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Stability of valuations and Kollár components

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Abstract. We prove that among all Kollár components obtained by plt blow ups of a klt singularity $o \in (X, D)$, there is at most one that is (log-)K-semistable. We achieve this by showing that if such a Kollár component exists, it uniquely minimizes the normalized volume function introduced in [Li18] among all divisorial valuations. Conversely, we show that any divisorial minimizer of the normalized volume function yields a K-semistable Kollár component. We also prove that for any klt singularity, the infimum of the normalized volume function is always approximated by the normalized volumes of Kollár components.

Keywords. Klt singularity, Kollár component, normalized volume, K-stability

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1. Introduction

Throughout this paper, we work over the field $\mathbb{C}$ of complex numbers. It has been well known by people working in higher-dimensional geometry that there is an analogue between the local objects, Kawamata log terminal (klt) singularities, and their global counterparts, log Fano varieties (cf. e.g. [Sho00, Xu14] etc.). From this comparison, since the stability theory of Fano varieties has been a central object of study in the last three decades, it is natural to expect that there is a local stability theory on singularities. The primary goal of this article is to develop such a theory. In other words, we want to investigate singularities using tools from the theory of K-stability, a notion first defined in [Tia97] and later algebraically formulated in [Don02]. This interaction between birational geometry and K-stability theory has proved to significantly fertilize both subjects (cf. [Oda12, Oda13, LX14, WX14, LWX19, Fuj18] etc.).

For the stability theory of log Fano varieties, a crucial ingredient is CM weight. Philosophically, the stability of log Fano varieties is equivalent to minimizing CM weight. In the stability theory of singularities, we fix a singularity (X, o) and look for 'the most stable' valuation $v \in \mathrm{Val}_{X,o}$ which is centered at o. Thus the first step in establishing a local stability theory for (X, o) would be to find the right counterpart of CM weight in the local setting. As a candidate the first named author [Li18] defined the normalized volume function $\widehat{\mathrm{vol}}_{(X,D),o}$ on the space of valuations centered at o. Its derivatives at the canonical divisorial valuation over a klt cone singularity along certain tangent directions associated to special test configurations are indeed CM weights. So in some sense, using the local picture, the normalized volume function carries more information than CM weight.

By the above discussion and inspired by the global theory, we focus on studying the valuation that minimizes the normalized volume function, which is conjectured to uniquely exist and ought to be thought of as a '(semi)stable' object. This picture is well understood in the case of Sasakian geometry where one only considers the valuations coming from the Reeb vector fields induced by a good torus action (e.g. [MSY08, CS19]). Here we can naturally compare the stability of the singularity with the stability for the base. However, this requires the extra cone structure. By investigating the minimizer of the normalized volume function on all valuations, our plan, as we mentioned, built on the previous work [Li18, Li17b, LL19], is to establish an intrinsic stability which only depends on the isomorphism class of the singularity. We recall that it was shown in [Li17b, LL19] that a Fano manifold X is K-semistable if and only if that among all valuations over the vertex o of the cone $C(X)$ given by a multiple of $-K_X$, the canonical valuation obtained by blowing up the vertex $o \in C(X)$ minimizes the normalized volume function.

This gives evidence to justify that at least for these singularities, our study is in the right direction.

For an *arbitrary* klt singularity, there is no direct way to associate a global object. Nevertheless, in differential geometry, when there is a 'canonical' metric, the metric tangent cone around the singularity is the stable object in the category of metric spaces. With a similar philosophy, we expect that the minimizer of the normalized volume function always gives *a degeneration* to a K-semistable Fano cone singularity in the Sasakian setting, and conversely any such degeneration should be provided by a minimizer of $\widehat{\mathrm{vol}}_{X,o}$. In the current paper, we work out this picture in the case that the minimizer is divisorial, by implementing the machinery of the minimal model program (MMP) (based on the foundational results in [BC+10]). So our treatment will be purely algebraic though it is strongly inspired by analytic results in the study of Kähler–Einstein/Sasaki–Einstein metrics.

One ingredient we introduce is the volume associated to a birational model and then we connect it to the normalized volume of a valuation. For studying the divisorial valuations, the class of models which play a central role here are the ones obtained in the construction of a *Kollár component* (cf. [Xu14]): for an arbitrary n-dimensional klt singularity (X, o), we can use the minimal model program to construct a birational model whose exceptional locus is an $(n-1)$-dimensional log Fano variety. In this paper, we will systematically develop the tools using Kollár components to understand the normalized volume function and its minimizers. In fact, Kollár components can be considered as the local analogue of special degenerations studied in [LX14]. We also observe that in the set-up of Sasakian geometry, a Reeb vector gives rise to a Kollár component if and only if it is rational, i.e., it is quasi-regular.

Therefore, to summarize, the aim of this paper is twofold. On the one hand, we aim to use the construction of Kollár components to get information on the space of valuations, especially for the minimizers of the normalized volume function. On the other hand, in the reverse direction, we want to use the stability viewpoint to study the construction of Kollár components in birational geometry, and search out a more canonical object under suitable assumptions.

We also expect that for any klt singularity (X, o), even when the minimizer is not necessarily divisorial, we can still use suitable birational models to degenerate (X, o) to a K-semistable (possibly irregular) singularity with a torus action of higher rank. However, that seems to involve a significant amount of new technical issues.

Below, we will give more details.

1.1. Kollár components

Definition 1.1 (Kollár component). Let $o \in (X, D)$ be a klt singularity. We say that a proper birational morphism $\mu\colon Y \to X$ provides a *Kollár component* S if μ is isomorphic over $X \setminus \{o\}$, and $\mu^{-1}(o)$ is an irreducible divisor S such that $(Y, S + \mu_*^{-1} D)$ is purely log terminal (plt) and $-S$ is $\mathbb{Q}$-Cartier and ample over X.

We easily see that the birational model Y is uniquely determined once the divisorial valuation S is fixed, and if we denote

$$(K_Y + S + \mu_*^{-1} D)|_S = K_S + \Delta_S \tag{1}$$

(see [Kol13, Definition 4.2]), then (S, Δ_S) is a klt log Fano pair.

Given any klt singularity $o \in (X, D)$, as the necessary minimal model program type result is established (see [BC+10]), we know that there always exists a Kollár component (see [Pro00] or [Xu14, Lemma 1]), but it is often not unique (nevertheless, see the discussion in 7.1.4 for some known special cases of uniqueness). From what we have discussed, instead of an arbitrary Kollár component, we want to study those which are 'the most stable', and show that they yield canonical objects if they exist. Indeed, we shall prove that if there is a K-semistable Kollár component, then it gives a unique minimizer of $\widehat{\mathrm{vol}}_{(X,D),o}$ among all Kollár components (actually even among all divisorial valuations).

Compared to the global theory of degeneration of Fano varieties, this fits into the philosophy that K-stability provides a canonical degeneration (cf. [LWX19, SSY16]) and it should minimize CM weight among all degenerations.

The following theorem is our main result. See Definition 2.4 for the definition of $\widehat{\mathrm{vol}}_{(X,D),o}$.

Theorem 1.2. *Let $o \in (X, D)$ be a klt singularity. A divisorial valuation ord_S is a minimizer of $\widehat{\mathrm{vol}}_{(X,D),o}$ if and only if the following conditions are satisfied:*

(i) *S is a Kollár component;*
(ii) *(S, Δ_S) is K-semistable.*

Moreover, such a minimizing divisorial valuation, if it exists, is unique among all divisorial valuations.

We do not know whether, up to rescaling, a valuation as in Theorem 1.2 is a unique minimizer of $\widehat{\mathrm{vol}}_{(X,D),o}$ among *all* valuations in $\mathrm{Val}_{X,o}$ (see [LX18] for further results).

More concretely, we will prove Theorem 1.2 by establishing the following four theorems. We will need different techniques to prove each of them.

First we prove

Theorem A. *Let $o \in (X, D)$ be an algebraic klt singularity. Let S be a Kollár component over X. If (S, Δ_S) is (log-)K-semistable, then $\widehat{\mathrm{vol}}_{(X,D),o}$ is minimized at the valuation* ord_S.

This extends the main theorem in [LL19] from cone singularities to a more general setting. For the proof, we need to degenerate a general klt singularity to a cone singularity induced by its Kollár components. However, instead of degenerating the valuation, we degenerate the associated valuative ideals. We will also use a result of [Liu18] which gives the infimum of normalized volumes using some normalized multiplicities. The latter was first considered in the work of de Fernex–Ein–Mustaţă [FEM04] and its behavior under degeneration of singularities can be studied as in [Mus02].

An extra subtlety is that we cannot directly use [LL19] since the result there was proved for the cone singularity over a $\mathbb{Q}$-Fano variety that specially degenerates to a

Kähler–Einstein $\mathbb{Q}$-Fano variety. It is conjectured that any K-semistable $\mathbb{Q}$-Fano variety has such a degeneration. Here we can indeed circumvent this difficulty in two different ways. First, we will show that it suffices to concentrate on torus equivariant data (see Section 4.2) and then use a similar argument to [Li17b] to complete the proof. Alternatively, we solve the question proposed in [LL19] and hence can use the strategy there to prove the version we need (see Proposition 5.3).

In Section 7, we use this criterion to find minimizers for various examples of singularities, including quotient singularities, A_k and E_k singularities etc.

Next, we turn to the result on uniqueness.

Theorem B. *If $o \in (X, D)$ is an algebraic klt singularity and S is a Kollár component over X such that (S, Δ_S) is K-semistable, then*

$$\widehat{\mathrm{vol}}_{(X,D),o}(\mathrm{ord}_S) < \widehat{\mathrm{vol}}_{(X,D),o}(\mathrm{ord}_{S'})$$

for any other divisorial valuation S'.

This is done by a detailed study of the geometry when equality holds. In the cone singularity case, we investigate when equality holds in [LL19]. It is a strong condition which enables us to compute the corresponding invariants including nef thresholds and pseudo-effective thresholds. The argument is partially inspired by [Liu18]. Once this is clear, the rest follows from an application of Kawamata's base point free theorem. The general case can be reduced to the case of cone singularity using a degeneration process which heavily relies on MMP techniques.

Now we consider the converse direction. For any klt singularity, a minimizer of the normalized volume function always exists by [Blu18]. The following theorem says that if a minimizer is divisorial, it always yields a Kollár component. We can indeed prove slightly more for a general rational rank 1 minimizer.

Theorem C. *Given an arbitrary algebraic klt singularity $o \in (X, D)$ where $X =$ $\mathrm{Spec}(R)$, let v be a valuation that minimizes $\widehat{\mathrm{vol}}_{(X,D),o}$. Assume the valuation group of v is isomorphic to $\mathbb{Z}$, i.e., v has rational rank 1, and one of the following two assumptions holds:*

(i) *v is a multiple of a divisorial valuation; or*
(ii) *the graded family of valuative ideals*

$$\mathfrak{a}_\bullet = \{\mathfrak{a}_k\} \quad where \quad \mathfrak{a}_k = \{f \in R \mid v(f) \geq k\}$$

is finitely generated, i.e., there exists $m \in \mathbb{N}$ such that $\mathfrak{a}_{mk} = (\mathfrak{a}_m)^k$ for any $k \in \mathbb{N}$.

Then up to rescaling, v is given by the divisorial valuation induced by a Kollár component S.

The above theorem is also independently proved in [Blu18] by a different argument. We note that a minimizer is conjectured to be quasi-monomial and the associated graded ring for a minimizer of the normalized volume function is conjectured to be always finitely generated (cf. [Li18, Conjecture 7.1]). So granted these conjectures, the above result should presumably characterize all the cases with minimizers of rational rank 1. After giving the definition of the volume of a model, the proof uses similar MMP arguments to define a process decreasing the volumes as in [LX14].

Next we turn to the stability of the minimizer. By using the techniques from toric degeneration (see [Cal02, AB04, And13]) and the relation between CM weight and normalized volume, we will prove

Theorem D. *In the notation of Theorem C, let $\mu\colon Y \to X$ be the morphism which extracts S, and write $(K_Y + S + \mu_*^{-1}D)|_S = K_S + \Delta_S$. Then (S, Δ_S) is a K-semistable log Fano pair.*

1.2. Approximation

In a slightly different direction, we also obtain results which describe the minimizer of the normalized volume function from the viewpoint of Kollár components. We show that for a general klt singularity, although the minimizer of its associated normalized volume function might not be given by one Kollár component, we can always approximate it by a sequence of such components.

Theorem 1.3. *Given an arbitrary algebraic klt singularity $o \in (X, D)$ and a minimizer v^{m} of $\widehat{\mathrm{vol}}_{(X,D),o}$, there always exists a sequence $\{S_j\}$ of Kollár components and positive numbers c_j such that*

$$\lim_{j\to\infty} c_j \cdot \mathrm{ord}_{S_j} = v^{\mathrm{m}} \quad \textit{in } \mathrm{Val}_{X,o} \quad \textit{and} \quad \lim_{j\to\infty} \widehat{\mathrm{vol}}(\mathrm{ord}_{S_j}) = \widehat{\mathrm{vol}}(v^{\mathrm{m}}).$$

Here $\mathrm{Val}_{X,o}$ consists of all valuations centered at o, and is endowed with the weakest topology as in [JM12, Section 4.1]. See Remark 2.5 for some discussion.

1.3. Equivariant K-semistability

By relating a Fano variety to the cone over it, we can compare the calculation in [Li17b] for a cone and in [Fuj19] for its base. Then an interesting by-product of our method is the following theorem.

Theorem E. *Let $T \cong (\mathbb{C}^*)^r$ be a torus. Let (V, Δ) be a log Fano variety with a T-action. Then (V, Δ) is K-semistable if and only if any T-equivariant special test configuration $(\mathcal{V}, \Delta^{\mathrm{tc}}) \to \mathbb{A}^1$ of (V, Δ) has nonnegative generalized Futaki invariant:* $\mathrm{Fut}(\mathcal{V}, \Delta^{\mathrm{tc}}) \geq 0$.

When V is smooth and $\Delta = 0$, this follows from the work of [DS16] with an analytic argument. Our proof is completely algebraic. It again uses the techniques of degenerating any ideal to an equivariant one and showing that it has a smaller invariant.

The paper is organized in the following way: In Section 2, we give some necessary background. In Section 3, we introduce one key new tool: the volume of a model. By combining the normalized volume function on valuations with the local volume defined in [Ful13], and applying the MMP, we prove Theorem 1.3 and Theorem C. In Section 4, we prove Theorem A, by connecting it to the infimum of the normalized multiplicities

$\mathrm{lct}(X, D; \mathfrak{a})^n \cdot \mathrm{mult}(\mathfrak{a})$ for all $\mathfrak{m}$-primary ideals $\mathfrak{a}$ centered at o. We note that this latter invariant indeed has also been studied in another context (cf. [FEM04]). In Section 5, we prove Theorem B. We first prove it for the cone singularity case, with the help of calculations from [LL19]. Then we use a degeneration argument to reduce the general case to the case of cone singularities. In Section 6, we prove Theorem D, which verifies the K-semistability of a minimizing Kollár component. In Section 7, we give some examples of how to apply our techniques to calculate the minimizer for various classes of klt singularities.

History. Since [Li18], there have been several papers relating to minimization of the normalized volume function (see [Li17b, LL19, Liu18, Blu18, LX18, LWX18, BL18]). In particular, after we posted the first version of our preprint, the existence of the minimizer was completely settled in [Blu18]. In the revision, we include his result in the exposition. We also get a complete characterization of K-semistability of $\mathbb{Q}$-Fano varieties using the normalized volume, improving previous results from [Li17b, LL19]. Another major improvement in this revision is that we can indeed show in Theorem D that any Kollár component which minimizes the normalized local volume is K-semistable.

2. Preliminaries

Notation and conventions. We follow the standard notation of [Laz04a, KM98, Kol13]. A *log Fano pair* (V, Δ) is a projective klt pair such that $-K_V - \Delta$ is ample.

For a local ring $(R, \mathfrak{m})$ and an $\mathfrak{m}$-primary ideal $\mathfrak{a}$, we denote by $l_R(R/\mathfrak{a})$ the length of $R/\mathfrak{a}$.

For a variety $\bullet$, we sometimes denote the product $\bullet \times \mathbb{A}^1$ by $\bullet_{\mathbb{A}^1}$.

We will use interchangeably the notations $\mathbb{A}^1$ with $\mathbb{C}$, and $\mathbb{G}_m$ with $\mathbb{C}^*$.

2.1. *K-semistability*

In this section, we give the definition of K-semistability of a log Fano pair following [Tia97, Don02] (see also [Oda13, LX14]).

First we need to define the notion of *test configuration*.

Definition 2.1. Let (V, Δ) be an $(n-1)$-dimensional log Fano pair. A ($\mathbb{Q}$-)*test configuration* of (V, Δ) consists of

- a pair $(\mathcal{V}, \Delta^{\mathrm{tc}})$ with a $\mathbb{G}_m$-action,
- a $\mathbb{G}_m$-equivariant ample $\mathbb{Q}$-line bundle $\mathcal{L} \to \mathcal{V}$,
- a flat $\mathbb{G}_m$-equivariant map $\pi : \mathcal{V} \to \mathbb{A}^1$, where $\mathbb{G}_m$ acts on $\mathbb{A}^1$ by multiplication in the standard way, $(t, a) \mapsto ta$,

such that for any $t \neq 0$, the restriction of $(\mathcal{V}, \Delta^{\mathrm{tc}}, \mathcal{L})$ over t is isomorphic to $(V, \Delta, -(K_V + \Delta))$, and Δ^{tc} does not have any vertical component, i.e., components of Δ^{tc} are the closures of components of Δ under the $\mathbb{G}_m$-action.

A test configuration $(\mathcal{V}, \Delta^{\mathrm{tc}}, \mathcal{L})$ is called *special* if the central fiber (V_0, Δ_0) is a log Fano variety with klt singularities and $\mathcal{L} \sim_{\mathbb{Q}} -(K_{\mathcal{V}} + \Delta^{\mathrm{tc}})$.

By [LX14], without loss of generality we will always assume that every test configuration considered is normal. Let $(\mathcal{V}, \Delta^{\mathrm{tc}}, \mathcal{L})$ be a test configuration of (V, Δ). Let $(\bar{\mathcal{V}}, \bar{\Delta}, \bar{\mathcal{L}}) \to \mathbb{P}^1$ be the natural compactification of $(\mathcal{V}, \Delta^{\mathrm{tc}}, \mathcal{L}) \to \mathbb{A}^1$ by adding a trivial fiber (V, Δ, L) over $\{\infty\} \in \mathbb{P}^1$. We call it a *compactified* test configuration. Then we can define the *generalized Futaki invariant*:

Definition 2.2. With the above notations, for any normal test configuration $(\mathcal{V}, \Delta^{\mathrm{tc}}, \mathcal{L})$, we define its *generalized Futaki invariant* to be

$$\mathrm{Fut}(\mathcal{V}, \Delta^{\mathrm{tc}}, \mathcal{L}) \; (= \mathrm{Fut}(\bar{\mathcal{V}}, \bar{\Delta}, \bar{\mathcal{L}})) = \frac{1}{n(-K_V - \Delta)^{n-1}}((n-1)\bar{\mathcal{L}}^n + n\bar{\mathcal{L}}^{n-1} \cdot K_{\bar{\mathcal{V}}/\mathbb{P}^1}). \tag{2}$$

In particular, for any special test configuration, we have

$$\mathrm{Fut}(\mathcal{V}, \Delta^{\mathrm{tc}}, \mathcal{L}) = \frac{(-K_{\bar{\mathcal{V}}/\mathbb{P}^1} - \bar{\Delta})^n}{n(-K_V - \Delta)^{n-1}}. \tag{3}$$

The above definition of the generalized Futaki invariant using the intersection formula is well known to be equivalent to the original one using the Riemann–Roch formula (cf. [Wan12, Oda13, LX14] etc.).

Definition 2.3. A log Fano pair (V, Δ) is *K-semistable* if for any test configuration $(\mathcal{V}, \Delta^{\mathrm{tc}}, \mathcal{L})$ of (V, Δ), we have

$$\mathrm{Fut}(\mathcal{V}, \Delta^{\mathrm{tc}}, \mathcal{L}) \geq 0.$$

2.2. *Normalized volume*

Let (X, o) be a normal algebraic singularity and $D \geq 0$ be a $\mathbb{Q}$-divisor such that $K_X + D$ is $\mathbb{Q}$-Cartier. Denote by $\mathrm{Val}_{X,o}$ the space of real valuations centered at o. For any $v \in \mathrm{Val}_{X,o}$, we can define the volume $\mathrm{vol}_{X,o}(v)$ following [ELS03] and the log discrepancy $A_{(X,D)}(v)$ following [JM12, BF+15] (if the context is clear, we will write $\mathrm{vol}(v)$ and $A(v)$). In particular, if S is a divisor over X, we have

$$A_{(X,D)}(S) := A_{(X,D)}(\mathrm{ord}_S) = a(S; X, D) + 1,$$

which is the same as the standard log discrepancy.

Definition 2.4. With notation as above, we define the *normalized volume*, denoted by $\widehat{\mathrm{vol}}_{(X,D),o}(v)$ (or $\widehat{\mathrm{vol}}_{(X,D)}(v)$ if o is clear or simply $\widehat{\mathrm{vol}}(v)$ if there is no confusion), to be

$$\mathrm{vol}_{X,o}(v) \cdot A_{(X,D)}(v)^n$$

if $A_{(X,D)}(v) < \infty$, and ∞ if $A_{(X,D)}(v) = \infty$. We define the *volume* of a klt singularity $o \in (X, D)$ to be

$$\mathrm{vol}(o, X, D) = \inf_{v \in \mathrm{Val}_{X,o}} \widehat{\mathrm{vol}}_{X,D}(v).$$

Remark 2.5. 1. The space $\mathrm{Val}_{X,o}$ is called the 'nonarchimedean link' of $o \in X$ in some literature (see [Thu07, Fan14]). It is well known that in the topological setting the Euclidean link captures a lot of (including all the topological) information about the singularity. We expect that the study of $\mathrm{Val}_{X,o}$ will also significantly improve our knowledge of the singularity.

One can try to investigate the normalized volume function more globally. For instance, it is interesting to ask, on a fixed model, how the function $\mathrm{vol}(o, X, D)$ changes when we vary o, including the case where o is not a closed point. In particular, we expect that there is a formula connecting the volume of a (not necessarily closed) point o and the volume of a general point o' in the closure $\overline{\{o\}}$. We note that this may give us a way to treat those valuations with centers containing the fixed point. It is also natural to ask how $\mathrm{vol}(o, X, D)$ changes when we modify the birational models. We hope to explore these interesting questions in the future.

2. The volume of klt singularities defined here is different from the volume of singularities defined in [BFF12] (see also [Zha14]). The volume in [BFF12] is defined using envelopes of log discrepancy b-divisors and vanishes for klt singularities. Intuitively, while [BFF12] computes the volume of log canonical classes, our definition of volume of klt singularities is for the anti-log-canonical classes.

In [Li18], it was shown that the space

$$\{v \in \mathrm{Val}_{X,o} \mid v(\mathfrak{m}) = 1,\ \widehat{\mathrm{vol}}(v) \le C\}$$

for any constant $C > 0$ is compact in the weak topology. However, in general the volume function vol is only upper semicontinuous on $\mathrm{Val}_{X,o}$.

Proposition 2.6. *If $\{v_i\}$ is a sequence of valuations such that $v_i \to v$ in the weak topology, then*

$$\mathrm{vol}(v) \ge \limsup_i \mathrm{vol}(v_i).$$

Proof. The valuation v determines a graded sequence of ideals

$$\mathfrak{a}_k = \mathfrak{a}_k(v) = \{f \in R \mid v(f) \ge k\}.$$

By [Mus02], we know that for any $\epsilon > 0$, there exists a sufficiently large k such that

$$\frac{1}{k^n}\mathrm{mult}(\mathfrak{a}_k) < \mathrm{vol}(v) + \epsilon.$$

Since R is Noetherian, we know that there exist finitely many generators f_p $(1 \le p \le j)$ of $\mathfrak{a}_k = (f_1, \dots, f_j)$. As $v(f_p) \ge k$, we know that for any δ, there exists sufficiently large i_0 such that for any $i \ge i_0$, $v_i(f_p) \ge k - \delta$. Thus

$$\mathfrak{a}^{(i)}_{k-\delta} = \{f \in R \mid v_i(f) \ge k - \epsilon\} \supset \mathfrak{a}_k.$$

Therefore,

$$\mathrm{vol}(v_i) \le \frac{1}{(k-\delta)^n}\mathrm{mult}(\mathfrak{a}^{(i)}_{k-\epsilon}) \le \frac{1}{(k-\delta)^n}\mathrm{mult}(\mathfrak{a}_k) \le \frac{k^n}{(k-\delta)^n}(\mathrm{vol}(v) + \epsilon). \qquad \square$$

We also have the following result.

Proposition 2.7. *Let* $(X, o) = (\mathrm{Spec}(R), \mathfrak{m})$ *be a singularity. Let* v *and* v' *be two real valuations in* $\mathrm{Val}_{X,o}$. *Assume*

$$\mathrm{vol}(v) = \mathrm{vol}(v') > 0 \quad \textit{and} \quad v(h) \geq v'(h) \quad \textit{for any } h \in R.$$

Then $v = v'$.

Proof. Assume that this is not true. We fix $f \in R$ such that

$$v(f) = l > v'(f) = s.$$

Denote $r = l - s > 0$. Fix $k \in \mathbb{R}_{>0}$. Consider

$$\mathfrak{a}_k := \{h \in R \mid v(h) \geq k\} \quad \text{and} \quad \mathfrak{b}_k := \{h \in R \mid v'(h) \geq k\}.$$

So by our assumption $\mathfrak{b}_k \subset \mathfrak{a}_k$, and we want to estimate

$$\dim(R/\mathfrak{b}_k) - \dim(R/\mathfrak{a}_k) = \dim(\mathfrak{a}_k/\mathfrak{b}_k).$$

Fix a positive integer $m < k/l$ and elements $g_m^{(1)}, \ldots, g_m^{(k_m)} \in \mathfrak{b}_{k-ml}$ whose images in $\mathfrak{b}_{k-ml}/\mathfrak{b}_{k-ml+r}$ form a $\mathbb{C}$-linear basis.

We claim that

$$\{f^m \cdot g_m^{(j)}\} \quad (1 \leq m \leq k/l,\ 1 \leq j \leq k_m)$$

are $\mathbb{C}$-linearly independent in $\mathfrak{a}_k/\mathfrak{b}_k$. Granted this, we know that since $\mathrm{vol}(v) > 0$ we have

$$\limsup_{k\to\infty} \frac{1}{k^n} \sum_{1\leq m\leq k/l} k_m = \limsup_{k\to\infty} \sum_{1\leq m\leq k/l} \frac{1}{k^n} \dim(\mathfrak{b}_{k-ml}/\mathfrak{b}_{k-ml+r}) > 0,$$

which implies $\mathrm{vol}(v) > \mathrm{vol}(v')$.

Now we prove the claim.

Step 1: For any $1 \leq m \leq k/l$, $1 \leq j \leq k_m$,

$$v(f^m \cdot g_m^{(j)}) = v(f^m) + v(g_m^{(j)}) \geq ml + v'(g_m^{(j)}) \geq ml + k - ml \geq k.$$

Thus $f^m \cdot g_m^{(j)} \in \mathfrak{a}_k$.

Step 2: If $\{f^m \cdot g_m^{(j)}\}$ $(1 \leq m \leq k/l,\ 1 \leq j \leq k_m)$ are not $\mathbb{C}$-linearly independent in $\mathfrak{a}_k/\mathfrak{b}_k$, then we have

$$\sum_m h_m = b \in \mathfrak{b}_k,$$

where there exist $c_j \in \mathbb{C}$ such that

$$h_m = f^m \cdot \sum_{1\leq j\leq k_m} c_j g_m^{(j)}$$

and some $h_m \neq 0$. Consider the maximal m such that $h_m \neq 0$. Then

$$v'(h_m) = v'\Big(f^m \cdot \sum_{1\le j\le k_m} c_j g_m^{(j)}\Big) = v'(f^m) + v'\Big(\sum_{1\le j\le k_m} c_j g_m^{(j)}\Big)$$
$$< ms + k - ml + r = k - (m-1)l + (m-1)s,$$

where the inequality follows from

$$\sum_{1\le j\le k_m} c_j g_m^{(j)} \notin \mathfrak{b}_{k-ml+r}.$$

However,

$$v'(h_m) = v'\Big(b - \sum_{j<m} h_j\Big) \ge \min\{v'(b), v'(h_1), \ldots, .v'(h_{m-1})\}$$
$$= \min_{1\le j\le m-1}\{k, js + k - jl\} = k - (m-1)l + (m-1)s,$$

which is a contradiction. □

Several results in our work depend on a relation between normalized volumes of valuations and normalized multiplicities of primary ideals. The latter quantity was first considered in the smooth case in [FEM04], and since then it has been studied in many other works, together with its positive characteristic version (see e.g. [TW04]). Its relevance to the normalized volume appeared in [Li18, Example 5.1]. In [Liu18] the following more precise observation was made.

Proposition 2.8 ([Liu18, Section 4.1]). *Let $(X, o) = (\mathrm{Spec}(R), \mathfrak{m})$ and let $D \ge 0$ be a $\mathbb{Q}$-divisor such that $o \in (X, D)$ is a klt singularity. Then*

$$\inf_v \widehat{\mathrm{vol}}_{(X,D),o}(v) = \inf_{\mathfrak{a}} \mathrm{lct}(X, D; \mathfrak{a})^n \cdot \mathrm{mult}(\mathfrak{a}), \tag{4}$$

where on the left hand side v runs over all real valuations centered at o, and on the right hand side $\mathfrak{a}$ runs over all $\mathfrak{m}$-primary ideals. Moreover, the left hand side can be replaced by $\inf_{v\in \mathrm{Div}_{X,o}} \widehat{\mathrm{vol}}_{(X,D),o}(v)$ where $\mathrm{Div}_{X,o}$ denotes the space of all divisorial valuations with center at o.

For the reader's convenience we provide a sketch of the proof.

Proof of Proposition 2.8. We first use the same argument as in [Li18, Example 5.1] to prove that the left hand side is greater than or equal to the right hand side. For any real valuation v, consider the graded family of valuative ideals

$$\mathfrak{a}_k = \mathfrak{a}_k(v) = \{f \in R \mid v(f) \ge k\}.$$

Then $v(\mathfrak{a}_k) \ge k$ and we can estimate

$$A_{(X,D)}(v)^n \cdot \frac{\mathrm{mult}(\mathfrak{a}_k)}{k^n} \ge \left(\frac{A_{(X,D)}(v)}{v(\mathfrak{a}_k)}\right)^n \cdot \mathrm{mult}(\mathfrak{a}_k) \ge \mathrm{lct}(X, D; \mathfrak{a}_k)^n \cdot \mathrm{mult}(\mathfrak{a}_k).$$

Since $\mathfrak{a}_\bullet = \{\mathfrak{a}_k\}$ is a graded family of $\mathfrak{m}$-primary ideals on X,

$$\mathrm{vol}(v) = \mathrm{mult}(\mathfrak{a}_\bullet) = \lim_{k\to\infty} \frac{l_R(R/\mathfrak{a}_k)}{k^n} = \lim_{k\to\infty} \frac{\mathrm{mult}(\mathfrak{a}_k)}{k^n}$$

(see e.g. [ELS03, Mus02, LM09, Cut12]). As $k \to \infty$, the left hand side of the previous estimate converges to $\widehat{\mathrm{vol}}(v)$ and we get one direction.

For the other direction, we follow the argument in [Liu18]. For any $\mathfrak{m}$-primary ideal $\mathfrak{a}$, we can choose a divisorial valuation v calculating $\mathrm{lct}(\mathfrak{a})$. Then v is centered at o. Assume $v(\mathfrak{a}) = k$, or equivalently $\mathfrak{a} \subseteq \mathfrak{a}_k(v)$. Then $\mathfrak{a}^l \subseteq \mathfrak{a}_k(v)^l \subseteq \mathfrak{a}_{kl}(v)$ for any $l \in \mathbb{Z}_{>0}$, so we can estimate

$$\begin{aligned}\mathrm{lct}(X, D; \mathfrak{a})^n \cdot \mathrm{mult}(\mathfrak{a}) &= \frac{A_{(X,D)}(v)^n}{k^n} \cdot \mathrm{mult}(\mathfrak{a}) = A_{(X,D)}(v)^n \cdot \frac{\mathrm{mult}(\mathfrak{a})l^n}{(kl)^n} \\ &= A_{(X,D)}(v)^n \cdot \frac{\mathrm{mult}(\mathfrak{a}^l)}{(kl)^n} \geq A_{(X,D)}(v)^n \cdot \frac{\mathrm{mult}(\mathfrak{a}_{kl})}{(kl)^n}.\end{aligned}$$

As $l \to \infty$, again the right hand side converges to

$$A_{(X,D)}(v)^n \cdot \mathrm{mult}(\mathfrak{a}_\bullet(v)) = \widehat{\mathrm{vol}}(v).$$

The last statement follows easily from the above proof. □

In [Blu18], it is proved that a minimizer always exists.

Theorem 2.9 ([Blu18]). *For any klt singularity $o \in (X, D)$, $\widehat{\mathrm{vol}}_{(X,D)}(v)$ always has a minimizer v^{m} in $\mathrm{Val}_{X,o}$.*

2.3. *Properties of Kollár components*

The concept of Kollár component is defined in Definition 1.1. They always exist by results from the MMP (see [Pro00] or [Xu14, Lemma 1]).

In this section, we establish some of their properties using the machinery of the minimal model program. The following statement is the local analogue of [LX14, Theorem 1.6], which can be obtained by following the proof of the existence of a Kollár component. (See e.g. the proof of [Xu14].)

Proposition 2.10. *Let $o \in (X, D)$ be a klt singularlty. Let $\mu\colon Z \to X$ be a model such that μ is an isomorphism over $X \setminus \{o\}$ and $(Z, E + \mu_*^{-1}D)$ is dlt where E is the divisorial part of $\mu^{-1}(o)$. Then we can choose a model $W \to Z$ and run MMP to obtain $W \dashrightarrow Y$ such that $Y \to X$ gives a Kollár component S that satisfies $a(S; Z, E) = -1$.*

We also have the following straightforward lemma.

Lemma 2.11. *If S is a Kollár component as the exceptional divisor of a plt blow up $\mu\colon Y \to X$, then*

$$\mathrm{vol}(\mathrm{ord}_S) = (-S|_S)^{n-1} \quad \text{and} \quad \widehat{\mathrm{vol}}(\mathrm{ord}_S) = (-(K_Y + S + \mu_*^{-1}D)|_S)^{n-1} \cdot A_{(X,D)}(S).$$

Proof. For any $k \geq 0$ such that kS is Cartier on Y, we have an exact sequence

$$0 \to \mathcal{O}_Y(-(k+1)S) \to \mathcal{O}_Y(-kS) \to \mathcal{O}_S(-kS) \to 0.$$

Because $-S$ is ample over X, we have the vanishing

$$R^1 f_*(\mathcal{O}_Y(-(k+1)S)) = 0,$$

from which we get

$$H^0(S, -kS|_S) \cong \frac{H^0(Y, -kS)}{H^0(Y, -(k+1)S)} = \frac{\mathfrak{a}_k(\mathrm{ord}_S)}{\mathfrak{a}_{k+1}(\mathrm{ord}_S)}$$

for any such k. Then the result follows easily from the Hirzebruch–Riemann–Roch formula and the asymptotic definition of $\mathrm{vol}(\mathrm{ord}_S)$.

As $K_Y + S + \mu_*^{-1} D \sim_{\mathbb{Q},X} A_{(X,D)}(S) \cdot S$, the second identity is implied by the first. □

Remark 2.12. Inspired by the above simple calculation, we can indeed extend the definition of normalized volumes to any model $f : Y \to (X, o)$ such that f is isomorphic over $X \setminus \{o\}$. See Section 3.

Lemma 2.13. *Let $f : (X', o') \to (X, o)$ be a finite morphism such that $f^*(K_X + D) = K_{X'} + D'$ for some effective $\mathbb{Q}$-divisors. Assume (X, D) and (X', D') are klt. If S is a Kollár component given by $Y \to X$ over o, then $Y' := Y \times_X X' \to X'$ induces a Kollár component S' over $o' \in (X', D')$.*

Conversely, if $X' \to X$ is Galois with Galois group G, then any G-invariant Kollár component S' over $o \in (X', D')$ is the pull back of a Kollár component over $o \in (X, D)$.

Proof. The first part is standard. Denote by $\mu' : Y' \to X'$ the birational morphism and set $S' = (f_Y^{-1}(S))_{\mathrm{red}}$ where $f_Y : Y' \to Y$ is the induced morphism. Then $(Y', \mu'^{-1}_* D' + S')$ is log canonical. If we restrict to T, a component of S',

$$(K_{Y'} + \mu'^{-1}_* D' + S')|_T = K_T + \Delta_T,$$

then (T, Δ_T) is klt, which by the Kollár–Shokurov connectedness theorem implies that $T = S'$.

For the converse, let

$$L \sim_{X'} -m(K_{Y'} + \mu'^{-1}_* D' + S')$$

be a divisor of general position for sufficiently divisible m and $H := \frac{1}{m} L$. Then $(Y', S' + \mu'^{-1}_* D' + H)$ is plt. Replacing H by $H_G := \frac{1}{|G|} \sum_{g \in G} g^* H$, we know that $(X', D' + \mu_* H_G)$ is G-invariant, and there exists a $\mathbb{Q}$-divisor $H_X \geq 0$ such that

$$f^*(K_X + D + H_X) = K_{X'} + D' + \mu_* H_G.$$

Therefore, $(X, D + H_X)$ is plt, and its unique log canonical place is a divisor S which is a Kollár component over $o \in (X, D)$ whose pull back gives the Kollár component S' over $o' \in (X', D')$. □

We now prove a change of volume formula for Kollár components under a finite map.

Lemma 2.14. *With the same notation as in Lemma 2.13,*

$$d \cdot \widehat{\mathrm{vol}}_{(X,D)}(\mathrm{ord}_S) = \widehat{\mathrm{vol}}_{(X',D')}(\mathrm{ord}_{S'}),$$

where d is the degree of $X' \to X$.

Proof. The pull back of S is S', which is irreducible by Lemma 2.13. Let the degree of $S' \to S$ be a and the ramified degree be r. We have the identities

$$ar = d \quad \text{and} \quad rA_{(X,D)}(\mathrm{ord}_S) = A_{X',D'}(\mathrm{ord}_{S'})$$

(see [KM98, 5.20]). By Lemma 2.11, we know that

$$\begin{aligned} d \cdot \widehat{\mathrm{vol}}_{(X,D)}(\mathrm{ord}_S) &= ar \cdot A_{(X,D)}(\mathrm{ord}_S) \cdot ((K_Y + S + \mu_*^{-1}D)|_S)^{n-1} \\ &= (rA_{(X,D)}(\mathrm{ord}_S)) \cdot \left(a \cdot ((K_Y + S + \mu_*^{-1}D)|_S)^{n-1}\right) \\ &= A_{(X',D')}(\mathrm{ord}_{S'}) \cdot \left(((K_{Y'} + S' + \mu_*'^{-1}D')|_{S'})^{n-1}\right) \\ &= \widehat{\mathrm{vol}}_{(X',D')}(\mathrm{ord}_{S'}), \end{aligned}$$

where for the third equality we use the projection formula for intersection numbers. □

2.4. Deformation to normal cones

Let $(X, o) = (\mathrm{Spec}(R), \mathfrak{m})$ be an algebraic singularity such that (X, D) is klt for a $\mathbb{Q}$-divisor $D \geq 0$. Let S be a Kollár component and $\Delta = \Delta_S$ be the different divisor defined by the adjunction $(K_Y + S + \mu_*^{-1}D)|_S = K_S + \Delta_S$ where $Y \to X$ is the extraction of S (see (1)).

For simplicity denote $v_0 := \mathrm{ord}_S$. Also denote

$$R^* := \bigoplus_{k=0}^{\infty} \mathfrak{a}_k(v_0)/\mathfrak{a}_{k+1}(v_0) = \bigoplus_{k=0}^{\infty} R_k^* \tag{5}$$

and its d-th truncation

$$R^{*(d)} = \bigoplus_{k=0}^{\infty} \mathfrak{a}_{dk}(v_0)/\mathfrak{a}_{dk+1}(v_0) = \bigoplus_{k=0}^{\infty} R_{dk}^* \qquad \text{for } d \in \mathbb{N}.$$

Now we give a more geometric description of $\mathrm{Spec}(R^*)$ and $\mathrm{Spec}(R^{*(d)})$ using the idea of degenerating $o \in (X, D)$ to an (orbifold) cone over the Kollár component S. Assume $\mu \colon Y \to X$ is the extraction of the Kollár component S of (X, o). Then $\mu_{\mathbb{A}^1} \colon Y \times \mathbb{A}^1 \to X \times \mathbb{A}^1$ has the exceptional divisor $S \times \mathbb{A}^1$. The divisor S is not necessarily Cartier, but only $\mathbb{Q}$-Cartier. Thus we can take the index 1 covering Deligne–Mumford stack $\pi \colon \mathfrak{Y} \to Y$ for S. So π is isomorphic over $Y \setminus S$ and $\pi^*(S) = \mathfrak{S}$ is Cartier on $\mathfrak{Y}$. Note that S and Y are coarse moduli spaces of $\mathfrak{S}$ and $\mathfrak{Y}$ respectively.

We consider the deformation to the normal cone construction for $\mathfrak{S} \subset \mathfrak{Y}$ (see [Ful84, Chapter 5]). More precisely, we consider the blow up $\tilde{\phi}_1 \colon \mathfrak{Z} \to \mathfrak{Y} \times \mathbb{A}^1$ along $\mathfrak{S} \times \{0\}$. Denote by $\mathfrak{P}$ the exceptional divisor and by $\mathfrak{S}'_{\mathbb{A}^1}$ the strict transform of $\mathfrak{S} \times \mathbb{A}^1$. We note that $\mathfrak{P}$ has a stacky structure along the 0 and ∞ section, but a scheme structure at other places. Then $\mathfrak{S}'_{\mathbb{A}^1} \subset \mathfrak{Z}$ is a Cartier divisor which is proper over $\mathbb{A}^1$ and can be contracted to a normal Deligne–Mumford stack $\tilde{\psi}_1 \colon \mathfrak{Z} \to \mathfrak{W}$ and in this way we get a flat family $\mathfrak{W} \to \mathbb{A}^1$ such that $\mathfrak{W}_t \cong X$ and $\mathfrak{W}_0 \cong \bar{\mathfrak{C}} \cup \mathfrak{Y}_0$, where $\mathfrak{Y}_0$ is the birational transform of $Y \times \{0\}$. If we denote $\mathfrak{W}^\circ := \mathfrak{W} \backslash \mathfrak{Y}_0$, then the fiber of $\mathfrak{W}^\circ$ over 0 is isomorphic to $\mathfrak{C}$ which is an affine orbifold cone over $\mathfrak{S}$ with polarization given by $\mathcal{O}_{\mathfrak{Y}}(-\mathfrak{S})|_{\mathfrak{S}}$. Moreover, $\bar{\mathfrak{C}}$ is the projective orbifold cone completing $\mathfrak{C}$. We will also denote by $\mathcal{W}$, $\mathcal{W}^\circ$, $\mathcal{Z}$, $\mathbb{P}$ the underlying coarse moduli spaces of $\mathfrak{W}$, $\mathfrak{W}^\circ$, $\mathfrak{Z}$, $\mathfrak{P}$ respectively. In particular, we have (see Figure 1)

$$\begin{aligned} \mathcal{Z} \times_{\mathbb{A}^1} (\mathbb{A}^1 \setminus \{0\}) &= Y \times (\mathbb{A}^1 \setminus \{0\}), & \mathcal{Z} \times_{\mathbb{A}^1} \{0\} &= \mathbb{P} \cup Y_0, \\ \mathcal{W} \times_{\mathbb{A}^1} (\mathbb{A}^1 \setminus \{0\}) &= X \times (\mathbb{A}^1 \setminus \{0\}), & \mathcal{W} \times_{\mathbb{A}^1} \{0\} &= \bar{C} \cup Y_0, \\ \mathcal{W}^\circ \times_{\mathbb{A}^1} (\mathbb{A}^1 \setminus \{0\}) &= X \times (\mathbb{A}^1 \setminus \{0\}), & \mathcal{W}^\circ \times_{\mathbb{A}^1} \{0\} &= C. \end{aligned} \tag{6}$$

Let d be a positive integer such that $d \cdot S$ is Cartier in Y. Then $\mathfrak{C}^{(d)}$ given by the cone over $\mathcal{O}_{\mathfrak{Y}}(-d \cdot \mathfrak{S})|_{\mathfrak{S}}$ is a degree d cyclic quotient of $\mathfrak{C}$, which is a usual ($\mathbb{A}^1$-)cone over $\mathfrak{S}$. We denote by C and $C^{(d)}$ the underlying coarse moduli spaces of $\mathfrak{C}$ and $\mathfrak{C}^{(d)}$. We also denote by S the coarse moduli space of $\mathfrak{S}$. The vertex of C is denoted by o_C.

For any k such that kS is Cartier, applying the exact sequence

$$0 \to \mathcal{O}_Y(-(k+1)S) \to \mathcal{O}_Y(-kS) \to \mathcal{O}_S(-kS) \to 0,$$

since $h^1(\mathcal{O}_Y(-(k+1)S)) = 0$ by the Grauert–Riemenschneider vanishing theorem, we get

$$H^0(S, \mathcal{O}(-kS|_S)) \cong H^0(\mathcal{O}_Y(-kS))/H^0(\mathcal{O}_Y(-(k+1)S)).$$

Notice that the right hand side is equal to

$$\frac{\mu_* \mathcal{O}_Y(-kS)}{\mu_* \mathcal{O}_Y(-(k+1)S)} = \frac{\mathfrak{a}_k(v_0)}{\mathfrak{a}_{k+1}(v_0)}.$$

In particular, $C^{(d)} = \mathrm{Spec}(R^{*(d)})$. Similarly, $C = \mathrm{Spec}(R^*)$.

There is also a degree d cyclic quotient morphism $h \colon \bar{C} \to \bar{C}^{(d)}$, and we know that

$$h^*(K_{\bar{C}^{(d)}} + C_1^{(d)} + C_2^{(d)}) = K_{\bar{C}} + C_D,$$

where C_D is the intersection of $\bar{C}$ with the birational transform of $D \times \mathbb{A}^1$, and $C_1^{(d)}$ (resp. $C_2^{(d)}$) on $\bar{C}^{(d)}$ is the induced cone over the branched $\mathbb{Q}$-divisor on S of $\mathfrak{S} \to S$ (resp. $\mu_*^{-1} D|_S$).

2.5. *Filtrations and valuations*

Here we recall some facts about $\mathbb{Z}$-graded filtration and its relation to valuations following [TW89]. A *filtration* on a ring R is a decreasing sequence $\mathcal{F} := \{\mathcal{F}^m\}_{m \in \mathbb{Z}}$ of ideals of R satisfying the following conditions:

(i) $\mathcal{F}^m \neq 0$ for every $m \in \mathbb{Z}$, $\mathcal{F}^m = R$ for $m \leq 0$ and $\bigcap_{m \geq 0} \mathcal{F}^m = (0)$.
(ii) $\mathcal{F}^{m_1} \cdot \mathcal{F}^{m_2} \subseteq \mathcal{F}^{m_1+m_2}$ for all $m_1, m_2 \in \mathbb{Z}$.

Notice that we can replace $\mathbb{Z}$ by any abelian group that is isomorphic to $\mathbb{Z}$. For a given filtration, we have the *Rees algebra* and *extended Rees algebra*:

$$\mathcal{R} := \mathcal{R}(\mathcal{F}) = \bigoplus_{k=0}^{\infty} (\mathcal{F}^k R)t^{-k}, \quad \mathcal{R}' := \mathcal{R}'(\mathcal{F}) = \bigoplus_{k=-\infty}^{\infty} (\mathcal{F}^k R)t^{-k}, \tag{7}$$

and the *associated graded ring*:

$$\mathrm{gr}_{\mathcal{F}} R = \mathcal{R}'/t\mathcal{R}' = \bigoplus_{k=0}^{\infty} (\mathcal{F}^k R/\mathcal{F}^{k+1} R)t^{-k}. \tag{8}$$

Assuming $\mathcal{R}'$ is finitely generated, $\mathcal{X} := \mathrm{Spec}_{\mathbb{C}[t]}(\mathcal{R}')$ can be seen as a $\mathbb{C}^*$-equivariant flat degeneration of $X = \mathrm{Spec}(R)$ into $\mathcal{X}_0 = \mathrm{Spec}_{\mathbb{C}}(\mathcal{R}'/t\mathcal{R}') = \mathrm{Spec}_{\mathbb{C}}(\mathrm{gr}_{\mathcal{F}} R)$. Denote $E = \mathrm{Proj}(\mathrm{gr}_{\mathcal{F}} R)$, $\tilde{X} = \mathrm{Proj}_R \mathcal{R}$. Then the natural map $\tilde{X} \to X$ is the filtered blow up associated with $\mathcal{F}$ such that E is the exceptional divisor. Moreover $\tilde{X}$ can be seen as a flat deformation of a natural filtered blow up on $\mathcal{X}_0$. Indeed, following [TW89, 5.15], we have a filtration $\mathcal{F}$ on $\mathcal{R}'$:

$$\mathcal{F}^m \mathcal{R}' = \left\{ \sum_{k=-\infty}^{\infty} (\mathcal{F}^{\max(k,m)} R)t^{-k} \right\}.$$

The objects associated to the corresponding Rees algebra and graded algebra over $\mathcal{R}'$ are

$$\tilde{\mathcal{X}} = \mathrm{Proj}_{\mathcal{R}'}\Big(\bigoplus_{r=0}^{\infty} (\mathcal{F}^r \mathcal{R}')T^{-r}\Big), \qquad \mathcal{E} = \mathrm{Proj}_{\mathbb{C}}\Big(\bigoplus_{r=0}^{\infty} (\mathcal{F}^r \mathcal{R}'/\mathcal{F}^{r+1}\mathcal{R}')T^{-r}\Big).$$

Moreover, since $\mathcal{R}'$ is finitely generated, there is an embedding $X \subset \mathbb{C}^p$ for some $p \in \mathbb{N}$ given by $x_i \mapsto f_i$ $(i = 1, \ldots, p)$ where $f_i \in \mathcal{F}^{k_i} R$ are such that $t^{-k_i} f_i$ $(i = 1, \ldots, p)$ and t generate $\mathcal{R}'$. Set $\deg(x_i) = k_i$ and let $\widehat{\mathbb{C}^p} \to \mathbb{C}^p$ be the weighted blow up with weights $(k_1, \ldots, k_p)$.

Then we have the following commutative diagram (see [TW89, Proposition 5.17]):

$$\begin{array}{ccccccc}
 & & E & \longrightarrow & \mathcal{E} & \longleftarrow & E \\
 & & \downarrow & & \downarrow & & \downarrow \\
\widehat{\mathbb{C}^p} & \longleftarrow & \tilde{X} & \longrightarrow & \tilde{\mathcal{X}} & \longleftarrow & \tilde{\mathcal{X}}_0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
\mathbb{C}^p & \longleftarrow & X & \longrightarrow & \mathcal{X} & \longleftarrow & \mathcal{X}_0
\end{array}$$

To relate the filtrations to valuations, we need the following well-known fact:

Lemma 2.15 (see [Tei14, p. 484, proof of Corollary 3.4]). *If the associated graded ring of $\mathcal{F}$ is an integral domain, then the filtration $\mathcal{F}$ is induced by a valuation.*

Proof. We define the order function $v\colon R \to \mathbb{Z}$ by $v(f) = \max\{m \mid f \in \mathcal{F}^m\}$. Then by the defining properties of filtrations, $v(f+g) \geq \min\{v(f), v(g)\}$ and $v(fg) \geq v(f) + v(g)$ for any $f, g \in R$. For any $f \in R$, let $[f]$ denote the image of $f \in R$

under the quotient map $\mathcal{F}^{v(f)} \to \mathcal{F}^{v(f)}/\mathcal{F}^{v(f)+1} \subset \mathrm{gr}_{\mathcal{F}} R$. Then $[f] \cdot [g] \neq 0$ by the assumption that $\mathrm{gr}_{\mathcal{F}} R$ is an integral domain. This translates to $v(fg) = v(f) + v(g)$, which implies v is indeed a valuation. □

Actually we can be more precise in a special case that we will deal with later. There is a natural $\mathbb{C}^*$-action on $\mathcal{X}_0$ associated to the natural $\mathbb{N}$-grading such that the quotient is isomorphic to E. Let $\mathcal{J} = \bigoplus_{k\geq 0} \mathcal{F}^{k+1} t^{-k} = t\mathcal{R}' \cap \mathcal{R}$ so that $\mathcal{R}/\mathcal{J} \cong \mathrm{gr}_{\mathcal{F}} R \cong \mathcal{R}'/t\mathcal{R}'$. Now we assume furthermore that E is a normal projective variety. This implies both $\mathcal{R}$ and $\mathcal{R}'$ are normal (see [TW89]). Let $\mathfrak{P}$ be the unique minimal prime ideal of $\mathcal{R}$ over $\mathcal{J}$ that corresponds to the cone over E, and w the valuation of $K(t)$ attached to $\mathfrak{P}$. Then the restriction of w to R is equal to $b \cdot \mathrm{ord}_E$. Assume $a = w(t)$. Thus the filtration $\mathcal{F}$ is equivalent to the filtration given by

$$(t^m \mathcal{R}') \cap R = \{f \in R \mid \mathrm{ord}_E(f) \geq ma/b\}.$$

Remark 2.16. There is a general Valuation Theorem concerning the relation between finitely generated filtrations and valuations proved by Rees [Ree88]. See also [BHJ17].

3. Volume of models

One very useful tool for us to study the minimizers of the normalized local volume is the concept of a local volume of a model. It is this concept that enables us to apply the machinery of the minimal model program to construct different models, especially those yielding Kollár components.

3.1. Local volume of models

In this section, we extend the definition of volume to volumes of birational models in the 'normalized' sense. We use the concept of local volumes as in [ELS03, Ful13]. Let us first recall the definition from [Ful13].

Definition 3.1 (Local volume). Let X be a normal algebraic variety of dimension $n \geq 2$ and let o be a point on X. For a fixed proper birational map $\mu\colon Y \to X$ and a Cartier divisor E on Y, we define the *local volume* of E at o to be

$$\mathrm{vol}_o^F(E) = \limsup_{m\to\infty} \frac{h_o^1(mE)}{m^n/n!} \quad \text{where} \quad h_o^1(mE) := \dim H^1_{\{o\}}(X, \mu_*\mathcal{O}_Y(mE)).$$

If E is a $\mathbb{Q}$-Cartier divisor, we define its volume to be

$$\mathrm{vol}_o^F(E) := \frac{\mathrm{vol}_o^F(mE)}{m^n}$$

for sufficiently divisible m.

Lemma 3.2. *Let $\mu\colon Y \to X$ be a birational morphism. If $E \geq 0$ is an exceptional $\mathbb{Q}$-divisor such that* $\mathrm{Supp}(E) \subset \mu^{-1}(o)$*, then*

$$\mathrm{vol}_o^F(-E) = \limsup_{k\to\infty} \frac{l_R(\mathcal{O}_X/\mathfrak{a}_k)}{k^n/n!}$$

where k is sufficiently divisible and $\mathfrak{a}_k = \mu_*(\mathcal{O}_Y(-kE))$.

Proof. This follows from [Ful13, Remark 1.1(ii)] (see also [Ful13, Remarks 1.31 and 1.32]). □

The right hand side of the above display is also the volume $\mathrm{vol}(\mathfrak{a}_\bullet)$ defined in [ELS03, Definition 3.1, Proposition 3.11]. In particular, given a prime divisor E over o with log discrepancy a, we see that

$$\mathrm{vol}_o^F(-aE) = \widehat{\mathrm{vol}}_{(X,D),o}(\mathrm{ord}_E).$$

Definition 3.3. Assume that $o \in (X, D)$ is a klt singularity, and $\mu\colon Y \to (X, o)$ is a birational morphism such that μ is an isomorphism over $X \setminus \{o\}$. Let $E = \sum_i G_i$ be the reduced divisor supported on the divisorial part of $\mu^{-1}(o)$. Then we define the *volume* $\mathrm{vol}_{(X,D),o}(Y)$ of Y (abbreviated as $\mathrm{vol}(Y)$ if $(X, D; o)$ is clear) to be

$$\mathrm{vol}_{(X,D),o}(Y) := \mathrm{vol}_o^F(-K_Y - E - \mu_*^{-1}(D)) = \mathrm{vol}_o^F\Big(\sum_i -a_i G_i\Big)$$

where $a_i = A_{(X,D)}(G_i)$ is the log discrepancy of G_i.

We will mainly combine the above definition with the following construction.

Definition 3.4. For a klt pair (X, D) with an ideal $\mathfrak{a}$, if c denotes its log canonical threshold $\mathrm{lct}(X, D; \mathfrak{a})$, then we say that $\mu\colon Y \to X$ is a *dlt modification* of $(X, D + c\cdot\mathfrak{a})$ if the following conditions are satisfied:

- if we denote the divisorial part of $\mu^*(\mathfrak{a})$ by $\mathcal{O}(-\sum m_i G_i)$ and write $\mu^*(K_X + D) = K_Y + D_Y$, then
$$D_Y + c\cdot\sum m_i G_i = \mu_*^{-1}(D) + E$$
where E is the reduced divisor on $\mathrm{Ex}(\mu)$;
- $(Y, D_Y + c\cdot\sum m_i G_i)$ is dlt.

By the argument in [OX12], it follows from the MMP results in [BC+10] that a dlt modification of $(X, D + c\cdot\mathfrak{a})$ always exists. More concretely, we can choose general elements $f_j \in \mathfrak{a}$ $(1 \leq j \leq l)$ which generate $\mathfrak{a}$ such that $c/l < 1$. If we let $D_j = (f_j = 0)$, then Y is the dlt modification of $\big(X, D + c\cdot\frac{1}{l}\sum_{j=1}^{l} D_j\big)$.

Lemma 3.5. *We can indeed assume that $-K_Y - \mu_*^{-1}D - E$ is nef over X.*

Proof. Since (X, D) is klt, we know that

$$K_Y + \mu_*^{-1}D + E \sim_{\mathbb{Q},X} \sum a_i G_i$$

with G_i all exceptional and $a_i = A_{(X,D)}(G_i) > 0$. Running a relative MMP of

$$\left(Y, \mu_*^{-1}\left(D + c \cdot \frac{1}{l}\sum_{j=1}^{l} D_j\right) + E - \epsilon \sum a_i G_i\right) \quad \text{over } X$$

with scaling by an ample divisor, we obtain a relative minimal model $Y \dashrightarrow Y'$ of

$$K_Y + \mu_*^{-1}\left(D + c \cdot \frac{1}{l}\sum_{j=1}^{l} D_j\right) + E \sim_{\mathbb{Q},X} -c \cdot \sum m_i G_i + \sum_i a_i G_i = 0.$$

So we have

$$K_Y + \mu_*^{-1}\left(D + c \cdot \frac{1}{l}\sum_{j=1}^{l} D_j\right) + E - \epsilon \sum a_i G_i = -\epsilon(K_Y + \mu_*^{-1} D + E),$$

and hence $-K_{Y'} - \mu'^{-1}_* D - E'$ is nef over X where $\mu' \colon Y' \to X$ and E' is the birational transform of E. Furthermore, since

$$K_Y + \mu_*^{-1} D + E \sim_{\mathbb{Q},X} -c \cdot \frac{1}{l}\mu_*^{-1}\sum_{j=1}^{l} D_j,$$

Y' also gives a minimal model of the dlt pair

$$\left(Y, \mu_*^{-1}(D + c(1+\epsilon) \cdot \frac{1}{l}\sum_{j=1}^{l} D_j) + E\right),$$

which implies $(Y', \mu'^{-1}_* D + E')$ is a dlt modification of $\left(X, D + c \cdot \frac{1}{l}\sum_{j=1}^{l} D_j\right)$. Therefore, we can replace Y by Y'. □

If E is irreducible, then $\mathrm{vol}_{(X,D),o}(Y) = \widehat{\mathrm{vol}}_{(X,D),o}(\mathrm{ord}_E)$. We can generalize Lemma 2.11 to the dlt case.

Lemma 3.6. *In the setting of Definition 3.3, if $-K_Y - \mu_*^{-1} D - E$ is nef over X then*

$$\mathrm{vol}_{(X,D),o}(Y) = \sum_i a_i \left((-K_Y - \mu_*^{-1} D - E)|_{E_i}\right)^{n-1}.$$

Proof. Let m be sufficiently divisible such that $L := m(K_Y + \mu_*^{-1} D + E)$ is Cartier. Denote by F the effective Cartier divisor $F := \sum_i m a_i G_i$. Then

$$0 \to \mathcal{O}_Y(-(k+1)L) \to \mathcal{O}_Y(-kL) \to \mathcal{O}_F(-kL) \to 0.$$

Since $-L$ is nef, we know that $R^1\mu_*(\mathcal{O}_Y(-(k+1)L)) = 0$. Thus

$$\mathrm{vol}_o^F(L) = \mathrm{vol}(L|_F),$$

and we conclude the proof by dividing both sides by m^n. □

Lemma 3.7. *Let $\mathfrak{a}$ be an $\mathfrak{m}$-primary ideal. Denote $c = \mathrm{lct}(X, D; \mathfrak{a})$ and let $(Y, E) \to X$ be a dlt modification of $(X, D + c \cdot \mathfrak{a})$. Then*

$$\mathrm{vol}_{(X,D),o}(Y) \le \mathrm{lct}(\mathfrak{a})^n \cdot \mathrm{mult}(\mathfrak{a}).$$

Proof. Write $K_Y + \mu_*^{-1}D + E = \mu^*(K_X + D) + \sum_i a_i G_i$, where E is the reduced divisor on $\mathrm{Ex}(\mu)$. If we denote by m_i the vanishing order of $\mu^*\mathfrak{a}$ along G_i, then since c is the log canonical threshold and every G_i computes the log canonical threshold, we know that $c \cdot m_i = a_i$. Thus

$$\mathfrak{a}^k \subset \mu_*\mathcal{O}_Y\Big(-\sum_i k m_i G_i\Big) =: \mathfrak{b}_k.$$

It suffices to show that

$$\mathrm{mult}(\mathfrak{b}_\bullet) = \mathrm{vol}_o^F\Big(-\sum m_i G_i\Big).$$

But this follows from Lemma 3.2. □

Lemma 3.8. *With the same assumptions as in Lemma 3.7, there exists a Kollár component S such that*

$$\widehat{\mathrm{vol}}_{(X,D),o}(\mathrm{ord}_S) \le \mathrm{vol}_{(X,D),o}(Y) \le \mathrm{lct}(\mathfrak{a})^n \cdot \mathrm{mult}(\mathfrak{a}).$$

Proof. It follows from Proposition 2.10 that we can choose a model $W \to Y$ and run MMP to obtain $W \dashrightarrow Y'$ such that $\mu' \colon Y' \to X$ gives a Kollár component S with $a(S; Y, E + \mu_*^{-1}D) = -1$. If we fix a common resolution $p\colon W' \to Y$ and $q\colon W' \to Y'$, then since $-(K_Y + E + \mu_*^{-1}D)$ is nef and $A_{Y,E+\mu_*^{-1}D}(S) = 0$, we know $-p^*(K_Y + E + \mu_*^{-1}D) + q^*(K_{Y'} + S + \mu_*'^{-1}D)$ is q-nef and q-exceptional. By the negativity lemma, we get

$$p^*(K_Y + E + \mu_*^{-1}D) \ge q^*(K_{Y'} + S + \mu_*'^{-1}D).$$

Thus

$$\widehat{\mathrm{vol}}(\mathrm{ord}_S) = \mathrm{vol}(-K_{Y'} - S - \mu_*'^{-1}D) \le \mathrm{vol}(-K_Y - E - \mu_*^{-1}D) = \mathrm{vol}(Y). \qquad \square$$

3.2. Approximating by Kollár components

We can now start proving our theorems.

Proof of Theorem 1.3. By Proposition 2.8, we know

$$\inf_v \widehat{\mathrm{vol}}_{(X,D),o}(v) = \inf_{\mathfrak{a}} \mathrm{lct}(\mathfrak{a})^n \cdot \mathrm{mult}(\mathfrak{a}).$$

By the above construction in Lemmas 3.7 and 3.8, for any $\mathfrak{m}$-primary ideal $\mathfrak{a}$, we know that there exists a Kollár component S such that

$$\widehat{\mathrm{vol}}(\mathrm{ord}_S) \le \mathrm{lct}(\mathfrak{a})^n \cdot \mathrm{mult}(\mathfrak{a}).$$

Let $\{\mathfrak{a}_k\}_{k\in\Phi}$ be the associated graded family of valuation ideals induced by v^{m} where $\Phi\subset\mathbb{R}$ is the value semigroup. For each $\mathfrak{a}_k$ ($k\in\Phi$), we denote

$$c_k := \mathrm{lct}(X, D; \mathfrak{a}_k).$$

Let $\mu_k\colon Y_k\to X$ be a dlt modification of $(X, D; c_k\cdot\mathfrak{a}_k)$ and E_k the exceptional divisor of Y_k over X. Assume the model we obtain from Lemma 3.8 is Y'_k with the Kollár component S_k.

We consider the valuation

$$v_k := \frac{c_k\cdot k}{A_{(X,D)}(S_k)}\mathrm{ord}_{S_k}.$$

Note that $A_{(X,D)}(v_k)=c_k\cdot k$ is uniformly bounded:

$$c_k\cdot k=\mathrm{lct}\left(X, D; \frac{1}{k}\mathfrak{a}_k\right)=\inf_{v'}\frac{A_{(X,D)}(v')}{\frac{1}{k}v'(\mathfrak{a}_k)}\le A_{(X,D)}(v^{\mathrm{m}})<\infty.$$

So by the Izumi type estimate in [Li18, Theorem 1.2], we know that

$$v_k(\mathfrak{m})\mathrm{ord}_o\le v_k\le cA_{(X,D)}(v_k)\cdot\mathrm{ord}_o\le c'\cdot\mathrm{ord}_o$$

for some positive constants c, c' and all k. By [Li18, Theorem 1.1] and the fact that $\widehat{\mathrm{vol}}(v_k)$ is bounded from above, we know that $v_k(\mathfrak{m})$ is bounded from below. In particular, by the compactness result of [JM12, Proposition 5.9] and Proposition 3.9, there is an infinite sequence $\{v_{k_i}\}_{k_i\in\Phi}$ with $k_i\to\infty$ which has a limit in $\mathrm{Val}_{X,o}$, denoted by

$$v'=\lim_{i\to\infty}v_{k_i}.$$

Then

$$A_{(X,D)}(v')\le\liminf_{i\to\infty}A_{(X,D)}(v_{k_i})=\liminf_{i\to\infty}c_{k_i}\cdot k_i\le A_{(X,D)}(v^{\mathrm{m}})$$

as $A_{(X,D)}$ is lower semicontinuous (see [JM12, Lemma 5.7]). We claim that for any f,

$$v'(f)\ge v^{\mathrm{m}}(f).$$

Assuming this is true, we have $\mathrm{vol}(v')\le\mathrm{vol}(v^{\mathrm{m}})$, which then implies $\widehat{\mathrm{vol}}(v')\le\widehat{\mathrm{vol}}(v^{\mathrm{m}})$. Because v^{m} is a minimizer of $\widehat{\mathrm{vol}}$, by Proposition 2.7 we must have $v'=v^{\mathrm{m}}$.

To verify the claim, we pick any $f\in R$ and let $v^{\mathrm{m}}(f)=p$. For a fixed k_j, choose l such that

$$(l-1)p<k_j\le lp.$$

Let $k=k_j$ in the previous construction. Then

$$\begin{aligned} v^{\mathrm{m}}(f)=p &\Longrightarrow v^{\mathrm{m}}(f^l)=pl\\ &\Longrightarrow f^l\in\mathfrak{a}_{pl}\\ &\Longrightarrow f^l\in\mathfrak{a}_{k_j}\\ &\Longrightarrow l\cdot\mathrm{ord}_{E_i}(f)\ge m_{k_j,i}\ \text{for any } i\\ &\Longrightarrow l\cdot\mathrm{ord}_{S_{k_j}}(f)\ge A_{(X,D)}(S_{k_j})\cdot\frac{1}{c_{k_j}}\\ &\Longrightarrow v_{k_j}(f)\ge k_j/l>p-p/l. \end{aligned}$$

The fourth arrow is because if $f^l \in \mathfrak{a}_{k_j}$, then f^l vanishes along $m_{k_j,i}G_{k_j,i}$; and the fifth because

$$K_{Y_{k_j}} + \mu_{k_j *}^{-1}D + E_{k_j} \sim_{\mathbb{Q},X} c_{k_j} \cdot \sum m_{k_j,i}G_{k_j,i},$$

and the pull back of $K_{Y_{k_j}} + \mu_{k_j *}^{-1}D + E_{k_j}$ is larger than the one from

$$K_{Y'_{k_j}} + \mu'^{-1}_{k_j *}D + S_{k_j} \sim_{\mathbb{Q},X} A_{(X,D)}(S_{k_j})S_{k_j}.$$

Thus $v'(f) = \lim v_{k_j}(f) \geq p = v^{\mathrm{m}}(f)$. □

Proposition 3.9. *Let $o \in (X, D)$ be a klt singularity. Let a and b be positive numbers. Then the subset $K_{a,b}$ of $\mathrm{Val}_{X,o}$ which consists of all valuations v with*

$$a \leq v(\mathfrak{m}) \quad \textit{and} \quad A_{(X,D)}(v) \leq b$$

is sequentially compact.

Proof. Let $\{v_i\}$ be a sequence contained in $K_{a,b}$. Let $\{\mathfrak{a}_{i,k}\}$ be the associated graded sequence of valuative ideals for $k \in \Phi_i$. We can find a countably generated field $F \subset \mathbb{C}$ such that $R = \mathrm{Spec}(R_F) \times_F \mathbb{C}$ for some finitely generated F-algebra R_F, and D, o are defined over F. Furthermore, we can assume that for each pair (i, k), $\mathfrak{a}_{i,k} = (\mathfrak{a}_{i,k})_F \times_F \mathbb{C}$ for some ideal $(\mathfrak{a}_{i,k})_F \subset R_F$. Let $X_F := \mathrm{Spec}(R_F)$ and let D_F be the divisor of D descending on X_F.

Now let $(v_i)_F$ be the restriction of v_i on R_F. By our definition,

$$\mathfrak{a}_{i,k} = \{f \in R_F \mid (v_i)_F(f) \geq k\},$$

and $(v_i)_F \in (K_{a,b})_F$ where $(K_{a,b})_F$ is defined for all $v \in \mathrm{Val}_{X_F,o}$ with $a \leq v(\mathfrak{m}_F)$ and $A_{X_F,D_F}(v) \leq b$. By [HLP14, Theorem 1.1], $\mathrm{Val}_{X_F,o}$ has the same topology as a set in some Euclidean space, thus $(K_{a,b})_F$ is sequentially compact as it is compact by [JM12, Proposition 5.9]. Therefore after passing to a subsequence, $(v_i)_F$ has a limit $(v_\infty)_F$, which can be extended to a valuation $v_\infty := (v_\infty)_F \otimes \mathbb{C}$. In fact, v_∞ is defined as follows: Any $f \in R$ can be written as $\sum_{j=1}^m f_j \otimes_F h_j$ with $0 \neq f_j \in R$ and $h_1, \ldots, h_m \in \mathbb{C}$ linearly independent over F. Then

$$v_\infty(f) = \min_{j=1}^{m} (v_\infty)_F(f_j).$$

We claim $v_i = (v_i|_{R_F}) \otimes_F \mathbb{C}$. In fact, for any f, if $v_i(f) = k$, then $f \in \mathfrak{a}_{i,k} = (\mathfrak{a}_{i,k})_F \otimes_F \mathbb{C}$, thus $(v_i|_{R_F}) \otimes_F \mathbb{C}(f) = k$.

To see that for any f, $v_\infty(f) = \lim v_i(f)$, we note that for some j,

$$v_\infty(f) = (v_\infty)_F(f_j) = \lim_i (v_i|_{R_F})(f_j) \geq \limsup_i v_i(f).$$

For the other direction, if we have a subsequence of i such that $\lim_i v_i(f) < v_\infty(f)$, then after passing to a subsequence again, we can find a j such that

$$\lim_i v_i(f) = \lim_i v_i(f_j) = \lim_i (v_\infty)_F(f_j) \geq v_\infty(f),$$

a contradiction. □

Remark 3.10. A referee pointed out that the sequential compactness of Berkovich space was studied in [Poi13]. The above result could also be derived from that work.

For a general klt singularity (X, o), the minimum is not always achieved by a Kollár component (see [Blu18, LX18]). Thus we have to take a limiting process. However, if the minimizer v is divisorial, then it should always yield a Kollár component. First we have the following result inspired by [Blu16] (it is also independently obtained in [Blu18]).

Lemma 3.11. *If* $\mathrm{ord}_E \in \mathrm{Val}_{X,o}$ *minimizes* $\widehat{\mathrm{vol}}_{(X,D)}$*, then the Rees algebra associated to* ord_E *is finitely generated.*

Proof. If $\{\mathfrak{a}_\bullet\}$ are the graded valuative ideals associated to ord_E, then

$$\begin{aligned}\widehat{\mathrm{vol}}(\mathrm{ord}_E) &= \lim_{k\to\infty} A_{(X,D)}(\mathrm{ord}_E)^n \cdot \frac{\mathrm{mult}(\mathfrak{a}_k)}{k^n}\\ &\geq \lim_{k\to\infty} \mathrm{lct}(X, D; \mathfrak{a}_k)^n \cdot \mathrm{mult}(\mathfrak{a}_k) \geq \widehat{\mathrm{vol}}(\mathrm{ord}_E)\end{aligned}$$

by Proposition 2.8 and our assumption that ord_E is a minimizer of $\widehat{\mathrm{vol}}_{(X,D)}$. So we conclude that (see [Mus02])

$$\mathrm{lct}(X, D; \mathfrak{a}_\bullet) := \lim_{k\to\infty} k \cdot \mathrm{lct}(X, D; \mathfrak{a}_k) = A_{(X,D)}(\mathrm{ord}_E),$$

which we denote by c. Therefore, we can choose ϵ so small that the discrepancy $a(E; X, D + (1-\epsilon)c \cdot \mathfrak{a}_\bullet)$ is in $(-1, 0)$,

On the other hand, we know

$$\mathrm{lct}(X, D; \mathfrak{a}_\bullet) = \lim_{m\to\infty} m \cdot \mathrm{lct}(X, D; \mathfrak{a}_m).$$

So for sufficiently large m, and all G,

$$a\Big(G; X, D + \frac{1}{m}(1-\epsilon)c \cdot \mathfrak{a}_m\Big) > -1.$$

We also have $a\big(E; X, D + \frac{1}{m}(1-\epsilon)c \cdot \mathfrak{a}_m\big) < 0$. Then similar to the discussion in 3.4, we can find a $\mathbb{Q}$-divisor Δ such that $\big(X, D + \frac{1}{m}(1-\epsilon)c \cdot \Delta\big)$ is klt and $a\big(E; X, D + \frac{1}{m}(1-\epsilon)c \cdot \Delta\big) < 0$. As a consequence we can apply [BC+10] to obtain a model $\mu\colon Y \to X$ such that $\mathrm{Ex}(\mu) = E$ and $-E$ is μ-ample, which implies finite generation. □

Proof of Theorem C. By Lemma 3.11, the assumption in (i) that v is a multiple of a divisorial valuation implies the assumption in (ii), thus we only need to treat (ii).

By the proof of Proposition 2.8,

$$A_{(X,D)}(v)^n \cdot \frac{\mathrm{mult}(\mathfrak{a}_k)}{k^n} \geq \left(\frac{A_{(X,D)}(v)}{v(\mathfrak{a}_k)}\right)^n \cdot \mathrm{mult}(\mathfrak{a}_k) \geq \mathrm{lct}(X, D; \mathfrak{a}_k)^n \cdot \mathrm{mult}(\mathfrak{a}_k).$$

By the finite generation assumption, $\mathfrak{a}_{kl} = \mathfrak{a}_k^l$ for sufficiently divisible k and any l. So replacing k by kl in the above display and letting $l \to \infty$, we find that

$$\widehat{\mathrm{vol}}_{(X,D),o}(v) \geq \mathrm{lct}(X, D; \mathfrak{a}_k)^n \cdot \mathrm{mult}(\mathfrak{a}_k) \geq \widehat{\mathrm{vol}}_{(X,D),o}(v).$$

Take $\mu\colon Y \to X$ to be the dlt modification of $(X, D + \mathrm{lct}(X, D, \mathfrak{a}_k) \cdot \mathfrak{a}_k)$ as given in Lemma 3.5. The above discussion then implies that

$$\mathrm{lct}(X, D; \mathfrak{a}_k)^n \cdot \mathrm{mult}(\mathfrak{a}_k) = \widehat{\mathrm{vol}}_{(X,D),o}(v) = \mathrm{vol}_{(X,D),o}(Y).$$

Moreover, it follows from Proposition 2.10 that we can choose a model $W \to Y$ and run MMP to obtain $W \dashrightarrow Y'$ such that $\mu'\colon Y' \to X$ gives a Kollár component S with $a(S; Y, \mu^{-1} D_* + E) = -1$. We only need to show that if Y' and Y are not isomorphic in codimension 1, then

$$\mathrm{vol}_{(X,D),o}(Y') < \mathrm{vol}_{(X,D),o}(Y).$$

This is the local analog of the argument in [LX14, Proposition 5]. We give the details for the reader's convenience.

Let $\pi : Y \to Y^c$ be the canonical model of $-K_Y - \mu_*^{-1} D - E$ over X, which exists because

$$-\epsilon(K_Y + \mu_*^{-1} D + E) \sim_{\mathbb{Q},X} K_Y + \mu_*^{-1}\Big(D + c \cdot \frac{1}{l} \sum D_j\Big) + E - \epsilon \sum_i A_{(X,D)}(G_i) G_i$$

is a klt pair for ϵ sufficiently small. The assumption that Y' and Y are not isomorphic in codimension 1 implies $Y^c \neq Y$.

Choose $p\colon \hat{Y} \to Y$ and $q\colon \hat{Y} \to Y'$ to be log resolutions of singularities from a common smooth variety $\hat{Y}$, and write

$$p^*(K_Y + \mu_*^{-1} D + E) = q^*(K_{Y'} + \mu'^{-1}_* D + S) + G.$$

By the negativity lemma (cf. [KM98, 3.39]), we conclude that $G \geq 0$. Since

$$K_Y + \mu_*^{-1} D + E \sim_{\mathbb{Q},X} \sum_i A_{(X,D)}(G_i) G_i$$

and

$$K_{Y'} + \mu'^{-1}_* D + S \sim_{\mathbb{Q},X} A_{(X,D)}(S) \cdot S,$$

we know that

$$p^*\Big(\sum_i A_{(X,D)}(G_i) G_i\Big) = q^*(A_{(X,D)}(S) \cdot S) + G.$$

For $0 \leq \lambda \leq 1$, let

$$L_\lambda = q^*(A_{(X,D)}(S) \cdot S) + \lambda G = \sum_i b_i(\lambda) F_i$$

where F_i runs over all divisor supports on $\hat{Y}_o := \hat{Y} \times_X \{o\}$, and $-L_\lambda|_{\hat{Y}_o}$ is nef. Define

$$f(\lambda) = \sum_i b_i(\lambda)(-L_\lambda|_{F_i})^{n-1}.$$

Then $f(\lambda)$ is nondecreasing as $G \geq 0$. By Lemmas 2.11 and 3.6, we know that

$$f(1) = \mathrm{vol}_{(X,D),o}(Y) \quad \text{and} \quad f(0) = \mathrm{vol}_{(X,D),o}(Y').$$

Since $Y \dashrightarrow Y'$ is not isomorphic in codimension 1, it must contract some component G_1 of E, and the coefficient of G_1 in G is

$$a := A_{(Y',\mu'^{-1}_{*}D+S)}(G_1) > 0.$$

Then

$$\begin{aligned}\frac{df(\lambda)}{d\lambda}\bigg|_{\lambda=1} &= n \cdot G \cdot (-p^*(K_Y + \mu_*^{-1}D + E))^{n-1} \\ &\geq n \cdot aG_1 \cdot (-\pi_*(K_Y + \mu_*^{-1}D + E))^{n-1} > 0.\end{aligned}$$

Thus $\mathrm{vol}_{(X,D),o}(Y') = f(0) < f(1) = \mathrm{vol}_{(X,D),o}(Y)$. □

With all these discussions, we also obtain the following result, which characterizes the equality condition in Proposition 2.8 and is a generalization of [FEM04, Theorem 1.4] (see Remark 3.13) for smooth points. See [Laz04b, 9.6] for more background.

Theorem 3.12. *Let* $(X, o) = (\mathrm{Spec}(R), \mathfrak{m})$. *Assume* (X, D) *is a klt singularity for a* $\mathbb{Q}$*-divisor* $D \geq 0$. *Then there exists an* $\mathfrak{m}$*-primary ideal* $\mathfrak{a}$ *that realizes the minimum of normalized volume, i.e.*

$$\mathrm{lct}(X, D; \mathfrak{a})^n \cdot \mathrm{mult}(\mathfrak{a}) = \inf_{v \in \mathrm{Val}_{X,o}} \widehat{\mathrm{vol}}(v),$$

if and only if there exists a Kollár component S that satisfies the following two conditions:

(a) ord_S *computes both* $\mathrm{lct}(X, D; \mathfrak{a})$ *and* $\inf_{v\in \mathrm{Val}_{X,o}} \widehat{\mathrm{vol}}(v)$.
(b) *There exists a positive integer k such that the only associated Rees valuation of* $\mathfrak{a}^k$ *is* ord_S.

Later we will verify Theorem B which says such a minimizing Kollár component S is unique.

Proof of Theorem 3.12. By the argument in Theorem C, we see that

$$\mathrm{lct}(X, D; \mathfrak{a})^n \cdot \mathrm{mult}(\mathfrak{a})$$

realizes the minimum of $\widehat{\mathrm{vol}}_{(X,D),o}$ if and only if there is a dlt modification $\mu\colon Y \to X$ of

$$(X, D; c \cdot \mathfrak{a}) \quad \text{where} \quad c = \mathrm{lct}(X, D; \mathfrak{a})$$

that only extracts a Kollár component S of (X, D) such that ord_S is a minimizer of $\widehat{\mathrm{vol}}_{(X,D),o}$.

Now we fix such an ideal $\mathfrak{a}$ and Kollár component S. Assume that $\mu^*\mathfrak{a}$ has vanishing order m along S. Since S is $\mathbb{Q}$-Cartier, we can choose a positive integer k such that mkS is Cartier. We claim that

$$\mu^*(\mathfrak{a}^k) = \mathcal{O}_Y(-mkS).$$

Granted this, we find that Y coincides with the normalized blow up $X^+ \to X$ of $\mathfrak{a}^k$, i.e., S is the only associated Rees valuation for $\mathfrak{a}^k$.

To verify the claim, since $-mkS$ is Cartier, we infer that

$$\mu^*(\mathfrak{a}^k) = \mathfrak{c} \cdot \mathcal{O}_Y(-mkS) \quad \text{for some ideal } \mathfrak{c} \subset \mathcal{O}_Y,$$

and we aim to show that $\mathfrak{c}$ is indeed trivial. If not, we take a normalized blow up $\phi\colon Y^+ \to Y$ of $\mathfrak{c}$, so $\phi^*\mathfrak{c} = \mathcal{O}_{Y^+}(-E)$ for some effective Cartier divisor E. Since $-S$ is ample over X, we can choose l so large that

$$-D := -\phi^*(mklS) - E$$

on Y^+ is ample over X.

Since

$$(\mu \circ \phi)^*\mathfrak{a}^{kl} = \mathcal{O}_{Y^+}(-\phi^*(mklS) - lE) \subset \mathcal{O}_{Y^+}(-\phi^*(mklS) - E),$$

we know that

$$\begin{aligned} \mathrm{mult}(\mathfrak{a}^{kl}) &\geq \mathrm{vol}_o^F(-\phi^*(mklS) - E) = \mathrm{vol}_o^F(-D) \\ &= mkl(-D|_{\phi^*S})^{n-1} + (-D|_E)^{n-1} \\ &> mkl(mkl(-S)|_S)^{n-1} = (mkl)^n \,\mathrm{vol}(\mathrm{ord}_S). \end{aligned}$$

Since $\mathrm{lct}(X, D; \mathfrak{a}) = \frac{1}{m} \cdot A_{(X,D)}(\mathrm{ord}_S)$, we can easily see the above inequality contradicts the assumption that

$$\mathrm{lct}(X, D; \mathfrak{a})^n \cdot \mathrm{mult}(\mathfrak{a}) = \widehat{\mathrm{vol}}(\mathrm{ord}_S).$$

Here the last inequality comes from a calculation similar to (but easier than) the proof of Theorem C.

For the converse direction, we assume conditions (a)–(b) hold. We assume that $\mathrm{ord}_S(\mathfrak{a}) = m$ and that for some integer k the only associated Rees valuation of $\mathfrak{a}^k$ is ord_S, i.e., the normalized blow up of $\mathfrak{a}^k$, denoted by $\mu\colon X^+ \to X$, has the property that $\mu^*(\mathfrak{a}^k) = \mathcal{O}_{X^+}(-mkS)$. Then we have

$$\mathfrak{a}_{mkl}(\mathrm{ord}_S) = \{f \in R \mid \mathrm{ord}_S(f) \geq mkl\} = \mu_*(\mu^*(\mathfrak{a}^k)^l) = \overline{\mathfrak{a}^{kl}}, \tag{9}$$

where $\overline{\mathfrak{a}^{kl}}$ is the integral closure of $\mathfrak{a}^{kl}$. By assumption, $\mathrm{lct}(X, D; \mathfrak{a}) = A_{(X,D)}(S)/m$. We claim that

$$\mathrm{mult}(\mathfrak{a}^k) = \lim_{k\to\infty} \frac{n! \cdot l_R(R/\overline{\mathfrak{a}^{kl}})}{l^n},$$

which together with (9) implies that

$$\begin{aligned} \mathrm{lct}(X, D; \mathfrak{a})^n \cdot \mathrm{mult}(\mathfrak{a}) &= \mathrm{lct}(X, D; \mathfrak{a}^k)^n \cdot \mathrm{mult}(\mathfrak{a}^k) \\ &= \frac{A_{(X,D)}(S)^n}{m^n} \lim_{l\to\infty} \frac{n! \cdot l_R(R/\mathfrak{a}_{ml}(\mathrm{ord}_S))}{l^n} \\ &= \widehat{\mathrm{vol}}(\mathrm{ord}_S) = \inf_v \widehat{\mathrm{vol}}(v). \end{aligned}$$

To verify the claim, if we denote by $\mathcal{J}(\mathfrak{a}^{kl}) = \mathcal{J}(X, D; \mathfrak{a}^{kl})$ the multiplier ideal, then

$$\operatorname{mult}(\mathfrak{a}^k) = \lim_{k\to\infty} \frac{n! \cdot l_R(R/\mathcal{J}(\mathfrak{a}^{kl}))}{l^n}$$

by the local Skoda Theorem [Laz04b, 9.6.39]. On the other hand, since (X, D) is klt,

$$\mathfrak{a}^{kl} \subseteq \overline{\mathfrak{a}^{kl}} \subseteq \mathcal{J}(\mathfrak{a}^{kl}).$$

Thus

$$\begin{aligned}\operatorname{mult}(\mathfrak{a}^k) &= \lim_{l\to\infty} \frac{n! \cdot l_R(R/\mathfrak{a}^{kl})}{l^n} \geq \lim_{l\to\infty} \frac{n! \cdot l_R(R/\overline{\mathfrak{a}^{kl}})}{l^n} \\ &\geq \lim_{l\to\infty} \frac{n! \cdot l_R(R/\mathcal{J}(\mathfrak{a}^{kl}))}{l^n} = \operatorname{mult}(\mathfrak{a}^k).\end{aligned}$$

Thus the inequalities have to be equalities and we are done. □

Remark 3.13. In the proof, we have indeed shown that if $\mathfrak{a}$ has the minimal normalized multiplicity and S is a Kollár component such that $\operatorname{ord}_S(\mathfrak{a}) = m$ as in the statement of the above theorem, then for any k such that mkS is Cartier on Y, the integral closure $\overline{\mathfrak{a}^k}$ coincides with the valuative ideal $\mathfrak{a}_{mk}$ of ord_S (see identity (9)).

4. K-semistability implies the minimum

4.1. Degeneration to initial ideals

Let $(X, o) = (\operatorname{Spec}(R), \mathfrak{m})$ be an algebraic singularity such that (X, D) is klt for a $\mathbb{Q}$-divisor $D \geq 0$. Given a Kollár component S, we consider the associated degeneration $\mathcal{W}^\circ/\mathbb{A}^1$ of X where $\mathcal{W}^\circ$ is the underlying coarse moduli space of $\mathfrak{W}^\circ = \mathfrak{W} \setminus \mathfrak{Y}_0$ defined in Section 2.4. We follow the notation of Section 2.4 and also denote by v_0 the valuation ord_S.

Suppose $\mathfrak{b}$ is an $\mathfrak{m}$-primary ideal on X. We will describe explicitly an ideal $\mathfrak{B}$ on $\mathcal{W}$ such that $\mathfrak{B} \otimes \mathcal{O}_{X\times\mathbb{C}^*}$ is the pull back of $\mathfrak{b}$ and $\mathfrak{B} \otimes \mathcal{O}_C$ is the ideal generated by initial forms of elements of $\mathfrak{b}$, by considering the closure of $\mathfrak{b} \times \mathbb{C}^*$ on $\mathcal{W}$. For this purpose we consider the extended Rees algebra associated to the Kollár component (see [Eis94, 6.5]):

$$\mathcal{R}' = \bigoplus_{k\in\mathbb{Z}} \mathcal{R}'_k := \bigoplus_{k\in\mathbb{Z}} \mathfrak{a}_k t^{-k} \subset R[t, t^{-1}]$$

where $\mathfrak{a}_k = \mathfrak{a}_k(\operatorname{ord}_S)$. Notice that if $k \leq 0$, then $\mathfrak{a}_k = R$. It is well known that the following identification holds true (recall that R^* was defined in (5)):

$$\mathcal{R}' \otimes_{\mathbb{C}[t]} \mathbb{C}[t, t^{-1}] \cong R[t, t^{-1}], \qquad \mathcal{R}' \otimes_{\mathbb{C}[t]} \mathbb{C}[t]/(t) \cong \bigoplus_{k=0}^{\infty} (\mathfrak{a}_k/\mathfrak{a}_{k+1}) t^{-k} \cong R^*.$$

Geometrically this exactly means $\mathcal{W}^\circ = \mathrm{Spec}(\mathcal{R}')$ and

$$\mathcal{W}^\circ \times_{\mathbb{A}^1} (\mathbb{A}^1 \setminus \{0\}) = X \times (\mathbb{A}^1 \setminus \{0\}), \quad \mathcal{W}^\circ \times_{\mathbb{A}^1} \{0\} = C.$$

Notice that there is a natural $\mathbb{G}_m$-action on $\mathcal{W}^\circ$ given by the $\mathbb{Z}$-grading.

For any $f \in R$, supposing $v_0(f) = k$, we define

$$\tilde{f} = t^{-k} f \in \mathfrak{a}_k t^{-k} \subset \mathcal{R}',$$

and denote

$$\mathbf{in}(f) = [f] = [f]_{\mathfrak{a}_{k+1}} \in \mathfrak{a}_k/\mathfrak{a}_{k+1} = R^*_k$$

where we use $[f]_{\mathfrak{a}}$ to denote the image of f in $R/\mathfrak{a}$. Then we define $\mathfrak{B}$ to be the ideal in $\mathcal{R}'$ generated by $\{\tilde{f} \mid f \in \mathfrak{b}\}$, and $\mathbf{in}(\mathfrak{b})$ to be the ideal of R^* generated by $\{\mathbf{in}(f) \mid f \in \mathfrak{b}\}$. The first two items of the following lemma are similar to (but not the same as) [Eis94, Theorem 15.17] and should be well known to experts. Notice that here we degenerate both the ambient space and the ideal. A version of the equality (10) was proved in [Li17b, Proposition 4.3].

Lemma 4.1. (i) *With the above notations,*

$$(\mathcal{R}'/\mathfrak{B}) \otimes_{\mathbb{C}[t]} \mathbb{C}[t, t^{-1}] \cong (R/\mathfrak{b})[t, t^{-1}], \quad (\mathcal{R}'/\mathfrak{B}) \otimes_{k[t]} k[t]/(t) \cong R^*/\mathbf{in}(\mathfrak{b}).$$

(ii) *The $\mathbb{C}[t]$-algebra $\mathcal{R}'/\mathfrak{B}$ is free and thus flat as a $\mathbb{C}[t]$-module. In particular,*

$$\dim_{\mathbb{C}}(R/\mathfrak{b}) = \dim_{\mathbb{C}}(R^*/\mathbf{in}(\mathfrak{b})). \tag{10}$$

(iii) $\mathbf{in}(\mathfrak{b})$ *is an $\mathfrak{m}_0$-primary homogeneous ideal, where $\mathfrak{m}_0 = \sum_{k>0} R^*_k$.*

Proof. (i) follows easily from the definition.

Next we prove (ii). Denote by $\mathfrak{c}_k = R^*_k \cap \mathbf{in}(\mathfrak{b})$ the k-th homogeneous piece of $\mathbf{in}(\mathfrak{b})$. We fix a basis $\{\mathbf{in}(f_i^{(k)}) \mid 1 \le i \le d_k\}$ of $R^*_k/\mathfrak{c}_k$. We want to show that

$$\mathcal{A}' := \{[\widetilde{f_i^{(k)}}] = [f_i^{(k)}]_{\mathfrak{B}} \mid 1 \le i \le d_k\} \subset \mathcal{R}'/\mathfrak{B}$$

is a $\mathbb{C}[t]$-basis of $\mathcal{R}'/\mathfrak{B}$.

We first verify that $\mathcal{A}'$ is a linearly independent set. To prove this, we just need to show that $\mathcal{A}'$ is a $\mathbb{C}[t, t^{-1}]$-linearly independent subset of $(R/\mathfrak{b})[t, t^{-1}]$. It is then enough to show that

$$\mathcal{A} := \{[f_i^{(k)}] = [f_i^{(k)}]_{\mathfrak{b}} \mid 1 \le i \le d_k\} \subset R/\mathfrak{b} \tag{11}$$

is $\mathbb{C}$-linearly independent, which can be verified directly as in [Li17b, Proposition 4.3]. See also [Eis94, Proposition 15.3].

So we just need to show that $\mathcal{A}'$ spans $\mathcal{R}'/\mathfrak{B}$. Equivalently, we need to show that for any $f \in R$, $[\tilde{f}] = [\tilde{f}]_{\mathfrak{B}} \in \mathcal{R}'/\mathfrak{B}$ is in the $\mathbb{C}[t]$-span of $\mathcal{A}'$. This can be shown again with the help of $\mathcal{A}$ in (11), that is, it is enough to prove that $\mathcal{A}$ spans $R/\mathfrak{b}$ as a $\mathbb{C}$-linear space. Indeed, assuming the latter, for any $f \in R$, there exists a linear combination $g = \sum_{i,k} c_{ik} f_i^{(k)}$ such that $f - g =: h \in \mathfrak{b}$. If $m = v_0(f)$, then

$$\tilde{f} = t^{-m} f = \sum_{i,k} c_{ik} t^{-m} f_i^{(k)} + t^{-m} h.$$

Because $t^{-m} h \in \mathfrak{B}$, the above indeed implies $[\tilde{f}]$ is in the $\mathbb{C}[t]$-span of $\mathcal{A}'$.

To prove that $\mathcal{A}$ indeed $\mathbb{C}$-spans $R/\mathfrak{b}$, we first claim that the following set is finite:

$$\{v_0(g) \mid g \in R - \mathfrak{b}\}.$$

Indeed, because $\mathfrak{b}$ is $\mathfrak{m}$-primary, there exists $N > 0$ such that $\mathfrak{m}^N \subseteq \mathfrak{b} \subseteq \mathfrak{m}$. So $R - \mathfrak{b} \subseteq R - \mathfrak{m}^N$. Now the claim follows from the fact that for any $f \in \mathfrak{m}^N$,

$$v_0(f) \leq c \cdot A(v_0) \cdot N$$

by Izumi's theorem, where c is a uniform constant not depending on f.

If there is $[f] \neq 0 \in R/\mathfrak{b}$ that is not in the span of $\mathcal{A}$, then we can choose a maximal $k = v_0(f)$ for which this happens. There are two cases:

- If $\mathbf{in}(f) \in R_k^* \setminus \mathfrak{c}_k$, then because $\mathbf{in}(f_i^{(k)})$ is a basis of $R_k^*/\mathfrak{c}_k$, there exist $t_j \in \mathbb{C}$ such that $\mathbf{in}(f) - \sum_{j=1}^{d_k} t_j \mathbf{in}(f_j^{(k)}) = \mathbf{in}(g) \in \mathfrak{c}_k$ for some $g \in \mathfrak{b}$. So we get

$$v_0\Big(f - \sum_{j=1}^{d_k} t_j f_j^{(k)} - g\Big) > k.$$

  By maximality of k, $[f - \sum_{j=1}^{d_k} t_j f_j^{(k)} - g] = [f] - \sum_{j=1}^{d_k} t_j [f_j^{(k)}]$ and hence $[f]$ is in the span of $\mathcal{A}$, a contradiction.
- If $\mathbf{in}(f) \in \mathfrak{c}_k = \mathbf{in}(\mathfrak{b}) \cap R_k^*$, then $\mathbf{in}(f) = \mathbf{in}(g)$ for some $g \in \mathfrak{b}$. So $v_0(f - g) > k$ and hence $[f - g]$ is in the span of $\mathcal{A}$ by the maximal property of k. But then $[f] = [f - g] + [g] = [f - g]$ is in the span of $\mathcal{A}$, a contradiction.

To prove part (iii) of the lemma, we need to show that there exists $N \in \mathbb{Z}_{>0}$ such that $\mathfrak{m}_0^N \subseteq \mathbf{in}(\mathfrak{b}) \subseteq \mathfrak{m}_0$. Because $\mathfrak{b}$ is $\mathfrak{m}$-primary, there exists $N_1 \in \mathbb{Z}_{>0}$ such that $\mathfrak{m}^{N_1} \subseteq \mathfrak{b} \subseteq \mathfrak{m}$. By Izumi's theorem, there exists $l \in \mathbb{Z}_{>0}$ such that $\mathfrak{a}_{lm} \subseteq \mathfrak{m}^m$ for any $m \in \mathbb{Z}_{>0}$. By letting $N = lN_1$, it is easy to see that $\mathfrak{m}_0^N \subseteq \mathbf{in}(\mathfrak{b}) \subseteq \mathfrak{m}_0$. □

Lemma 4.2. *If $\mathfrak{b}_\bullet = \{\mathfrak{b}_k\}$ is a graded family of ideals of R, then $\mathbf{in}(\mathfrak{b}_\bullet) := \{\mathbf{in}(\mathfrak{b}_k)\}$ is also a graded family of ideals of R^*.*

Proof. We just need to show that

$$\mathbf{in}(\mathfrak{b}_k) \cdot \mathbf{in}(\mathfrak{b}_l) \subseteq \mathbf{in}(\mathfrak{b}_{k+l}).$$

If $v_0(f) = k$ and $v_0(g) = l$, then $v_0(fg) = k + l$, and we have

$$\mathbf{in}(f) \cdot \mathbf{in}(g) = [f]_{\mathfrak{a}_{k+1}} \cdot [g]_{\mathfrak{a}_{l+1}} = [fg]_{\mathfrak{a}_{k+l+1}} = \mathbf{in}(f \cdot g). \qquad \square$$

Lemma 4.3. *If* $\mathfrak{b}_\bullet$ *is a graded family of ideals, then*

$$\mathrm{lct}(\mathfrak{b}_\bullet)^n \cdot \mathrm{mult}(\mathfrak{b}_\bullet) \geq \mathrm{lct}(\mathbf{in}(\mathfrak{b}_\bullet)^n) \cdot \mathrm{mult}(\mathbf{in}(\mathfrak{b}_\bullet)). \tag{12}$$

Proof. By the flatness of $\mathfrak{B}$ and the lower semicontinuity of log canonical thresholds, we have $\mathrm{lct}(\mathfrak{b}_k) \geq \mathrm{lct}(\mathbf{in}(\mathfrak{b}_k))$. Therefore, by (10),

$$\mathrm{lct}(\mathfrak{b}_k)^n \cdot l_R(R/\mathfrak{b}_k) \geq \mathrm{lct}(\mathbf{in}(\mathfrak{b}_k)^n) \cdot l_{R^*}(R^*/\mathbf{in}(\mathfrak{b}_k)).$$

Letting $k \to \infty$, we get (12). □

4.2. Equivariant K-semistability and minimizer

In this section, we will take a detour to show that the discussion in Section 4.1 can be used to study equivariant K-semistability. Here a $\mathbb{Q}$-Fano variety (V, Δ) with an action of an algebraic group G is called *G-equivariantly K-semistable* (resp. *Ding semistable*) if for any G-equivariant test configuration, the generalized Futaki (resp. Ding) invariant is nonnegative. Let $T = (\mathbb{C}^*)^r$ be a torus. First we improve the two approximating results to the equivariant case.

Proposition 4.4. *Let* $(X, o) = (\mathrm{Spec}(R), \mathfrak{m})$ *and* $D \geq 0$ *a* $\mathbb{Q}$*-divisor such that* $o \in (X, D)$ *is a klt singularity. Assume* $o \in (X, D)$ *admits a* T*-action. Then*

$$\min_v \widehat{\mathrm{vol}}_{(X,D),o}(v) = \inf_{\mathfrak{a}} \mathrm{lct}(X, D; \mathfrak{a})^n \cdot \mathrm{mult}(\mathfrak{a}) = \inf_S \widehat{\mathrm{vol}}_{(X,D),o}(\mathrm{ord}_S) \tag{13}$$

where on the left hand side v *runs over all the valuations centered at* o*, in the middle term* $\mathfrak{a}$ *runs over all* T*-equivariant* $\mathfrak{m}$*-primary ideals, and in the right term* S *runs over all* T*-equivariant Kollár components.*

Proof. Let $\mathfrak{a}_\bullet = \{\mathfrak{a}^k\}$ be a graded sequence for an $\mathfrak{m}$-primary ideal $\mathfrak{a}$. Assume $T \cong (\mathbb{C}^*)^r$. Fixing a lexicographic order on $\mathbb{Z}^r$, we can degenerate the ideal $\mathfrak{a}^k$ to its initial ideal $\mathbf{in}(\mathfrak{a}^k)$.

Lemma 4.3 implies that for $\mathfrak{b}_\bullet = \{\mathfrak{b}_k\} =: \{\mathbf{in}(\mathfrak{a}^k)\}$,

$$\mathrm{lct}(X, D; \mathfrak{b}_\bullet)^n \cdot \mathrm{mult}(\mathfrak{b}_\bullet) \leq \mathrm{lct}(X, D; \mathfrak{a}_\bullet)^n \cdot \mathrm{mult}(\mathfrak{a}_\bullet).$$

Since

$$\mathrm{lct}(X, D; \mathfrak{b}_\bullet)^n \cdot \mathrm{mult}(\mathfrak{b}_\bullet) = \lim_m \mathrm{lct}(X, D; \mathfrak{b}_m)^n \cdot \mathrm{mult}(\mathfrak{b}_m),$$

the first equality is a corollary of Proposition 2.8.

For the second equality, we just need to show that the construction in Section 3.1 can be established T-equivariantly. This is standard, and relies on two facts: first, we can always take an equivariant log resolution of $(X, D, \mathfrak{a})$ (see [Kol07]); second, as T is a connected group, for any curve C in a T-variety and any $t \in T$, $t \cdot C$ will always be numerically equivalent to C; as the minimal model program only depends on the numerical class $[C]$, we know that any MMP sequence is automatically T-equivariant. Therefore, for any T-equivariant $\mathfrak{m}$-primary ideal $\mathfrak{a}$, we can find a T-equivariant dlt modification $Y \to X$ and then a T-invariant Kollár component S such that

$$\operatorname{lct}(X, D; \mathfrak{a})^n \cdot \operatorname{mult}(\mathfrak{a}) \geq \operatorname{vol}(Y) \geq \widehat{\operatorname{vol}}(\operatorname{ord}_S). \qquad \square$$

In [Li17b] (see also [LL19]), it was proved that the canonical valuation on the affine cone minimizing $\widehat{\operatorname{vol}}_X$ implies V is K-semistable. Conversely, if V is K-semistable then the canonical valuation minimizes $\widehat{\operatorname{vol}}_X$ among all $\mathbb{C}^*$-invariant valuations. The argument extends easily to the logarithmic case. Proposition 4.4 allows us to extend the minimization result to all valuations in Val_X. ([LL19] proved the same result, but under the assumption that V degenerates to a Fano with Kähler–Einstein metric.) For the reader's convenience, we sketch the argument from [Li17b, LL19].

Theorem 4.5. *Let (V, Δ) be a projective log Fano variety and $o \in (X, D)$ the affine cone over (V, Δ) induced by some ample Cartier divisor $L = -r^{-1}(K_V + \Delta)$. Then the canonical valuation v_0 obtained by blowing up the vertex minimizes $\widehat{\operatorname{vol}}_{(X,D)}$ on $\operatorname{Val}_{X,o}$ if and only if (V, Δ) is log-K-semistable.*

Proof. First we assume that (V, Δ) is log-K-semistable and prove the volume minimizing property of ord_V. By Proposition 4.4, we only need to prove that for any $\mathbb{C}^*$-invariant divisorial valuation v over (X, o),

$$\widehat{\operatorname{vol}}(v_0) \leq \widehat{\operatorname{vol}}(v).$$

Let $Y \to X$ be the blow up at o with exceptional divisor still denoted by V. Denote by $\mathcal{I}_V$ the ideal sheaf of $V \subset Y$ and define (see [Li17b, Lemma 4.2])

$$c_1 := c_1(\mathcal{I}_V) = \min\{v(\phi) \mid \phi \in \mathcal{I}_V(U),\ U \cap \operatorname{center}_Y(U) \neq \emptyset\}.$$

Let $R = \bigoplus_{k=0}^{\infty} R_k = \bigoplus_{k=0}^{\infty} H^0(V, kL)$ be such that $X = \operatorname{Spec}(R)$. On R, we define a graded filtration

$$\mathcal{F}R^{(t)} = \bigoplus_{k=0}^{\infty} \mathcal{F}^{kt} R_k \quad \text{with} \quad \mathcal{F}^x R_k := \{f \in R_k \mid v(f) \geq x\}.$$

The volume of $\mathcal{F}R^{(t)}$ is defined to be

$$\operatorname{vol}(\mathcal{F}R^{(t)}) := \limsup_{m\to\infty} \frac{\dim_{\mathbb{C}}(\mathcal{F}^{mt} R_m)}{m^n/n!}.$$

By [Li17b, (21) and (22)], we get a formula for $\operatorname{vol}(v)$:

$$\operatorname{vol}(v) = \lim_{m\to\infty} \frac{n!}{m^n} \dim_{\mathbb{C}}(R/\mathfrak{a}_m(v)) = \frac{L^{n-1}}{c_1^n} - \int_{c_1}^{\infty} \operatorname{vol}(\mathcal{F}R^{(t)}) \frac{dt}{t^{n+1}}$$
$$= -\int_{c_1}^{\infty} \frac{d\operatorname{vol}(\mathcal{F}R^{(t)})}{t^n}.$$

Then we consider the function

$$\begin{aligned}\Phi(\lambda, s) &= \frac{L^{n-1}}{(\lambda c_1 s + (1-s))^n} - n \int_{c_1}^{\infty} \mathrm{vol}(\mathcal{F}R^{(t)}) \frac{\lambda s\, dt}{(1 - s + \lambda s t)^{n+1}} \\ &= \int_{c_1}^{\infty} \frac{-d\, \mathrm{vol}(\mathcal{F}R^{(t)})}{((1-s) + \lambda s t)^n}.\end{aligned}$$

$\Phi(\lambda, s)$ satisfies the following properties:

1. For any $\lambda \in (0, \infty)$, we have

$$\Phi(\lambda, 1) = \mathrm{vol}(\lambda v) = \lambda^{-n}\, \mathrm{vol}(v), \quad \Phi(\lambda, 0) = \mathrm{vol}(v_0) = L^{n-1}.$$

2. For any $\lambda \in (0, \infty)$, $\Phi(\lambda, s)$ is continuous and convex with respect to $s \in [0, 1]$.
3. The directional derivative of $\Phi(\lambda, s)$ at $s = 0$ is equal to

$$\Phi_s(\lambda, 0) = n\lambda L^{n-1}\left(\lambda^{-1} - c_1 - \frac{1}{L^{n-1}} \int_{c_1}^{\infty} \mathrm{vol}(\mathcal{F}R^{(t)})\, dt\right).$$

Let $\lambda_* = r/A_{(X,D)}(v)$. Note that $A_{(X,D)}(v_0) = r$. So by item 1, we have

$$\Phi(\lambda_*, 1) = \frac{\widehat{\mathrm{vol}}(v)}{r^n}, \quad \Phi(\lambda_*, 0) = L^{n-1} = \frac{\widehat{\mathrm{vol}}(v_0)}{r^n}.$$

By item 2, we just need to prove $\Phi_s(\lambda_*, 0) \geq 0$. Let $\bar{v} = v|_{\mathbb{C}(V)}$ be the restriction of v under the inclusion $\mathbb{C}(V) \hookrightarrow \mathbb{C}(X)$. It is known that $\bar{v} = b \cdot \mathrm{ord}_E$ where $b \geq 0$ by [BHJ17, proof of Lemma 4.1] and ord_E is a divisorial valuation on $\mathbb{C}(V)$. Moreover v is the $\mathbb{C}^*$-invariant extension of $\bar{v}$ to $\mathbb{C}(X)$ (cf. [BHJ17, Lemma 4.2], [Li17b, Appendix 4.2.1]):

$$v(f) = \min\left\{c_1 k + \bar{v}(f_k) \;\middle|\; f = \sum_k f_k \in R \text{ with } f_k \neq 0 \in R_k\right\}. \tag{14}$$

If $\phi\colon \tilde{V} \to V$ is a model that contains E as a divisor, then v can also be obtained as a quasi-monomial valuation on the model $\tilde{Y} \to Y$ where $\tilde{Y} = Y \times_V \tilde{V}$ (see [Li17b, Definition 6.12]). Using this description, it is easy to show that

$$\lambda_*^{-1} - c_1 = \frac{A_{(X,D)}(v)}{r} - c_1 = \frac{A_{(V,\Delta)}(\bar{v})}{r} = \frac{b \cdot A_{(V,\Delta)}(E)}{r}.$$

By a change of variables we get

$$\int_{c_1}^{\infty} \mathrm{vol}(\mathcal{F}R^{(t)})\, dt = \int_0^{\infty} \mathrm{vol}(\mathcal{F}_{\bar{v}} R^{(t)})\, dt$$

where

$$\mathcal{F}_{\bar{v}} R^{(t)} = \bigoplus_k H^0(V, L^{\otimes k} \otimes \mathfrak{a}_{kt}) \quad \text{and} \quad \mathfrak{a}_{kt} = \{f \in \mathcal{O}_V \mid \bar{v}(f) \geq kt\}.$$

So we get

$$\begin{aligned}\Phi_s(\lambda_*,0) &= n\lambda_* L^{n-1}\left(A_{(V,\Delta)}(\bar v) - \frac{r}{L^{n-1}}\int_0^\infty \mathrm{vol}(\mathcal{F}_{\bar v}R^{(t)})\,dt\right)\\ &= n\lambda_* L^{n-1} b\left(A_{(V,\Delta)}(E) - \frac{r}{L^{n-1}}\int_0^\infty \mathrm{vol}(\mathcal{F}_{\mathrm{ord}_E}R^{(t)})\,dt\right).\end{aligned}$$

By applying Fujita's result [Fuj18] (see also [Fuj19, Li17b, LL19]), we get $\Phi_s(\lambda_*,0)\geq 0$.

Conversely, if ord_V is volume minimizing, then the above calculation shows that

$$A_{(V,\Delta)}(\mathrm{ord}_E) - \frac{r}{L^{n-1}}\int_0^\infty \mathrm{vol}(\mathcal{F}_{\mathrm{ord}_E}R^{(t)})\,dt \tag{15}$$

is nonnegative for any divisorial valuation ord_E over V. By the valuative criterion for (log-)K-semistability in [Fuj19, Li17b, LL19], this implies (V,Δ) is indeed log-K-semistable. □

An alternative way to prove the first implication of Theorem 4.5 is using Proposition 5.3 and the arguments of [LL19, Section 4.2]. With all the techniques we have, we can now prove Theorem E.

Proof of Theorem E. Let (X,D) be the affine cone of $L=-r^{-1}(K_V+\Delta)$ over (V,Δ) for r^{-1} being a sufficiently divisible positive integer. We consider minimizing the normalized local volume at the T-equivariant singularity o which is the vertex. We aim to show that if (V,Δ) is T-equivariantly log-K-semistable then ord_V minimizes $\widehat{\mathrm{vol}}_{(X,D)}$. This then implies that (V,Δ) is log-K-semistable by Theorem 4.5 .

Following the proof of Proposition 4.4, we assume that $T=(\mathbb{C}^*)^r$ and fix a lexicographic order on $\mathbb{Z}^r$. Then by taking initial ideals, we can always associate a graded sequence of T-equivariant ideals to a given primary ideal. On the other hand,

$$\inf_{\mathfrak{a}} \mathrm{lct}(X,D;\mathfrak{a})^n\cdot \mathrm{mult}(\mathfrak{a}) = \inf_{\mathfrak{a}_\bullet} \mathrm{lct}(X,D;\mathfrak{a}_\bullet)^n\cdot \mathrm{mult}(\mathfrak{a}_\bullet) = \min_{v\in \mathrm{Val}_{X,o}} \widehat{\mathrm{vol}}(v).$$

So we can find a sequence $\{\mathfrak{a}_i\}$ of T-equivariant ideals such that

$$\inf_i \mathrm{lct}(\mathfrak{a}_i)^n\cdot \mathrm{mult}(\mathfrak{a}_i) = \min_{v\in \mathrm{Val}_{X,o}} \widehat{\mathrm{vol}}(v).$$

Using the equivariant resolution and running an MMP process as in Section 3.2, we can find a sequence of T-equivariant Kollár components S_i such that

$$\inf_i \widehat{\mathrm{vol}}(\mathrm{ord}_{S_i}) = \min_{v\in \mathrm{Val}_{X,o}} \widehat{\mathrm{vol}}(v).$$

For any T-equivariant Kollár component S_i, we consider $v=\mathrm{ord}_{S_i}\in \mathrm{Val}_{X,o}$. Denote its induced divisorial valuation on V by $b\cdot \mathrm{ord}_E$.

Arguing as in the proof of Theorem 4.5, in order to conclude $\widehat{\mathrm{vol}}(v_*)\leq \widehat{\mathrm{vol}}(\mathrm{ord}_{S_i})$, we want to show that $\Phi_s(\lambda_*,0)\geq 0$, where

$$\Phi_s(\lambda_*,0) = n\lambda_* L^{n-1} b\left(A_{(V,\Delta)}(E) - \frac{r}{L^{n-1}}\int_0^\infty \mathrm{vol}(\mathcal{F}_{\mathrm{ord}_E}R^{(t)})\,dt\right)$$

for the T-equivariant divisorial valuation E over (V,Δ).

Now we use the assumption that (V, Δ) is T-equivariantly K-semistable. Following the argument in [BBJ15, Fuj19], we deduce that (V, Δ) is T-equivariantly Ding-semistable. Indeed, for any special test configuration, the Futaki invariant is the same as the Ding invariant. Using the fact that T-equivariant MMP decreases the Ding invariant by [BBJ15, Fuj19], we find that the Ding invariant for any T-equvariant test configuration is nonnegative. Applying the argument in [Fuj18] (see [Li17b, Fuj19]), we conclude that $\Phi_s(\lambda_*, 0) \geq 0$ as desired. □

4.3. *Proof of Theorem A*

Let $(X, o) = (\mathrm{Spec}(R), \mathfrak{m})$ be an algebraic singularity such that (X, D) is klt for a $\mathbb{Q}$-divisor $D \geq 0$. Let S be a Kollár component and $\Delta = \Delta_S$ the different divisor defined by the adjunction $(K_Y + S + \mu_*^{-1}D)|_S = K_S + \Delta_S$ where $\mu\colon Y \to X$ is the extraction of S. We follow the notation of Sections 2.4 and 4.1. In this section, we will prove Theorem A which states that if (S, Δ_S) is K-semistable, then ord_S minimizes $\widehat{\mathrm{vol}}_X$ over $\mathrm{Val}_{X,o}$.

Lemma 4.6. *Let $\mathfrak{b}_\bullet$ be a graded sequence of $\mathfrak{m}_0$-primary ideals whose reduced support is $o_C \in C$. If (S, Δ_S) is K-semistable, then*

$$\mathrm{lct}(\mathfrak{b}_\bullet)^n \cdot \mathrm{mult}(\mathfrak{b}_\bullet) \geq \widehat{\mathrm{vol}}_{(C,C_D),o_C}(\mathrm{ord}_S).$$

Proof. Using the result in [JM12], we have

$$\mathrm{lct}(\mathfrak{b}_\bullet)^n \cdot \mathrm{mult}(\mathfrak{b}_\bullet) = \lim_{k\to\infty} (k \cdot \mathrm{lct}(\mathfrak{b}_k))^n \cdot \frac{\mathrm{mult}(\mathfrak{b}_k)}{k^n} = \lim_{k\to\infty} \mathrm{lct}(\mathfrak{b}_k)^n \cdot \mathrm{mult}(\mathfrak{b}_k).$$

By Proposition 2.8, it suffices to show that $\widehat{\mathrm{vol}}_{(C,C_D),o_C}(\mathrm{ord}_S)$ is equal to

$$\min_v \widehat{\mathrm{vol}}_{(C,C_D),o_C}(v)$$

with v running over valuations centered at o_C.

It follows from Theorem 4.5 that if we choose d sufficiently divisible such that $C^{(d)} = C(S, H)$ is constructed as the cone over S with an ample Cartier divisor H proportional to $-(K_S + \Delta_S)$, then the canonical valuation $\mathrm{ord}_{S^{(d)}}$ is a minimizer of $\widehat{\mathrm{vol}}_{(C^{(d)},C_1^{(d)}+C_2^{(d)})}$. By Proposition 4.7, this implies the same for C. □

Proposition 4.7. *With the above notations,* ord_S *minimizes* $\widehat{\mathrm{vol}}_{(C,C_D)}$ *if and only if* $\mathrm{ord}_{S^{(d)}}$ *minimizes* $\widehat{\mathrm{vol}}_{(C^{(d)},C_1^{(d)}+C_2^{(d)})}$.

Proof. The degree d cover $h\colon C \to C^{(d)}$ is a fiberwise map with respect to the cone structures and the Galois group $G =: \mathbb{Z}/d$ is naturally a subgroup of $\mathbb{C}^*$. Let E be a Kollár component over $C^{(d)}$. By Lemma 2.13 we know $h^*(E)$ is a Kollár component over C, and it follows from Lemma 2.14 (or [Li17b, Lemma 6.9]) that

$$d \cdot \widehat{\mathrm{vol}}(\mathrm{ord}_E) = \widehat{\mathrm{vol}}(h^*E).$$

So if ord_S minimizes $\widehat{\mathrm{vol}}_{(C,C_D)}$, then the corresponding canonical valuation also minimizes $\widehat{\mathrm{vol}}_{(C^{(d)},C_1^{(d)}+C_2^{(d)})}$.

For the converse, let E be a T-invariant Kollár component over C. Since it is G-invariant, by Lemma 2.13 we know that it is a pull back of a Kollár component F over $C^{(d)}$. Assume that the canonical valuation minimizes $\widehat{\mathrm{vol}}_{(C^{(d)},C_1^{(d)}+C_2^{(d)})}$. Then over C, we see that $\widehat{\mathrm{vol}}(\mathrm{ord}_S) \le \widehat{\mathrm{vol}}(\mathrm{ord}_E)$ for any T-equivariant Kollár component E. Therefore ord_S is a minimizer of $\widehat{\mathrm{vol}}_{(C,C_D)}$ by Proposition 4.4. □

Theorem A is a consequence of Theorem 4.5 and the following proposition.

Proposition 4.8. *Any Kollár component S over $o \in (X, D)$ induces a $\mathbb{C}^*$-equivariant degeneration to an 'orbifold' cone $o_C \in (C, C_D)$ with a Kollár component $S_0 \cong S$ which is the canonical valuation with respect to the orbifold cone structure, and*

$$\widehat{\mathrm{vol}}_{(X,D),o}(\mathrm{ord}_S) = \widehat{\mathrm{vol}}_{(C,C_D),o_C}(\mathrm{ord}_{S_0}), \qquad \mathrm{vol}(o, X, D)) \ge \mathrm{vol}(o_C, C, C_D).$$

Proof. We use the same notations as in Section 2.4. In particular, we denote by $\mathcal{Z}$ (resp. $\mathcal{W}$) the coarse moduli space of $\mathfrak{Z}$ (resp. $\mathfrak{W}$). Let $\phi\colon \mathcal{Z} \to X_{\mathbb{A}^1}$ $(= X \times \mathbb{A}^1)$ be the birational morphism and $S'_{\mathbb{A}^1}$ the birational transform of $S_{\mathbb{A}^1} \subset Y_{\mathbb{A}^1}$ on $\mathcal{Z}$. Write $aS'_{\mathbb{A}^1} \sim_{\mathbb{Q},\mathcal{W}} K_{\mathcal{Z}} + \phi_*^{-1}D_{\mathbb{A}^1} + S'_{\mathbb{A}^1}$. Restricting over a general fiber and taking the coarse moduli spaces, we obtain

$$aS \sim_{\mathbb{Q},X} K_Y + S + \mu_*^{-1}(D),$$

so $a = A_{(X,D)}(S)$. Similarly, over the central fiber, we get

$$aS_0 \sim_{\mathbb{Q},C} K_{Y_0} + S_0 + (\mu_0^{-1})_* C_D,$$

where $\mu_0\colon Y_0 \to C$ is the blow up of the vertex o_C with exceptional divisor $S_0 \cong S$. Thus $a = A_{C,C_D}(S_0)$.

We also know that

$$\mathrm{vol}_{X,o}(\mathrm{ord}_S) = (-S|_S)^{n-1} = (-S_0|_{S_0})^{n-1} = \mathrm{vol}_{C,o_C}(\mathrm{ord}_{S_0}).$$

Combining all the above, we find that for any ideal $\mathfrak{b}$ on X, if we let $\mathfrak{b}_\bullet = \{\mathfrak{b}^k\}$, then

$$\begin{aligned}\widehat{\mathrm{vol}}_{(X,D),o}(\mathrm{ord}_S) &= \mathrm{vol}_{X,o}(\mathrm{ord}_S) \cdot A_{(X,D)}(S)^n = \mathrm{vol}_{C,o_C}(\mathrm{ord}_{S_0}) \cdot A_{(C,C_D)}(S_0)^n \\ &\le \mathrm{lct}(\mathbf{in}(\mathfrak{b}_\bullet)^n) \cdot \mathrm{mult}(\mathbf{in}(\mathfrak{b}_\bullet)) \le \mathrm{lct}(\mathfrak{b})^n \cdot \mathrm{mult}(\mathfrak{b})\end{aligned}$$

where the last two inequalities follow from Lemmas 4.6 and 4.3. Thus we conclude that

$$\widehat{\mathrm{vol}}_{(X,D),o}(\mathrm{ord}_S) \le \inf_{\mathfrak{b}} \mathrm{lct}(\mathfrak{b})^n \cdot \mathrm{mult}(\mathfrak{b}) = \inf_v \widehat{\mathrm{vol}}_{(X,D),o}(v)$$

where the second equality follows from Proposition 2.8. □

5. Uniqueness

In this section, we will prove Theorem B on the uniqueness of the minimizers among all Kollár components. There are two steps: first we prove this for cone singularities; then for a general singularity, we combine the deformation construction with some results from the minimal model program to essentially reduce the situation to the case of cone singularities.

5.1. Case of cone singularity

We first settle the case of cone singularities. It can be proved using Proposition 5.9 and [Li17b, Theorem 3.4]. Here we give a different proof, which analyzes the geometry in more detail. A similar argument in the global case appears in [Liu18, proof of Theorem 3], where a characterization of quotients of $\mathbb{P}^n$ was given as those achieving the maximal possible volumes among all K-semistable $\mathbb{Q}$-Fano varieties with only quotient singularities.

Let (V, Δ) be an $(n-1)$-dimensional log Fano variety and $-(K_V + \Delta) = rH$ for some $r \in \mathbb{Q}$ and an ample Cartier divisor H. We assume $r \leq n$. Let $X^0 := C(V, H)$ be the affine cone over the base V with vertex o and let X be the projective cone and D be the cone divisor over Δ on X.

Consider a Kollár component S over $o \in (X, D)$ with the extraction morphism $\mu\colon Y \to X$. Let $\mu_{\mathbb{A}^1}\colon Y_{\mathbb{A}^1} \to X_{\mathbb{A}^1}$ be the extraction of $S_{\mathbb{A}^1}$. We carry out the process of deformation to normal cones as in Section 2.4 with respect to S. Here X is a projective variety instead of a local singularity, but the construction is exactly the same. We denote by $\mathcal{Z}$ (resp. $\mathcal{W}$) the coarse moduli space of $\mathfrak{Z}$ (resp. $\mathfrak{W}$), so there are morphisms $\psi_1\colon \mathcal{Z} \to \mathcal{W}$, $\phi_1\colon \mathcal{Z} \to Y_{\mathbb{A}^1}$ and $\pi\colon \mathcal{W} \to X_{\mathbb{A}^1}$. We denote $\phi = \mu_{\mathbb{A}^1} \circ \phi_1$.

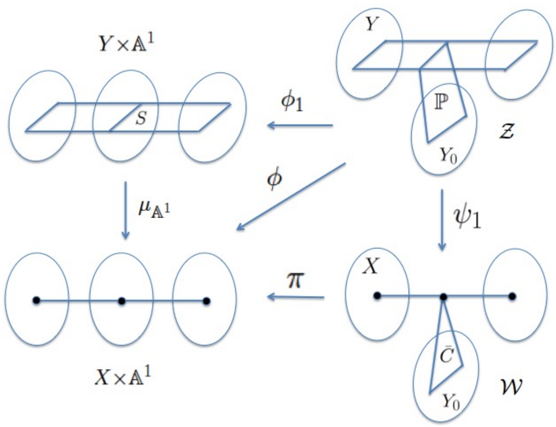

Fig. 1. Degeneration associated to a Kollár component.

Denote by $\mathbb{P}$ the irreducible exceptional divisor for ϕ_1. We have the following equalities:

1. $K_{Y_{\mathbb{A}^1}} + (\mu_{\mathbb{A}^1})_*^{-1} D_{\mathbb{A}^1} = \mu_{\mathbb{A}^1}^*(K_{X_{\mathbb{A}^1}} + D_{\mathbb{A}^1}) + aS_{\mathbb{A}^1}$ with $a = A_{(X,D)}(S) - 1$;
2. $K_{\mathcal{Z}} + \phi_*^{-1} D_{\mathbb{A}^1} = \phi_1^*(K_{Y_{\mathbb{A}^1}} + (\mu_{\mathbb{A}^1})_*^{-1} D_{\mathbb{A}^1}) + \mathbb{P}$;
3. $K_{\mathcal{Z}} + \phi_*^{-1} D_{\mathbb{A}^1} = \psi_1^*(K_{\mathcal{W}} + (D_{\mathbb{A}^1})_{\mathcal{W}}) + aS'_{\mathbb{A}^1}$, where $(D_{\mathbb{A}^1})_{\mathcal{W}} := \psi_{1*}\phi_*^{-1}(D_{\mathbb{A}^1})$ and $S'_{\mathbb{A}^1} = (\phi_1^{-1})_*(S_{\mathbb{A}^1})$.

The first two equalities imply

$$\begin{aligned} K_{\mathcal{Z}} + \phi_*^{-1}(D_{\mathbb{A}^1}) &= \phi_1^*(K_{Y_{\mathbb{A}^1}} + (\mu_{\mathbb{A}^1})_*^{-1} D_{\mathbb{A}^1}) + \mathbb{P} \\ &= \phi_1^* \mu_{\mathbb{A}^1}^*(K_{X_{\mathbb{A}^1}} + D_{\mathbb{A}^1}) + a\phi_1^* S_{\mathbb{A}^1} + \mathbb{P} \\ &= \phi^*(K_{X_{\mathbb{A}^1}} + D_{\mathbb{A}^1}) + a S'_{\mathbb{A}^1} + (a+1)\mathbb{P}. \end{aligned}$$

So $A_{X_{\mathbb{A}^1}, D_{\mathbb{A}^1}}(\mathbb{P}) = a + 2 = A_{(X,D)}(S) + 1$. This implies

$$K_{\mathcal{W}} + (D_{\mathbb{A}^1})_{\mathcal{W}} = \pi^*(K_{X_{\mathbb{A}^1}} + D_{\mathbb{A}^1}) + A_{(X,D)}(S)\bar{C}.$$

Denote by $\hat{L} = \mathcal{O}_X(V_\infty)$ the line bundle over the projective cone X, where V_∞ is the divisor at infinity which is isomorphic to V. Then $-K_X - D = (1+r)\hat{L}$ and

$$K_{\mathcal{W}} + (D_{\mathbb{A}^1})_{\mathcal{W}} = -(1+r)\rho^*\hat{L} + A_{(X,D)}(S)\bar{C}$$

where $\bar{C}$ is the orbifold cone over C and $\rho\colon \mathcal{W} \to X$ the composite of $\pi: \mathcal{W} \to X_{\mathbb{A}^1}$ with the projection $X_{\mathbb{A}^1} \to X$.

As in [LL19], we define the cone angle parameter $\beta = \frac{r}{n}$ and let $\delta = r\frac{n+1}{n}$. Then

$$-(K_X + D + (1-\beta)V_\infty) \sim_{\mathbb{Q}} (1+r)\hat{L} - \left(1 - \frac{r}{n}\right)\hat{L} = r\frac{n+1}{n}\hat{L} = \delta\hat{L}. \tag{16}$$

Denote by $\mathbb{V}_\infty$ the birational transform of $(V_\infty)_{\mathbb{A}^1}$ on $\mathcal{W}$. We also get

$$\begin{aligned} K_{\mathcal{W}} + (D_{\mathbb{A}^1})_{\mathcal{W}} + (1-\beta)\mathbb{V}_\infty &= \pi^*(K_{X_{\mathbb{A}^1}} + D_{\mathbb{A}^1} + (1-\beta)(V_\infty)_{\mathbb{A}^1}) + A_{(X,D)}(S)\bar{C} \\ &= -\delta\rho^*\hat{L} + A_{(X,D)}(S)\bar{C}. \end{aligned} \tag{17}$$

The above construction works for any Kollár component. From now on we assume that (V, Δ) is K-semistable and S minimizes the normalized volume, i.e.

$$\widehat{\mathrm{vol}}(\mathrm{ord}_S) = \widehat{\mathrm{vol}}(\mathrm{ord}_{V_0}) = r^n(H^{n-1}) \tag{18}$$

where V_0 denotes the exceptional divisor obtained by blowing up the vertex of the cone, and we aim to show $S = V_0$. We note that by Theorem 4.5, $\widehat{\mathrm{vol}}(\mathrm{ord}_{V_0})$ is the minimal normalized volume. Then we have

$$\mathrm{vol}(\mathrm{ord}_S) = \frac{\widehat{\mathrm{vol}}(\mathrm{ord}_S)}{A_{(X,D)}(S)^n} = \frac{r^n(H^{n-1})}{A_{(X,D)}(S)^n}.$$

In Section 4.2, we have used the filtration induced by a valuation (see also [BHJ17, Fuj18]). Here we use the same construction but for sections on the projective cone instead of the base.

Definition 5.2 (Filtration by valuation). For a fixed a valuation $v \in \mathrm{Val}_{X,o}$, let $\hat{R}_m = H^0(X, m\hat{L})$. Define $\mathcal{F}^x \hat{R}_m := \mathcal{F}^x_v \hat{R}_m \subset \hat{R}_m$ to be a decreasing filtration (with respect to x) given by

$$\mathcal{F}^x \hat{R}_m = H^0(X, m\hat{L} \otimes \mathfrak{a}_x) \quad \text{where} \quad \mathfrak{a}_x = \{f \in \mathcal{O}_X \mid v(f) \geq x\}.$$

On $\bigoplus_{m=0}^{\infty} \hat{R}_m$, we define $\mathcal{F}\hat{R}^{(t)} := \mathcal{F}_v \hat{R}^{(t)} = \bigoplus \mathcal{F}^{kt} \hat{R}_k$. Then the volume is defined to be

$$\mathrm{vol}(\mathcal{F}\hat{R}^{(t)}) := \limsup_{m \to \infty} \frac{\dim_{\mathbb{C}}(\mathcal{F}^{mt} \hat{R}_m)}{m^n/n!}.$$

The following proposition answers the question in [LL19, Section 6].

Proposition 5.3. *With the above notation, if the base (V, Δ) is log-K-semistable, then $(X, D + (1-\beta)V_\infty)$ is log-K-semistable. As a consequence,*

$$A_{(X,D)}(S) - \frac{\delta}{\hat{L}^n} \int_0^{\infty} \mathrm{vol}(\mathcal{F}_{\mathrm{ord}_S} \hat{R}^{(x)})\, dx \geq 0.$$

Proof. It is enough to verify that the generalized Futaki invariant is nonnegative for any compactified special test configuration $\pi: (\mathcal{X}, \mathcal{D}+(1-\beta)\mathcal{V}) \to \mathbb{P}^1$ of $(X, D+(1-\beta)V_\infty)$ over $\mathbb{P}^1$ (see Section 2.1), where $\mathcal{V}$ is the closure of $V_\infty \times (\mathbb{P}^1 \setminus \{0\})$. Let Δ_∞ $(= \Delta) = V_\infty \cap D$ and let Δ^{tc} be the closure of $\Delta_\infty \times (\mathbb{P}^1 \setminus \{0\})$. Then $\mu: (\mathcal{V}, \Delta^{\mathrm{tc}}) \to \mathbb{P}^1$ is a compactified test configuration of (V, Δ). As $(1+r)V_\infty \sim_{\mathbb{Q}} -(K_X + D)$, we know that there exists $k \in \mathbb{Q}$ such that

$$\begin{aligned}
&(1+r)\mathcal{V} \sim_{\mathbb{Q}} -K_{\mathcal{X}} - \mathcal{D} + \pi^* \mathcal{O}_{\mathbb{P}^1}(k), \\
&K_{\mathcal{V}} + \Delta^{\mathrm{tc}} = (K_{\mathcal{X}} + \mathcal{D} + \mathcal{V})|_{\mathcal{V}} = -r\mathcal{V}|_{\mathcal{V}} + \mu^* \mathcal{O}_{\mathbb{P}^1}(k).
\end{aligned}$$

The adjunction formula holds because $\mathcal{X}$ is smooth along the codimension 2 points over 0 and so there is no different divisor. Since $\beta = \frac{r}{n}$ and $\delta = r\frac{1+n}{n}$, we have

$$-(K_{\mathcal{X}/\mathbb{P}^1} + \mathcal{D} + (1-\beta)\mathcal{V}) \sim_{\mathbb{Q}} \delta \cdot \mathcal{V} + \pi^* \mathcal{O}_{\mathbb{P}^1}(-2-k).$$

Then the generalized Futaki invariant of $(\mathcal{X}, \mathcal{D} + (1-\beta)\mathcal{V})/\mathbb{P}^1$ is

$$\begin{aligned}
\mathrm{Fut}(\mathcal{X}) &= -\frac{1}{(n+1)(\delta\hat{L})^n}(-K_{\mathcal{X}/\mathbb{P}^1} - \mathcal{D} - (1-\beta)\mathcal{V})^{n+1} \\
&= -\frac{1}{\hat{L}^n}\pi^*\mathcal{O}_{\mathbb{P}^1}(-2-k) \cdot \mathcal{V}^n - \frac{\delta}{(n+1)\hat{L}^n}\mathcal{V}^{n+1}.
\end{aligned}$$

On the other hand, the generalized Futaki invariant of $(\mathcal{V}, \Delta^{\mathrm{tc}})/\mathbb{P}^1$ is

$$\begin{aligned}
\mathrm{Fut}(\mathcal{V}) &= -\frac{1}{nr^{n-1}H^{n-1}}((-K_{\mathcal{V}/\mathbb{P}^1} - \Delta^{\mathrm{tc}})|_{\mathcal{V}})^n \\
&= -\frac{1}{nr^{n-1}H^{n-1}}(r\mathcal{V}|_{\mathcal{V}} - \mu^*\mathcal{O}_{\mathbb{P}^1}(k) + \mu^* K_{\mathbb{P}^1})^n \\
&= -\frac{1}{r^{n-1}H^{n-1}} r^{n-1} \mu^* \mathcal{O}_{\mathbb{P}^1}(-2-k) \cdot (\mathcal{V}|_{\mathcal{V}})^{n-1} - \frac{r}{nH^{n-1}}(\mathcal{V}|_{\mathcal{V}})^n \\
&= -\frac{1}{H^{n-1}}\pi^*\mathcal{O}_{\mathbb{P}^1}(-2-k) \cdot \mathcal{V}^n - \frac{r}{nH^{n-1}}\mathcal{V}^{n+1}.
\end{aligned}$$

Because $H^{n-1} = \int_{[V]} H^{n-1} = \int_{[X]} \hat{L}^n = \hat{L}^n$, we have

$$\mathrm{Fut}(\mathcal{V}) = \mathrm{Fut}(\mathcal{X}).$$

Finally, recall that log-K-semistability is equivalent to log-Ding-semistability (see e.g. [Fuj19]). Then the second statement is obtained by applying [LL19, Proposition 4.5] to $(X, D + (1-\beta)V_\infty)$ and $\hat{L} = -\frac{1}{\delta}(K_X + D + (1-\beta)V_\infty)$. □

The following key calculations are proved in [LL19, proof of Proposition 4.5].

Proposition 5.4 ([LL19]). *Suppose (V, Δ) is log-K-semistable. If S is a Kollár component realizing the minimum of $\widehat{\mathrm{vol}}$ over (X, o), then the graded filtration induced by S satisfies the following two conditions:*

(i) *The following identity holds:*

$$A_{(X,D)}(S) - \frac{\delta}{\hat{L}^n}\int_0^\infty \mathrm{vol}(\mathcal{F}\hat{R}^{(x)})\,dx = 0.$$

(ii) *Denote $\tau := \sqrt[n]{\hat{L}^n/\mathrm{vol}(\mathrm{ord}_S)}$. Then*

$$\mathrm{vol}(\mathcal{F}\hat{R}^{(x)}) = \mathrm{vol}_Y(\mu^*\hat{L} - xS) = \hat{L}^n - \mathrm{vol}(\mathrm{ord}_S)x^n \quad \textit{for any } x \in [0, \tau].$$

Lemma 5.5. *We have $\tau = A_{(X,D)}(S)/r$.*

Proof. Combining (i) and (ii) in Proposition 5.4, we find that

$$A_{(X,D)}(S) - \frac{r(1+n)}{n \cdot \hat{L}^n}\int_0^\tau (\hat{L}^n - \mathrm{vol}(\mathrm{ord}_S)x^n)\,dx = A_{(X,D)}(S) - r\cdot\tau = 0. \quad □$$

Arguing as in [Fuj18] (see also [Liu18]), we deduce

Lemma 5.6. *τ is the nef threshold of $\mu^*\hat{L}$ with respect to the divisor S, i.e.*

$$\tau = \sup\{x \mid \mu^*\hat{L} - xS \textit{ is ample}\}.$$

Proof. When the point is smooth, this follows from [Fuj18, Theorem 2.3(2)]. Exactly the same argument can be used to treat the current case. □

Theorem 5.7. *If S is a Kollár component that minimizes the normalized volume, then S is the canonical component V_0.*

We first show the following statements.

Lemma 5.8. (i) *$\rho^*\hat{L} - \tau\bar{C}$ is semiample, and contracts Y to S_∞ ($\cong S$) $\subset \bar{C}$ as the divisor at infinity of the orbifold projective cone $\bar{C} = \bar{C}(S, -S|_S)$.*

(ii) *$A_{(X,D)}(S) = r$ and there is a special test configuration $\mathcal{X}$ of $(X, D+(1-\beta)V_\infty; \hat{L})$ whose central fiber X_0 is $(\bar{C}, C_D + (1-\beta)S_\infty; \hat{L}_0)$ where C_D is the intersection of $\bar{C}$ with $(D\times\mathbb{A}^1)_{\mathcal{W}}$. Moreover, $(\bar{C}, C_D + (1-\beta)S_\infty; \hat{L}_0) \cong (X, D + (1-\beta)V_\infty; \hat{L})$.*

Proof. The proof of (i) is along the lines of [Liu18, proof of Lemma 33]. First we observe the following restrictions of $\rho^*\hat{L} - x\bar{C}$:

- $(\rho^*\hat{L} - x\bar{C})|_{X_t} = \hat{L}$, $t \neq 0$. Recall that $X_t \cong X$ for $t \in \mathbb{C}^*$.
- $(\rho^*\hat{L} - x\bar{C})|_{Y_0} = \mu^*\hat{L} - xS$.
- $(\rho^*\hat{L} - x\bar{C})|_{\bar{C}} = -x\bar{C}|_{\bar{C}} = xY_0|_{\bar{C}} = xS_\infty = x\mathcal{O}_{\bar{C}}(1)$.

So by Lemma 5.6, it is easy to see that $\rho^*\hat{L} - x\bar{C}$ is ample when $x \in (0, \tau)$. To show that $\rho^*\hat{L} - \tau\bar{C}$ is semiample, we use (17) to calculate

$$\begin{aligned} m(\rho^*\hat{L} - x\bar{C}) - K_{\mathcal{W}} - (D_{\mathbb{A}^1})_{\mathcal{W}} &= m(\rho^*\hat{L} - x\bar{C}) + (1+r)\rho^*\hat{L} - A_{(X,D)}(S)\bar{C} \\ &= (m+1+r)\left(\rho^*\hat{L} - \frac{mx + A_{(X,D)}(S)}{m+1+r}\bar{C}\right). \end{aligned}$$

Notice that

$$\frac{mx + A_{(X,D)}(S)}{m+1+r} < \tau = \frac{A_{(X,D)}(S)}{r}$$

if and only if

$$x < \left(1 + \frac{1}{m}\right)\frac{A_{(X,D)}(S)}{r}.$$

Since this is satisfied for

$$x = \tau = A_{(X,D)}(S)/r \qquad \text{for any } m > 0,$$

the semiampleness of $\rho^*\hat{L} - \tau\bar{C}$ holds by the base-point-free theorem [KM98, Theorem 3.13]. Next we claim that

$$H^0(Y, m(\mu^*\hat{L} - \tau S)) \cong H^0(S, -m\tau S) \tag{19}$$

for any m sufficiently divisible. To see this, we consider the exact sequence

$$0 \to \mathcal{O}_Y(m(\mu^*\hat{L} - \tau S) - S) \to \mathcal{O}_Y(m(\mu^*\hat{L} - \tau S)) \to \mathcal{O}_Y(m(\mu^*\hat{L} - \tau S)) \otimes \mathcal{O}_S \to 0, \tag{20}$$

and its associated long exact sequence of cohomology groups. By the above discussion, and since

$$m(\mu^*\hat{L} - \tau S) - S - K_Y = m\left(\mu^*\hat{L} - \frac{A_{(X,D)}(S)}{r}S\right) + (1+r)\mu^*\hat{L} - A_{(X,D)}(S)S$$

is ample, it follows from the Kawamata–Viehweg vanishing theorem that

$$H^1(Y, m(\mu^*\hat{L} - \tau S) \otimes \mathcal{O}(-S)) = 0 \quad \text{for any } m \geq 0.$$

We also have

$$H^0(Y, m(\mu^*\hat{L} - \tau S) \otimes \mathcal{O}(-S)) = 0 \quad \text{for any } m \geq 0,$$

as τ is also the pseudo-effective threshold. Thus we know $|m(\rho^*\hat{L} - \tau\bar{C})|$ contracts the fiber $\mathcal{W} \times_{\mathbb{A}^1} \{0\}$ to $\bar{C}$ for sufficiently divisible m. This finishes the proof of (i). We denote by $\theta\colon \mathcal{W} \to \mathcal{X}$ the induced morphism and there is an ample line bundle $\hat{\mathcal{L}}$ on $\mathcal{X}$ such that $\theta^*\hat{\mathcal{L}} = \rho^*\hat{L} - \tau\bar{C}$.

Next we prove (ii). Let $(D_{\mathbb{A}^1})_{\mathcal{X}}$ be the push forward of $(D_{\mathbb{A}^1})_{\mathcal{W}}$ on $\mathcal{X}$. Then $-K_{\mathcal{X}} - (D_{\mathbb{A}^1})_{\mathcal{X}}$ and $(1+r)\hat{\mathcal{L}}$ coincide outside X_0, and they must be relatively linearly equivalent on the whole $\mathcal{X}$ because X_0 is irreducible. In particular, they are linearly equivalent when restricted to X_0.

Since

$$(K_Y + \mu_*^{-1}D + S)|_S = K_S + \Delta_S \sim_{\mathbb{Q}} A_{(X,D)}(S) \cdot S|_S,$$

we know that

$$-K_{\mathcal{X}} - (D_{\mathbb{A}^1})_{\mathcal{X}}|_{X_0} = -K_{\bar{C}} - C_D \sim_{\mathbb{Q}} (1 + A_{(X,D)}(S))S_\infty.$$

Similarly, we have $\hat{\mathcal{L}}|_{X_0} \sim_{\mathbb{Q}} \tau S$ with $\tau = A_{(X,D)}(S)/r$. Therefore,

$$1 + A_{(X,D)}(S) = (1+r)\frac{A_{(X,D)}(S)}{r},$$

which implies $A_{(X,D)}(S) = r$ and $\tau = 1$.

The degree of V_∞ under $\hat{\mathcal{L}}$ is

$$\hat{\mathcal{L}}|_{X_0}^{n-1} \cdot V_\infty = \hat{L}^{n-1} \cdot V_\infty = \hat{L}^n,$$

while the degree of S is

$$\hat{\mathcal{L}}|_{X_0}^{n-1} \cdot S = \tau^{-1}\hat{\mathcal{L}}|_{X_0}^n = \hat{L}^n = \hat{L}_0^n.$$

The restriction $\theta|_{V_\infty}\colon V_\infty \to S$ is finite since

$$(\rho^*\hat{L} - \tau\bar{C})|_{V_\infty} = \hat{L}|_{V_\infty}$$

is ample. And the degree is 1 by the above calculation on degrees, which implies this is an isomorphism. We claim that Y is indeed the $\mathbb{P}^1$-bundle over V_∞ induced by blowing up the vertex of X, S is a section, and the morphism θ is just contracting the $\mathbb{P}^1$-bundle. Granted this, we then indeed have an isomorphism from $(X, D + (1-\beta)V_\infty; \hat{L})$ to $(\bar{C}, C_D + (1-\beta)S_\infty; \hat{L}_0)$.

To see the claim, let l be a curve contracted by θ; we want to show that it is the birational transform of a ruling line of X. To see this, since $(\rho^*(\hat{L}) - \bar{C}) \cdot l = 0$, we know that $\rho^*(\hat{L}) \cdot l = 1$. So the image $\rho_* l$ of l in X is a line, and it passes through the vertex. Therefore, it is a ruling of the cone. □

By the above proof, let $\mathcal{V}$ be the birational transform of $(V_\infty)_{\mathbb{A}^1}$ on $\mathcal{X}$, and $\mathcal{H}$ the extension of $H_{\mathbb{A}^1} \setminus \{0\}$ on $\mathcal{X}$. Then

$$\mathcal{X} = \operatorname{Proj}_{\mathcal{V}}\Big(\bigoplus_{k=0}^{\infty} \mathcal{S}_k\Big) \quad \text{where} \quad \mathcal{S}_k = \bigoplus_{i=0}^{k} (H^0(\mathcal{V}, i\mathcal{H}) \cdot u^{k-i}).$$

From this we easily see that S and V give the same component over the vertex.

5.2. The general case

In this section, we prove Theorem B in the general case. We first show that the cone case we proved in Section 5.1 can be generalized to the case of an orbifold cone. Let $T = \mathbb{C}^*$.

Proposition 5.9. *Let $o \in (X, D)$ be a klt T-singularity. Assume that a minimizer v of $\widehat{\mathrm{vol}}_{(X,D),o}$ is given by a rescaling of ord_S for a Kollár component S. Then v is T-invariant.*

Proof. Let $\mathfrak{a}$ be an ideal whose normalized blow up gives the model of extracting the Kollár component S (see the proof of Theorem 3.12). Denote the degeneration of $\mathfrak{a}_\bullet := \{\mathfrak{a}^p\}$ induced by the T-action by $\mathfrak{b}_\bullet := \{\mathbf{in}(\mathfrak{a}^p)\}$ (which in general is not necessarily equal to but only contains $\mathbf{in}(\mathfrak{a})^p$) by a sequence $\{\mathfrak{B}_\bullet\}$ of flat families of ideals over $\mathbb{A}^1$.

Because ord_S is a minimizer of $\widehat{\mathrm{vol}} = \widehat{\mathrm{vol}}_{(X,D),o}$, we have

$$\mathrm{mult}(\mathfrak{a}) \cdot \mathrm{lct}(X, D; \mathfrak{a})^n = \widehat{\mathrm{vol}}(\mathrm{ord}_S) \leq \mathrm{mult}(\mathfrak{b}_\bullet) \cdot \mathrm{lct}(X, D; \mathfrak{b}_\bullet)^n.$$

But $\mathrm{mult}(\mathfrak{a}) = \mathrm{mult}(\mathfrak{b}_\bullet)$, and $\mathrm{lct}(X, D, \mathfrak{a}) \geq \mathrm{lct}(X, D, \mathfrak{b}_\bullet)$ by the lower semicontinuity of log canonical thresholds. So we know that

$$\mathrm{lct}(X, D, \mathfrak{a}) = \mathrm{lct}(X, D, \mathfrak{b}_\bullet) = \lim_{k\to\infty} \mathrm{lct}\left(X, D, \frac{1}{k}\mathfrak{b}_k\right) =: c.$$

Since S computes the log canonical threshold of $\mathfrak{a}$, we have $A(S; X, D + c \cdot \mathfrak{a}) = 0$. As a consequence, we can choose ϵ sufficiently small, and k sufficiently large, such that the log discrepancy satisfies

$$A\left(S; X, D + \frac{c-\epsilon}{k}\mathfrak{a}^k\right) < 1 \quad \text{and} \quad \left(X, D + \frac{c-\epsilon}{k}\mathfrak{b}_k\right) \text{ is klt.}$$

This implies that $\left(X_{\mathbb{A}^1}, D_{\mathbb{A}^1} + \frac{c-\epsilon}{k}\mathfrak{B}_k\right)$ is klt and $A\left(S_{\mathcal{Y}}; X_{\mathbb{A}^1}, D_{\mathbb{A}^1} + \frac{c-\epsilon}{k}\mathfrak{B}_k\right) < 1$, where $S_{\mathcal{Y}}$ is the divisor birational to $S \times \mathbb{A}^1$. Thus by [BC+10] we can construct a model $\mu_{\mathcal{Y}} : \mathcal{Y} \to X_{\mathbb{A}^1}$ extracting only the irreducible divisor $S_{\mathcal{Y}}$, which gives S over the generic fiber. Furthermore, we can assume $-S_{\mathcal{Y}}$ is ample over $X_{\mathbb{A}^1}$.

Let $Y \to X$ and $Y_0 \to X$ be models given by a general fiber and the special fiber of $\mathcal{Y} \to X_{\mathbb{A}^1}$ over $\mathbb{A}^1$, thus $\mathrm{vol}(Y) = \mathrm{vol}(Y_0)$, which is the minimal normalized volume. In particular, by the proof of Theorem C, the special fiber S_0 of $S_{\mathcal{Y}}$ over 0 is irreducible and yields a T-equivariant minimizer. By Theorem E (we note its proof does not need any part of the argument for uniqueness) both (S, Δ_S) and (S_0, Δ_{S_0}) are K-semistable log Fano pairs. Take the degeneration $\mathcal{X}$ of X to the orbifold cone (C, D_C) (over (S, Δ_S)) over $\mathbb{A}^1$. A priori, $\mathcal{X}$ is not known to be T-equivariant.

On the other hand, arguing as above (for more details see the proof of Theorem B below), we know that S_0 degenerates to a $\mathbb{C}^*$-equivariant minimizer of (C, D_C), i.e., there is a $\mathbb{C}^*$-equivariant model $\mu' : \mathcal{Y}' \to \mathcal{X}$ over $\mathbb{C}$ with an exceptional divisor $\mathcal{S}_{\mathcal{Y}'}$, whose general fiber yields the Kollár component of S_0 over X, and the special fiber is a $\mathbb{C}^*$-equivariant model Y_0' over (C, D_C) whose volume $\mathrm{vol}(Y_0') = \widehat{\mathrm{vol}}_{X,D}(\mathrm{ord}_{S_0})$ is the minimal normalized volume over (C, D_C). Thus $Y_0' \to (C, D_C)$ yields an equivariant Kollár component F. Since any $\mathbb{C}^*$-equivariant Kollár component can be obtained by descent from an orbifold cone C_0 to the $\mathbb{C}^*$-cone $C_0^{(d)}$ for a sufficiently divisible d, by Theorem 5.7, it follows that F is the canonical component of (C, D_C). Then we take the degeneration

$$\mathfrak{X} := \operatorname{Spec}\Big(\bigoplus_{m\in\mathbb{Z}} \mu'_*\mathcal{O}_{\mathcal{Y}'}(-m\mathcal{S}_{\mathcal{Y}'})\Big)\cdot t^m \to \mathbb{A}^1 \quad \text{of } \mathcal{X}.$$

$\mathfrak{X}$ is a $(\mathbb{C}^*)^2$-family over $\mathbb{A}^2$ with central fiber X_0 admitting a $(\mathbb{C}^*)^2$-action. From the above argument, we know that along $\{0\}\times\mathbb{C}$, the degeneration of (C, D_C) to (C, D_C) is indeed trivial. This implies that the graded ring induced by ord_S is the same as the graded ring induced by ord_{S_0}, thus $S = S_0$. □

Proposition 5.10. *In the notation of Section 2.4, S is the unique minimizer among all Kollár components for $\widehat{\operatorname{vol}}_{(C,C_D)}$ if and only if the same holds for $C^{(d)}$ on $\widehat{\operatorname{vol}}_{(C^{(d)},C_1^{(d)}+C_2^{(d)})}$.*

Proof. By Proposition 5.9, any minimizing Kollár component E of $\widehat{\operatorname{vol}}_{(C,C_D)}$ is T-invariant. Therefore it is $G = \mathbb{Z}/d\mathbb{Z}$-invariant. So E is the pull back of a Kollár component on $C^{(d)}$ by Lemma 2.13, which can only be the canonical component obtained by blowing up the vertex by our assumption and Lemma 2.14. □

Proof of Theorem B. By Theorem 5.7 and Proposition 5.10, for the coarse moduli space of an orbifold cone over a K-semistable log Fano pair, the only Kollár component which minimizes the normalized volume function is the canonical component.

Now we consider the case of a general klt singularity $o \in (X, D)$. Because (S, Δ_S) is K-semistable, by Theorem A, ord_S minimizes $\widehat{\operatorname{vol}}_{(X,D),o}$. Let us assume that there is another divisor F over (X, o) such that

$$\widehat{\operatorname{vol}}_{(X,D),o}(\operatorname{ord}_F) = \widehat{\operatorname{vol}}_{(X,D),o}(\operatorname{ord}_S).$$

Then by Theorem C, F is indeed a Kollár component.

As in Section 2.4, let $\pi : \mathcal{W} \to X \times \mathbb{A}^1$ be the flat family which degenerates X to $W_0 = Y_0 \cup \bar{C}$, where $Y_0 \cong Y$ extracts S over X and $\bar{C}$ is the coarse moduli space of the orbifold cone over $S = \bar{C} \cap Y_0$. Then as in the proof Proposition 5.9, let $\mathfrak{a}$ be an ideal whose normalized blow up gives the model of extracting the Kollár component F (see the proof of Theorem 3.12). Denote the degeneration of $\{\mathfrak{a}_\bullet\} := \{\mathfrak{a}^p\}$ by $\mathfrak{b}_\bullet := \{\mathbf{in}(\mathfrak{a}^p)\}$.

Denote by S_0 the induced Kollár component over (C, C_D, o) (see Proposition 4.8). We then have

$$\operatorname{mult}(\mathfrak{a})\cdot \operatorname{lct}(X, D, \mathfrak{a})^n = \widehat{\operatorname{vol}}_X(\operatorname{ord}_F) = \widehat{\operatorname{vol}}_{\bar{C}}(\operatorname{ord}_{S_0}) \le \operatorname{mult}(\mathfrak{b}_\bullet)\cdot \operatorname{lct}(\bar{C}, \bar{C}_D, \mathfrak{b}_\bullet)^n,$$

where the last inequality is from the assumption that $S_0 \cong S$ is K-semistable and Theorem A. On the other hand, we have $\operatorname{mult}(\mathfrak{a}) = \operatorname{mult}(\mathfrak{b}_\bullet)$ and $\operatorname{lct}(X, D, \mathfrak{a}) \ge \operatorname{lct}(\bar{C}, \bar{C}_D, \mathfrak{b}_\bullet)$ (see the proof of Lemma 4.3). So we know that

$$\operatorname{lct}(X, D, \mathfrak{a}) = \operatorname{lct}(\bar{C}, \bar{C}_D, \mathfrak{b}_\bullet) = \lim_{k\to\infty} \operatorname{lct}\Big(\bar{C}, \bar{C}_D, \frac{1}{k}\mathfrak{b}_k\Big),$$

which we denote by c. In particular, we can choose ϵ sufficiently small, and k sufficiently large, such that the log discrepancy satisfies

$$A(F; X, D + (c - \epsilon)\mathfrak{a}) < \delta \qquad \text{for sufficiently small } \delta > 0$$

and $\left(C, C_D + (c-\epsilon)\frac{1}{k}\mathfrak{b}_k\right)$ is klt. Thus similar to the proof of Proposition 5.9, by [BC+10] we can construct a model $\psi_1' \colon \mathcal{Z}' \to \mathcal{W}$ extracting only the irreducible divisor $F_{\mathcal{Z}'}$ which gives F over the generic fiber. Furthermore, we can assume $-F_{\mathcal{Z}'}$ is ample over $\mathcal{W}$.

We claim that the special fiber $Z_0' \to W_0$ is a normal model which also only extracts a Kollár component over $\bar{C}$. In fact, let $\nu \colon (Z_0')^{\mathrm{n}} \to Z_0'$ be the normalization and $\rho \colon (Z_0')^{\mathrm{n}} \to W_0$ be the composite morphism. Locally over the vertex v of $\bar{C}$, we have

$$\begin{aligned} \nu^*((K_{\mathcal{Z}'} + Z_0' + F_{\mathcal{Z}'} + (\phi'^{-1})_* D_{\mathbb{A}^1})|_{Z_0'}) &=: K_{(Z_0')^{\mathrm{n}}} + G + \rho_*^{-1} C_D \\ &\geq K_{(Z_0')^{\mathrm{n}}} + \mathrm{Ex}(\rho) + \rho_*^{-1} C_D \end{aligned} \tag{21}$$

by [Kol13, Proposition 4.5]. Denote the pull back of F_0 on $(Z_0')^{\mathrm{n}}$ by $\tilde{F}_0$. Then

$$\begin{aligned} \widehat{\mathrm{vol}}_{(C,C_D),o_C}(\mathrm{ord}_{S_0}) &= \widehat{\mathrm{vol}}_{(X,D),o}(\mathrm{ord}_F) \\ &= \left(-(K_{\mathcal{Z}'} + F_{\mathcal{Z}'} + (\phi'^{-1})_* D_{\mathbb{A}^1})|_F\right)^{n-1} \\ &= \left(-\nu^*((K_{\mathcal{Z}'} + F_{\mathcal{Z}'} + (\phi'^{-1})_* D_{\mathbb{A}^1})|_{Z_0'})|_{\tilde{F}_0}\right)^{n-1} \\ &\geq \left(-\nu^*((K_{\mathcal{Z}'} + F_{\mathcal{Z}'} + (\phi'^{-1})_* D_{\mathbb{A}^1})|_{Z_0'})|_{(\tilde{F}_0)_{\mathrm{red}} = \mathrm{Ex}(\rho)}\right)^{n-1} \\ &\geq \mathrm{vol}_{(C,C_D),o_C}((Z_0')^{\mathrm{n}}) \qquad \text{(by Definition 3.3 and (21)).} \end{aligned}$$

Hence the volume of the model $(Z_0')^{\mathrm{n}}$ is equal to the minimum of the normalized volume $\widehat{\mathrm{vol}}_{(C,C_D),o_C}$. It follows from the argument in the proof of Theorem C that $\tilde{F}_0$ is reduced and yields a Kollár component over o_C. This implies that ν is isomorphic along the generic point of $\tilde{F}_0$, and thus Z_0' is regular along the generic point of F_0. Since Z_0' is Cohen–Macaulay, we conclude that Z_0' is normal by Serre's criterion. By the proof in the cone case (Theorem 5.7), F_0 has to be the same as the canonical component S_0.

Then by the same argument as in the last paragraph of the proof of Proposition 5.9, we conclude that $\mathrm{ord}_F = \mathrm{ord}_S$. □

6. Minimizing Kollár component is K-semistable

In this section, we aim to prove that a Kollár component is minimizing only if it is K-semistable. The method used in the proof is motivated by the one in the study of toric degenerations (see e.g. [Cal02, Section 3.2], [AB04, Proposition 2.2] and [And13, Proposition 3]). In particular the argument allows us to reduce two-step degenerations to a one-step degeneration.

Proof of Theorem D. Let (X, D, o) be a klt singularity with $X = \mathrm{Spec}(R)$. Assume that S is a Kollár component that minimizes $\widehat{\mathrm{vol}}_{(X,D),o}$ and appears as the exceptional divisor in a plt blow up $\mu\colon Y \to X$. Let Δ_S be the divisor on S satisfying $K_Y + (\mu^{-1})_*D + S|_S = K_S + \Delta_S$. By Theorems 4.5 and 4.7, to show that (S, Δ_S) is K-semistable, it suffices to show that the canonical component is a minimizer of $\widehat{\mathrm{vol}}_{(C,C_D)}$, where (C, C_D) is the degeneration associated to S (see the degeneration construction in Sections 2.4 and 5.1).

By Proposition 4.4, we only need to show that

$$\widehat{\mathrm{vol}}_{(C,C_D)}(\mathrm{ord}_{S_0}) \le \widehat{\mathrm{vol}}_{(C,C_D)}(\mathrm{ord}_F)$$

for any $\mathbb{C}^*$-invariant Kollár component F over the vertex $o_C \in (C, C_D)$. Let $(\mathcal{C}, \mathcal{E})$ be the associated special degeneration which degenerates (C, C_D) to a pair (C_0, E_0) where C_0 is an orbifold cone over (F, Δ_F) (see Section 2.4). Then we have a $\mathbb{Z}_{\ge 0} \times \mathbb{Z}_{\ge 0}$-valued order function, which yields a rank 2 valuation, defined on R:

$$w\colon R \to \mathbb{Z}_{\ge 0} \times \mathbb{Z}_{\ge 0}, \qquad f \mapsto (\mathrm{ord}_S(f), \mathrm{ord}_F(\mathbf{in}(f))). \tag{22}$$

We give $\mathbb{Z}_{\ge 0} \times \mathbb{Z}_{\ge 0}$ the following lexicographic order: $(m_1, u_1) < (m_2, u_2)$ if and only if $m_1 < m_2$, or $m_1 = m_2$ and $u_1 < u_2$. If we denote by Γ the valuative monoid and denote the associated graded ring by

$$\mathrm{gr}_w R = \bigoplus_{(m,u)\in\Gamma} R_{\ge(m,u)}/R_{>(m,u)},$$

then it is easy to see that $C_0 = \mathrm{Spec}_{\mathbb{C}}(\mathrm{gr}_w R)$. We will also denote

$$R^* = \bigoplus_{m\in\mathbb{N}} R_{\ge m}/R_{>m} = \bigoplus_{m\in\mathbb{N}} R^*_m.$$

Then $\mathrm{Spec}(R^*) = C$ and $\mathrm{gr}_w R = \mathrm{gr}_{\mathrm{ord}_F} R^*$. Moreover, we define the extended Rees ring of R^* with respect to the filtration associated to ord_F (see Section 4.1):

$$\mathcal{A}' := \bigoplus_{k\in\mathbb{Z}} \mathcal{A}_k := \bigoplus_{k\in\mathbb{Z}} \mathfrak{b}_k t^{-k} \subset R^*[t, t^{-1}]$$

where $\mathfrak{b}_k = \{f \in R^* \mid \mathrm{ord}_F(f) \ge k\}$. Then the flat family $\mathcal{C} \to \mathbb{A}^1$ is given by $\mathrm{Spec}_{\mathbb{C}[t]}(\mathcal{A}')$. In particular,

$$\mathcal{A}' \otimes_{\mathbb{C}[t]} \mathbb{C}[t, t^{-1}] \cong R^*[t, t^{-1}], \quad \mathcal{A}' \otimes_{\mathbb{C}[t]} \mathbb{C}[t]/(t) \cong \mathrm{gr}_w R = \mathrm{gr}_{\mathrm{ord}_F} R^*.$$

Pick a set of homogeneous generators $\bar{f}_1, \dots, \bar{f}_p$ for $\mathrm{gr}_w R$ with $\deg(\bar{f}_i) = (m_i, u_i)$. Lift them to generators $f_1, \dots, f_p$ for R^* such that $f_i \in R^*_{m_i}$. Set $P = \mathbb{C}[x_1, \dots, x_p]$ and give P the grading by $\deg(x_i) = (m_i, u_i)$ so that the surjective map

$$\rho_0\colon P \to \mathrm{gr}_w R \quad \text{given by} \quad x_i \mapsto f_i$$

is a map of graded rings. Let $\bar{g}_1, \dots, \bar{g}_q \in P$ be a set of homogeneous generators of $\mathrm{Ker}(\rho_0)$ and assume $\deg(\bar{g}_j) = (n_j, v_j)$.

Since $\bar{g}_j(\bar{f}_1, \dots, \bar{f}_p) = 0 \in \mathrm{gr}_w R$, it follows that

$$\bar{g}_j(f_1, \dots, f_p) \in (R_{n_j})_{>v_j} \quad \text{for each } j.$$

By the flatness of $\mathcal{A}'$ over $\mathbb{C}[t]$, there exist liftings $g_j \in \bar{g}_j + (P_{n_j})_{>v_j}$ of the relation $\bar{g}_j$ such that

$$g_j(f_1, \ldots, f_p) = 0 \quad \text{for } 1 \leq j \leq q.$$

So the g_j's form a Gröbner basis of J with respect to the order function ord_F, where J is the kernel of the surjection $P \to R^*$. In other words, if we let $K = (\bar{g}_1, \ldots, \bar{g}_q)$ denote the kernel of $P \to \mathrm{gr}_w R$, then K is the initial ideal of J with respect to the order determined by ord_F. As a consequence,

$$\mathcal{A}' = P[\tau]/(\tilde{g}_1, \ldots, \tilde{g}_q) \quad \text{where} \quad \tilde{g}_j = \tau^{v_j} g_j(\tau^{-u_1}x_1, \ldots, \tau^{-u_p}x_p).$$

Now we lift $f_1, \ldots, f_p$ further to generators $F_1, \ldots, F_p$ of R. Then

$$g_j(F_1, \ldots, F_p) \in R_{>n_j}.$$

Let $\mathcal{R}'$ be the extended Rees algebra associated to ord_S on R (see Section 4.1). By the flatness of $\mathcal{R}'$ over $\mathbb{C}[t]$, there exist $G_j \in g_j + P_{>n_j}$ such that

$$G_j(F_1, \ldots, F_p) = 0.$$

Let I be the kernel of $P \to R$. Then the G_j's form a Gröbner basis with respect to the valuation ord_S and the associated initial ideal is J. As a consequence,

$$\mathcal{R}' = P[\zeta]/(\tilde{G}_1, \ldots, \tilde{G}_q) \quad \text{where} \quad \tilde{G}_j = \zeta^{n_j} G_j(\zeta^{-m_1}x_1, \ldots, \zeta^{-m_p}x_p).$$

In summary, we have a $(\mathbb{C}^*)^2$-action on $\mathbb{C}^p$ generated by two one-parameter subgroups $\lambda_0(t) = t^{\mathbf{m}}$ and $\lambda'(t) = t^{\mathbf{u}}$ where $\mathbf{m}, \mathbf{u} \in \mathbb{N}^p$; λ_0 degenerates (X, D) to (C, C_D), and λ' degenerates (C, C_D) further to (C_0, E_0).

Lemma 6.1. *For $0 < \epsilon \ll 1$ and $\epsilon \in \mathbb{Q}$, there is a family (parametrized by ϵ) of subgroups $\lambda_\epsilon \colon \mathbb{C}^* \to (\mathbb{C}^*)^2$ such that $\lambda_\epsilon(t)$ degenerates X to C_0 as $t \to 0$.*

Proof. Let (n'_j, v'_j) be a degree of any homogeneous component of $G_j - \bar{g}_j$ and consider the difference $(n'_j, v'_j) - (n_j, v_j)$. Note that $(n'_j, v'_j) > (n_j, v_j)$. Denote by $B \subset \mathbb{Z} \times \mathbb{Z}$ the finite set consisting of such differences $(n'_j, v'_j) - (n_j, v_j)$, together with 0 and the two generators of $\mathbb{N} \times \mathbb{N}$. Let M be a positive integer that is larger than all absolute values of coordinates of pairs B and let ϵ be so small that $1 > M\epsilon$. After tensoring with $\mathbb{Q}$, we can define

$$\pi_\epsilon = e_0^* + \epsilon e_1^* \colon \mathbb{Q}^2 \to \mathbb{Q}$$

where e_0^* and e_1^* denote the first and second projections on the product $\mathbb{Z}^2 = \mathbb{Z} \times \mathbb{Z}$.

For $\epsilon > 0$ suficiently small, we define $\lambda_\epsilon \colon \mathbb{C}^* \to \mathrm{GL}(p, \mathbb{C})$ to be the one-parameter subgroup corresponding to the prime integral vector $N_\epsilon \cdot \pi_\epsilon$ in $\mathbb{Q}_{>0} \cdot \pi_\epsilon$:

$$\lambda_\epsilon(t) \cdot z = (t^{N_\epsilon(m_1+\epsilon u_1)}z_1, \ldots, t^{N_\epsilon(m_p+\epsilon u_p)}z_p) \quad \text{for any } (z_1, \ldots, z_p) \in \mathbb{C}^p.$$

Note that in this setting, π_0 corresponds to ord_S.

Now to see that λ_ϵ degenerates X to C_0, note that for any monomial $x^{\mathbf{p}}$ with bidegree $(m, u) = (\mathbf{p}\cdot\mathbf{m}, \mathbf{p}\cdot\mathbf{u})$, its degree under λ_ϵ is given by $N_\epsilon \cdot \pi_\epsilon(m, u)$. Then from our construction, we have

$$\pi_\epsilon(n_j', v_j') > \pi_\epsilon(n_j, v_j)$$

where (n_j', v_j') is the degree of any homogeneous part of $G_j - \bar{g}_j$ such that $(n_j', v_j') > (n_j, v_j)$. Thus the initial term of G_j with respect to the weight function π_ϵ is exactly $\bar{g}_j$. $\square$

Fix any $\lambda_\epsilon\colon \mathbb{C}^* \to (\mathbb{C}^*)^2$ for $0 < \epsilon \ll 1$ as above. Then $\mathbb{C}^*$ acts on C_0 via λ_ϵ and $C_0 \setminus \{o_{C_0}\}$ is a $\mathbb{C}^*$-Seifert bundle (see [Kol04]) where o_{C_0} is the vertex of C_0. We claim that the quotient $(C_0 \setminus \{o_{C_0}\})/\lambda_\epsilon$ (which we will simply denote by C_0/λ_ϵ) yields a Kollár component S_ϵ over (C_0, E_0). Furthermore, it induces Kollár components over (C, C_D) and (X, D) which are both isomorphic to S_ϵ and such that the associated degenerations degenerate (C, C_D) and (X, D) to (C_0, E_0). By abuse of notation we will also denote those Kollár components over (C, C_D) and (X, D) by S_ϵ.

Assuming this claim is true, we have

$$\widehat{\mathrm{vol}}_{(X,D)}(\mathrm{ord}_{S_\epsilon}) = \widehat{\mathrm{vol}}_{(C_0,E_0)}(\mathrm{ord}_{S_\epsilon}) = \widehat{\mathrm{vol}}_{(C,C_D)}(\mathrm{ord}_{S_\epsilon}).$$

In the rational coweight cone $N_{\mathbb{Q}} \cong \mathbb{Q}^2$, the one-parameter subgroup $\lambda_0(t)$, which degenerates (X, D) to (C, C_D), corresponds to the coweight vector $(1, 0)$ and the one-parameter subgroup $\lambda'(t)$, which degenerates (C, C_D) to (C_0, E_0), corresponds to the coweight vector $(0, 1)$. By construction, π_ϵ corresponds to the coweight $(1, \epsilon)$ and induces the one-parameter subgroup $\lambda_\epsilon\colon \mathbb{C}^* \to \mathrm{GL}(p, \mathbb{C})$ which preserves (C_0, E_0).

Consider the valuation $\mathrm{wt}_{\lambda_t} \in \mathrm{Val}_{C_0, o_{C_0}}$ induced by the coweight vector of the form $(1, t) \in N_{\mathbb{R}}$ for $t \in [0, \infty)$. It is just the valuation associated to the Reeb vector field on C_0 that generates λ_t (see [LX18]). Define $f(t) = \widehat{\mathrm{vol}}_{(C_0,E_0)}(\mathrm{wt}_{\lambda_t})$. Then $f(t)$ is a smooth convex function on $[0, \infty)$ (see [MSY08, C.2] and [LX18, Proposition 2.21]), and

$$\begin{aligned} f(\epsilon) &= \widehat{\mathrm{vol}}_{(C_0,E_0)}(\mathrm{ord}_{S_\epsilon}) = \widehat{\mathrm{vol}}_{(X,D)}(\mathrm{ord}_{S_\epsilon}) \\ &\geq \widehat{\mathrm{vol}}_{(X,D)}(\mathrm{ord}_{S}) = \widehat{\mathrm{vol}}_{(C_0,E_0)}(\mathrm{ord}_{S_0}) = f(0). \end{aligned}$$

The inequality is because ord_S is assumed to be a minimizer of $\widehat{\mathrm{vol}}_{(X,D),o}$. By convexity, this implies that $f(t)$ is an increasing function of t. Recall that the coweight $(0, 1) = \lim_{t\to\infty} t^{-1}(1, t)$ corresponds to the Kollár component F_0 which is the degeneration of F over C_0. By the rescaling invariance of $\widehat{\mathrm{vol}}$, we have $\lim_{t\to\infty} f(t) = \widehat{\mathrm{vol}}_{(C_0,E_0)}(\mathrm{ord}_{F_0})$ (cf. Remark 4.8 or [LX18, proof of Theorem 3.5]). So indeed

$$\widehat{\mathrm{vol}}_{(C,C_D)}(\mathrm{ord}_F) = \widehat{\mathrm{vol}}_{(C_0,E_0)}(\mathrm{ord}_{F_0}) \geq \widehat{\mathrm{vol}}_{(C_0,E_0)}(\mathrm{ord}_{S_0}) = \widehat{\mathrm{vol}}_{(C,C_D)}(\mathrm{ord}_{S_0}).$$

It remains to verify the claim on S_ϵ. For that we define a filtration by

$$\begin{aligned} \mathcal{F}^N R &= \mathrm{Span}_{\mathbb{C}}\Big\{ F_1^{a_1} \ldots F_p^{a_p} \;\Big|\; \pi_\epsilon\Big(\sum_{i=1}^{p} a_i (m_i, u_i)\Big) \geq N \Big\} \\ &= \{ g \in R \mid \text{there exists } G \in P \text{ such that } G|_X = g \text{ and } \deg_{\pi_\epsilon}(G) \geq N \}. \end{aligned}$$

Then $\{\mathcal{F}^N R\}$ is the filtration induced by the weighted blow up $\widehat{\mathbb{C}^p} \to \mathbb{C}^p$. The associated graded ring of $\{\mathcal{F}^N R\}$ is isomorphic to $\mathrm{gr}_w R$ with the grading given by the weight function $\pi_\epsilon \circ w$. Because $\mathrm{gr}_w R$ is a normal integral domain, by Lemma 2.15 the above filtration is induced by a valuation w_ϵ on R, which is a divisorial valuation. Indeed, denote by $\hat{X}$ the strict transform of X under the weighted blow up (i.e., filtered blow up) $\widehat{\mathbb{C}^p} \to \mathbb{C}^p$. Then, by the discussion in Section 2.5, the exceptional divisor $\hat{X} \to X$ is isomorphic to $S_\epsilon = C_0/\lambda_\epsilon := (C_0 \setminus \{o_{C_0}\})/\mathbb{C}^*$ and $w_\epsilon = c \cdot \mathrm{ord}_{S_\epsilon}$ for some $c > 0$. By Proposition 6.2,

$$(S_\epsilon, \Delta_\epsilon) = (C_0, E_0)/\lambda_\epsilon := (C_0 \setminus \{o_{C_0}\}, E_0 \setminus \{o_{C_0}\})/\mathbb{C}^*$$

is indeed a klt log Fano pair and a Kollár component over $o \in (X, D)$. □

Proposition 6.2. *With the above notation, for any $0 < \epsilon \ll 1$ with $\epsilon \in \mathbb{Q}_+$, let $(S_\epsilon, \Delta_\epsilon) = (C_0, E_0)/\lambda_\epsilon$. Then S_ϵ is a Kollár component over $o \in (X, D)$ and $o_C \in (C, C_D)$.*

Proof. For $0 < \epsilon \ll 1$ with $\epsilon \in \mathbb{Q}_+$, λ_ϵ is associated to a $\mathbb{C}^*$-action. We have a log orbifold $\mathbb{C}^*$-bundle $\pi : (C_0^\circ, E_0^\circ) := (C_0 \setminus \{o_{C_0}\}, E_0 \setminus \{o_{C_0}\}) \to (S_\epsilon, \Delta_\epsilon)$. The Chern class of this orbifold $\mathbb{C}^*$-bundle, denoted by $c_1(C_0^\circ/S_\epsilon)$, is contained in $\mathrm{Pic}(S_\epsilon)$ and is ample. One can extract S_ϵ over C_0 to get a birational morphism $\mu\colon Y_\epsilon \to C_0$ with exceptional divisor isomorphic to S_ϵ. (We note that this is an example of the Dolgachev–Pinkham–Demazure construction, see e.g. [Kol04].)

Because C_0 has a $\mathbb{Q}$-Gorenstein klt singularity at o_{C_0}, by [Kol04, 40–42] we know that $c_1(C_0^\circ/S_\epsilon) = -r^{-1}(K_{S_\epsilon} + \Delta_\epsilon)$ for $r \in \mathbb{Q}_{>0}$ and $(S_\epsilon, \Delta_\epsilon)$ has klt singularities. So $(S_\epsilon, \Delta_\epsilon)$ is a Kollár component over $v \in (C_0, E_0)$.

To transfer this to (X, o), we notice that by our construction the graded ring $\mathrm{gr}_{w_\epsilon} R$ is isomorphic to

$$\mathrm{gr}_{\mathrm{wt}_{\lambda_\epsilon}} \mathbb{C}[C_0] = \mathrm{gr}_{\mathrm{wt}_{\lambda_\epsilon}} (\mathrm{gr}_w R) \; (\cong \mathrm{gr}_w R).$$

The exceptional divisor of the filtered blow up over X associated to w_ϵ is isomorphic to that associated to $\mathrm{wt}_{\lambda_\epsilon}$ over C_0, which is $\mathrm{Proj}(\mathrm{gr}_{\mathrm{wt}_{\lambda_\epsilon}} (\mathrm{gr}_w R))$ and isomorphic to $(S_\epsilon, \Delta_\epsilon)$. Since $(S_\epsilon, \Delta_\epsilon)$ is klt, by the inversion of adjunction we know that the filtered blow up is indeed a plt blow up and hence S_ϵ is a Kollár component over (X, D, o).

The same argument also applies to (C, C_D). □

7. Examples

In this section, we find the minimizer for some examples of klt singularities $(X, o) = (\mathrm{Spec}(R), \mathfrak{m})$. We note that by Proposition 2.8 and Theorem 3.12, this also explicitly yields the sharp lower bound of normalized multiplicities, i.e.,

$$\inf_{\mathfrak{a}} \mathrm{lct}(X, \mathfrak{a})^n \cdot \mathrm{mult}(\mathfrak{a})$$

for all $\mathfrak{m}$-primary ideals $\mathfrak{a}$ and gives the equality condition, which generalizes the results in [FEM04] on a smooth point.

Example 7.1. Below, we explicitly compute the minimizer for quotient, A_k, E_k and weakly exceptional singularities.

1. Let $(X, o) = (\mathbb{C}^n, 0)/G$ be an n-dimensional quotient singularity. Let $E \cong \mathbb{P}^{n-1}$ be the exceptional divisor over $\mathbb{C}^n$ obtained by blowing up 0. Denote by S the valuation over (X, o) which is the quotient of E by G. Applying Lemma 2.13 to the pull back of Kollár components on X, we find that

$$\widehat{\mathrm{vol}}_{X,o}(\mathrm{ord}_S) \le \widehat{\mathrm{vol}}_{X,o}(\mathrm{ord}_F)$$

for any Kollár component F over (X, o). So ord_S minimizes $\widehat{\mathrm{vol}}_{X,o}$ with

$$\mathrm{vol}(o, X) = \widehat{\mathrm{vol}}_{X,o}(\mathrm{ord}_S) = n^n/|G|.$$

For $n = 2$, this is proved in [LL19, Example 4.9].

2. Consider the n-dimensional A_{k-1} singularity:

$$X = A^n_{k-1} := \{z_1^2 + \cdots + z_n^2 + z_{n+1}^k = 0\}.$$

We consider cases when $k > \frac{2(n-1)}{n-2}$ (for other cases, see [LL19, Example 4.7]). We want to show that the valuation corresponding to the weight $w_* = (n-1, \ldots, n-1, n-2)$ is a minimizer among all valuations in $\widehat{\mathrm{vol}}_{(X,D),o}$. In [Li18, Example 2.8], it is computed to be a minimizer among all valuations obtained by weighted blow ups on the ambient space $\mathbb{C}^{n+1}$.

We notice that under the weighted blow up corresponding to w_*, we have a birational morphism $Y \to X$ with exceptional divisor S isomorphic to the weighted hypersurface

$$S := \{Z_1^2 + \cdots + Z_n^2 = 0\} \subset \mathbb{P}(n-1, \ldots, n-1, n-2) =: \mathbb{P}_{w_*}.$$

Because $\mathbb{P}_{w_*} \cong \mathbb{P}(1, \ldots, 1, n-2)$, it is easy to see that S is isomorphic to $\bar{C}(Q, -K_Q)$ where $Q = Q^{n-2} = \{Z_1^2 + \cdots + Z_n^2 = 0\} \subset \mathbb{P}^{n-1}$ (notice that $K_Q^{-1} = (n-2)H$). On the other hand, because $\mathbb{P}_{w_*}$ is not well-formed, we have codimension 1 orbifold locus along the infinity divisor $Q_\infty \subset S$ with isotropy group $\mathbb{Z}/(n-1)\mathbb{Z}$. So the corresponding Kollár component is the log Fano pair $\left(S, \left(1 - \frac{1}{n-1}\right) Q_\infty\right)$. Because Q_∞ has KE, by [LL19] there is a conical KE on the pair $\left(S, \left(1 - \frac{1}{n-1}\right) Q_\infty\right)$. So by Theorems A and B, ord_S is indeed a global minimizer of $\widehat{\mathrm{vol}}$ that is the unique minimizer among all Kollár components. Notice that for any higher order klt perturbation of these singularities, w_* is also a minimizer.

3. We can also use Theorem A to verify that the valuations in [Li18, Example 2.8] for E_k ($k = 6, 7, 8$) are indeed minimizers of $\widehat{\mathrm{vol}}_{(X,D),o}$, which are unique among Kollár components. To avoid repetition, we will only do this for E_7 singularities. The arguments for the other two cases are similar. So consider the $(n+1)$-dimensional E_7 singularity

$$X^{n+1} = \{z_1^2 + z_2^2 + \cdots + z_n^2 + z_{n+1}^3 z_{n+2} + z_{n+2}^3 = 0\} \subset \mathbb{C}^{n+2}.$$

(a) If $n + 1 = 2$, then X^2 is a quotient singularity $\mathbb{C}^2/E_7$ and so we get the unique polystable component by [LL19, Example 4.9] and example 1 above.

(b) If $n+1=3$, then $X^3=\{z_1^2+z_2^2+z_3^3z_4+z_4^3=0\}\subset\mathbb{C}^4\cong\{w_1w_2+w_3^3w_4+w_4^3=0\}\subset\mathbb{C}^4$ by the change of variables. This singularity has a $(\mathbb{C}^*)^2$-action and is an example of T-variety of complexity 1. By the recent work in [CS19, Theorem 7.1(II)], X^3 indeed has a Ricci flat cone Kähler metric associated to the canonical $\mathbb{C}^*$-action associated to w_*. So by [LL19, Theorem 1.7], the unique K-polystable Kollár component is given by the orbifold $X^3/\langle w_*\rangle$.

(c) If $n+1=4$, then under the weighted blow up corresponding to $w_*=(9,9,9,5,6)$, we have a birational morphism $\hat{X}\to X$ with exceptional divisor E isomorphic to the weighted hypersurface

$$E=\{z_1^2+z_2^2+z_3^2+z_5^3=0\}\subset\mathbb{P}(9,9,9,5,6)=\mathbb{P}(w_*).$$

Since $\mathbb{P}(w_*)$ is not well-formed, we have

$$E\cong\{z_1^2+z_2^2+z_3^2+z_5^3=0\}\subset\mathbb{P}(3,3,3,5,2)=\mathbb{P}'$$

with orbifold locus of isotropy group $\mathbb{Z}/3\mathbb{Z}$ along

$$V=\{z_1^2+z_2^2+z_3^2+z_5^3=0\}\subset\mathbb{P}(3,3,3,2).$$

Alternatively, E is a weighted projective cone over the weighted hypersurface. It is easy to see that as an orbifold, $(V,\Delta)\cong(\mathbb{P}^2,(1-1/3)Q)$ where $Q=\{z_1^2+z_2^2+z_3^2=0\}\subset\mathbb{P}^2$. By [LS14], there exists an orbifold Kähler–Einstein metric on (V,Δ). Notice that $-(K_V+\Delta)=3L-\frac{4}{3}L=\frac{5}{3}L$ where L is the hyperplane bundle of $\mathbb{P}^2$. Denoting by H the hyperplane bundle of $\mathbb{P}'$, we have $H|_V=L/3$. If V is considered as a divisor on E, then

$$V|_V=(\{z_4=0\}\cap E)=5H|_V=\tfrac{5}{3}L.$$

So $-(K_V+\Delta)=V|_V$. Then by [LL19, Theorem 1.7], there exists an orbifold Kähler–Einstein metric on E because the cone angle at infinity is $\beta=1/3$. Thus the unique log-K-semistable (actually log-K-polystable) Kollár component is given by the pair $(E,(1-1/3)V)$.

(d) If $n+1=5$, then under the weighted blow up corresponding to $w_*=(3,3,3,3,2,2)$, we have a birational morphism $\hat{X}\to X$ with exceptional divisor E isomorphic to the weighted hypersurface

$$E=\{z_1^2+z_2^2+z_3^2+z_4^2+z_6^3=0\}\subset\mathbb{P}(3,3,3,3,2,2)=:\mathbb{P}(w_*).$$

This is a weighted projective cone over the weighted hypersurface

$$V=\{z_1^2+z_2^2+z_3^2+z_4^2+z_6^3=0\}\subset\mathbb{P}(3,3,3,3,2).$$

As orbifolds, $(V,\Delta)=(\mathbb{P}^3,(1-1/3)Q)$. By [LS14, Li17a], (V,Δ) is log-K-semistable and degenerates to a conical Kähler–Einstein pair. So by [LL19], we know that $(E,(1-\beta)V_\infty)$ is log-K-semistable. To determine β, we notice that

$$-(K_V+\Delta)=4L-\tfrac{4}{3}L=\tfrac{8}{3}L=4\cdot\tfrac{2}{3}L=4\cdot V_\infty|_V.$$

So $\beta = 1$ and we conclude that the unique (strictly) K-semistable Kollár component is indeed the $\mathbb{Q}$-Fano variety E.

(e) If $n + 1 \geq 6$, then under the weighted blow up corresponding to $w_* = (n-1, \ldots, n-1, n-2, n-2)$, we have a birational morphism $\hat{X} \to X$ with exceptional divisor E isomorphic to the weighted hypersurface

$$E = \{z_1^2 + \cdots + z_n^2 = 0\} \subset \mathbb{P}(n-1, \ldots, n-1, n-2, n-2) =: \mathbb{P}(w_*).$$

This is the weighted projective cone over

$$V = \{z_1^2 + \cdots + z_n^2 = 0\} \subset \mathbb{P}(n-1, \ldots, n-1, n-2).$$

By the discussion in the above A_{k-1}^n singularity case, we know that as an orbifold, $(V, \Delta) = \left(\bar{C}(Q, -K_Q), \left(1 - \frac{1}{n-1}\right)Q_\infty\right)$, which has an orbifold Kähler–Einstein metric. Notice that

$$-(K_V + \Delta) = (n(n-1) + n - 2)H|_V - 2(n-1)H|_V = n(n-2)H|_V.$$

By [LL19, Theorem 1.7], the $\mathbb{Q}$-Fano variety E indeed has an orbifold Kähler–Einstein metric ($\beta = n/n = 1$ at infinity) and hence by Theorem A is the unique K-semistable (actually K-polystable) Kollár component.

We remark that in the case of D_{k+1} singularities, since the valuations computed in [Li18, Example 2.8] could be irrational, the result in this paper does not directly tell whether it is a minimizer in $\mathrm{Val}_{X,o}$. This irregular situation is studied in [LX18] (see also [LL19, Section 6]).

4. A notion of weakly exceptional singularity is introduced in [Pro00]. As the name suggests, this is a weaker notion than the exceptional singularity introduced in [Sho00], which forms a special class of singularities in the theory of local complements. In our language, a singularity (X, o) is *weakly exceptional* if it has a unique Kollár component S. We know that if a singularity is weakly exceptional, then the log α-invariant for the log Fano (S, Δ_S) is at least 1 (see [Pro00, Theorem 4.3] and [CS14]). In particular, (S, Δ_S) is K-semistable (see [OS14, Theorem 1.4] or [Ber13, Theorem 3.12]). And by Theorems A and B, ord_S is the unique minimizer of $\widehat{\mathrm{vol}}(S)$ among all Kollár components. See [CS14] for examples of weakly exceptional singularities.

Finally, we point out that there are examples of minimizers from Sasaki–Einstein metrics; see [LL19, LX18] for details. The articles [LX18, LWX18] also apply minimization of normalized volumes to Donaldson–Sun's conjecture about metric tangent cones on Gromov–Hausdorff limits of Kähler–Einstein Fano manifolds.

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