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Abstract

This work is a continuation of the author's previous paper [*Greatest lower bounds on the Ricci curvature of toric Fano manifolds*, Adv. Math. **226** (2011), 4921–4932]. On any toric Fano manifold, we discuss the behavior of the limit metric of a sequence of metrics which are solutions to a continuity family of complex Monge–Ampère equations in the Kähler–Einstein problem. We show that the limit metric satisfies a singular complex Monge–Ampère equation. This gives a conic-type singularity for the limit metric. Information on conic-type singularities can be read off from the geometry of the moment polytope.

1. Introduction

Let (X, J) be a Fano manifold, that is, K_X^{-1} is ample. Fix a reference Kähler form $\omega \in 2\pi c_1(X)$. In local coordinates $\{z_i\}$ we can write

$$\omega = \sqrt{-1} \sum_{i,j=1}^{n} g_{i\bar{j}}\, dz^i \wedge\, d\bar{z}^j,$$

where $g = (g_{i\bar{j}})$ is positive definite and defines a Kähler metric on X. For simplicity, we will just call ω the Kähler metric. Its Ricci curvature $\mathrm{Ric}(\omega)$ is defined by the formula

$$\mathrm{Ric}(\omega) = -\sqrt{-1} \sum_{i,j=1}^{n} \frac{\partial^2}{\partial z^i \partial \bar{z}^j} \log \det(g_{k\bar{l}})\, dz^i \wedge\, d\bar{z}^j.$$

The $(1,1)$-form $\mathrm{Ric}(\omega)$ represents the cohomology class $2\pi c_1(X)$. By the $\partial\bar{\partial}$-lemma in Kähler geometry, there exists $h_\omega \in C^\infty(X)$ such that

$$\mathrm{Ric}(\omega) - \omega = \sqrt{-1}\, \partial\bar{\partial} h_\omega \quad \text{and} \quad \int_X e^{h_\omega} \omega^n = \int_X \omega^n. \tag{1}$$

In order to solve for the Kähler–Einstein metric, we consider the following family of Monge–Ampère equations in the continuity method:

$$(\omega + \sqrt{-1}\, \partial\bar{\partial}\phi)^n = e^{h_\omega - t\phi} \omega^n. \tag{$*_t$}$$

Note that when $t = 1$ this is equivalent to the Kähler–Einstein equation

$$\mathrm{Ric}(\omega + \sqrt{-1}\, \partial\bar{\partial}\phi) = \omega + \sqrt{-1}\, \partial\bar{\partial}\phi.$$

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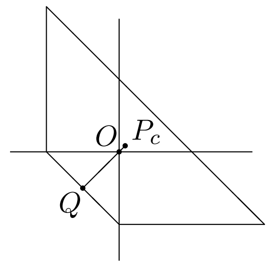

Figure 1. $\mathrm{Bl}_p\mathbb{P}^2$.

Define $\mathcal{S} = \{t : (*_t) \text{ is solvable}\}$. Then one can prove that: (i) $0 \in \mathcal{S}$ (see [Yau78a]); (ii) $\mathcal{S}$ is open in $[0, 1]$ (see [Tia87]); and (iii) $\mathcal{S}$ is closed if and only if $1 \in \mathcal{S}$, i.e. there is a Kähler–Einstein metric.

Now define

$$R(X) = \sup\{t : (*_t) \text{ is solvable}\}.$$

Tian [Tia92] first showed that, on certain manifolds, one cannot solve $(*_t)$ for t sufficiently close to 1. Equivalently, for such a Fano manifold, $R(X) < 1$. More recently, Székelyhidi proved the following result.

Proposition 1 [Szé11]. *We have the following characterization of $R(X)$:*

$$R(X) = \sup\{t : \exists \text{ a Kähler metric } \omega \in 2\pi c_1(X) \text{ such that } \mathrm{Ric}(\omega) > t\omega\}.$$

In particular, $R(X)$ is independent of the reference metric ω. In [Li11], we determined $R(X)$ for any toric Fano manifold.

A toric Fano manifold $X_\triangle$ is determined by a reflexive lattice polytope $\triangle$ (for details on toric manifolds see [Oda88]). For example, let $\mathrm{Bl}_p\mathbb{P}^2$ denote the manifold obtained by blowing up one point on $\mathbb{P}^2$. Then $\mathrm{Bl}_p\mathbb{P}^2$ is a toric Fano manifold and is determined by the polytope shown in Figure 1.

Any such polytope $\triangle$ contains the origin $O \in \mathbb{R}^n$. We denote the barycenter of $\triangle$ by P_c. If $P_c \neq O$, the ray $P_c + \mathbb{R}_{\geqslant 0} \cdot \overrightarrow{P_cO}$ intersects the boundary $\partial\triangle$ at point Q.

Theorem 1.1 [Li11]. *If $P_c \neq O$, then*

$$R(X_\triangle) = \frac{|\overline{OQ}|}{|\overline{P_cQ}|}.$$

Here $|\overline{OQ}|$ and $|\overline{P_cQ}|$ are the lengths of the line segments $\overline{OQ}$ and $\overline{P_cQ}$. If $P_c = O$, then there is a Kähler–Einstein metric on $X_\triangle$ and $R(X_\triangle) = 1$.

The next natural question is what the limit metric looks like as $t \to R(X)$. For the special example $X = \mathrm{Bl}_p\mathbb{P}^2$, which is also the projective compactification of the total space of the line bundle $\mathcal{O}(-1) \to \mathbb{P}^2$, Székelyhidi [Szé11] constructed a sequence of Kähler metrics ω_t with $\mathrm{Ric}(\omega_t) \geqslant t\omega_t$ and ω_t converging to a metric with conic singularity along the divisor D_∞ of conic angle $2\pi \times 5/7$, where D_∞ is the divisor at infinity added in projective compactification. Shi and Zhu [SZ11] proved that rotationally symmetric solutions to the continuity equations $(*_t)$ converge to a metric with conic singularity of conic angle $2\pi \times 5/7$ in the Gromov–Hausdorff sense, which seems to be the first strict result on the limit behavior of solutions to $(*_t)$.

Note that by the theory of Cheeger *et al.* [CCT02], the limit metric in the Gromov–Hausdorff sense should have complex codimension-one conic-type singularities since we only have the positive lower Ricci bound.

For the more general toric case, if we use a special toric metric, which is just the Fubini–Study metric in the projective embedding given by the vertices of the polytope, then, after transforming by some biholomorphic automorphism, we can prove that there is a sequence of Kähler metrics which solve the equation ($*_t$) and converge to a limit metric satisfying a singular complex Monge–Ampère equation (see also the equivalent real version in Theorem 3.1). This generalizes the result of [SZ11] for the special reference Fubini–Study metric.

To be precise, let $\{p_\alpha : \alpha = 1, \dots, N\}$ be all the vertex lattice points of $\triangle$, and let $\{s_\alpha : \alpha = 1, \dots, N\}$ be the corresponding holomorphic sections of $K^{-1}_{X_\triangle}$. Then we take the reference metric to be

$$\omega = \omega_{\mathrm{FS}} := \sqrt{-1}\,\partial\bar\partial \log \sum_{\alpha=1}^{N} |s_\alpha|^2,$$

which is the pull-back of the Fubini–Study metric of $\mathbb{CP}^{N-1}$ under the Kodaira embedding induced by $\{s_\alpha\}$. Now, with the same notation as in Theorem 1.1, let $\mathcal{F}$ be the minimal face of $\triangle$ containing Q. Let $\{p_k^{\mathcal{F}}\}$ be the vertex lattice points of $\mathcal{F}$; then they correspond to a sublinear system $\mathfrak{L}_{\mathcal{F}}$ of $|-K_{X_\triangle}|$. We let $\mathrm{Bs}(\mathfrak{L}_{\mathcal{F}})$ denote the base locus of this sublinear system. Also, let $\sum_\alpha{}'$ denote the sum $\sum_{p_k^{\mathcal{F}}}$. Then we have the following theorem.

Theorem 1.2. *After some biholomorphic transformation $\sigma_t : X_\triangle \to X_\triangle$, there is a subsequence $t_i \to R(X)$ such that the $\sigma_{t_i}^*\omega_{t_i}$ converge to a Kähler current $\omega_\infty = \omega + \sqrt{-1}\,\partial\bar\partial\psi_\infty$, with $\psi_\infty \in L^\infty(X_\triangle) \cap C^\infty(X_\triangle \backslash \mathrm{Bs}(\mathfrak{L}_{\mathcal{F}}))$, which satisfies a complex Monge–Ampère equation of the form*

$$(\omega + \sqrt{-1}\,\partial\bar\partial\psi_\infty)^n = e^{-R(X)\psi_\infty}\Big(\sum_\alpha{}' b_\alpha \|s_\alpha\|^2\Big)^{-(1-R(X))}\Omega. \tag{2}$$

Here $\Omega = e^{h_\omega}\omega^n$ is a smooth volume form. For each vertex lattice point $p_\alpha^{\mathcal{F}}$ of $\mathcal{F}$, b_α is a constant satisfying $0 < b_\alpha \leqslant 1$, and $\|\cdot\| = \|\cdot\|_{\mathrm{FS}}$ is (up to constant multiplication) the Fubini–Study metric on $K^{-1}_{X_\triangle}$. In particular,

$$\mathrm{Ric}(\omega_{\psi_\infty}) = R(X)\omega_{\psi_\infty} + (1 - R(X))\sqrt{-1}\,\partial\bar\partial \log\Big(\sum_\alpha{}' b_\alpha |s_\alpha|^2\Big). \tag{3}$$

The above equation reveals the conic-type singularities for the limit metric. We can read off the location of conic singularities and conic angles from the geometry of the polytope. See § 3.3 for the method and discussions. In particular, this can provide a toric explanation of the special case $\mathrm{Bl}_p\mathbb{P}^2$ mentioned earlier (see Example 1).

Note that although we can prove that the limit metric is smooth outside the singular locus, to prove geometrically that it is a conic metric along codimension-one strata of the singular set, we need to establish a more delicate estimate, which we will discuss in the future. There are also difficulties in studying the behavior of the limit metric around higher-codimensional strata (see Remark 4 and Example 2).

Finally, we remark that, in view of the special case $\mathrm{Bl}_p\mathbb{P}^2$ in [SZ11] and the results in [LS12], we expect the following statement to be true: the Gromov–Hausdorff limit of $(X_\triangle, \omega_{t_i})$ is the metric completion of $(X_\triangle \backslash \mathrm{Bs}(\mathfrak{L}_{\mathcal{F}}), \omega_\infty)$.

2. Consequence of estimates of Wang and Zhu

The proof of Theorem 1.1 is based on the methods of Wang and Zhu [WZ04].

For a reflexive lattice polytope $\triangle$ in $\mathbb{R}^n = \mathbb{Z}^n \otimes_{\mathbb{Z}} \mathbb{R}$, we have a Fano toric manifold $X_\triangle \supset (\mathbb{C}^*)^n$ with a $(\mathbb{C}^*)^n$ action. In the following, for simplicity we will sometimes just write X for $X_\triangle$.

Let $(S^1)^n \subset (\mathbb{C}^*)^n$ be the standard real maximal torus. Let $\{z_i\}$ be the standard coordinates of the dense orbit $(\mathbb{C}^*)^n$, and let $x_i = \log |z_i|^2$. We have a standard lemma about the toric Kähler metric, whose proof we omit; see, for example, [WZ04].

Lemma 1. *Any $(S^1)^n$-invariant Kähler metric ω on X has a potential $u = u(x)$ on $(\mathbb{C}^*)^n$, i.e. $\omega = \sqrt{-1}\,\partial\bar\partial u$. The potential u is a proper convex function on $\mathbb{R}^n$ and satisfies the momentum map condition*

$$Du(\mathbb{R}^n) = \triangle.$$

Also,

$$\frac{(\sqrt{-1}\,\partial\bar\partial u)^n/n!}{(dz_1/z_1)\wedge(d\bar z_1/\bar z_1)\wedge\cdots\wedge(dz_n/z_n)\wedge(d\bar z_n/\bar z_n)} = \det\left(\frac{\partial^2 u}{\partial x_i \partial x_j}\right) =: \det(u_{ij}). \tag{4}$$

Let $\{p_\alpha : \alpha = 1, \ldots, N\}$ be all the *vertex lattice points* of $\triangle$. Each p_α corresponds to a holomorphic section $s_\alpha \in H^0(X_\triangle, K^{-1}_{X_\triangle})$. We can embed $X_\triangle$ into $\mathbb{P}^N$ using $\{s_\alpha\}$. Let us first find the appropriate potential on $(\mathbb{C}^*)^n$ for the pull-back of the Fubini–Study metric (see [Li11, § 2] or [WZ04]).

Recall that for any section s of K_X^{-1}, the Fubini–Study metric as a Hermitian metric on K_X^{-1} is defined, up to multiplication by a positive constant, as

$$\|s\|^2_{\mathrm{FS}} = e^{-\tilde C}\frac{|s|^2}{\sum_\beta |s_\beta|^2}. \tag{5}$$

The right-hand side is well-defined by means of local trivializations; $\tilde C$ is some normalizing constant, which we choose so as to simplify the computations later.

First, let $\tilde s_0$ be the section corresponding to the origin $0 \in \triangle$. On the open dense orbit $(\mathbb{C}^*)^n$, by standard toric geometry we can assume that

$$\frac{s_\alpha}{\tilde s_0} = \prod_{i=1}^n z_i^{p_{\alpha,i}}. \tag{6}$$

So the Fubini–Study norm of $\tilde s_0$ is

$$\|\tilde s_0\|^2_{\mathrm{FS}} = e^{-\tilde C}\frac{|\tilde s_0|^2}{\sum_{\alpha=1}^N |s_\alpha|^2} = e^{-\tilde C}\left(\sum_{\alpha=1}^N \prod_{i=1}^n |z_i|^{2p_{\alpha,i}}\right)^{-1} = e^{-\tilde C}\left(\sum_{\alpha=1}^N e^{\langle p_\alpha, x\rangle}\right)^{-1} =: e^{-\tilde u_0}.$$

In other words, we define

$$\tilde u_0 = \log\left(\sum_{\alpha=1}^N e^{\langle p_\alpha, x\rangle}\right) + \tilde C. \tag{7}$$

Now we can choose $\tilde C$ by the normalization condition:

$$\int_{\mathbb{R}^n} e^{-\tilde u_0}\,dx = \mathrm{Vol}(\triangle) = \frac{c_1(X_\triangle)^n}{n!} = \frac{1}{(2\pi)^n}\int_{X_\triangle}\frac{\omega^n}{n!}. \tag{8}$$

On the other hand, $\mathrm{Ric}(\omega)$ is the curvature form of the Hermitian line bundle K_M^{-1} with Hermitian metric determined by the volume form ω^n. Note that we can take $\tilde{s}_0 = z_1(\partial/\partial z_1) \wedge \cdots \wedge z_n(\partial/\partial z_n)$. Since $\partial/\partial \log z_i = (\partial/\partial \log|z_i| - \sqrt{-1}\,\partial/\partial\theta_i)/2 = \partial/\partial \log|z_i|^2 = \partial/\partial x_i$ when acting on any $(S^1)^n$-invariant function on $(\mathbb{C}^*)^n$, we have

$$\begin{aligned}\|\tilde{s}_0\|^2_{\omega^n} &= \left\| z_1\frac{\partial}{\partial z_1} \wedge \cdots \wedge z_n\frac{\partial}{\partial z_n}\right\|^2_{\omega^n} = \det\left(\frac{\partial^2\tilde{u}_0}{\partial \log z_i\, \overline{\partial \log z_j}}\right)\\ &= \det\left(\frac{\partial^2\tilde{u}_0}{\partial \log|z_i|^2\, \partial \log|z_j|^2}\right) = \det(\tilde{u}_{0,ij}).\end{aligned}$$

It is easy to see from the definition of h_ω in (1) and the normalization condition (8) that

$$\frac{e^{h_\omega}\omega^n/n!}{(dz_1/z_1)\wedge(d\bar{z}_1/\bar{z}_1)\wedge\cdots\wedge(dz_n/z_n)\wedge(d\bar{z}_n/\bar{z}_n)} = e^{h_\omega}\|\tilde{s}_0\|^2_{\omega^n} = \|\tilde{s}_0\|^2_{\mathrm{FS}} = e^{-\tilde{u}_0}. \tag{9}$$

Remark 1. We use only vertex lattice points because, roughly speaking, later in Lemma 5 vertex lattice points alone help us to determine which sections become degenerate when doing a biholomorphic transformation and taking the limit; see Remark 3. We expect results similar to Theorem 1.2 to hold for general toric reference Kähler metrics.

So, divide both sides of $(*_t)$ by the meromorphic volume form $n!\,((dz_1/z_1)\wedge(d\bar{z}_1/\bar{z}_1)\wedge\cdots\wedge(dz_n/z_n)\wedge(d\bar{z}_n/\bar{z}_n))$, We can rewrite the equations $(*_t)$ as a family of real Monge–Ampère equations on $\mathbb{R}^n$,

$$\det(u_{ij}) = e^{-(1-t)\tilde{u}_0 - tu}, \tag{$**_t$}$$

where u is the potential for $\omega + \sqrt{-1}\,\partial\bar{\partial}\phi$ on $(\mathbb{C}^*)^n$ and is related to ϕ in $(*_t)$ by

$$\phi = u - \tilde{u}_0.$$

For simplicity, let

$$w_t(x) = tu(x) + (1-t)\tilde{u}_0.$$

Then w_t is also a proper convex function on $\mathbb{R}^n$ satisfying $Dw_t(\mathbb{R}^n) = \triangle$. Thus it has a unique absolute minimum at a point $x_t \in \mathbb{R}^n$. Let

$$m_t = \inf\{w_t(x) : x \in \mathbb{R}^n\} = w_t(x_t).$$

Then the main estimate of Wang and Zhu [WZ04] is as follows.

Proposition 2 ([WZ04]; see also [Don08]). (i) *There exists a constant C, independent of $t < R(X_\triangle)$, such that*

$$|m_t| < C.$$

(ii) *There exists $\kappa > 0$ and a constant C, both independent of $t < R(X_\triangle)$, such that*

$$w_t \geqslant \kappa|x - x_t| - C. \tag{10}$$

Proposition 3 [WZ04]. *Existence of a uniform bound on $|x_t|$, for any $0 \leqslant t \leqslant t_0$, is equivalent to being able to solve $(**_t)$ or, equivalently, $(*_t)$ for t up to t_0. More precisely, this condition is equivalent to obtaining uniform C^0 estimates for the solution ϕ_t in $(*_t)$ for $t \in [0, t_0]$.*

By the above proposition, we have the following lemma.

Lemma 2. *If $R(X_\triangle) < 1$, then there exists a subsequence $\{x_{t_i}\}$ of $\{x_t\}$ such that*

$$\lim_{t_i \to R(X_\triangle)} |x_{t_i}| = +\infty.$$

By the properness of $\tilde{u}_0$ and the compactness of $\triangle$, we immediately get the following result.

Lemma 3. *If $R(X_\triangle) < 1$, then there exists a subsequence of $\{x_{t_i}\}$, which we still denote by $\{x_{t_i}\}$, and $y_\infty \in \partial\triangle$ such that*

$$\lim_{t_i \to R(X_\triangle)} D\tilde{u}_0(x_{t_i}) = y_\infty. \tag{11}$$

To determine $R(X_\triangle)$, we use the key identity

$$\frac{1}{\mathrm{Vol}(\triangle)} \int_{\mathbb{R}^n} D\tilde{u}_0 e^{-w}\, dx = -\frac{t}{1-t} P_c. \tag{12}$$

Remark 2. This identity is a toric form of a general formula for solutions of equations ($*_t$):

$$-\frac{1}{V}\int_X \mathrm{div}_\Omega(v)\omega_t^n = \frac{t}{1-t} F_{2\pi c_1(X)}(v).$$

Here $\Omega = e^{h_\omega}\omega^n$, v is any holomorphic vector field, $\mathrm{div}_\Omega(v) = \mathcal{L}_v\Omega/\Omega$ is the divergence of v with respect to Ω, and

$$F_{2\pi c_1(X)}(v) = \frac{1}{V}\int_X v(h_\omega)\omega^n$$

is the Futaki invariant in class $2\pi c_1(X)$ (see [Fut83]).

By the uniform linear growth of w_t in (10), the left-hand side of (12) is roughly $D\tilde{u}_0(x_t)$. As long as this is bounded away from the boundary of the polytope, we can control the point x_t. So, as t goes to $R(X_\triangle)$, since x_t goes to infinity in $\mathbb{R}^n$, the left-hand side goes to a point on $\partial\triangle$, which is roughly y_∞. To make a precise statement, assume that the reflexive polytope $\triangle$ is defined by inequalities

$$\lambda_r(y) \geqslant -1, \quad r = 1, \ldots, K, \tag{13}$$

where the $\lambda_r(y) = \langle v_r, y\rangle$ are fixed linear functions. We also identify the minimal face of $\triangle$ where y_∞ lies:

$$\begin{aligned} \lambda_r(y_\infty) &= -1 \quad \text{for } r = 1, \ldots, K_0, \\ \lambda_r(y_\infty) &> -1 \quad \text{for } r = K_0+1, \ldots, K. \end{aligned} \tag{14}$$

Then Theorem 1.1 follows from the next proposition.

Proposition 4 [Li11]. *If $P_c \neq O$, then*

$$-\frac{R(X_\triangle)}{1-R(X_\triangle)} P_c \in \partial\triangle.$$

More precisely,

$$\lambda_r\left(-\frac{R(X_\triangle)}{1-R(X_\triangle)} P_c\right) \geqslant -1, \tag{15}$$

with equality holding if and only if $r = 1, \ldots, K_0$. So $-(R(X_\triangle)/(1-R(X_\triangle)))P_c$ and y_∞ lie on the same faces defined by (14).

3. Discussion of conic-type limit metrics

3.1 Equation for the limit metric

We first fix the reference metric to be the Fubini–Study metric

$$\omega = \sqrt{-1}\,\partial\bar\partial\tilde u_0 = \sqrt{-1}\,\partial\bar\partial\log\Big(\sum_\alpha |s_\alpha|^2\Big).$$

We seek the limit of ω_t as $t \to R(X)$ under a suitable transformation, where

$$\omega_t = \omega + \sqrt{-1}\,\partial\bar\partial\phi_t$$

is the solution to the continuity equation ($*_t$). We use notation from the previous section. So, in toric coordinates,

$$\omega_t = \frac{\partial^2 u}{\partial\log z_i\,\partial\log z_j}\,d\log z_i\wedge d\log z_j = -\sqrt{-1}\,u_{ij}\,dx_i\,d\theta_j,$$

where $u = u_t$ is the solution to the real Monge–Ampère equation ($**_t$).

Let $\sigma = \sigma_t$ be the holomorphic transformation given by

$$\sigma_t(x) = x + x_t.$$

Assume $x_t = (x_t^1, \dots, x_t^n)$; then in complex coordinates we have

$$\sigma_t(\{z_i\}) = \{e^{x_t^i/2} z_i\}.$$

By the analysis of the previous section, we perform the following transformation:

$$\begin{aligned} U(x) &= \sigma_t^* u(x) - u(x_t) = u(x+x_t) - u(x_t),\\ \tilde U_t(x) &= \sigma_t^*\tilde u_0(x) - \tilde u_0(x_t) = \tilde u_0(x+x_t) - \tilde u_0(x_t). \end{aligned} \tag{16}$$

Note that $w_t(x) = tu + (1-t)\tilde u_0$. Then $U = U_t(x)$ satisfies the Monge–Ampère equation

$$\det(U_{ij}) = e^{-tU-(1-t)\tilde U - w(x_t)}. \tag{$**_t'$}$$

By Proposition 4, we know that $Q = -(R(X_\triangle)/(1-R(X_\triangle)))P_c$ lies on the boundary of $\triangle$. Let $\mathcal F$ be the minimal face of $\triangle$ which contains Q. Now we make the following observation.

Proposition 5. *There is a subsequence $t_i \to R(X)$ such that the $\tilde U_{t_i}$ converge locally uniformly to a convex function of the form*

$$\tilde U_\infty := \log\Big(\sum_{p_\alpha\in\mathcal F} b_\alpha e^{\langle p_\alpha, x\rangle}\Big), \tag{17}$$

where $0 < b_\alpha \leqslant 1$ are some constants. For simplicity, we will use $\sum_\alpha{}' = \sum_{p_\alpha\in\mathcal F}$ to denote the sum over all the vertex lattice points *contained in $\mathcal F$.*

Proof. By (7) and (16), we have

$$\tilde U(x) = \log\Big(\sum_\alpha e^{\langle p_\alpha, x+x_t\rangle}\Big) - \log\Big(\sum_\alpha e^{\langle p_\alpha, x_t\rangle}\Big) = \log\Big(\sum b(p_\alpha, t)e^{\langle p_\alpha, x\rangle}\Big), \tag{18}$$

where

$$b(p_\alpha, t) = \frac{e^{\langle p_\alpha, x_t\rangle}}{\sum_\beta e^{\langle p_\beta, x_t\rangle}}.$$

Since $0 < b(p_\alpha, t) < 1$, we can assume that there is a subsequence $t_i \to R(X)$ such that for any vertex lattice point p_α,

$$\lim_{t\to R(X)} b(p_\alpha, t) = b_\alpha. \tag{19}$$

We need to prove that $b_\alpha \neq 0$ if and only if $p_\alpha \in \mathcal{F}$. To do this, we first note that

$$D\tilde{u}_0(x_t) = \frac{\sum_\alpha p_\alpha e^{\langle p_\alpha, x_t\rangle}}{\sum_\beta e^{\langle p_\beta, x_t\rangle}} = \sum_\alpha b(p_\alpha, t) p_\alpha. \tag{20}$$

By Lemma 3, $D\tilde{u}_0(x_t) \to y_\infty$. So, upon letting $t \to R(X)$ in (20) and using (19), we get

$$y_\infty = \sum_\alpha b_\alpha p_\alpha.$$

By Proposition 4, $y_\infty \in \partial\triangle$ lies on the same faces as Q does, i.e. $\mathcal{F}$ is also the minimal face containing y_∞, so we must have $b_\alpha = 0$ if $p_\alpha \notin \mathcal{F}$. Therefore we only need to show that if $p_\alpha \in \mathcal{F}$, then $b_\alpha \neq 0$.

If $\dim \mathcal{F} = k$, then there exist $k+1$ vertex lattice points $\{p_1, \ldots, p_{k+1}\}$ of $\mathcal{F}$ such that the corresponding coefficients b_i, $i = 1, \ldots, k+1$, are nonzero, i.e. $\lim_{t\to R(X)} b(p_i, t) = b_i > 0$.

Remark 3. Here is why we need to assume that the p_α are all vertex lattice points.

Let p be any vertex point of $\mathcal{F}$; then

$$p = \sum_{i=1}^{k+1} c_i p_i \quad \text{where} \quad \sum_{i=1}^{k+1} c_i = 1.$$

It follows that

$$b(p,t) = \frac{e^{\langle \sum_{i=1}^{k+1} c_i p_i, x_t\rangle}}{\sum_\beta e^{\langle p_\beta, x_t\rangle}} = \prod_{i=1}^{k+1} \left(\frac{e^{\langle p_i, x_t\rangle}}{\sum_\beta e^{\langle p_\beta, x_t\rangle}}\right)^{c_i} = \prod_{i=1}^{k+1} b(p_i, t)^{c_i} \xrightarrow{t\to R(X)} \prod_{i=1}^{k+1} b_i^{c_i} > 0. \qquad \square$$

We can now state a real version of Theorem 1.2.

THEOREM 3.1. *There is a subsequence $t_i \to R(X)$ such that the $U_{t_i}(x)$ converge to a smooth entire solution of the equation*

$$\det(U_{ij}) = e^{-R(X)U(x) - (1-R(X))\tilde{U}_\infty(x) - c} \tag{$**'_\infty$}$$

on $\mathbb{R}^n$, where $c = \lim_{t_i \to R(X)} w(x_{t_i})$ is some constant.

3.2 Transformation to a complex Monge–Ampère equation

The proof of Theorem 3.1 could be done using the theory of real Monge–Ampère equations, but here we take a different approach and rewrite ($**'_t$) as a family of complex Monge–Ampère equations. This will allow us to apply some standard estimates from the theory of complex Monge–Ampère equations.

We rewrite the formula for $\tilde{U}(x)$ in (18) as

$$e^{\tilde{U}} = \frac{\sum_\alpha b(p_\alpha, t) e^{\langle p_\alpha, x\rangle}}{\sum_\beta e^{\langle p_\beta, x\rangle}} \sum_\beta e^{\langle p_\beta, x\rangle} = \frac{\sum_\alpha b(p_\alpha, t) |s_\alpha|^2}{\sum_\beta |s_\beta|^2} e^{-\tilde{C} + \tilde{u}_0} = \left(\sum_\alpha b(p_\alpha, t) \|s_\alpha\|^2\right) e^{\tilde{u}_0}, \tag{21}$$

where s_α is the holomorphic section of K_X^{-1} corresponding to the lattice point p_α. Here and in what follows, $\|\cdot\| := \|\cdot\|_{\mathrm{FS}}$ is the Fubini–Study metric on K_X^{-1}. Recall that, by (5), for any section s we have

$$\|s\|^2_{\mathrm{FS}} = e^{-\tilde{C}} \frac{|s|^2}{\sum_\beta |s_\beta|^2}.$$

The second equality in (21) holds because $e^{\langle p_\alpha, x\rangle} = |s_\alpha/\tilde{s}_0|^2$ by (6). We also used the definition of $\tilde{u}_0$ in (7).

Equation ($**'_t$) can then be rewritten as

$$\det(U_{ij}) = e^{-t\psi} e^{-\tilde{u}_0} \left(\sum_\alpha b(p_\alpha, t)\|s_\alpha\|^2\right)^{-(1-t)} e^{-w(x_t)}$$

where

$$\psi = \psi_t = U - \tilde{u}_0. \tag{22}$$

By (4) and (9), ($**'_t$) can finally be written as the complex Monge–Ampère equation

$$(\omega + \sqrt{-1}\,\partial\bar{\partial}\psi)^n = e^{-t\psi} \left(\sum_\alpha b(p_\alpha, t)\|s_\alpha\|^2\right)^{-(1-t)} e^{h_\omega - w(x_t)} \omega^n. \tag{$***_t$}$$

Similarly, for $\tilde{U}_\infty$ in (17) we write

$$e^{\tilde{U}_\infty} = \frac{\sum_\alpha {}'b_\alpha e^{\langle p_\alpha, x\rangle}}{\sum_\beta e^{\langle p_\beta, x\rangle}} \sum_\beta e^{\langle p_\beta, x\rangle} = \left(\sum_\alpha {}'b_\alpha \|s_\alpha\|^2\right) e^{\tilde{u}_0},$$

and the limit equation ($**'_\infty$) becomes:

$$(\omega + \sqrt{-1}\,\partial\bar{\partial}\psi)^n = e^{-R(X)\psi} \left(\sum_\alpha {}'b_\alpha \|s_\alpha\|^2\right)^{-(1-R(X))} e^{h_\omega - c} \omega^n. \tag{$***_\infty$}$$

Therefore, we reformulate Theorem 3.1 as Theorem 1.2 stated in the introduction.

3.3 Discussion of the conic behavior of limit metrics

For any lattice point $p_\alpha \in \triangle$, let $D_{p_\alpha} = \{s_\alpha = 0\}$ be the zero divisor of the corresponding holomorphic section s_α. By toric geometry, we have

$$D_{p_\alpha} = \{s_\alpha = 0\} = \sum_{i=1}^K (\langle p_\alpha, v_i\rangle + 1) D_i.$$

Here v_i is the primitive inward normal vector to the ith codimension-one face, and D_i is the toric divisor corresponding to this face.

Recall that $\mathcal{F}$ is the minimal face containing Q. Let $\{p_k^{\mathcal{F}}\}$ be all the vertex lattice points of $\mathcal{F}$. They correspond to a sublinear system $\mathfrak{L}_{\mathcal{F}}$ of $|K_X^{-1}|$. The base locus of $\mathfrak{L}_{\mathcal{F}}$ is given by the schematic intersection

$$\mathrm{Bs}(\mathfrak{L}_{\mathcal{F}}) = \bigcap_k D_{p_k^{\mathcal{F}}}.$$

The fixed components in $\mathrm{Bs}(\mathfrak{L}_{\mathcal{F}})$ are

$$D^{\mathcal{F}} = \sum_{i=1}^r a_i D_i, \tag{23}$$

where

$$\mathbb{N} \ni a_i = 1 + \min_k \langle p_k^{\mathcal{F}}, v_i\rangle > 0 \quad \text{for } i = 1, \ldots, r.$$

For $i = 1, \ldots, K$, we always have $a_i = 1 + \min_k \langle p_k^{\mathcal{F}}, v_i \rangle \geqslant 0$. In (23), the coefficients a_i are the ones with $a_i \neq 0$.

Pick any generic point p on $D^{\mathcal{F}}$; then p lies on only one component of $D^{\mathcal{F}}$. Without loss of generality, assume $p \in D_1$, and in a neighborhood $\mathcal{U}_p$ of p choose local coordinates $\{z_i\}$ such that D_1 is defined by $z_1 = 0$. Then the singular Monge–Ampère equation (2) locally becomes

$$(\omega + \sqrt{-1}\,\partial\bar{\partial}\psi)^n = |z_1|^{-2a_1(1-R(X))}\Omega, \tag{24}$$

with Ω being a nonvanishing smooth volume form in $\mathcal{U}_p$.

So, locally around a generic point p, we have

$$\mathrm{Ric}(\omega_\psi) = 2\pi(1 - R(X))a_1(\{z_1 = 0\}) + \mathrm{Ric}(\Omega), \tag{25}$$

where $\{z_1 = 0\}$ is the current of integration along divisor $\{z_1 = 0\}$.

Note that we have the following singular conic metric in $\mathcal{U}_p$:

$$\eta = \frac{dz_1 \wedge d\bar{z}_1}{|z_1|^{2\alpha}} + \sum_{i=2}^{n} dz_i \wedge d\bar{z}_i.$$

This metric η has conic singularity along $\{z_1 = 0\}$ with conic angle $2\pi(1-\alpha)$, and it satisfies

$$\frac{\eta^n}{n!} = \frac{dz_1 \wedge d\bar{z}_1 \wedge \cdots \wedge dz_n \wedge d\bar{z}_n}{|z_1|^{2\alpha}} \quad \text{and} \quad \mathrm{Ric}(\eta) = 2\pi\alpha(\{z_1 = 0\}).$$

Comparing this with (24) and (25), we would expect the limit Kähler metric around p to have conic singularity along D_1 with conic angle equal to $2\pi(1 - (1 - R(X))a_1)$, and we would expect the same to hold for generic points on D_i, i.e. the limit metric should have conic singularity along D_i of conic angle equal to $2\pi(1 - (1 - R(X))a_i)$.

Remark 4. At present, it seems difficult to speculate on the behavior of limit metrics around higher-codimensional strata of $D^{\mathcal{F}}$. See the discussion in Example 2. We hope to return to this problem in the future.

4. Proof of Theorem 1.2

We are now in the general setting of complex Monge–Ampère equations: ($*{*}*_\infty$) is a complex Monge–Ampère equation with poles in the right-hand side; the ($*{*}*_t$) can be seen as regularizations of ($*{*}*_\infty$). We ask whether the solutions of ($*{*}*_t$) converge to a solution of ($*{*}*_\infty$). Beginning with Yau's work [Yau78a], similar problems have been considered by many people. Owing to the considerable progress made by Kołodziej [Koł98], complex Monge–Ampère equations can be solved with very general, usually singular, right-hand sides. Kołodziej's result was also proved by first regularizing the singular Monge–Ampère equation and then taking the limit back to get a solution of the original equation.

We will derive several a priori estimates to prove Theorem 1.2. For the C^0 estimate, the upper bound follows from how we transform the potential function in (16). The lower bound follows from a Harnack estimate for the transformed potential function, which we will prove using Tian's argument from [Tia89]. For the proofs of partial C^2 estimates, higher-order estimates and convergence of solutions, we will use arguments similar to those used by Ruan and Zhang [RZ11] and Demailly and Pali [DP10].

4.1 C^0 estimate

We first derive the C^0 estimate for $\psi = U - \tilde{u}_0$. Let $\bar{v} = \bar{v}(x)$ be a piecewise linear function defined by

$$\bar{v}(x) = \max_{p_\alpha} \langle p_\alpha, x \rangle.$$

Then u_0 is asymptotic to $\bar{v}$ and it is easy to see that $|\bar{v} - \tilde{u}_0| \leqslant C$. So we only need to show that $|U(x) - \bar{v}(x)| \leqslant C$. Here and in the following, C is some constant independent of $t \in [0, R(X))$.

One side is easy. Since $DU(\mathbb{R}^n) = \triangle$ and $U(0) = 0$, for any $x \in \mathbb{R}^n$ we have that $U(x) = U(x) - U(0) = DU(\xi) \cdot x \leqslant \bar{v}(x)$ where ξ is some point between 0 and x. So

$$\psi = (U - \bar{v}) + (\bar{v} - \tilde{u}_0) \leqslant C.$$

To prove the lower bound for ψ, we only need to prove a Harnack inequality.

Proposition 6. *For ψ and t as above, we have*

$$\sup_X (-\psi) \leqslant n \sup_X \psi + C(n) t^{-1}. \tag{26}$$

For this, we use the same idea of proof as in [Tia89]. First, we rewrite ($*{*}*_t$) as

$$(\omega + \sqrt{-1}\,\partial\bar{\partial}\psi)^n = e^{-t\psi + F - B_t}\omega^n, \tag{27}$$

where

$$B_t = (1-t)\log\biggl(\sum_\alpha b(p_\alpha, t)\|s_\alpha\|^2\biggr), \quad F = h_\omega - w(x_t).$$

Now consider a new continuous family of equations

$$(\omega + \sqrt{-1}\,\partial\bar{\partial}\theta_s)^n = e^{-s\theta_s + F - B_t}\omega^n. \tag{27_s}$$

Define $S = \{s' \in [0,t] : (27_s) \text{ is solvable for } s \in [s', t]\}$. We want to prove that $S = [0,t]$. Since (27) has a solution ψ, we have that $t \in S$ and so S is nonempty. It is therefore sufficient to show that S is both open and closed.

For openness, we begin by estimating the first eigenvalue of the metric g_θ associated with the Kähler form $\omega_\theta = \omega + \sqrt{-1}\,\partial\bar{\partial}\theta$ for the solution θ of (27_s):

$$\begin{aligned}
\operatorname{Ric}(\omega_\theta) &= s\sqrt{-1}\,\partial\bar{\partial}\theta - \sqrt{-1}\,\partial\bar{\partial}F + \sqrt{-1}\,\partial\bar{\partial}B_t + \operatorname{Ric}(\omega) \\
&= s\sqrt{-1}\,\partial\bar{\partial}\theta + \omega + (1-t)(\sigma^*\omega - \omega) = s(\sqrt{-1}\,\partial\bar{\partial}\theta + \omega) + (t-s)\omega + (1-t)\sigma^*\omega \\
&= s\omega_\theta + (t-s)\omega + (1-t)\sigma^*\omega.
\end{aligned} \tag{28}$$

In particular, $\operatorname{Ric}(\omega_\theta) > s\omega_\theta$. So, by Bochner's formula, the first nonzero eigenvalue $\lambda_1(g_{\theta_s})$ is greater than s. This gives invertibility of the linearization operator $(-\Delta_s) - s$ of equation (27_s), and the openness of the solution set S follows.

To prove closedness, we need to derive an a priori estimate. First, define the functionals

$$I(\theta_s) = \frac{1}{V}\int_X \theta_s(\omega^n - \omega_{\theta_s}^n), \quad J(\theta_s) = \int_0^1 \frac{I(x\theta_s)}{x}\,dx.$$

Then we have the following estimates.

Lemma 4 [BM87, Tia89]. *For the functionals $I(\theta_s)$ and $J(\theta_s)$ introduced above, the following estimates hold:*

$$(n+1)J(\theta_s)/n \leqslant I(\theta_s) \leqslant (n+1)J(\theta_s); \tag{i, 29}$$

$$\frac{d}{ds}(I(\theta_s)-J(\theta_s)) = -\frac{1}{V}\int_X \theta_s(\Delta_s\dot{\theta}_s)\omega_{\theta_s}^n. \tag{ii}$$

Using $\lambda_1(g_{\theta_s}) > s$, Lemma 4(ii) gives the following consequence.

Lemma 5 [BM87, Tia87]. *The difference $I(\theta_s) - J(\theta_s)$ is monotonically increasing.*

Let us recall Bando and Mabuchi's estimate for Green functions.

Proposition 7 [BM87]. *For every m-dimensional compact Riemannian manifold (X, g) with $\operatorname{diam}(X, g)^2\operatorname{Ric}(g) \geqslant -(m-1)\alpha^2$, there exists a positive constant $\gamma = \gamma(m, \alpha)$ such that*

$$G_g(x, y) \geqslant -\gamma(m, \alpha)\operatorname{diam}(X, g)^2/V_g. \tag{30}$$

Here the Green function $G_g(x, y)$ is normalized to satisfy

$$\int_M G_g(x, y)\, dV_g(x) = 0.$$

Bando and Mabuchi used this estimate to prove the key estimate below.

Proposition 8 [BM87]. *Let*

$$\mathcal{H}^s = \{\theta \in C^\infty(X) : \omega_\theta = \omega + \sqrt{-1}\,\partial\bar{\partial}\theta > 0,\ \operatorname{Ric}(\omega_\theta) \geqslant s\omega_\theta\}.$$

Then, for any $\theta \in \mathcal{H}^s$, we have

$$\sup_X(-\theta) \leqslant \frac{1}{V}\int_X(-\theta)\omega_\theta^n + C(n)s^{-1}, \tag{i, 31}$$

$$\operatorname{Osc}(\theta) \leqslant I(\theta) + C(n)s^{-1}. \tag{ii, 32}$$

Proposition 9. *Equation (27_s) is solvable for $0 \leqslant s \leqslant t$.*

Proof. From (27_s), there exists $x_s \in X$ such that $-s\theta_s(x_s) + F(x_s) - B_t(x_s) = 0$, so $|\theta_s(x_s)| = |F - B_t|(x_s)/s \leqslant C_t s^{-1}$. By (32) and the fact that $I \leqslant (n+1)(I-J)$ (from (29)), we get that

$$\sup_X \theta_s \leqslant \operatorname{Osc}(\theta) + \theta(x_s) \leqslant (n+1)(I-J)(\theta) + C(n)s^{-1} + C_t s^{-1}.$$

By Lemma 5, for any $\delta > 0$ we get a uniform estimate for $\sup_X \theta_s$ and hence also for $\inf_X \theta_s$, when $s \in [\delta, t]$. So $\|\theta_s\|_{C^0} \leqslant C\delta^{-1}$. We can use Yau's estimate to get C^2 and higher-order estimates. Thus we can solve (27_s) for $s \in [\delta, t]$, for any $\delta > 0$.

On the other hand, by Yau's theorem, we can solve (27_s) for $s = 0$. Moreover, by the implicit function theorem, we can solve (27_s) for $s \in [0, \tau)$ for τ sufficiently small. We can pick δ such that $\delta < \tau$, so we get a solution of (27_s) for $s \in [\delta, \tau)$ in two ways. These two results must coincide by the recent work of Berndtsson [Ber11] on the uniqueness of solutions for the twisted Kähler–Einstein equation (28). Thus the proof is complete. □

One can then use the same argument as in [Tia87, Tia89] to prove the following proposition.

Proposition 10 [Tia89]. *For the solution θ to (27_s), we have*

$$-\frac{1}{V}\int_X \theta\omega_\theta^n \leqslant \frac{n}{V}\int_X \theta\omega^n \leqslant n\sup_X \theta. \tag{33}$$

Proof. First, by taking derivatives of equation (27_s), we get

$$\Delta_s\dot{\theta} = -\theta - s\dot{\theta}.$$

So

$$\begin{aligned}\frac{d}{ds}(I-J)(\theta_s) &= -\int_X \theta\frac{d}{ds}\omega_\theta^n = -\frac{d}{ds}\left(\int_X \theta\omega_\theta^n\right) + \int_X \dot{\theta}\omega_\theta^n \\ &= -\frac{d}{ds}\left(\int_X \theta\omega_\theta^n\right) - \frac{1}{s}\int_X \theta\omega_\theta^n = -\frac{1}{s}\frac{d}{ds}\left(s\int_X \theta\omega_\theta^n\right)\end{aligned}$$

and hence

$$\frac{d}{ds}(s(I-J)(\theta_s)) - (I-J)(\theta_s) = -\frac{d}{ds}\left(s\int_X \theta\omega_\theta^n\right). \tag{34}$$

By Proposition 9, θ_s can be solved for $s\in[0,t]$, and $\theta_t = \psi = \psi_t$, so we can integrate to get

$$t(I-J)(\psi) - \int_0^t (I-J)(\theta_s)\,ds = -t\int_X \psi\,\omega_\psi^n.$$

Divide both sides by t to get

$$(I-J)(\psi) - \frac{1}{t}\int_0^t (I-J)(\theta_s)\,ds = -\int_X \psi\,\omega_\psi^n.$$

By Lemma 4(i), we obtain

$$\frac{n}{n+1}\int_X \psi(\omega^n - \omega_\psi^n) = \frac{n}{n+1}I(\psi) \geqslant -\int_X \psi\,\omega_\psi^n.$$

Equation (33) follows from this inequality immediately. □

Combining (33) with Bando and Mabuchi's estimate (31) when $s=t$, we then prove the Harnack estimate (26). So we can derive the lower bound on ψ from the upper bound on ψ, and hence the C^0 estimate is obtained.

Remark 5. Professor Jian Song showed me that, by modifying the above argument, one can prove Harnack's inequality using a weaker statement than the one in Proposition 9, namely that (27_s) can be solved for $s\in(0,t]$. In this way, one can avoid having to use Berndtsson's uniqueness result. We present his nice argument here for comparison. First, by the concavity of the log function and using $(27)_s$, we have

$$\frac{1}{V}\left(-s\int_X \theta_s\omega^n + \int_X (F-B_t)\omega^n\right) \leqslant \log\left(\frac{1}{V}\int_X e^{-s\theta_s+F-B_t}\omega^n\right) = \log\left(\frac{1}{V}\int_X \omega_{\theta_S}^n\right) = 0.$$

So

$$-s\int_X \theta_s\omega^n \leqslant \int_X (B_t - F)\omega^n \leqslant C, \tag{35}$$

where C is a constant independent of both s and t. Now we integrate (34) from any s to t, to get

$$t(I-J)(\psi)-s(I-J)(\theta_s)-\int_s^t (I-J)(\theta_s)\,ds=-t\int_X \psi\omega_\psi^n+s\int_X \theta_s(\omega_{\theta_s}^n-\omega^n)+s\int_X \theta_s\omega^n.$$

Using positivity of $I-J$, (35), and Lemmas 4 and 5, we get

$$\begin{aligned}t(I-J)(\psi) &\geqslant -t\int_X \psi\omega_\psi^n - sI(\theta_s)-C\\ &\geqslant -t\int_X \psi\omega_\psi^n - s(n+1)(I-J)(\theta_s)-C\\ &\geqslant -t\int_X \psi\omega_\psi^n - s(n+1)(I-J)(\psi)-C.\end{aligned}$$

Now, letting $s\to 0$, we obtain

$$(I-J)(\psi)\geqslant -\int_X \psi\omega_\psi^n-\frac{C}{t},$$

and we can argue as before to get the Harnack inequality.

4.2 Partial C^2 estimate

Equation ($*_t$) is equivalent to

$$\mathrm{Ric}(\omega_\phi)=t\omega_\phi+(1-t)\omega.$$

From our transformation (16), we get

$$\mathrm{Ric}(\omega_\psi)=t\omega_\psi+(1-t)\sigma^*\omega. \tag{36}$$

In particular, $\mathrm{Ric}(\omega_\psi)>t\omega_\psi$. We will use an argument similar to that in [RZ11, proof of Lemma 5.2].

Let $f=\mathrm{tr}_{\omega_\psi}\omega$ and let Δ' be the complex Laplacian associated with the Kähler metric ω_ψ. As in [Yau78b], we can calculate

$$\Delta' f=g'^{i\bar l}g'^{k\bar j}R'_{k\bar l}g_{i\bar j}+g'^{i\bar j}g'^{k\bar l}T^\alpha_{i,k}T^{\bar\beta}_{\bar j,\bar l}g_{\alpha\bar\beta}-g'^{i\bar j}g'^{k\bar l}S_{i\bar j k\bar l}.$$

Here the tensor $T^\alpha_{i,j}=\tilde\Gamma^\alpha_{ij}-\Gamma^\alpha_{ij}$ is the difference of the Levi-Civita connections $\tilde\Gamma$ and Γ associated with g_ω and $g'=g_{\omega_\psi}$, respectively, $R'_{k\bar j}$ is the Ricci curvature of ω_ψ, and $S_{i\bar j k\bar l}$ is the curvature of the reference metric ω. Let ∇' be the gradient operator associated with g_{ω_ψ}; then

$$\begin{aligned}\Delta'\log f &= \frac{\Delta' f}{f}-\frac{|\nabla' f|^2_{\omega_\psi}}{f^2}\\ &=\frac{g'^{i\bar l}g'^{k\bar j}R'_{k\bar l}g_{i\bar j}}{f}-\frac{g'^{i\bar j}g'^{k\bar l}S_{i\bar j k\bar l}}{f}+\frac{g'^{i\bar j}g'^{k\bar l}T^\alpha_{i,k}T^{\bar\beta}_{\bar j,\bar l}g_{\alpha\bar\beta}}{f}-\frac{g'^{p\bar q}g'^{i\bar j}g'^{k\bar l}T^\alpha_{ip}T^{\bar\beta}_{\bar l\bar q}g_{\alpha\bar j}g_{k\bar\beta}}{f^2}\\ &=\frac{\sum_i\mu_i^{-2}R_{i\bar i}}{\sum_i\mu_i^{-1}}-\frac{\sum_{i,j}\mu_i^{-1}\mu_j^{-1}S_{i\bar i j\bar j}}{\sum_i\mu_i^{-1}}+\frac{\sum_{i,k,\alpha}\mu_i^{-1}\mu_k^{-1}|T^\alpha_{ik}|^2}{\sum_i\mu_i^{-1}}-\frac{\sum_p\mu_p^{-1}|\sum_i\mu_i^{-1}T^i_{ip}|^2}{(\sum_i\mu_i^{-1})^2}\\ &\geqslant t-C\sum_i\mu_i^{-1}=t-Cf.\end{aligned}\tag{37}$$

In the third equality in (37), for any fixed point $P\in X$ we choose a coordinate near P such that $g_{i\bar j}=\delta_{ij}$ and $\partial_k g_{i\bar j}=0$. We can assume that $g'=g_{\omega_\psi}$ is also diagonalized so that

$$g'_{i\bar j}=\mu_i\delta_{ij}\quad\text{with } \mu_i=1+\psi_{i\bar i}.$$

For the last inequality in (37), we used $\mathrm{Ric}(\omega_\psi) > t\omega_\psi$ and the inequality

$$\begin{aligned}\sum_p \mu_p^{-1}\Big|\sum_i \mu_i^{-1}T^i_{ip}\Big|^2 &= \sum_p \mu_p^{-1}\Big|\sum_i \mu_i^{-1/2}T^i_{ip}\mu_i^{-1/2}\Big|^2\\ &\leqslant \Big(\sum_{p,i}\mu_p^{-1}\mu_i^{-1}|T^i_{ip}|^2\Big)\Big(\sum_i \mu_i^{-1}\Big)\\ &\leqslant \Big(\sum_{p,i,\alpha}\mu_p^{-1}\mu_i^{-1}|T^\alpha_{ip}|^2\Big)\Big(\sum_i \mu_i^{-1}\Big).\end{aligned}$$

So

$$\Delta'(\log f - \lambda\psi) \geqslant t - Cf - \lambda\,\mathrm{tr}_{\omega_\psi}(\omega_\psi - \omega) = (\lambda - C)f - (\lambda n - t) = C_1 f - C_2$$

for some constants $C_1 > 0$ and $C_2 > 0$, if we choose λ to be sufficiently large. So at the maximum point P of the function $\log f - \lambda\psi$, we have

$$0 \geqslant \Delta'(\log f - \lambda\psi)(P) \geqslant C_1 f(P) - C_2.$$

Therefore,

$$f(P) = \mathrm{tr}_{\omega_\psi}(\omega)(P) \leqslant C_3.$$

So, for any point $x \in X$,

$$\mathrm{tr}_{\omega_\psi}\omega(x) \leqslant C_3 e^{\lambda(\psi(x)-\psi(P))} \leqslant C_3 e^{\lambda(\mathrm{osc}(\psi))}.$$

By the C^0 estimate for ψ, we get the estimate $\mathrm{tr}_{\omega_\psi}\omega \leqslant C_4$. Hence $\omega_\psi \geqslant C_4\omega$, i.e. $\mu_i \geqslant C_4$.

Now, by (27),

$$\prod_j \mu_j = \frac{\omega_\psi^n}{\omega^n} = e^{-t\psi+F-B}$$

with $F = h - w(x_t)$ and $B = (1-t)\log\big(\sum_\alpha b(p_\alpha, t)\|s_\alpha\|^2\big)$. So, by the C^0 estimate of ψ, we get

$$\mu_i = \frac{\prod_j \mu_j}{\prod_{j\neq i}\mu_j} \leqslant \frac{e^{-t\psi+F-B}}{C_4^{n-1}} \leqslant C_5 e^{-B}.$$

In conclusion, we have the partial C^2 estimate

$$C_4\omega \leqslant \omega_\psi \leqslant C_5 e^{-B}\omega. \tag{38}$$

Remark 6. The partial C^2 upper bound $\omega_\psi \leqslant C_5 e^{-B}\omega$ can also be proved by the maximum principle. In fact, let

$$\Lambda = \log(n + \Delta\psi) - \lambda\psi + B, \tag{39}$$

where $\Delta = \Delta_\omega$ is the complex Laplacian with respect to the reference metric ω. Then, by standard calculations as in Yau [Yau78a], we obtain

$$\begin{aligned}\Delta'\Lambda &\geqslant \Big(\inf_{i\neq j} S_{i\bar i j\bar j} + \lambda\Big)\sum_i \frac{1}{1+\psi_{i\bar i}} + \Big(\Delta F - \Delta B - t\Delta\psi - n^2\inf_{i\neq j} S_{i\bar i j\bar j}\Big)\frac{1}{n+\Delta\psi} - \lambda n + \Delta' B\\ &= \Big(\inf_{i\neq j} S_{i\bar i j\bar j} + \lambda\Big)\sum_i \frac{1}{1+\psi_{i\bar i}} + \Big(\Delta F + nt - n^2\inf_{i\neq j} S_{i\bar i j\bar j}\Big)\frac{1}{n+\Delta\psi}\\ &\quad + \sum_i B_{i\bar i}\Big(\frac{1}{1+\psi_{i\bar i}} - \frac{1}{n+\Delta\psi}\Big) - (\lambda n + t).\end{aligned} \tag{40}$$

Since $1/(n+\Delta\psi) \leqslant 1/(1+\psi_{i\bar{i}})$ for each i, we have $1/(n+\Delta\psi) \leqslant (1/n)\sum_i 1/(1+\psi_{i\bar{i}})$. So the second term on the right of (40) is bounded below by $-C_0\sum_i 1/(1+\psi_{i\bar{i}})$ for some positive constant $C_0 > 0$.

For the third term, observe from (16) and (21) that

$$\sqrt{-1}\,\partial\bar{\partial}B = (1-t)(\sigma^*\omega - \omega) \geqslant -(1-t)\omega.$$

So, since again $1/(n+\Delta\psi) \leqslant 1/(1+\psi_{i\bar{i}})$, we have that

$$B_{i\bar{i}}\left(\frac{1}{1+\psi_{i\bar{i}}} - \frac{1}{n+\Delta\psi}\right) \geqslant -(1-t)\left(\frac{1}{1+\psi_{i\bar{i}}} - \frac{1}{n+\Delta\psi}\right) \geqslant -(1-t)\frac{1}{1+\psi_{i\bar{i}}}.$$

By the above discussion, at the maximal point P_t of Λ we have

$$0 \geqslant \Delta'\Lambda \geqslant \left(\lambda + \inf_{i\neq j} S_{i\bar{i}j\bar{j}} - C_0 - (1-t)\right)\sum_i \frac{1}{1+\psi_{i\bar{i}}} - (\lambda n + t) = C_2\sum_i \frac{1}{1+\psi_{i\bar{i}}} - C_3 \tag{41}$$

for some constants $C_2 > 0$ and $C_3 > 0$, upon choosing λ sufficiently large.

Now we use the following inequality from [Yau78a]:

$$\begin{aligned}\sum_i \frac{1}{1+\psi_{i\bar{i}}} &\geqslant \left(\frac{\sum_i(1+\psi_{i\bar{i}})}{\prod_j(1+\psi_{j\bar{j}})}\right)^{1/(n-1)} = (n+\Delta\psi)^{1/(n-1)}\exp\left(\frac{B-F+t\psi}{n-1}\right)\\ &= \exp\left(\frac{\Lambda}{n-1}\right)\exp\left(\frac{-F+(t+\lambda)\psi}{n-1}\right).\end{aligned} \tag{42}$$

By (41) and (42), we get the bound

$$e^{\Lambda(P_t)} \leqslant C_4 e^{-(t+\lambda)\psi(P_t)}.$$

So for any $x \in X = X_\triangle$ we have the estimate

$$(n+\Delta\psi)e^{-\lambda\psi}e^B \leqslant e^{\Lambda(P_t)} \leqslant C_4 e^{-(t+\lambda)\psi(P_t)}.$$

Since we have a C^0 estimate for ψ, we get a partial C^2 upper estimate

$$(n+\Delta\psi)(x) \leqslant C_4 e^{-(t+\lambda)\psi(P_t)} e^{\lambda\psi(x)} e^{-B} \leqslant C_5\left(\sum_\alpha b(p_\alpha,t)\|s_\alpha\|^2\right)^{-(1-t)}. \tag{43}$$

In particular,

$$1+\psi_{i\bar{i}} \leqslant C_5 e^{-B},$$

which is same as $\omega_\psi \leqslant C_5 e^{-B}$.

4.3 Higher-order estimates and completion of the proof of Theorem 1.2

For any compact set $K \subset X\backslash D$, we first get the gradient estimate by an interpolation inequality:

$$\max_K |\nabla\psi| \leqslant C_K\left(\max_K \Delta\psi + \max_K |\psi|\right). \tag{44}$$

Next, by the complex version of Evans–Krylov theory [Tia88], we have a uniform $C_{2,K} > 0$ such that $\|\psi\|_{C^{2,\alpha}(K)} \leqslant C_{2,K}$ for some $\alpha \in (0,1)$. Now take the derivative of the equation

$$\log\det(g_{i\bar{j}} + \psi_{i\bar{j}}) = \log\det(g_{i\bar{j}}) - t\psi + F - B$$

to get

$$g'^{i\bar{j}}\psi_{i\bar{j},k} = -t\psi_k + F_k - B_k + g^{i\bar{j}}g_{i\bar{j},k} - g'^{i\bar{j}}g_{i\bar{j},k}. \tag{45}$$

By (38), (44) and the inequality $\|\psi\|_{C^{2,\alpha}(K)} \leqslant C_{2,K}$, (45) is a linear elliptic equation with C^α coefficients. By Schauder's estimate, we get $\|\psi_k\|_{C^{2,\alpha}} \leqslant C$, i.e. $\|\psi\|_{C^{3,\alpha}} \leqslant C$. Then we can iterate in (45) to get $\|\psi\|_{C^{r,\alpha}} \leqslant C$ for any $r \in \mathbb{N}$. So we see that $(\psi = \psi(t))_{t<R(X)} \subset C^\infty(X\backslash D)$ is precompact in the smooth topology.

Now we can finish the proof of Theorem 1.2 by using an argument from [DP10].

Proof of Theorem 1.2. The uniform estimate $\|\psi\|_{L^\infty}$ implies the existence of a L^1-convergent sequence $(\psi_j = \psi_{t_j})_j$, $t_j \uparrow R(X)$, with limit $\psi_\infty \in \mathcal{PSH}(\omega) \cap L^\infty(X)$. We can assume that a.e.-convergence holds too. The precompactness of the family $(\psi_j) \subset C^\infty(X\backslash D)$ in the smooth topology implies the convergence of the limits over $X\backslash D$:

$$\begin{aligned}(\omega + \sqrt{-1}\,\partial\bar\partial\psi_\infty)^n &= \lim_{t_j\to R(X)} (\omega + \sqrt{-1}\,\partial\bar\partial\psi_j)^n \\ &= \lim_{t_j\to R(X)} e^{-t_j\psi_{t_j}} \left(\sum_\alpha b(p_\alpha, t_j)\|s_\alpha\|^2\right)^{-(1-t_j)} e^{h_\omega - w(x_{t_j})}\omega^n \\ &= e^{-R(X)\psi_\infty} \left(\sum_\alpha {}'b_\alpha\|s_\alpha\|^2\right)^{-(1-R(X))} e^{h_\omega - c}\omega^n.\end{aligned}$$

The fact that ψ_∞ is a bounded potential implies that the global complex Monge–Ampère measure $(\omega + \sqrt{-1}\,\partial\bar\partial\psi_\infty)^n$ does not carry any mass on complex analytic sets. This follows from pluripotential theory [Kli91] because complex analytic sets are pluripolar. We conclude that ψ_∞ is a global bounded solution of the complex Monge–Ampère equation $(***_\infty)$, which belongs to the class $\mathcal{PSH}(\omega) \cap L^\infty(X) \cap C^\infty(X\backslash D)$. □

5. Example

Example 1. $X_\triangle = \mathrm{Bl}_p\mathbb{P}^n$. The polytope $\triangle$ is defined by

$$x_i \geqslant -1 \quad \text{for } i = 1, \dots, n, \qquad \sum_i x_i \geqslant -1 \quad \text{and} \quad -\sum_i x_i \geqslant -1.$$

Using the symmetry of the polytope, we can calculate that

$$\mathrm{Vol}(\triangle) = \frac{1}{n!}((n+1)^n - (n-1)^n),$$
$$P_c = \left(x_i = \frac{2(n-1)^n}{(n+1)((n+1)^n - (n-1)^n)}\right), \quad Q = \left(x_i = -\frac{1}{n}\right).$$

So

$$R(X_\triangle) = \frac{|\overline{OQ}|}{|\overline{P_cQ}|} = \left(1 + \frac{|\overline{OP_c}|}{|\overline{OQ}|}\right)^{-1} = \frac{(n+1)((n+1)^n - (n-1)^n)}{(n+1)^{(n+1)} + (n-1)^{(n+1)}}.$$

Here $\mathcal{F}$ is the $(n-1)$-dimensional simplex with vertices

$$P_i = (-1, \dots, \overset{i\text{th place}}{n-2}, \dots, -1), \quad i = 1, \dots, n.$$

Let e_j be the jth coordinate unit vector; then $\langle P_i, e_j\rangle = -1$ for $i \neq j$, $\langle P_i, e_i\rangle = n-2$, and $\langle P_i, \pm(1, \dots, 1)\rangle = \mp 1$. So P_i corresponds to a holomorphic section s_i with $\{s_i = 0\} = (n-1)D_i + 2D_\infty$, where D_i is the toric divisor corresponding to the codimension-one face with inward normal e_i and D_∞ is the toric divisor corresponding to the simplex face with vertices $Q_i = (-1, \dots, n, \dots, -1)$.

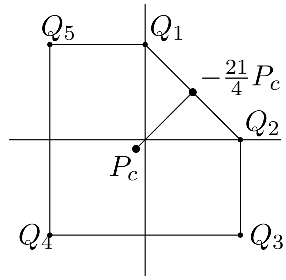

Figure 2. $\mathrm{Bl}_{p,q}\mathbb{P}^2$.

It is easy to see that $\mathrm{Bs}(\mathfrak{L}_{\mathcal{F}}) = 2D_\infty$. If we view $X_\triangle$ as the projective compactification of $\mathcal{O}(-1) \to \mathbb{P}^{n-1}$, then D_∞ is just the divisor added at infinity. So the limit metric should have conic singularity along D_∞ with conic angle

$$\theta = 2\pi \times (1 - (1 - R(X)) \times 2) = 2\pi \frac{(n+1)^{n+1} - (3n+1)(n-1)^n}{(n+1)^{n+1} + (n-1)^{n+1}}.$$

In particular, if $n = 2$, i.e. $X_\triangle = \mathrm{Bl}_p\mathbb{P}^2$, which is the case of Figure 1 in the introduction, then

$$R(X_\triangle) = \tfrac{6}{7}, \quad \theta = 2\pi \times \tfrac{5}{7}.$$

This agrees with the results of [Szé11, SZ11]. In fact, the results in [Szé11, SZ11] can easily be generalized to $\mathrm{Bl}_p\mathbb{P}^n$, giving the same results as obtained here.

Example 2. $X_\triangle = \mathrm{Bl}_{p,q}\mathbb{P}^2$, $P_c = \frac{2}{7}(-\frac{1}{3}, -\frac{1}{3})$ and $-\frac{21}{4}P_c \in \partial\triangle$, so $R(X_\triangle) = \frac{21}{25}$. The polytope determining $\mathrm{Bl}_{p,q}\mathbb{P}^2$ is depicted in Figure 2.

Here $\mathcal{F} = \overline{Q_1Q_2}$, where Q_1 corresponds to the holomorphic section s_1 with $\{s_1 = 0\} = 2D_1 + D_2$ and Q_2 corresponds to s_2 with $\{s_2 = 0\} = D_1 + 2D_2$. The fixed components in $\mathrm{Bs}(\mathfrak{L}_{\mathcal{F}})$ are $D_1 + D_2$, where D_1 and D_2 are the divisors corresponding to the faces $\overline{Q_4Q_3}$ and $\overline{Q_4Q_5}$, respectively. So at a generic point of D_1 (or D_2), the conic angle along D_1 (or D_2) should be

$$2\pi \times (1 - (1 - \tfrac{21}{25}) \times 1) = 2\pi \times \tfrac{21}{25}.$$

Around the point $p = D_1 \cap D_2$, if we choose local coordinates near p such that $D_1 = \{z_1 = 0\}$ and $D_2 = \{z_2 = 0\}$, the ideal defining the base locus would be $(z_1^2 z_2, z_1 z_2^2) = (z_1)(z_2)(z_1, z_2)$. The limit singular Monge–Ampère equation locally looks like

$$(\omega + \sqrt{-1}\,\partial\bar{\partial}\psi)^n = \frac{\Omega}{|z_1|^{2\alpha}|z_2|^{2\alpha}(|z_1|^2 + |z_2|^2)^\alpha},$$

where Ω is a nonvanishing smooth volume form near p and $\alpha = 1 - R(X) = 4/25$. The author does not yet know of a candidate singular Kähler metric as the local model. See Remark 4.

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