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Greatest lower bounds on Ricci curvature for toric Fano manifolds

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Abstract

In this short note, based on the work of Wang and Zhu (2004) [8], we determine the greatest lower bounds on Ricci curvature for all toric Fano manifolds.

Keywords: Toric Fano manifold; Ricci curvature

1. Introduction

On Fano manifolds X, i.e. K_X^{-1} is ample, the Kähler–Einstein equation

$$Ric(\omega) = \omega$$

is equivalent to the complex Monge–Ampère equation:

$$(\omega + \partial\bar{\partial}\phi)^n = e^{h_\omega - \phi}\omega \tag{$*$}$$

where ω is a fixed Kähler metric in $c_1(X)$, and h_ω is the normalized Ricci potential:

$$Ric(\omega) - \omega = \partial\bar{\partial}h_\omega, \qquad \int_X e^{h_\omega}\omega^n = \int_X \omega^n \tag{1}$$

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In order to solve this equation, the continuity method is used. So we consider a family of equations with parameter t:

$$(\omega + \partial\bar{\partial}\phi_t)^n = e^{h_\omega - t\phi}\omega^n \qquad (*)_t$$

Define $S_t = \{t\colon (*)_t \text{ is solvable}\}$. It was known that the set S_t is open. To solve $(*)$, the crucial thing is to obtain the closedness of this set. So we need some a priori estimates. By Yau's C^2 and Calabi's higher order estimates (see [9,7]), we only need uniform C^0-estimates for solutions ϕ_t of $(*)_t$. In general one cannot solve $(*)$, and so cannot get the C^0-estimates, due to the well-known obstruction of Futaki invariant [2].

It was first showed by Tian [6] that we may not be able to solve $(*)_t$ on certain Fano manifold for t sufficiently close to 1. Equivalently, for such a Fano manifold, there is some $t_0 < 1$, such that there is no Kähler metric ω in $c_1(X)$ which can have $Ric(\omega) \geqslant t_0\omega$. It is now made more precise.

Define

$$R(X) = \sup\bigl\{t\colon (*)_t \text{ is solvable}\bigr\}$$

Recently, Székelyhidi proved

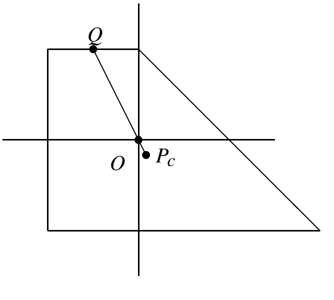
Proposition 1. *(See [4].)*

$$R(X) = \sup\bigl\{t\colon \exists \text{ a Kähler metric } \omega \in c_1(X) \text{ such that } Ric(\omega) > t\omega\bigr\}$$

In particular, $R(X)$ is independent of $\omega \in c_1(X)$.

Let $Bl_{p_1,\dots,p_k}\mathbb{P}^n$ denote the manifold obtained by blowing up $\mathbb{P}^n$ at points $p_1,\dots,p_k$. Székelyhidi showed in [4] that $R(Bl_p\mathbb{P}^2) = \frac{6}{7}$ and $\frac{1}{2} \leqslant R(Bl_{p,q}\mathbb{P}^2) \leqslant \frac{21}{25}$.

Let $\Lambda \simeq \mathbb{Z}^n$ be a lattice in $\mathbb{R}^n = \Lambda \otimes_{\mathbb{Z}} \mathbb{R}$. A toric Fano manifold $X_\triangle$ is determined by a reflexive lattice polytope $\triangle$ (for details on toric manifolds, see [3]). For example, $Bl_p\mathbb{P}^2$ is a toric Fano manifold and is determined by the following polytope.



In this short note, we determine $R(X_\triangle)$ for every toric Fano manifold $X_\triangle$ in terms of the geometry of polytope $\triangle$.

Any such polytope $\triangle$ contains the origin $O \in \mathbb{R}^n$. We denote the barycenter of $\triangle$ by P_c. If $P_c \neq O$, the ray $P_c + \mathbb{R}_{\geqslant 0} \cdot \overrightarrow{P_c O}$ intersects the boundary $\partial\triangle$ at point Q. Our main result is

Theorem 1. *If $P_c \neq O$,*

$$R(X_\triangle) = \frac{|\overline{OQ}|}{|\overline{P_cQ}|}$$

Here $|\overline{OQ}|$, $|\overline{P_cQ}|$ are lengths of line segments $\overline{OQ}$ and $\overline{P_cQ}$. If $P_c = O$, then there is Kähler–Einstein metric on $X_\triangle$ and $R(X_\triangle) = 1$.

Remark 1. Note for the toric Fano manifold, P_c is just Futaki invariant (see [8]). So the second statement follows from Wang and Zhu [8]. We will repeat the proof in the next section.

Our method is based on Wang–Zhu's [8] theory for proving the existence of Kähler–Ricci solitons on toric Fano manifolds. In view of the analysis in [8], if $R(X_\triangle) < 1$, then as $t \to R(X_\triangle)$, the blow-up happens exactly because the minimal points of a family of proper convex functions go to infinity, or, equivalently, the images of minimal points under the momentum map of a fixed metric tend to the boundary of the toric polytope. The key identity relation in Section 2, (11) and some uniform a priori estimates enable us to read out $R(X_\triangle)$ in terms of geometry of $\triangle$.

This note is inspired by the Székelyhidi's paper [4] and Donaldson's survey [1]. The author thanks Professor Gang Tian for constant encouragement.

2. Consequence of Wang–Zhu's theory

First we recall the set-up of Wang and Zhu [8]. For a reflexive lattice polytope $\triangle$ in $\mathbb{R}^n = \Lambda \otimes_{\mathbb{Z}} \mathbb{R}$, we have a Fano toric manifold $(\mathbb{C}^*)^n \subset X_\triangle$ with a $(\mathbb{C}^*)^n$ action. Let $\{z_i\}$ be the standard coordinates of the dense orbit $(\mathbb{C}^*)^n$, and $x_i = \log|z_i|^2$. Let $\{p_\alpha\}_{\alpha=1,\ldots,N}$ be the lattice points contained in $\triangle$. We take the fixed Kähler metric ω to be given by the potential (on $(\mathbb{C}^*)^n$)

$$\tilde{u}_0 = \log\left(\sum_{\alpha=1}^{N} e^{\langle p_\alpha, x\rangle}\right) + C \tag{2}$$

C is some constant determined by normalization condition:

$$\int_{\mathbb{R}^n} e^{-\tilde{u}_0}\, dx = Vol(\triangle) = \frac{1}{n!}\int_{X_\triangle} \omega^n = \frac{c_1(X_\triangle)^n}{n!} \tag{3}$$

By standard toric geometry, each lattice point p_α contained in $\triangle$ determines, up to a constant, a $(\mathbb{C}^*)^n$-equivariant section s_α in $H^0(X, K_X^{-1})$. We can embed $X_\triangle$ into $P(H^0(X, K_X^{-1})^*)$ using these sections. Let $\tilde{s}_0$ be the section corresponding to the origin $0 \in \triangle$, then its Fubini–Study norm is

$$|\tilde{s}_0|^2_{FS} = \frac{|\tilde{s}_0|^2}{\sum_{\alpha=1}^{N} |s_\alpha|^2} = \left(\sum_{\alpha=1}^{N}\prod_{i=1}^{n} |z_i|^{2p_{\alpha,i}}\right)^{-1} = \left(\sum_{\alpha=1}^{N} e^{\langle p_\alpha, x\rangle}\right)^{-1} = e^C e^{-\tilde{u}_0}$$

So the Kähler metric $\omega = \frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}\tilde{u}_0$ is the pull back of Fubini–Study metric.

On the other hand, $Ric(\omega)$ is the curvature of Hermitian line bundle K_M^{-1} with Hermitian metric determined by the volume form ω^n. Note that on the open dense orbit $(\mathbb{C}^*)^n$, we can take $\tilde{s}_0 = z_1\frac{\partial}{\partial z_1}\wedge\cdots\wedge z_n\frac{\partial}{\partial z_n}$. Since $\frac{\partial}{\partial \log z_i} = \frac{1}{2}(\frac{\partial}{\partial \log|z_i|} - \sqrt{-1}\frac{\partial}{\partial\theta_i}) = \frac{\partial}{\partial \log|z_i|^2} = \frac{\partial}{\partial x_i}$ when acting on any $(S^1)^n$ invariant function on $(\mathbb{C}^*)^n$, we have

$$|\tilde{s}_0|^2_{\omega^n} = \left|z_1\frac{\partial}{\partial z_1}\wedge\cdots\wedge z_n\frac{\partial}{\partial z_n}\right|^2_{\omega^n} = \det\left(\frac{\partial^2\tilde{u}_0}{\partial\log z_i\,\partial\overline{\log z_j}}\right)$$
$$= \det\left(\frac{\partial^2\tilde{u}_0}{\partial\log|z_i|^2\partial\log|z_j|^2}\right) = \det(\tilde{u}_{0,ij})$$

It's easy to see from definition of h_ω (1) and normalization condition (3) that

$$e^{h_\omega} = e^{-C}\frac{|\tilde{s}_0|^2_{FS}}{|\tilde{s}_0|^2_{\omega^n}} = e^{-\tilde{u}_0}\det(\tilde{u}_{0,ij})^{-1}$$

Then using the torus symmetry, $(*)_t$ can be translated into real Monge–Ampère equation [8] on $\mathbb{R}^n$:

$$\det(u_{ij}) = e^{-(1-t)\tilde{u}_0 - tu} = e^{-w_t} \qquad (**)_t$$

Here and in the following we denote

$$w_t = (1-t)\tilde{u}_0 + tu$$

The solution u_t of $(**)_t$ is related to Kähler potential ϕ_t in $(*)_t$ by the identity:

$$u = \tilde{u}_0 + \phi_t \qquad (4)$$

where ϕ_t is viewed as a function of $x_i = \log|z_i|^2$ by torus symmetry.

Every strictly convex function f appearing in $(**)_t$ ($f = \tilde{u}_0$, u, $w_t = (1-t)\tilde{u}_0 + tu_t$) must satisfy $Df(\mathbb{R}^n) = \triangle^\circ$ ($\triangle^\circ$ means the interior of $\triangle$). Since 0 is (the unique lattice point) contained in $\triangle^\circ = Df(\mathbb{R}^n)$, the strictly convex function f is proper.

Wang–Zhu's [8] method for solving $(**)_t$ consists of two steps. The *first step* is to show some uniform a priori estimates for w_t. For $t < R(X_\triangle)$, the proper convex function w_t obtains its minimum value at a unique point $x_t \in \mathbb{R}^n$. Let

$$m_t = \inf\{w_t(x):\ x \in \mathbb{R}^n\} = w_t(x_t)$$

Proposition 2. *(See [8], see also [1].)*

1. *There exists a constant C, independent of $t < R(X_\triangle)$, such that*

$$|m_t| < C$$

2. *There exist $\kappa > 0$ and a constant C, both independent of $t < R(X_\triangle)$, such that*

$$w_t \geqslant \kappa |x - x_t| - C \tag{5}$$

For the reader's convenience, we record the proof here.

Proof. Let $A = \{x \in \mathbb{R}^n;\ m_t \leqslant w(x) \leqslant m_t + 1\}$. A is a convex set. By a well-known lemma due to Fritz John, there are a unique ellipsoid E of minimum volume among all the ellipsoids containing A, and a constant α_n depending only on dimension, such that

$$\alpha_n E \subset A \subset E$$

$\alpha_n E$ means the α_n-dilation of E with respect to its center. Let T be an affine transformation with $\det(T) = 1$, which leaves $x' =$ the center of E invariant, such that $T(E) = B(x', R)$, where $B(x', R)$ is the Euclidean ball of radius R. Then

$$B\bigl(x', \alpha_n R\bigr) \subset T(A) \subset B\bigl(x', R\bigr)$$

We first need to bound R in terms of m_t. Since $D^2 w = tD^2 u + (1-t)D^2\tilde{u}_0 \geqslant tD^2 u$, by $(**)_t$, we see that

$$\det(w_{ij}) \geqslant t^n e^{-w}$$

Restrict to the subset A, it's easy to get

$$\det(w_{ij}) \geqslant C_1 e^{-m_t}$$

Let $\tilde{w}(x) = w(T^{-1}x)$, since $\det(T) = 1$, $\tilde{w}$ satisfies the same inequality

$$\det(\tilde{w}_{ij}) \geqslant C_1 e^{-m_t}$$

in $T(A)$.

Construct an auxiliary function

$$v(x) = C_1^{\frac{1}{n}} e^{-\frac{m_t}{n}} \frac{1}{2}\bigl(|x - x'|^2 - (\alpha_n R)^2\bigr) + m_t + 1$$

Then in $B(x', \alpha_n R)$,

$$\det(v_{ij}) = C_1 e^{-m_t} \leqslant \det(\tilde{w}_{ij})$$

On the boundary $\partial B(x', \alpha_n R)$, $v(x) = m_t + 1 \geqslant \tilde{w}$. By the Comparison Principle for Monge–Ampère operator, we have

$$\tilde{w}(x) \leqslant v(x) \quad \text{in } B\bigl(x', \alpha_n R\bigr)$$

In particular

$$m_t \leqslant \tilde{w}(x') \leqslant v(x') = C_1^{\frac{1}{n}} e^{-\frac{m_t}{n}} \frac{1}{2}\left(-\frac{R^2}{n^2}\right) + m_t + 1$$

So we get the bound for R:

$$R \leqslant C_2 e^{\frac{m_t}{2n}}$$

So we get the upper bound for the volume of A:

$$Vol(A) = Vol(T(A)) \leqslant C R^n \leqslant C e^{\frac{m_t}{2}}$$

By the convexity of w, it's easy to see that $\{x;\ w(x) \leqslant m_t + s\} \subset s \cdot \{x;\ w(x) \leqslant m_t + 1\} = s \cdot A$, where $s \cdot A$ is the s-dilation of A with respect to point x_t. So

$$Vol(\{x;\ w(x) \leqslant m_t + s\}) \leqslant s^n Vol(A) \leqslant C s^n e^{\frac{m_t}{2}} \tag{6}$$

The lower bound for volume of sublevel sets is easier to get. Indeed, since $|Dw(x)| \leqslant L$, where $L = \max_{y \in \triangle} |y|$, we have $B(x_t, s \cdot L^{-1}) \subset \{x;\ w(x) \leqslant m_t + s\}$. So

$$Vol(\{x;\ w(x) \leqslant m_t + s\}) \geqslant C s^n \tag{7}$$

Now we can derive the estimate for m_t. First note the identity:

$$\int_{\mathbb{R}^n} e^{-w}\, dx = \int_{\mathbb{R}^n} \det(u_{ij})\, dx = \int_{\triangle} d\sigma = Vol(\triangle) \tag{8}$$

Second, we use the co-area formula

$$\begin{aligned} \int_{\mathbb{R}^n} e^{-w}\, dx &= \int_{\mathbb{R}^n} \int_{w}^{+\infty} e^{-s}\, ds\, dx = \int_{-\infty}^{+\infty} e^{-s}\, ds \int_{\mathbb{R}^n} 1_{\{w \leqslant s\}}\, dx \\ &= \int_{m_t}^{+\infty} e^{-s}\, Vol(\{w \leqslant s\})\, ds \\ &= e^{-m_t} \int_0^{+\infty} e^{-s}\, Vol(\{w \leqslant m_t + s\})\, ds \end{aligned} \tag{9}$$

Using the bound for the volume of sublevel sets (6) and (7) in (9), and comparing with (8), it's easy to get the bound for $|m_t|$.

Now we prove the estimate (5) following the argument of [1]. We have seen $B(x_t, L^{-1}) \subset \{w \leqslant m_t + 1\}$, and $Vol(\{w \leqslant m_t + 1\}) \leqslant C$ by (6) and uniform bound for m_t. Then we must have $\{w \leqslant m_t + 1\} \subset B(x_t, R(C, L))$ for some uniformly bounded radius $R(C, L)$. Otherwise, the convex set $\{w \leqslant m_t + 1\}$ would contain a convex subset of arbitrarily large volume. By the convexity of w, we have $w(x) \geqslant \frac{1}{R(C,L)}|x - x_t| + m_t - 1$. Since m_t is uniformly bounded, the estimate (5) follows. □

The *second step* is trying to bound $|x_t|$. In Wang–Zhu's [8] paper, they proved the existence of Kähler–Ricci soliton on toric Fano manifold by solving the real Monge–Ampère equation corresponding to Kähler–Ricci soliton equation. But now we only consider the Kähler–Einstein equation, which in general can't be solved because there is the obstruction of Futaki invariant.

Proposition 3. *(See [8].) The uniform bound of $|x_t|$ for any $0 \leqslant t \leqslant t_0$, is equivalent to that we can solve $(**)_t$, or equivalently solve $(*)_t$, for t up to t_0. More precisely (by the discussion in introduction), this condition is equivalent to the uniform C^0-estimates for the solution ϕ_t in $(*)_t$ for $t \in [0, t_0]$.*

Again we sketch the proof here.

Proof. If we can solve $(**)_t$ (or equivalently $(*)_t$) for $0 \leqslant t \leqslant t_0$, then $\{w(t) = (1-t)\tilde{u}_0 + tu;\ 0 \leqslant t \leqslant t_0\}$ is a smooth family of proper convex functions on $\mathbb{R}^n$. By implicit function theorem, the minimal point x_t depends smoothly on t. So $\{x_t\}$ are uniformly bounded in a compact set.

Conversely, assume $|x_t|$ is bounded. First note that $\phi_t = u - \tilde{u}_0 = \frac{1}{t}(w_t(x) - \tilde{u}_0)$.

As in Wang and Zhu [8], we consider the enveloping function:

$$v(x) = \max_{p_\alpha \in \Lambda \cap \triangle} \langle p_\alpha, x\rangle$$

Then $0 \leqslant \tilde{u}_0(x) - v(x) \leqslant C$, and $Dw(\xi) \cdot x \leqslant v(x)$ for all $\xi, x \in \mathbb{R}^n$. We can assume $t \geqslant \delta > 0$. Then using uniform boundedness of $|x_t|$

$$\begin{aligned}\phi_t(x) &= \frac{1}{t}\big(w_t(x) - \tilde{u}_0\big) = \frac{1}{t}\big[\big(w_t(x) - w_t(x_t)\big) - v(x) + \big(v(x) - \tilde{u}_0(x)\big) + w_t(x_t)\big] \\ &\leqslant \delta^{-1}\big(Dw_t(\xi) \cdot x - v(x) - Dw_t(\xi) \cdot x_t\big) + C \leqslant C'\end{aligned}$$

Thus we get the estimate for $\sup_t \phi_t$. Then one can get the bound for $\inf_t \phi_t$ using the Harnack inequality in the theory of Monge–Ampère equations. For details see [8, Lemma 3.5] (see also [5]). □

By the above proposition, we have

Lemma 2. *If $R(X_\triangle) < 1$, then there exists a subsequence $\{x_{t_i}\}$ of $\{x_t\}$, such that*

$$\lim_{t_i \to R(X_\triangle)} |x_{t_i}| = +\infty$$

The observation now is that

Lemma 3. *If $R(X_\triangle) < 1$, then there exist a subsequence of $\{x_{t_i}\}$ which we still denote by $\{x_{t_i}\}$, and $y_\infty \in \partial\triangle$, such that*

$$\lim_{t_i \to R(X_\triangle)} D\tilde{u}_0(x_{t_i}) = y_\infty \tag{10}$$

This follows easily from the properness of $\tilde{u}_0$ and compactness of $\triangle$.

We now use the key relation (see [8, Lemma 3.3], and also [1, p. 29])

$$0 = \int_{\mathbb{R}^n} Dw(x)e^{-w}\,dx = \int_{\mathbb{R}^n} \big((1-t)D\tilde{u}_0 + tDu\big)e^{-w}\,dx$$

Since

$$\int_{\mathbb{R}^n} Due^{-w}\,dx = \int_{\mathbb{R}^n} Du\det(u_{ij})\,dx = \int_{\triangle} y\,d\sigma = Vol(\triangle)P_c$$

where P_c is the barycenter of $\triangle$, so

$$\frac{1}{Vol(\triangle)}\int_{\mathbb{R}^n} D\tilde{u}_0 e^{-w}\,dx = -\frac{t}{1-t}P_c \tag{11}$$

The idea to determine $R(X_\triangle)$ is as follows. First by strictly linear growth of w_t obtained in Proposition 2(2), the left-hand side of (11) is roughly $D\tilde{u}_0(x_t)$. By properness of $\tilde{u}_0$, as long as this is bounded away from the boundary of the polytope, we can control the point x_t. So as t goes to $R(X_\triangle)$, since x_t goes to infinity in $\mathbb{R}^n$, the left-hand side goes to a point on $\partial\triangle$, which is roughly y_∞. We will prove a precise statement in the next section by using the defining function of $\triangle$. Some similar argument was given in the survey [1, p. 30].

3. Proof of Theorem 1

We now assume the reflexive polytope $\triangle$ is defined by inequalities:

$$\lambda_r(y) \geqslant -1, \quad r = 1, \ldots, K \tag{12}$$

$\lambda_r(y) = \langle v_r, y\rangle$ are fixed linear functions. We also identify the minimal face of $\triangle$ where y_∞ lies:

$$\begin{aligned} &\lambda_r(y_\infty) = -1, \quad r = 1, \ldots, K_0 \\ &\lambda_r(y_\infty) > -1, \quad r = K_0+1, \ldots, K \end{aligned} \tag{13}$$

Clearly, Theorem 1 follows from

Proposition 4. *If $P_c \neq O$,*

$$-\frac{R(X_\triangle)}{1-R(X_\triangle)}P_c \in \partial\triangle$$

Precisely,

$$\lambda_r\left(-\frac{R(X_\triangle)}{1-R(X_\triangle)}P_c\right) \geqslant -1 \tag{14}$$

Equality holds if and only if $r = 1, \dots, K_0$. *So* $-\frac{R(X_\triangle)}{1-R(X_\triangle)}P_c$ *and* y_∞ *lie on the same faces* (13).

Proof. By (11) and defining function of $\triangle$, we have

$$\begin{aligned}\lambda_r\left(-\frac{t}{1-t}P_c\right)+1 &= \frac{1}{Vol(\triangle)}\int\limits_{\mathbb{R}^n}\lambda_r(D\tilde{u}_0)e^{-w}\,dx+1\\ &= \frac{1}{Vol(\triangle)}\int\limits_{\mathbb{R}^n}\big(\lambda_r(D\tilde{u}_0)+1\big)e^{-w}\,dx\end{aligned} \tag{15}$$

The inequality (14) follows from (15) by letting $t \to R(X_\triangle)$. To prove the second statement, by (15) we need to show

$$\lim_{t_i\to R(X_\triangle)}\frac{1}{Vol(\triangle)}\int\limits_{\mathbb{R}^n}\lambda_r(D\tilde{u}_0)e^{-w_{t_i}}\,dx+1\begin{cases}=0, & r=1,\dots,K_0\\ >0, & r=K_0+1,\dots,N\end{cases} \tag{16}$$

For any $\epsilon > 0$, by the uniform estimate (5) and fixed volume (8), and since $D\tilde{u}_0(\mathbb{R}^n) = \triangle^\circ$ is a bounded set, there exists R_ϵ, independent of $t \in [0, R(X_\triangle))$, such that

$$\frac{1}{Vol(\triangle)}\int\limits_{\mathbb{R}^n\setminus B_{R_\epsilon}(x_t)}\lambda_r(D\tilde{u}_0)e^{-w_t}\,dx<\epsilon \quad \text{and} \quad \frac{1}{Vol(\triangle)}\int\limits_{\mathbb{R}^n\setminus B_{R_\epsilon}(x_t)}e^{-w_t}\,dx<\epsilon \tag{17}$$

Now (16) follows from the following claim.

Claim 1. *Let* $R > 0$*, there exists a constant* $C > 0$*, which only depends on the polytope* $\triangle$*, such that for all* $\delta x \in B_R(0) \subset \mathbb{R}^n$*,*

$$e^{-CR}\big(\lambda_r\big(D\tilde{u}_0(x_{t_i})\big)+1\big)\leqslant\lambda_r\big(D\tilde{u}_0(x_{t_i}+\delta x)\big)+1\leqslant e^{CR}\big(\lambda_r\big(D\tilde{u}_0(x_{t_i})\big)+1\big) \tag{18}$$

Assuming the claim, we can prove two cases of (16). First by (10) and (13), we have

$$\lim_{t_i\to R(X_\triangle)}\lambda_r\big(D\tilde{u}_0(x_{t_i})\big)+1=\lambda_r(y_\infty)+1=\begin{cases}0, & r=1,\dots,K_0\\ a_r>0, & r=K_0+1,\dots,N\end{cases} \tag{19}$$

1. $r = 1, \dots, K_0$. $\forall\epsilon > 0$, first choose R_ϵ as in (17). By (18) and (19), there exists $\rho_\epsilon > 0$, such that if $|t_i - R(X_\triangle)| < \rho_\epsilon$, then for all $\delta x \in B_{R_\epsilon}(0) \subset \mathbb{R}^n$,

$$0\leqslant\lambda_r\big(D\tilde{u}_0(x_{t_i}+\delta x)\big)+1<e^{CR_\epsilon}\big(\lambda_r(D\tilde{u}_0)(x_{t_i})+1\big)<\epsilon$$

in other words, $\lambda_r(D\tilde{u}_0(x_{t_i}+\delta x))+1\to 0$ uniformly for $\delta x\in B_{R_\epsilon}(0)$, as $t_i\to R(X_\triangle)$. So when $|t_i-R(X_\triangle)|<\rho_\epsilon$,

$$\begin{aligned}\frac{1}{Vol(\triangle)}\int\limits_{\mathbb{R}^n}\lambda_r(D\tilde{u}_0)e^{-w}\,dx+1&=\frac{1}{Vol(\triangle)}\int\limits_{\mathbb{R}^n\setminus B_{R_\epsilon}(x_{t_i})}\lambda_r(D\tilde{u}_0)e^{-w}\,dx\\&\quad+\frac{1}{Vol(\triangle)}\int\limits_{\mathbb{R}^n\setminus B_{R_\epsilon}(x_{t_i})}e^{-w}\,dx\\&\quad+\frac{1}{Vol(\triangle)}\int\limits_{B_{R_\epsilon}(x_{t_i})}\big(\lambda_r(D\tilde{u}_0)+1\big)e^{-w}\,dx\\&\leqslant 2\epsilon+\epsilon\frac{1}{Vol(\triangle)}\int\limits_{B_{R_\epsilon}(x_{t_i})}e^{-w}\,dx\leqslant 3\epsilon\end{aligned}$$

The first case in (16) follows by letting $\epsilon\to 0$.

2. $r=K_0+1,\dots,N$. We fix $\epsilon=\frac{1}{2}$ and $R_{\frac{1}{2}}$ in (17). By (18) and (19), there exists $\rho>0$, such that if $|t_i-R(X_\triangle)|<\rho$, then for all $\delta x\in B_{R_{\frac{1}{2}}}(0)\subset\mathbb{R}^n$,

$$\lambda_r\big(D\tilde{u}_0(x_{t_i}+\delta x)\big)+1>e^{-CR_{\frac{1}{2}}}\big(\lambda_r\big(D\tilde{u}_0(x_{t_i})\big)+1\big)>e^{-CR_{\frac{1}{2}}}\frac{a_r}{2}>0$$

$$\begin{aligned}\frac{1}{Vol(\triangle)}\int\limits_{\mathbb{R}^n}\lambda_r(D\tilde{u}_0)e^{-w}\,dx+1&\geqslant\frac{1}{Vol(\triangle)}\int\limits_{B_{R_{\frac{1}{2}}}(x_{t_i})}\big(\lambda_r(D\tilde{u}_0)+1\big)e^{-w}\,dx\\&\geqslant e^{-CR_{\frac{1}{2}}}\frac{a_r}{2}\frac{1}{Vol(\triangle)}\int\limits_{B_{R_{\frac{1}{2}}}(x_{t_i})}e^{-w}\,dx\\&\geqslant e^{-CR_{\frac{1}{2}}}\frac{a_r}{2}\frac{1}{2}>0\end{aligned}$$

Now we prove the claim. We can rewrite (18) using the special form of $\tilde{u}_0$ (2):

$$D\tilde{u}_0(x)=\sum_\alpha\frac{e^{\langle p_\alpha,x\rangle}}{\sum_\beta e^{\langle p_\beta,x\rangle}}p_\alpha=\sum_\alpha c_\alpha(x)p_\alpha$$

Here the coefficients c_α satisfy

$$0\leqslant c_\alpha(x)=\frac{e^{\langle p_\alpha,x\rangle}}{\sum_\beta e^{\langle p_\beta,x\rangle}},\qquad\sum_{\alpha=1}^N c_\alpha(x)=1$$

So

$$\lambda_r\big(D\tilde{u}_0(x)\big)+1=\sum_\alpha c_\alpha(x)\big(\lambda_r(p_\alpha)+1\big)=\sum_{\{\alpha:\ \lambda_r(p_\alpha)+1>0\}} c_\alpha(x)\big(\lambda_r(p_\alpha)+1\big)$$

Since $\lambda_r(p_\alpha)+1\geqslant 0$ is a fixed value, to prove the claim, we only need to show the same estimate for $c_\alpha(x)$.

But now

$$c_\alpha(x_{t_i}+\delta x)=\frac{e^{\langle p_\alpha,x_{t_i}\rangle}e^{\langle p_\alpha,\delta x\rangle}}{\sum_\beta e^{\langle p_\beta,x_{t_i}\rangle}e^{\langle p_\beta,\delta x\rangle}}\leqslant e^{|p_\alpha|R}\cdot e^{\max_\beta|p_\beta|\cdot R}\frac{e^{\langle p_\alpha,x_{t_i}\rangle}}{\sum_\beta e^{\langle p_\beta,x_{t_i}\rangle}}$$
$$\leqslant e^{CR}\frac{e^{\langle p_\alpha,x_{t_i}\rangle}}{\sum_\beta e^{\langle p_\beta,x_{t_i}\rangle}}=e^{CR}c_\alpha(x_{t_i})$$

And similarly

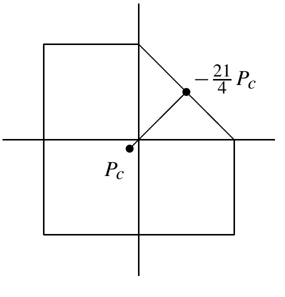
$$c_\alpha(x_{t_i}+\delta x)\geqslant e^{-CR}c_\alpha(x_{t_i})$$

So the claim holds and the proof is completed. □

4. Example

Example 1. $X_\triangle=Bl_p\mathbb{P}^2$. See the figure in the Introduction. $P_c=\frac{1}{4}(\frac{1}{3},-\frac{2}{3})$, $-6P_c\in\partial\triangle$, so $R(X_\triangle)=\frac{6}{7}$.

Example 2. $X_\triangle=Bl_{p,q}\mathbb{P}^2$, $P_c=\frac{2}{7}(-\frac{1}{3},-\frac{1}{3})$, $-\frac{21}{4}P_c\in\partial\triangle$, so $R(X_\triangle)=\frac{21}{25}$.



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