$6j$ SYMBOLS FOR $U_q(\mathfrak{sl}_2)$ AND NON-EUCLIDEAN TETRAHEDRA

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Abstract. We relate the semiclassical asymptotics of the $6j$ symbols for the quantized enveloping algebra $U_q(\mathfrak{sl}_2)$ at q a root of unity (resp. q real positive) to the geometry of spherical (resp. hyperbolic) tetrahedra.

1. Introduction

Let $r > 2$ be an integer, $q = \exp(\pi i/r)$ and $U_q(\mathfrak{sl}_2)$ the quantized enveloping algebra for the Lie algebra $\mathfrak{sl}_2$. The category of finite dimensional representations of $U_q(\mathfrak{sl}_2)$ has a semisimple subquotient $\mathcal{F}(U_q(\mathfrak{sl}_2))$ called the *fusion category* [4, Section 3.3]. The isomorphism classes of simple objects in $\mathcal{F}(U_q(\mathfrak{sl}_2))$ have canonical representatives V_j labelled by half-integers $j \in [0, (r-2)/2]$. Given $j_1, j_2 \in [0, (r-2)/2] \cap \mathbb{Z}/2$, the tensor product $V_{j_1} \otimes V_{j_2}$ is isomorphic to the direct sum of objects V_{j_3} where the sum is over j_3 satisfying the *quantum Clebsch-Gordan inequalities*

$$\max(j_1 - j_2, j_2 - j_1) \leq j_3 \leq \min(j_1 + j_2, r - 2 - j_1 - j_2) \tag{1}$$

and the parity condition $j_1 + j_2 + j_3 \in \mathbb{Z}$. Geometrically, the condition (1) means that there exists a triangle in the unit sphere with edge lengths $j_a 2\pi/(r-2), a = 1, 2, 3$. Generalizations of these inequalities to Lie algebras of higher rank are described in [1],[7],[6],[37].

This paper concerns a generalization of this relationship in a different direction, namely from triangles to tetrahedra. The *quantum* $6j$ *symbol* is a function of a 6-tuple $j_{ab}, 1 \leq a \leq b \leq 4$, defined as follows: Since the tensor product of simple modules is multiplicity-free there is a projective basis for $\mathrm{Hom}_{U_q(\mathfrak{sl}_2)}(V_{j_{14}}, V_{j_{12}} \otimes V_{j_{23}} \otimes V_{j_{34}})$ associated to each way of parenthesizing, parameterized by half-integers j_{13} resp. j_{24}. The $6j$ symbols $\left\{ \begin{array}{ccc} j_{12} & j_{23} & j_{13} \\ j_{34} & j_{14} & j_{24} \end{array} \right\}$ are the coefficients in the change of basis matrix. $6j$ symbols for $q = 1$ were introduced as a tool in atomic spectroscopy by Racah [33], and then studied mathematically by Wigner [42]. $6j$ symbols for $U_q(\mathfrak{sl}_2)$ were introduced by Kirillov and Reshetikhin [22], who used them to generalize the Jones knot invariant. Turaev and Viro used them to define three-manifold invariants [39], or what physicists call quantum gravity with cosmological

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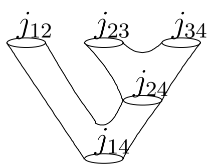
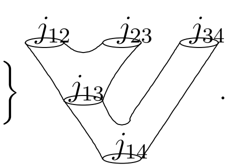
constant [5]. Roughly speaking these invariants are obtained by combinatorial integration of a product of $6j$ symbols: for the Turaev-Viro invariant, a $6j$ symbol is attached to each tetrahedron in a triangulation, while in the Jones invariant a $6j$ symbol is attached to each crossing. Various outstanding conjectures concern the asymptotics of these invariants: The Witten conjecture relates the asymptotics of the Turaev-Viro invariant (the norm-square of the Reshetikhin-Turaev invariant) with Chern-Simons invariants of flat $SO(4)$ bundles [14] while the volume conjecture of Murakami-Murakami relates the asymptotics of the colored Jones polynomial with the hyperbolic volume of the knot complement [30]. A natural question is whether non-Euclidean geometry appears in the asymptotics of the $6j$ symbol; we will show that this is indeed the case.

The connection of the $6j$ symbols to geometry arises as follows. By a theorem of Finkelberg [13], $\mathcal{F}(U_q(\mathfrak{sl}_2))$ is isomorphic to the tensor category of level $r-2$ representations of the affine Lie algebra $\widehat{\mathfrak{sl}_2}$. The product for the latter category uses as its definition the space of genus zero conformal blocks for Wess-Zumino-Witten (WZW) conformal field theory. A picture is perhaps the best way of getting across the idea of the role the quantum $6j$ symbols play in WZW:

(2) 

$= \sum_{j_{13}} \left\{ \begin{array}{ccc} j_{12} & j_{23} & j_{13} \\ j_{34} & j_{14} & j_{24} \end{array} \right\}$ 

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The vector space of conformal blocks is isomorphic to the space of holomorphic sections of the determinant line bundle on the moduli space of flat $SU(2)$-bundles, see Pauly [31]. The $6j$ symbol is equal to the Hermitian pairing between two holomorphic sections. According to geometric quantization, the sections should be interpreted as quantum[1]-mechanical states, and one should try to show that in the semiclassical limit the states concentrate to Lagrangian submanifolds. Assuming this holds, the leading term in the asymptotics of the pairing is a sum over intersection points of the Lagrangians, involving Poisson brackets of the functions defining the Lagrangians and the pairing between the sections in the fiber of the determinant line bundle, see Borthwick-Paul-Uribe [9]. The particular case of interest is the moduli space of flat $SU(2)$ bundles on the four-holed two-sphere, with holonomies around the four boundary components fixed by the labels $j_{12}, j_{23}, j_{34}, j_{14}$. The Lagrangians consists of flat bundles for which the holonomy around the intermediate circle correspond to j_{13}, j_{24} respectively, as in (2). Using the diffeomorphism $SU(2) \to S^3$, the intersection points correspond to spherical tetrahedra with edge lengths $l_{ab} := 2\pi j_{ab}/(r-2)$. The Poisson bracket turns out to be the determinant

[1]In this subject the term *quantum* is used rather confusingly in two senses: $q \neq 1$ and $\hbar \neq 0$. From our point of view, $-\ln(q)^2$ describes the curvature or, in physics language, the cosmological constant. The true quantum parameter is $\hbar$, which appears in the guise of $1/k$ where quantum states are sections of the k-th tensor power of the determinant bundle.

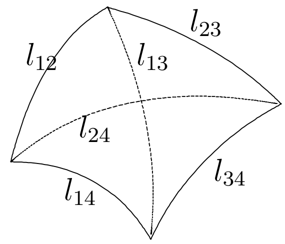


FIGURE 1. The spherical tetrahedon τ

of the Gram matrix of the tetrahedron, and the phase shift is the area of a holomorphic disk in the moduli space which can be computed using Schläfli's formula. These ideas lead to the conjecture that if τ is non-degenerate then

$$(3) \qquad \left\{ \begin{array}{ccc} kj_{12} & kj_{23} & kj_{13} \\ kj_{34} & kj_{14} & kj_{24} \end{array} \right\}_{q=\exp(\pi i/r(k))} \sim \frac{2\pi \cos(\phi(k)+\pi/4)}{r(k)^{3/2} \det(\cos(l_{ab}))^{1/4}}$$

as $k \to \infty$ where $r(k) = k(r-2)+2$,

$$\phi(k) = \frac{r(k)}{2\pi}\left(\sum_{a<b} \theta_{ab}(k) l_{ab}(k) - 2\,\mathrm{Vol}(\tau(k))\right)$$

$\tau(k)$ is the spherical tetrahedron with edge lengths

$$l_{ab}(k) = 2\pi \frac{kj_{ab}+\frac{1}{2}}{r(k)},$$

and $\theta_{ab}(k)$ are its exterior dihedral angles. The formula (3) generalizes one for the case $q=1$ given by the physicists Wigner [42, p.356], Ponzano and Regge [32], and proved by J. Roberts [34], see also [11],[41]. The numerator of the formula was conjectured by Mizoguchi and Tada [27]. See also Roberts' discussion [35].

Our proof of (3) is not based on the above ideas. Instead, we show (following Schulten-Gordon [36] and [27]) that both sides satisfy a second order difference equation as one label is varied. It follows that each side is a linear combination of the two linearly independent solutions to the equation. Taking the Euclidean limit and applying Roberts' theorem [34] shows equality of the coefficients. This method, while less geometric, has the advantage that it works for non-integral r and gives a formula for degenerate tetrahedra with non-degenerate faces. In fact there are seven different cases depending on the type of degeneracy of the tetrahedron τ; these are pictured in Figure 2. The second column, titled Maslov diagram, shows the geometry of the relevant Lagrangian submanifolds (or, for the more degenerate cases, isotropic submanifolds). The possible degenerations in the table are

$$(a) \to (b) \to (c) \to ((d) \text{ or } (e)) \to (f) \to (g).$$

We refer to case (a) as the non-degenerate case, and case (b) as the tangent case. There is an overlap between cases (d) and (e), when the vertices are colinear but

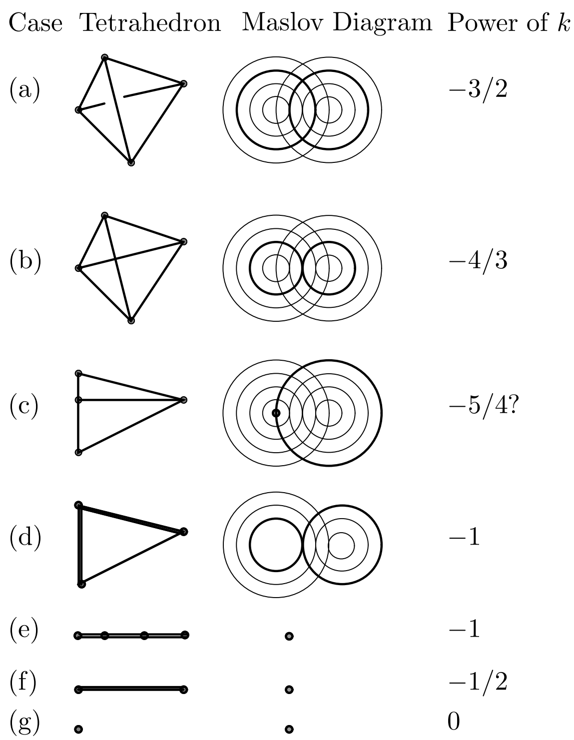


FIGURE 2. Seven types of tetrahedra and the corresponding Lagrangians

one edge length vanishes; perhaps cases (d) and (e) should not really be considered separate. We managed to cover them almost completely. The exception is the case (c) that exactly one face is degenerate: although one can compute the asymptotics, we did not find a simple formula or manage to construct non-trivial cases where the configuration occurs. We also failed to cover the eighth "classically forbidden" case that the tetrahedron does not exist, where one expects (and numerically experiments show) exponential decay of the $6j$ symbols. The main theorem (case (c) is intentionally missing) follows:

Theorem 1.0.1. *Let* $r > 2, j_{12}, \ldots, j_{34} \in [0, (r-2)/2] \cap \mathbb{Z}/2$ *and*

$$s(k) := \left\{ \begin{array}{ccc} kj_{12} & kj_{23} & kj_{13} \\ kj_{34} & kj_{14} & kj_{24} \end{array} \right\}_{q=\exp(\pi i/r(k))}.$$

(a) *If* τ *exists and is non-degenerate, then*

$$s(k) \sim \frac{2\pi \cos(\phi(k) + \pi/4)}{r(k)^{3/2} \det(\cos(l_{ab}))^{1/4}}.$$

(b) *If τ exists, is degenerate and all faces are non-degenerate then*

$$s(k) \sim r(k)^{-\frac{4}{3}} 2^{\frac{2}{3}} 3^{-\frac{1}{3}} \pi^{4/3} \Gamma\left(\frac{2}{3}\right)^{-1} \frac{\cos(k\sum\theta_{ab} j_{ab})}{(A_1A_2A_3A_4)^{1/6}}$$

where $A_a = \det(\cos(l_{bc})_{b,c\neq a})^{1/2}$ and $\cos(k\sum\theta_{ab}j_{ab}) = \pm 1$ is a sign depending on the sum of the j_{ab}'s for which $\theta_{ab} = \pi$.

(d) *If τ exists and has exactly one edge length, say l_{ab} vanishing, then*

$$s(k) \sim (-1)^{k(j_{bc}+j_{cd}+j_{bd})} \pi r(k)^{-1} (\sin(l_{ac})\sin(l_{bd}))^{-1/2}.$$

Note that τ need not have any non-degenerate faces.

(e) *If τ exists, all faces are degenerate, the vertices lie on a geodesic of length at most π in order a, b, c, d and the edge lengths l_{ac}, l_{bd} are non-vanishing, then*

$$s(k) \sim (-1)^{2kj_{ad}} \pi r(k)^{-1} (\sin(l_{ac})\sin(l_{bd}))^{-1/2}.$$

Note the edge length l_{bc} may vanish.

(f) *If τ exists, all faces are degenerate, either*

(i) *all non-zero edge lengths are equal and less than π, or*

(ii) *one edge length is zero and the opposite edge length is π*

and $l_{ab} \in (0, \pi)$ then

$$s(k) \sim (-1)^{2kj_{ab}} \pi^{1/2} r(k)^{-1/2} \sin(l_{ab})^{-1/2}.$$

(g) *If τ exists, but all edge lengths are 0 or π then $s(k) = 1$.*

Some of these formulas hold for slightly more general sequences of labels, see Theorems 4.5.2,5.0.4 below. Most of the results were checked numerically. In Figure 3 we show a numerical comparison generated by Maple between the $6j$ symbols and the asymptotic formula Theorem 1.0.1 (a) for the sequence of $6j$ symbols $\left\{\begin{array}{ccc} 40 & 48 & 50 \\ 52 & 54 & j \end{array}\right\}$ as j varies from 0 to 108, with $r = 200$.[2] The phase function ϕ was computed by numerical integration, using Schläfli's formula 2.4.1 (m). The amplitudes $\pm 2\pi r(k)^{-\frac{3}{2}} \det(\cos(l_{ab}))^{-\frac{1}{4}}$ are also shown , as well as the functions from case (b) of the main theorem governing the degenerate limits . The code used to generate the graph is available at http://www.math.rutgers.edu/∼ctw/6j.html.

An outline of the paper is as follows. Section 2 contains background. The conjecture on the geometry of conformal blocks in the semiclassical limit is explained in Section 3. Sections 4 to 7 contain a proof of the main theorem. Section 8 contains a short discussion of the classically forbidden case. The results for the hyperbolic case, that is, the case q real and positive, are stated without proof in Section 9. Various questions we were not able to resolve are listed in Section 10.

Acknowledgements The project was suggested by J. Roberts at the end of [34], and we thank him for his encouragement. Discussions with L. Rozansky, F. Luo, I. Korepanov, M. Leingang, M. McDuffee and I. Rivin were also helpful. The reference [36] and the idea of using recursion to prove the formulas were pointed

[2]For the purposes of Maple, dominant weights were labelled by non-negative *integers*.

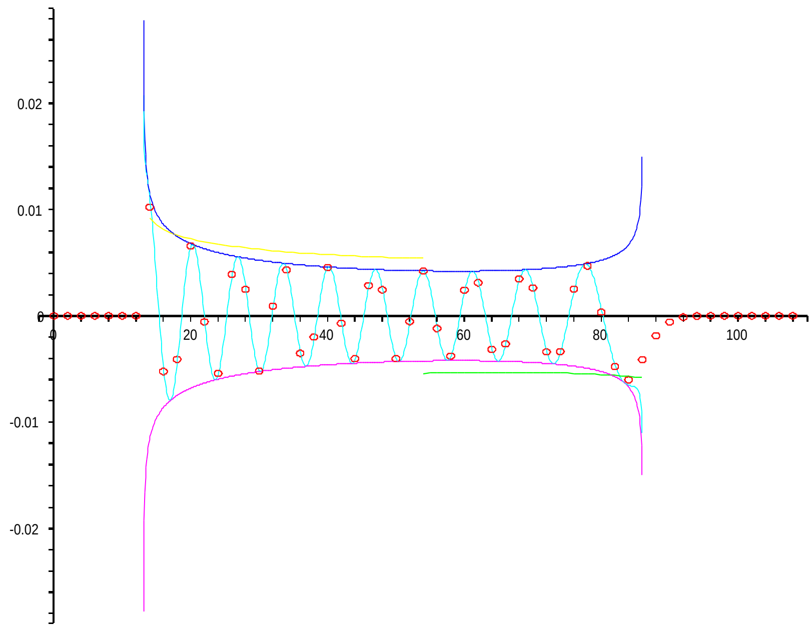


FIGURE 3. Quantum $6j$ symbols versus the asymptotic formulas

out to us by N. Reshetikhin. Related work on the asymptotics of Turaev-Viro has been carried out by Frohman and Kania-Bartoszynska [15] and J. Murakami, unpublished.

2. BACKGROUND

2.1. The quantum group $U_q(\mathfrak{sl}_2)$. The quantum group $U_q(\mathfrak{sl}_2)$ was introduced by Kulish and Reshetikhin in [23]. Let q be a complex number not equal to $0, 1$ or -1. For any integer n, the *quantum integer* $[n]$ is defined by

$$[n] = \frac{q^n - q^{-n}}{q - q^{-1}}.$$

Note that $[n] \to n$ as $q \to 1$, and also

$$[n] = \frac{\sin(n\pi/r)}{\sin(\pi/r)} \qquad \text{if } q = \exp(\pi i/r).$$

Let $\mathfrak{sl}_2$ denote the Lie algebra of 2×2 traceless matrices, with basis E, F, H and relations $[H,E]=E,\ \ [H,F]=-F,\ \ [E,F]=2H$. The irreducible representations of $\mathfrak{sl}_2$ are $V_j,\ j=0,\frac{1}{2},1,\ldots,$ where V_j is the space of degree $2j$ homogeneous polynomials in two variables z, w. A basis for V_j is given by $v_{j,j}, v_{j,j-1},\ldots,v_{j,-j}$ where the *weight vector* $v_{j,m}$ is defined by

$$v_{j,m}=z^a w^b,\quad a-b=2m,\quad a+b=2j.$$

The action of $\mathfrak{sl}_2$ in this basis is given by

$$Ev_{j,m}=(j-m)v_{j,m+1},\quad Fv_{j,m}=(j+m)v_{j,m-1},\quad Hv_{j,m}=mv_{j,m}.$$

The quantum group $U_q(\mathfrak{sl}_2)$ is the algebra with generators E, F, K, K^{-1} and relations

$$[E,F]=\frac{K^2-K^{-2}}{q-q^{-1}},\quad KE=qEK,\quad KF=q^{-1}FK.$$

The action of $U_q(\mathfrak{sl}_2)$ on V_j is defined by

$$Ev_{j,m}=[j-m]v_{j,m+1},\quad Fv_{j,m}=[j+m]v_{j,m-1},\quad Kv_{j,m}=q^m v_{j,m}.$$

The coproduct on $U_q(\mathfrak{sl}_2)$ defined by

$$E\mapsto E\otimes K+K^{-1}\otimes E,\quad F\mapsto F\otimes K+K^{-1}\otimes F,\quad K^{\pm1}\mapsto K^{\pm1}\otimes K^{\pm1}$$

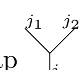
makes the category $\mathcal{R}(U_q(\mathfrak{sl}_2))$ of finite dimensional $U_q(\mathfrak{sl}_2)$ modules a tensor category. $\mathcal{R}(U_q(\mathfrak{sl}_2))$ has a semisimple subquotient $\mathcal{F}(U_q(\mathfrak{sl}_2))$ called the fusion category for $U_q(\mathfrak{sl}_2)$ [4, Section 3.3]. Any simple object in $\mathcal{F}(U_q(\mathfrak{sl}_2))$ is isomorphic to V_j for some j satisfying

$$j=\begin{cases}0,1/2,1,\ldots & \text{if } q \text{ is not a root of unity ;}\\ 0,1/2,1,\ldots,(r-2)/2 & \text{if } q=\exp(\frac{\pi i}{r})\ .\end{cases}$$

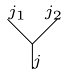
If $q=\exp(\pi i/r)$ the branching rule for the product is

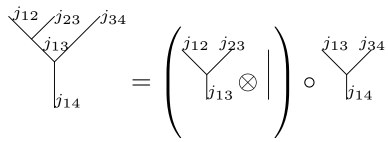
$$V_{j_1}\otimes V_{j_2}=\bigoplus_j V_j$$

where j satisfies the quantum Clebsch-Gordan rules (1). If q is not a root of unity, the branching rule is that for representations of $\mathfrak{sl}_2$.

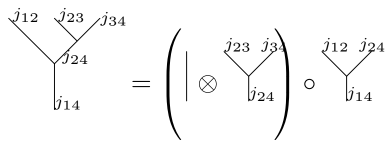
2.2. **The 6j symbols.** 6j symbols for $\mathfrak{sl}_2$ were introduced as a tool in atomic spectroscopy by Racah [33], and then studied mathematically by Wigner; see [42, Chapter 24]. 6j symbols for $U_q(\mathfrak{sl}_2)$ were introduced by Kirillov and Reshetikhin [22], who used them to generalize the Jones knot invariant. The material below can be found in the book by Carter, Flath, and Saito [10]. If $\mathrm{Hom}_{U_q(\mathfrak{sl}_2)}(V_j, V_{j_1}\otimes V_{j_2})$ is non-trivial we fix as a generator the map 

defined in [10, p.96]. (A small difference is that in [10] the map is from $V_{\frac{1}{2}}^{\otimes j}$ to $V_{\frac{1}{2}}^{\otimes j_1}\otimes V_{\frac{1}{2}}^{\otimes j_2}$.) If $\mathrm{Hom}_{U_q(\mathfrak{sl}_2)}(V_j, V_{j_1}\otimes V_{j_2})$

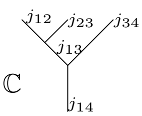
is trivial, define  $= 0$. Let $j_{ab}, 1 \le a < b \le 4$ be non-negative half-integers. Define



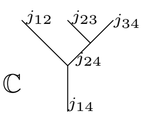
and



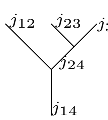
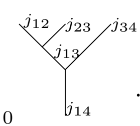
where $|$ denotes the identity. By associativity of the tensor product we have a basis for $\mathrm{Hom}_{U_q(\mathfrak{sl}_2)}(V_{j_{14}}, V_{j_{12}} \otimes V_{j_{23}} \otimes V_{j_{34}})$ for each way of parenthesizing:

$$\mathrm{Hom}_{U_q(\mathfrak{sl}_2)}(V_{j_{14}}, (V_{j_{12}} \otimes V_{j_{23}}) \otimes V_{j_{34}}) = \bigoplus_{j_{13}} \mathbb{C}$$ 

and

$$\mathrm{Hom}_{U_q(\mathfrak{sl}_2)}(V_{j_14}, V_{j_{12}} \otimes (V_{j_{23}} \otimes V_{j_{34}})) = \bigoplus_{j_{24}} \mathbb{C}$$ 

where the sum is over j_{13} resp. j_{24} satisfying the quantum Clebsch-Gordan inequalities for each vertex. The $6j$ symbol $\left\{ \begin{array}{ccc} j_{12} & j_{23} & j_{13} \\ j_{34} & j_{14} & j_{24} \end{array} \right\}_0$ is defined by

 $$= \sum_{j_{13}} \left\{ \begin{array}{ccc} j_{12} & j_{23} & j_{13} \\ j_{34} & j_{14} & j_{24} \end{array} \right\}_0$$  .

Assume that q is real or $|q| = 1$, so that the quantum integers are real. One can modify the definition slightly so that the symbols have tetrahedral symmetry:

$$\left\{ \begin{array}{ccc} j_{12} & j_{23} & j_{13} \\ j_{34} & j_{14} & j_{24} \end{array} \right\} = \frac{(-1)^{j_{12}+j_{23}+j_{34}+j_{14}}}{[2j_{13}+1]} \sqrt{\left| \frac{\theta(123)\theta(134)}{\theta(234)\theta(124)} \right|} \left\{ \begin{array}{ccc} j_{12} & j_{23} & j_{13} \\ j_{34} & j_{14} & j_{24} \end{array} \right\}_0$$

where

$$\theta(abc) = \frac{[j_{ab} + j_{bc} - j_{ac}]![j_{ab} - j_{bc} + j_{ac}]![-j_{ab} + j_{bc} + j_{ac}]![j_{ab} + j_{bc} + j_{ac} + 1]!}{(-1)^{j_{ab}+j_{bc}+j_{ac}}[2j_{ab}]![2j_{bc}]![2j_{ac}]!}$$

and the quantum factorial is defined by

$$[n]! = [n][n-1]\dots[1]. \tag{4}$$

That is,

$$\begin{Bmatrix} j_{12} & j_{23} & j_{13} \\ j_{34} & j_{14} & j_{24} \end{Bmatrix} = \begin{Bmatrix} j_{\sigma(1)\sigma(2)} & j_{\sigma(2)\sigma(3)} & j_{\sigma(1)\sigma(3)} \\ j_{\sigma(3)\sigma(4)} & j_{\sigma(1)\sigma(4)} & j_{\sigma(2)\sigma(4)} \end{Bmatrix}$$

for any $\sigma \in S_4$. Equivalently, the $6j$ symbol is invariant under the exchange of any two columns, and any two entries in the first row with the corresponding entries in the second. A word on sign conventions: In the case q arbitrary one defines the $6j$ symbols without taking the absolute value of the θ values. This leads to a difference of sign $(-1)^{\sum_{a<b} j_{ab}}$ from our convention, which was chosen because it agrees with the usual definition of classical $6j$ symbols. For readers familiar with spin networks, the $6j$ symbol is the value of a tetrahedral graph with labels j_{ab}, divided by the square root of the absolute value of the product of the theta graphs for the faces. This graph is *dual* to the edge graph of the geometric tetrahedron, that is, edges j_{ab}, j_{bc}, j_{ac} are joined at a vertex, for any $1 \le a < b < c \le 4$.

The $6j$ symbols satisfy the *pentagon* or *Biedenharn-Elliot relation*

$$\begin{Bmatrix} j_{23} & j_{34} & j_{24} \\ j_{14} & j_{12} & j_{13} \end{Bmatrix} \begin{Bmatrix} j_{23} & j_{34} & j_{24} \\ j_{45} & j_{25} & j_{35} \end{Bmatrix} = \tag{5}$$

$$\sum_{j_{15}} (-1)^z [2j_{15}+1] \begin{Bmatrix} j_{13} & j_{34} & j_{14} \\ j_{45} & j_{15} & j_{35} \end{Bmatrix} \begin{Bmatrix} j_{12} & j_{24} & j_{14} \\ j_{45} & j_{15} & j_{25} \end{Bmatrix} \begin{Bmatrix} j_{12} & j_{23} & j_{13} \\ j_{35} & j_{15} & j_{25} \end{Bmatrix}$$

where $z = j_{12}+j_{13}+j_{14}+j_{15}+j_{23}+j_{24}+j_{25}+j_{34}+j_{35}+j_{45}$. This identity is obtained by applying the definition of the $6j$ symbols to Figure 4.

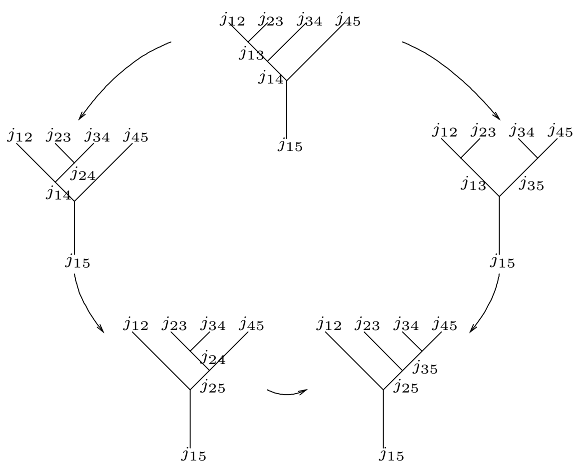


FIGURE 4. The pentagon identity

Racah gave a hypergeometric formula whose generalization to arbitrary q is

$$\left\{ \begin{array}{ccc} j_{12} & j_{23} & j_{13} \\ j_{34} & j_{14} & j_{24} \end{array} \right\} = \Delta(123)\Delta(134)\Delta(234)\Delta(124) \sum (-1)^z [z+1]! f(z)^{-1} \tag{6}$$

where

$$\begin{aligned} f(z) = {} & [z - j_{12} - j_{23} - j_{13}]![z - j_{13} - j_{34} - j_{14}]![z - j_{23} - j_{34} - j_{24}]! \\ & [z - j_{12} - j_{24} - j_{14}]![j_{12} + j_{23} + j_{34} + j_{14} - z]![j_{12} + j_{13} + j_{34} + j_{24} - z]! \\ & [j_{23} + j_{13} + j_{14} + j_{24} - z]!, \end{aligned} \tag{7}$$

$$\Delta(abc) = \left(\frac{[j_{ab} + j_{bc} - j_{ac}]![j_{ab} - j_{bc} + j_{ac}]![-j_{ab} + j_{bc} + j_{ac}]!}{[j_{ab} + j_{bc} + j_{ac} + 1]!} \right)^{1/2}$$

and the sum is over integers z such that the factorials in $f(z)$ are defined. Because a large amount of cancellation occurs, (6) is useful for studying asymptotics only in special cases.

2.3. **Asymptotics for $q = 1$.** The asymptotic values of the $6j$ symbols for $q = 1$ were considered by Wigner [42, p.355]. Consider a system of particles A,B,C with total angular momenta j_{12}, j_{23}, j_{34}. Given that the total angular momentum of the combined AB system is j_{13}, and the total angular momentum of the combined ABC system is j_{14}, the probability of measuring j_{24} for the total angular momentum of the combined BC system is the square of the $6j$ symbol. Wigner computed the classical probability and arrived at the conjecture

$$\left\{ \begin{array}{ccc} j_{12} & j_{23} & j_{13} \\ j_{34} & j_{14} & j_{24} \end{array} \right\}^2 \approx \frac{1}{24\pi \operatorname{Vol}(\tau)}$$

where τ is the Euclidean tetrahedron with lengths $l_{ab} = j_{ab} + \frac{1}{2}$, if it exists, and the approximation is meant to hold after averaging out local oscillations. The Wigner conjecture was refined by Ponzano and Regge [32] as follows. Let θ_{ab} denote the exterior dihedral angles of τ, and

$$\phi = \sum_{a<b} l_{ab} \theta_{ab}.$$

Ponzano and Regge conjectured in the semiclassical limit

$$\left\{ \begin{array}{ccc} j_{12} & j_{23} & j_{13} \\ j_{34} & j_{14} & j_{24} \end{array} \right\} \sim \frac{\cos(\phi + \pi/4)}{\sqrt{12\pi \operatorname{Vol}(\tau)}} \tag{8}$$

assuming that τ exists and is non-degenerate, and also formulas for degenerate tetrahedra with non-degenerate faces and the classically forbidden case. The Ponzano-Regge conjectures were further refined by Schulten and Gordon [36], see Section 4 below. The mathematical meaning of the semiclassical limit is not entirely clear. An obvious interpretation (but perhaps not the only one) is the asymptotics of the sequence $\left\{ \begin{array}{ccc} kj_{12} & kj_{23} & kj_{13} \\ kj_{34} & kj_{14} & kj_{24} \end{array} \right\}$, $k \to \infty$. In this setting the Ponzano-Regge formula (8) has recently been proved by J. Roberts [34], see also [11],[41]. The conjectures for the other cases follow from the recursion in [36] and its mathematical justification by Geronimo, Bruno and Assche [18, Theorem 3.8]. The remaining cases (c)-(f) can be handled using Stirling's formula applied to (6), see also Sections 6 and 7 below.

2.4. **Non-Euclidean tetrahedra.** Let E^n, S^n, H^n denote Euclidean, spherical, or hyperbolic n-space respectively. Let σ be an n-simplex in E^n, S^n or H^n. Let $l_{ab} = l_{ba},\ 0 \leq a, b \leq n$ denote the lengths of the edges, and l the matrix with coefficients l_{ab}. The study of which edge lengths occur is called *distance geometry* in Blumenthal [8]. Let $G_0(l)$ (also denoted $G_0(\sigma)$ denote the *Cayley-Menger* $(n+2)\times(n+2)$-matrix (actually a slight modification) written in block-diagonal form

$$G_0(l) = \begin{bmatrix} 0 & 1 & 1 & 1 & \dots & 1 \\ 1 & 0 & -\frac{1}{2}l_{01}^2 & -\frac{1}{2}l_{02}^2 & \dots & -\frac{1}{2}l_{0n}^2 \\ 1 & -\frac{1}{2}l_{01}^2 & 0 & -\frac{1}{2}l_{12}^2 & \dots & -\frac{1}{2}l_{1n}^2 \\ \vdots & \vdots & \vdots & & & \vdots \\ 1 & -\frac{1}{2}l_{0n}^2 & -\frac{1}{2}l_{1n}^2 & & \dots & 0 \end{bmatrix}. \tag{9}$$

For a spherical or hyperbolic simplex σ we define the *Gram matrices* $G_\pm(l)$ (also denoted $G_\pm(\sigma)$) by

$$G_+(l)_{ab} = \cos(l_{ab}), \quad G_-(l)_{ab} = -\cosh(l_{ab})$$

and $|G_\pm(l)|$ its determinant. We will need later the following facts:

Proposition 2.4.1. (a) *If an n-simplex σ with edge lengths l_{ab} exists in E^n, then σ has volume given by $(n!\,\mathrm{Vol}(\sigma))^2 = -|G_0(l)|$.*

(b) *An n-simplex σ with edge lengths l_{ab} exists in E^n if and only if all principal minors in $G_0(l)$ containing the 11-entry are non-positive. In this case, σ is non-degenerate if and only if $|G_0(l)| < 0$.*

(c) *An n-simplex with edge lengths l_{ab} exists in S^n if and only if $G_+(l)$ is positive semidefinite. In this case, the simplex is non-degenerate if and only if $|G_+(l)| > 0$.*

(d) *An n-simplex with edge lengths l_{ab} exists in H^n if and only if all principal minors of $G_-(l)$ are non-positive. In this case, the simplex is non-degenerate if and only if $|G_-(l)| < 0$.*

(e) *If a tetrahedron τ with edge lengths l_{ab} exists, it is unique up to isometry (not necessarily orientation-preserving) in E^n resp. S^n, H^n. If τ is non-degenerate, there are two equivalence classes of tetrahedra up to orientation-preserving isometry, called* mirror tetrahedra.

(f) *In the case $n = 1$ we have*

$$|G_0(l)| = -l_{01}^2, \quad |G_+(l)| = \sin^2(l_{01}), \quad |G_-(l)| = -\sinh^2(l_{01}).$$

(g) *In the case $n = 2$, there are factorizations,*

$$|G_0(l)| = -4s(s-l_{01})(s-l_{12})(s-l_{02})$$
$$|G_+(l)| = 4\sin(s)\sin(s-l_{01})\sin(s-l_{12})\sin(s-l_{02})$$
$$|G_-(l)| = -4\sinh(s)\sinh(s-l_{01})\sinh(s-l_{12})\sinh(s-l_{02})$$

where $s = (l_{01} + l_{12} + l_{02})/2$ is the semiperimeter. (These quantities appear in the numerators of the Heron area formulas.)

(h) *A Euclidean, resp. spherical, resp. hyperbolic triangle exists if and only if* (10)*, resp.* (11) *and* (10)*, resp.* (10) *hold:*

$$l_{ac} \le l_{ab} + l_{bc}, \quad l_{ab} \le l_{ac} + l_{bc}, \quad l_{bc} \le l_{ab} + l_{ac}. \tag{10}$$

$$l_{ab} + l_{bc} + l_{ac} \le 2\pi. \tag{11}$$

(i) *In the case $n = 3$, a tetrahedron τ with edge lengths l_{ab} exists in E^3 resp. S^3, H^3 if and only if l_{ab} satisfy the triangle inequalities for each face and*

$$\det G_0(l) \le 0, \ \textit{resp.} \det G_+(l) \ge 0, \quad \textit{resp.} \det G_-(l) \le 0.$$

(j) *In the limit of small lengths, the determinants of $G_\pm$ approach the determinant of the Cayley-Menger matrix, that is,*

$$\lim_{\epsilon \to 0} \frac{\det G_\pm(\epsilon l)}{\epsilon^{2n}} = \mp \det G_0(l).$$

(k) *A spherical, resp. hyperbolic n-simplex with edge lengths l exists if and only if $0 \le l \le \arccos(-1/n)$, resp $0 \le l$.*

(l) *The volumes of non-degenerate n-simplices σ in E^n, S^n or H^n satisfy an identity due to Schläfli*

$$(n-1)\kappa \mathrm{d}\,\mathrm{Vol}(\sigma) = \sum \mathrm{Area}(F) \mathrm{d}\theta_F$$

where the sum is over $(n-2)$-dimensional faces F of the simplex σ, θ_F is the exterior dihedral angle, and $\kappa = 0, 1, -1$ resp. is the curvature.

(m) *For non-degenerate tetrahedra τ in $S^3, H^3, or E^3$,*

$$2\kappa \mathrm{d}\,\mathrm{Vol}(\tau(s)) = \sum_{a<b} l_{ab} \mathrm{d}\theta_{ab}.$$

(n) *The derivative of dihedral angle with respect to opposite edge length is*

$$\left(\frac{\partial \theta_{ab}}{\partial l_{cd}}\right)^{-1} = \left\{ \begin{array}{ll} -||G_0(l)||^{-1/2} l_{cd} l_{ab} & \textit{in the Euclidean case} \\ -|G_+(l)|^{-1/2} \sin(l_{cd}) \sin(l_{ab}) & \textit{in the spherical case} \\ -||G_-(l)||^{-1/2} \sinh(l_{cd}) \sinh(l_{ab}) & \textit{in the hyperbolic case} \end{array} \right\}$$

Proof. (a-d) are all discussed in [8]. Note that the non-degenerate cases of (c,d) can be treated uniformly using a theorem of Jacobi [17, p.303], as was pointed out to us by F. Luo. The case of degenerate hyperbolic simplices is slightly problematic, and we do not know a reference; it can be proved by induction on n. We will not use this case in the paper. (e-j) are all straight-forward. (k) follows from the fact that for all edge lengths equal to l we have

$$|G_+(l)| = (1-\cos(l))^{n-1}(1+n\cos(l)), \ \ |G_-(l)| = -(\cosh(l)-1)^{n-1}(n\cosh(l)+1).$$

For (l),(m) see [26, p. 281]. We remark that a formula for the volume of a hyperbolic tetrahedron in terms of its edge lengths is given in Murakami-Ushijima [24], see also Mohanty [29]. (n) $||G_0(l)||$ refers to the absolute value of the determinant. The Euclidean case is a computation of Wigner [42, p.355]. We prove the claim for the spherical case. Let $V_1, \ldots, V_4$ be matrices in $SU(2) \cong S^3$, and $E_{ij} = V_i V_j^{-1}$. We may assume

$$V_a = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad V_b = \begin{bmatrix} e^{il_{ab}} & 0 \\ 0 & e^{-il_{ab}} \end{bmatrix}.$$

Let $D_\theta = \operatorname{diag}(\exp(i\theta/2), \exp(-i\theta/2))$. Rotation around E_{ab} by angle θ corresponds to conjugation by D_θ. We compute

$$\frac{\mathrm{d}}{\mathrm{d}\theta}|_{\theta=0} \operatorname{Tr}(D_\theta V_c D_\theta^{-1} V_d^{-1}) = -2 \det \begin{bmatrix} V_c^3 & V_d^3 \\ V_c^4 & V_d^4 \end{bmatrix}$$

where $V_i^j, i, j = 1, 2, 3, 4$ are the components of V_i as a vector in $\mathbb{R}^4$, using the diffeomorphism $SU(2) \to S^3$. Let V be the matrix with coefficients V_i^j. Since

$$(V^T V)_{ij} = V_i \cdot V_j = \cos(l_{ij}) = G_+(l)_{ij}$$

we have $\det(V)^2 = \det(G_+(l))$. Since V is in block-diagonal form,

$$\det(V) = \det \begin{bmatrix} V_a^1 & V_b^1 \\ V_a^2 & V_b^2 \end{bmatrix} \det \begin{bmatrix} V_c^3 & V_d^3 \\ V_c^4 & V_d^4 \end{bmatrix} \tag{12}$$

$$= -\begin{bmatrix} 1 & \cos(l_{ab}) \\ 0 & \sin(l_{ab}) \end{bmatrix} \frac{1}{2} \frac{\mathrm{d} \operatorname{Tr}(E_{cd}))}{\mathrm{d}\theta} \tag{13}$$

$$= -\sin(l_{ab}) \sin(l_{cd}) \left(\frac{\partial \theta_{ab}}{\partial l_{cd}} \right)^{-1} \tag{14}$$

which proves the spherical part of the lemma. The hyperbolic case is similar, using singular values instead of eigenvalues. □

The following gives a particularly useful formula for the dihedral angle of a spherical tetrahedron. Let $v_0, \ldots, v_n$ in $\mathbb{R}^n$. For any $\{i_1, \ldots, i_k\} \subset \{1, \ldots, n\}$ we denote by $\sigma(i_1, i_2, \ldots, i_k)$ the $n-k$ simplex spanned by v_j for $j \neq i_1, \ldots, i_k$.

Lemma 2.4.2. *For $a, b \in \{1, \ldots, n\}$ distinct, let θ_{ab} be the exterior dihedral angle between $\sigma(a), \sigma(b)$. $n \operatorname{Vol}(\sigma) \operatorname{Vol}(\sigma(a,b)) = (n-1) \operatorname{Vol}(\sigma(a)) \operatorname{Vol}(\sigma(b)) \sin(\theta_{ab})$.*

Proof. Let h_a denote the orthogonal projection of v_a to onto the orthogonal complement of the span of $\sigma(a,b)$ for $i = a, b$.

$$n(n-1) \operatorname{Vol}(\sigma) = \|h_a\| \|h_b\| \sin\theta \operatorname{Vol}(\sigma(a,b)).$$

Using $(n-1) \operatorname{Vol}(\sigma(a)) = \|h_b\| \operatorname{Vol}(\sigma(a,b))$ and similarly for $\sigma(b)$ proves the claim. □

Corollary 2.4.3. *Let τ be a spherical tetrahedron with vertices $v_a, v_b, v_c, v_d \in S^3$.*

$$|G_+(\tau)|^{\frac{1}{2}} \sin l_{ab} = |G_+(\tau(d))|^{\frac{1}{2}} |G_+(\tau(c))|^{\frac{1}{2}} \sin \theta_{ab}, \tag{15}$$

where $G_+(\tau(c))$ denotes the Gram matrix with vertices a, b, d etc. Furthermore,

$$2|G_+(\tau(d))|^{1/2} |G_+(\tau(c))|^{1/2} \cos\theta_{ab} = \frac{d|G_+(\tau)|}{d \cos l_{cd}}. \tag{16}$$

Proof. (15) is a special case of Lemma 2.4.2. (16) is the derivative of (15) with respect to $\cos l_{cd}$. We start with squaring (15):

$$|G_+(\tau)| \sin^2 l_{ab} = |G_+(\tau(d))||G_+(\tau(c))| \sin^2 \theta_{ab}. \tag{17}$$

Taking the derivative gives

$$\frac{d|G_+(\tau)|}{\mathrm{d}\cos l_{cd}} \sin^2 l_{ab} = |G_+(\tau(d))||G_+(\tau(c))| 2\sin\theta_{ab}\cos\theta_{ab} \frac{\mathrm{d}\theta_{ab}}{\mathrm{d}\cos l_{cd}}.$$

Substituting (15) into the right-hand side gives

$$|G_+(\tau(d))|^{1/2}|G_+(\tau(c))|^{1/2} 2\cos\theta_{ab}|G_+(\tau)|^{1/2}\sin l_{ab}\frac{\mathrm{d}\theta_{ab}}{\mathrm{d}\cos l_{cd}}. \tag{18}$$

We compute

$$\frac{\mathrm{d}\theta_{ab}}{\mathrm{d}\cos l_{cd}} = \frac{\mathrm{d}\theta_{ab}}{\mathrm{d}l_{cd}}\frac{\mathrm{d}l_{cd}}{\mathrm{d}\cos l_{cd}} = -\left(\frac{\sin l_{ab}\sin l_{cd}}{|G_+(\tau)|^{1/2}}\right)\left(\frac{-1}{\sin l_{cd}}\right).$$

Substitute this derivative in the equation (18), we get

$$2|G_+(\tau(d))|^{1/2}|G_+(\tau(c))|^{1/2}\cos\theta_{ab}\sin^2 l_{ab}.$$

Thus,

$$\frac{\mathrm{d}|G_+(\tau)|}{\mathrm{d}\cos l_{cd}} = 2|G_+(\tau(d))|^{1/2}|G_+(\tau(c))|^{1/2}\cos\theta_{ab},$$

as required. □

We remark that

$$\frac{\mathrm{d}|G_+(\tau)|}{\mathrm{d}\cos l_{cd}} = -2\begin{vmatrix} 1 & \cos(l_{ab}) & \cos(l_{ad}) \\ \cos(l_{ab}) & 1 & \cos(l_{bd}) \\ \cos(l_{ac}) & \cos(l_{bc}) & \cos(l_{cd}) \end{vmatrix}.$$

3. Geometry of conformal blocks

In this section we explain the conjectural geometry behind the formula (3). Let p_1, p_2, p_3, p_4 be distinct points on $\mathbb{P}^1$. Given $l_{a(a+1)} \in [0, \pi], a = 1, \ldots, 4$ let

$$t_a = \begin{bmatrix} e^{il_{a(a+1)}} & 0 \\ 0 & e^{-il_{a(a+1)}} \end{bmatrix}.$$

Let $\mathcal{M} = \mathcal{M}(l_{12}, l_{23}, l_{34}, l_{14})$ denote the moduli space of flat $SU(2)$-bundles on $\mathbb{P}^1 - \{p_1, p_2, p_3, p_4\}$ with holonomy around p_a conjugate to t_a. The fundamental group of $\mathbb{P}^1 - \{p_1, p_2, p_3, p_4\}$ is generated by the loops γ_j around p_j, with the single relation $\gamma_1\gamma_2\gamma_3\gamma_4 = 1$. Hence

$$\mathcal{M} = \{(e_1, e_2, e_3, e_4) \in SU(2)^4,\ e_1e_2e_3e_4 = I,\ e_a \sim t_a, \quad a = 1, 2, 3, 4\}/SU(2).$$

Under the diffeomorphism $SU(2) \to S^3$, the set of matrices conjugate to t_a is mapped to the set of points in S^3 at distance $l_{a(a+1)}$ from $(1, 0, 0)$. The 4-tuple

$$v_1 = I, \quad v_2 = e_1, \quad v_3 = e_1e_2, \quad v_4 = e_1e_2e_3$$

maps to the vertices of a closed 4-gon in S^3 with edge lengths $l_{a(a+1)}$. From now on, we assume $l_{a(a+1)} < \pi$. Then $\mathcal{M} = \mathcal{M}(l_{12}, l_{23}, l_{34}, l_{41})$ is the moduli space of closed 4-gons in S^3 with edge lengths $l_{a(a+1)}$, where we identify two 4-gons if they are related by an isometry of S^3. $\mathcal{M}$ is non-empty if and only if the inequalities

$$l_{14} \le l_{12} + l_{23} + l_{34}, \quad l_{12} + l_{23} + l_{34} \le 2\pi + l_{14}$$

and their cyclic permutations hold; this can be seen either directly or from [1],[7] by computing Gromov-Witten invariants of $\mathbb{P}^1$. If one of these equalities holds with equality then the 4-gon is contained in a geodesic on S^2 and $\mathcal{M}$ consists of a single point. If one of the lengths is zero then $\mathcal{M}$ is a moduli space of triangles and again consists of single point. Otherwise, $\mathcal{M}$ is homeomorphic to S^2. If in addition $\pm l_{12} \pm l_{23} \pm l_{34} \pm l_{14}$ is not a multiple of 2π for all choices of sign then $\mathcal{M}$ is non-singular and symplectomorphic to S^2.

There are two canonical coordinate systems on $\mathcal{M}$, described as follows. Let λ_{13}, resp. λ_{24}, denote the distance from the vertex v_1 to v_3, resp. v_2 to v_4. In the language of flat bundles, the functions λ_{13}, λ_{24} measure the eigenvalues of the holonomy of the connection around a circle separating the four points into two groups of two. Let θ_{13}, resp. θ_{24}, denote the corresponding dihedral angle. Let $\mathcal{M}^*$ denote the subset of 4-gons that are not contained in a geodesic, that is, the subset of irreducible bundles. Let

$$\mathcal{M}^*_{13} \subset \mathcal{M}^*, \quad \mathcal{M}^*_{24} \subset \mathcal{M}^*$$

denote the subset on which $0 < \lambda_{13} < \pi$ and the faces containing the edge $v_1 v_3$ are non-degenerate, resp. $0 < \lambda_{24} < \pi$ and the faces containing $v_2 v_4$ are non-degenerate. A special case of a result of Goldman [19] states that $\lambda_{13}, \theta_{13}$, respectively $\lambda_{24}, \theta_{24}$, are smooth on $\mathcal{M}^*_{13}$, resp. $\mathcal{M}^*_{24}$ and form a set of action-angle coordinates. That is,

$$\omega|_{\mathcal{M}^*_{13}} = \mathrm{d}\theta_{13} \wedge \mathrm{d}\lambda_{13}, \quad \omega|_{\mathcal{M}^*_{24}} = \mathrm{d}\theta_{24} \wedge \mathrm{d}\lambda_{24}.$$

If $l_{12} = l_{23}$ and $l_{34} = l_{41}$ then $\lambda_{13}^{-1}(0)$ is a subset of $\mathcal{M}$ homeomorphic to S^1, meeting the complement of $\mathcal{M}^*$ in the two configurations where all vertices are colinear. Let $l_{13}^{\min}$ denote the minimum value of λ_{13}. If $l_{13}^{\min} \neq 0$, the set defined by $\lambda_{13} = l_{13}^{\min}$ is a point. If $l_{13}^{\min} = 0$, the set $\lambda_{13} = l_{13}^{\min}$ is a circle.

By a theorem of Mehta-Seshadri [25] and Furuta-Steer [16], $\mathcal{M}$ has the structure of a normal projective variety. By a theorem of Pauly [31], $\mathcal{M}$ has a positive line bundle $\mathcal{L}$ whose sections $H^0(\mathcal{M}, \mathcal{L}^{r-2})$ may be identified with the space of genus zero WZW conformal blocks at level $r-2$ with weights $j_{a(a+1)} = (r-2)l_{a(a+1)}/2\pi$, assuming these are half-integral. Let $\overline{\mathcal{C}}_{0,4}$ denote the moduli space of stable 4-pointed genus zero curves; the open subset $\mathcal{C}_{0,4}$ of 4-pointed smooth curves is an open subset identified with $\mathbb{C} - \{0,1\}$ by the map $(\mathbb{P}^1; 0, 1, \infty, z) \mapsto z$. Let X_{13}, resp. X_{24}, denote the point in $\overline{\mathcal{C}}_{0,4}$ represented by a nodal curve with two components and points p_1, p_2 on one component and p_3, p_4 on the other, resp. p_1, p_4 on one component and p_2, p_3 on the other. Let $\mathcal{V}$ denote the bundle with fiber $\mathcal{V}_{(\mathbb{P}^1, p_1, p_2, p_3, p_4)} = H^0(\mathcal{L}^{r-2}(\mathbb{P}^1, p_1, p_2, p_3, p_4))$. The bundle $\mathcal{V}$ has a projectively flat

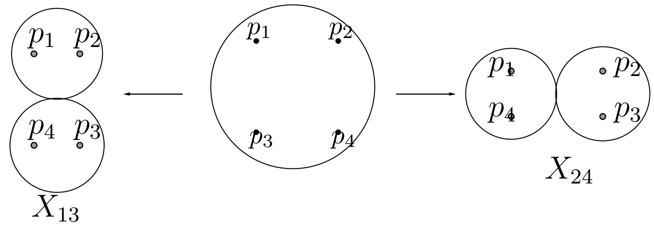


FIGURE 5. Two degenerations of $(\mathbb{P}^1; p_1, p_2, p_3, p_4)$ in $\overline{\mathcal{C}}_{0,4}$.

connection ∇ [38],[3]. The fiber over X_{13}, resp. X_{24} has a canonical projective basis $\{s_{13}(j_{13})\}$ resp. $\{s_{24}(j_{24})\}$, whose elements are the limits of the eigenvectors for the monodromy. The $6j$ symbols are the entries in the change of basis matrix for the bases $\{s_{13}(j_{13})\}, \{s_{24}(j_{24})\}$ after parallel transport to a common point in $\overline{C}_{0,4}$. (However, we only know how to give a canonical normalization to the basis elements using the quantum group definition.) Define

$$\Lambda_{13} = \lambda_{13}^{-1}\left(\frac{2\pi j_{13}}{r-2}\right), \quad \Lambda_{24} = \lambda_{24}^{-1}\left(\frac{2\pi j_{24}}{r-2}\right).$$

$\Lambda_{13}, \Lambda_{24}$ are *Bohr-Sommerfeld* Lagrangians. That is, the restriction of $\mathcal{L}^{r-2}$ to Λ_{ab} has a flat section $s_{ab}^{\infty}(j_{ab})$. Extend the flat section to a section $s_{ab}^{\infty}(j_{ab})$ defined in a neighborhood by parallel transport in the perpendicular directions, followed by multiplication by the Gaussian function $\exp(-4\pi(r-2)d_{ab}^2)$ where d_{ab} is the geodesic distance to Λ_{ab}. $s_{ab}^{\infty}(j_{ab})$ is not in general holomorphic. We conjecture that as $k \to \infty$, after replacing $(r-2)$ with $k(r-2)$ one has

$$s_{k,ab}(kj_{ab}) \sim s_{k,ab}^{\infty}(kj_{ab}). \tag{19}$$

Part of the problem is to give a precise sense in which this asymptotics holds.

We briefly discuss a possible approach to proving (19). The vector field $\frac{\partial}{\partial\theta_{13}}$ has (where defined) a canonical lift to a connection-preserving vector field on $\mathcal{L}$. However, $\frac{\partial}{\partial\theta_{13}}$ does not preserve the holomorphic structure, and so even if globally defined does not define a circle action on $H^0(\mathcal{L}^{r-2})$. However, we conjecture that in the limit $(\mathbb{P}^1, p_1, p_2, p_3, p_4) \to X_{13}$ in $\overline{\mathcal{C}}_{0,4}$, the complex structure is invariant; see Daskalopolous-Wentworth [12]. This would imply that the Verlinde sections $s_{13}(j_{13})$ are asymptotically eigenvectors for the circle action with weight j_{13}. (19) would follow from standard arguments discussed in [34]. We also remark that the leading order part of ∇ as discussed in [3, (4.23)]) is a first order equation and should not change the large k asymptotics.

The determinant line bundle $\mathcal{L}$ is conjectured to admit a Hermitian metric so that the induced Hermitian metric on $\mathcal{V}$ is compatible with the projectively flat connection [3], see also [21]. It would follow that the $6j$ symbol equals the Hermitian pairing of the sections $(s_{13}(j_{13}), s_{24}(j_{24}))$. Since $s_{k,ab}(kj_{ab})$ are asymptotically Gaussian concentrating to Λ_{ab}, one naively expects the pairing to have asymptotics

$$\sum_{\tau\in\Lambda_{13}\cap\Lambda_{24}} \frac{(k(r-2))^{-1}(s_{k,13}(kj_{13})(\tau), s_{k,24}(kj_{24})(\tau))e^{\pm\pi i/4}}{\sqrt{(k\omega)_\tau(\frac{\partial}{\partial\theta_{13}}, \frac{\partial}{\partial\theta_{24}})}}. \tag{20}$$

We denote the two points of intersection by τ_+, τ_-. The pairings $\omega_{\tau_\pm}(\frac{\partial}{\partial\theta_{13}}, \frac{\partial}{\partial\theta_{24}})$ are by 2.4.1 (n)

$$\omega_{\tau_\pm}\left(\frac{\partial}{\partial\theta_{13}}, \frac{\partial}{\partial\theta_{24}}\right) = \frac{\partial l_{13}}{\partial\theta_{24}} = -\frac{|G_+(l)|^{1/2}}{\sin(\mu_{cd})\sin(\mu_{ab})}.$$

Moving along the manifold Λ_{13} from τ_+ to τ_- and then Λ_{24} from τ_- to τ_+ produces a path γ in the Maslov diagram in Figure 2, case (a). Let $\mathrm{Area}(\gamma)$ denote the symplectic area enclosed by γ. Then

$$(s_{13}(j_{13}), s_{24}(j_{24}))(\tau_+) = \exp(i\,\mathrm{Area}(\gamma))(s_{13}(j_{13}), s_{24}(j_{24}))(\tau_-).$$

(20) becomes

$$(s_{k,13}(kj_{13}), s_{k,24}(kj_{24})) \sim \frac{\cos(\frac{1}{2}\mathrm{Area}(\gamma) + \frac{\pi}{4})}{(k(r-2))^{3/2}\sqrt{\omega(\frac{\partial}{\partial\theta_{13}}, \frac{\partial}{\partial\theta_{24}})}}.$$

We claim that

$$\mathrm{Area}(\gamma) = 2\sum_{a<b} l_{ab}\theta_{ab} + \pi(l_{12}+l_{23}+l_{34}+l_{14}).$$

As we vary l_{13} and l_{24},

$$d\,\mathrm{Area}(\gamma) = 2(\theta_{13}\mathrm{d}l_{13} + \theta_{24}\mathrm{d}l_{24}) = 2(\mathrm{d}\sum_{a<b}\theta_{ab}l_{ab} - 2\,\mathrm{Vol}(\tau))$$

by Schläfli's identity 2.4.1 (m). Hence

$$\mathrm{Area}(\gamma) = 2\sum_{a<b}\theta_{ab}l_{ab} + c(l_{12}, l_{23}, l_{34}, l_{14})$$

where $c(l_{12}, l_{23}, l_{34}, l_{14})$ is a constant depending on $l_{12}, l_{23}, l_{34}, l_{14}$. For l_{13}, l_{24} corresponding to a degenerate tetrahedron with four exterior edges, we have

$$\theta_{a(a+1)} = \pi, \quad a = 1,2,3,4, \quad \theta_{13} = \theta_{24} = 0.$$

This implies

$$c(l_{12}, l_{23}, l_{34}, l_{14}) = 2\pi(l_{12}+l_{23}+l_{34}+l_{14}).$$

Putting everything together, this gives in the transversal case the naive prediction

$$(s_{k,13}(kj_{13}), s_{k,24}(kj_{24})) \sim \frac{2\pi\cos\left(\frac{k(r-2)}{2\pi}(\sum\theta_{ab}l_{ab} - 2\,\mathrm{Vol}(\tau)) + \frac{\pi}{4}\right)}{(k(r-2))^{3/2}\det(\cos(l_{ab}))^{1/4}}$$

up to sign. Including half-forms in the quantization scheme should produce shifts giving Theorem 1.0.1 (a). (Readers familiar with the representation theory of affine Lie algebras will recognize the numbers $\frac{1}{2}, 2$ in the formulas for l_{ab} and $r(k)$ as the half-sum of positive weights, resp. dual Coxeter number of $\mathfrak{sl}_2$.) Unfortunately we lack the techniques to carry this out. In the follow section we introduce a recursion which reduces the formula to the $q = 1$ case; the recursion is needed anyway to handle the first degenerate case, so even if a geometric proof could be found for case (a) the proof of the Theorem 1.0.1 we give is presumably shorter.

4. Non-degenerate tetrahedra

First we show that both sides satisfy a second order difference equation as one label is varied; the same strategy was followed by Mizoguchi-Tada [27] without knowledge of the denominator in the asymptotic formula. This implies that each side is a linear combination of the two linearly independent solutions. To prove that the coefficients are equal we show that in the limit as the labels j_{ab} are made small in relation to the level $r-2$ our equation reduces to the Ponzano-Regge formula (8).

4.1. The Schulten-Gordon recursion.

Proposition 4.1.1.

$$P(j_{23}) \left\{ \begin{array}{ccc} j_{12} & j_{23} & j_{13} \\ j_{34} & j_{14} & j_{24} \end{array} \right\} + [2j_{23}]Q(j_{23}+1) \left\{ \begin{array}{ccc} j_{12} & j_{23}+1 & j_{13} \\ j_{34} & j_{14} & j_{24} \end{array} \right\} + [2j_{23}+2]Q(j_{23}) \left\{ \begin{array}{ccc} j_{12} & j_{23}-1 & j_{13} \\ j_{34} & j_{14} & j_{24} \end{array} \right\} = 0, \tag{21}$$

where

$$\begin{aligned} P(j_{23}) = {} & [2j_{23}][2j_{23}+1][2j_{23}+2][j_{14}+j_{13}+j_{34}+1][j_{13}+j_{34}-j_{14}] \\ & - [2j_{23}][j_{12}+j_{13}-j_{23}][j_{12}+j_{23}-j_{13}+1][j_{24}+j_{34}-j_{23}][j_{24}+j_{23}-j_{34}+1] \\ & - [2j_{23}+2][j_{12}+j_{23}+j_{13}+1][j_{13}+j_{23}-j_{12}][j_{24}+j_{34}+j_{23}+1][j_{34}+j_{23}-j_{24}] \end{aligned}$$

and

$$\begin{aligned} Q(j_{23}) = {} & ([j_{12}+j_{23}-j_{13}][j_{12}+j_{13}-j_{23}+1][j_{24}+j_{34}-j_{23}+1][j_{24}+j_{34}-j_{23}] \\ & [j_{12}+j_{23}+j_{13}+1][j_{13}+j_{23}-j_{12}][j_{24}+j_{34}+j_{23}+1][j_{34}+j_{23}-j_{24}])^{\frac{1}{2}}. \end{aligned}$$

The proof is a combination of pentagon identities (5), as in [36], and is omitted. In the limit $r \to \infty$ the recursion (21) simplifies dramatically. Let Δ_{ab} denote the discrete Laplacian

$$(\Delta_{ab} f)(j_{ab}) = f(j_{ab}+1) - 2f(j_{ab}) + f(j_{ab}-1).$$

Let τ denote the tetrahedron with lengths $l_{ab} = 2\pi(j_{ab}+\frac{1}{2})/r$, if it exists, and θ_{ab} the exterior dihedral angle around the edge e_{ab}. Set

$$f(j_{ab}) = \left(\frac{|G_+(l)|^{\frac{1}{2}}}{\sin(\theta_{ab})} \right)^{\frac{1}{2}} \left\{ \begin{array}{ccc} j_{12} & j_{23} & j_{13} \\ j_{34} & j_{14} & j_{24} \end{array} \right\}.$$

Let $T_S \subset \mathbb{R}^6$ denote the subset of edge lengths of possibly degenerate tetrahedra, with all faces non-degenerate. T_S is defined by the inequalities

$$|G_+(\tau(d))| > 0, 1 \leq d \leq 4 \tag{22}$$

$$|G_+(\tau)| \geq 0.$$

Theorem 4.1.2. *Let K be a compact subset of T_S and $1 \leq a < b \leq 4$. There exists a constant C and a second order difference operator $\epsilon(r)$ with coefficients bounded by Cr^{-2} for r sufficiently large such that*

$$(\Delta_{ab} + 2 - 2\cos\theta_{ab} + \epsilon(r))f(j_{ab}) = 0 \tag{23}$$

on the relative interior of K.

Remark 4.1.3. One may extend this recursion to the subset of $\mathbb{R}^6$ of 6-tuples l_{ab} satisfying only (22) by defining θ_{ab} to be the imaginary number given by (16).

Proof. Let cd be the opposite edge of ab. A somewhat long computation involving standard trigonometric identities shows that

$$\left| P(j_{ab}) \left(\sin\frac{\pi}{r}\right)^5 - \frac{1}{4}\frac{\mathrm{d}}{\mathrm{d}\cos(l_{cd})}|G_+(l)| \right| < Cr^{-2}. \tag{24}$$

Substituting this into the recursion (21) and using the factorization 2.4.1 (g) gives

$$\frac{1}{4}\sin(l_{ab})\frac{\mathrm{d}|G_+(l)|}{\mathrm{d}\cos(l_{cd})} \begin{Bmatrix} j_{ac} & j_{ab} & j_{bc} \\ j_{bd} & j_{cd} & j_{ad} \end{Bmatrix} + \frac{1}{4}\sin(l_{ab} - \frac{\pi}{r})\cdot \tag{25}$$
$$[|G_+(l_{ac}, l_{ab} + \frac{2\pi}{r}, l_{bc})||G_+(l_{bd}, l_{ab} + \frac{2\pi}{r}, l_{ad})||G_+(l_{ac}, l_{ab}, l_{bc})||G_+(l_{bd}, l_{ab}, l_{ad})|]^{\frac{1}{4}}$$
$$\begin{Bmatrix} j_{ac} & j_{ab}+1 & j_{bc} \\ j_{bd} & j_{cd} & j_{ad} \end{Bmatrix} + \frac{1}{4}\sin(l_{ab} + \frac{\pi}{r}) \cdot \begin{Bmatrix} j_{ac} & j_{ab}-1 & j_{bc} \\ j_{bd} & j_{cd} & j_{ad} \end{Bmatrix}$$
$$[|G_+(l_{ac}, l_{ab} - \frac{2\pi}{r}, l_{bc})||G_+(l_{bd}, l_{ab} - \frac{2\pi}{r}, l_{ad})||G_+(l_{ac}, l_{ab}, l_{bc})||G_+(l_{bd}, l_{ab}, l_{ad})|]^{\frac{1}{4}} = O(r^{-2})$$

on the relative interior of K. (Here $G_+(l_{ac}, l_{ab}, l_{bc})$ etc. refer to Gram matrices of the faces.) We divide (25) by

$$\left(\sin(l_{ab} - \frac{2\pi}{r})\sin(l_{ab})\sin(l_{ab} + \frac{2\pi}{r})\right)^{\frac{1}{2}} [|G_+(l_{ac}, l_{ab}, l_{bc})G_+(l_{ab}, l_{bd}, l_{ad})|]^{\frac{1}{4}}.$$

Using the equality

$$\sin(l_{ab})(\sin(l_{ab} - 2\pi/r)\sin(l_{ab})\sin(l_{ab} + 2\pi/r))^{-\frac{1}{2}} = \sin(l_{ab})^{-\frac{1}{2}}(1 + O(r^{-2}))$$

and similar identities one obtains the vanishing of

(26)
$$
\begin{aligned}
&-\sin(l_{ab})^{-\frac{1}{2}}[|G_+(l_{ac},l_{ab},l_{bc})||G_+(l_{ab},l_{bd},l_{ad})|]^{-\frac{1}{4}}\frac{\mathrm{d}|G_+(l)|}{\mathrm{d}\cos(l_{cd})}\left\{\begin{array}{ccc} j_{ac} & j_{ab} & j_{bc} \\ j_{bd} & j_{cd} & j_{ad}\end{array}\right\}\\
&+\sin(l_{ab}+\frac{2\pi}{r})^{-\frac{1}{2}}[|G_+(l_{ac},l_{ab}+\frac{2\pi}{r},l_{bc})||G_+(l_{bd},l_{ab}+\frac{2\pi}{r},l_{ad})|]^{\frac{1}{4}}\left\{\begin{array}{ccc} j_{ac} & j_{ab}+1 & j_{bc} \\ j_{bd} & j_{cd} & j_{ad}\end{array}\right\}\\
&+\sin(l_{ab}-\frac{2\pi}{r})^{-\frac{1}{2}}[G_+(l_{ac},l_{ab}-\frac{2\pi}{r},l_{bc})G_+(l_{bd},l_{ab}-\frac{2\pi}{r},l_{ad})]^{\frac{1}{4}}\left\{\begin{array}{ccc} j_{ac} & j_{ab}-1 & j_{bc} \\ j_{bd} & j_{cd} & j_{ad}\end{array}\right\}
\end{aligned}
$$

up to terms of order r^{-2}. Substituting (16) and (15) into the above gives the result. □

Remark 4.1.4. An examination of the proof shows that the constant C depends linearly on a lower bound for $(|G_+(l_{ac},l_{ab},l_{bc})||G_+(l_{ab},l_{bd},l_{ad})|)^{\frac{1}{4}}$.

4.2. **Second-order recursion for the asymptotic formula.** We now show that the right-hand side of (3) is also an approximate solution to (23). Set

$$
\left\{\begin{array}{ccc} j_{12} & j_{23} & j_{13} \\ j_{34} & j_{14} & j_{24}\end{array}\right\}^{\infty}_{q=\exp(\pi i/r)} := \frac{2\pi\cos(\phi+\frac{\pi}{4})}{r^{\frac{3}{2}}|G_+(l)|^{\frac{1}{4}}}. \tag{27}
$$

Let Δ^r_{ab} be the second-order discrete Laplacian at step size $2\pi/r$,

$$
(\Delta^r_{ab}f)(l) = f(l-2\pi/r) - 2f(l) + f(l+2\pi/r).
$$

Let T^+_S denote the subset of $\mathbb{R}^6$ defined by the inequalities (22) and $|G_+(\tau)|>0$, that is, the set of edge lengths of non-degenerate tetrahedra.

Theorem 4.2.1. *Let K be a compact subset of T^+_S. There exists a constant C and a second-order difference operator $\epsilon(r)$ with coefficients bounded by Cr^{-2} for r sufficiently large such that*

$$
(\Delta^r_{ab}+2-2\cos(\theta_{ab})+\epsilon(r))\left(\frac{|G_+(l)|^{\frac{1}{2}}}{\sin\theta_{ab}}\right)^{\frac{1}{2}}\left\{\begin{array}{ccc} j_{12} & j_{23} & j_{13} \\ j_{34} & j_{14} & j_{24}\end{array}\right\}^{\infty}_{q=\exp(\pi i/r)}
$$

vanishes on the relative interior of K.

Proof. Let $\theta=\theta_{ab}$. By Taylor's theorem

$$
\phi(l_{ab}\pm\frac{2\pi}{r}) = \phi(l_{ab}) \pm \frac{\partial\phi}{\partial l_{ab}}\frac{2\pi}{r}(l_{ab}) + \frac{1}{2}\frac{\partial^2\phi}{\partial {l_{ab}}^2}\left(\frac{2\pi}{r}\right)^2(l_{ab}+\epsilon) \tag{28}
$$

for some $\epsilon\in[-\frac{2\pi}{r},\frac{2\pi}{r}]$. Because K is compact, $\frac{\partial^2\phi}{\partial {l_{ab}}^2}$ is bounded on an open neighborhood of K. By 2.4.1 (m),

$$
\frac{\partial\phi}{\partial l_{ab}} = \frac{r}{2\pi}\theta,\quad \frac{\partial^2\phi}{\partial {l_{ab}}^2} = \frac{r}{2\pi}\frac{\partial\theta}{\partial l_{ab}}.
$$

Thus

$$\cos(\phi(l_{ab} \pm 2\pi/r)) = \cos(\phi(l_{ab}) \pm \theta) - \frac{\pi}{r}\frac{\partial\theta}{\partial l_{ab}}\sin(\phi(l_{ab}) \pm \theta) + O(r^{-2}). \tag{29}$$

Similarly

$$\sin^{-1/2}(\theta(l_{ab} \pm 2\pi/r)) = \sin^{-1/2}(\theta(l_{ab})) \mp \frac{\pi}{r}\sin^{-3/2}(\theta(l_{ab}))\cos(\theta)\frac{\partial\theta}{\partial l_{ab}} + O(r^{-2}).$$

Expanding $\cos(\phi(l_{ab}) \pm \theta), \sin(\phi(l_{ab}) \pm \theta)$ one finds that the $O(1)$ and $O(r^{-1})$ terms cancel, which completes the proof. □

Remark 4.2.2. An examination of the proof shows that the constant C depends linearly on an upper bound for $\frac{\partial^2\theta_{ab}}{\partial {l_{ab}}^2}$.

4.3. **Euclidean limit of the $6j$ symbol.** Next we investigate the limit of both sides of (3) as the level is taken to infinity much faster than the labels.

Proposition 4.3.1. *Let $j_{ab} \in [0, (r-2)/2]$ and δ_{ab} be half-integers, for $1 \le a < b \le 4$, and $p \ge 3$ an integer. Let $j_{ab}(k) = kj_{ab} + \delta_{ab}$. Then*

$$\left\{\begin{matrix} j_{12}(k) & j_{23}(k) & j_{13}(k) \\ j_{34}(k) & j_{14}(k) & j_{24}(k) \end{matrix}\right\}_{q=\exp(\pi i/r(k^p))} \sim \left\{\begin{matrix} j_{12}(k) & j_{23}(k) & j_{13}(k) \\ j_{34}(k) & j_{14}(k) & j_{24}(k) \end{matrix}\right\}_{q=1}.$$

The proof relies on the

Lemma 4.3.2. *There exist $C, c > 0$ such that if $n/r < c$ then $|\ln([n]!/n!)| < Cn^3r^{-2}$.*

Proof. Applying the logarithm to (4) gives

$$\ln[n]! = \sum_{j=1}^{n} \ln\left(\sin\left(\frac{\pi j}{r}\right)\right) - \ln\left(\sin\left(\frac{\pi}{r}\right)\right).$$

Note that $|\sin(x) - x| < \frac{x^3}{6}$ and $\ln(\sin(x)) = \ln(x) + (\sin(x) - x)/y$ for some $y \in [x, \sin(x)]$. For x sufficiently small,

$$\sin(x) \in [x - \frac{x^3}{6}, x], \qquad y > x/2$$

which implies

$$|\ln(\sin(x)) - \ln(x)| < \frac{x^3}{6}\frac{2}{x} = \frac{x^2}{3}.$$

Hence

$$|\ln[n]! - \ln n!| < \sum_{j=1}^{n}\frac{1}{3}((\frac{\pi j}{r})^2 + (\frac{\pi}{r})^2) < \frac{n}{3}((\frac{\pi n}{r})^2 + (\frac{\pi}{r})^2) < \frac{2\pi^2 n^3}{3r^2}.$$

□

Proof of 4.3.1: By Lemma 4.3.2 and Racah's formula (6), since the number of terms in the sum grows linearly with k. □

4.4. **Euclidean limit of the asymptotic formula.** In this section we compare the quantities

$$\left\{ \begin{matrix} j_{12} & j_{23} & j_{13} \\ j_{34} & j_{14} & j_{24} \end{matrix} \right\}^{\infty}_{q=\exp(\pi i/r)} := \frac{2\pi\cos(\phi + \frac{\pi}{4})}{r^{\frac{3}{2}}|G_+(l)|^{\frac{1}{4}}}$$

and

$$\left\{ \begin{matrix} j_{12} & j_{23} & j_{13} \\ j_{34} & j_{14} & j_{24} \end{matrix} \right\}^{\infty}_{q=1} := \frac{2\pi\cos(\phi_0 + \frac{\pi}{4})}{||G_0(j+\frac{1}{2})||^{1/4}}$$

in the case that the corresponding tetrahedra are non-degenerate. We will prove

Proposition 4.4.1. *Let $j_{ab} \in [0,(r-2)/2], 1 \le a < b \le 4$ be such that a non-degenerate Euclidean tetrahedron with edge lengths j_{ab} exists. Let δ_{ab} be half-integers, and $p > 3/2$. Let $j_{ab}(k) = kj_{ab} + \delta_{ab}$.*

$$\left\{ \begin{matrix} j_{12}(k) & j_{23}(k) & j_{13}(k) \\ j_{34}(k) & j_{14}(k) & j_{24}(k) \end{matrix} \right\}^{\infty}_{q=\exp(\pi i/r(k^p))} \sim \left\{ \begin{matrix} j_{12}(k) & j_{23}(k) & j_{13}(k) \\ j_{34}(k) & j_{14}(k) & j_{24}(k) \end{matrix} \right\}^{\infty}_{q=1}.$$

Proof. By 2.4.1 (j) $r(k^p)^{3/2}|G_+(l)| \sim ||G_0(j+\frac{1}{2})||$. Since the edge lengths are $O(k^{1-p})$, the volume $\mathrm{Vol}(\tau)$ is $O(k^{3-3p})$. Let θ^E_{ab} denote the dihedral angles of the Euclidean tetrahedron with edge lengths l_{ab}, if it exists. It follows from Corollary 2.4.3 that $\cos(\theta_{ab}) = \cos(\theta^E_{ab})(1+O(k^{2-2p}))$. Since $\theta_{ab} \to \theta^E_{ab}$ as $k \to \infty$, $\theta_{ab} - \theta^E_{ab} < Ck^{2-2p}$. Hence

$$\sum l_{ab}(\theta_{ab} - \theta^E_{ab}) < Ck^{3-3p}$$

which implies that

$$\phi(k) - \phi_0(k) < Ck^{3-2p}.$$

For $p > 3/2$, this approaches zero as $k \to \infty$. □

4.5. **All edge lengths equal.** The second order difference equation corresponds to a system of first-order difference equations in the standard way. In fact, if v is the vector formed from all 2^6 sets of labels of the form $j_{ab}+\delta_{ab}$, where $\delta_{ab} = 0$ or 1, then the vector of the corresponding 2^6 $6j$ symbols satisfies a first order recursion in all six variables j_{ab}. We call the first-order systems corresponding to (21),(23) the first-order Schulten-Gordon difference equation and first-order asymptotic difference equation, respectively. By Theorems 4.1.2 and 4.2.1, to show Theorem 1.0.1 (a) it suffices to prove the claim for all components of v and a single set of values of j_{ab}. We do so for the case that all edge lengths are equal. By 2.4.1 (k), a spherical tetrahedron with edge lengths equal to l exists for $0 \le l \le \arccos(-1/3)$. Let $h = h(k)$ be a function of k so that $h > k^p$, for some $p > 3$. Choose $r \gg 0$ so that there exists $j \in (0,(r-2)/4) \cap \mathbb{Z}/2$. For sufficiently large k, there exists an approximation $\gamma_k(t)$ to the linear path $lt, t \in [k/h(k), 1]$ such that

(a) $\gamma_k(t) + \delta_{ab}/r(h(k))$ is the set of edge lengths of a non-degenerate spherical tetrahedron, for any choice of δ_{ab};
(b) γ_k is a lattice path for $\mathbb{Z}/2r(h(k))$, that is, obtained by concatenating paths between lattice points whose difference is a standard basis vector divided by $2r(h(k))$;

(c) $\gamma_k(k/h(k))_{ab} = (kj + \frac{1}{2})/h(k)$ and $\gamma_k(1)_{ab} = (h(k)j + \frac{1}{2})/h(k)$;
(d) the number of lattice points in γ_k is $O(h(k))$.

Let $l(k,m), m = 0, \ldots, n(k)$ be the lattice points occurring in γ. Let $A(m)$ be the operator in the m-th term of the first-order Schulten-Gordon difference equation, and $A^\infty(m)$ the operator in the m-th term in the asymptotic difference equation. On an open cone containing l, we have an estimate

$$(|G_+(l_{ac}, l_{ab}, l_{bc})||G_+(l_{ab}, l_{bd}, l_{ad})|)^{\frac{1}{4}} > C\|l\|^2.$$

By Remark 4.1.4 $A(m)^{-1}A^\infty(m) = I + \epsilon(m)$ where $\epsilon(m)$ has coefficients bounded by

$$Ch^{-2}\|l(k,m)\|^{-2}. \tag{30}$$

Let

$$v(k,m) = A(m)A(m-1)\ldots A(1)v(k,0)$$

denote the solution to the Schulten-Gordon difference equation, and

$$v^\infty(k,m) = A^\infty(m)A^\infty(m-1)\ldots A^\infty(1)v(k,0)$$

the solution to the asymptotic equation. Then

$$\begin{aligned} v^\infty(k,m) &= A(m)(I+\epsilon(m))A(m-1)(I+\epsilon(m-1))\ldots A(1)(I+\epsilon(1))v(k,0) \\ &= (I+\epsilon'(m))(I+\epsilon'(m-1))\ldots(I+\epsilon(1))v(k,m) \end{aligned}$$

where $\epsilon'(1), \ldots, \epsilon'(m)$ are operators with coefficients also bounded by (30). The norm of the product of error factors satisfies the estimate

$$\left\| \ln \left(\prod_{m=1}^{n(k)} (I + \epsilon'(m)) \right) \right\| \leq C \sum_{m=1}^{n(k)} \|hl(k,m)\|^{-2}$$

which goes to zero as $k \to \infty$. It follows that the components of $v^\infty(k,m)$ are asymptotic to those of $v(k,m)$ as $k \to \infty$. (The convergence of approximate solutions of second order difference equations to actual solutions is discussed in much greater generality in [18].)

A similar estimate holds for the right-hand side of (3). On the cone generated by a small neighborhood of l^0_{ab} we have a bound

$$\frac{\partial^2 \theta_{l_{ab}}}{\partial {l_{ab}}^2}(l) < C\|l\|^{-2}.$$

Using Remark 4.2.2, the error factor approaches 1 as $k \to \infty$, that is, the asymptotic $6j$ symbols approach an exact solution of the asymptotic difference equation.

To show that $v(k, n(k))$ is asymptotic to $v^\infty(k, n(k))$ it remains to show that each component of $v(k,0)$ is asymptotic to the corresponding component of $v^\infty(k,0)$. By Propositions 4.3.1, 4.4.1 it suffices to show that for any $\delta_{ab} \in \{0,1\}$,

$$\left\{ \begin{array}{ccc} kj+\delta_{12} & kj+\delta_{23} & kj+\delta_{13} \\ kj+\delta_{34} & kj+\delta_{14} & kj+\delta_{24} \end{array} \right\}_{q=1} \sim \left\{ \begin{array}{ccc} kj+\delta_{12} & kj+\delta_{23} & kj+\delta_{13} \\ kj+\delta_{34} & kj+\delta_{14} & kj+\delta_{24} \end{array} \right\}^\infty_{q=1}.$$

For $\delta_{ab} = 0$ this is Roberts theorem [34]. To handle the remaining cases, we need a generalization:

Theorem 4.5.1. *Let $j_{12}(k), \ldots, j_{34}(k) \in \mathbb{Z}/2$ be functions of k such that $l_{ab}(k) = 2\pi(j_{12}(k) + \frac{1}{2})/k$ converge as $k \to \infty$ to $l_{ab} \in (0, \pi)$ the edge lengths of a non-degenerate Euclidean tetrahedron.*

$$\left\{ \begin{array}{ccc} j_{12}(k) & j_{23}(k) & j_{13}(k) \\ j_{34}(k) & j_{14}(k) & j_{24}(k) \end{array} \right\}_{q=1} \sim \left\{ \begin{array}{ccc} j_{12}(k) & j_{23}(k) & j_{13}(k) \\ j_{34}(k) & j_{14}(k) & j_{24}(k) \end{array} \right\}^{\infty}_{q=1}.$$

Proof. It seems that this follows by the same method used by Roberts [34]. Alternatively, the $6j$ symbols for $q = 1$ as a function of any j_{ab} satisfy the Schulten-Gordon second-order difference equation up to an error term depending linearly on an upper bound for $||G_0(l_{ab}, l_{ac}, l_{bc})||G_0(l_{ab}, l_{ad}, l_{bd})||^{-1/4}$. Let T_E^+ be the subset of $\mathbb{R}^6$ consisting of edge lengths of non-degenerate tetrahedra, and $K \subset T_E^+$ a compact subset. On kK, the $6j$ symbols solve the difference equation up to an error operator of order $O(k^{-2})$, and so equal an exact solution up to $O(k^{-1})$. Any exact solution is a linear combination

$$2\pi r^{-3/2}||G_0(l)||^{-1/4}(c_1 \cos(\phi_0 + \pi/4) + c_2 \sin(\phi_0 + \pi/4));$$

We wish to show that $c_1 = 1, c_2 = 0$. Let $ab = 12$ and suppose that $j_{ab}(k) = kj_{ab}$ for $ab \neq 12$. Roberts' theorem applied to $\left\{ \begin{array}{ccc} kj_{12} & kj_{23} & kj_{13} \\ kj_{34} & kj_{14} & kj_{24} \end{array} \right\}$ gives

$$c_1 \cos(\phi_0 + \pi/4) + c_2 \sin(\phi_0 + \pi/4) = \cos(\phi_0 + \pi/4). \tag{31}$$

Roberts' theorem applied to $\left\{ \begin{array}{ccc} k(j_{12}+1) & kj_{23} & kj_{13} \\ kj_{34} & kj_{14} & kj_{24} \end{array} \right\}$ gives

$$c_1 \cos(\phi_0 + \theta_{12} + \pi/4) + c_2 \sin(\phi_0 + \theta_{12} + \pi/4) = \cos(\phi_0 + \theta_{12} + \pi/4). \tag{32}$$

Since $\theta_{12} \in (0, \pi)$, the equations (31),(32) are linearly independent, hence $c_1 = 1$ and $c_2 = 0$. Applying the same argument to the variation in j_{23}, j_{13} etc. proves the theorem. □

This completes the proof of Theorem 1.0.1 (a). The argument used for Theorem 4.5.1 applies to the $6j$ symbols for $q = \exp(\pi i/r)$ and gives the following generalization of Theorem 1.0.1 (a).

Theorem 4.5.2. *Let $r > 2, j_{12}(k), \ldots, j_{34}(k) \in [0, (r-2)/2] \cap \mathbb{Z}/2$ be functions of k such that $l_{ab}(k) = 2\pi(j_{12}(k) + \frac{1}{2})/r(k)$ converge as $k \to \infty$ to $l_{ab} \in (0, \pi)$ the edge lengths of a non-degenerate spherical tetrahedron.*

$$\left\{ \begin{array}{ccc} j_{12}(k) & j_{23}(k) & j_{13}(k) \\ j_{34}(k) & j_{14}(k) & j_{24}(k) \end{array} \right\}_{q=\exp(\pi i/r(k))} \sim \left\{ \begin{array}{ccc} j_{12}(k) & j_{23}(k) & j_{13}(k) \\ j_{34}(k) & j_{14}(k) & j_{24}(k) \end{array} \right\}^{\infty}_{q=\exp(\pi i/r(k))}.$$

5. Degenerate tetrahedra with non-degenerate faces

We apply the difference equation of the previous section to a sequence of labels in which the corresponding tetrahedra degenerates to a degenerate tetrahedron with non-degenerate faces. The two possible limiting cases are shown below in Figure 6. Let l^0_{ab}, θ^0_{ab} denote the lengths, resp. angles of the degeneration. Following

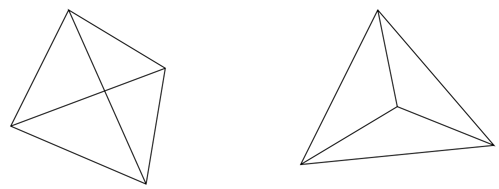


Figure 6. Two limiting cases

Schulten and Gordon [36] we define

$$\phi_0 = \frac{r(k)}{2\pi} \sum_{a<b} \theta^0_{ab} l_{ab} \qquad \theta^0_{ab} = \begin{cases} 0, & \text{if } \theta_{ab} \leq \pi/2\ ; \\ \pi, & \text{if } \theta_{ab} > \pi/2\ . \end{cases}$$

In the two limiting cases shown above, we have

$$\phi_0 = \pi(j_{12} + j_{23} + j_{34} + j_{41} + 2), \quad \text{resp. } \pi(j_{12} + j_{23} + j_{31} + \frac{3}{2}).$$

Hence ϕ_0 is an integer, resp. half integer, times π. By Schäfli's formula 2.4.1 (m)

$$\mathrm{d}(\phi - \phi_0) = \frac{r(k)}{2\pi} \sum (\theta_{ab} - \theta^0_{ab}) \mathrm{d} l_{ab} \tag{33}$$

which implies $\phi - \phi_0 < 0$, resp. $\phi - \phi_0 > 0$ for tetrahedra near the first, resp. second, limiting case.

We apply the LG-WKB method to arrive at a conjectural formula. Consider a solution f to the recursion relation

$$(\Delta_{ab} + 2 - 2\cos(\theta_{ab} - \theta^0_{ab})) f(j_{ab}) = 0.$$

According to the ansatz described in [36], in the semiclassical limit the solution must solve the differential equation

$$\left(\frac{\partial^2}{\partial j_{ab}} - (\theta_{ab} - \theta^0_{ab})^2 \right) \sqrt{\frac{\sin(\theta_{ab} - \theta^0_{ab})}{\theta_{ab} - \theta^0_{ab}}} f(j_{ab}) = 0. \tag{34}$$

Let $F(x)$ denote a solution to the Airy equation $F''(x) = xF(x)$. Define

$$f(j_{ab}) = \left(\frac{\sin(\theta_{ab} - \theta^0_{ab})}{\theta_{ab} - \theta^0_{ab}} \right)^{-1/2} A(j_{ab}) F(\Omega(j_{ab})).$$

Inserting this into (34) gives

$$\left(\frac{A''}{A} + \Omega'^2 \Omega + (\theta_{ab} - \theta^0_{ab})^2 \right) F(\Omega(j_{ab})) + \left(2\frac{A'}{A}\Omega' + \Omega'' \right) F'(\Omega(j_{ab})) = 0.$$

Assuming that $A''/A \approx 0$, the solution is

$$\Omega(j_{ab}) = -\frac{3}{2}\left(\int(\theta_{ab}-\theta^0_{ab})\mathrm{d}j_{ab}\right)^{2/3} = -\frac{3}{2}(\phi(j_{ab})-\phi^0(j_{ab}))^{2/3}$$

$$A(j_{ab}) = C|\phi(j_{ab})-\phi^0(j_{ab})|^{1/4}\theta_{ab}^{-1/2}.$$

We need to choose F, f to agree with the solution in the transversal case. Introduce the regular and irregular Airy functions

$$\mathrm{Ai}(x) = \frac{1}{\pi}\int_0^\infty \cos(t^3/3+xt)\mathrm{d}t, \quad \mathrm{Bi}(x) = \frac{1}{\pi}\int_0^\infty (\exp(-t^3/3+xt)+\sin(t^3/3+xt))\mathrm{d}t.$$

For large x one has the asymptotics

$$\text{(35)} \quad \mathrm{Ai}(-x) \sim \pi^{-1/2}x^{-1/4}\cos(\xi-\frac{\pi}{4}), \quad \mathrm{Bi}(-x) \sim \pi^{-1/2}x^{-1/4}(-\sin(\xi-\frac{\pi}{4}))$$

where $\xi = \frac{2}{3}x^{3/2}$. It follows that the function

$$\text{(36)} \qquad f = \frac{2\pi^{3/2}Z^{1/4}}{r^{3/2}\sin(\theta_{ab})^{1/2}}\begin{cases}\cos(\phi_0)\,\mathrm{Ai}(-Z)-\sin(\phi_0)\,\mathrm{Bi}(-Z) & \phi-\phi_0<0\\ \cos(\phi_0)\,\mathrm{Bi}(-Z)-\sin(\phi_0)\,\mathrm{Ai}(-Z) & \phi-\phi_0>0\end{cases}$$

where

$$Z = (\frac{3}{2}|\phi-\phi_0|)^{2/3}$$

satisfies the asymptotic recursion and matches up with the non-degenerate solution.

Lemma 5.0.3. *In the limit $\theta_{ab} \to \theta^0_{ab}$,*

$$\frac{|\phi-\phi_0|}{|G_+(l)|^{3/2}} \to \frac{3r}{2\pi A_1A_2A_3A_4}.$$

Proof. Using (15), at $l_{ab} = l^0_{ab}$ we have

$$\begin{aligned}
\frac{\partial}{\partial l_{ab}}\frac{|G_+(l)|^{3/2}}{A_1A_2A_3A_4} &= \frac{3}{A_1A_2A_3A_4}|G_+(l)|\frac{\partial}{\partial l_{ab}}|G_+(l)|^{1/2}\\
&= \frac{3}{A_1A_2A_3A_4}|G_+(l)|^{1/2}\frac{A_aA_b\sin(\theta_{ab})}{\sin(l_{ab})}\frac{\partial}{\partial l_{ab}}\frac{A_cA_d\sin(\theta_{cd})}{\sin(l_{cd})}\\
&= \frac{3|G_+(l)|^{1/2}}{A_1A_2A_3A_4}\frac{A_aA_b\sin(\theta_{ab})}{\sin(l_{ab})}\frac{A_cA_d\cos(\theta_{cd})}{\sin(l_{cd})}\frac{\partial\theta_{cd}}{\partial l_{ab}}\\
&= 3|G_+(l)|^{1/2}\frac{\sin(\theta_{ab})\cos(\theta_{cd})}{\sin(l_{ab})\sin(l_{cd})}\frac{\sin(l_{ab})\sin(l_{cd})}{|G_+(l)|^{1/2}}\\
&= 3\cos(\theta_{cd})\sin(\theta_{ab}) \cong 3|\theta_{ab}-\theta^0_{ab}|.
\end{aligned}$$

The Lemma now follows from L'Hopital's rule and (33). □

The values of the Airy functions at zero are

$$\mathrm{Ai}(0) = (3^{2/3}\Gamma(\frac{2}{3}))^{-1}, \quad \mathrm{Bi}(0) = (3^{\frac{1}{6}}\Gamma(\frac{2}{3}))^{-1}.$$

Therefore, the limit of the $6j$ symbol as $j_{ab} \to j^0_{ab}$ is $(-1)^{j_{12}+j_{23}+j_{34}+j_{41}}$ resp. $(-1)^{j_{12}+j_{23}+j_{13}}$ times

$$\begin{aligned}\lim_{j_{ab}\to j^0_{ab}} \frac{\sin(\theta_{ab})^{1/2}}{|G_+(l)|^{1/4}} f(j_{ab}) &= \lim_{j_{ab}\to j^0_{ab}} \frac{2\pi^{3/2}((\frac{3}{2}|\phi-\phi_0|)^{2/3})^{1/4}}{r^{3/2}|G_+(l)|^{1/4}} \operatorname{Ai}(0) \\ &= \left(\frac{2\pi}{r}\right)^{3/2}\left(\frac{3}{2}\right)^{1/6}\left(\frac{3r}{2\pi A_aA_bA_cA_d}\right)^{\frac{1}{6}} 3^{-2/3}\Gamma\left(\frac{2}{3}\right)^{-1} \\ &= r^{-4/3}2^{2/3}3^{-1/3}\pi^{4/3}(A_aA_bA_cA_d)^{-1/6}\Gamma\left(\frac{2}{3}\right)^{-1}\end{aligned}$$

which is (b) in Theorem 1.0.1.

The LG-WKB method for finite difference equations near a turning point is studied by Geronimo, Bruno and Assche in [18, Theorem 3.8]. The assumptions [18, p.106] that guarantee that an approximate solution converges to the solution given by the ansatz translate to the conditions

$$\left(\frac{\partial}{\partial l_{ab}}\right)^i (\theta_{ab}-\theta^0_{ab})^2 \in C^0((0,\pi)),\ i=0,1,2,3; \tag{37}$$

$$\left(\frac{\partial}{\partial l_{ab}}\right)^i \frac{(\theta_{ab}-\theta^0_{ab})^2}{l_{ab}-l^0_{ab}} \in C^0((0,\pi)),\ i=0,1,2. \tag{38}$$

To see that these conditions holds, note that by (16) $\cos^2(\theta_{ab}-\theta^0_{ab})$ is a smooth function of l_{ab}. Hence so is $(\theta_{ab}-\theta^0_{ab})^2$ for θ_{ab} sufficiently close to $0,\pi$, which proves (37). (38) follows since $(\theta_{ab}-\theta^0_{ab})^2$ has a simple zero at $l_{ab}=l^0_{ab}$. Condition (iii) on [18, p.106] is achieved by choosing the sign of the parameter t appropriately, for $t=\pm(l_{ab}-l^0_{ab})$ sufficiently small.

Unfortunately we do not know whether there are any configurations of the type described in Theorem 1.0.1 (b), that is, degenerate tetrahedra with non-degenerate faces and all edge lengths commensurate with π. However, the following generalization of Theorem 1.0.1 (b) is non-vacuous:

Theorem 5.0.4. *Let $j_{ab}(k) \in [0,(r-2)/2]\cap\mathbb{Z}/2$ be a sequence of labels such that $l_{ab}(k) = (j_{ab}(k)+\frac{1}{2})/r(k)$ converge to the edge lengths l_{ab} of a degenerate tetrahedron with non-degenerate faces and $l_{ab}(k)-l_{ab} = O(k^{-1})$. The sequence $\begin{Bmatrix} j_{12}(k) & j_{23}(k) & j_{13}(k) \\ j_{34}(k) & j_{14}(k) & j_{24}(k)\end{Bmatrix}$ is asymptotic to the expression in Theorem 1.0.1 (b) as $k\to\infty$.*

Proof. By the discussion above, the sequence has asymptotics given by (36). By Schläfli's formula and Taylor's theorem, $\phi-\phi^0 < C(\theta_{ab}-\theta^0_{ab}) = O(k^{-1/2})$. Hence the argument in the Airy function in (36) goes to zero, which completes the proof. □

6. One face degenerate

If one face, say 123, is degenerate then there are two possibilities: either one length, say j_{13}, is the sum of the other two, $j_{12} + j_{23}$, or $j_{12} + j_{23} + j_{23} = r - 2$. In the first case the Racah sum (6) has a single term

$$\left\{ \begin{array}{ccc} j_{12} & j_{23} & j_{12}+j_{23} \\ j_{34} & j_{14} & j_{24} \end{array} \right\} = \left\{ \frac{[2j_{12}]![2j_{23}]![j_{12}+j_{23}+j_{34}+j_{14}+1]!}{[2j_{12}+2j_{23}+1]![-j_{12}-j_{23}+j_{34}+j_{14}]!} \right\}^{\frac{1}{2}} \tag{39}$$

$$(-1)^{j_{12}+j_{23}+j_{34}+j_{14}} \left\{ \frac{[j_{12}+j_{23}+j_{34}-j_{14}]![j_{12}+j_{23}-j_{34}+j_{14}]!}{[j_{23}+j_{34}-j_{24}]![j_{23}-j_{34}+j_{24}]![j_{23}+j_{34}+j_{24}+1]!} \right\}^{\frac{1}{2}}$$

$$\left\{ \frac{[j_{34}+j_{24}-j_{23}]![j_{14}+j_{24}-j_{12}]!}{[j_{12}+j_{24}+j_{14}+1]![j_{12}-j_{14}+j_{24}]![j_{12}+j_{14}-j_{24}]!} \right\}^{\frac{1}{2}}.$$

The asymptotics of the $6j$ symbol are therefore determined by the asymptotics of quantum factorials which have been investigated by Moak [28], see also [40]. The following can be derived from [28, (2.12-16)].

Proposition 6.0.5. *(q-Stirling) Set $q = \exp(\pi i/kr)$. Then as $k \to \infty$,*

$$[kn]! \sim \sqrt{2\pi}[kn]^{kn+\frac{1}{2}} I(n\pi/r)^{-kn} \tag{40}$$

where

$$I(x) := \sin(x) \exp\left(-x^{-1} \int_0^x \ln(\sin(y))\mathrm{d}y\right), \quad \lim_{x\to 0} I(x) = e.$$

We conjecture that the factors involving $I(n\pi/r)$ cancel; in the case $q = 1$ these factors are replaced by factors of e which do cancel. Assuming this, one obtains

$$s(k) \sim 2^{-1/4} r(k)^{-\frac{5}{4}} \pi \left\{ \frac{\sin(l_{12})^{\frac{1}{2}} \sin(l_{23})^{\frac{1}{2}} I(l_{13}, l_{34}, l_{41})}{\sin(l_{12}+l_{23})^{\frac{3}{2}} I(l_{12}, l_{24}, l_{41}) I(l_{23}, l_{34}, l_{41})} \right\}^{1/2} \tag{41}$$

times $(-1)^{k(j_{13}+j_{34}+j_{14})}$ where

$$I(l_{ab}, l_{bc}, l_{ac}) = \frac{H(l_{ab}, l_{bc}, l_{ac})^{1/2} \sin(\frac{1}{2}(l_{ab}+l_{bc}+l_{ac}))}{\sin(\frac{1}{2}(l_{bc}+l_{ac}-l_{ab}))}$$

and

$$H(l_{ab}, l_{bc}, l_{ac}) = \sin(\tfrac{l_{ab}+l_{bc}+l_{ac}}{2}) \sin(\tfrac{l_{ab}+l_{bc}-l_{ac}}{2}) \sin(\tfrac{l_{ab}-l_{bc}+l_{ac}}{2}) \sin(\tfrac{-l_{ab}+l_{bc}+l_{ac}}{2}) .$$

If $j_{12} = j_{23}$, (41) simplifies to

$$s(k) \sim \frac{r(k)^{-5/4}\pi}{\sin(l_{12})^{3/4} \sin(l_{24})^{1/2}}.$$

The analogous formula for $q = 1$ was checked numerically.

We do not know whether there exist configurations of the type described in Figure 2(c) in the spherical case with edge lengths commensurable to and less than π. Neither to do we have a uniform formula, similar to (36), which describes the asymptotics of the $6j$ symbols in the neighborhood of such a degeneration. The case $j_{12}+j_{23}+j_{13}=r-2$ seems even more mysterious.

7. Colinear vertices or degenerate edges

Suppose that every face of a tetrahedron τ is degenerate. We may assume without loss of generality that the vertices lie on a geodesic with length at most π in the order $1,2,3,4$ so that $j_{ab}+j_{bc}=j_{ac}$ for any $1\le a<b<c\le 4$. In this case the $6j$ symbol equals

$$(-1)^{2kj_{14}}([2kj_{13}+1][2kj_{24}+1])^{-1/2}\sim(-1)^{2kj_{14}}\pi r(k)^{-1}(\sin(l_{13})\sin(l_{24}))^{-1/2}.$$

In the case that one of the lengths is zero one obtains from (6) the formula

$$\begin{aligned}\left\{\begin{array}{ccc} kj_{12} & kj_{23} & kj_{13}\\ kj_{34} & kj_{12} & 0\end{array}\right\} &= (-1)^{kj_{12}+kj_{23}+kj_{13}}([2kj_{12}+1][2kj_{23}+1])^{-1/2}\\ &\sim (-1)^{k(j_{12}+j_{23}+j_{13})}\pi r(k)^{-1}\sin(l_{12})^{-\frac{1}{2}}\sin(l_{23})^{-\frac{1}{2}}.\end{aligned}$$

Similarly, one obtains the last two formulas in Theorem 1.0.1.

8. The classically forbidden case

Naturally one expects $6j$ symbols are exponentially decreasing in k if τ does not exist; this was proved in the case $q=1$ in [34]. The difference equation of Section 4 together with the results of Geronimo, Bruno and Assche [18] prove that the $6j$ symbols are exponentially decreasing as a function of a label j_{ab} after it passes into the classically forbidden regime. However, it is not clear to us whether this implies exponential decay as a function of k for arbitrary classically forbidden tetrahedra.

9. The hyperbolic case

In the case $q=\exp(\pi/r)$ (no factor of i) the asymptotics are related to hyperbolic tetrahedra. If it exists, let $\tau(k)$ denote the hyperbolic tetrahedron with edge lengths $l_{ab}(k)=2\pi(kj_{ab}+\frac{1}{2})/r(k)$ and exterior dihedral angles $\theta_{ab}(k)$. Let τ denote the hyperbolic tetrahedron with edge lengths $l_{ab}=2\pi j_{ab}/(r-2)$ and exterior dihedral angles θ_{ab}. One has identities similar to the spherical case, e.g.

$$|G_-(\tau)|^{\frac{1}{2}}=\frac{A_aA_b\sin(\theta_{cd})}{\sinh(l_{cd})}.$$

where $A_a=|\det(-\cosh(l_{bc}|_{b,c\neq a}))|^{1/2}$. We state the results in the hyperbolic case as follows. They are identical to the spherical case, except for the replacements sin with sinh, cos with $-\cosh$, and $\mathrm{Vol}(\tau)$ with $-\mathrm{Vol}(\tau)$). The proofs are similar.

Theorem 9.0.6. *Let $j_{ab}, 1\le a<b\le 4$ be non-negative half-integers, $r>2$ and*

$$s(k):=\left\{\begin{array}{ccc} kj_{12} & kj_{23} & kj_{13}\\ kj_{34} & kj_{14} & kj_{24}\end{array}\right\}_{q=\exp(\pi/r(k))}.$$

(a) *If τ exists and is non-degenerate, then*

$$s(k) \sim \frac{2\pi \cos(\phi(k) + \pi/4)}{r(k)^{3/2} |\det(-\cosh(l_{ab}))|^{1/4}}$$

where

$$\phi(k) = \frac{r(k)}{2\pi} \left(\sum_{a<b} \theta_{ab}(k) l_{ab}(k) + 2 \operatorname{Vol}(\tau(k)) \right).$$

(b) *If τ exists, has zero volume but all faces have non-zero area then*

$$s(k) \sim r(k)^{-\frac{4}{3}} \pi^{\frac{4}{3}} 2^{\frac{2}{3}} 3^{-\frac{1}{3}} \Gamma(\frac{2}{3})^{-1} \frac{\cos(k \sum \theta_{ab} j_{ab})}{(A_1 A_2 A_3 A_4)^{1/6}}.$$

(d) *If τ exists and has exactly one edge length, say l_{ab} vanishing, then*

$$s(k) \sim (-1)^{k(j_{bc}+j_{cd}+j_{bd})} \pi r(k)^{-1} (\sinh(l_{ac}) \sinh(l_{bd}))^{-1/2}.$$

(e) *If τ exists, all faces have zero area but all edge lengths are non-vanishing then τ lies on a geodesic, say with vertices in order a, b, c, d then*

$$s(k) \sim (-1)^{2kj_{ad}} \pi r(k)^{-1} (\sinh(l_{ac}) \sinh(l_{bd}))^{-1/2}.$$

(f) *If τ exists, all faces are degenerate, and all non-zero edge lengths are equal then (supposing without loss of generality that $l_{ad} \neq 0$)*

$$s(k) \sim (-1)^{2kj_{ad}} (\frac{\pi}{r(k)})^{1/2} \sinh(l_{ad})^{-1/2}.$$

(g) *If τ exists, but all edge lengths are vanishing then $s(k) = 1$.*

10. Questions

(a) Is there a uniform formula which includes all cases, just as the Schulten-Gordon formula (36) includes cases (a) and (b)? Such a formula (or least uniform estimates) would be needed to apply these results to the asymptotics of the quantum invariants such as Turaev-Viro and Jones, since one would presumably have to show that the contributions from the non-degenerate case are dominant.

(b) Is there a geometric description of the tensor category of representations of $U_q(\mathfrak{sl}_2)$ in terms of the moduli space of hyperbolic bundles [2], for q positive real, similar to the description for q a primitive root of unity using unitary bundles, which gives a geometric explanation of the hyperbolic formulas?

(c) Are there results for other values of q, for instance, q negative real, or $q = \exp(\pi i/r)$ but $j_{ab} \notin [0, (r-2)/2]$? Numerical experiments show that in some of these cases the $6j$ symbols are rapidly increasing.

(d) Are there similar formulas for $6j$ symbols related to the Kashaev-Reshetikhin invariants [20]?

(e) There are analogs of the $6j$ symbols for other groups. For example, for any four-tuple of dominant weights $j_{12}, j_{23}, j_{34}, j_{41}$ such that the tensor products $V_{j_{ab}} \otimes V_{j_{bc}}$ are multiplicity-free, one has two canonical bases for the space of invariants $V_{j_{12}} \otimes V_{j_{23}} \otimes V_{j_{34}} \otimes V_{j_{14}}$ given by the two ways of pairing. Are there explicit asymptotic formulas for these symbols?

References

[1] S. Agnihotri and C. Woodward. Eigenvalues of products of unitary matrices and quantum Schubert calculus. *Math. Res. Lett.*, 5(6):817–836, 1998.

[2] A. Y. Alekseev and A. Z. Malkin. The hyperbolic moduli space of flat connections and the isomorphism of symplectic multiplicity spaces. *Duke Math. J.*, 93(3):575–595, 1998.

[3] S. Axelrod, S. Della Pietra, and E. Witten. Geometric quantization of Chern-Simons gauge theory. *J. Differential Geom.*, 33:787–902, 1991.

[4] B. Bakalov and A. Kirillov, Jr. *Lectures on tensor categories and modular functors*, volume 21 of *University Lecture Series*. American Mathematical Society, Providence, RI, 2001.

[5] J. Barrett. Quantum gravity as topological quantum field theory. *J. Math. Phys.*, 36(11):6161–6179, 1995.

[6] P. Belkale. The Quantum Generalization of the Horn Conjecture. math.AG/0303013.

[7] P. Belkale. Local systems on $\mathbb{P}^1 - S$ for S a finite set. *Compositio Math.*, 129(1):67–86, 2001.

[8] L. M. Blumenthal. *Theory and applications of distance geometry.* Oxford Clarendon Press, 1953.

[9] D. Borthwick, T. Paul, and A. Uribe. Legendrian distributions with applications to relative Poincaré series. *Invent. Math.*, 122(2):359–402, 1995.

[10] J. S. Carter, D. E. Flath, and M. Saito. *The classical and quantum 6j-symbols.* Princeton University Press, Princeton, NJ, 1995.

[11] L.-C. Chen, M. E. H. Ismail, and P. Simeonov. Asymptotics of Racah coefficients and polynomials. *J. Phys. A*, 32(3):537–553, 1999.

[12] G. Daskalopoulos and R. Wentworth. Local degeneration of the moduli space of vector bundles and factorization of rank two theta functions. I. *Math. Ann.*, 297(3):417–466, 1993.

[13] M. Finkelberg. An equivalence of fusion categories. *Geom. Funct. Anal.*, 6(2):249–267, 1996.

[14] D. S. Freed and R. E. Gompf. Computer calculation of Witten's 3-manifold invariant. *Comm. Math. Phys.*, 141(1):79–117, 1991.

[15] C. Frohman and J. Kania-Bartoszynska. Quantum invariants away from roots of unity. preprint.

[16] M. Furuta and B. Steer. Seifert fibred homology 3-spheres and the Yang-Mills equations on Riemann surfaces with marked points. *Adv. Math.*, 96(1):38–102, 1992.

[17] F. R. Gantmacher. *The theory of matrices. Vol. 1.* AMS Chelsea Publishing, Providence, RI, 1998. Translated from the Russian by K. A. Hirsch, Reprint of the 1959 translation.

[18] J. S. Geronimo, O. Bruno, and W. Van Assche. WKB and turning point theory for second-order difference equations. In *Spectral methods for operators of mathematical physics*, volume 154 of *Oper. Theory Adv. Appl.*, pages 101–138. Birkhäuser, Basel, 2004.

[19] W. M. Goldman. Invariant functions on Lie groups and Hamiltonian flows of surface group representations. *Invent. Math.*, 85:263–302, 1986.

[20] R. Kashaev and N. Reshetikhin. Invariants of links with flat connections in their complements.II. holonomy r-matrices related to quantized universal enveloping algebras at roots of 1. math.AT/0202212.

[21] A. A. Kirillov, Jr. On an inner product in modular tensor categories. *J. Amer. Math. Soc.*, 9(4):1135–1169, 1996.

[22] A. N. Kirillov and N. Yu. Reshetikhin. Representations of the algebra $U_q(\mathrm{sl}(2))$, q-orthogonal polynomials and invariants of links. In *Infinite-dimensional Lie algebras and groups (Luminy-Marseille, 1988)*, pages 285–339. World Sci. Publishing, Teaneck, NJ, 1989.

[23] P. P. Kuliš and N. Ju. Rešetikhin. Quantum linear problem for the sine-Gordon equation and higher representations. *Zap. Nauchn. Sem. Leningrad. Otdel. Mat. Inst. Steklov. (LOMI)*, 101:101–110, 207, 1981. Questions in quantum field theory and statistical physics, 2.
[24] J. Murakami and A. Ushijima. A volume formula for hyperbolic tetrahedra in terms of edge lengths. math.MG/0402087.
[25] V. B. Mehta and C. S. Seshadri. Moduli of vector bundles on curves with parabolic structure. *Math. Ann.*, 248:205–239, 1980.
[26] J. Milnor. *Collected papers. Vol. 1.* Publish or Perish Inc., Houston, TX, 1994. Geometry.
[27] S. Mizoguchi and T. Tada. 3-dimensional gravity and the Turaev-Viro invariant. *Progr. Theoret. Phys. Suppl.*, (110):207–227, 1992. Recent developments in string and field theory (Kyoto, 1991).
[28] D. S. Moak. The q-analogue of Stirling's formula. *Rocky Mountain J. Math.*, 14(2):403–413, 1984.
[29] Y. Mohanty. *Hyperbolic polyhedra: volume and scissors congruence*. PhD thesis, University of California at San Diego, 2002.
[30] H. Murakami and J. Murakami. The colored Jones polynomials and the simplicial volume of a knot. *Acta Math.*, 186(1):85–104, 2001.
[31] C. Pauly. Espaces de modules de fibrès paraboliques et blocs conformes. *Duke Math. J.*, 84:217–235, 1996.
[32] G. Ponzano and T. Regge. Semiclassical limit of Racah coefficients. In *Spectroscopic and group theoretical methods in physics*, pages 1–58. Amsterdam, 1968. North-Holland.
[33] G. Racah. Theory of complex spectra. I. *Phys. Rev. 94*, 62:438–462, 1942.
[34] J. Roberts. Classical $6j$-symbols and the tetrahedron. *Geom. Topol.*, 3:21–66 (electronic), 1999.
[35] J. Roberts. Asymptotics and 6j-symbols. *Geom. Topol. Monogr.*, 4:245–261, 2002.
[36] K. Schulten and R. G. Gordon. Semiclassical approximations to $3j$- and $6j$-coefficients for quantum-mechanical coupling of angular momenta. *J. Mathematical Phys.*, 16(10):1971–1988, 1975.
[37] C. Teleman and C. Woodward. Parabolic bundles, products of conjugacy classes and Gromov-Witten invariants. *Ann. Inst. Fourier (Grenoble)*, 53(3):713–748, 2003.
[38] A. Tsuchiya, K. Ueno, and Y. Yamada. Conformal field theory on universal family of stable curves with gauge symmetries. In *Integrable systems in quantum field theory and statistical mechanics*, volume 19 of *Adv. Stud. Pure Math.*, pages 459–566. Academic Press, 1989.
[39] V. G. Turaev and O. Ya. Viro. State sum invariants of 3-manifolds and quantum $6j$-symbols. *Topology*, 31(4):865–902, 1992.
[40] K. Ueno and M. Nishizawa. Quantum groups and zeta-functions. In *Quantum groups (Karpacz, 1994)*, pages 115–126. PWN, Warsaw, 1995. hep-th/940814.
[41] J. K. G. Watson. Asymptotic approximations for certain $6j$ and $9j$ symbols. *J. Phys. A*, 32(39):6901–6902, 1999.
[42] E. P. Wigner. *Group theory: And its application to the quantum mechanics of atomic spectra*. Academic Press, New York, 1959.

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