# THE NUMBER OF WAYS OF MAKING CHANGE AND PICK'S THEOREM

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I wrote this note up for my third grade daughter, after trying to help her with a homework problem. This month, she has been learning about making change and polygons at the same time. I found it funny that the curriculum does not connect the two ideas, so I tried to write a little note explaining the connection. Not everything is at the third grade level; a little bit I wrote up for myself as well. Strangely, the formula for the number of ways of making change for some number of dollars using only coins less than a dollar is not produced by a google search; there are many explanations of how to do it but no one seems to mention that there is a simple formula, see Theorem 3 below. Still, I tried to avoid formulas as much as possible. I hope that a reader at any level would learn something new!

## 1. The number of ways of making change for fifty cents

The United States Mint produces 1, 5, 10, and 25 cent coins, called pennies, nickels, dimes and quarters; there are also half-dollar and dollar coins that are less widely used. How many ways are there of making change for fifty cents using pennies, nickels, dimes and quarters?

Let's start with an easier question: how many ways of making change for thirty cents are there using only pennies, nickels, and dimes? The number of pennies is determined by the number of nickels and dimes, because the total has to add up to thirty cents. So, it's enough to look at the number of nickels and dimes; the amount of money made from the nickels and dimes should be at most thirty cents. A table of the possibilities is in Figure 1. The table shows 16 ways. Let's plot the number of dimes and nickels, with one dot for each way. On the lowest row, there are the six ways of making change for thirty cents using zero dimes; the next row has the ways with one dime, etc.

Each dot lies in the triangle given by requiring that the total in dimes and nickels is less than thirty cents. The vertices of the triangle are the ways for making change that use only dimes, or only nickels, or only pennies.

Is there a way of finding the number of ways without making a table? Suppose that we take another triangle like the first one, and flip it over so that it makes a rectangle, as in Figure 3. This rectangle has 8 times 4 dots, that is, 32 dots. So the each triangle has 16 dots. In general the same trick shows the following:

| dimes | nickels | dimes | nickels |
|:-:|:-:|:-:|:-:|
| 0 | 0 | 1 | 1 |
| 0 | 1 | 1 | 2 |
| 0 | 2 | 1 | 3 |
| 0 | 3 | 1 | 4 |
| 0 | 4 | 2 | 2 |
| 0 | 5 | 2 | 1 |
| 0 | 6 | 2 | 0 |
| 1 | 0 | 3 | 0 |

FIGURE 1. Sixteen ways to make change for thirty cents using dimes, nickels, and pennies
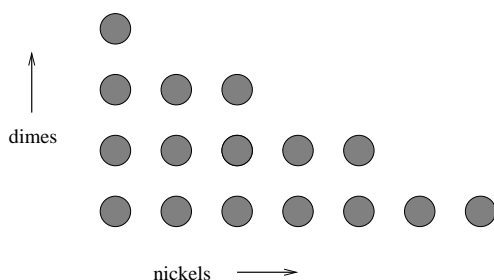
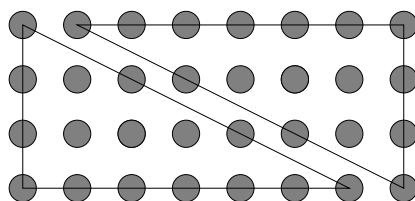

FIGURE 2. Change for thirty cents



FIGURE 3. Doubling the ways gives a rectangle

**Theorem 1.** *Suppose that some amount of money can be changed into only dimes. Then the number of ways of making change using only pennies, dimes, and nickels, is the maximum number of dimes, plus one, squared.*

For example, the number of ways of making change for fifty cents, using only pennies, nickels, and dimes, is five, plus one, squared, or 36.

**Problem:** Make a table for the number of ways of making change for twenty cents using only pennies, nickels and dimes. Graph the results with dots, like we did above. Draw the triangle that the dots fit into.

**Problem:** How many ways of making change for a dollar are there, using only pennies dimes and nickels? (Hint: use the Theorem.)

What about other amounts of money? Counting ways of making change for multiples of five is similar. For example, suppose we want to make change for 25 cents, using only dimes and nickels. The possibilities are shown below.

| dimes | nickels | dimes | nickels |
|-------|---------|-------|---------|
| 0 | 0 | 1 | 0 |
| 0 | 1 | 1 | 1 |
| 0 | 2 | 1 | 2 |
| 0 | 3 | 1 | 3 |
| 0 | 4 | 2 | 0 |
| 0 | 5 | 2 | 1 |

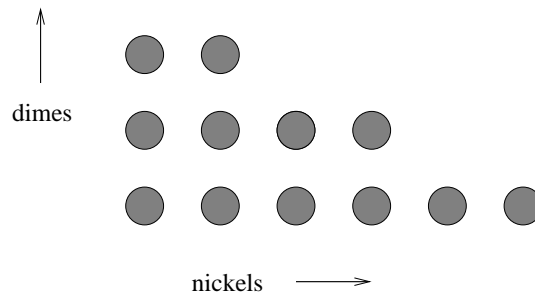Let's plot the number of dimes and nickels on a graph.



FIGURE 4. Change for twenty five cents

Notice that this graph is obtained from the graph for thirty cents by subtracting one dot for each row, see Figure 5.
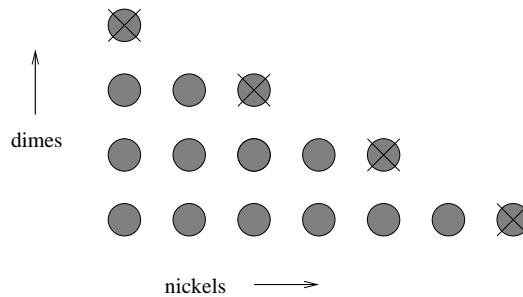


FIGURE 5. Deleting dots to get from thirty to twenty-five

The same trick works for any amount of money:

**Theorem 2.** *The amount of money needed to make change for any amount that can be made with only nickels, but not dimes, is the number of ways to make change for that amount plus five cents, minus the max number of dimes, minus one.*

In our example, we get that the number of ways of making change for twenty-five cents is the number of ways of making thirty cents minus the max number of dimes, minus one, that is, 16 minus 3 minus 1, that is, 12.

**Problem:** How many ways of making change for ninety-five cents are there, using only pennies, nickels and dimes? (Hint: you already found the number of ways to make change for a dollar.)

Let's sum up what we've found for making change for $0, 25$ and $50$ cents: for $0$, there is 1 way of making change. For 25 cents, there are 12 ways. For 50, there are 36 ways. We can now figure out how many ways of making change using quarters also. To make fifty cents, we can use either $0, 1$, or 2 quarters. After we use the quarters, there are either zero, twenty-five, or fifty cents left to make up. So there are $1 + 12 + 36 = 49$ ways of making change for fifty cents, or 50 if you are allowed to use a single half-dollar coin. It's easy from this to work out the number of ways of making change for a dollar (without making a huge table!).

## 2. The number of ways of making change for some number of half-dollars

There is a fairly simple formula for the number of ways of making change for any number of dollars, using pennies, nickels, dimes, quarters, and half-dollars. One possible way is to use the formulas we gave so far, and sum over the number of ways of using half-dollars and quarters.

**Theorem 3.** *The number $n(h)$ of ways of making change for any number $h$ of half-dollars, using pennies, nickels, dimes, quarters and half-dollars, is*

$$n(h) = (6 + 55h + 119h^2 + 95h^3 + 25h^4)/6.$$

*Proof.* There are $h+1$ ways of making $h$ half-dollars using only half-dollars and quarters, and $h + 1$ ways of making change for $2h + 1$ quarters using half-dollars and quarters. I asked Mathematica to find the answer for me, it already knows about Bernoulli numbers and there is no point in doing by hand. Putting in

$\mathtt{Sum[(h - i + 1) * (5 * i + 1)^2, i, 0, h] + Sum[(h - 1 - i + 1) * (3 + 5 * i)(4 + 5 * i), i, 0, h - 1]}$

gave the formula above.                                                                     $\square$

For example, this formula gives 292 ways of making change for a dollar using coins less than a dollar. The coefficients in formulas like this are called *Ehrhart coefficients*. I haven't looked up exactly what the denominations of bills are printed in the United States, but I assume it's a finite set. (According to Wikipedia, paper currency is not produced by the Mint but rather the Bureau of Engraving and Printing.) So there is also a formula for the number of ways of making change for $d$ dollars, using all bills and coins currently in circulation, but I haven't worked out what it is.

## 3. Pick's theorem

The formulas for the number of ways of making change above using pennies, nickels and dimes are special cases of a formula of Pick for the number of dots in any triangle whose vertices are dots; such a triangle is called a *lattice triangle*. In the example in Figure 6, there are 11 dots in the triangle. (We include dots on the boundary.)
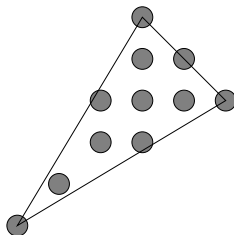
FIGURE 6. Dots in a triangle

**Theorem 4** (Pick's theorem). *The number of dots in any lattice triangle is the area plus half of the dots on the boundary plus one.*

In the example where we made change for money, the area of the triangle was $d^2$, because it was half the area of the rectangle with sides $d$ and $2d$. The number of dots on the boundary was $2d$, so the number of total dots is $d^2 + 2d + 1$ which is $(d+1)^2$ as we said before.

Pick's theorem is not hard to prove. First, you notice that the the number of dots minus half the dots on the boundary minus one, is exactly the number of "fractional dots" in the triangle, where each dot is counted with a fraction of it that lies in the triangle. The triangle in Figure 7 has five full dots, five half dots, and three less-than-half dots.
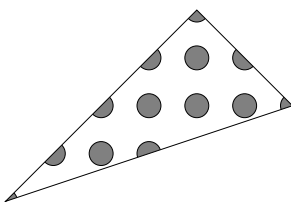
FIGURE 7. Fractional dots

(The three less-than-half dots always make up one half dot together by Euclid I.32, which is one less than three half-dots would make. We assume that the dots are sufficiently small so that if the center of the dot is completely inside the triangle, then the whole dot is.) The number of fractional dots is *additive*, that is, if a triangle can be broken up into smaller triangles, then the number of fractional dots in the bigger triangle
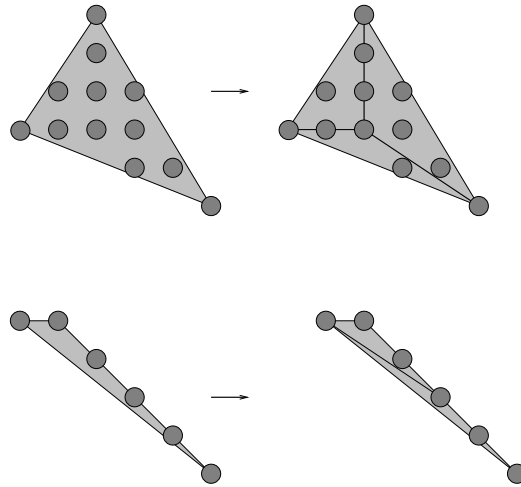
FIGURE 8. Breaking up lattice triangles into smaller ones

is the sum of number of fractional dots in the smaller triangles. Any triangle with more than three dots in it can be broken up into smaller triangles, as in the examples in Figure 8. So it suffices to show that Pick's theorem is true for lattice triangles containing just three dots, that is:

**Lemma 5.** *The area of a lattice triangle containing just three dots is one-half.*

Equivalently, the area of a parallelogram containing just four dots is one. To prove this, note that you can tile the plane using any such parallelogram, with one tile for each dot. See Figure 9.
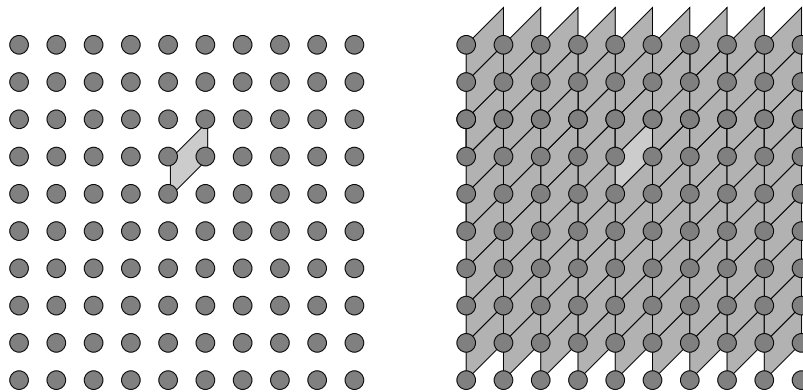


FIGURE 9. Tiling the plane with four-dot parallelograms

The number of dots, hence tiles, in a square of size $d$ is about $d^2$, which is the area, plus a correction term about the size of $d$. Since this is true for all sizes $d$, the area of each tile must be 1.

There are many generalization of Pick's theorem. For example, there is a Mordell-Pommersheim formula for the number of dots in a tetrahedron, or a formulas of Khovanskii and others for the number of dots in any simple convex polyhedron. See Brion-Vergne [1] for a fairly up-to-date explanation, and references. (Note that this article is available on the web by googling "Brion Vergne JAMS").

## References

[1] M. Brion and M. Vergne. Lattice points in simple polytopes. *J. Amer. Math. Soc.*, 10:371–392, 1997.