lemma 4.4 (Delzant)** *Let $M$ be a compact connected multiplicity-free Hamiltonian $G$-space containing a dense subset on which $G$ acts locally freely. Let $\Phi : M \to \mathfrak{g}^*$ denote the moment map, $\Delta$ the moment polytope, $S \subset T$ the principal isotropy subgroup of $T$ acting on $Y_+$, and $Q$ the quotient $Q = T/S$. Since $S$ is discrete, the map $T \to Q$ induces an identification of $\mathfrak{t}$ and $\mathfrak{q}$. If $F$ is a face of $\Delta$ lying in* int $\mathfrak{t}_+^*$, *and $p \in \Phi^{-1}(F)$, then the isotropy subgroup $Q_p$ is connected with Lie algebra $\mathfrak{q}_p$ equal to $F^0$, and $\Delta$ is Delzant at $\Phi(p)$ with respect to the lattice* $\ker \exp : \mathfrak{q} \to Q$.

Proof - The proof goes exactly as in the abelian case, with $M$ replaced by $Y_+$, until we come to the Morse theoretic lemma of Atiyah and Guillemin and Sternberg, which requires compactness. Let $\langle \Phi, v_i \rangle$ have a local minimum at a point $p \in Y_+$, and let $U$ be a neighborhood of $p$ in $Y_+$ so that $\langle \Phi, v_i \rangle$ restricted to $U$ takes its minimum at $p$. Since $T$ is abelian we can assume that $U$ is $T$-invariant. Suppose that $\langle \Phi, v_i \rangle$ does not have a global minimum at $p$. Then there exists a sequence $x_n$ of points in $\Delta$ approaching $\Phi(p)$, with $\langle x_n - \Phi(p), v_i \rangle < 0$. Let $q_n \in \Phi^{-1}(x_n)$. By compactness of $M$, we can assume that $q_n$ approaches a point $q \in M$. By continuity $\Phi(q) = x$. Since $q_n \notin U$, the point $q$ does not lie in $U$. But $U \cap \Phi^{-1}(x)$ is closed, and therefore $U \cap \Phi^{-1}(x)$ is both closed and open in $\Phi^{-1}(x)$, which implies that $\Phi^{-1}(x)$ is not connected, contradicting a theorem of Kirwan [20]. $\square$

Now let $x$ be a point of $\Delta$ contained in $\sigma$, $F$ a facet of $\Delta$ meeting $x$, $v$ the normal vector to $F$, and $p \in \Phi^{-1}(F)$. Since $F$ is interior, by Lemma 4.4

$$Q_p = \exp(\mathbb{R}v).$$

Let $(q, t) = \psi(p)$ be the image of $p$ in $\Phi^{-1}(\sigma) \times \mathfrak{t}_\sigma^*$. By the equivariance of $\psi$,

$$Q_q = \exp(\mathbb{R}v).$$

Suppose that $[w]$ lies in the Weyl group $W_\sigma$ of $T_\sigma$ in $K_\sigma$, and let $w \in K_\sigma$ be a representative of $[w]$. The stabilizer $Q_{wq}$ of $wq$ equals

$$Q_{wq} = \exp(\mathbb{R}\ wv).$$

We can assume that $t$ is small enough so that $(wq, t)$ lies in the domain of $\psi^{-1}$. Let $p' \in U$ be defined by

$$p' = \psi^{-1}(wq, t).$$

Then $p' \in \Phi^{-1}(\text{int } \mathfrak{t}_+^*)$ and

$$Q_{p'} = \exp(\mathbb{R}\ wv)$$

so that by Lemma 4.4, $\Phi(p')$ lies in a facet, $F'$, with normal vector $\pm wv$.

It remains to check that $F'$ meets $x$. Let $\{p_s, s \in [0, 1)\}$ be a path of points in $M$ with $p_0 = p$ such that $\Phi(p_s) \in F$ and $\Phi(p_s) \to x$. Define $(q_s, t_s) = \psi(p_s)$. Because $t_s \to 0$, $\Phi(q_s) \to x$. By compactness we can assume that $q_s \to q_1$ for some $q_1 \in \Phi^{-1}(x)$. Therefore $wq_s \to wq_1$, and if we define $p'_s = \psi^{-1}(wq_s, t_s)$, then $\Phi(p'_s) \to \Phi(wq_1) = x$. By Lemma 4.4, since $G_{p'_s}$ is constant, the points $\Phi^{-1}(p'_s)$ all lie in $F'$.

## 5 Consequences of reflectivity

In this section we show that reflectivity leads to a number of restrictions on the polytope, including a restriction discovered by Guillemin [9] which is (4) in the proposition below.

**Proposition 5.1** *Let $\Delta \subset \mathfrak{t}^*_+$ be a reflective polytope of maximal dimension, $x$ a point in $\Delta$, $\sigma$ the Weyl wall containing $x$, and $r(\sigma)$ the set of simple roots perpendicular to $\sigma$.*

*(1) Let $r \in W_\sigma$ be a simple reflection and $v \in V(x)$ such that $rv \neq v$. Then $rv \in -V(x)$.*

*(2) If $\alpha \in r(\sigma)$ and $v \in V(x)$ then $(\alpha, v) \geq 0$.*

*(3) Any $v \in V(x)$ is perpendicular to all but at most one element of $r(\sigma)$.*

*(4) The elements of $r(\sigma)$ are orthogonal. Consequently, $\mathfrak{k}_\sigma$ is a product of $\mathfrak{su}(2)$'s.*

Proof of (1) - Let $\lambda = (v, x)$ so that $(v, y) \geq \lambda$ for all $y \in \Delta$. By assumption, $rv \in \pm V(x)$. Suppose that $rv \in +V(x)$. Then

$$(rv, x) = (v, rx) = (v, x) = \lambda$$

so $(rv, y) \geq \lambda$ for any $y \in \Delta$. If $r$ is the simple reflection corresponding to $\alpha \in r(\sigma)$, then

$$(rv, y) = \left(v - 2\frac{(\alpha, v)}{(\alpha, \alpha)}\alpha, y\right) \geq \lambda. \tag{21}$$

If $(\alpha, v) > 0$, let $y$ lie in the facet $F$ with normal vector $v$, so that $(v, y) = \lambda$. Since by assumption $F$ is interior, $(\alpha, y) > 0$ which implies $(rv, y) < \lambda$ - a contradiction.

On the other hand, if $(\alpha, v) < 0$, then $(rv, y) = \lambda$ only if $(v, y) = \lambda$ and $(\alpha, y) = 0$, which implies, for instance, that the facet with normal vector $rv$ is contained in $\ker \alpha$ and therefore is not interior.

Proof of (2) - If $(\alpha, v) \neq 0$ then $rv \neq v$ and therefore by (1) $rv \in -V(x)$. Since $(-rv, x) = -(v, x) = -\lambda$, we have

$$(-rv, y) = \left(2\frac{(\alpha, v)}{(\alpha, \alpha)}\alpha - v, y\right) \geq -\lambda.$$

for all $y \in \Delta$. Suppose that $(\alpha, v) < 0$. The inequalities $(v, y) \geq \lambda$ and $(\alpha, y) \geq 0$ imply that $(-rv, y) \leq -\lambda$. Therefore, $(-rv, y) = \lambda$ for all $y \in \Delta$, which is impossible since $\Delta$ is of maximal dimension.

Proof of (3) and (4) - Let $\alpha_1, \alpha_2 \in r(\sigma)$ and let $r_1 \in W_\sigma$ be the reflection corresponding to $\alpha_1$. There must exist a vector $v \in V(x)$ such that $(v, \alpha_1) \neq 0$. If not, then

$$(x - t\alpha_1, v) = (x, v)$$

for all $v \in V(x)$, so for $t > 0$ small, $x - t\alpha_1 \in \Delta$, which is impossible since $(x - t\alpha_1, a_1) < 0$ and $\Delta$ is contained in $\mathfrak{t}^*_+$.

By (1), $-r_1 v \in V(x)$, and by definition

$$(r_1 v, \alpha_2) = (2\frac{(\alpha_1, v)}{(\alpha_1, \alpha_1)}\alpha_1 - v, \alpha_2) \geq 0$$

by (2). Since $\alpha_1$ and $\alpha_2$ are simple and distinct, $(\alpha_1, \alpha_2) \leq 0$, and by (2) $(\alpha_1, v)$ and $(\alpha_2, v)$ are at least 0. Therefore $(r_1 v, \alpha_2)$, $(\alpha_1, \alpha_2)$, and $(v, \alpha_2)$. must all be zero. □

Proposition 5.1 (4) implies that if $M$ is a Hamiltonian $G$-space satisfying the assumptions of Proposition 4.2, then its moment polytope $\Delta$ intersects only those walls $\sigma$ such that the Lie algebra $\mathfrak{k}_\sigma$ is a product of $\mathfrak{su}(2)$'s. This was first proved by Guillemin [9] by a different method. We will give another proof, if $M$ is torsion-free, in Corollary 6.5.

The following corollary gives the form of the set $V(x)$ for any point $x \in \Delta$.

**Corollary 5.2** *Let $\Delta$ be a reflective polytope of maximal dimension, and let $\sigma$ be a Weyl wall meeting $\Delta$. Then*

*(1) For each $\alpha_i \in r(\sigma)$ there exists a vector $\beta_i \in \mathfrak{z}_\sigma$ such that $(\alpha_i \pm \beta_i)/2$ are normal vectors to facets $F_\pm$ of $\Delta$ and the intersection $\overline{F_+} \cap \overline{F_-}$ equals $\Delta \cap \ker \alpha_i$. If $\Delta$ is simple, then $\beta_i$ is unique.*

*(2) For any $x \in \sigma$, the elements of $V(x)$ not of the form $\alpha_i \pm \beta_i$, for some $\alpha_i \in r(\sigma)$ and $\beta_i \in \mathfrak{z}_\sigma$, lie in $\mathfrak{z}_\sigma$.*

Proof - By the proof of (3) and (4) of Proposition 5.1, there exists $v \in V(x)$ such that $(\alpha_i, v) \neq 0$. By rescaling, we can assume that

$$v = (\alpha_i + \beta_i)/2$$

for some $\beta_i \in \mathfrak{t}$ such that $(\beta_i, \alpha_i) = 0$. By Proposition 5.1, (3), if $i \neq j$ then

$$0 = (\alpha_j, v) = (\alpha_j, (\alpha_i + \beta_i)/2).$$

which by (4), equals $(\alpha_j, \beta_i/2)$. Hence, $\beta_i \in \mathfrak{z}_\sigma$. This also shows that any $v \in V(x)$ such that $(v, \alpha_i) \neq 0$ must be of the form $(\alpha_i + \beta_i)/2$, for some $\alpha_i \in r(\sigma), \beta_i \in \mathfrak{z}_\sigma$. If $v$ is not of this form, then $(\alpha_j, v) = 0$, for all $j$, that is, $v \in \mathfrak{z}_\sigma$.

Suppose that $v = (\alpha_i + \beta_i)/2$ and let $r$ be the reflection corresponding to $\alpha_i$. By 5.1 (1)

$$-rv = (\alpha_i - \beta_i)/2 \in V(x).$$

Let $\lambda = (v, x)$ so that $(-rv, x) = -\lambda$. Note that

$$\begin{aligned} \overline{F_+} \cap \overline{F_-} &= \{y \in \Delta | (y, (\alpha_i \pm \beta_i)/2) = \pm\lambda\} \\ &= \{y \in \Delta | (y, \alpha_i) = 0 \text{ and } (y, \beta_i/2) = \lambda\} \end{aligned}$$

which is contained in $\ker \alpha_i$. On the other hand, if $y \in \Delta \cap \ker \alpha_i$, then

$$(y, \pm\beta_i/2) \geq \pm\lambda,$$

so that $(y, \beta_i/2) = \lambda$ and $y \in \overline{F_+} \cap \overline{F_-}$. If $\Delta$ is simple, then since $\operatorname{span}(\alpha_i \pm \beta_i, \alpha_i \pm \beta_i') = \operatorname{span}(\alpha_i, \beta_i, \beta_i')$, $\beta_i$ must be unique. □

## 6 The Delzant condition

For each Weyl wall $\sigma \subset \mathfrak{t}_+^*$, let $G_\sigma$ be its isotropy subgroup, and $K_\sigma = (G_\sigma, G_\sigma)$ its semisimple part. We first define (see Lemma 2.2)

**Definition 6.1** A compact connected transversal Hamiltonian $G$-space is *torsion-free* if and only if

(a) $G$ acts freely on a dense set,

(b) $K_\sigma$ acts freely on $\Phi^{-1}(\sigma)$ for all Weyl walls $\sigma$.

(c) $K_\sigma$ is simply connected.

The purpose of this section is to prove

**Proposition 6.2** *Let $M$ be a compact connected symplectic manifold with a transversal multiplicity-free action of $G$ which is free on a dense subset. Let $\Delta$ be the moment polytope of the action, and suppose that $K_\sigma$ is simply connected for all Weyl walls $\sigma$ such that $\Delta \cap \sigma \neq \emptyset$. Then $M$ is torsion-free if and only if $\Delta$ is Delzant.*

**Remark 6.3** In particular $\Delta$ is simple. There is a stronger result, proved in [30]: $\Delta$ is simple if $M$ is compact, connected, transversal, and multiplicity-free. If $M$ is not transversal, $\Delta$ is not necessarily simple.

Proof - First, assume $M$ is torsion-free, so that $K_\sigma$ acts freely on $\Phi^{-1}(\sigma)$ for all walls $\sigma$. Consider the Hamiltonian action of $Z_\sigma$ on $Y_\sigma$. Let $S_\sigma$ be the principal isotropy subgroup, that is, the kernel of the action $Z_\sigma \to \mathrm{Diff}(Y_\sigma)$, and let $T_\sigma$ denote the maximal torus of $K_\sigma$.

**Lemma 6.4** *There exists an isomorphism $\gamma : S_\sigma \to T_\sigma$ and an action $\rho : T_\sigma \to \mathrm{Diff}(\Phi^{-1}(\sigma))$ commuting with the action of $K_\sigma$ such that $S_\sigma$ acts by $\rho \circ \gamma$.*

Proof - Since $Z_\sigma$ is abelian, its orbits in $Y_\sigma$ are isotropic submanifolds. A generic orbit is isomorphic to $Q_\sigma := Z_\sigma / S_\sigma$, and therefore

$$\begin{aligned} \dim Q_\sigma &\leq \frac{1}{2}(\dim Y_\sigma) \\ &\leq \dim Z_\sigma - \dim T_\sigma \end{aligned}$$

by (20). (Equality holds if and only if $\dim K_\sigma = 3 \dim T_\sigma$.) Hence,

$$\dim S_\sigma \geq \dim T_\sigma. \tag{22}$$

Let $q \in \Phi^{-1}(\sigma)$ be any point, and let $K_\sigma q$ denote the fiber of $\Phi^{-1}(\sigma)$ containing $q$. Since $S_\sigma$ acts trivially on the base $Y_\sigma$, $K_\sigma q$ is invariant under the action of $S_\sigma$. Since the action of $S_\sigma$ commutes with that of $T_\sigma$, $S_\sigma$ acts on the quotient $(K_\sigma q)/T_\sigma$. Because the Euler characteristic of $(K_\sigma q)/T_\sigma$ (which is the order of $W_\sigma$) is non-zero, the action of $S_\sigma$ has a fixed point $[p] \in (K_\sigma q)/T_\sigma$. Let $p \in K_\sigma q$ be a representative of $[p]$, that is, $[p] = T_\sigma p$. The action of $S_\sigma$ on $K_\sigma q = K_\sigma p$ is equal to the action on $T_\sigma p$, extended to $K_\sigma p$ by requiring $K_\sigma$-equivariance. Define a map $\gamma : S_\sigma \to T_\sigma$ by $sp = \gamma(s)p$. Since $S_\sigma$ commutes with $T_\sigma$, the map $\gamma$ is a homomorphism. To show that $\gamma$ is an isomorphism, note that if $K_\sigma p$ is a generic fiber, then $S_\sigma$ acts freely on $K_\sigma p$. Indeed, by the Guillemin local model, since $S_\sigma$ acts generically freely on $\Phi^{-1}(\mathfrak{t}^*)$, $S_\sigma$ acts generically freely on $\Phi^{-1}(\sigma)$. In fact, by choosing an $S_\sigma$-invariant connection on $\Phi^{-1}(\sigma)$, the fibers of $\Phi^{-1}(\sigma)$ are $(S_\sigma \times K_\sigma)$-diffeomorphic, so $S_\sigma$ acts freely on every fiber. Therefore, $\gamma$ is an injection. By (22), $\gamma$ is an isomorphism.

Let $\lambda$ denote the action of $S_\sigma$ on $\Phi^{-1}(\sigma)$ given by

$$\lambda(s)q = s\gamma(s)^{-1}q$$

for $s \in S_\sigma$ and $q \in \Phi^{-1}(\sigma)$. A computation shows that $T_\sigma p$ is the fixed point set of $\lambda$ on $K_\sigma p$. Since the fibers of $\Phi^{-1}(\sigma)$ are $(K_\sigma \times S_\sigma)$-diffeomorphic, the fixed point set of $\lambda$ is a principal $T_\sigma$-bundle

$$\Lambda \subset \Phi^{-1}(\sigma).$$

Therefore

$$\Phi^{-1}(\sigma) \cong K_\sigma \times_{T_\sigma} \Lambda. \tag{23}$$

We define the right action $\rho$ of $T_\sigma$ on $\Phi^{-1}(\sigma)$ by extending the action of $T_\sigma$ on $\Lambda$ by $K_\sigma$-equivariance. By construction, $S_\sigma$ acts by $\rho \circ \gamma$. □

**Corollary 6.5** *(1)* $\dim K_\sigma = 3 \dim T_\sigma$.

*(2)* $\dim Y_\sigma = 2 \dim Q_\sigma$.

Since $K_\sigma$ is by assumption simply-connected, (1) implies that $K_\sigma$ must be a product of $SU(2)$'s. See proposition 5.1, (4). (2) says that $Y_\sigma$ is multiplicity-free under the (effective) action of $Q_\sigma$. In summary, for all walls $\sigma$, $\Phi^{-1}(\sigma)$ *is a principal* $K_\sigma$*-bundle over the multiplicity-free space* $Y_\sigma$.

Suppose that $x \in \Delta$ is contained in a Weyl wall $\sigma$. We wish to show that $V(x)$ is a lattice sub-basis for $L \subset t$.

**Lemma 6.6** *Let $v$ be any vector in $V(x)$. If $\alpha_i \in r(\sigma)$ is a simple root such that $(v, \alpha_i) \neq 0$, then $v = \alpha_i \pm d\gamma^{-1}\alpha_i$.*

Proof - By Corollary 5.2 there exists $\beta \in \mathfrak{z}_\sigma$ such that $v = (\alpha_i + \beta)/2$. Let $p$ lie in the fixed point set of $\exp(\mathbb{R}v)$ acting on $\Phi^{-1}(\sigma)$. For some $k \in K_\sigma$, we have $kp \in \Lambda$. Let

$$v' = \mathrm{Ad}(k)v = (\mathrm{Ad}(k)\alpha_i + \beta_i)/2$$

so that $kp$ is fixed by $\exp(\mathbb{R}v')$ and

$$v'' = (\mathrm{Ad}(k)\alpha_i + d\gamma(\beta_i))/2.$$

Since the action $\lambda$ is trivial on $\Lambda$, the point $kp$ is also fixed by $\exp(\mathbb{R}v'')$, and since $v'' \in \mathfrak{k}_\sigma$ and $K_\sigma$ acts freely, the vector $v''$ must vanish. Since $d\gamma(\beta_i) \in \mathfrak{t}_\sigma$ we have that $k \in N(T_\sigma)$ and since $\mathfrak{k}_\sigma$ is a product of $\mathfrak{su}(2)$'s this implies that $\mathrm{Ad}(k)\alpha_i = \pm\alpha_i$, as required. □

Define $\beta_i := d\gamma^{-1}\alpha_i$, $V_Z(x) := V(X) \cap \mathfrak{z}_\sigma$ and

$$V_K(x) := \{(\alpha_i \pm \beta_i)/2 \mid i = 1, \ldots, \mathrm{codim}(\sigma)\}.$$

so that by Lemma 6.6 $V(x) = V_K(x) \cup V_Z(x)$. We claim that $V_K(x)$ is a basis for the lattice $L \cap (\mathfrak{t}_\sigma + \mathfrak{s}_\sigma)$. Since each fiber of $\Phi^{-1}(\sigma)$ is isomorphic to $SU(2)^k$, and $T_\sigma$ (resp. $S_\sigma$) acts by the left (resp. right) action of $U(1)^k$

$$\exp(\sum_{i=1}^{k} t_i\alpha_i + \sum_{j=1}^{k} s_j\beta_j) = \mathrm{Id}$$

if and only if $t_i, s_j \in \mathbb{Z}/2$ and $t_i + s_i \in \mathbb{Z}$ for all $i$, which proves the claim.

**Lemma 6.7** *The Lie algebra $\mathfrak{g}_p$ contains $V_Z(x)$.*

Proof - Let $u \in V_Z(x)$ be a normal vector to a facet $F$ of $\Delta$, and let $p_j \in F$ be a sequence of points such that $\Phi(p_j) \to x$. By compactness, we can assume that $p_j \to q$, where $q \in \Phi^{-1}(x)$. By Delzant's lemma,

$$\mathfrak{g}_{p_j} = \mathbb{R}u$$

and therefore by continuity

$$\exp(tu)q = q \text{ for all } t \in \mathbb{R}.$$

Since $M$ is multiplicity-free, $p = gq$ for some $g \in G_\sigma$, and since $u \in \mathfrak{z}_\sigma$, $\mathrm{Ad}(g)u = u \in \mathfrak{g}_p$. □

Let $L'$ denote the lattice

$$L' := \ker\ \exp : q_\sigma \to Q_\sigma$$

and let $\pi : \mathfrak{z}_\sigma \to q_\sigma$ denote the projection of $\mathfrak{z}_\sigma$ onto $q_\sigma$. We think of $q_\sigma^*$ sitting inside $\mathfrak{z}_\sigma^*$ as the subspace annihilating $\mathfrak{s}_\sigma$, so that $q_\sigma^*$ contains $\Delta \cap \sigma$. Let $V_\sigma(x) \subset q_\sigma$ denote the normal vectors to facets of $\Delta \cap \sigma$ at $x$.

**Lemma 6.8** *The projection $\pi$ gives an isomorphism between $V_Z(x)$ and $V_\sigma(x)$. Furthermore, $V_\sigma(x)$ is a lattice sub-basis of $L' \subset q_\sigma$.*

Proof - Let $v \in V_Z(x)$ be the normal vector to a face $F$ of $\Delta$ meeting $x$. Then by Lemma 4.4, $\Phi^{-1}(F)$ is contained in a codimension 2 component, $C$, of the fixed point set of $\exp(\mathbb{R}v)$ acting on $Y_+$. Since $v \in \mathfrak{z}_\sigma$, $C$ is a $K_\sigma$-bundle over $Q(C)$, where $Q : \Phi^{-1}(\sigma) \to Y_\sigma$ is the projection. Furthermore, $Q(C)$ is contained in the fixed point set, $(Y_\sigma)_v$, of $\exp(\mathbb{R}v)$ acting on $Y_\sigma$. If $(Y_\sigma)_v$ is codimension 0, then $v \in \mathfrak{s}_\sigma$, which is a contradiction since $S_\sigma$ acts freely on $\Phi^{-1}(\sigma)$. Therefore, $Q(C)$ is a codimension 2 component of $(Y_\sigma)_v$. Delzant's Lemma 3.2, and Corollary 3.3 apply to the action of $Q_\sigma$ on $Y_\sigma$ even though $Y_\sigma$ is not compact. (The proof is the same as that of Lemma 4.4.) Therefore, $\pi(v) \in V_\sigma(x)$. If elements $v, w \in V_Z(x)$ have the same projection $\pi(v) = \pi(w)$, then by Lemma 6.7, $v - w \in \mathfrak{s}_\sigma \cap \mathfrak{g}_p$. Since $S_\sigma$ acts freely, $v - w = 0$. It remains to show that $V_Z(x)$ is mapped onto $V_\sigma(x)$. Let $F$ be any facet of $\Delta \cap \sigma$. We can write $\overline{F} = \overline{F'} \cap \Delta \cap \sigma$ where $\overline{F'}$ is a facet of $\Delta$. If $F'$ has normal vector $v$ lying in $V_K(x)$, then $\overline{F'}$ contains $\Delta \cap \sigma$, so $\overline{F'} \cap \Delta \cap \sigma = \Delta \cap \sigma$. Therefore, $v \in V_Z(x)$. $\Box$

Now suppose that $u \in L \cap \operatorname{span}(V(x))$ and that $u = u_k + u_z$ with $u_k \in \mathfrak{t}_\sigma$ and $u_z \in \mathfrak{z}_\sigma$. Since $\exp(u_z)$ equals $\exp(u_k)^{-1}$,

$$\exp(u_z) \in T_\sigma \cap Z_\sigma \subset S_\sigma.$$

By Lemma 6.8 there exist a unique element $u' \in \operatorname{span}_{\mathbb{Z}}(V_Z(x))$ such that $u_z - u' \in \mathfrak{s}_\sigma$, and $u'$ can be written uniquely as a linear combination of elements of $V_Z(x)$, with integer coefficients. By the first part of the proof, we can write

$$u - u' = u_k + (u_z - u') \in \mathfrak{t}_\sigma + \mathfrak{s}_\sigma$$

uniquely as a linear combination with integer coefficients of elements of $V_K(x)$, which shows that $V(x)$ is a lattice sub-basis.

Let $\Delta$ be Delzant, and suppose that there exists some $g \in K_\sigma$ and some point $p \in \Phi^{-1}(\sigma)$ such that $gp = p$. We claim that $g = \mathrm{Id}$. Since $hgh^{-1}$ fixes $hp$, for any $h \in G_\sigma$, we can assume that $g \in T_\sigma$. Let $\psi$ be the map in Lemma 4.1, and let $t \in \operatorname{int}\,(\mathfrak{t}_\sigma)^*_+$ be small enough so that $(p,t)$ lies in the domain of $\psi$. If we define $q = \psi(p,t)$, then by the equivariance of $\psi$, $gq = q$. Let $F \subset \Delta$ be the open face containing $\Phi(q)$, which meets $\Phi(p)$. By Lemma 4.4, $g$ lies in $T_q = \exp(F^0))$. By Corollary 5.2, the normal vectors to facets meeting $\Phi(p)$ are of the form

$$(\alpha_1 \pm \beta_1)/2, \ldots, (\alpha_k \pm \beta_k)/2, v_1, \ldots, v_l \tag{24}$$

where $\beta_i, v_i \in \mathfrak{z}_\sigma$ and $k = \operatorname{codim}\,\sigma$. By Corollary 5.2 the intersection of the facets $F_{\pm,i}$ with normal vectors $\alpha_i \pm \beta_i$ is contained in $\ker \alpha_i$, so $\Phi(q)$ cannot be contained in both $\overline{F_{+,i}}$ and $\overline{F_{-,i}}$. Without loss of generality, the normal vectors to facets meeting $\Phi(q)$ are

$$(\alpha_1 - \beta_1)/2, \ldots, (\alpha_{k'} - \beta_{k'})/2, v_1, \ldots, v_{l'}$$

for some $k' < k$, $l' < l$, such that $k' + l' = \operatorname{codim}\,F$. The vectors

$$\alpha_1, (\alpha_1 - \beta_1)/2, \ldots, \alpha_k, (\alpha_k - \beta_k)/2, v_1, \ldots, v_l$$

generate the same lattice as the lattice generated by (24), which since $\Delta$ is Delzant is a lattice sub-basis. Therefore, the vectors

$$\alpha_1, \ldots, \alpha_k, (\alpha_1 - \beta_1)/2, \ldots, (\alpha_{k'} - \beta_{k'})/2, v_1, \ldots, v_{l'}$$

generate a lattice sub-basis. It follows that the intersection $T_\sigma \cap T_q = \{\mathrm{Id}\}$ which implies that $g = \mathrm{Id}$. $\Box$

## 7 The orbit-type decomposition of $M$

In the multiplicity-free case, the quotient $M/G$ is homeomorphic to the moment polytope $\Delta$, so it's natural to look for a relationship between the orbit-type decomposition of $M/G$ and the decomposition of $\Delta$ into faces.

We define an equivalence relation on the set of faces of $\Delta$ as follows. A facet $F_0$ with normal vector $v_0$ is called *reflected* from a facet $F_1$ with normal vector $v_1$ if the intersection $\overline{F_0} \cap \overline{F_1}$ is non-empty and the $v_0 = wv_1$, for some $w \in W$ which fixes the intersection $\overline{F_0} \cap \overline{F_1}$. We say that a face $F_0$ is reflected from a face $F_1$ if any facet containing $F_0$ is reflected from some facet containing $F_1$. We say that two faces $F_0$ and $F_l$ are reflection-equivalent if there exist a sequence of faces $F_1, \ldots, F_{l-1}$ such that $F_j$ is reflected from $F_{j+1}$ for $j = 0, \ldots, l-1$.

**Example 7.1** For polytope $\Delta = [\lambda_3, \lambda_2] \times [\lambda_2, \lambda_1]$ shown in Figure 2, there are six equivalence classes. One contains the left facet, the top facet, and the zero dimensional face $\{(\lambda_2, \lambda_2)\}$. The rest contain a single face.

The purpose of this section is to prove

**Theorem 7.2** *Let $M$ be a torsion-free, transversal, multiplicity-free, compact, connected Hamiltonian $G$-space with moment map $\Phi : M \to \mathfrak{g}^*$ and moment polytope $\Delta$. A connected component of a stratum of the orbit-type decomposition of $M$ is a union of sets $\Phi^{-1}(F)$ over faces $F$ of $\Delta$ reflection-equivalent to some fixed face, $F_0$.*

For any $x \in \Delta$, let $V_K^{\pm}(x)$ be the set of vectors in $V(x)$ of the form $(\alpha_i \pm \beta_i)/2$, so that $V_K(x) = V_K^+(x) \cup V_K^-(x)$. The span of $V_K^-(x)$ is then the image of $\mathfrak{t}_\sigma$ under the map

$$d\lambda : \mathfrak{t}_\sigma \to \mathfrak{t}_\sigma + \mathfrak{s}_\sigma$$

$$t \mapsto (t, d\gamma^{-1}(t)).$$

We extend $d\lambda$ to $\mathfrak{t}$ by setting $d\lambda = 0$ on $\mathfrak{z}_\sigma$, and prove the following generalization of Delzant's Lemma 3.2.

**Lemma 7.3** *The stabilizer, $G_p$, of any point $p$ in $\Phi^{-1}(x)$ is connected and its Lie algebra, $\mathfrak{g}_p$, is $K_\sigma$-conjugate to* $\mathrm{span}(V_Z(x) \cup V_K^-(x))$.

Proof - By Lemma 4.1, the level sets of $\pi \circ \Phi$ are equivariantly diffeomorphic, so $G_p = G_q$ for some $q \in Y_+$, and the first claim follows from Lemma 4.4.

To prove the second assertion, suppose that $x$ lies in a Weyl wall $\sigma$. We can assume that $p \in \Lambda$. By definition, $\Lambda$ is the fixed point set of the action $\lambda$, so $\mathfrak{g}_p$ contains $\mathrm{span}(V_K^-(x)) = \mathrm{Im}\ d\lambda$. By Lemma 6.7, $\mathfrak{g}_p$ contains $V_Z(x)$.

It remains to show that $\mathfrak{g}_p$ is contained in $\mathrm{span}(V_K^-(x) \cup V_Z(x))$. Since for any $v \in \mathfrak{g}_p$, the vector $(d\lambda)v$ lies in $\mathrm{span} V_K^-(x)$ it suffices to show that $v - (d\lambda)v$ lies in $\mathrm{span}(V_Z(x))$. Note that $v - (d\lambda)v$ lies in $\mathfrak{g}_p \cap \mathfrak{z}_\sigma$. Indeed, let $v = v_k + v_z$ with $v_k \in \mathfrak{t}_\sigma$ and $v_z \in \mathfrak{z}_\sigma$. Then

$$v - (d\lambda)v = (v_k + v_z) - (v_k - (d\gamma)v_k) = v_z - (d\gamma)v_k.$$

Therefore, it suffices to show that $\mathfrak{g}_p \cap \mathfrak{z}_\sigma$ is contained in $\mathrm{span}(V_Z(x))$. Since $\mathfrak{g}_p$ contains $\mathrm{span}(V_Z(x))$, it suffices to show that the dimension of $\mathfrak{g}_p \cap \mathfrak{z}_\sigma$ is at most that of $\mathrm{span}(V_Z(x))$. Let $\pi$ be the quotient map $\pi : \mathfrak{z}_\sigma \to q_\sigma$. Since $p \in \Lambda$,

$$\mathfrak{g}_p \cap \mathfrak{s}_\sigma = \{0\},$$

so that $\pi$ restricted to $\mathfrak{g}_p \cap \mathfrak{z}_\sigma$ is an injection. By Corollary 6.8, $\pi$ is also injective on $\mathrm{span}(V_Z(x))$, so it suffices to show that the dimension of $\pi(\mathfrak{g}_p \cap \mathfrak{z}_\sigma)$ is at most that of $\pi(\mathrm{span}(V_Z(x)) = \mathrm{span}(V_\sigma(x))$. If $Q : \Phi^{-1}(\sigma) \to Y_\sigma$ is the projection, then

$$\pi(\mathfrak{g}_p \cap \mathfrak{z}_\sigma) \subset (q_\sigma)_{Q(p)} = \mathrm{span}(V_\sigma(x))$$

by Lemma 3.2 applied to the action of $Q_\sigma$ on $Y_\sigma$. □

Proof of Theorem 7.2 - Let $F_0$ and $F_1$ be faces of $\Delta$, and let $x$ and $y$ be points in $F_0$ and $F_1$, respectively. It suffices to consider the case that $F_0$ is contained in $\overline{F_1}$, so that $V(x) \supset V(y)$, and to show that $F_0$ is reflected from $F_1$ if and only if $\Phi^{-1}(F_0)$ and $\Phi^{-1}(F_1)$ have the same orbit-type. Let $\sigma$ be the Weyl wall containing $F_0$. The faces $F_0$ and $F_1$ are reflected if and only if for any $v \in V(x)$, the vector $wv$ lies in $V(y)$ for some $w \in W_x$. That is, $V(x) = W_x V(y)$, which holds if and only if $V_Z(x) \subset V_Z(y)$ and for some choice of $V_K^-(x)$, the difference $V_K^-(x) - V_K^-(y)$ lies in $V_Z(y)$. That is,

$$V_K^-(x) \cup V_Z(x) = V_K^-(y) \cup V_Z(y).$$

By Lemma 7.3 $F_0$ is reflected from $F_1$ if and only if the sets $\Phi^{-1}(F_0)$ and $\Phi^{-1}(F_1)$ have the same orbit-type. □

**Example 7.4** Let $M$ be the 10-dimensional coadjoint orbit of $G_2$ through the point $x$ pictured below, so that $M \cong G_2/(SU(2) \times U(1))$. We show that the orbit-type decomposition of the action of $SU(3) \subset G_2$ on $M$ has three components. The moment polytope $\Delta$ can be computed using, for instance, a formula of Guillemin and Prato [11], and is pictured as the shaded region in the figure below on the right. A local model for $M$ in a neighborhood of $x$ shows that $SU(3)$ acts generically freely. By 1.4, since $\Delta$ is reflective and simple, $M$ is transversal. Since $\Delta$ is Delzant, by Proposition 6.2, $M$ is torsion-free. By Theorem 7.2, the orbit-type decomposition of $\Delta$ consists of only three strata: $\{x\}$, $\partial\Delta - \{x\}$, and int $\Delta$. These facts can be verified using the embedding of $G_2$ in $SO(7)$.

## 8 Construction I

The purpose of this section is to show how the technique of symplectic cutting using components of the moment map (used in Lerman's version of the construction of multiplicity-free torus actions)

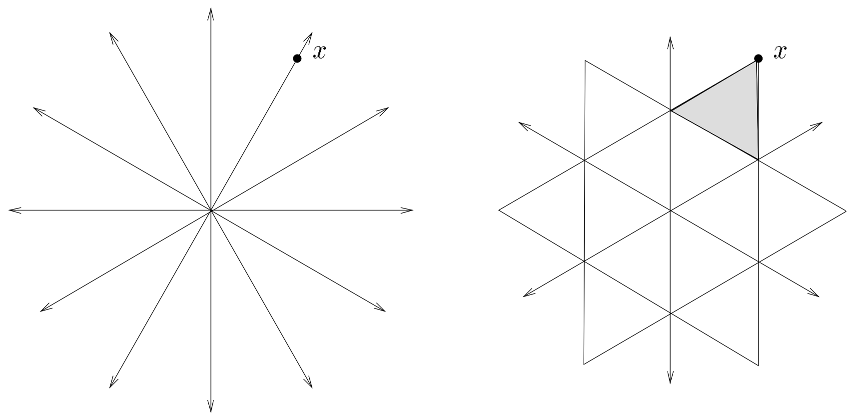


Figure 7: The point $x \in \mathrm{Lie}(G_2)^*$, and the moment polytope $\Delta \subset \mathfrak{su}(3)^*$.

generalizes to the non-abelian case. Let

$$Q : \mathfrak{g}^* \to \mathfrak{t}^*_+$$

be the quotient map which assigns to each $x \in \mathfrak{g}^*$ the unique point in $Gx \cap \mathfrak{t}^*_+$. The map $Q$ is continuous, and smooth on $\mathfrak{g}^*_{reg}$. Let

$$\tilde{\Phi} = Q \circ \Phi.$$

If $v \in \mathfrak{g}$, then $\langle \tilde{\Phi}, v \rangle$ is a $G$-invariant function on $M$, which is continuous but not necessarily smooth everywhere. Regardless, we will use the following proposition to show that we can cut with a component $\langle \tilde{\Phi}, v \rangle$ of $\tilde{\Phi}$ as long as the "cut hyperplane" $H$ defined by $( \ , v) = \lambda$ is perpendicular to any Weyl wall $\sigma$ meeting $H \cap \Delta$, or equivalently, $v \in \mathfrak{z}_\sigma$ for any such wall $\sigma$. Recall that if $\sigma$ is any Weyl wall, then $U_\sigma \subset \mathfrak{g}^*_\sigma$ is called a cross-section for $\sigma$ if for any point $x \in U_\sigma$, $G_x \subset G_\sigma$. By the Guillemin-Sternberg cross-section theorem, [16, Theorem 26.2] $\Phi^{-1}(U_\sigma)$ is a Hamiltonian $G_\sigma$-space.

**Proposition 8.1** *For any Weyl wall $\sigma$ and vector $v \in \mathfrak{z}_\sigma$, $\langle \tilde{\Phi}, v \rangle$ is smooth on $G\Phi^{-1}(U_\sigma)$. Furthermore, the Hamiltonian vector field $\tilde{X}_v$ associated to $\langle \tilde{\Phi}, v \rangle$ is equal to the Hamiltonian vector field $X_v$ associated to $\langle \Phi, v \rangle$ on $\Phi^{-1}(U_\sigma)$.*

This implies that the flow of $\tilde{X}_v$ is equal to the flow of $X_v$ on $\Phi^{-1}(U_\sigma)$. Since the flow of $\tilde{X}_v$ is $G$-equivariant, this implies that the flow of $\tilde{X}_v$ is periodic, if $v$ is rational, wherever it is defined.

Proof - Suppose that $\Phi(p) \in \sigma$. Since $v \in \mathfrak{z}_\sigma$,

$$\langle \tilde{\Phi}, v \rangle = \langle \Phi, v \rangle \quad \text{on } \Phi^{-1}(U_\sigma). \tag{25}$$

Since

$$G\Phi^{-1}(U_\sigma) = G \times_{G_\sigma} \Phi^{-1}(U_\sigma)$$

the function $\langle\tilde{\Phi}, v\rangle$ is smooth on $G\Phi^{-1}(U_\sigma)$. Since $\langle\tilde{\Phi}, v\rangle$ is $G$-invariant, the flow of $\tilde{X}_v$ preserves $\Phi^{-1}(U_\sigma)$, and since the flow of $X_v$ also preserves $\Phi^{-1}(U_\sigma)$, the two vector fields are the Hamiltonian vector fields corresponding to the same function $\langle\tilde{\Phi}, v\rangle = \langle\Phi, v\rangle$ on $\Phi^{-1}(U_\sigma)$. Hence

$$X_v = \tilde{X}_v \text{ on } \Phi^{-1}(U_\sigma),$$

and the flow of $\tilde{X}_v$ is the flow of $X_v$ on $\Phi^{-1}(U_\sigma)$ extended to $G\Phi^{-1}(U_\sigma)$ by requiring $G$-equivariance. □

Because symplectic cutting is a local operation, we can (following [24]) define the symplectic cut of $M$ at $\lambda$ as long as $f$ is smooth in a neighborhood $U$ of $f^{-1}(\lambda)$. Indeed, the symplectic cut $U_{\geq\lambda}$ is well-defined, and has a dense subset $U_{>\lambda}$ which is equivariantly symplectomorphic to $f^{-1}(\lambda, \infty) \cap U$. We define $M_{\geq\lambda}$ to be the union of $U_{\geq\lambda}$ and $f^{-1}(\infty, \lambda) \subset M$ modulo the identification of $U_{>\lambda}$ with $f^{-1}(\lambda, \infty) \cap U \subset M$.

**Remark 8.2** A $G$-invariant complex structure on $M$ is not necessarily preserved by the flow of $\tilde{X}_v$. In particular, a symplectic cut of $M$ using $\langle\tilde{\Phi}, v\rangle$ does not in general inherit a complex structure from $M$.

Proposition 8.1 allows us to generalize the construction of multiplicity-free torus actions. We say that a polytope $\Delta_1 \subset \Delta$ is a clean sub-polytope of $\Delta$ if for every facet $F$ of $\Delta$ meeting $\Delta_1$, $F \cap \Delta_1$ is a facet of $\Delta_1$. Equivalently, $V(x) \subset V_1(x)$ (up to positive scaling) for all $x \in \Delta_1$, where $V_1(x)$ is the set of normal vectors to facets of $\Delta_1$ meeting $x$.

**Theorem 8.3** *Let $M$ be as in Theorem 1.3, and let $\Delta_1$ be Delzant, reflective, and clean subpolytope of $\Delta$. Then there exists a connected, compact, transversal, torsion-free, multiplicity-free space $M_1$ with moment polytope $\Delta_1$, obtained from $M$ by symplectic cutting.*

Proof - Let $v_1, \ldots, v_d$ be the normal vectors to the facets $F_1, \ldots, F_d$ of $\Delta_1$ that are not of the form $F \cap \Delta_1$ where $F$ is a facet of $\Delta$. Let $\lambda_1, \ldots, \lambda_d \in \mathbb{R}$ be constants such that $\overline{F_i} = \Delta \cap \{x | \langle x, v_i\rangle = \lambda_i\}$. Fix $i$, and let $x$ be a point in $\overline{F_i}$ which is contained in a Weyl wall $\sigma$. Since $\Delta_1$ is a clean sub-polytope, $V_K(x) \supset (V_K)_1(x)$ and since these two sets are of order $2\text{codim}(\sigma)$, they are equal. Hence,

$$v_i \in V_1(x) - (V_K)_1(x) = (V_Z)_1(x).$$

That is, $v_i \in \mathfrak{z}_\sigma$. Therefore, the function

$$f_i = \langle\tilde{\Phi}, v_i\rangle$$

is smooth near $f_i^{-1}(\lambda_i)$.

To show that the symplectic cut of $M$ using $f = (f_1, \ldots, f_d)$ at $\lambda = (\lambda_1, \ldots, \lambda_d)$ is smooth, let $T^k$ be the torus whose Lie algebra is $\text{span}(v_1, \ldots, v_k)$. By (7) without loss of generality it suffices to show that $T^k$ acts freely on

$$\bigcap_{i=1}^{k} f_i^{-1}(\lambda_i) \cap \bigcap_{i=k+1}^{d} f_i^{-1}(-\infty, \lambda_i). \tag{26}$$

Let $p$ lie in (26) and suppose that $x = \Phi(p)$. Since $T^k$ acts $K_\sigma$-equivariantly, we can assume that $p \in \Lambda$. By Lemma 7.3

$$\mathfrak{g}_p = \mathrm{span}((V_Z)(x) \cup V_K^-(x)).$$

Since $\Delta_1$ is Delzant, $V_1(x)$ is a lattice sub-basis, and therefore

$$(V_1(x) - V(x)) \cup V_Z(x) \cup V_K^-(x) \subset V_1(x)$$

is a lattice sub-basis. It follows that

$$T^k \cap G_p = \{id\}. \tag{27}$$

Symplectic cutting leaves a dense subset unchanged, so that $M_{\geq\lambda}$ is multiplicity-free. To show that $M_{\geq\lambda}$ is torsion-free and transversal, we must prove that $G_p^\lambda \cap K_\sigma = \{id\}$, where $G_p^\lambda = G_p \times T^k$ is the stabilizer of $G$ acting on $\Phi_{\geq\lambda}^{-1}(x)$. We can assume that $G_p \subset T$. Because $r(\sigma) \cup V_K^-(x)$ generates the same lattice as $V_K(x)$, and $\Delta_1$ is Delzant

$$(V_1(x) - V(x)) \cup V_Z(x) \cup V_K^-(x) \cup r(\sigma)$$

is a lattice sub-basis, and so

$$(T^k \times G_p) \cap T_\sigma = \{id\}.$$

Since the fibers of $\Phi$ are compact and connected, $M_{\geq\lambda}$ is compact and connected also. □

Although this theorem allows us to construct infinitely many families of multiplicity-free, torsion-free, transversal spaces it is not enough to construct all of them, except in exceptional cases, e.g., when $G$ is abelian, or equal to $U(2)$, $SU(3)$, or $SO(4)$. For instance, in the case $G = U(2)$, it's not hard to see that any reflective polytope is a clean sub-polytope of the polytope in Figure 2 of the action of $U(2)$ a coadjoint orbit of $U(3)$.

**Example 8.4** Let $M$ be the generic coadjoint orbit of $U(4)$ with distinct eigenvalues $i\lambda_1, i\lambda_2, i\lambda_3$ and $i\lambda_4$, and let $G = U(3)$ be embedded in $U(4)$ by $A \to \mathrm{diag}(1, A)$. Choose the positive Weyl chamber to be int $\mathfrak{t}_+^* = \{(\mu_1, \mu_2, \mu_3) \in \mathbb{R}^3 | \mu_1 < \mu_2 < \mu_3\}$. By a generalization of the argument in Section 2.1, the moment polytope of $M$ is the box $\Delta = [\lambda_1, \lambda_2] \times [\lambda_2, \lambda_3] \times [\lambda_3, \lambda_4]$. Let $f = \langle \tilde{\Phi}, -e_1 - e_2 + e_3 \rangle$ and $\lambda = 2\lambda_2 + \lambda_4 - \epsilon$, for $\epsilon > 0$ small. The polytopes $\Delta$ and $\Delta_{\leq\lambda} = \{x \in \Delta | f(x) \leq \lambda\}$ are shown in Figure 8. By Theorem 8.3, there exists a space $M_{\leq\lambda}$ with moment polytope $\Delta_{\leq\lambda}$.

**Theorem 8.5** *Let $\Delta \subset \mathfrak{t}_+^*$ be a reflective Delzant polytope, and $\sigma$ a Weyl wall intersecting $\Delta$ such that $\Delta$ is contained in a cross-section for $\sigma$. That is, for any $x \in \Delta$, $G_x \subset G_\sigma$. Then there exists a transversal, torsion-free, multiplicity-free, compact, connected $G$-space $M$ with moment polytope $\Delta$.*

Proof - Let $x$ be any point in $\Delta \cap \sigma$ and in the notation of Corollary 5.2 let

$$V_K(x) = \{(\alpha_1 \pm \beta_1)/2, \ldots, (\alpha_k \pm \beta_k)/2\}.$$

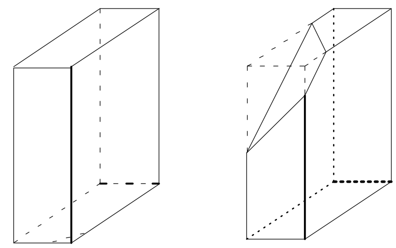

Figure 8: The polytopes $\Delta$ and $\Delta_{\leq\lambda}$.

Define a map $d\gamma : \mathfrak{t}_\sigma \to \mathfrak{z}_\sigma$ by $(d\gamma)\alpha_i = \beta_i$. We claim that $d\gamma$ induces an injection $\gamma : T_\sigma \to Z_\sigma$. Since $\Delta$ is Delzant, there exists a choice of scalars $c_i^\pm \in \mathbb{R}_+$ such that

$$\{c_1^\pm(\alpha_1 \pm \beta_1), \ldots, c_k^\pm(\alpha_k \pm \beta_k)\}$$

is a basis for $L \cap \operatorname{span}(V_K(x))$. In particular the vectors $c_i^\pm(\alpha_i \pm \beta_i)$ are in minimal form. Since $c_i(\alpha_i - \beta_i)$ is a lattice element, $c_i(\alpha_i + \beta_i)$ is a lattice element also, because $L$ is invariant under $W$. Therefore $c_i^+ = c_i^-$. Consider the three-dimensional subalgebra

$$\mathfrak{g}_i := \mathbb{R}\alpha_i + \mathfrak{g}_{\alpha_i}$$

and let $G_i = \exp(\mathfrak{g}_i)$, which is isomorphic to either $SU(2)$ or $SO(3)$. We claim that the latter is impossible. Indeed, since the center of $SO(3)$ is trivial, the 1-parameter subgroup $\exp(\mathbb{R}\alpha_i)$ intersects $\exp(\mathbb{R}\beta_i)$ trivially, so that

$$\exp(c_i^\pm\alpha_i) = \exp(c_i^\pm\beta_i) = \mathrm{Id}.$$

That is, $c_i^\pm\alpha_i \in L$, which implies that $\{c_i^\pm(\alpha_i \pm \beta_i)\}$ cannot be a lattice sub-basis. Therefore, $G_i \cong SU(2)$, and since $\alpha_i$ is a minimal lattice element, $c_i^\pm = 1/2$. It follows that $\{\alpha_1, \ldots, \alpha_k\}$ and $\{\beta_1, \ldots \beta_k\}$ are lattice sub-bases, which proves the claim.

Let $S_\sigma = \gamma(T_\sigma)$. Since $S_\sigma$ is connected, we can find a subgroup $Q_\sigma \subset Z_\sigma$ such that $Z_\sigma = S_\sigma \times Q_\sigma$. Define

$$\tilde{N} = T^*(K_\sigma \times Q_\sigma).$$

Let $K_\sigma \times Q_\sigma$ act on $\tilde{N}$ by left translation, and let $S_\sigma$ act by $\gamma^{-1} : S_\sigma \to T_\sigma$ composed with the right action of $T_\sigma$ on $\tilde{N}$. We claim that the action of $K_\sigma \times Z_\sigma$ on $\tilde{N}$ descends to a transversal, torsion-free, multiplicity-free action of $G_\sigma \cong K_\sigma \times_{K_\sigma \cap Z_\sigma} Z_\sigma$. Note that the kernel of the action of $K_\sigma \times S_\sigma \times Q_\sigma$ is the set

$$\{(g, \gamma(g)^{-1}, \mathrm{Id}), \quad g \in Z(K_\sigma)\}.$$

Indeed, an element $(g, g', g'') \in K_\sigma \times S_\sigma \times Q_\sigma$ which acts trivially must have $g'' = \mathrm{Id}$, and $g$ acting $K_\sigma$-equivariantly. Thus, $(g, g', g'')$ acts trivially if and only if $g \in Z(K_\sigma)$, $g' = \gamma(g)^{-1}$, and $g'' = \mathrm{Id}$, which shows that the action descends to an action of $G_\sigma$. Clearly, the action of $K_\sigma$ on $\tilde{N}$ is free.

Since for any Weyl wall $\tau$, $K_\tau \subset K_\sigma$, $\tilde{N}$ is transversal and torsion-free. A dimension count shows that $\tilde{N}$ is multiplicity-free.

Let $\Phi_{\tilde{N}}$ denote the moment map for the action of $G_\sigma$ on $\tilde{N}$. We claim that the convex polyhedral set $\Delta_{\tilde{N}} := \Phi_{\tilde{N}} \cap \mathfrak{t}^*_+$ is given by

$$\Delta_{\tilde{N}} = \bigcap_{v \in V_K(x)} \{y \in \mathfrak{t}^* \mid (y, v) \geq (x, v)\}.$$

It suffices to consider the case $\tilde{N} = T^*SU(2)$ and $G_\sigma = U(2)$. In this case with respect to the standard basis for $\mathfrak{t} \cong \mathbb{R}^2$, the unique positive root is $\alpha = (1, -1)$, and $\beta = (1, 1)$, so that

$$\{\frac{1}{2}(\alpha \pm \beta)\} = \{(1, 0), (0, -1)\}$$

and we have to show that the moment "polytope" of $T^*SU(2)$ under the action of $U(2)$ defined above is the lower right quadrant. We identify $T^*(SU(2))$ with $SU(2) \times \mathfrak{su}(2)^*$ by left translation. The moment map for the left action of $SU(2)$ is $\Phi_L(g, v) = \mathrm{Ad}^*(g)v$ so that for any $v \in \mathfrak{u}(1)^*$, we have

$$\Phi_L^{-1}(v) = \{(g, \mathrm{Ad}^*(g)^{-1}(v)) \mid g \in SU(2)\}.$$

On the other hand, the moment map for the right action is $\Phi_R(g, v) = v$ and so a moment map for the right action of $T_\sigma$ is

$$\Phi_T(g, v) = \pi(v)$$

where $\pi : \mathfrak{su}(2)^* \to \mathfrak{u}(1)^*$ is the projection onto the Cartan subalgebra $\mathfrak{u}(1)^*$. Therefore,

$$\Phi_T(\Phi_L^{-1}(v)) = \{\pi(\mathrm{Ad}^*(g)^{-1}(v)) \mid g \in SU(2)\} = [-v, +v]$$

and the moment "polytope" is the set

$$\{v(1, -1) + w(1, 1) \mid v \geq 0, |w| \leq v\}$$

which is the lower right quadrant.

Since $\Delta_{\tilde{N}}$ contains $\Delta$ cleanly, by symplectic cutting as in the proof of Theorem 8.3, there exists a Hamiltonian $G_\sigma$-space $N$ with moment polytope $\Delta$. (Although $\tilde{N}$ is not compact, its moment map is proper and has connected fibers, and Theorem 8.3 extends to this case without difficulty. The details are left to the reader.)

The existence of a Hamiltonian $G$-space $M$ with moment polytope $\Delta$ follows from the following lemma, and the assumption that $\Delta$ is contained in a cross-section for $\sigma$:

**Lemma 8.6** *Let $H \subset G$ be a connected subgroup containing $T$ and let $(N, \omega)$ be a compact Hamiltonian $H$-space with moment polytope $\Delta$. Then if $\mathfrak{g}_x \subset \mathfrak{h}$ for all $x \in \Delta$, there exists a Hamiltonian $G$-structure on $M = G \times_H N$ with moment polytope $\Delta$.*

Proof - Identify $T^*G$ with $G \times \mathfrak{g}*$ by left-translation, and let $\omega_{can}$ be the canonical symplectic form on $G \times \mathfrak{g}^*$. Let $i$ be the inclusion of $G \times \mathfrak{h}^*$ in $G \times \mathfrak{g}^*$. Then $i^*\omega_{can}$ is symplectic at points $(g, v) \in G \times \mathfrak{h}^*$ such that $G_v \subset H$. The moment map $\Phi_H$ for the action of $H$ on $(G \times \mathfrak{h}^*, -\omega_{can})$ is $\Phi_H(g, v) = -v$. Let $N \times G \times \mathfrak{h}^*$ be the product with closed two-form $\omega_{prod} = \pi_1^*\omega - \pi_2^*\omega_{can}$. The moment map for the action of $H$ on the product is $\Phi = \Phi_N - \Phi_H$. Therefore

$$\Phi^{-1}(0) = \{(m, g, \Phi(m)) \mid m \in M, g \in G\}.$$

The assumption that $\mathfrak{g}_x \subset \mathfrak{h}$ for $x \in \Delta$ guarantees that the $\omega_{prod}$ is non-degenerate at $\Phi^{-1}(0)$, so that the quotient $\Phi^{-1}(0) \cong G \times_H M$ is by the Marsden-Weinstein-Meyer theorem a symplectic manifold. □

# 9 Uniqueness

Suppose that $M_1$ and $M_2$ are as in Theorem 1.3 and have the same moment polytope $\Delta$. We want to construct an equivariant symplectomorphism $\varphi : M_1 \cong M_2$.

## 9.1 Local equivariant symplectomorphism

First consider the situation when $\Delta \cap \sigma$ is closed, so that $\Phi_i^{-1}(\sigma)$ are compact. By Lemma 6.4 the generic stabilizers $(S_\sigma)_i$ of $Z_\sigma$ acting on $\Phi_i^{-1}(\sigma)/K_\sigma$ are connected. Their Lie algebras are $(\mathfrak{s}_\sigma)_i = (\Delta \cap \sigma)^0 \subset \mathfrak{z}_\sigma$, so

$$(S_\sigma)_1 = (S_\sigma)_2 = S_\sigma.$$

By Corollary 6.5, the actions of $Q_\sigma := Z_\sigma/S_\sigma$ on $(Y_\sigma)_i := \Phi_i^{-1}(\sigma)/K_\sigma$ are multiplicity-free, and their moment polytopes are $\Delta \cap \sigma$. By Delzant's theorem there exists an $Q_\sigma$-equivariant symplectomorphism $\varphi : (Y_\sigma)_1 \cong (Y_\sigma)_2 =: Y_\sigma$.

The next step is to show that $\Phi_2^{-1}(\sigma) \cong \Phi_1^{-1}(\sigma)$ as $K_\sigma$-principal bundles. Let $c_i \in H^2(Y_\sigma, \mathfrak{t}_\sigma)$ denote the Chern class of the bundles $\Lambda_i$ in equation (23). By definition, $c_i = [d\alpha_i]$ where $\alpha_i$ is a connection form for the $T_\sigma$-action. It suffices to show that $c_1 = \varphi^* c_2$. For $t \in \mathfrak{t}_\sigma$ small let $\mu_i(t) : Y_\sigma \to q_\sigma^*$ denote the moment map for the action of $Q_\sigma$ on $(Y_\sigma, \omega_\sigma + \langle d\alpha_i, t\rangle)$, where $\omega_\sigma$ denotes the symplectic form on $Y_\sigma$ induced from $\omega$. By Lemma 3.5, it suffices to show that $\mu_1(t) = \mu_2(t)$ on the fixed point set $(Y_\sigma)_{(Q_\sigma)}$, for all $t$ in a neighborhood $U$ of 0. Since by Lemma 3.2, the image of the fixed point set $\mu_i(t)((Y_\sigma)_{(Q_\sigma)})$ is the set of vertices of $\Delta \cap \sigma$, and since $\mu_1(0) = \mu_2(0)$, it suffices to show that the moment polytopes $\mu_1(t)(Y_\sigma)$ and $\mu_2(t)(Y_\sigma)$ are equal for all $t$ in $U$. We can assume that $\alpha_i$ are restrictions of $K_\sigma$-connection forms on $\Phi_i^{-1}(\sigma)$ to $\Lambda_i$, and that we have local equivariant symplectomorphisms

$$\psi_i : \Phi_i^{-1}(\sigma) \times \mathfrak{k}_\sigma^* \to \Phi_i^{-1}(\mathfrak{g}_\sigma^*).$$

For any $t \in \mathfrak{t}_\sigma^*$, define $i_t : \Lambda_i \to \Lambda_i \times \{t\}$ by

$$i_t(p) = (p, t).$$

Then $\Phi_i \circ \psi \circ i_t$ descends to $\mu_i$. It therefore suffices to show that the sets $(\Phi_i \circ \psi \circ i_t)(\Lambda_i)$ are equal for $i = 1, 2$ and $t$ small. By definition $\Lambda_i \subset \Phi_i^{-1}(\sigma)$ is the fixed point set of the sub-torus with Lie algebra $\text{span}(\alpha_1 - \beta_1, \ldots, \alpha_k - \beta_k)$. By Lemma 4.4

$$(\Phi_i \circ \psi_i)(\Lambda_i \times \text{int}(\mathfrak{t}_\sigma)^*_+) = \overline{F} \cap \text{int } \mathfrak{t}^*_+$$

where $\overline{F}$ is the intersection of the closed facets with normal vectors $\alpha_1 - \beta_1, \ldots, \alpha_k - \beta_k$. Therefore,

$$\mu_i(t)(Y_\sigma) = \overline{F} \cap \pi^{-1}(t).$$

Hence, the bundles $\Lambda_1 \cong \Lambda_2 =: \Lambda$ are isomorphic.

**Example 9.1** In Example 8.4, let $\sigma$ be the wall given by $\mu_1 = \mu_2$, and take $\beta = \mu_1 + \mu_2$. Then $\overline{F}$ is the facet given by $\mu_2 = \lambda_2$, and $\overline{F} \cap \pi^{-1}(t)$ projected onto $\mathfrak{z}^*_\sigma / \mathfrak{s}^*_\sigma$ is constant (resp. non-constant) when the polytope is $\Delta$ (resp. $\Delta_{\leq\lambda}$.) Therefore, the bundle $\Lambda$ is trivial (resp. non-trivial) for $M$ (resp. $M_{\leq\lambda}$).

It remains to show that the actions $\tau_i : T \to \text{Diff}(\Lambda)$ of $T$ on $\Lambda$ are the same. Define a map $\tau : T \to \text{Diff}(\Lambda)$ by

$$\tau(t) = \tau_1^{-1}(t)\tau_2(t)$$

for $t \in T$. Since $\Lambda_1$ and $\Lambda_2$ are isomorphic $T_\sigma$-bundles, the map $\tau$ vanishes on $T_\sigma$, and for $t \in T_\sigma$ we define $tp := \tau_1(p) = \tau_2(p)$. Since $\tau_1$ and $\tau_2$ cover the same action on $\Lambda/T_\sigma$, for any $p \in \Lambda$ and $t \in T$ there exists an element $\theta_p(t) \in T_\sigma$ such that $\theta_p(t)p = \tau(t)p$. Since the action $\tau$ commutes with the action of $T_\sigma$, the map $\theta_p$ is a homomorphism, and since the set of homomorphisms from $T$ to $T_\sigma$ is discrete, $\theta_p := \theta$ does not depend on $p$. Let $d\theta : \mathfrak{t} \to \mathfrak{t}_\sigma$ denote the differential of $\theta$, and let $(\mathfrak{g}_p)_i$ denote the infinitesimal stabilizers of a point $p \in \Lambda$ with respect to $\tau_i$. Then $(\mathfrak{g}_p)_1 = (\text{Id} + d\theta)(\mathfrak{g}_p)_2$. By Corollary 3.3 (2), for any point $x \in \Delta \cap \sigma$, we have

$$V(x) = (\text{Id} + d\theta)V(x). \tag{28}$$

Let $x$ be a vertex of $\Delta$ contained in $\sigma$ so that $\text{span}(V(x)) = \mathfrak{t}$. Suppose that for some $v \in V(x)$, $d\theta(v) \neq 0$, so that

$$(\text{Id} + d\theta)^n v = v + nd\theta v \in V(x)$$

for all $n \in \mathbb{Z}$. Since this is impossible, $d\theta = 0$. Therefore $\Phi_1^{-1}(\sigma)$ and $\Phi_2^{-1}(\sigma)$ are isomorphic as $G_\sigma$-spaces with 2-form, $\omega_\sigma$. We write $\Phi^{-1}(\sigma) := \Phi_1^{-1}(\sigma) \cong \Phi_2^{-1}(\sigma)$.

By (15) there exist local $G_\sigma$-equivariant symplectomorphisms $\psi_i : \Phi_i^{-1}(U_\sigma) \cong \Phi^{-1}(\sigma) \times \mathfrak{k}^*_\sigma$. where $U_\sigma \subset \mathfrak{g}^*_\sigma$ is a neighborhood of $\sigma$. Since $\Phi_i^{-1}(U_\sigma)$ are multiplicity-free $G_\sigma$-spaces, the equation $\varphi := \psi_2^{-1} \circ \psi_1$ defines a $G_\sigma$-equivariant symplectomorphism of $\Phi_1^{-1}(G_\sigma V)$ and $\Phi_2^{-1}(G_\sigma V)$, where $V$ is some neighborhood of $\Delta \cap \sigma$ in $\Delta$. Since $G\Phi_i^{-1}(V) = G \times_{G_\sigma} \Phi_i^{-1}(V)$, the map $\varphi$ extends to a map $\varphi : G\Phi_i^{-1}(V) \cong G\Phi_i^{-1}(V)$ by requiring $G$-equivariance, and $\varphi$ is easily checked to be symplectic.

## 9.2 The non-compact case

In general, $\Delta \cap \sigma$ is not closed. In this section we use E. Lerman's symplectic cutting trick to compactify the sets $\Phi_i^{-1}(\Delta \cap \sigma)$, which reduces this case to the previous one. See [25] for further details and applications of this technique.

Let $\alpha$ be a simple root, and

$$\Delta_+ = \{y \in \Delta | (y, \alpha) \geq \epsilon\},$$

$$\Delta_- = \{y \in \Delta | (y, \alpha) \leq \epsilon\}.$$

**Lemma 9.2** *For $\epsilon > 0$ sufficiently small, $\Delta_\pm$ are reflective and Delzant.*

Proof - Let F be the facet of $\Delta_+$ defined by

$$F = \{y \in \Delta | (y, \alpha) = \epsilon\}.$$

The facets of $\Delta$ are interior, so if $\epsilon$ is sufficiently small, the facets of $\Delta_+$ are the facets of $\Delta$, plus $F$. In particular, the facets of $\Delta_+$ are interior.

To show $\Delta$ is Delzant, let $x$ be a vertex of $\Delta_+$, and let $V_+(x)$ be the set of normal vectors to facets of $\Delta_+$ meeting $x$. Either $V_+(x) = V(x)$ or $V_+(x) = V(x) \cup \{\alpha\}$. If the latter is the case, let $\overline{F'}$ be the intersection of the closed facets of $\Delta$ containing $x$. We can assume that $\epsilon$ is smaller than $(\alpha, y)$ for any point $y$ in a closed face of $\Delta$ not meeting $\ker \alpha$. Therefore, $\overline{F'}$ meets $\ker \alpha$. Let $y$ be a vertex of $\Delta$ contained in $\overline{F'} \cap \ker \alpha$. Then $(\alpha \pm \beta)/2 \in V(y)$. Since $\#V(x) = \#V(y) - 1$, either one or both of $(\alpha \pm \beta)/2$ lie in $V(x)$. B y Corollary 5.2, if both lie in $V(x)$ then $x \in \ker \alpha$ which is impossible. Suppose without loss of generality that $(\alpha + \beta)/2 \in V(x)$. Then

$$V_+(x) = \{(\alpha + \beta)/2, \alpha, v_3, \ldots, v_n\}$$

and

$$V(y) = \{(\alpha + \beta)/2, (\alpha - \beta)/2, v_3, \ldots, v_n\}$$

for some vectors $v_3, \ldots, v_n \in \mathfrak{t}$, which shows that $V_+(x)$ and $V(y)$ generate the same lattice.

It remains to show that $V_+(x) \cup -V_+(x)$ is $W_x$-invariant. Because $\Delta$ is reflective, $V(x) \cup -V(x)$ is $W_x$-invariant, so it suffices to show that $\alpha$ is $W_x$-invariant. Suppose that $(\alpha', x) = 0$ for some simple root $\alpha'$. Since $\ker \alpha' \cap \Delta$ is a closed face, by the definition of $\epsilon$,

$$\ker \alpha \cap \ker \alpha' \cap \Delta \neq \emptyset$$

and so by Corollary 6.5, or Proposition 5.1 (4), $(\alpha, \alpha') = 0$. It follows that $\alpha$ is $W_x$-invariant. □

The idea now is to chop off $\Delta$ near the boundary of $\sigma$, and to apply the previous subsection to the resulting symplectic manifold. For some small $\epsilon(\sigma) > 0$ define

$$\overline{\Delta_{\epsilon(\sigma)}} = \{z \in \Delta | (z, \alpha) \geq \epsilon(\sigma) \text{ for } \alpha \notin r(\sigma)\}.$$

By induction using the previous lemma, $\overline{\Delta_{\epsilon(\sigma)}}$ is reflective and Delzant. By Theorem 8.3 there exist spaces $\overline{M_i^{\epsilon(\sigma)}}$ with moment polytope $\overline{\Delta_{\epsilon(\sigma)}}$ obtained from $M_i$ by symplectic cutting. Let

$$\Delta_{\epsilon(\sigma)} = \{z \in \Delta | (z, \alpha) > \epsilon(\sigma), \alpha \notin r(\sigma)\}$$

which is an open subset of $\Delta$, and let

$$M_i^{\epsilon(\sigma)} = \tilde{\Phi}_i^{-1}(\Delta_{\epsilon(\sigma)}).$$

The sets $M_i^{\epsilon(\sigma)}$ are open subsets of $M_i$ which are equivariantly symplectomorphic to dense subsets of $\overline{M_i^{\epsilon(\sigma)}}$. Let $\Phi_{i,\epsilon}$ be the moment maps for the actions of $G$ on $\overline{M_i^{\epsilon(\sigma)}}$. By the previous subsection, there exists a neighborhood $\overline{V_\sigma}$ of $\overline{\Delta_\epsilon(\sigma)} \cap \sigma$ in $\overline{\Delta_\epsilon(\sigma)}$ and an equivariant symplectomorphism

$$\varphi_\sigma : \tilde{\Phi}_{1,\epsilon}^{-1}(\overline{V_\sigma}) \cong \tilde{\Phi}_{2,\epsilon}^{-1}(\overline{V_\sigma}).$$

Let $V_\sigma = \overline{V_\sigma} \cap \Delta_{\epsilon(\sigma)}$. The map $\varphi_\sigma$ restricts to an equivariant symplectomorphism of the sets $\tilde{\Phi}_i^{-1}(V_\sigma) \subset M_i$ for $i = 1, 2$. For any point $x \in \Delta \cap \sigma$, there exists an $\epsilon > 0$ sufficiently small such that $\tilde{\Phi}^{-1}(x) \subset \tilde{\Phi}_i^{-1}(V_\sigma)$, which completes the proof that $M_1$ and $M_2$ are locally equivariantly symplectomorphic.

### 9.3 Local to Global

The existence of a global equivariant symplectomorphism follows from a Čech cohomology argument, as in the abelian case. Let $\{V_i\}$ be a good cover of $\Delta$ such that for each $i$ there exists an equivariant symplectomorphism $\varphi_i : \Phi_1^{-1}(V_i) \to \Phi_2^{-1}(V_i)$. As in the abelian case, the maps

$$\psi_{ij} = \varphi_i^{-1}\varphi_j$$

define a cocycle $\psi$ in $\check{C}^1(\Delta, \mathrm{Diff}(\omega, \Phi_1, G))$, where $\mathrm{Diff}(\omega, \Phi_1, G)$ is the sheaf which assigns to each open subset $V \subset \Delta$ the group of equivariant symplectomorphisms of $\tilde{\Phi}_1^{-1}(V)$ which intertwine the moment map. To define a global equivariant symplectomorphism $M_1 \to M_2$ it suffices to show that the cohomology class

$$[\psi] \in \check{H}^1(\Delta, \mathrm{Diff}(\omega, \Phi_1, G))$$

is trivial.

Let $L$ be the kernel of the map $\exp : \mathfrak{t} \to T$, and $\mathcal{L}'$ be the sheaf which assigns to any open $V \subset \Delta$ the subset of $L$ which is invariant under $W_x$ for all $x \in V$. More precisely,

$$\mathcal{L}'(V) = L \cap (\bigcap_{x \in V} \mathfrak{z}_x). \tag{29}$$

where $\mathfrak{z}_x$ denotes the Lie algebra of the center $Z_x$ of the isotropy subgroup $G_x$.

**Lemma 9.3** *Let $M$ be a compact connected symplectic manifold with a transversal, torsion-free, multiplicity-free action of $G$. Then the sequence of sheaves $0 \to \mathcal{R} \oplus \mathcal{L}' \to C_G^\infty \to \mathrm{Diff}(\omega, \Phi, G) \to 0$ is exact.*

Proof - First, we show that $C^\infty_G$ surjects onto $\mathrm{Diff}(\omega, \Phi, G)$. Let $x \in \Delta$ be any point, and let $V \subset \Delta$ be a contractible neighborhood of $x$. Let $\varphi$ be an element of $\mathrm{Diff}(\omega, \Phi, G)(V)$, and let $\sigma$ be the Weyl wall containing $x$. We can assume that $V$ is small enough so that $V$ is a cross-section for $\sigma$, that is, for any point $y \in V$, $G_y \subset G_\sigma$, and that $K_\sigma$ acts freely on $\Phi^{-1}(G_\sigma V)$.

We claim that there exists a smooth map $\theta : \tilde{\Phi}^{-1}(V) \to G$ such that $\varphi(p) = \theta(p)p$ for all $p \in \tilde{\Phi}^{-1}(V)$. Let $B = \Phi^{-1}(G_\sigma V)/K_\sigma$, and let $\varphi_B : B \cong B$ denote the diffeomorphism of $B$ induced by $\varphi$. Since

$$B/Z_\sigma = \Phi^{-1}(G_\sigma V)/G_\sigma \cong V,$$

and $\varphi$ preserves $\Phi$, $\varphi$ induces the identity on $B/Z_\sigma$. It follows from the Haefliger-Salem-Schwarz lemma [17, Theorem 3.1] that there exists a smooth map $\theta_Z : B \to Z_\sigma$ such that

$$\varphi_B([p]) = \theta_Z([p])[p]$$

for all $[p] \in B$. Replace $\theta_Z$ by its lift to $\Phi^{-1}(G_\sigma V)$. The map which assigns to any $p \in \Phi^{-1}(G_\sigma V)$ the point $\theta_Z^{-1}(p)\varphi(p)$ induces identity map on $B$. Since $K_\sigma$ acts freely, for all $p \in \Phi^{-1}(G_\sigma V)$,

$$\theta_Z^{-1}(p)\varphi(p) = \theta_K(p)p$$

for a unique $\theta_K(p) \in K_\sigma$. The assignment $p \to \theta_K(p)$ defines a smooth map

$$\theta_K : \Phi^{-1}(G_\sigma V) \to K_\sigma.$$

We define

$$\theta(p) = \theta_K(p)\theta_Z(p),$$

so that $\varphi_D(p) = \theta(p)p$ for all $p \in \Phi^{-1}(G_\sigma V)$. The map $\theta$ extends to $\tilde{\Phi}^{-1}(V) = \Phi^{-1}(GV)$ by requiring that $\theta(gp) = g\theta(p)g^{-1}$.

The next step is to show that $\theta$ lifts to a $G$-equivariant map $\bar{\theta} : \tilde{\Phi}^{-1}(V) \to \mathfrak{g}$. We claim that the map $\theta_\Delta : V \to T$ defined by $\theta_\Delta(y) = \theta(p)$ for any $p \in \Phi^{-1}(y)$ is well-defined. Because $\varphi$ is $G$-equivariant, and $K_y$ acts freely on $\Phi^{-1}(y)$, $\theta(p)$ must lie in $Z_y$, and so $\theta$ is constant on $\Phi^{-1}(y) = G_y p$, which proves the claim. Since $V$ is simply connected, we can lift $\theta_\Delta$ to $\mathfrak{t}$, that is, there exists a continuous map $\overline{\theta_\Delta} : V \to \mathfrak{t}$ such that

$$\exp(\overline{\theta_\Delta})(y) = \theta_\Delta(y)$$

for all $y \in V$. In constructing the lift, we can require that $\overline{\theta_\Delta}(x) \subset \mathfrak{z}_x$. Since any Weyl wall $\tau$ meeting $V$ contains $\sigma$, and $\mathfrak{z}_\tau \supset \mathfrak{z}_x$, we have $\overline{\theta_\Delta}(\tau) \subset \mathfrak{z}_\tau$. Define $\bar{\theta} : \Phi^{-1}(V) \to \mathfrak{t}$ by

$$\bar{\theta}(p) = \overline{\theta_\Delta}(\Phi(p)).$$

The map $\bar{\theta}$ extends to $\tilde{\Phi}^{-1}(V)$ by requiring $G$-equivariance. Since $\bar{\theta}$ is $G$-equivariant, we can define a $G$-invariant vector field $X \in \mathrm{Vect}(M)$ by $X(p) = \bar{\theta}(p)^\#_p$. By the same argument as in the abelian case, $X$ is the Hamiltonian vector field of some $G$-invariant function $f \in C^\infty_G(\tilde{\Phi}^{-1}(V))$.

To show exactness at $C^\infty_G$, let $V \subset \Delta$ be a connected open subset, and let $l$ of $L$ be invariant under $W_x$ for all $x \in V$. By Proposition 8.1 the function $f_l = \langle \tilde{\Phi}, l \rangle$ is $G$-invariant and smooth, and the time one exponential of the Hamiltonian vector field associated to $f_l$ is the identity. Conversely, let $f \in C^\infty_G(\tilde{\Phi}^{-1}(V))$, and let $X_f$ be its Hamiltonian vector field. Suppose that the time one exponential of $X_f$ is the identity. Since $X_f \Phi = 0$, $X_f$ is tangent to the level sets of $\Phi$. If $Tp$ is a generic orbit of $T$ in $\Phi^{-1}(V \cap \text{int } \mathfrak{t}^*_+)$, then

$$X_f = l(p)^\#$$

for a unique $l(p) \in L$. Since $l(p)$ varies continuously with $p$, $l$ must be constant. By continuity, $f = f_l$. Since $f$ is smooth, $l$ must be $W_x$-invariant for all $x \in V$. This is because by the Guillemin local model $\Phi^{-1}(\mathfrak{t}^*)$ is locally equivariant symplectomorphic to $\Phi^{-1}(\sigma) \times \mathfrak{t}^*_\sigma$ near $\Phi^{-1}(o)$, and $\mathfrak{t}^*_\sigma \cong \mathbb{R}^{\text{codim } \sigma}$. For $l' \in \mathfrak{t}_\sigma$, the map $\langle \tilde{\Phi}^{-1}, l' \rangle$ locally has the form

$$\langle \tilde{\Phi}^{-1}, l' \rangle (p, t) = |\langle l', t \rangle|,$$

which is not smooth. On the other hand, if $l' \in \mathfrak{z}_\sigma$, then by 8.1, $\langle \tilde{\Phi}, l' \rangle$ is smooth at $\Phi^{-1}(x)$. It follows that $l \in \mathfrak{z}_\sigma$. The difference $f - f_l$ is a constant $r \in \mathbb{R}$. □

To show that $\psi$ is cohomologically trivial, it suffices to show that the cohomology groups $\check{H}^i(\Delta, \mathcal{L}')$ vanish for $i > 1$. For each simple root $\alpha$, let $\mathcal{L}_\alpha$ be sheaf which assigns to an open subset $V \subset \Delta$ the group $\mathcal{L}(V) = \mathbb{Z}$ if $V \cap \ker \alpha \neq 0$, and $\mathcal{L}_\alpha(V) = \{0\}$ otherwise. That is, $\mathcal{L}_\alpha$ is the push-forward of the constant sheaf $\mathbb{Z}$ on $\Delta \cap \ker \alpha$ to $\Delta$. We define a morphism of sheaves

$$\pi_\alpha : \mathcal{L} \to \mathcal{L}_\alpha$$

as follows. For any open set $V \subset \Delta$ and $l \in \mathcal{L}(V) = L$ let

$$\pi_\alpha(l) = \begin{matrix} \frac{(l,\alpha)}{2(\alpha,\alpha)} & \text{if } V \cap \ker \alpha \neq 0 \\ 0 & \text{otherwise.} \end{matrix}$$

We claim that

$$0 \to \mathcal{L}' \to \mathcal{L} \to \oplus_\alpha \mathcal{L}_\alpha \to 0. \tag{30}$$

is a short exact sequence of sheaves. It suffices to show that the sequence is exact on open sets $V \subset \Delta$ which are cross-sections for some Weyl wall $\sigma$ meeting $\Delta$. By Section 6, the quotient lattice

$$\mathcal{L}(V)/\mathcal{L}'(V) = L/(L \cap \mathfrak{z}_\sigma) \cong \bigoplus_{\alpha_i \in r(\sigma)} \mathbb{Z}[\alpha_i]/2$$

where $[\alpha_i]$ denotes the equivalence class of $\alpha_i$ in $L/(L \cap \mathfrak{z}_\sigma)$. The claim follows.

The higher cohomology groups of $\mathcal{L}$ are zero since $\Delta$ is contractible. The cohomology groups of $\mathcal{L}_\alpha$ are isomorphic to the cohomology groups $\check{H}^i(\Delta \cap \ker \alpha, \mathbb{Z})$, which are also zero for $i > 0$. By the long exact sequence in cohomology, $\check{H}^i(\Delta, \mathcal{L}') = 0$ for $i > 1$, as required.

## 10 Construction II

**Lemma 10.1** *Let $M_+$ and $M_-$ be as in theorem 1.3 with moment polytopes $\Delta_+$ and $\Delta_-$, such that $\Delta_0 = \Delta_+ \cap \Delta_-$ is a Delzant, reflective, clean sub-polytope of both $\Delta_+$ and $\Delta_-$. Suppose that the boundaries $\partial(\Delta_0, \Delta_\pm)$ of $\Delta_0$ in $\Delta_\pm$ are disjoint. Then there exists a compact, connected, transversal, torsion-free, multiplicity-free space $M$ with moment polytope $\Delta = \Delta_+ \cup \Delta_-$.*

Proof - By Theorem 8.3 there exist spaces $\overline{M_{+,0}}$ and $\overline{M_{-,0}}$ with moment polytope $\Delta_0$ obtained from $M_+$ and $M_-$ by symplectic cutting. By uniqueness, there exists an equivariant symplectomorphism

$$\overline{\varphi} : \overline{M_{+,0}} \to \overline{M_{-,0}}.$$

Let

$$U_+ = \tilde{\Phi}_+^{-1}(\Delta_+ - \partial(\Delta_0, \Delta_-)) \quad \text{and} \quad U_- = \tilde{\Phi}_-^{-1}(\Delta_- - \partial(\Delta_0, \Delta_+))$$

so that $U_\pm$ are open subset of $M_\pm$. Let

$$M_{\pm,0} = \tilde{\Phi}_\pm^{-1}(\Delta_0 - \partial(\Delta_0, \Delta_+) - \partial(\Delta_0, \Delta_-)).$$

The map $\overline{\varphi}$ restricts to an equivariant symplectomorphism $\varphi : M_{+,0} \cong M_{-,0}$. The sets $M_{\pm,0}$ are equivariantly symplectomorphic to open subsets of $U_\pm$. Let $M$ be the space formed by taking the disjoint union of $U_+$ and $U_-$ and identifying the sets $M_{\pm,0}$ using $\varphi$. $\square$

In particular, this gives a very indirect proof that $\Delta = \Delta_+ \cup \Delta_-$ is a convex polytope (which is not necessarily true if $\partial(\Delta_+ \cap \Delta_-, \Delta_\pm)$ are not disjoint. ) In the figure below we give an example of this construction. The shaded regions are the polytope $\Delta_0$.

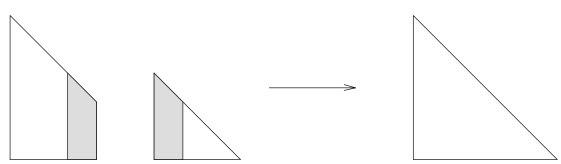

Figure 9: The polytopes $\Delta_\pm$ and $\Delta$.

Let $\Delta$ be any Delzant reflective polytope. We wish to construct a transversal, torsion-free, multiplicity-free, compact, connected $G$-space $M$, having moment polytope $\Delta$. Let $\alpha_1, \ldots, \alpha_n$ be an ordering of the simple roots. For each $k = 0, \ldots, n$ and subset $I \subset \{1, \ldots, k\}$, let

$$\Delta_I^k = \left\{ x \in \Delta \,\middle|\, \begin{array}{ll} (x, \alpha_i) \leq \epsilon, & i \in I \\ (x, \alpha) \geq \epsilon/2, & \text{otherwise} \end{array} \quad \text{for} \quad i = 1, \ldots, k \right\}.$$

By construction, $\Delta_I^n$ is contained in a cross-section for the Weyl wall $\sigma$, where $\sigma$ is the intersection of $\ker \alpha_i$ for $i \in I$. Furthermore, by induction using Lemma 9.2, $\Delta_I^n$ are reflective and Delzant.

By Theorem 8.5, there exist spaces $M_I^n$ with moment polytopes $\Delta_I^n$. Let $I \subset \{1, \ldots, k\}$ and $I' = I \cup \{k+1\}$. If there exist spaces $M_I^k$ and $M_{I'}^k$ with moment polytopes $\Delta_I^{k+1}$ and $\Delta_{I'}^{k+1}$, then Lemma 10.1 implies that there exists a space $M_I^k$ with moment polytope $\Delta_I^k$. By induction, there exist spaces $M_I^k$ with moment polytopes $\Delta_I^k$ (with the properties listed in Theorem 1.3) for all $k = n, \ldots, 0$. The polytope $\Delta_\emptyset^0$ is just $\Delta$, so we have proved the last part of Theorem 1.3.

**Remark 10.2** In this remark we consider the effect of dropping (c) in the definition of torsion-free 6.1. Let

$$L_K = L \cap \mathfrak{t}_\sigma, \quad L_S = L \cap \mathfrak{s}_\sigma, \quad L_Q = L \cap q_\sigma$$

with $\mathfrak{t}_\sigma, \mathfrak{s}_\sigma, q_\sigma$ as above. Lemma 6.4 implies that (1) the map $\mathfrak{t}_\sigma \to \mathfrak{s}_\sigma$ given by $\alpha_i \to \beta_i$ induces an isomorphism of $L_K$ and $L_S$, and since $G$ acts generically freely we must have (2) $Z(K_\sigma) = T_\sigma \cap S_\sigma$. Conversely, given a convex polytope $\Delta \subset \mathfrak{t}_+^*$ satisfying (1) and (2) at each Weyl wall, the same construction as before produces a transversal multiplicity-free action with moment polytope $\Delta$. The uniqueness part of Theorem 1.3, with the Delzant condition replaced by (1) and (2), also seems likely to be true. For example, it would suffice to prove that the sheaf $\mathcal{L}'$ described in (29) always has trivial cohomology in dimension 2.

# A The definition of a multiplicity-free action

The term "multiplicity-free" comes from representation theory. Let $V$ be a finite-dimensional complex representation of a compact connected Lie group $G$. We say that $V$ is multiplicity-free if each irreducible occurs in $V$ with multiplicity zero or one, or equivalently by Schur's lemma, if the algebra $\mathrm{End}_G(V)$ of $G$-equivariant endomorphisms is abelian. Let $M$ be a Hamiltonian $G$-space with moment map $\Phi : M \to \mathfrak{g}^*$, where $\mathfrak{g}$ is the Lie algebra of $G$. In geometric quantization, one tries to find a representation of (subalgebras of) the Poisson algebra $C^\infty(M)$ as operators on a "quantized space" $Q(M)$. In analogue with the representation theory, we say (following Guillemin and Sternberg in [15]) that $M$ is multiplicity-free if the Poisson algebra $C_G^\infty(M)$ of $G$-invariant smooth functions is an abelian Lie algebra. The purpose of this appendix is to prove

**Proposition A.1** *(1) Let $G$ be a compact, connected Lie group, and $M$ a compact, connected Hamiltonian $G$-space. Then $C_G^\infty(M)$ is an abelian Poisson algebra (that is, has a trivial Poisson structure) if and only if all of the reduced spaces $M_a = \Phi^{-1}(a)/G_a$ are points. If either condition is true, we say $M$ is multiplicity-free.*

*(2) If in addition $G$ acts locally freely on a dense set then $M$ is multiplicity-free if and only if $\dim(M) = \dim(G) + \mathrm{rank}(G)$.*

First suppose that $G$ acts freely on $\Phi^{-1}(a)$. By the Marsden-Weinstein-Meyer theorem the reduced space $M_a$ is a symplectic manifold, and the restriction map $r_a : C_G^\infty(M) \to C^\infty(M_a)$ is Poisson and surjective. Therefore, if $C_G^\infty(M)$ is abelian, the algebra $C^\infty(M_a)$ must be abelian, and $M_a$ must be discrete. By Kirwan's theorem [20], $M_a$ is connected, and therefore a point.

In general $M_a$ is not smooth, and one needs a lemma of Arms, Cushman, and Gotay (see [28]) which says that for arbitrary $a$, the algebra $C^\infty(M_a) := C^\infty_G(M)/I_a$, where $I_a$ is the ideal of functions vanishing on $\Phi^{-1}(a)$, is a non-degenerate Poisson algebra. That is, the Poisson bracket vanishes only on pointwise constant functions on $M_a$. If $C^\infty_G(M)$ is abelian, then $C^\infty_G(M)/I_a$ is abelian and therefore $M_a$ must be a point. On the other hand, if all the reduced spaces are points then $r_a(\{f,g\}) = 0$ for all $a \in \mathfrak{g}^*$ so that $\{f,g\} = 0$.

If $G$ acts locally freely (that is, with discrete stabilizer) on a dense subset of $M$, then for a dense set of values of $a$, $\Phi^{-1}(a)$ is smooth and has codimension $\dim G$. Therefore, the generic reduced space is a point if and only if $\dim M = (\dim + \mathrm{rank})(G)$. From above, the generic reduced space is a point if and only if $C^\infty(M_a)$ is abelian for generic $a$, which by continuity holds if and only if $C^\infty_G(M)$ is abelian.

## References


[1] M. F. Atiyah, Convexity and commuting hamiltonians, *Bull. London Math. Soc.* **14** (1982), 1-15.

[2] M. F. Atiyah and R. Bott, The moment map and equivariant cohomology, *Topology* **23** No. 1 (1984), 1-28.

[3] V. V. Batyrev, Dual polyhedra and mirror symmetry for Calabi-Yau hypersurfaces in toric varieties, *Jour. Alg. Geom.* **3** (1994), 493-535.

[4] M. Brion, Sur l'image de l'application moment, (Séminaire d'algèbre M. P. Malliavin 1987), Springer Lecture Notes **1296**.

[5] M. Brion, Spherical varieties: An introduction, in: *Topological methods in algebraic transformation groups*, (H. Kraft, T. Petrie, and G. Schwarz eds.) Prog. Math. **80**, Birkhauser, Basel-Boston-Berlin, 1989, 11-26.

[6] A. Brønsted, *An introduction to convex polytopes*, Graduate Texts in Mathematics **90**, Springer-Verlag, New York, 1983.

[7] T. Delzant, Hamiltoniens périodiques et images convexes de l'application moment, *Bul. Soc. math. France* **116** (1988), 315-339.

[8] T. Delzant, Classification des actions hamiltoniennes complètement intégrables de rang deux, *Ann. Global Anal. Geom.* **8** No. 1 (1990), 87-112.

[9] V. Guillemin, Transversal multiplicity-free spaces, unpublished note.

[10] V. Guillemin, *Moment maps and combinatorial invariants of Hamiltonian $T^n$-spaces*, Birkhäuser, Basel-Boston-Berlin, 1994.

[11] V. Guillemin and E. Prato, On Kostant, Heckman, and Steinberg formulas for symplectic manifolds, *Adv. in Math.* **82** No. 2 (1990), 160-179.

[12] V. Guillemin and S. Sternberg, Convexity properties of the moment mapping I and II, *Invent. Math.* **67** (1982), 491-513.

[13] V. Guillemin and S. Sternberg, Geometric quantization and multiplicities of group representations, *Invent. Math.* **67** (1982), 515-538.

[14] V. Guillemin and S. Sternberg, The Gelfand-Cetlin system and quantization of the complex flag manifolds, *Jour. Funct. Anal.* **52** (1983), 106-128.

[15] V. Guillemin and S. Sternberg, Multiplicity-free spaces, *J. Diff Geom.* **19** (1984), 31-56.

[16] V. Guillemin and S. Sternberg, *Symplectic Techniques in Physics*, Cambridge University Press, Cambridge, 1984.

[17] A. Haefliger and E. Salem, Actions of tori on orbifolds, *Ann. Global Anal. Geom.* **9** (1991), 37-59.

[18] G. Heckman, Projections of orbits and asymptotic behavior of multiplicities for compact connected Lie groups, *Inv. Math.* **67** (1982), 333-356.

[19] P. Iglesias, Les $SO(3)$-variétés symplectiques et leur classification en dimension 4, *Bull. Soc. math. France*, **119** (1991), 371-396.

[20] F. Kirwan, *Cohomology of Quotients in Symplectic and Algebraic Geometry*, Princeton University Press, Princeton, 1984.

[21] F. Kirwan, Convexity of the moment map III, *Invent. Math.* **77** (1984), 547-552.

[22] A. Khovanskii and S. Pukhlikov, Théorème de Riemann-Roch pour les intégrales et les sommes de quasi-polynomes sur les polyédres virtuels, *Alg. i analyz* **4** (1992), 188-216.

[23] F. Knop, The Luna-Vust theory of spherical embeddings, in: *Proceedings of the Hyderabad Conference on Algebraic Groups, December, 1989*, Madras, Manoj Prakashan, 1991, 225-249.

[24] E. Lerman, Symplectic cuts, *Math. Res. Letters* (1995), to appear.

[25] E. Lerman, E. Meinrenken, S. Tolman, C. Woodward, Non-abelian convexity by symplectic cuts, in preparation.

[26] E. Meinrenken, Symplectic surgery and the Spin$_c$-Dirac operator, M.I.T. preprint, March 1995, dg-ga/9504002.

[27] R. Sjamaar, personal communication, January 1995.

[28] R. Sjamaar and E. Lerman, Stratified symplectic spaces and reduction, *Ann. of Math.* **134** (1991), 375-422.

[29] R. Souza, Multiplicity-free Actions and the Image of the Moment Map, Ph.D. Thesis, M.I.T., 1990.

[30] C. Woodward, The moment polytope of a transversal multiplicity-free group action, in preparation.

[31] C. Woodward, Multiplicity-free Hamiltonian actions need not be Kähler, June 1995 preprint, dg-ga/9506009.

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