

Floer cohomology of Gelfand-Zeitlin fibers

2026 Les Diablerets Eckhardfest Workshop

Chris Woodward, Yao Xiao Rutgers University - New Brunswick



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Abstract: Yao and I compute the Floer cohomology of Lagrangian Gelfand-Zeitlin fibers arising from square tilings, generalizing work of my former student Castronovo (who studied the generic fiber) and Nohara-Ueda (who studied the $U(2)$ -fiber in $Gr(2, 4)$). Partly using ChatGPT we prove homological mirror symmetry $Gr(k, n)$ for $n < 10$.

The main result is an explicit formula for the disk potential and Floer differential for such a Gelfand-Cetlin fiber, on Borel generators.

Same application: For $n < 10$, the Fukaya category of $\mathrm{Gr}(k, n)$ is split generated by (possibly special) Gelfand-Zeitlin fibers, except for $k = 4, n = 8$, and the categorical version of mirror symmetry (hms) holds for these Grassmannians. For $k = 4, n = 8$, hms holds and the Fukaya category is generated by special Gelfand-Zeitlin fibers and a Lagrangian torus corresponding to a non-rectangular cluster chart.

MIRROR SYMMETRY

Rough idea of mirror symmetry for a symplectic manifold X , à la Strominger-Yau-Zaslow:

Find a Lagrangian torus fibration, probably singular.

The mirror $X^\vee$ is roughly the space of Lagrangian fibers with local systems.

Counts of holomorphic disks bounding the Lagrangian determine the *Landau-Ginzburg potential* $W : X^\vee \rightarrow \mathbb{C}$ and complex structure.

The mirror of X is the pair $(X^\vee, W)$.

APPROACHES TO CONSTRUCTING MIRRORS

In general, part of the problem is constructing Lagrangian fibrations.

There are various workarounds. E.g. Gross-Siebert assume an anti-canonical divisor with a maximally degenerate point.

Varolgunes constructs a mirror via relative symplectic homology, but needs an involutive cover.

Various recent work constructs Lagrangian fibrations in low-dimensional cases, e.g. del Pezzo surfaces, Fano three-folds.

MIRROR SYMMETRY FOR GRASSMANNIANS

In the case of the Grassmannian, there is a natural Lagrangian fibration-with-singularities, given by the Gelfand-Zeitlin system.

Unfortunately, it is rather hard to count the pseudoholomorphic disks in the fibers. Nohara-Ueda-Yamada did a partial computation for the generic fiber, using "small toric degeneration". But this technique doesn't work for all fibers.

TROPICAL/DEGENERATION TECHNIQUES

Venugopalan and I wrote a book ("Tropical Fukaya algebras") showing how symplectic cutting can be used to do count pseudoholomorphic disks as sums of tropical graphs (generalizing work by Brett Parker and others on counting pseudoholomorphic spheres) and a follow-up paper on almost toric manifolds.

The goal of this talk is to explain how this works in the special case of Grassmannians and extend the work of Nohara-Ueda-Yamada to (possibly singular) fibers.

THE GRASSMANNIAN

Let $0 < k < n$ be integers.

The Grassmannian $\text{Gr}(k, n)$ is the set of k -dimensional subspaces of $\mathbb{C}^n$.

Equivalently, $\text{Gr}(k, n)$ is the set of $k \times n$ matrices of rank k modulo row-equivalence.

GRASSMANNIANS

Grassmann published his work on linear algebra, such as the dimension formula

$\dim(V + W) = \dim(V) + \dim(W) - \dim(V \cap W)$, but his work was ignored until much later. His publisher wrote him "Your book *Die Ausdehnungslehre* has been out of print for some time. Since your work hardly sold at all, roughly 600 copies were used in 1864 as waste paper and the remaining few odd copies have now been sold out, with the exception of the one copy in our library."

Grassmann: "My hope of obtaining an academic position missed its mark" "no one can blame the publisher" "
(Translation by Kannenberg)

GRASSMANNIANS

In modern language, Grassmannians are manifolds.

Charts are provided by Gauss-Jordan elimination, e.g. for $\text{Gr}(2, 4)$

$$\begin{bmatrix} 1 & 0 & a & b \\ 0 & 1 & c & d \end{bmatrix}$$

is one chart; there are $\binom{4}{2}$ positions of the pivots and so 6 such charts.

SYMPLECTIC STRUCTURE ON THE GRASSMANNIAN

We may identify $\text{Gr}(k, n)$ as a symplectic leaf of the Lie-Poisson structure on $\mathfrak{u}(n)^\vee$ by the map

$$\text{row-space}(A) \mapsto iA^*A$$

where A is a $k \times n$ matrix whose rows are a unitary basis of the subspace.

The image is a coadjoint orbit, namely, the set of skew-Hermitian matrices with eigenvalues $0, i$ of multiplicity $(n - k), k$.

GELFAND-ZEITLIN MAP

The Grassmannian has a natural fibration-with-singularities by Lagrangian tori.

The action of $U(n)$ on $\text{Gr}(k, n)$ is Hamiltonian with moment map given by the inclusion in $\mathfrak{u}(n)^\vee$. Consider the chain of embeddings

$$U(1) \subset U(2) \dots \subset \dots U(n-1).$$

We can identify the codomain with Hermitian matrices and take the eigenvalues to get a continuous not-everywhere-smooth map

$$\Psi : X \rightarrow \mathbb{R}^{n(n-1)/2}.$$

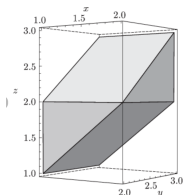
Guillemin-Sternberg: The symplectic cross-section theorem implies that the components $\lambda_{i,j}$, which are the eigenvalues of the upper left $i \times i$ minors, Poisson commute where smooth.

INTERLACING INEQUALITIES

Gelfand-Naimark: These eigenvalues satisfy the interlacing inequalities

$$\lambda_{i,j} \geq \lambda_{i-1,j} \geq \lambda_{i,j+1}.$$

Denote the set of λ satisfying these inequalities $GC(k, n)$, the *Gelfand-Zeitlin polytope*. (Figure from Smirnov shows a slice of the polytope $GC(2, 4)$ at $\lambda_{2,2} = 1/2$)



GELFAND-ZEITLIN DIAGRAMS

We write the eigenvalues $\lambda_{i,j}$ diagonally so that the columns and rows are non-increasing. We can abbreviate the diagram to keep track of only the bottom left $k \times (n - k)$ matrix $\mu_{i,j}$.

[illegible]

Cho-Kim-Oh, Bouloc-Miranda-Zung, Carlson-Lane: The fiber over any μ is an isotropic submanifold, given by an iterated sphere bundle.

TOPOLOGY OF GELFAND-ZEITLIN FIBERS

What do the other fibers look like?

Cho-Kim-Oh, Bouloc-Miranda-Zung: Gelfand-Zeitlin fibers are iterated sphere bundles. If the pattern corresponds to a tiling by squares, then the fibers are Lagrangian and diffeomorphic to products of unitary groups.

Example:

$$\mu = \begin{array}{cc|c} 1 & 1 & \\ \hline 4/6 & 4/6 & 0 \\ 4/6 & 4/6 & 0 \\ 2/6 & 2/6 & 0 \\ 2/6 & 2/6 & 0 \end{array}$$

corresponds to a tiling by 2×2 squares and gives $U(2) \times U(2)$ fiber.

THE HERMITIAN LAGRANGIAN GRASSMANNIAN

If $X = Gr(n, 2n)$ then there is a pattern with a single $n \times n$ square, e.g.

$$\mu = \begin{array}{cc|c} 1 & 1 & \\ \hline 1/2 & 1/2 & 0 \\ 1/2 & 1/2 & 0 \end{array}$$

The Gelfand-Zeitlin fiber is a copy of $U(n)$. In fact, this was studied by Arnold under the name of the Hermitian Lagrangian Grassmannian:

Thm (essentially Arnold): The Gelfand-Zeitlin fiber L with one block is the set of subspaces W of $\mathbb{C}^{2n}$ such that $(JW)^\perp = W$, where J is the matrix for the standard complex structure matrix, and $\perp$ is the Hermitian perpendicular.

BOREL'S DESCRIPTION OF THE COHOMOLOGY OF $U(n)$

We will want to think of Floer cohomology as a "quantization" of ordinary cohomology. The cohomology of the Gelfand-Zeitlin fibers $L \cong \prod U(d_j)$ is described in Borel's thesis as an exterior algebra on generators of odd degree.

One can obtain this description directly from Nicolaescu's description of the Morse complex of the function

$$f : U(n) \rightarrow \mathbb{R}, A \mapsto \sum_i c_i \operatorname{Re}(A_{ii})$$

for c_i generic.

The critical points are diagonal entries with entries ± 1 , and this Morse function is perfect, hence $\chi(U(n)) = 2^n$.

BOREL GENERATORS VIA MAURER-CARTAN

Denote the *Borel generators* with a single -1 on the diagonal by $e_1, \dots, e_n$.

Nicolaescu: The Morse basis for $H(U(n))$ is equivalent to Borel's basis, and the generators e_i are cohomologous (up to constants) to

$$\omega_i := \text{Tr}((g^{-1}dg)^{2i-1}) \in \Omega(U(n)).$$

In fact, the unstable manifolds are the Harvey-Lawson limits of ω_i , as currents.

VERSIONS OF MIRROR SYMMETRY

Versions of mirror symmetry:

Closed string: $QH(X)$ is the space $\text{Fun}(\text{Crit}(W)) =: \text{Jac}(W)$ of functions on the critical locus of W , or something similar.

Open string: The Fukaya category of X is equivalent to the category of derived matrix factorizations of W .

In the case of Grassmannians, the story is partially understood.

CLOSED STRING MIRROR SYMMETRY

Rietsch: $QH(X) = Jac(W)$ for some W introduced by Dale Peterson, see also Eguchi, Hori, and Xiong (and is also the Verlinde algebra, and is twisted K-theory of a group,), Marsh-Rietsch

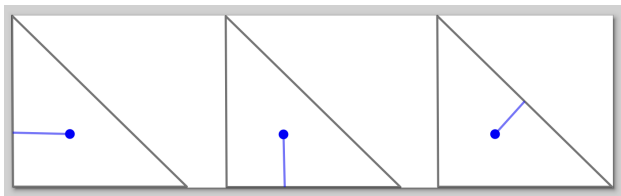
Later it was proved that W is equal, on an open set, to the disk potential for a generic Gelfand-Zeitlin fiber.

Nohara-Ueda-Yamada, Castronovo: $W(y) = \sum y^\nu$ where the sum is over facets of the Gelfand-Zeitlin polytope and ν is the normal vector to the facet, and y is the coordinate on local systems.

CLOSED STRING MIRROR SYMMETRY FOR GRASSMANNIANS

Example: $X = \mathbb{CP}^2$. The Gelfand-Zeitlin fibration is the usual fibration as a toric variety. Let L be the fiber over the barycenter of the moment polytope. There are three isolated (up to isomorphism) disks (approximate moment images shown below)

$$z \mapsto [z, 1, 1], [1, z, 1], \text{ or } [1, 1, z].$$



CP^2 CONTINUED

Local systems on $L = \{(a, b)\} = (\mathbb{C}^\times)^2$.

The mirror potential is $W(a, b) = a + b + 1/(ab)$ and the Jacobian ring is $\mathbb{C}[a^{\pm 1}, b^{\pm 1}]$ /first derivatives of W which is

$$\begin{aligned} QH(X) = Jac(W) &= \mathbb{C}[a, b] / \langle a\partial_a W = a - 1/(ab), b\partial_b W = b - 1/(ab) \rangle \\ &= \mathbb{C}[a, b] / \langle a - b, a - 1/ab \rangle = \mathbb{C}[a] / \langle a^3 - 1 \rangle. \end{aligned}$$

To get $H(X)$ we just take the homogenous part of the relation $a^3 = 0$. This recomputes the cohomology ring of CP^2 , just from knowing the disks.

OPEN STRING MIRROR SYMMETRY

Open string mirror symmetry for $\text{Gr}(k, n)$ is known only in some cases, e.g. Castronovo for n prime and $G(2, 4)$ by Nohara-Ueda.

Note that $QH(X) = \text{Jac}(W)$ only holds if one includes some critical points which have some variables equal to zero, which do not correspond to local systems. These critical points do not fit inside the Strominger-Yau-Zaslow picture, because these points are not in any dual torus in the fibration.

Three possibilities: (a) Look at other "cluster charts" corresponding to other Lagrangian fibrations of the Grassmannian.

(b) Look at the "singular fibers"

or maybe some combination of (a+b) is needed.

CLUSTER CHARTS ON GRASSMANNIANS

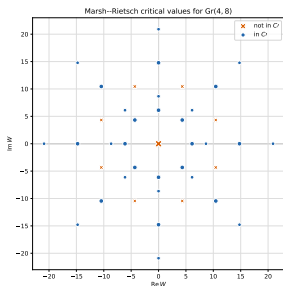
Castronovo studied in a later paper the other cluster charts on the dual Grassmannian (which is a cluster variety) arising from other toric degenerations.

Castronovo showed that these charts give rise to other Lagrangian fibrations, which give rise to non-Hamiltonian-isotopic Lagrangian tori (generalizing a result of Vianna for the projective plane.)

For $\text{Gr}(k, n)$ with n prime he proved open string mirror symmetry.

MISSING EIGENVALUES

$\text{Crit}(W)$ for the Marsh-Rietsch potential W is the spectrum of $c_1 \star$ on $QH(\text{Gr}(k, n))$. However there are eigenvalues not in any of the cluster charts, i.e. a non-empty *deep locus* studied by Castronovo-Gorsky-Simental-Speyer. Critical values for $\text{Gr}(4, 8)$ are below:



Our results show that the “missing” critical values arise from special Gelfand-Zeitlin fibers.

THE FUKAYA CATEGORY

Open string mirror symmetry is a statement about the Fukaya category. The Fukaya category of a symplectic manifold X has objects given by Lagrangians with some additional structure such as a flat $\mathbb{C}^\times$ -bundle. (Many flavors.)

Morphisms from L_1 to L_2 are chains on $L_1 \cap L_2$. Composition maps

$$m_d : \mathrm{Hom}(L_{d-1}, L_d) \times \dots \times \mathrm{Hom}(L_0, L_1) \rightarrow \mathrm{Hom}(L_0, L_d)$$

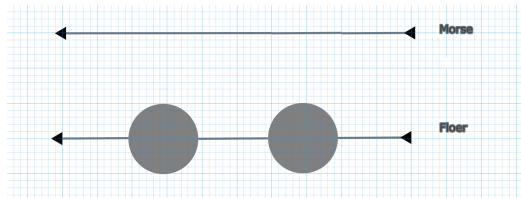
count pseudoholomorphic polygons, weighted by holonomy.

FLOER HOMOLOGY

An important first question is to find the non-trivial objects in the Fukaya category. (Many Lagrangian isomorphic to the empty object.) For this it suffice to compute the self-hom groups.

Choose $d = 1$ and $L_0 = L_1$. If m_0 is central, then $m_1^2 = 0$ and $HF(L) = H(\text{Hom}(L, L))$ is the *Lagrangian Floer cohomology* of L . This centrality always holds if L is *monotone*, i.e. the relative first Chern class is a positive multiple of the symplectic class, which I will always assume.

One can think of the complex $\text{Hom}(L, L)$ as the Morse complex, with the differential deformed by including contributions from holomorphic disks.



The Floer operator m_1 on degree one generators is obtained by differentiating the potential W (Divisor equation.)

Example: Let $X = \mathbb{CP}^2$ with area one. There are two generators $a_1, b_1 \in H(L)$ with $L \cong S^1 \times S^1$ the central fiber of the moment map. Since $W(a, b) = a + b + 1/(ab)$ from the three Blaschke disks,

$$m_1(a_1) = a\partial_a W = a - 1/(ab), \quad m_1(b_1) = b\partial_b W = b - 1/(ab)$$

For vanishing of the differential, it's enough to check vanishing on the degree one generators.

LOCAL SYSTEMS ON THE CLIFFORD TORUS

GENERATING THE FUKAYA CATEGORY

The differential vanishes iff

$$a = b, a^3 = 1 \implies a = b =^3 \sqrt{1}$$

so there are three value of (a, b) for which one gets a non-trivial object in the Fukaya category. These are the three generators of the Fukaya category in this case.

(Compare with the derived category of sheaves generated by $\mathcal{O}(0), \mathcal{O}(1), \mathcal{O}(2)$.)

SYMPLECTIC CUTTING

Our goal is to generalize this computation to Gelfand-Zeitlin fibers corresponding to square tilings.

The main technique is multidirectional symplectic cutting. The effect of cutting on quantization studied in the 90s.

MULTIDIRECTIONAL SYMPLECTIC FIELD THEORY (NECK STRETCHING)

Symplectic cutting i.e. multidirectional sft allows one to go from local to global.

In graduate school I learned the following formula from Eckhard: If $\mathcal{P} = \{P\}$ is a polyhedral decomposition e.g. of the Gelfand-Zeitlin polytope, and X_P are the cut spaces, then

$$\chi(X) = \sum_P (-1)^{\text{codim}(P)} \chi(X_P)$$

where χ denotes index of some spin-c dirac operators.



MULTIDIRECTIONAL SYMPLECTIC FIELD THEORY (NECK STRETCHING)

The analog in pseudoholomorphic curve theory is (multidirectional) symplectic field theory, following works of Hofer et al, E. Ionel, B. Parker,, Woodward-Venugopalan in symplectic geometry, Jun Li, Abramovich-Chen-Gross-Siebert in algebraic geometry.

These go under the name of *relative maps* (for the single cut version) or *exploded* or *log stable* or *broken* maps for the multi-directional case (many versions.)

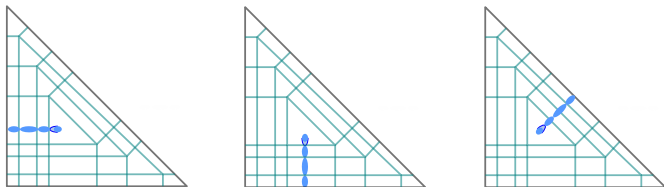
THE DUAL COMPLEX

The cutting formula in pseudoholomorphic curve has an additional assumption: The polyhedral decomposition $\mathcal{P}$ has to admit a *dual complex*: A collection of polytope $P^\vee$ satisfying the dual relation that $P^\vee$ is a face of $Q^\vee$ whenever Q is a face of P . The dual complex $B^\vee$ is obtained by gluing together the polytopes $P^\vee$.

Example: If the decomposition is into $(-\infty, 0)$, $\{0\}$, $(0, \infty)$ then a dual complex is $B^\vee = \{-1\} \cup [-1, 1] \cup \{1\}$.

WHAT DO BROKEN DISKS LOOK LIKE?

Typically the limits of broken maps are strings of spheres and disks



The broken maps contributing to the differentials often are just strings, i.e. have no trivalent vertices in the graph.

SUM OVER TROPICAL GRAPHS

Suppose $n(X)$ is some kind of pseudoholomorphic curve count. Then n can be written as a sum over tropical graphs Γ in the dual complex $B^\vee$:

$$n(X) = \sum_{\Gamma \rightarrow B^\vee} n(X, \Gamma)$$

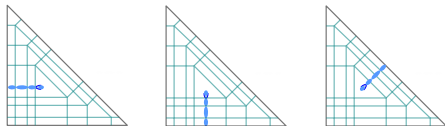
counting collections of pseudoholomorphic maps $(u_v : C_v \rightarrow X_{P(v)})$ satisfy matching conditions at the edges. In good cases (with caveats, in general) $n(X, \Gamma)$ can be written as a product over local contributions

$$n(X, \Gamma) = (\# \operatorname{Aut}(\Gamma))^{-1} \sum_{v \in \operatorname{Vert}(\Gamma)} n(X, \Gamma, v).$$

This is a local-to-global formula, i.e. replacement for Mayer-Vietoris.

REDUCTION TO THE CASE OF A FIXED POINT SET OF AN INVOLUTION

Main Lemma: Let $\mathcal{P}$ be a sufficiently fine polyhedral decomposition. Let u be a broken map contributing to the differential of a Borel generator $m_1(e_{i,j})$. Then u consists of a disk u_0 and a string of spheres $u_i, i > 0$, with the last intersecting the anticanonical divisor once.



Idea of Proof: Various area formulas.

LOCAL PICTURE

Say two integrable systems $(X_k, \omega_k, f_{1,k}, \dots, f_{n,k})$ are *isomorphic* if there is a symplectomorphism intertwining the functions $f_{1,k}, \dots, f_{n,k}$.

Lemma: Let μ be a Gelfand-Zeitlin pattern tiled by squares, and $L_\mu \cong \prod_j U(d_j)$ the corresponding Gelfand-Zeitlin fiber. Then $\text{Gr}(k, n)$ is locally isomorphic to $\prod_j \text{Gr}(d_j, 2d_j)$.

Idea of proof: Show that both have a local model given by the *double Gelfand-Zeitlin system* studied by Crooks-Weitsman (use left and right actions on $\prod_j T^*U(d_j)$).

It would be nice if this were implied by some kind of Delzant theorem, but I couldn't find any relevant one.

COMPUTATION OF THE LOCAL INVARIANTS

The count of pseudoholomorphic spheres in the string is a standard computation in toric geometry: Each corresponds to the closure of a one-parameter subgroup, and is unique for the given constraint.

The disk count, by the local model, is count of pseudoholomorphic disks with boundary in the Hermitian Lagrangian considered by Nohara-Ueda.

Arnold's Lagrangian Grassmannian L is the fixed point set of the involution $W \mapsto (JW)^\perp$, so pseudoholomorphic disks can be doubled to pseudoholomorphic spheres in the Grassmannian, and these can be classified by *Blaschke-Potapov factors*. The total count is ± 1 .

FLOER COHOMOLOGY OF GELFAND-ZEITLIN FIBERS

To sum up: We've used symplectic cutting to reduce the count to the case that the Lagrangian is the fixed point set of an anti-symplectic involution. The following is the result:

Let μ be a Gelfand-Zeitlin pattern tiled by squares of size d_j . The space of flat $\mathbb{C}^\times$ -bundles on L_μ is $(\mathbb{C}^\times)^{\#blocks} = \{(y_j)\}$. We also consider the 1's and 0's as a *fake block*; these are not squares; declare the corresponding block variable to be 1.

Main Theorem (W-Y): The Floer complex $\text{Hom}(L, L)$ is additively the exterior algebra on Borel generators $e_{i,j}, i \leq d_j$ with differential

$$m_1(e_{i,j}) = \sum_k \pm (y_j/y_k)^{\pm 1}$$

for each adjacent (true or fake) block variable y_k with difference i/n

EXAMPLE: THE EQUATOR IN THE TWO-SPHERE

Consider $L_{1/2} \subset G(1, 2)$ corresponding to the pattern

$$\nu = \frac{1}{1/2} \bigg| 0$$

There is 1 block with variable y_1 , and two adjacent fake blocks with differences $1/2$ with variables 1.

So $m_1(e_{1,1}) = y_1 - 1/y_1$ which is the same result we got before by counting disks.

OPEN STRING MIRROR SYMMETRY FOR $\text{Gr}(2, 6)$

Let $X = \text{Gr}(2, 6)$ and

$$\mu = \begin{array}{cc|c} 1 & 1 & \\ \hline 5/6 & 4/6 & 0 \\ 3/6 & 3/6 & 0 \\ 3/6 & 3/6 & 0 \\ 2/6 & 1/6 & 0 \end{array}$$

The fiber is of the form below and so has a single 2×2 square and four 1×1 squares::

$$\mu = \begin{array}{cc|c} 1 & 1 & \\ \hline a & b & 0 \\ c & c & 0 \\ c & c & 0 \\ d & e & 0 \end{array}$$

DIFFERENTIALS FOR $\mathrm{Gr}(2, 6)$

We write $a_1, b_1, c_1, d_1, e_1 \in CF(L_\mu)$ for the degree one Borel generators and $c_3 \in CF(L_\mu)$ the degree 3 Borel generator. The differentials are

$$m_1(a_1) = -1/a + a/b$$

$$m_1(b_1) = -a/b + b/c$$

$$m_1(c_1) = b/c + c/d$$

$$m_1(d_1) = c/d + d/e$$

$$m_1(e_1) = -d/e + e.$$

$$m_1(c_3) = c/a + e/c.$$

All the differentials including e_3 vanish if

$$1/a = a/b = b/c = -c/d = d/e = e/c = \alpha$$

Since the product of these ratios is -1 , we must have

$$\alpha^6 = -1.$$

We get six local systems for which the Floer cohomology is non-zero.

Together with the pattern with three 2×2 blocks and the pattern with six 1×1 blocks, one can show that one gets $\chi(\mathrm{Gr}(2, 6)) = 15$ local systems with non-vanishing cohomology and proves open string mirror symmetry for this Grassmannian, which was not known before.

OPEN QUESTIONS

- (1) There are other Lagrangian Gelfand-Zeitlin fibers, other than those corresponding to square tilings, which we don't know how to treat, because we have no local model. Is there a Delzant theorem for integrable systems of this type?
- (2) We can prove homological mirror symmetry for $\text{Gr}(k, n)$ for $n < 10$, partly using ChatGPT. The problem in general is one of combinatorics: How does one tell which square tilings give non-trivial Floer cohomology?
- (3) What is the relationship of the potentials of Gelfand-Zeitlin fibers to the Peterson-Marsh-Rietsch potential on the boundary (e.g. deep locus)?