On rotationally symmetric Kähler-Ricci solitons

Chi Li

ABSTRACT: In this note, using Calabi's method, we construct rotationally symmetric Kähler-Ricci solitons on the total space of direct sum of fixed hermitian line bundle and its projective compactification, where the curvature of hermitian line bundle is Kähler-Einstein. These examples generalize the construction of Koiso, Cao and Feldman-Ilmanen-Knopf.

1 A little motivation

In [1], the authors constructed some examples of gradient Kähler-Ricci soliton. Among them is the shrinking soliton on the $Bl_0\mathbb{C}^m$. They also glue this to an expanding soliton on $\mathbb{C}^m$ to extend the Ricci flow across singular time.

Recently, La Nave and Tian [7] studied the formation of singularity along Kähler-Ricci flow by symplectic quotient. The idea is explained by the following example.

Let $\mathbb{C}^*$ act on $\mathbb{C}^{m+n}$ by

$$t\cdot(x_1,\cdots,x_m,y_1,\cdots,y_n)=(t\,x_1,\cdots,t\,x_m,t^{-1}\,y_1,\cdots,t^{-1}\,y_n)$$

$S^1\subset\mathbb{C}^*$ preserves the standard Kähler structure on $\mathbb{C}^{m+n}$:

$$\omega=\sqrt{-1}\Big(\sum_{i=1}^m dx_i\wedge d\bar{x}_i+\sum_{\alpha=1}^n dy_\alpha\wedge d\bar{y}_\alpha\Big)$$

Let $z=(x,y)=(x_1,\cdots,x_m,y_1,\cdots,y_n)$. The momentum map of this Hamiltonian action is

$$m(z)=\sum_{i=1}^m|x_i|^2-\sum_{\alpha=1}^n|y_\alpha|^2=|x|^2-|y|^2$$

The topology of symplectic quotient $X_a=m^{-1}(a)/S^1$ changes as a across 0.

Let $\mathcal{O}_{\mathbb{P}^N}(-1)$ be the tautological line bundle on the complex projective space $\mathbb{P}^N$. We will use $Y_{N,R}$ to represent the total space of holomorphic vector bundle $(\mathcal{O}_{\mathbb{P}^N}(-1)^{\oplus R}\to\mathbb{P}^N)$.

1. (a>0) $\forall z=(x,y)\in X_a$, $m(z)=|x|^2-|y|^2=a>0$, so $x\neq 0$.

   $X_a\simeq Y_{m-1,n}\simeq\{\mathbb{C}^{m+n}-\{x=0\}\}/\mathbb{C}^*$. The isomorphism is given by

   $$(x_1,\cdots,x_m,y_1,\cdots,y_n)\mapsto([x_1,\cdots,x_m],y_1\cdot x,\cdots,y_n\cdot x)$$

   There is an induced Kähler metric on X_a. Choose a coordinate chart $u_1=\frac{x_2}{x_1},\cdots,u_{m-1}=\frac{x_m}{x_1},\xi_1=x_1y_1,\cdots,\xi_n=x_1y_n$. The C^* action is then trivialized to: $(x_1,u,\xi)\mapsto(tx_1,u,\xi)$. The Kähler potential can be obtained by some Legendre transformation (see [2]). Specifically, the potential for the standard flat Kähler metric on $\{\mathbb{C}^{m+n}-\{x=0\}\}$ is

   $$\phi=|x|^2+|y|^2=|x_1|^2(1+|u|^2)+\frac{|\xi|^2}{|x_1|^2}=e^{r_1}(1+|u|^2)+e^{-r_1}|\xi|^2$$

$$\begin{tikzcd} & Bl_{\mathbb{P}^{m-1}}Y_{m-1,n} = Bl_{\mathbb{P}^{n-1}}Y_{n-1,m} \arrow[dl] \arrow[dd] \arrow[dr] & \\ Y_{m-1,n} \arrow[dr] & & Y_{n-1,m} \arrow[dl] \\ & C(\mathbb{P}^{m-1}\times\mathbb{P}^{n-1}) & \end{tikzcd}$$

where $r_1 = \log|x_1|^2$. ϕ is a convex function of r_1. $a = \frac{\partial\phi}{\partial r_1}$ is the momentum map of the S^1 action. In the induced coordinate chart (u, ξ), the Kähler potential of the induced metric on the symplectic quotient is the Legendre transform of ϕ with respect to r_1:

$$\Phi_a = a\log(1+|u|^2) + \sqrt{a^2+4(1+|u|^2)|\xi|^2} - a\log(a+\sqrt{a^2+4(1+|u|^2|)|\xi|^2}) + (\log 2)\,a \quad (1)$$

2. (a<0) By symmetry, $X_a \simeq Y_{n-1,m} \simeq \{\mathbb{C}^{m+n} - \{y=0\}\}/\mathbb{C}^*$. Choose a coordinate chart $v_1 = \frac{y_2}{y_1}, \cdots, v_{n-1} = \frac{y_n}{y_1}, \eta_1 = y_1x_1, \cdots, \eta_m = y_1x_m$. The Kähler potential has the same expression as (1) but replacing a by $-a$, u by v, and ξ by η.

3. (a=0) $X_a \cong$ affine cone over the Segre embedding of $\mathbb{P}^{m-1}\times\mathbb{P}^{n-1} \hookrightarrow \mathbb{P}^{mn-1}$:

$$(x_1, \cdots, x_m, y_1, \cdots, y_n) \mapsto \{x_iy_\alpha\}$$

Away from the vertex of the affine cone, choose a coordinate chart: $u_1 = \frac{x_2}{x_1}, \cdots, u_{m-1} = \frac{x_m}{x_1}, v_1 = \frac{y_2}{y_1}, \cdots, v_{n-1} = \frac{y_n}{y_1}$, $\zeta = x_1y_1$. The Kähler potential is given by

$$\Phi_0 = 2\sqrt{(1+|u|^2)(1+|v|^2)|\zeta|^2}$$

Note that Φ_0 is obtained from Φ_a by coordinate change $\xi_1 = \zeta, \xi_2 = v_1\zeta, \cdots, \xi_n = v_{n-1}\zeta$, and let a tend to 0.

This is a simple example of flip when $m \neq n$, or flop when $m = n$, in the setting of symplectic geometry. $X_{<0}$ is obtained from $X_{>0}$ by first blowing up the zero section $\mathbb{P}^{m-1}$, and then blowing down the exceptional divisor $E \cong \mathbb{P}^{m-1}\times\mathbb{P}^{n-1}$ to $\mathbb{P}^{n-1}$. Note that when $n = 1$, this process is just blow-down of exceptional divisor in $Bl_0\mathbb{C}^m$.

One hopes to have a Kähler metric on a larger manifold $\mathcal{M}$ such that induced Kähler metrics on symplectic quotient would satisfy the Kähler-Ricci flow equation as the image of momentum varies. See [7] for details.

Our goal to construct a Kähler-Ricci soliton on $Y = Y_{m-1,n}$ and its projective compactification, and this generalizes constructions of [6], [5] [1]. The construction follows these previous constructions closely, but we need to modify them to fit our setting. The higher dimensional analogs have the new phenomenon of contracting higher codimension subvariety to highly singular point. To continue the flow, surgery are needed. The surgeries in these cases should be the naturally appearing flips.

The organization of this note is as follows. In section 2, we put the construction in a more general setting where the base manifold is Kähler-Einstein, and state the main results: Theorem 2.1 and Theorem 2.2. In section 3, by the rotational symmetry, we reduce the Kähler-Ricci soliton equation to an ODE. In section 4, we analyze the condition in order for the general solution of the ODE to give a smooth Kähler metric near zero section. In section 5.1, we get the condition for the metric to be complete near infinity. In section 5.2, we prove theorem 2.1, i.e. construct

Kähler-Ricci solitons in the noncompact complete Kähler manifold and study its behavior as time approaches the singular time. Finally in section 6, we prove theorem 2.2 by constructing the compact shrinking soliton on projective compactification.

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2 General setup and the result

Let M be a Kähler manifold of **dimension d**. Kähler-Ricci soliton on M is a Kähler metric ω satisfying the equation

$$Ric(\omega) = \lambda\omega + L_V\omega \tag{2}$$

where V is a holomorphic vector field. The Kähler-Ricci soliton is called gradient if $V = \nabla f$ for some potential function f. If $\sigma(t)$ is the 1-parameter family of automorphisms generated by V, then

$$\omega(t) = (1 - \lambda t)\sigma\left(-\frac{1}{\lambda}\log(1 - \lambda t)\right)^* \omega \tag{3}$$

is a solution of Kähler-Ricci flow equation:

$$\frac{\partial\omega(t)}{\partial t} = -Ric(\omega(t))$$

We will construct gradient Kähler-Ricci solitons on the total space of special vector bundle $L^{\oplus n} \to M$ and its projective compactification $\mathbb{P}(\mathbb{C} \oplus L^{\oplus n}) = \mathbb{P}(L^{-1} \oplus \mathbb{C}^{\oplus n})$. Here M is a Kähler-Einstein manifold:

$$Ric(\omega_M) = \tau\omega_M$$

L has an Hermitian metric h, such that

$$c_1(L, h) = -\sqrt{-1}\partial\bar{\partial}\log h = -\epsilon\omega_M$$

In the following, we always consider the case $\epsilon \geq 0$.

We consider the Kähler metric of the form considered by Calabi [3]:

$$\omega = \pi^*\omega_M + \partial\bar{\partial}P(s) \tag{4}$$

Here s is the norm square of vectors in L. Under local trivialization of holomorphic local section e_L,

$$s(\xi e_L) = a(z)|\xi|^2,\ \xi = (\xi^1, \cdots, \xi^n)$$

P is a smooth function of s we are seeking for.

Using the form (4), we can determine λ immediately. Let M be the zero section. By adjoint formula,

$$-K_Y|_M = -K_M + \wedge^n N_M|_M = -K_M + nL$$

Note that $\omega|_M = \omega_M$, so by restricting both sides of (2) to M, and then taking cohomology, we see that

$$\tau[\omega_M] - n\epsilon[\omega_M] = c_1(Y)|_M = \lambda[\omega_M] \tag{5}$$

So $\lambda = \tau - n\epsilon$.

Remark 1. *If we rescale the Kähler-Einstein metric: $\omega_M \to \kappa\omega_M$, then $\tau \to \tau/\kappa$, $\epsilon \to \epsilon/\kappa$, $\lambda \to \lambda/\kappa$.*

The main theorem is

Theorem 2.1. *On the total space of $L^{\oplus n}$, there exist rotationally symmetric solitons of types depending on the sign of $\lambda = \tau - n\epsilon$. If $\lambda > 0$, there exists a unique shrinking soliton. If $\lambda = 0$, there exists a family of steady solitons. If $\lambda < 0$, there exists a family of expanding solitons. (The solitons are rotationally symmetric in the sense that it's of the form of (4))*

Remark 2. *If we take $M = \mathbb{P}^{m-1}$, $L = \mathcal{O}(-1)$, $\omega_M = \omega_{FS}$, then $\tau = m$, $\epsilon = 1$. Then we get to the situation in section 1. So depending on the sign of $\lambda = m - n$, there exist either a unique rotationally symmetric shrinking KR soliton when $m > n$, or a family of rotationally symmetric steady KR solitons when m=n, or a family of rotationally symmetric expanding KR solitons when $m < n$.*

We also have the compact shrinking soliton:

Theorem 2.2. *Using the above notation, assume $\lambda = \tau - n\epsilon > 0$, then on the space $\mathbb{P}(\mathbb{C} \oplus L^{\oplus n}) = \mathbb{P}(L^{-1} \oplus \mathbb{C}^{\oplus n})$, there exists a unique shrinking Kähler-Ricci soliton.*

3 Reduction to ODE

The construction of solitons is straightforward by reducing the soliton equation to an ODE.

First, in local coordinates, (4) is expressed as

$$\omega = (1 + \epsilon P_s s)\omega_M + a(P_s \delta_{\alpha\beta} + P_{ss} a \overline{\xi^\alpha} \xi^\beta) \nabla \xi^\alpha \wedge \overline{\nabla \xi^\beta} \tag{6}$$

Here

$$\nabla \xi^\alpha = d\xi^\alpha + a^{-1} \partial a \, \xi^\alpha$$

Note that $\{dz^i, \nabla \xi^\alpha\}$ are dual to the basis consisting of horizontal and vertical vectors:

$$\nabla_{z^i} = \frac{\partial}{\partial z^i} - a^{-1} \frac{\partial a}{\partial z^i} \sum_\alpha \xi^\alpha \frac{\partial}{\partial \xi^\alpha}, \quad \frac{\partial}{\partial \xi^\alpha}$$

ω is positive if and only if

$$1 + \epsilon P_s s > 0, \; P_s > 0, \text{ and } P_s + P_{ss} s > 0 \tag{7}$$

$$\omega^{d+n} = (1 + \epsilon P_s s)^d \omega_M^d a^n P_s^{n-1} (P_s + P_{ss} s) \prod_{\alpha=1}^n d\xi^\alpha \wedge d\bar{\xi}^\alpha$$

Since we assume $Ric(\omega_M) = \tau \omega_M = (\lambda + n\epsilon)\omega_M$,

$$\partial\bar{\partial} \log \det \omega^{d+n} + \lambda(\omega_M + \partial\bar{\partial} P) = \partial\bar{\partial} \left[d \cdot \log(1 + \epsilon P_s s) + (n-1) \log P_s + \log((P_s s)_s) + (\tau - n\epsilon) P \right]$$

Let $r = \log s$, then $\partial_r = s\partial_s$. Define

$$\begin{aligned} Q \quad &:= \quad d \cdot \log(1 + \epsilon P_s s) + (n-1) \log P_s + \log((P_s s)_s) + (\tau - n\epsilon) P \\ &= \quad d \cdot \log(1 + \epsilon P_r) + (n-1) \log P_r + \log P_{rr} - nr + (\tau - n\epsilon) P \end{aligned} \tag{8}$$

To construct a gradient Kähler-Ricci soliton (2), it is sufficient to require that $Q(t)$ is a potential function for the holomorphic vector field $-V$. Notice that, for the radial holomorphic vector field:

$$V_{rad} = \sum_{\alpha=1}^n \xi^\alpha \frac{\partial}{\partial \xi^\alpha} \tag{9}$$

$$i_{V_{rad}}\omega = (P_s + P_{ss}s)a\sum_{\beta}\xi^{\beta}\overline{\nabla\xi^{\beta}} = (P_s s)_s\bar{\partial}s$$

Now

$$-i_V\omega = \bar{\partial}Q(s) = Q_s\bar{\partial}s = \frac{Q_s}{(P_s s)_s}i_{V_{rad}}\omega$$

which means $-V = \frac{Q_s}{(P_s s)_s}V_{rad}$, so $\frac{Q_s}{(P_s s)_s}$ is a holomorphic function. Since $s = a(z)|\xi|^2$ is not holomorphic, $\frac{Q_s}{(P_s s)_s}$ has to be a constant μ. We assume $\mu \neq 0$, since $V \neq 0$. So we get the equation: $Q_s = \mu(P_s s)_s$. Multiplying by s on both sides, this is equivalent to

$$Q_r = \mu P_{rr} \tag{10}$$

Remark 3. *Note that, if $\mu = 0$, then we go back to Calabi's construction of Kähler-Einstein metrics in [3].*

Define $\phi(r) = P_r(r)$ and substitute (8) into (10), then we get

$$d\frac{\epsilon\phi_r}{1+\epsilon\phi} + (n-1)\frac{\phi_r}{\phi} + \frac{\phi_{rr}}{\phi_r} + (\tau - n\epsilon)\phi - n = \mu\phi_r \tag{11}$$

Since $\phi_r = P_{rr} = (P_s s)_s s = (P_s + P_{ss}s)s > 0$ by (7), we can solve r as a function of ϕ: $r = r(\phi)$. Define $F(\phi) = \phi_r(r(\phi))$, then $F'(\phi) = \phi_{rr}r'(\phi) = \frac{\phi_{rr}}{\phi_r}$. So the above equation change into an ODE

$$F'(\phi) + d\frac{\epsilon F(\phi)}{1+\epsilon\phi} + (n-1)\frac{F(\phi)}{\phi} - \mu F(\phi) = n - (\tau - n\epsilon)\phi = n(1+\epsilon\phi) - \tau\phi \tag{$*$}$$

Remark 4. *We will explain how this equation is related to the ODE in [1],(25). In our notation, in [1], $M = \mathbb{P}^d$, $L = O_{\mathbb{P}^d}(-k)$, $n = 1$. For the shrinking soliton case, $d+1-k > 0$, $\omega_M = (d+1-k)\omega_{FS}$, $\tau = \frac{d+1}{d+1-k}$, $\epsilon = \frac{k}{d+1-k}$, $\lambda = \tau - \epsilon = 1$. Let $r = k\tilde{r}$, $P(r) = \tilde{P}(\tilde{r}) - (d+1-k)\tilde{r} = \tilde{P}(\frac{r}{k}) - \frac{d+1-k}{k}r$, $\phi(r) = P_r(r) = \tilde{P}_{\tilde{r}}(\tilde{r})\frac{1}{k} - \frac{d+1-k}{k} = \frac{1}{k}(\tilde{\phi}(\tilde{r}) - (d+1-k))$, $F(\phi) = \phi_r(r(\phi)) = \frac{1}{k^2}\tilde{\phi}_{\tilde{r}}(\tilde{r}(\tilde{\phi})) = \frac{1}{k^2}\tilde{F}(\tilde{\phi})$, $F'_\phi(\phi) = \frac{1}{k}\tilde{F}'_{\tilde{\phi}}(\tilde{\phi})$. Substitute these expressions into ($*$), then we get the ODE*

$$\tilde{F}'_{\tilde{\phi}} + \left(\frac{d}{\tilde{\phi}} - \frac{\mu}{k}\right)\tilde{F} - ((d+1) - \tilde{\phi}) = 0$$

So we see this is exactly the ODE in [1], (25). The expanding soliton case is the similar.

We can solve ($*$) by multiplying the integral factor: $(1+\epsilon\phi)^d\phi^{n-1}e^{-\mu\phi}$:

$$\phi_r = F(\phi) = \nu(1+\epsilon\phi)^{-d}\phi^{1-n}e^{\mu\phi} - (1+\epsilon\phi)^{-d}\phi^{1-n}e^{\mu\phi}\int h(\phi)e^{-\mu\phi}d\phi \tag{12}$$

where

$$h(\phi) = \tau(1+\epsilon\phi)^d\phi^n - n(1+\epsilon\phi)^{d+1}\phi^{n-1} \tag{13}$$

is a polynomial of ϕ with degree $d+n$. Note the identity:

$$\int h(\phi)e^{-\mu\phi}d\phi = -\sum_{k=0}^{+\infty}\frac{1}{\mu^{k+1}}h^{(k)}(\phi)e^{-\mu\phi}$$

Since $h(\phi)$ is a polynomial of degree d+n, the above sum is a finite sum. So

$$F(\phi) = (1+\epsilon\phi)^{-d}\phi^{1-n}\left(\nu e^{\mu\phi} + \sum_{k=0}^{d+n}\frac{1}{\mu^{k+1}}h^{(k)}(\phi)\right) \tag{14}$$

4 Boundary condition at zero section

Since $\lim_{r\to-\infty}\phi(r)=\lim_{s\to 0}P_s s=0$, we have the boundary condition

$$\lim_{\phi\to 0}F(\phi)=\lim_{r\to-\infty}\phi_r=\lim_{s\to 0}(P_s s)_s s=0 \tag{15}$$

So $\phi^{n-1}(1+\epsilon\phi)^d F(\phi)=O(\phi^n)$. Now the l-th term of Taylor expansion of $\phi^{n-1}(1+\epsilon\phi)^d F(\phi)$ at $\phi=0$ is

$$(\phi^{n-1}(1+\epsilon\phi)^d F(\phi))^{(l)}\Big|_{\phi=0}=\nu\mu^l+\sum_{k=0}^{+\infty}\frac{1}{\mu^{k+1}}h^{(k+l)}(0)=\mu^l\left(\nu+\sum_{k=l}^{d+n}\frac{1}{\mu^{k+1}}h^{(k)}(0)\right) \tag{16}$$

Note that by (13) $h^{(k)}(\phi)=0$ for $k>d+n$, and $h^{(k)}(0)=0$ for $k<n-1$. The vanishing of the 0-th(constant) term in expansion gives the equation:

$$\nu+\sum_{k=n-1}^{d+n}\frac{1}{\mu^{k+1}}h^{(k)}(0)=0 \tag{17}$$

Using relation (17) we see that, when $l<n$, $(\phi^{n-1}(1+\epsilon\phi)^d F(\phi))^{(l)}\big|_{\phi=0}=0$, and

$$(\phi^{n-1}(1+\epsilon\phi)^d F(\phi))^{(n)}\Big|_{\phi=0}=\mu^n\left(\nu+\sum_{k=n}^{d+n}\frac{1}{\mu^{k+1}}h^{(k)}(0)\right)=-h^{(n-1)}(0)=n!>0$$

So we see that (15) and (17) are equivalent, and if they are satisfied,

$$\phi^{n-1}(1+\epsilon\phi)^d F(\phi)=\phi^n+O(\phi^{n+1}),\ \text{or } F(\phi)=\phi+O(\phi^2)$$

So $F(\phi)>0$ for ϕ near 0.

We can rewrite the relation (17) more explicitly:

$$\begin{aligned}\nu &= \sum_{k=0}^{+\infty}\frac{1}{\mu^{k+1}}\left[n((1+\epsilon\phi)^{d+1}\phi^{n-1})^{(k)}-\tau((1+\epsilon\phi)^d\phi^n)^{(k)}\right]\Bigg|_{\phi=0}\\ &= \sum_{k=n-1}^{d+n}\frac{1}{\mu^{k+1}}\left(n\binom{k}{n-1}(n-1)!\frac{(d+1)!}{(d+n-k)!}\epsilon^{k-n+1}-\tau\binom{k}{n}n!\frac{d!}{(d+n-k)!}\epsilon^{k-n}\right)\\ &= \sum_{k=n-1}^{d+n}C_k\frac{1}{\mu^{k+1}}\epsilon^{k-n}\end{aligned} \tag{18}$$

Here

$$C_k=\frac{k!d!}{(k-n+1)!(d+n-k)!}(n\epsilon(d+1)-\tau(k-n+1)) \tag{19}$$

$$C_{n-1}=n!\epsilon,\quad C_{d+n}=-(d+n)!(\tau-n\epsilon)$$

So, when k starts from $n-1$ to $d+n$, C_k change signs from positive to negative if and only if $\lambda=\tau-n\epsilon>0$. We need the following simple lemma later.

Lemma 1. *Let $P(x)=\sum_{i=0}^{l}a_ix^i-\sum_{j=l+1}^{N}a_jx^j$ be a polynomial function. Assume $a_i>0$ for $0\le a_i\le N$. Then there exists a unique root for $P(x)$ on $[0,\infty)$.*

Proof. First $P(0)=a_0>0$. Since $a_N<0$, when x is large enough $P(x)<0$. So there exists at least one root on $[0,\infty)$. Assume there are more than one root, than it's easy to see that $P'(x)$ has at least two roots on $[0,\infty)$. Note that $P'(x)$ has the same form as $P(x)$, so $P''(x)$ has at least two roots on $[0,\infty)$. By induction, $P^{(l)}(x)$ has at least two roots on $[0,\infty)$, but $P^{(l)}(x)$ has only negative coefficients, so it has no root at all. This contradiction proves the lemma. □

5 Complete noncompact case

We prove theorem 2.1 in this section.

5.1 Condition at infinity

As $\phi \to +\infty$,

$$F(\phi) = \nu(1+\epsilon\phi)^{-d}\phi^{1-n}e^{\mu\phi} + \frac{\tau - n\epsilon}{\mu}\phi + O(1) \tag{20}$$

Let $\phi = b_1 > 0$ be the first positive root for $F(\phi) = 0$, then $F'(b_1) \le 0$. By $(*)$, $F'(b_1) = n - (\tau - n\epsilon)b_1$. So if $\lambda = \tau - n\epsilon \le 0$, there exits no such b_1. If $\lambda = \tau - n\epsilon > 0$, we integrate (12) to get

$$r = r(\phi) = \int_{\phi_0}^{\phi} \frac{1}{F(u)} du + r(\phi_0) \tag{21}$$

then the metric is defined for $-\infty < r < r(b_1)$.

We can also calculate the length of radial curve extending to infinity. In a fixed fibre, the radial vector

$$\frac{\partial}{\partial r} = \frac{1}{2}|\xi|\frac{\partial}{\partial|\xi|} = \frac{1}{2}\sum_{\alpha=1}^{n}\xi^{\alpha}\frac{\partial}{\partial\xi^{\alpha}} = \frac{1}{2}V_{rad}$$

$$\left|\frac{\partial}{\partial r}\right|^2 = \frac{1}{4}g_{\omega}(V_{rad}, V_{rad}) = C(P_s s + P_{ss}s^2) = C\phi_r$$

The completeness implies that the length of the radial curve extending to infinity is infinity:

$$\int_{-\infty}^{r(b_1)} \sqrt{\phi_r}dr = \int_0^{b_1} \sqrt{\phi_r}\phi_r^{-1}d\phi = \int_0^{b_1} \phi_r^{-\frac{1}{2}}d\phi = \int_0^{b_1} F(\phi)^{-\frac{1}{2}}d\phi = +\infty \tag{22}$$

If $0 < b_1 < +\infty$,(22) means $F(\phi) = c(\phi - b_1)^2 + O((\phi - b_1)^3)$, i.e. $F'(b_1) = F(b_1) = 0$.

But this can't happen: $b_1 = \frac{n}{\tau - n\epsilon}$, and $c = -(\tau - n\epsilon)$. First we have $b_1 > 0$. Second, $c \ge 0$ since $F(\phi) > 0$ when $\phi < b_1$. But they contradict with each other.

In conclusion, there can't be any finite value positive root for $F(\phi)$.

5.2 Existence and asymptotics

1. ($\lambda = \tau - n\epsilon > 0$) The solution is a shrinking Kähler-Ricci soliton. If $\mu < 0$, then when ϕ becomes large, the dominant term in $F(\phi)$(20) is $\frac{\lambda}{\mu}\phi < 0$, so there exists $0 < b < +\infty$ such that $F(b) = 0$. But this is excluded by former discusions. So we must have $\mu > 0$.

   If $\nu < 0$ the dominant term is $\nu\phi^{1-n}(1+\epsilon\phi)^{-m}e^{\mu\phi} < 0$, so there is $0 < b < +\infty$, such that $F(b) = 0$. Again, this is impossible. If $\nu > 0$, when ϕ becomes large, the dominant term is $\nu\phi^{1-d-n}e^{\mu\phi}$,

   $$\int_{\phi_0}^{+\infty} F(s)^{-\frac{1}{2}}ds \le C\int_{\phi_0}^{+\infty} \nu^{-\frac{1}{2}}\phi^{\frac{d+n-1}{2}}e^{-\frac{\mu}{2}\phi}d\phi < +\infty$$

   This contradicts (22). So we must have $\nu = 0$. This gives us an equation for μ via (18). Since when $\lambda = \tau - n\epsilon > 0$, C_k change signs exactly once, by lemma 1, there exists a unique μ such that $\nu(\mu) = 0$ in (18).

   We now verify this μ guarantees the positivity of ϕ_r. Since the dominant term in (20) is $\frac{\lambda}{\mu}\phi > 0$, $F(\phi) \overset{\phi\to+\infty}{\longrightarrow} +\infty$. We have also $F(\phi) > 0$ for ϕ near 0. If $\phi = b_1 > 0$ is the first root and $\phi = b_2 > 0$ is the last root of $F(\phi)$, then $b_1 \le b_2$, and

   $$F'(b_1) = -\lambda b_1 + n \le 0, \quad F(b_2) = -\lambda b_2 + n \ge 0$$

So $b_1 \geq \frac{n}{\lambda} \geq b_2$, this implies $b_1 = b_2$ and $F'(b_1) = 0$. We have ruled out this possibility before. In conclusion, $F(\phi) > 0$ for all $\phi > 0$, or equivalently $\phi_r > 0$ for all r.

Now in (20), $\phi = \frac{\tau - n\epsilon}{\mu}\phi + O(1)$ as $\phi \to +\infty$, so by (21), the maximum value of r defined for the solution is

$$r_{max} = \int_0^{+\infty} \frac{1}{F(u)} du = +\infty \tag{23}$$

So we already get the soliton on the whole manifold. In the following, we study the limit of flow as time approaches singularity time.

Define $p = \frac{\lambda}{\mu} = \frac{\tau - n\epsilon}{\mu}$,

$$r(\phi) - r(\phi_0) = \int_{\phi_0}^{\phi} \frac{du}{F(u)} = \int_{\phi_0}^{\phi} \frac{du}{pu} + \int_{\phi_0}^{\phi} \frac{pu - F(u)}{puF(u)} ds = \frac{1}{p}(\log\phi - \log\phi_0) + G(\phi_0, \phi)$$

$$\phi(r) = \phi_0 e^{-pr(\phi_0)} e^{-G(\phi_0,\phi)} e^{pr} = \phi_0 e^{-pr(\phi_0)} e^{-G(\phi_0,\phi(r))} s^p \tag{24}$$

The holomorphic vector field $-\frac{V}{2} = \frac{\mu}{2}\sum_\alpha \xi^\alpha \frac{\partial}{\partial \xi^\alpha}$ generates the 1-parameter family of automorphisms: $\sigma(\tilde{t}) \cdot (u, \xi) = (u, e^{\frac{\tilde{t}\mu}{2}}\xi)$. Let

$$\tilde{t}(t) = -\frac{1}{\lambda}\log(1 - \lambda t) \tag{25}$$

$$\begin{aligned} \lim_{t \to \frac{1}{\lambda}} (1 - \lambda t)\sigma(\tilde{t})^*\phi &= \phi_0 e^{-pr(\phi_0)} e^{-\lim_{\phi \to +\infty} G(\phi_0,\phi)} \lim_{t \to \frac{1}{\lambda}} (1 - \lambda t)((1 - \lambda t)^{-\frac{\mu}{\lambda}} s)^p \\ &= \phi_0 e^{-pr(\phi_0)} e^{-G(\phi_0,+\infty)} s^p = D_0 s^p \end{aligned}$$

So

$$\lim_{t \to \frac{1}{\lambda}} (1 - \lambda t)\sigma(\tilde{t})^*\phi_r = \lim_{t \to \frac{1}{\lambda}} (1 - \lambda t)\sigma(\tilde{t})^* F(\phi) = pD_0 s^p$$

(6) can be rewritten as

$$\omega = (1 + \epsilon\phi)\pi^*\omega_{FS} + (\phi|\xi|^{-2}\delta_{\alpha\beta} + (\phi_r - \phi)|\xi|^{-4}\overline{\xi^\alpha}\xi^\beta)\nabla\xi^\alpha \wedge \overline{\nabla\xi^\beta} \tag{26}$$

$$\begin{aligned} \lim_{t \to \frac{1}{\lambda}} (1 - \lambda t)\sigma(\tilde{t})^*\omega &= D_0 \left[s^p \epsilon\omega_M + s^p(|\xi|^{-2}\delta_{\alpha\beta} + (p - 1)|\xi|^{-4}\overline{\xi^\alpha}\xi^\beta)\nabla\xi^\alpha \wedge \overline{\nabla\xi^\beta} \right] \\ &= D_0 \partial\bar{\partial}\left(\frac{1}{p}s^p\right) \end{aligned}$$

Remark 5. *One sees that as $t \to \frac{1}{\lambda}$, the flow shrinks the base (zero section of the vector bundle). In the model case, $M = \mathbb{P}^{m-1}$, $L = \mathcal{O}(-1)$, the flow contracts the manifold to the affine cone of the Segre embedding $\mathbb{P}^{m-1} \times \mathbb{P}^{n-1} \hookrightarrow \mathbb{P}^{mn-1}$. This is the same phenomenon as that appears for the symplectic quotients at the beginning of this note.*

2. ($\lambda = \tau - n\epsilon = 0$) the solution is a steady Kähler-Ricci soliton.

$$F(\phi) = \nu(1 + \epsilon\phi)^{-d}\phi^{1-n}e^{\mu\phi} - n(1 + \epsilon\phi)^{-d}\phi^{1-n}\sum_{k=0}^{d+n-1}\frac{1}{\mu^{k+1}}\left((1 + \epsilon\phi)^d\phi^{n-1}\right)^{(k)}$$

If $\mu > 0$, then $v(\mu) > 0$ by (18). So the dominant term in (20) is $\nu(1 + \epsilon\phi)^{-d}\phi^{1-n}e^{\mu\phi}$, again this would contradict (22).

So $\mu < 0$ and the dominant term in (14) is the constant term $-\frac{n}{\mu} > 0$. As $\phi \to +\infty$,

$$F(\phi) = -\frac{n}{\mu} - \frac{n(d+n-1)}{\mu^2}\frac{1}{\phi} + O\left(\frac{1}{\phi^2}\right) = c_1 - c_2\frac{1}{\phi} + O\left(\frac{1}{\phi^2}\right)$$

So by (21), $r_{max} = +\infty$ and the soliton is defined on the whole manifold.

For the asymptotic behavior, let

$$\int \frac{du}{c_1 - \frac{c_2}{u}} = \frac{1}{c_1}u + \frac{c_2}{c_1^2}\log(c_1 u - c_2) = R(u)$$

$$\begin{aligned} r(\phi) - r(\phi_0) &= \int_{\phi_0}^{\phi} \frac{du}{F(u)} = \int_{\phi_0}^{\phi} \frac{du}{c_1 - \frac{c_2}{u}} + \int_{\phi_0}^{\phi} \left(\frac{1}{F(u)} - \frac{1}{c_1 - \frac{c_2}{u}}\right) du \\ &= R(\phi) - R(\phi_0) + G(\phi_0, \phi) \end{aligned}$$

Since $c_1 > 0$ and $c_2 > 0$ ($\mu < 0$, $d > 0$, $n \geq 1$), $R(u)$ is an increasing function for $u \gg 0$, and has an inverse function denoted by R^{-1}. Let $\tilde{G}(r) = -G(\phi_0, \phi(r)) + R(\phi_0) - r(\phi_0)$, then $\tilde{G}$ is a bounded smooth function of r. We have

$$\phi(r) = R^{-1}(r + \tilde{G}(r))$$

The condition (22) is always satisfied. There is a family of steady Kähler-Ricci solitons.

Remark 6. *If we let $d = 0$, then we get expanding solitons on $\mathbb{C}^n$. (17) becomes $\nu\mu^n = n!$. The equation becomes*

$$(|\mu|\phi)_r = (-1)^n n!(|\mu|\phi)^{1-n}e^{-|\mu|\phi} + n!\sum_{k=0}^{n-1}\frac{(-1)^k}{(n-1-k)!(|\mu|\phi)^k}$$

In particular, if $n = 1$, the equation becomes

$$\phi_r = F(\phi) = \nu e^{\mu\phi} - \frac{1}{\mu}$$

$$\phi(r) = -\frac{1}{\mu}\log(\mu\nu + Ce^r) = -\frac{1}{\mu}\log(1 + C|z|^2), \quad \phi_r = -\frac{C|z|^2}{\mu(1+C|z|^2)}$$

$$\omega = \frac{\phi_r}{|z|^2}dz \wedge d\bar{z} = -\frac{Cdz \wedge d\bar{z}}{\mu(1 + C|z|^2)} \overset{w=\sqrt{C}z}{=} \frac{1}{-\mu}\frac{dw \wedge d\bar{w}}{1 + |w|^2}$$

This is cigar steady soliton.

3. ($\lambda = \tau - n\epsilon < 0$) the solution is an expanding Kähler-Ricci soliton. By similar argument, we see that $\mu < 0$. The situation is similar to the shrinking soliton case. Now $t \to \frac{1}{\lambda} < 0$, or equivalently $\tilde{t} \to -\infty$ (25),

$$\phi(r) = \phi_0 e^{-pr(\phi_0)}e^{-G(\phi_0,\phi(r))}s^p$$

$$\lim_{t\to\frac{1}{\lambda}}(1 - \lambda t)\sigma(\tilde{t})^*\phi_r = \lim_{t\to\frac{1}{\lambda}}(1 - \lambda t)\sigma(\tilde{t})^*F(\phi) = pD_0 s^p$$

$$\lim_{t\to\frac{1}{\lambda}}(1 - \lambda t)\sigma(\tilde{t})^*\omega = D_0\partial\bar{\partial}\left(\frac{1}{p}s^p\right)$$

The condition (22) is always satisfied. So there is a family of expanding Kähler-Ricci solitons.

Remark 7. *One can apply the same argument in [1] to get the Gromov-Hausdorff convergence and continuation of flow through singularity time.*

6 Compact shrinking soliton

We prove theorem 2.2 in this section. First we show the considered manifold is Fano. For some results on Fano manifolds with a structure of projective space bundle, see [9].

Lemma 2. *If $\lambda = \tau - n\epsilon > 0$, then $\mathbb{P}(\mathbb{C} \oplus L^{\oplus n})$ is Fano.*

Proof. Let $E = \mathbb{C} \oplus L^{\oplus n}$ and $X = \mathbb{P}(E)$. We have the formula for anti-canonical bundle:

$$K_X^{-1} = (n+1)\mathcal{O}(1) + \pi^*(K_M^{-1} + L^{\otimes n})$$

$\mathcal{O}(1)$ is the relative hyperplane bundle. Since $c_1(L^{-1}) = \epsilon[\omega_M] \geq 0$, one can prove $\mathcal{O}(1)$ is nef on X [8]. $c_1(K_M^{-1} + L^{\otimes n}) = (\tau - n\epsilon)[\omega_M] > 0$, so $K_M^{-1} + L^{\otimes n}$ is an ample line bundle on M. So $\mathcal{O}(1)$ and $\pi^*(K_M^{-1} + L^{\otimes n})$ are different rays of the cone of numerically effective divisors in $Pic(\mathbb{P}(E)) = \mathbb{Z}Pic(M) + \mathbb{Z}\mathcal{O}(1)$. So K_X^{-1} is ample, i.e. X is Fano. □

The construction of shrinking soliton was developed for the n=1 case, see [4], [6], [5], [1]. We will give a simple direct argument under our setting. Note here we will use Tian-Zhu's theory [11] to get the uniqueness of Kähler-Ricci soliton.

First we need to know the expression for the metric near infinity. By change of coordinate

$$[1, \xi_1, \xi_2, \cdots, \xi_n] = [\eta, 1, u_2, \cdots, u_n]$$

So the coordinate change is given by

$$\xi_1 = \frac{1}{\eta}, \xi_2 = \frac{u_2}{\eta}, \cdots, \xi_n = \frac{u_n}{\eta} \quad \Longleftrightarrow \quad \eta = \frac{1}{\xi_1}, u_2 = \frac{\xi_2}{\xi_1}, \cdots, u_n = \frac{\xi_n}{\xi_1}$$

Since

$$\xi_1 \frac{\partial}{\partial \xi_1} = -\eta \frac{\partial}{\partial \eta} - \sum_{i=2}^{n} u_\alpha \frac{\partial}{\partial u_\alpha}, \quad \xi_\alpha \frac{\partial}{\partial \xi_\alpha} = u_\alpha \frac{\partial}{\partial u_\alpha}$$

So the radial vector $\sum_{i=\alpha}^n \xi_\alpha \frac{\partial}{\partial \xi_\alpha} = -\eta \frac{\partial}{\partial \eta}$ is a holomorphic vector field on $\mathbb{P}(\mathbb{C} \oplus L^{\oplus n})$. The dual 1-forms transform into

$$\nabla \xi^1 = -\frac{1}{\eta^2}(d\eta - \eta a^{-1}\partial a) = -\frac{1}{\eta^2}\omega_0, \quad \nabla \xi^\alpha = \frac{du_\alpha}{\eta} - \frac{u_\alpha}{\eta^2}(d\eta - \eta a^{-1}\partial a) = \frac{du_\alpha}{\eta} - \frac{u_\alpha}{\eta^2}\omega_0$$

Note that the dual basis for the basis $\{dz_i, \omega_0, du_\alpha\}$ is

$$\nabla_{z_i} = \frac{\partial}{\partial z_i} + a^{-1}\frac{\partial a}{\partial z_i}\eta \frac{\partial}{\partial \eta}, \quad \frac{\partial}{\partial \eta}, \quad \frac{\partial}{\partial u_\alpha}$$

So making coordinate change,

$$\begin{aligned}
\omega \quad = \quad & (1 + \epsilon P_s s)\omega_M + \sum_{\alpha=2}^{n}\sum_{\beta=2}^{n} a\left(P_s \delta_{\alpha\beta} + P_{ss} a \frac{\bar{u}_\alpha u_\beta}{|\eta|^2}\right)\left(\frac{1}{\eta}du_\alpha - \frac{u_\alpha}{\eta^2}\omega_0\right) \wedge \left(\frac{1}{\bar{\eta}}d\bar{u}_\beta - \frac{\bar{u}_\beta}{\bar{\eta}^2}\overline{\omega_0}\right) \\
& - \sum_{\beta=2}^{n} P_{ss} a^2 \frac{u_\beta}{|\eta|^2}\frac{\omega_0}{\eta^2} \wedge \left(\frac{1}{\bar{\eta}}d\bar{u}_\beta - \frac{\bar{u}_\beta}{\bar{\eta}^2}\overline{\omega_0}\right) - \sum_{\alpha=2}^{n} P_{ss} a^2 \frac{\bar{u}_\alpha}{|\eta|^2}\left(\frac{1}{\eta}du_\alpha - \frac{u_\alpha}{\eta^2}\omega_0\right) \wedge \frac{\overline{\omega_0}}{\bar{\eta}^2} \\
& + a(P_s + P_{ss} a \frac{1}{|\eta|^2})\frac{\omega_0 \wedge \overline{\omega_0}}{|\eta|^4} \\
= \quad & (1 + \epsilon P_s s)\omega_M + \sum_{\alpha=2}^{n}\sum_{\beta=2}^{n}(P_s \delta_{\alpha\beta} + P_{ss} s \frac{\bar{u}_\alpha u_\beta}{1 + |u|^2})\frac{s}{1 + |u|^2} du_\alpha \wedge d\bar{u}_\beta \\
& - \sum_{\alpha=2}^{n} \frac{\bar{u}_\alpha \eta}{a(1 + |u|^2)^2}(P_s + P_{ss}s)s^2 du_\alpha \wedge \overline{\omega_0} - \sum_{\beta=2}^{n} \frac{u_\beta \bar{\eta}}{a(1 + |u|^2)^2}(P_s + P_{ss}s)s^2 \omega_0 \wedge d\bar{u}_\beta \qquad (27) \\
& + \frac{1}{a(1 + |u|^2)}(P_s + P_{ss}s)s^2 \omega_0 \wedge \overline{\omega_0}
\end{aligned}$$

In the above calculation, we used many times the relation $s = a|\xi|^2 = \frac{a(1+|u|^2)}{|\eta|^2}$.

Lemma 3. *The closing condition for compact shrinking soliton is: there exists a $b_1 > 0$, such that*

$$F(b_1) = 0, F'(b_1) = -1 \tag{28}$$

Proof. Define $\tilde{s} = s^{-1} = \frac{|\eta|^2}{a(1+|u|^2)}$. Under the condition (28), then near b_1, $\phi_r = F(\phi) = -(\phi - b_1) + O((\phi - b_1)^2)$. So up to the main term, $\phi - b_1 \sim -C_0 e^{-r} = -C_0 \frac{1}{s} = -C_0 \tilde{s}$ for some $C_0 > 0$, $(P_s s)_s s^2 = \phi_s s^2 \sim C_0$, $P_{ss} s^2 = (P_s s)_s s - P_s s = \phi_s s - \phi \sim -b_1 + \frac{2C_0}{s} = -b_1 + 2C_0 \tilde{s}$, . So we first see that the coefficients in (27) are smooth near infinity divisor defined by $\eta = 0$ (or equivalently $\tilde{s} = 0$). We only need to show ω is positive definite everywhere. In fact, we only need to check when $\tilde{s} = 0$. When $\tilde{s} = 0$, we have

$$\omega = (1 + \epsilon b_1)\omega_M + \sum_{\alpha=2}^{n} \sum_{\beta=2}^{n} (b_1 \delta_{\alpha\beta} - b_1 \frac{\bar{u}_\alpha u_\beta}{1 + |u|^2}) \frac{1}{1 + |u|^2} du_\alpha \wedge d\bar{u}_\beta + \frac{C_0}{a(1 + |u|^2)} \omega_0 \wedge \overline{\omega_0} \tag{29}$$

So ω is positive definite. So it defines a smooth Kähler metric on the projective compactification. $\square$

By $(*)$, this condition determines

$$b_1 = \frac{n+1}{\tau - n\epsilon} = \frac{n+1}{\lambda} \tag{30}$$

In fact, b_1 is a cohomological constant. Indeed.

$$[\omega] = \frac{1}{\lambda} c_1(X) = [\omega_M] + \frac{n+1}{\lambda} \mathcal{O}_X(1)$$

Note that $D_\infty = M \times \mathbb{P}^n$, and

$$[\omega]|_{D_\infty} = [\omega_M] + \frac{n+1}{\lambda}(-L + \mathcal{O}_{\mathbb{P}^n}(1))$$

On the otherhand, by (29), we see that

$$[\omega]|_{D_\infty} = (1 + \epsilon b_1)[\omega_M] + C_0[\mathcal{O}_{\mathbb{P}^n}(1)]$$

Comparing the above expression and using $c_1(L) = -\epsilon[\omega_M]$, we get (30).

Since $F(0) = 0$, this condition is equivalent to

$$0 = F(b_1) - F(0) = \int_0^{b_1} h(\phi) e^{-\mu\phi} d\phi = T(\mu) \tag{31}$$

$$\begin{aligned}
T(0) &= \int_0^{b_1} h(\phi) d\phi = \int_0^{b_1} (1 + \epsilon\phi)^d \phi^{n-1}((\tau - n\epsilon)\phi - 1) \\
&= \int_0^{b_1} \sum_{k=0}^{d} \binom{d}{k} \epsilon^k (\lambda \phi^{k+n} - \phi^{k+n-1}) d\phi \\
&= \sum_{k=0}^{d} \binom{d}{k} \epsilon^k b_1^{k+n} \left(\frac{\lambda b_1}{k+n+1} - \frac{n}{k+n} \right) \\
&= \sum_{k=0}^{d} \binom{d}{k} \epsilon^k b_1^{k+n} \frac{k}{(k+n+1)(k+n)} > 0
\end{aligned}$$

On the other hand,

$$T(\mu) = \frac{1}{\mu^{d+n+1}} \sum_{k=0}^{d+n} \mu^{d+n-k} (h^{(k)}(0) - h^{(k)}(b_1) e^{-\mu b_1})$$

Since $h^{(i)}(0) = 0$ for $0 \le i \le n-2$ and $h^{(n-1)}(0) = -n! < 0$, and $\lim_{\mu \to +\infty} e^{-\mu b_1} = 0$. It's easy to see that $T(\mu) < 0$ for μ sufficiently large. So there is a zero point for $T(\mu)$ on $(0, \infty)$. The uniqueness is difficult to see directly, but because different solutions of (31) would give proportional vector fields and hence proportional potential functions, by using Tian-Zhu's invariant [11], we indeed have the uniqueness.

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Department of Mathematics, Princeton University, Princeton, NJ 08544, USA
E-mail address: chil@math.princeton.edu