ON SHARP RATES AND ANALYTIC COMPACTIFICATIONS OF ASYMPTOTICALLY CONICAL KÄHLER METRICS

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Abstract

Let X be a complex manifold, and let $S \hookrightarrow X$ be an embedding of a complex submanifold. Assuming that the embedding is $(k-1)$-linearizable or $(k-1)$-comfortably embedded, we construct via the deformation to the normal cone a diffeomorphism F from a small neighborhood of the zero section in the normal bundle N_S to a small neighborhood of S in X such that F is in a precise sense holomorphic up to the $(k-1)$th order. Using this F, we obtain optimal estimates on asymptotic rates for asymptotically conical (AC) Calabi–Yau (CY) metrics constructed by Tian and Yau. Furthermore, when S is an ample divisor satisfying an appropriate cohomological condition, we relate the order of comfortable embedding to the weight of the deformation of the normal isolated cone singularity arising from the deformation to the normal cone. We also give an example showing that the condition of comfortable embedding depends on the splitting liftings. We then prove an analytic compactification result for the deformation of the complex structure on an affine cone that decays to any positive order at infinity. This can be seen as an analytic counterpart of Pinkham's result on deformations of cone singularities with negative weights.

Contents

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1. Introduction and main results

Our original motivation for this paper was to understand the optimal convergence rate of asymptotically conical (AC) Calabi–Yau (CY) Kähler metrics on noncompact Kähler manifolds. However, this led us to the study of embeddings of complex submanifolds and deformations of isolated normal singularities. We start the discussion with the embedding problem.

Let S be a complex submanifold of an ambient complex manifold X. The comparison between neighborhoods of S inside X with neighborhoods of S inside the normal bundle N_S is a classical subject in complex geometry (see, e.g., [6], [7], [14], [16] for details). It is clear that, although in general N_S has a different holomorphic structure than that of any neighborhood of S inside X, N_S can be viewed as a first-order approximation of a small neighborhood of S. More precisely, we will denote by $S(k)$ the ringed analytic space $(S, \mathcal{O}_X/\mathcal{I}_S^{k+1})$, which is called the kth infinitesimal neighborhood of S inside X. Recall the following definition.

Definition 1.1
The submanifold S is k-linearizable inside X if its kth infinitesimal neighborhood $S(k)$ in X is isomorphic to its kth infinitesimal neighborhood $S_N(k)$ in N_S. Here we identify S with the zero section S_0 of $N_S =: N$.

Our first preliminary result is that there is a diffeomorphism from a neighborhood of $S \subset X$ to a neighborhood of $S_0 \subset N_S$ that is in some sense the most holomorphic one. Although the existence of such a diffeomorphism may be known to experts after the celebrated work of Grauert [14] (cf. [3], [11], [18], [33]), here we would like to give an almost explicit construction using the work of Abate, Bracci, and Tovena in [1] together with the deformation to the normal cone construction. Let $\tilde{g}_0$ be a smooth Riemannian metric on a neighborhood W_0 of S_0 inside N_S. Denote by $\|\cdot\|_{\tilde{g}_0}$ the C^0-norms of tensors on W_0 with respect to $\tilde{g}_0$, and denote by $\tilde{r}$ the distance function to S_0 with respect to $\tilde{g}_0$.

PROPOSITION 1.2
Assume that S is a smooth submanifold of X. If $S \hookrightarrow X$ is $(k-1)$-linearizable, then there exist a small neighborhood W_0 of $S_0 \hookrightarrow N_S$ and a diffeomorphism $F : W_0 \to F(W_0) \subset W$, where W is a small neighborhood of $S \subset X$, such that for any $j \geq 0$, there exists a constant $C_j > 0$ and F satisfies

$$\left\|\nabla^j_{\tilde{g}_0}(F^* J - J_0)\right\|_{\tilde{g}_0} \leq C_j \tilde{r}^{k-j} \quad \text{on } W_0. \tag{1.1}$$

Our next result deals with a special situation that arises in Tian and Yau's construction in [32] of an asymptotically conical (AC) Calabi–Yau (CY) metric on the complement of some divisor inside a Fano manifold. To state the result, we need to use the notion of conical metrics on affine cones. In this paper, by an affine cone $C(D,L)$ we will mean the normal affine variety obtained by contracting the zero section of a negative line bundle L^{-1} over a smooth projective manifold D. We will also consider the compactified cone $\bar{C}(D,L) = C(D,L) \cup D_\infty$ obtained by adding the divisor D_∞ at infinity. These varieties can be expressed using pure algebras (x has degree 1 in the second graded ring):

$$C := C(D,L) = \operatorname{Spec} \bigoplus_{m=0}^{\infty} H^0(D, mL),$$

$$\bar{C} := \bar{C}(D,L) = \operatorname{Proj} \bigoplus_{m=0}^{\infty} \Big(\bigoplus_{r=0}^{m} H^0(D, L^r) \cdot x^{m-r} \Big).$$

Now let h be a Hermitian metric on the negative line bundle $L^{-1} \to D$ with negative Chern curvature. Since $C = C(D,L)$ is obtained from L^{-1} by contracting the zero section, h can be considered as a nonnegative function on the cone C. For any $\delta > 0$, there is a complete Kähler cone metric on $C(D,L)$ whose Kähler form on the regular part $C \setminus \{o\}$ is given by

$$\omega_0^{(\delta)} := \sqrt{-1}\partial\bar{\partial} h^{\delta}. \tag{1.2}$$

It is easy to verify that the associated Kähler metric tensor $g_0^{(\delta)}$ is indeed a Riemannian cone metric (see Section 5.1).

In the following proposition, we need to use the notion of *comfortable embedding*, which is a property that appeared in the study of embeddings of complex submanifolds in [14]. It refines the notion of linearizability in Definition 1.1 and was explicitly introduced in [1]. We refer to Definition A.4 for its definition.

PROPOSITION 1.3
Let X be an n-dimensional projective manifold, and let D be a smooth divisor such that N_D is ample over D. Let $\omega_0 = \omega_0^{(\delta)}$ be a cone metric on $C(D, N_D)$ as defined in (1.2). Assume that the embedding $D \hookrightarrow X$ is $(k-1)$-comfortable. Then there exists a diffeomorphism away from compact sets $F_K : C(D, N_D) \setminus B_R(\underline{o}) \to (X \setminus D) \setminus K$ such that

$$\big\| \nabla^j_{\omega_0} (F_K^* J - J_0) \big\|_{\omega_0} \le r^{-\frac{k}{\delta} - j} \quad \text{for any } j \ge 0, \tag{1.3}$$

where J (resp., J_0) denotes the complex structure on $X \setminus D$ (resp., $C(D, N_D) \setminus \{o\}$).

Note that the norm used in (1.1) is with respect to $\tilde{g}_0$, while the norm used in (1.3) is with respect to the cone metric ω_0 (or g_0) (see Section 5.1 for the comparison between these two Kähler metrics). This difference corresponds to the difference between the linearizable and comfortable embeddings.

The next corollary follows from Proposition 1.3 combined with the regularity theory developed by Conlon and Hein in [10] (see (5.9)). In many cases, Proposition 1.3 improves the regularity in [11] (see also [12, Remark 1.2]).

COROLLARY 1.4
With the same notation as in Proposition 1.3, let X be an n-dimensional Fano manifold, and assume that $-K_X = \alpha D$ with $\alpha > 1$. Denote $\delta = \frac{\alpha-1}{n}$. Suppose that D has a Kähler–Einstein metric and that D is $(k-1)$-comfortably embedded into X. Then the metric ω_{TY} constructed by Tian and Yau (see Section 5.2) satisfies

$$\left\| \nabla^j_{\omega_0}(F_K^* \omega_{\mathrm{TY}} - \omega_0) \right\|_{\omega_0} \leq r^{-\min\{2, \frac{k}{\delta}\} - j} \quad \text{for any } j \geq 0.$$

If, moreover, we assume that the Kähler class is contained in the compactly supported cohomology $H^2_c(X \backslash D)$, then we get

$$\left\| \nabla^j_{\omega_0}(F_K^* \omega_{\mathrm{TY}} - \omega_0) \right\|_{\omega_0} \leq r^{-\min\{2n, \frac{k}{\delta}\} - j} \quad \text{for any } j \geq 0.$$

The special number $\delta = \frac{\alpha-1}{n}$ in the above corollary is the exponent in the Calabi ansatz for Kähler–Ricci flat cone metrics (see (5.7) in Section 5.1).

Under appropriate assumptions, our next result relates the order of embedding of $D \to X$ to the order and the weight of a deformation of $C(D, N_D)$. To construct the deformation that we like to use, let X be a projective manifold of dimension greater than 2, and let D be a smooth ample divisor on X. Let $\mathcal{X}$ denote the flat family that is obtained by first blowing up $D \times \{0\}$ inside $X \times \mathbb{C}$ and then blowing down the strict transform of $X \times \{0\}$. Let $\mathcal{D}$ be the strict transform of $D \times \mathbb{C}$. It is easy to see that $\mathcal{D} \cong D \times \mathbb{C}$. Assume that the central fiber $\mathcal{X}_0$ coincides with $\bar{C}(D, N_D)$ so that $\mathcal{X}^\circ = \mathcal{X} \setminus \mathcal{D}$ is a flat deformation $\mathcal{X}^\circ \to \mathbb{C}$ of $C(D, N_D)$. We remark that this assumption is always satisfied when X is Fano and $-K_X = \alpha D$ with $\alpha > 1$.

Denote by $m(X, D)$ the maximum positive integer m such that the embedding $D \hookrightarrow X$ is $(m-1)$-comfortably embedded. Let $\mathrm{Ord}(\mathcal{X}^\circ)$ denote the order of deformation (see Definition 2.11), and let $w(\mathcal{X}^\circ)$ be the weight of the reduced Kodaira–Spencer class $\mathbf{KS}^{\mathrm{red}}_{\mathcal{X}^\circ}$ (see Definition 2.12).

THEOREM 1.5
In the setting of the above paragraph, we have the identities

$$m(X, D) = \mathrm{Ord}(\mathcal{X}^\circ) = -w(\mathcal{X}^\circ). \tag{1.4}$$

Notice that the integer $m(X, D)$ in the above theorem was considered in [1, Remark 4.6]. If $\dim D \geq 2$ and D is ample, then, by Remark A.7, $m(X, D)$ is also the maximal order of linearizability. In other words, $D \subset X$ is $(m(X, D)-1)$-linearizable but not $m(X, D)$-linearizable. When $\dim D = 1$, we expect the conclusion of Theorem 1.5 to also be true. In fact, a parallel analytic result will be shown in Theorem 1.6 without the restriction on dimension. On the other hand, we will calculate the example of diagonal embedding $\mathbb{P}^1 \hookrightarrow \mathbb{P}^1 \times \mathbb{P}^1$ explicitly to see some new phenomena about the embedding of submanifolds in Proposition 4.9. In particular, this example shows that the condition of comfortable embedding depends on the choice of splitting liftings, and thus answers a question by Abate, Bracci, and Tovena negatively.

Combining Theorem 1.5 with Proposition 1.3, we can give algebraic interpretations of ad hoc calculations in [10] on the asymptotic rates of holomorphic volume forms. See the examples in Section 5.2.

Finally, we ask if any deformation of complex structures on C that decays at infinity comes from this construction. We have a good understanding of the algebraic version of this problem thanks to the work of Pinkham. His results in particular imply that any (formal) deformation of C with negative weight can be extended to a (formal) deformation of $\bar{C}$ (see Theorem A.14). For the application to the study of AC Kähler metrics, we prove an analytic compactification result, which can be seen as the analytic counterpart of Pinkham's result. Note that a similar compactification result in the asymptotically cylindrical Calabi–Yau case has recently appeared in [18]. See Remark 6.2 for some comparison.

To state this result in a general form, let h be a Hermitian metric on any negative line bundle $L^{-1} \to D$ with negative Chern curvature, and use the notation $\omega_0 := \omega_0^{(\delta)}$ in (1.2). Let U_ϵ denote a neighborhood of the infinity end of $C(D, L)$. Equivalently, U_ϵ is a punctured neighborhood of the embedding $D = D_\infty \hookrightarrow \overline{C}(D, L)$. Denote by J_0 the standard complex structure on $C(D, L)$, and denote by $\overline{U}_\epsilon = U_\epsilon \cup D$ the compactification of U_ϵ in $\overline{C}(D, L)$.

THEOREM 1.6
Assume that J is a complex structure on $U_\epsilon = \overline{U_\epsilon} \backslash D$ such that there exists $\lambda > 0$ such that

$$\left\| \nabla_{g_0}^k (J - J_0) \right\|_{\omega_0} \leq r^{-\lambda - k}, \quad \textit{for any } k \geq 0.$$

Then the complex analytic structure on U_ϵ extends to a complex analytic structure on $\overline{U}_\epsilon$. Moreover, if we denote by $m = \lceil \delta\lambda \rceil$ the minimal integer which is greater than or equal to $\delta\lambda$, then in the compactification $(\overline{U}_\epsilon, J)$ the divisor D is $(m-1)$-comfortably embedded.

This can be seen as a converse to the first part of Proposition 1.3 and implies that the estimate in Proposition 1.3 is sharp.

Remark 1.7
Because our proof uses only local information near the divisor, the argument in the proof should apply in the more general orbifold case. In fact, Conlon and Hein [12] recently used the compactification obtained in Theorem 1.6 to prove that any asymptotically conical Calabi–Yau metric with quasiregular metric tangent cone at infinity comes from Tian and Yau's construction.

We end this Introduction with the organization of this paper. A more detailed summary of materials will be given at the beginning of each section. In Section 2, we recall the standard Kodaira–Spencer theory of infinitesimal deformations and generalize it to a higher-order setting. We also explain how the (higher-order) abstract deformations and embedded deformations are related via Schlessinger's exact sequence. In Section 3, we relate the order of embedding to the order of deformation of neighborhoods of complex submanifolds. This is achieved by writing down explicitly a reduced Kodaira–Spencer class and relating it to obstructions to extension of embeddings (in Proposition 3.3). In Section 4, we treat the case when the submanifold is an ample divisor and prove Theorem 1.5. In Section 5, we apply the result in Section 4 to estimate the asymptotic rates of complex structures on asymptotic conical Kähler manifolds in order to prove Proposition 1.3. In Section 6, we adapt Newlander and Nirenberg's work to prove an analytic compactification result for asymptotically conical complex manifolds. In the Appendix, we collect some background results, including Abate, Bracci, and Tovena's work on embedding of submanifolds, and theory of infinitesimal deformations of normal affine varieties with isolated singularities.

2. Preliminaries on deformation theory

Our primary object of interest will be a normal affine variety Z with an isolated singularity o. We would like to explain what it means for a deformation of Z to be trivial up to a certain order and to classify the next order of deformations in terms of a Kodaira–Spencer class in $\mathbf{T}^1_Z$. This is done in Section 2.3, following Artin and Schlessinger relying on manipulations with defining equations. We will show that these concepts are "identical" to certain analogous concepts in the deformation theory of the complex manifold $Z \backslash K$, where K is a small pseudoconvex neighborhood of o. We will define such concepts in Section 2.2 following essentially Kodaira–Spencer. The desired identification is proved in Proposition 2.15. For this purpose we will introduce a notation of "p-trivial embeddings," which connects the two primary concepts to each other. We will be working in the category of analytic varieties.

2.1. Infinitesimal deformations via coordinate changes and embedded deformations
In this subsection, we recall how to get the first-order Kodaira–Spencer class for an analytic family by using the variation of holomorphic coordinate changes (see [22]) and its relation to embedded deformations. Suppose that $\mathcal{Y} \to \mathbb{B}$ is an analytic family of complex manifolds over the unit disk $\mathbb{B} = \{z \in \mathbb{C}; |z| < 1\}$.

Definition 2.1
An *atlas covering* $\mathcal{Y}_0$ is a collection of coordinate charts $\{\mathcal{U}_\alpha, \Phi_\alpha = (z_\alpha, t)\}_{\alpha\in\mathcal{A}}$ such that

(1) for each $\alpha \in \mathcal{A}$, $\mathcal{U}_\alpha \subset \mathcal{Y}$ is biholomorphic to polydisk $\mathbb{B}^{n+1}$, and $\mathcal{Y}_0 \subset \bigcup_\alpha \mathcal{U}_\alpha$, that is, $\mathcal{Y}_0 = \bigcup_\alpha (\mathcal{U}_\alpha \cap \mathcal{Y}_0)$;
(2) there is a biholomorphic map $\Phi_\alpha = (z_\alpha, t) : \mathcal{U}_\alpha \to \Phi_\alpha(\mathcal{U}_\alpha) \subset \mathbb{C}^n \times \mathbb{C}$ such that t is the coordinate on $\mathbb{B}$; in particular, $U_\alpha := \mathcal{Y}_0 \cap \mathcal{U}_\alpha = \{t = 0\}$.

Remark 2.2

(1) Since we only care about the behavior near the central fiber $\mathcal{Y}_0$, the base $\mathbb{B}$ is not very important. For example, we will frequently shrink $\mathbb{B}$ to become $\mathbb{B}_\epsilon = \{t \in \mathbb{C}; |t| < \epsilon\}$ for any $0 < \epsilon \ll 1$ in the following discussion.
(2) Since we can always shrink $\mathcal{U}_\alpha$, the assumption that $\mathcal{U}_\alpha$ is biholomorphic to polydisk $\mathbb{B}^{n+1}$ is just for the simplicity of the argument.

We first recall two ways to get the first-order Kodaira–Spencer class for a holomorphic family of complex manifolds by using the variation of holomorphic coordinate changes.

(1) *(Čech cohomology)* Suppose that the coordinate changes are given by

$$z^i_\alpha = F^i_{\alpha\beta}(z_\beta, t), \qquad t|_{\mathcal{U}_\alpha} = t|_{\mathcal{U}_\beta}. \tag{2.1}$$

Then we can deduce that

$$\begin{aligned} &F^i_{\alpha\beta}\big(F_{\beta\gamma}(z_\gamma, t), t\big) = F^i_{\alpha\gamma}(z_\gamma, t) \\ &\Longrightarrow \sum_{j=1}^n \frac{\partial F^i_{\alpha\beta}(z_\beta, t)}{\partial z^j_\beta} \frac{\partial F^j_{\beta\gamma}(z_\gamma, t)}{\partial t} + \frac{\partial F^i_{\alpha\beta}(z_\beta, t)}{\partial t}\Big|_{t=0} = \frac{\partial F^i_{\alpha\gamma}(z_\gamma, t)}{\partial t}\Big|_{t=0}. \end{aligned}$$

So if we denote

$$\theta_{\beta\gamma} = \sum_{i=1}^n \frac{\partial F^i_{\beta\gamma}(z_\gamma, t)}{\partial t}\Big|_{t=0} \frac{\partial}{\partial z^i_\beta} = \sum_{i=1}^n \frac{\partial z^i_\beta(z_\gamma, t)}{\partial t}\Big|_{t=0} \frac{\partial}{\partial z^i_\beta}, \tag{2.2}$$

then it satisfies the cocycle condition $\theta_{\beta\gamma} = \theta_{\alpha\gamma} - \theta_{\alpha\beta}$ so that $\{\theta_{\alpha\beta}\} \in \check{H}^1(\{U_\alpha\}, \Theta_{\mathcal{Y}_0})$, where $U_\alpha = \mathcal{U}_\alpha \cap \mathcal{Y}_0$ and $\Theta_{\mathcal{Y}_0}$ is the tangent sheaf on $\mathcal{Y}_0$.

The class defined by $\theta = \{\theta_{\alpha\beta}\}$ in $H^1(\mathcal{Y}_0, \Theta_{\mathcal{Y}_0})$ is the classical Kodaira–Spencer class associated to the analytic family $\mathcal{Y} \to \mathbb{B}$.

(2) *(Dolbeault cohomology)* It is well known that the above θ can be represented by using Dolbeault cohomology. For this purpose take $\{\rho_\alpha\}$ to be a partition of unity for the covering $\{\mathcal{U}_\alpha\}$ and define

$$\xi_\alpha = \sum_{i=1}^n \sum_\gamma \rho_\gamma \frac{\partial F^i_{\alpha\gamma}(z_\gamma, t)}{\partial t}\Big|_{t=0} \frac{\partial}{\partial z^i_\alpha}.$$

It is easy to verify that $\theta_{\alpha\beta} = \xi_\alpha - \xi_\beta$, so that $\bar\partial \xi_\alpha = \bar\partial \xi_\beta$ is a globally defined $\Theta_{\mathcal{Y}_0}$-valued closed $(0,1)$-form and it represents a cohomology class, still denoted by θ, in $H^{(0,1)}_{\bar\partial}(\mathcal{Y}_0, \Theta_{\mathcal{Y}_0})$. On the other hand, θ measures the first-order variation of the complex structure. We can follow the method in Kodaira's book [22, Section 2.3] to define a differentiable vector field $\mathbb{V}$. First notice that by the chain rule

$$\Big(\frac{\partial}{\partial t}\Big)_\beta = \sum_{i=1}^n \frac{\partial F^i_{\alpha\beta}(z_\beta, t)}{\partial t} \frac{\partial}{\partial z^i_\alpha} + \Big(\frac{\partial}{\partial t}\Big)_\alpha,$$

$$\frac{\partial}{\partial z^j_\beta} = \sum_{i=1}^n \frac{\partial F^i_{\alpha\beta}(z_\beta, t)}{\partial z^j_\beta} \frac{\partial}{\partial z^i_\alpha}.$$

We can define a differentiable vector field locally on $\mathcal{U}_\alpha$ for fixed α by

$$\begin{aligned}
\mathbb{V} &= \sum_\beta \rho_\beta \Big(\frac{\partial}{\partial t}\Big)_\beta \\
&= \sum_\beta \rho_\beta \sum_{i=1}^n \frac{\partial F^i_{\alpha\beta}(z_\beta, t)}{\partial t} \frac{\partial}{\partial z^i_\alpha} + \Big(\frac{\partial}{\partial t}\Big)_\alpha \\
&= \sum_{i=1}^n \Big(\sum_\beta \rho_\beta \frac{\partial F^i_{\alpha\beta}(z_\beta, t)}{\partial t}\Big) \frac{\partial}{\partial z^i_\alpha} + \Big(\frac{\partial}{\partial t}\Big)_\alpha.
\end{aligned}$$

Then $\mathbb{V}$ is a globally defined vector field in a neighborhood of $\mathcal{Y}_0$. Let $\sigma(t)$ be the flow associated with $\mathbb{V}$ which exists for sufficiently small t. We have the identity

$$\frac{d}{dt}\big(\sigma(t)^* J\big) = (\mathcal{L}_{\mathbb{V}} J)(\partial_{\bar z^j})\, d\bar z^j = \bar\partial \mathbb{V}.$$

Notice that $\bar\partial \mathbb{V}|_{t=0} = \bar\partial \xi_\alpha = \theta \in H^{(0,1)}_{\bar\partial}(\mathcal{Y}_0, \Theta_{\mathcal{Y}_0}) \cong H^1(\mathcal{Y}_0, \Theta_{\mathcal{Y}_0})$.

Assume that a holomorphic family of complex manifolds $\mathcal{Y} \to \mathbb{B}$ is embedded into $\mathbb{C}^N \times \mathbb{B}$. Then the Kodaira–Spencer class can also be obtained by using the relation between embedded deformations and abstract deformations. In the following discussion we assume that $Y = \mathcal{Y}_0$ is smooth. First there is an exact sequence of sheaves

$$0 \to \mathcal{I}_Y/\mathcal{I}_Y^2 \to \Omega^1_{\mathbb{C}^N}|_Y \to \Omega^1_Y \to 0,$$

where Ω^1 denotes the cotangent sheaf. The dual of this sequence is given by

$$0 \to \Theta_Y \to \Theta_{\mathbb{C}^N}|_Y \to N_Y \to 0,$$

where $N_Y = N_{Y|\mathbb{C}^N}$ is the normal sheaf of Y as a complex submanifold of $\mathbb{C}^N$. Then there is a long exact sequence

$$\begin{aligned} &0 \to H^0(Y, \Theta_Y) \\ &\quad \to H^0(Y, \Theta_{\mathbb{C}^N}) \to H^0(Y, N_Y) \xrightarrow{\delta_Y} H^1(Y, \Theta_Y) \to H^1(Y, \Theta_{\mathbb{C}^N}|_Y). \end{aligned} \tag{2.3}$$

Choose an atlas covering $\mathcal{Y}_0$, denoted by $\{\mathcal{U}_\alpha, (z^i_\alpha, t)\}$, such that the embedding $\mathcal{U}_\alpha \to \mathbb{C}^N \times \mathbb{B}$ is given by holomorphic functions

$$w^b = w^b_\alpha(z^i_\alpha, t), \quad 1 \le b \le N.$$

Note that we will use $\{w^b; b = 1, \dots, N\}$ to denote the coordinates of $\mathbb{C}^N$ and use w^b_α (i.e., depending on α) to denote w^b as functions of the coordinates $\{z^i_\alpha, t\}$. Then there is a locally defined section $v_\alpha \in H^0(U_\alpha, \Theta_{\mathbb{C}^N}|_{U_\alpha})$ given by

$$v_\alpha = \sum_{b=1}^N \frac{\partial w^b_\alpha}{\partial t}\Big|_{t=0} \frac{\partial}{\partial w^b}.$$

Let $[v_\alpha] \in H^0(U_\alpha, N_Y|_{U_\alpha})$ denote the induced local section under the natural projection $\Theta_{\mathbb{C}^N}|_{\mathcal{Y}_0} \to N_{\mathcal{Y}_0}$.

LEMMA 2.3
The local section $\{[v_\alpha]\}$ can be glued together to become a global section $\mathfrak{v}$ in $H^0(Y, N_Y)$. Moreover, $\delta_Y(\mathfrak{v}) = \theta$, where δ_Y is the connecting morphism in (2.3) and θ is the classical Kodaira–Spencer class defined in (2.2).

Proof
Notice that we have the relation

$$w^b = w^b_\beta(z_\beta, t) = w^b_\beta\big(z^i_\beta(z^j_\alpha, t), t\big) = w^b_\alpha(z^j_\alpha, t).$$

Taking derivatives on both sides with respect to t at $t=0$, we get

$$\sum_{b=1}^{N}\frac{\partial w_\alpha^b}{\partial t}\Big|_{t=0}\frac{\partial}{\partial w^b}=\sum_{b=1}^{N}\sum_{i=1}^{n}\frac{\partial w_\beta^b}{\partial z_\beta^i}\frac{\partial z_\beta^i}{\partial t}\Big|_{t=0}\frac{\partial}{\partial w^b}+\sum_{b=1}^{N}\frac{\partial w_\beta^b}{\partial t}\Big|_{t=0}\frac{\partial}{\partial w^b}.$$

Denote by $\iota_Y : Y \to \mathbb{C}^N$ the induced embedding. Then the above equality is equivalent to

$$v_\alpha - v_\beta = \sum_{b=1}^{N}\Big(\sum_{i=1}^{n}\frac{\partial z_\beta^i(z_\alpha,t)}{\partial t}\Big|_{t=0}\frac{\partial w^b}{\partial z_\beta^i}\Big)\frac{\partial}{\partial w^b}=(\iota_Y)_*(\theta_{\beta\alpha}),$$

where we used the identity (2.2). Since $\theta_{\beta\alpha}\in\Theta_{\mathcal{Y}_0}(U_\alpha\cap U_\beta)$, we get $[v_\alpha]=[v_\beta]$. By the definition of the connecting morphism δ_Y in (2.3), we indeed have $\delta_Y(\mathfrak{v})=\theta$. $\square$

2.2. *p-trivial atlas and p-trivial embeddings*

We can generalize the above discussion to higher-order deformations. Let us introduce a condition that will be important in the following discussion.

Definition 2.4

Assume that there is an atlas $\mathcal{U}=\{\mathcal{U}_\alpha,\Phi_\alpha=(z_\alpha,t)\}$ covering $\mathcal{Y}_0$ with coordinate change functions $z_\alpha^i=F_{\alpha\beta}^i(z_\beta,t)$ on $\mathcal{U}_\alpha\cap\mathcal{U}_\beta$. We say that $\mathcal{U}$ is p-trivial if $F_{\alpha\beta}^i(z_\beta,t)-F_{\alpha\beta}^i(z_\beta,0)$ vanishes up to order p at $t=0$:

$$\frac{\partial^l(F_{\alpha\beta}^i(z_\beta,t)-F_{\alpha\beta}^i(z_\beta,0))}{\partial t^l}\Big|_{t=0}=0,\quad \text{for } 0\le l\le p.$$

Notice that, since the $l=0$ case is automatically true, this p-trivial condition is equivalent to

$$\frac{\partial^l F_{\alpha\beta}^i(z_\beta,t)}{\partial t^l}\Big|_{t=0}=0,\quad \text{for } 1\le l\le p. \tag{2.4}$$

If this is the case, then we define the $(p+1)$-order Kodaira–Spencer (Čech) class, denoted by $\theta_{p+1}(\mathcal{U})$ or simply by θ_{p+1} if the atlas is clear, as the (Čech) cohomology defined by the cocycle

$$\begin{aligned}(\theta_{p+1})_{\alpha\beta}&=\frac{1}{(p+1)!}\sum_{i=1}^{n}\frac{\partial^{p+1}F_{\alpha\beta}^i(z_\beta,t)}{\partial t^{p+1}}\Big|_{t=0}\frac{\partial}{\partial z_\alpha^i}\\&\in H^0(\mathcal{U}_\alpha\cap\mathcal{U}_\beta\cap\mathcal{Y}_0,\Theta_{\mathcal{Y}_0}).\end{aligned}\tag{2.5}$$

LEMMA 2.5

(1) *We have that* $\theta_{p+1} := \theta_{p+1}(\mathcal{U})$ *is well defined, that is,* $\theta_{p+1} = \{(\theta_{p+1})_{\alpha\beta}\}$ *satisfies the cocycle condition* $(\theta_{p+1})_{\beta\gamma} = (\theta_{p+1})_{\alpha\gamma} - (\theta_{p+1})_{\alpha\beta}$.

(2) *If we have another* p*-trivial atlas* $\tilde{\mathcal{U}} = \{\tilde{\mathcal{U}}_\alpha, \tilde{\Phi}_\alpha = (\tilde{z}_\alpha, t)\}$*, then* $\tilde{\theta}_{p+1} = \theta_{p+1}(\tilde{\mathcal{U}})$ *defines the same Čech cohomology class as* θ_{p+1}.

(3) *Assume that there exists a* $(p-1)$*-trivial atlas covering* $\mathcal{Y}_0$ *and* $\theta_p = 0 \in H^1(\mathcal{Y}_0, \Theta_{\mathcal{Y}_0})$*. Then for any relatively compact open subset* $\mathcal{K} \Subset \mathcal{Y}$ *such that* $\mathcal{K}_0 = \pi^{-1}(0) \cap \mathcal{K}$ *is a relatively compact open set of* $\mathcal{Y}_0 = \pi^{-1}(0)$*, there exists a* p*-trivial atlas covering* $\mathcal{K}_0$.

Proof

Using the cocycle condition of $\{F_{\alpha\beta}\}$ and the vanishing condition (2.4), we can take higher-order derivatives with respect to t to get

$$\begin{aligned} &F^i_{\alpha\beta}\big(F_{\beta\gamma}(z_\gamma,t),t\big) = F^i_{\alpha\gamma}(z_\gamma,t) \\ &\qquad\Longrightarrow \sum_{j=1}^n \frac{\partial F^i_{\alpha\beta}(z_\beta,t)}{\partial z^j_\beta}\frac{\partial F^j_{\beta\gamma}(z_\gamma,t)}{\partial t} + \frac{\partial F^i_{\alpha\beta}(z_\beta,t)}{\partial t} = \frac{\partial F^i_{\alpha\gamma}(z_\gamma,t)}{\partial t} \\ &\Longrightarrow \sum_{j=1}^n \frac{\partial F^i_{\alpha\beta}(z_\beta,t)}{\partial z^j_\beta}\frac{\partial^{p+1} F^j_{\beta\gamma}(z_\gamma,t)}{\partial t^{p+1}} + O(t) + \frac{\partial^{p+1} F^i_{\alpha\beta}(z_\beta,t)}{\partial t^{p+1}} = \frac{\partial^{p+1} F^i_{\alpha\gamma}(z_\gamma,t)}{\partial t^{p+1}}. \end{aligned}$$

From this it is clear that $\theta_{p+1} = \{(\theta_{p+1})_{\beta\alpha}\}$ satisfies the cocycle condition.

To prove the second item, we first choose a common refinement of $\mathcal{U}$ and $\tilde{\mathcal{U}}$ and assume that we have the same collection of open sets: $\mathcal{U}_\alpha = \tilde{\mathcal{U}}_\alpha$ for $\alpha \in \mathcal{A}$. Suppose that the coordinate function $\tilde{\mathcal{U}}_\alpha$ is denoted by $\tilde{\Phi}_\alpha = (\tilde{z}_\alpha, t)$. We then have the following relation on the composition of coordinate functions:

$$z_\alpha = z_\alpha(\tilde{z}_\alpha,t) = z_\alpha\big(\tilde{z}_\alpha(\tilde{z}_\beta,t),t\big) = z_\alpha\big(\tilde{z}_\alpha\big(\tilde{z}_\beta(z_\beta,t),t\big),t\big) = z_\alpha(z_\beta,t).$$

Taking derivatives on both sides with respect to t we get

$$\begin{aligned} \frac{\partial z^i_\alpha(z_\beta,t)}{\partial t} = \sum_{j=1}^n \frac{\partial z^i_\alpha(\tilde{z}_\alpha,t)}{\partial \tilde{z}^j_\alpha}\Big(\sum_{k=1}^n \frac{\partial \tilde{z}^j_\alpha(\tilde{z}_\beta,t)}{\partial \tilde{z}^k_\beta}\frac{\partial \tilde{z}^k_\beta(z_\beta,t)}{\partial t} + \frac{\partial \tilde{z}^j_\alpha(\tilde{z}_\beta,t)}{\partial t}\Big) \\ + \frac{\partial z^i_\alpha(\tilde{z}_\alpha,t)}{\partial t}. \end{aligned} \tag{2.6}$$

Note that we used the Einstein summation rule. On the other hand, we have

$$\tilde{z}_\beta = \tilde{z}_\beta\big(z_\beta(\tilde{z}_\beta,t),t\big) \Longrightarrow \sum_{j=1}^n \frac{\partial \tilde{z}^k_\beta(z_\beta,t)}{\partial z^j_\beta}\frac{\partial z^j_\beta(\tilde{z}_\beta,t)}{\partial t} + \frac{\partial \tilde{z}^k_\beta(z_\beta,t)}{\partial t} = 0. \tag{2.7}$$

Combining (2.6)–(2.7) and using the chain rule, we get

$$\sum_{i=1}^{n} \frac{\partial z_\alpha^i(z_\beta,t)}{\partial t}\frac{\partial}{\partial z_\alpha^i} - \sum_{j=1}^{n} \frac{\partial \tilde z_\alpha^j(\tilde z_\beta,t)}{\partial t}\frac{\partial}{\partial \tilde z_\alpha^j}$$

$$= \sum_{i=1}^{n} \frac{\partial z_\alpha^i(\tilde z_\alpha,t)}{\partial t}\frac{\partial}{\partial z_\alpha^i} - \sum_{j=1}^{n} \frac{\partial z_\beta^j(\tilde z_\beta,t)}{\partial t}\frac{\partial}{\partial z_\beta^j}. \tag{2.8}$$

At $t=0$, this shows that $\theta_1 - \tilde\theta_1$ is indeed a coboundary. For p-trivial atlases $\mathcal{U}$ and $\tilde{\mathcal{U}}$, we can take higher-order Lie derivatives $(\mathcal{L}_{\partial_t})^{p+1}$ on both sides of (2.8) at $t=0$ to get

$$\frac{\partial^{p+1} z_\alpha^i(z_\beta,t)}{\partial t^{p+1}}\Big|_{t=0}\frac{\partial}{\partial z_\alpha^i} - \frac{\partial^{p+1}\tilde z_\alpha^j(\tilde z_\beta,t)}{\partial t^{p+1}}\Big|_{t=0}\frac{\partial}{\partial \tilde z_\alpha^j}$$

$$= \sum_{i=1}^{n} \frac{\partial^{p+1} z_\alpha^i(\tilde z_\alpha,t)}{\partial t}\Big|_{t=0}\frac{\partial}{\partial z_\alpha^i} - \frac{\partial^{p+1} z_\beta^j(\tilde z_\beta,t)}{\partial t^{p+1}}\Big|_{t=0}\frac{\partial}{\partial z_\beta^j}. \tag{2.9}$$

So, using the definition in (2.5), $\theta_{p+1} - \tilde\theta_{p+1}$ is indeed a coboundary.

Finally, we prove the third item. Assume that $\mathcal{U} = \{\mathcal{U}_\alpha, \Phi_\alpha = (z_\alpha, t)\}_{\alpha\in\mathcal{A}}$ is a $(p-1)$-trivial atlas. Then by the definition of θ_p and the assumption, we have

$$\theta_p = \frac{1}{p!}\sum_{i=1}^{n} \frac{\partial^p z_\alpha^i(z_\beta,t)}{\partial t^p}\Big|_{t=0}\frac{\partial}{\partial z_\alpha^i} = \sum_{i=1}^{n} c_\alpha^i\frac{\partial}{\partial z_\alpha^i} - c_\beta^i\frac{\partial}{\partial z_\beta^i}. \tag{2.10}$$

Define the new coordinates $\tilde z_\alpha^i = z_\alpha^i + t^p c_\alpha^i$ which are genuine coordinate charts on an open neighborhood of $\mathcal{K}_0$ inside $\mathcal{Y}$, since $\mathcal{K}\subset\mathcal{Y}$ and $\mathcal{K}_0\subset\mathcal{Y}_0$ are relatively compact open subsets:

$$\tilde z_\alpha^i = z_\alpha^i(z_\beta,t) + t^p c_\alpha^i = z_\alpha^i(\tilde z_\beta^j - t^p c_\beta^j, t) + t^p c_\alpha^i = \tilde z_\alpha^i(\tilde z_\beta, t).$$

Taking the p-order derivative with respect to t on both sides, we get

$$\frac{1}{p!}\frac{\partial^p \tilde z_\alpha^i(\tilde z_\beta,t)}{\partial t^p}\Big|_{t=0} = -\sum_{j=1}^{n}\frac{\partial z_\alpha^i}{\partial z_\beta^j}\cdot c_\beta^j\Big|_{t=0} + \frac{1}{p!}\frac{\partial^p z_\alpha^i(z_\beta,t)}{\partial t^p}\Big|_{t=0} + c_\alpha^i.$$

Notice that $\frac{\partial}{\partial \tilde z_\alpha^i} = \frac{\partial}{\partial z_\alpha^i}$ at $t=0$, so we get by (2.10) that

$$\frac{1}{p!}\sum_{i=1}^{n}\frac{\partial^p \tilde z_\alpha^i}{\partial t^p}\Big|_{t=0}\frac{\partial}{\partial \tilde z_\alpha^i} = -\sum_{j=1}^{n} c_\beta^j\frac{\partial}{\partial z_\beta^j} + \sum_{i=1}^{n} c_\alpha^i\frac{\partial}{\partial z_\alpha^i} + \frac{1}{p!}\sum_{i=1}^{n}\frac{\partial^p z_\alpha^i(z_\beta,t)}{\partial t^p}\Big|_{t=0}\frac{\partial}{\partial z_\alpha^i}$$

$$= 0.$$

So the new atlas $\{\mathcal{U}_\alpha, \tilde\Phi = (\tilde z_\alpha, t)\}$ is a p-trivial atlas covering $\mathcal{K}_0$. $\square$

To make a connection with embedded deformations, we introduce the following definition.

Definition 2.6
Let $\mathcal{Y} \to \mathbb{B}$ be a holomorphic family of complex manifolds that can be embedded into $\mathbb{C}^N \times \mathbb{B}$. We say that an embedding $\iota_{\mathcal{Y}} : \mathcal{Y} \to \mathbb{C}^N \times \mathbb{B}$ is p-trivial (along $\mathcal{Y}_0 =: Y$) if there exists an atlas $\mathcal{U} = \{\mathcal{U}_\alpha, \Phi_\alpha = (z^i_\alpha, t)\}_{\alpha \in \mathcal{A}}$ covering $\mathcal{Y}_0$ such that, for each $\alpha \in \mathcal{A}$, if the embedding $\mathcal{U}_\alpha \to \mathbb{C}^N \times \mathbb{B}$ is represented by the functions $w^b = w^b_\alpha(z_\alpha, t)$, then the following vanishing conditions are satisfied:

$$\frac{\partial^l w^b_\alpha(z_\alpha, t)}{\partial t^l}\bigg|_{t=0} = 0, \quad 1 \le l \le p. \tag{2.11}$$

In this case, we say that $\mathcal{U}$ is an atlas adapted for the p-trivial embedding, or simply a p-adapted atlas.

To state the next result, we introduce additional notation. Let $\pi : \mathcal{Y} \to \mathbb{B}$ be a holomorphic family of complex manifolds over the unit disk. For any $0 < \epsilon < 1$ and any subset $\mathcal{K} \subseteq \mathcal{Y}$, denote $\mathbb{B}_\epsilon = \{t \in \mathbb{B}; |t| < \epsilon\}$ and

$$\mathcal{Y}_\epsilon = \mathcal{Y} \times_{\mathbb{B}} \mathbb{B}_\epsilon = \pi^{-1}(\mathbb{B}_\epsilon), \qquad \mathcal{K}_\epsilon = \pi^{-1}(\mathbb{B}_\epsilon) \cap \mathcal{K}. \tag{2.12}$$

LEMMA 2.7
With the above notation, if there exists a p-trivial embedding $\mathcal{Y}_\epsilon \hookrightarrow \mathbb{C}^N \times \mathbb{B}_\epsilon$ for some $0 < \epsilon \ll 1$, then there exists a p-trivial atlas covering $\mathcal{Y}_0$.

Conversely, assume that there exists a p-trivial atlas covering $\mathcal{Y}_0$. Then for any relatively compact open subset $\mathcal{K} \Subset \mathcal{Y}$, there is a p-trivial embedding $\mathcal{K}_\epsilon \hookrightarrow \mathbb{C}^N \times \mathbb{B}_\epsilon$ for $0 < \epsilon \ll 1$. More precisely, given an embedding $\mathcal{Y} \to \mathbb{C}^N \times \mathbb{B}$ and a relatively compact open set $\mathcal{K} \Subset \mathcal{Y}$, there exist $0 < \epsilon \ll 1$, a neighborhood $\mathcal{W}_\epsilon$ of $\mathcal{K}_\epsilon$ inside $\mathbb{C}^N \times \mathbb{B}$, and a biholomorphism Φ of the form $\Phi(w,t) = (\Psi_t(w), t)$, $\Psi_0 = \mathrm{Id}$, from $\mathcal{W}_\epsilon$ onto its image in $\mathbb{C}^N \times \mathbb{B}$ such that $\Phi|_{\mathcal{K}_\epsilon}$ is a p-trivial embedding.

Proof
Assume that there is a p-trivial embedding with p-adapted atlas $\{\mathcal{U}_\alpha, \Phi_\alpha = (z^i_\alpha, t)\}_{\alpha \in \mathcal{A}}$. We prove that the p-adapted atlas is a p-trivial atlas defined in Definition 2.4. In other words, we want to show that

$$\frac{\partial^l (z_\alpha(z_\beta, t) - z_\alpha(z_\beta, 0))}{\partial t^l}\bigg|_{t=0} = 0, \quad \text{for } 0 \le l \le p.$$

We prove this by induction. The case of $l = 0$ is automatically true. Assume that this is proved for the $(l-1)$th-order derivative for some $1 \le l \le p$. Then we take the

lth-order derivative on both sides of (with respect to t at $t=0$),

$$w^b = w^b_\alpha(z_\alpha, t) = w^b_\alpha\big(z_\alpha(z_\beta, t), t\big) = w^b_\beta(z_\beta, t),$$

and we use the $(l-1)$-trivial atlas and the l-adapted property to get

$$\begin{aligned} 0 &= \frac{\partial^l w^b_\beta(z_\beta, t)}{\partial t^l}\Big|_{t=0} \\ &= \sum_{i=1}^n \frac{\partial w^b_\alpha(z_\alpha, t)}{\partial z^i_\alpha} \frac{\partial^l z^i_\alpha(z_\beta, t)}{\partial t^l}\Big|_{t=0} + \frac{\partial^l w^b_\alpha(z_\alpha, t)}{\partial t^l}\Big|_{t=0} \\ &= \sum_{i=1}^n \frac{\partial w^b_\alpha(z_\alpha, t)}{\partial z^i_\alpha} \frac{\partial^l z^i_\alpha(z_\beta, t)}{\partial t^l}\Big|_{t=0}. \end{aligned}$$

Because the $N \times n$ matrix

$$M_{bi} = \frac{\partial w^b_\alpha(z_\alpha, t)}{\partial z^i_\alpha}$$

has rank n and zero kernel, we get $\frac{\partial^l z^i_\alpha(z_\beta,t)}{\partial t^l}|_{t=0} = 0$. So the atlas is l-trivial. This completes the induction argument and shows that the p-adapted atlas is indeed p-trivial.

Conversely, we choose a p-trivial atlas $\mathcal{U} = \{\mathcal{U}_\alpha, \Phi_\alpha = (z_\alpha, t)\}_{\alpha \in \mathcal{A}}$ covering $\mathcal{Y}_0$ and an embedding which, for each $\alpha \in \mathcal{A}$, is represented by $w^b = w^b_\alpha(z_\alpha, t)$. Then we have the relation

$$w^b = w^b_\beta(z_\beta, t) = w^b_\beta\big(z_\beta(z_\alpha, t), t\big) = w^b_\alpha(z_\alpha, t).$$

Taking the derivative on both sides at $t=0$ and using the p-trivial condition of the atlas, we get

$$\frac{\partial^l w^b_\alpha}{\partial t^l}\Big|_{t=0} = \sum_{i=1}^n \frac{\partial w^b_\beta}{\partial z^i_\beta} \frac{\partial^l z^i_\beta(z_\alpha, t)}{\partial t^l}\Big|_{t=0} + \frac{\partial^l w^b_\beta}{\partial t^l}\Big|_{t=0} = \frac{\partial^l w^b_\beta}{\partial t^l}\Big|_{t=0}, \quad 1 \le l \le p.$$

So we see that for each $1 \le l \le p$, there is a globally defined vector field

$$v^{(l)} = \sum_{b=1}^N \frac{\partial^l w^b_\beta}{\partial t^l}\Big|_{t=0} \frac{\partial}{\partial w^b} \in H^0(Y, \Theta_{\mathbb{C}^N}|_Y).$$

We claim that the given embedding can be modified to become a p-trivial embedding on any relatively compact open subset. We do this by induction as follows. Assume that we already get an $(l-1)$-trivial embedding for some $1 \le l \le p$. Let $\sigma^{(l)}(w, s)$

be the flow generated by an extension of holomorphic vector field $-v^{(l)}/l!$ to $\mathbb{C}^N$. Note that $\sigma^{(l)}(w,s)$ exists on a relatively compact open subset for $|s|$ sufficiently small.

Set $\Phi(w,t) = (\sigma^{(l)}(w,t^l),t) =: (\Psi_t(w),t)$. Then Φ is a biholomorphism defined on a relatively compact open neighborhood $\mathcal{W}_\epsilon$ of $\mathcal{K}_\epsilon$ when ϵ is sufficiently small. Define a new embedding $\tilde{\iota}_{\mathcal{W}_\epsilon} := \Phi \circ \iota_{\mathcal{Y}}|_{\mathcal{W}_\epsilon}$. Then there is a new representation $\tilde{w}^b = \tilde{w}^b(w_\alpha(z_\alpha,t),t) = \tilde{w}^b_\alpha(z_\alpha,t)$. We can then take the derivative with respect to t by using the $(l-1)$-trivial condition to see that $\tilde{\iota}_{\mathcal{Y}}$ is indeed an l-trivial embedding:

$$\begin{aligned}
&\sum_{b=1}^N \frac{\partial^l \tilde{w}^b(z_\alpha,t)}{\partial t^l}\Big|_{t=0}\frac{\partial}{\partial w^b}\\
&\quad= \sum_{b=1}^N\sum_{c=1}^N \frac{\partial \tilde{w}^b}{\partial w^c}\frac{\partial^l w^c(z_\alpha,t)}{\partial t^l}\Big|_{t=0}\frac{\partial}{\partial w^b} + \sum_{b=1}^N \frac{\partial^l \tilde{w}^b(w,t)}{\partial t^l}\Big|_{t=0}\frac{\partial}{\partial w^b}\\
&\quad= \sum_{c=1}^N \frac{\partial^l w^c_\alpha(z_\alpha,t)}{\partial t^l}\Big|_{t=0}\frac{\partial}{\partial w^c} - v^{(l)} = 0.
\end{aligned}$$

□

The first statement of following lemma generalizes Lemma 2.3.

LEMMA 2.8

(1) *If there is a p-trivial embedding $\iota_{\mathcal{Y}}: \mathcal{Y} \to \mathbb{C}^N \times \mathbb{B}$ with p-adapted atlas $\{U_\alpha, \Phi_\alpha = (z_\alpha,t)\}$, then we can define a global section $\mathfrak{v}_{p+1} := \mathfrak{v}_{p+1}(\iota_{\mathcal{Y}},\Phi_\alpha) \in H^0(\mathcal{Y}_0, N_{\mathcal{Y}_0})$ such that*

$$\begin{aligned}
\mathfrak{v}_{p+1}(U_\alpha) &= \frac{1}{(p+1)!}\Big[\sum_{b=1}^N \frac{\partial^{p+1} w^b_\alpha(z_\alpha,t)}{\partial t^{p+1}}\Big|_{t=0}\frac{\partial}{\partial w^b}\Big]\\
&\in H^0(\mathcal{U}_\alpha \cap \mathcal{Y}_0, N_{\mathcal{Y}_0}),
\end{aligned} \tag{2.13}$$

where we used the natural morphism $\Theta_{\mathbb{C}^N}|_{\mathcal{Y}_0} \to N_{\mathcal{Y}_0}$ (remember that $(w^b)_{b=1}^N$ denotes the standard coordinates on $\mathbb{C}^N$). Furthermore, $\delta_Y(\mathfrak{v}_{p+1}) = \theta_{p+1}$ where δ_Y is the connecting morphism $\delta_Y: H^0(\mathcal{Y}_0, N_{\mathcal{Y}_0}) \to H^1(\mathcal{Y}_0, \Theta_{\mathcal{Y}_0})$ introduced in (2.3) and θ_{p+1} is the reduced Kodaira–Spencer cocycle associated to the p-adapted atlas.

(2) *Assume that there is another p-adapted atlas $\{\tilde{\mathcal{U}}_\alpha, \tilde{\Phi}_\alpha = (\tilde{z}^i_\alpha,t)\}$ for the same embedding $\iota_{\mathcal{Y}}$. If we denote $\tilde{\mathfrak{v}}_{p+1} = \mathfrak{v}_{p+1}(\iota_{\mathcal{Y}},\tilde{\Phi}_\alpha)$, then $\delta_Y(\mathfrak{v}_{p+1} - \tilde{\mathfrak{v}}_{p+1}) = 0$.*

Proof

By the proof of Lemma 2.7, a p-adapted atlas is p-trivial. So we can use the p-trivial condition to take the $(p+1)$th-order derivative with respect to t at $t=0$ on both sides of the identity

$$w^b = w^b_\beta(z_\beta, t) = w^b_\beta\big(z^i_\beta(z^j_\alpha, t), t\big) = w^b_\alpha(z_\alpha, t)$$

to get

$$\frac{\partial^{p+1} w^b_\alpha(z_\alpha, t)}{\partial t^{p+1}} = \sum_{i=1}^{n} \frac{\partial w^b_\beta}{\partial z^i_\beta} \frac{\partial^{p+1} z^i_\beta}{\partial t^{p+1}} + \frac{\partial^{p+1} w^b(z_\beta, t)}{\partial t^{p+1}}. \tag{2.14}$$

If we define

$$v_\alpha = \frac{1}{(p+1)!} \sum_{b=1}^{N} \frac{\partial^{p+1} w^b(z_\alpha, t)}{\partial t^{p+1}} \frac{\partial}{\partial w^b}\Big|_{t=0},$$

then $v_\alpha - v_\beta = \iota_{Y*}(\theta_{p+1})_{\beta\alpha}$. So $\{[v_\alpha]\}_{\alpha\in\mathcal{A}}$ can be glued to become a global section $\mathfrak{v}_p \in H^0(Y, N_Y)$ using the fact that $N_Y = \Theta_{\mathbb{C}^N}/\Theta_Y$.

For the second item, we use (2.14) to get the following identities:

$$\begin{aligned}\delta(\mathfrak{v}_{p+1} - \tilde{\mathfrak{v}}_{p+1})(U_\alpha \cap U_\beta) &= \iota_{Y*}\big((\theta_{p+1})_{\beta\alpha}\big) - \iota_{Y*}\big((\tilde{\theta}_{p+1})_{\beta\alpha}\big)\\ &= \iota_{Y*}\big((\theta_{p+1})_{\beta\alpha} - (\tilde{\theta}_{p+1})_{\beta\alpha}\big).\end{aligned}$$

By Lemma 2.5(2), more specifically identity (2.8), we know that $\theta_{p+1} - \tilde{\theta}_{p+1} = 0 \in H^1(Y, \Theta_Y)$. So the proof is complete. □

LEMMA 2.9

Assume that there exists a $(p-1)$-trivial embedding $\iota_{\mathcal{Y}} : \mathcal{Y} \to \mathbb{C}^N \times \mathbb{B}$ with $\mathfrak{v}_p(\iota_{\mathcal{Y}}) = 0 \in H^0(\mathcal{Y}_0, N_{\mathcal{Y}_0})$ (see Lemma 2.8 for the definition of $\mathfrak{v}_p$). Then for any relatively compact open subset $\mathcal{K} \Subset \mathcal{Y}$, there is a p-trivial embedding $\mathcal{K}_\epsilon \hookrightarrow \mathbb{C}^N \times \mathbb{B}_\epsilon$ for $0 < \epsilon \ll 1$.

Proof

We need to prove that there exists an atlas satisfying the condition (2.11). By assumption, there is an atlas $\mathcal{U} = \{\mathcal{U}_\alpha, \Phi_\alpha = (z_\alpha, t)\}_{\alpha\in\mathcal{A}}$ covering $\mathcal{Y}_0$ such that the following condition is satisfied: for each $\alpha \in \mathcal{A}$, if the embedding $\iota_{\mathcal{Y}}|_{\mathcal{U}_\alpha} : \mathcal{U}_\alpha \to \mathbb{C}^N \times \mathbb{B}$ is represented by the function $w^b = w^b_\alpha(z_\alpha, t)$, then we have $\frac{\partial^l w^b_\alpha(z_\alpha,t)}{\partial t^l}|_{t=0} = 0$ ($b = 1, \ldots, N$ and $1 \le l \le p-1$), and (see (2.13))

$$\frac{1}{p!} \sum_{b=1}^{N} \frac{\partial^p w^b_\alpha(z_\alpha, t)}{\partial t^p}\Big|_{t=0} \frac{\partial}{\partial w^b} \in \Theta_{\mathcal{Y}_0}(\mathcal{U}_\alpha \cap \mathcal{Y}_0). \tag{2.15}$$

So we get functions $d_\alpha^i(z_\alpha)$ satisfying

$$\sum_{b=1}^{N} \frac{\partial^p w_\alpha^b(z_\alpha,t)}{\partial t^p}\Big|_{t=0} \frac{\partial}{\partial w^b} = \sum_{i=1}^{n} d_\alpha^i \frac{\partial}{\partial z_\alpha^i}$$
$$= \sum_{i=1}^{n} d_\alpha^i \frac{\partial w_\alpha^b(z_\alpha,0)}{\partial z_\alpha^i} \frac{\partial}{\partial w^b} \in \Theta_{\mathcal{Y}_0}(\mathcal{U}_\alpha \cap \mathcal{Y}_0). \qquad (2.16)$$

Define the new functions $\tilde{z}_\alpha^i = z_\alpha^i + d_\alpha^i \frac{t^p}{p!}$ which are coordinates on $\mathcal{K}_\epsilon$ for $0 < \epsilon \ll 1$. Taking the derivative all the way up to order p on both sides of

$$w^b = w_\alpha^b(z_\alpha,t) = w_\alpha^b\big(z_\alpha(\tilde{z}_\alpha,t),t\big) = \tilde{w}_\alpha^b(\tilde{z}_\alpha,t)$$

at $t = 0$, we get

$$\frac{\partial^p w^b(\tilde{z}_\alpha,t)}{\partial t^p}\Big|_{t=0} = \sum_{i=1}^{n} \frac{\partial w_\alpha^b(z_\alpha,0)}{\partial z_\alpha^i} \frac{\partial^p z_\alpha^i(\tilde{z}_\alpha,t)}{\partial t^p} + \frac{\partial^p w_\alpha^b(z_\alpha,t)}{\partial t^p}\Big|_{t=0}$$
$$= -\sum_{i=1}^{n} d_\alpha^i \frac{\partial w_\alpha^b(z_\alpha,0)}{\partial z_\alpha^i} + \frac{\partial^p w_\alpha^b(z_\alpha,t)}{\partial t^p}\Big|_{t=0} = 0.$$

So we see that the atlas $\{\mathcal{U}_\alpha, (\tilde{z}_\alpha, t)\}$ is indeed a p-adapted atlas. □

LEMMA 2.10

Let $\pi : \mathcal{Y} \to \mathbb{B}$ be a holomorphic family of complex manifolds embedded into $\mathbb{C}^N \times \mathbb{C}$. Let $\mathcal{K} \subset \mathcal{Y}$ be a relatively compact open set such that there exist a bounded open set $\mathcal{W} \subset \mathbb{C}^N \times \mathbb{C}$ and $H_1, \dots, H_d \in \mathcal{O}(\mathcal{W})$ satisfying

$$\mathcal{K} = \big\{(w,t) \in \mathcal{W} : H_1(w,t) = \cdots = H_d(w,t) = 0\big\}. \qquad (2.17)$$

Then for all $p \geq 1$, the following are equivalent (see (2.12) for notation):

(1) *There exists $0 < \epsilon \ll 1$ such that there exists a p-trivial atlas on $\mathcal{K}_\epsilon$.*

(2) *There exist $0 < \epsilon \ll 1$ and a biholomorphism Φ of the form $\Phi(w,t) = (\Psi_t(w), t)$, $\Psi_0 = \mathrm{Id}$, from $\mathcal{W}_\epsilon$ onto its image in $\mathbb{C}^N \times \mathbb{C}$ such that $\Phi|_{\mathcal{K}_\epsilon}$ is a p-trivial embedding.*

(3) *There exist $0 < \epsilon \ll 1$ and a biholomorphism Φ of the form $\Phi(w,t) = (\Psi_t(w,t), t)$, $\Psi_0 = \mathrm{Id}$, from $\mathcal{W}_\epsilon$ onto its image in $\mathbb{C}^N \times \mathbb{C}$ such that*

$$\Phi(\mathcal{K}_\epsilon) = \big\{(w,t) \in \Phi(\mathcal{W}_\epsilon) : F_1(w,t) = \cdots = F_d(w,t) = 0\big\}$$

for some holomorphic functions $F_1, \dots, F_d$ with $F_m(w,t) = F_m(w,0) + t^{p+1} G_m(w,t)$.

Proof

The equivalence of (1) and (2) has been proved in Lemma 2.7. We now prove that (2) implies (3). So assume that $\Phi|_{\mathcal{K}_\epsilon}$ is a p-trivial embedding with a p-adapted atlas $\{\mathcal{U}_\alpha, (z_\alpha, t)\}$. Set $F_m = H_m \circ \Phi^{-1}$. Then the ideal sheaf of $\Phi(\mathcal{K}_\epsilon)$ is generated by $\{F_1(w,t), \dots, F_d(w,t)\}$. We will prove by induction that there exists a sequence of open sets $\mathcal{W} = \mathcal{W}^{(0)} \supseteq \mathcal{W}^{(1)} \supseteq \cdots \supseteq \mathcal{W}^{(p+1)}$ and holomorphic functions $F_m^{(l)}(w,t)$ on $\mathcal{W}^{(l)}$ such that

(i) $\mathcal{Y} \cap \mathcal{W}^{(l)}$ is generated by $F_m^{(l)}(w,t)$;

(ii) there exist holomorphic functions $G_{m,l}(w,t)$ on $\mathcal{W}^{(l)}$ such that

$$F_m^{(l)}(w,t) = F_m^{(l)}(w,0) + t^l G_{m,l}(w,t). \tag{2.18}$$

For $l = 1$, let $\mathcal{W}^{(1)} = \mathcal{W}$, let $F_m^{(1)}(w,t) = F_m(w,t)$, and let $G_{m,1}(w,t) = \frac{1}{t}(F_m(w,t) - F_m(w,0))$. Then assume that the statement is true for $1 \le l \le p$. We have the identity

$$F_m^{(l)}\big(w_\alpha^1(z_\alpha,t), \dots, w_\alpha^N(z_\alpha,t), t\big) \equiv 0. \tag{2.19}$$

Taking the derivative with respect to t l times and using the identities (2.11) and (2.19) we get

$$G_{m,l}\big(w_\alpha(z_\alpha,t),t\big)\Big|_{t=0} = 0. \tag{2.20}$$

Because the ideal sheaf of $\mathcal{K}_0 \cap \mathcal{W}^{(l)}$ is generated by $\{F_1^{(l)}(w,0), \dots, F_d^{(l)}(w,0)\}$, there exists $h_{m,l,r}(w)$ such that

$$G_{m,l}(w,0) = \sum_{m=1}^{d} F_r^{(l)}(w,0) h_{m,l,r}(w). \tag{2.21}$$

Now define

$$\begin{aligned} F_m^{(l+1)}(w,t) &= F_m^{(l)}(w,t) - t^l \sum_{r=1}^{d} F_r^{(l)}(w,t) h_{m,r,l}(w) \\ &= F_m^{(l)}(w,0) + t^l G_{m,l}(w,t) - t^l \sum_{r=1}^{d} F_r^{(l)}(w,t) h_{m,r,l}(w). \end{aligned}$$

We then have

$$\frac{\partial^l F^{(l+1)}(w,t)}{\partial t^l}\Big|_{t=0} = 0.$$

So we know that $F_{m,l}^{(l+1)}(w,t)$ has the expansion

$$F_m^{(l+1)}(w,t) = F_m^{(l+1)}(w,0) + t^{l+1} G_{m,l+1}(w,t) \tag{2.22}$$

over an open subset $\mathcal{W}^{(l+1)}$ of $\mathcal{W}^{(l)}$. Note that $\{F_r^{(l+1)}\}$ generates the same ideal as the $\{F_r^{(l)}\}$. Indeed $\{F_r^{(l+1)}\}$ is obtained by multiplying a holomorphic matrix of the form $\mathrm{Id}_{d\times d} + O(t^l)$ to $\{F_r^{(l)}\}$. Because $l \geq 1$, this matrix has a holomorphic inverse for $|t| \ll 1$. So it is easy to see that $F_m^{(l+1)}$ satisfies the wanted properties.

Conversely, we assume that (3) holds, and we consider the biholomorphism Φ of (3). Choose an arbitrary atlas $\mathcal{U} = \{\mathcal{U}_\alpha, \Phi_\alpha = (z_\alpha, t)\}$ covering $\mathcal{K}_\epsilon$. We want to use induction to prove that there exists an l-adapted atlas for the embedding $\Phi|_{\mathcal{K}_\epsilon}$ for $1 \leq l \leq p$. Assume that this has been proved for $l-1$. This is trivially true when $l = 1$. Then note that

$$(F_r + t^{p+1} G_r)\big(w^b(z_\alpha, t)\big) = 0, \quad \text{for } 1 \leq r \leq d. \tag{2.23}$$

Taking the lth-order derivative on both sides of (2.23) and using the $(l-1)$-adapted property $\frac{\partial^j w_\alpha^b}{\partial t^j}|_{t=0} = 0$ for $1 \leq j \leq l-1$, we get

$$\sum_{b=1}^{N} \frac{\partial F_r}{\partial w^b} \frac{\partial^l w_\alpha^b}{\partial t^l}\Big|_{t=0} = 0, \quad \text{for } 1 \leq r \leq d.$$

Since $\{F_r\}$ are defining functions of $\mathcal{K}_0$, this means that the vector field $\sum_{b=1}^{N} \frac{\partial^l w_\alpha^b}{\partial t^l} \frac{\partial}{\partial w^b}|_{t=0}$ is tangent to $\mathcal{K}_0$. So there exists $c_\alpha^i = c_\alpha^i(z_\alpha)$ such that

$$\sum_{b=1}^{N} \frac{\partial^l w_\alpha^b}{\partial t^l} \frac{\partial}{\partial w^b}\Big|_{t=0} = \sum_{i=1}^{n} c_\alpha^i \frac{\partial}{\partial z_\alpha^i} = \sum_{b=1}^{N} \sum_{i=1}^{n} c_\alpha^i \frac{\partial w_\alpha^b}{\partial z_\alpha^i} \frac{\partial}{\partial w^b}\Big|_{t=0}. \tag{2.24}$$

Now define a new coordinate function

$$\tilde{z}_\alpha^i = z_\alpha^i + \frac{t^l}{l!} c_\alpha^i(z_\alpha).$$

Then we get a new representation of the embedding on $\mathcal{U}_\alpha$:

$$\tilde{w}^b = w^b(z_\alpha, t) = w^b\big(z_\alpha(\tilde{z}_\alpha, t), t\big).$$

Taking lth-order derivatives on both sides, by (2.24) we get

$$\frac{\partial^l \tilde{w}^b}{\partial t^l}\Big|_{t=0} = -\sum_{i=1}^{n} \frac{\partial w_\alpha^b}{\partial z_\alpha^i} c_\alpha^i + \frac{\partial^l w_\alpha^b}{\partial t^l}\Big|_{t=0} = 0.$$

So by induction, we indeed get a p-adapted atlas on $\mathcal{K}_\epsilon$ for $0 < \epsilon \ll 1$. ☐

2.3. Higher-order deformation of a normal isolated singularity via higher-order deformation of a regular part

Let $Z \subset \mathbb{C}^N$ be an affine algebraic variety with exactly one singularity $o \in Z$, and we can assume that this singularity is the origin $0 \in \mathbb{C}^N$. Assume that there is a holomorphic family of complex analytic varieties $\mathcal{Z} \to \mathbb{B}$ which is a deformation of the analytic germ $(\mathcal{Z}_0, o) = (Z, o)$. For any $k \geq 0$, this induces a deformation over the analytic space $\mathbb{B}(k) = (\mathbb{B}, \mathcal{O}_{\mathbb{B}}/\mathcal{I}_0^{k+1})$, where $\mathcal{I}_0 = (t)$ is the ideal sheaf of the point $0 \in \mathbb{B}$. Indeed, we have the flat morphism $\mathcal{Z}(k) := \mathcal{Z} \times_{\mathbb{B}} \mathbb{B}(k) \to \mathbb{B}(k)$.

Definition 2.11

The order of the deformation $(\mathcal{Z}, (\mathcal{Z}_0, o)) \to (\mathbb{B}, 0)$ is defined to be the natural number

$$\mathrm{Ord}\big((\mathcal{Z}, (\mathcal{Z}_0, o))/(\mathbb{B}, 0)\big) = \max\big\{k+1; \mathcal{Z}(k) \to \mathbb{B}(k) \text{ is trivial}\big\}.$$

If the pointed base $(\mathbb{B}, 0)$ and the point $o \in \mathcal{Z}$ are clear, we shall just write $\mathrm{Ord}(\mathcal{Z})$ for $\mathrm{Ord}((\mathcal{Z}, (\mathcal{Z}_0, o))/(\mathbb{B}, 0))$.

It is well known that higher-order deformation theory in the algebraic category (see [4], [17, Theorem 10.1]) can also be developed in the analytic category (cf. [15, Proposition 1.29]). Given a deformation of certain order, the space of possible deformations to the next order is a principal homogeneous space under $\mathbf{T}^1_Z$, that is, an affine space without preferred origin. More precisely, suppose that there is a flat family $\mathcal{Z}(k) \to \mathbb{B}(k)$ and an extension to $\mathcal{Z}^*(k+1) \to \mathbb{B}(k+1)$ of $\mathcal{Z}(k)$ with $\mathcal{Z}^*(k) = \mathcal{Z}^*(k+1) \times_{\mathbb{B}(k+1)} \mathbb{B}(k) = \mathcal{Z}(k)$. Then the set of $(k+1)$th-order deformations that extend the kth-order deformation $\mathcal{Z}(k) \to \mathbb{B}(k)$ can be identified with $\mathbf{T}^1_Z$. In the special case at hand, there is a preferred origin given by the trivial deformation and this allows us to define a reduced Kodaira–Spencer class.

Definition 2.12

Suppose that there is a flat family $\mathcal{Z} \to \mathbb{B}$ of complex analytic varieties with $(\mathcal{Z}_0, o) = (Z, o)$. Assume that $\mathcal{Z}(k) \to \mathbb{B}(k)$ is trivial for a fixed $k \geq 0$. If the trivial deformation $\mathcal{Z}^*(k+1) := Z \times \mathbb{B}(k+1)$ is used as the basepoint so that $\mathcal{Z}^*(k) = Z \times \mathbb{B}(k)$ coincides with $\mathcal{Z}(k)$, then the corresponding class representing $\mathcal{Z}(k+1)$ in $\mathbf{T}^1_Z$ is defined to be the $(k+1)$th-order Kodaira–Spencer class of $\mathcal{Z} \to \mathbb{B}$ and is denoted by $\mathbf{KS}^{(k+1)}_{\mathcal{Z}}$. If $p+1 = \mathrm{Ord}(\mathcal{Z})$, then we define the reduced Kodaira–Spencer class as $\mathbf{KS}^{\mathrm{red}}_{\mathcal{Z}} = \mathbf{KS}^{(p+1)}_{\mathcal{Z}}$.

LEMMA 2.13

With the same notation as above, if $\mathrm{Ord}(\mathcal{Z}) \geq p+1$*, then there exist a small neighborhood* $\mathcal{W}$ *of* $o \in \mathbb{C}^N \times \mathbb{C}$ *and a biholomorphism* Φ *of the form* $\Phi(w,t) = (\Psi_t(w), t)$*,*

$\Psi_0 = \mathrm{Id}$, *from* $\mathcal{W}$ *onto its image in* $\mathbb{C}^N \times \mathbb{C}$ *such that the ideal sheaf of* $\Phi(\mathcal{Z})$ *in* $\Phi(\mathcal{W})$ *is generated by* $F_1(w,t), \dots, F_d(w,t)$ *satisfying* $F_m(w,t) = F_m(w,0) + t^{p+1} G_m(w,t)$ *on* $\mathcal{W}$ *with* $G_m(w,t)$ *analytic in* $\mathcal{W}$.

Proof

By assumption, there exists an isomorphism of quotients of power series rings

$$\begin{aligned}\phi : \mathbb{C}\{w^1, \dots, w^N, t\}\}/\big(F_1(w,t), \dots, F_d(w,t), t^{p+1}\big) \\ \to \mathbb{C}\{\hat{w}^1, \dots, \hat{w}^N, t\}/\big(f_1(\hat{w}), \dots, f_d(\hat{w}), t^{p+1}\big),\end{aligned}$$

where $F_1(w,t), \dots, F_d(w,t)$ are defining equations of the germ $(\mathcal{Z}, (o,0)) \subset (\mathbb{C}^N \times \mathbb{C}, (o,0))$. We will change the embedding of $(\mathcal{Z}, (o,0))$ several times during the proof but will continue to use $F_m(w,t)$ to denote the defining equations of $(\mathcal{Z}, (o,0))$ in each step.

Assume that ϕ is represented by functions $w^b = B_b(\hat{w}^1, \dots, \hat{w}^N, t)$. Then we have

$$F_r\big(B_b(\hat{w},t),t\big) = \sum_{l=1}^{d} f_l(\hat{w}) h_{r,l}(\hat{w},t) + t^{p+1} u_r(\hat{w},t), \quad r = 1, \dots, d, \tag{2.25}$$

where $h_{r,l}$ and u_r are holomorphic near $o \in \mathbb{C}^N \times \mathbb{C}$. We can assume that $B_b(\hat{w}_1, \dots, \hat{w}^N, 0) = \hat{w}^b$ and $F_r(B_b(\hat{w},0),0) = F_r(\hat{w},0) =: f_r(\hat{\omega})$ so that $h_{r,l}(\hat{w},0) = \delta_{rl}$. Multiplying (2.25) by the inverse matrix $(h_{r,l})^{-1}$ (which exists for $|t|$ sufficiently small) and replacing F_r, we can assume that $h_{r,l}(\hat{w},t) = \delta_{rl}$ so that the following identities hold:

$$F_r\big(B_b(\hat{w},t),t\big) = f_r(\hat{w}) + t^{p+1} u_r(\hat{w},t). \tag{2.26}$$

We will prove by induction that there exist a small open neighborhood $\mathcal{W}$ of $(o,0) \in \mathbb{C}^N \times \mathbb{C}$ and a biholomorphism Φ of the form $\Phi(w,t) = (\Psi_t(w),t)$, $\Psi_0 = \mathrm{Id}$, from $\mathcal{W}$ onto its image $\mathbb{C}^N \times \mathbb{C}$ such that $\Phi(\mathcal{Z} \cap \mathcal{W})$ is defined by equations $F_r(w,t) = 0$, where the following hold for any $0 \le l \le p$:

$$\frac{\partial^l (F_r(w,t) - f_r(w))}{\partial t^l}\Big|_{t=0} = 0 \qquad \text{and} \qquad \frac{\partial^l (B_b(\hat{\omega},t) - \hat{w})}{\partial t^l}\Big|_{t=0} = 0. \tag{2.27}$$

This clearly implies the statement of the lemma.

The identity (2.27) holds for $l = 0$. Assume (2.27) for $l-1$. Taking the derivative for both sides of (2.26) with respect to t l times and evaluating at $t = 0$, we get

$$\sum_{b=1}^{N} \frac{\partial F_r(w,0)}{\partial w^b} \frac{\partial^l B_b(\hat{w},t)}{\partial t^l} + \frac{\partial^l F_r(w,t)}{\partial t^l}\Big|_{t=0} = 0. \tag{2.28}$$

Note that

$$v := \sum_{b=1}^{N} v^b \frac{\partial}{\partial w^b}\Big|_{\mathcal{Z}_0} = -\sum_{b=1}^{N} \frac{\partial^l B_b(\hat{w},t)}{\partial t^l}\Big|_{t=0} \frac{\partial}{\partial w^b} \in H^0(\mathcal{Z}_0, \Theta_{\mathbb{C}^N}|_{\mathcal{Z}_0}) \tag{2.29}$$

is a globally defined vector field on $\mathcal{Z}_0$. Let $\sigma(w,s)$ be the one-parameter subgroup generated by a holomorphic extension of v. Then $\sigma(w,s)$ exists for $|s|$ sufficiently small on an open neighborhood of $o \in \mathcal{Z}_0 \subset \mathbb{C}^N \times \{0\}$. Set

$$\begin{aligned} \tilde{w} &= \tilde{w}(w,t) = \sigma(w, t^l/l!), \\ \tilde{F}_r(\tilde{w},t) &= F_r\big(w(\tilde{w},t),t\big), \\ \tilde{B}(\hat{w},t) &= \tilde{w}\big(w(\hat{w},t),t\big). \end{aligned} \tag{2.30}$$

In particular, $\frac{\partial^l \tilde{w}^b(w,t)}{\partial t^l}|_{t=0} = v^b = -\frac{\partial^l w^b(\tilde{w},t)}{\partial t^l}|_{t=0}$ for $b = 1, \ldots, N$. Then we get, since $l \geq 1$,

$$\begin{aligned} &\frac{\partial^l}{\partial t^l}\big(\tilde{F}_r(\tilde{w},t) - F_r(\tilde{w},0)\big)\Big|_{t=0} \\ &\quad = \frac{\partial^l}{\partial t^l} F_r\big(w(\tilde{w},t),t\big)\Big|_{t=0} \\ &\quad = \sum_{b=1}^{N} \frac{\partial F_r(w,0)}{\partial w^b} \frac{\partial^l w^b(\tilde{w},t)}{\partial t^l} + \frac{\partial^l F_r(w,t)}{\partial t^l}\Big|_{t=0} \\ &\quad = \sum_{b=1}^{N} \frac{\partial F_r(w,0)}{\partial w^b} \frac{\partial^l B_b(\hat{w},t)}{\partial t^l} + \frac{\partial^l F_r(w,t)}{\partial t^l}\Big|_{t=0} = 0 \quad \text{(by (2.28))}. \end{aligned}$$

Moreover, we have the vanishing

$$\begin{aligned} \frac{\partial^l}{\partial t^l} \tilde{B}_b(\hat{w},t)\Big|_{t=0} - \hat{w} &= \frac{\partial^l}{\partial t^l} \tilde{w}^b\big(B(\hat{w},t),t\big)\Big|_{t=0} \\ &= \sum_{c=1}^{N} \frac{\partial \tilde{w}^b(w,0)}{\partial w^c} \frac{\partial^l}{\partial t^l} B_c(\hat{w},t) + \frac{\partial^l}{\partial t^l} \tilde{w}^b(w,t)\Big|_{t=0} \\ &= \frac{\partial^l B_b(\hat{w},t)}{\partial t^l}\Big|_{t=0} - \frac{\partial^l B_b(\hat{w},t)}{\partial t^l}\Big|_{t=0} = 0. \end{aligned}$$

So the induction argument completes. □

If $\mathrm{Ord}(\mathcal{Z}) \geq p+1$, then by Lemma 2.13, after changing the embedding of $\mathcal{Z}$, there exists a small open neighborhood $\mathcal{W}$ of $(o,0) \in \mathbb{C}^N \times \mathbb{C}$ such that $\mathcal{I}_{\mathcal{Z}}(\mathcal{W})$ is generated by $\{F_i(w,t) = F_i(w,0) + t^{p+1}G_i(w,t)\}$. In particular $\mathcal{I}_{\mathcal{Z}_0}(\mathcal{W} \cap \mathcal{Z}_0)$ is generated by $\{f_1, \ldots, f_d\}$ where $f_i(w) := F_i(w,0)$ for $i = 1, \ldots, d$. Set $g_i(w) = G_i(w,0)$. The flatness condition of $\mathcal{Z} \to \mathbb{B}$ implies that $\{g_i\}$ (and hence $\{G_i\}$) determines a well-defined morphism (see [15, Proposition II.1.25] and [4, Section 6])

$$\bar{g} : \mathcal{I}_Z \to \mathcal{O}_{\mathbb{C}^N}/\mathcal{I}_Z, \qquad \sum_{r=1}^{d} f_r h_r \mapsto \sum_{r=1}^{d} g_r h_r. \tag{2.31}$$

We have $\bar{g} \in \mathrm{Hom}_{\mathcal{O}_{\mathbb{C}^N}}(\mathcal{I}_Z, \mathcal{O}_{\mathbb{C}^N}/\mathcal{I}_Z) = \mathrm{Hom}_{\mathcal{O}_Z}(\mathcal{I}_Z/\mathcal{I}_Z^2, \mathcal{O}_Z) = H^0(Z, N_Z)$. So if $\mathrm{Ord}(\mathcal{Z}) \geq p+1$, then there is a well-defined class

$$-\psi_Z(\bar{g}) \in \mathbf{T}^1_Z, \tag{2.32}$$

where $\psi_Z : H^0(Z, N_Z) \to \mathbf{T}^1_Z$ was defined by Schlessinger (see (A.5)). This class is exactly the Kodaira–Spencer class $\mathbf{KS}^{(p+1)}_{\mathcal{Z}}$ of Definition 2.12. Notice that here we are working in the analytic category as in [31] and [15].

From now on assume that Z has a normal isolated singularity at o, and denote $U = Z \setminus \{o\}$. Schlessinger showed in [31] that the (infinitesimal) embeddable deformations can be determined by deformations of U and that $\mathbf{T}^1_Z$ is a subspace of $H^0(U, \Theta_U)$ (see Proposition A.10 and (A.7)). More precisely, there are two exact sequences:

$$\begin{aligned} H^0(U, \Theta_{\mathbb{C}^N}|_U) \to H^0(U, N_U) \overset{\psi_U}{\to} \mathbf{T}^1_Z \to 0, \\ 0 \to \mathbf{T}^1_Z \xrightarrow{\tau_U} H^1(U, \Theta_U) \to H^1(U, \Theta_{\mathbb{C}^N}|_U). \end{aligned} \tag{2.33}$$

Fix an embedding $o \in Z \hookrightarrow \mathbb{C}^N$, and let $\{w_i\}_{i=1}^N$ be the standard coordinates of $\mathbb{C}^N$ with $w_i(o) = 0$. Choose a smooth, strictly plurisubharmonic function φ on $\mathbb{C}^N$ such that the following conditions are satisfied:

(1) $\varphi|_U > 0$ is a strict plurisubharmonic function on $U = Z \backslash o$;

(2) for any $\epsilon > 0$ and $c > 0$, the subset $\{p \in U; \epsilon < \varphi(p) < c\}$ is relatively compact in U;

(3) for $c > 0$, the subset $K_c := \{p \in U; \varphi(p) \leq c\}$ satisfies that ∂K is compact and strongly pseudoconvex.

Now assume that (Z, o) is the germ of the vertex of an affine cone $Z = C(D, L)$ and that $\mathcal{Z}$ is a $\mathbb{C}^*$-equivariant deformation of Z. We can then assume that the embedding of $\mathcal{Z}$ into $\mathbb{C}^N \times \mathbb{C}$ is $\mathbb{C}^*$-equivariant and that the morphisms in the sequences (2.33) are $\mathbb{C}^*$-equivariant. Moreover, we can choose φ to be an S^1-invariant function so that the compact set K_c becomes S^1-invariant. Fix $0 < c_1 \ll c_2 < +\infty$.

LEMMA 2.14

With the same notation as in the above paragraph, set $\mathcal{F} = \Theta_U$ *or* N_U. *Then for* $i \geq 1$, *the natural morphism* $R : H^i(Z\backslash K_{c_1}, \mathcal{F}) \to H^i((Z\backslash K_{c_1}) \cap \mathring{K}_{c_2}, \mathcal{F})$ *induced by the inclusion is an isomorphism.*

Proof

Since we are working with Čech cohomology, we first construct coverings by S^1-invariant Stein open sets in the following way. Let $\pi : Z \setminus \{o\} \to D$ be the natural projection realizing $Z \setminus \{o\}$ as a $\mathbb{C}^*$-bundle over D. Choose a Stein covering $\{U_\alpha^D\}$ of D, and set $U_\alpha = \pi^{-1}(U_\alpha^D) \cap (Z\backslash K_{c_1})$. Similarly, we get an S^1-invariant Stein covering $\{U'_\alpha\}$ of $(Z\backslash K_{c_1}) \cap \mathring{K}_{c_2}$.

We first argue that R is injective. Represent the cohomology classes by Čech cocycles with respect to the above S^1-invariant Stein coverings. If $[\xi] = [\{\xi_{\alpha_1 \cdots \alpha_i}\}] \in H^i(Z\backslash K_{c_1}, \mathcal{F})$ satisfies $R([\xi]) = 0 \in H^i((Z\backslash K_{c_1}) \cap \mathring{K}_{c_2}, \Theta_Z)$, then $\xi = \delta(\eta)$ is a coboundary where $\eta = \{\eta_{\alpha_1\alpha_2\cdots\alpha_{i-1}}\}$ is a cochain (over $(Z\backslash K_{c_1}) \cap \mathring{K}_{c_2}$).

By using the result in [20, Proposition 3.4], we can decompose each component of η into weight pieces. More precisely, we can write $\eta = \sum_k \eta_k$, where $\eta_k = \{(\eta_k)_{\alpha_1\cdots\alpha_{i-1}}\}$ has weight k under the S^1-action. Note that $\mathcal{F}$ is associated to a $\mathbb{C}^*$-equivariant vector bundle over $\pi^{-1}(U_\alpha^D)$. So each $(\eta_k)_{\alpha_1\cdots\alpha_{i-1}}$ is represented by holomorphic functions over $U'_{\alpha_1} \cap \cdots \cap U'_{\alpha_{i-1}}$ with respect to a $\mathbb{C}^*$-equivariant trivialization of $\mathcal{F}$. Since homogeneous holomorphic functions on an annulus in $\mathbb{C}$ uniquely extend to holomorphic functions on $\mathbb{C}^*$, it is easy to see that η extends uniquely to a holomorphic cochain of $\mathcal{F}$ with respect to the covering $\{U_\alpha\}$ such that $\xi = \delta(\eta)$ also holds on $Z \setminus K_{c_1}$. So ξ is also a coboundary over $Z \setminus K_{c_1}$ and hence represents zero in $H^i(Z \setminus K_{c_1}, \mathcal{F})$.

By using exactly the same argument, which again depends on the weight decomposition (using [20, Proposition 3.4]) and the holomorphicity of cochains, we also prove that each cocycle over $(Z \setminus K_{c_1}) \cap \mathring{K}_{c_2}$ extends to a cocycle over $Z \setminus K_{c_1}$. So the surjectivity of the morphism R is also true. □

With the same notation as in the above discussion, set $Y := (Z\backslash K_{c_1}) \cap \mathring{K}_{c_2}$ and $Y' := Z\backslash K_{c_1}$. By [2, Théorème 15], for any locally free sheaf $\mathcal{F}$ (whose depth is always n), the natural restriction morphism $H^0(U, \mathcal{F}) \to H^0(Y', \mathcal{F}|_{Y'})$ is an isomorphism and $H^1(U, \mathcal{F}) \to H^1(Y', \mathcal{F}|_{Y'})$ is injective (since $n \geq 2$). Combining this with the above lemma, we get that the restriction morphism $\mu_0 : H^0(U, \mathcal{F}|_Y) \to H^0(Y, \mathcal{F}|_Y)$ is an isomorphism and $\mu_1 : H^1(U, \mathcal{F}) \to H^1(Y, \mathcal{F}|_Y)$ is injective.

Now we have the following commutative diagram:

$$\begin{tikzcd} & H^0(U,N_U) \arrow[dl, "\psi_U"'] \arrow[d, "\delta_U"] \arrow[r, "\mu_0", "\cong"'] & H^0(Y,N_Y) \arrow[d, "\delta_Y"] \\ \mathbf{T}^1_Z \arrow[r, hook, "\tau_U"] & H^1(U,\Theta_U) \arrow[r, hook, "\mu_1"] & H^1(Y,\Theta_Y) \end{tikzcd} \tag{2.34}$$

Note that ψ_U and τ_U are defined via Schlessinger's result in Proposition A.10 and that δ_U and δ_Y are connecting morphisms as in (2.3) (see also (A.9)).

PROPOSITION 2.15

With the above notation, let $\mathcal{Y} \to \mathbb{B}$ be the holomorphic family of complex manifolds that is induced by $\mathcal{Z} \to \mathbb{B}$. The following conditions are equivalent:

(1) $\operatorname{Ord}(\mathcal{Z}) \geq p+1$ *and hence there is a well-defined* $\mathbf{KS}^{(p+1)}_{\mathcal{Z}} \in \mathbf{T}^1_Z$.

(2) *There is a p-trivial embedding of $\mathcal{Y}$ and hence there is a well-defined* $\mathfrak{v}_{p+1} \in H^0(Y,N_Y)$.

(3) *There is a p-trivial atlas covering $\mathcal{Y}_0$ and hence there is a well-defined* $\theta_{p+1} \in H^1(Y,\Theta_Y)$.

If one of the above conditions holds true, then we have the following identities:

$$\begin{aligned} \delta_Y(\mathfrak{v}_{p+1}) = \theta_{p+1} &= \mu_1 \circ \tau_U(\mathbf{KS}^{(p+1)}_{\mathcal{Z}}) \qquad \textit{and} \\ \mathbf{KS}^{(p+1)}_{\mathcal{Z}} &= \psi_U \circ \mu_0^{-1}(\mathfrak{v}_{p+1}). \end{aligned} \tag{2.35}$$

Proof

Notice that the equivalence of (2) and (3) was already proved in Lemma 2.7. So we only need to prove the equivalence of (1) and (2).

Assume that $\operatorname{Ord}(\mathcal{Z}) \geq p+1$. Then by Lemma 2.13, after changing the embedding of $\mathcal{Z}$, we can choose an open neighborhood $\mathcal{W}$ of $(o,0) \in \mathbb{C}^N \times \mathbb{C}$ such that $\mathcal{I}_{\mathcal{Z}}(\mathcal{W})$ is generated by $\{F_1(w,t) = F_1(w,0) + t^{p+1}G_1(w,t), \ldots, F_d(w,t) = F_d(w,0) + t^{p+1}G_d(w,t)\}$. By Lemma 2.10, condition (2) holds, that is, we get a p-trivial embedding and a p-adapted atlas.

Now we verify the identities in (2.35) by using this p-adapted atlas. Set $f_r(w) = F_r(w,0)$ and $g_r(w) = G_r(w,0)$. Taking $(p+1)$th derivatives with respect to t on both sides of the equation

$$(f_r + t^{p+1}G_r)\big(w^b(z_\alpha,t)\big) = 0,$$

we get

$$\sum_{b=1}^{N} \frac{\partial f_r}{\partial w^b} \frac{1}{(p+1)!} \frac{\partial^{p+1} w^b}{\partial t^{p+1}}\bigg|_{t=0} + g_r = 0.$$

Comparing with the definition of $\mathfrak{v}_{p+1}$ in (2.13) and the definition of $\bar{g}$ in (2.31), this says that $-\bar{g}|_Y = \mathfrak{v}_{p+1} \in H^0(Y, N_Y)$. It is clear that $\mathfrak{v}_{p+1} = -\mu_0(\bar{g}|_U)$ so that $-\bar{g}|_U = \mu_0^{-1}(\mathfrak{v}_{p+1})$ since μ_0 is an isomorphism. On the other hand, we have $-\psi_U(\bar{g}|_U) = \mathbf{KS}_{\mathcal{Z}}^{(p+1)}$. So we get

$$\psi_U \circ \mu_0^{-1}(\mathfrak{v}_{p+1}) = \mathbf{KS}_{\mathcal{Z}}^{(p+1)}.$$

The identity $\delta_Y(\mathfrak{v}_{p+1}) = \theta_{p+1}$ was proved in Lemma 2.8. The other identity is a consequence now:

$$\begin{aligned}\mu_1 \circ \tau_U(\mathbf{KS}_{\mathcal{Z}}^{(p+1)}) &= \mu_1 \circ \tau_U \circ \psi_U \circ \mu_0^{-1}(\mathfrak{v}_{p+1}) \\ &= \mu_1 \circ \delta_U \circ \mu_0^{-1}(\mathfrak{v}_{p+1}) = \delta_Y(\mathfrak{v}_{p+1}) = \theta_{p+1}.\end{aligned}$$

We are left to prove that (2) implies (1). Now assume that (2) holds but, on the contrary, $\mathrm{Ord}(\mathcal{Z}/\mathbb{B}) = l + 1$ with $l < p$. Then by using the defining functions $\{F_r(w, 0) + t^{l+1} G_r(w, t)\}$ from Lemma 2.13, we have $\psi_U(\bar{g}) = -\mathbf{KS}_{\mathcal{Z}}^{(l+1)} \neq 0 \in \mathbf{T}^1_{\mathcal{Z}}$. So $\delta_U(\bar{g}) = \tau_U \circ \psi_U(\bar{g}) \neq 0 \in H^1(U, \Theta_U)$ since τ_U is injective. By the discussion before Proposition 2.15, μ_1 is injective. So $\mu_1 \circ \delta_U(\bar{g}) \neq 0$. Hence

$$\theta_{l+1} = \delta_Y(\mathfrak{v}_{l+1}) = -\mu_1 \circ \delta_U(\bar{g}) \neq 0.$$

On the other hand, we assumed that there is a p-trivial embedding $\tilde{\imath}_Y$ with $p > l$. So by choosing a p-adapted atlas, the corresponding class $\tilde{\mathfrak{v}}_{l+1} := \mathfrak{v}_{l+1}(\tilde{\imath}_Y) = 0$. So $\delta_Y(\tilde{\mathfrak{v}}_{l+1}) = 0$. By Lemma 2.8(1) and Lemma 2.5(2), $\delta_Y(\mathfrak{v}_{l+1}) = \delta_Y(\tilde{\mathfrak{v}}_{l+1}) \in H^1(Y, \Theta_Y)$. So we get a contradiction. □

3. Embeddings of submanifolds and deformations

In Section 3.1, we will construct the "most holomorphic" diffeomorphism between a neighborhood of a complex submanifold to a neighborhood of the zero section of its normal bundle. In particular, this allows us to get Proposition 1.2. We do this by first using the "deformation to normal cone" to construct a "holomorphic family of neighborhoods" as the deformation of a neighborhood of the zero section of the normal bundle. We also construct a $(k-2)$-trivial (resp., $(k-1)$-trivial) atlas on this family under the assumption that the embedding is $(k-1)$-linearizable (resp., $(k-1)$-comfortable). Then we use a method similar to that used in Section 2.1 to get the wanted diffeomorphism. Our main goal in this section is a technical Proposition 3.3 which relates the reduced Kodaira–Spencer class of the "holomorphic family of neighborhoods" to the obstructions to splitting embedding and comfortable embedding.

3.1. Construction of comparison diffeomorphism and $(k-1)$*-trivial atlas*

As mentioned above, the construction of diffeomorphism F in Propositions 1.2 and 1.3 uses a construction in algebraic geometry called *deformation to the normal cone* (see [13, Chapter 5]). This is a way to degenerate a neighborhood of $S \hookrightarrow X$ to a neighborhood of $S \hookrightarrow N_S$. The construction is simply to blow up the submanifold $S \times \{0\} \subset X \times \mathbb{C}$ which gives a total family $\tilde{\mathcal{X}} = \mathrm{Bl}_{S\times\{0\}}(X \times \mathbb{C})$ with the projection $\pi : \tilde{\mathcal{X}} \to \mathbb{C}$. The central fiber $\tilde{\mathcal{X}}_0 = \mathrm{Bl}_S X \cup E$ is the union of two components. The exceptional divisor $E = \mathbb{P}(N_S \oplus \mathbb{C})$ is the projective compactification of the normal bundle N_S of $S \subset X$. In this way, we can view $S \hookrightarrow X$ as an analytic deformation of $S_0 \hookrightarrow N_S$. More precisely, we will construct an analytic family $\mathcal{W}$ as an open neighborhood of $\mathcal{S} \cong S \times \mathbb{C} \hookrightarrow \tilde{\mathcal{X}}$. In other words, $\mathcal{W}$ is considered as a deformation of a neighborhood of $S \to X$.

The main result of this subsection is the following proposition, which contains the statement of Proposition 1.2. For the construction in its proof, we refer to Section A.1 in the Appendix for preliminary results from [1] that will enable us to read out the precise order of holomorphicity of the diffeomorphism constructed.

PROPOSITION 3.1

Assume that S is a smooth submanifold of X. If $S \hookrightarrow X$ is $(k-1)$-linearizable, then the following statements are true.

(1) *There is a holomorphic family of complex manifolds $\mathcal{W}$ such that $\mathcal{W}_0$ is a neighborhood of $S_0 \hookrightarrow N_S$ and $\mathcal{W}_1 =: W$ is a small neighborhood of $S \subset X$, and there is a $(k-2)$-trivial atlas covering $\mathcal{W}_0$.*

(2) *There is a diffeomorphism $F : \mathcal{W}_0 \to F(\mathcal{W}_0) \subset W$ where $W = \mathcal{W}_1$ such that for any $j \geq 0$, there exists a constant C_j such that F satisfies*

$$\left\| \nabla^j_{\tilde{g}_0} (F^* J - J_0) \right\|_{\tilde{g}_0} \leq C_j \tilde{r}^{k-j} \quad \text{on } W_0. \tag{3.1}$$

If $S \hookrightarrow X$ is furthermore $(k-1)$-comfortable, then the above properties can be improved as follows.

(3) *There is a $(k-1)$-trivial atlas covering $\mathcal{W}_0$.*

(4) *There is a local decomposition of $\Phi := F^* J - J_0$ into four types of components (see (3.9))*

$$\Phi = \Phi^h_v + \Phi^v_h + \Phi^v_v + \Phi^h_h$$

such that, for any $j \geq 0$, the following estimates hold over W_0 for a uniform constant C_j:

$$\begin{aligned} &\|\nabla^j_{\tilde{g}_0} \Phi^v_h\|_{\tilde{\omega}_0} \leq C_j \tilde{r}^{k+1-j}, \qquad \|\nabla^j_{\tilde{g}_0} \Phi^v_v\|_{\tilde{\omega}_0} \leq C_j \tilde{r}^{k+1-j}, \\ &\|\nabla^j_{\tilde{g}_0} \Phi^h_h\|_{\tilde{\omega}_0} \leq C_j \tilde{r}^{k-j}, \qquad \|\nabla^j_{\tilde{g}_0} \Phi^h_v\|_{\tilde{\omega}_0} \leq C_j \tilde{r}^{k-j}. \end{aligned} \tag{3.2}$$

The improved estimates (3.2) will be used to prove Proposition 1.3 in Section 5.

Proof

Assume that the embedding $S \hookrightarrow X$ is $(k-1)$-linearizable. By Theorem A.9 in Section A.1, we can find coordinate charts $\{V_\alpha, (z_\alpha)\}$ of X near the submanifold S such that $S \cap V_\alpha = \{z_\alpha^1 = \cdots = z_\alpha^m = 0\}$ and the transition functions on $V_\alpha \cap V_\beta$ are given by

$$\begin{cases} z_\beta^r = \sum_{s=1}^m (a_{\beta\alpha})_s^r (z_\alpha'') z_\alpha^s + R_k^r & \text{for } r = 1, \ldots, m, \\ z_\beta^p = \phi_{\beta\alpha}^p (z_\alpha'') + R_k^p & \text{for } p = m+1, \ldots, n, \end{cases} \tag{3.3}$$

where we have denoted by $z'' = (z_\alpha^{m+1}, \ldots, z_\alpha^n)$ the tangent variables, which can also serve as coordinates on S. Here $R_k^r, R_k^p \in \mathcal{I}_S^k$. We also consider coordinate charts $\{V_\alpha \times \mathbb{C}, (z_\alpha, t)\}$ on $X \times \mathbb{C}$ so that $S \times \{0\} = \{z_\alpha^1 = \cdots = z_\alpha^m = t = 0\}$.

Consider the blowup $\pi : \tilde{\mathcal{X}} := \mathrm{Bl}_{S\times\{0\}}(X \times \mathbb{C}) \to X \times \mathbb{C}$ with the exceptional divisor $E = \mathbb{P}(N_S \oplus \mathbb{C})$. Here E is the projective compactification of the normal bundle $N_S \to S$ and S_0 sits inside $N_S \subset E \subset \tilde{\mathcal{X}}_0 \subset \tilde{\mathcal{X}}$ as the zero section of $N_S \to S$. The subset $\pi^{-1}(V_\alpha \times \mathbb{C}) \subset \tilde{\mathcal{X}}$ is defined as the following subvariety of $V_\alpha \times \mathbb{C} \times \mathbb{P}^m$:

$$\begin{aligned} \big\{ \big(z_\alpha^r, z_\alpha^p, t, [Z_\alpha^r, T]\big); (z_\alpha^r, z_\alpha^p) \in V_\alpha, t \in \mathbb{C}, z_\alpha^r Z_\alpha^s - z_\alpha^s Z_\alpha^r = 0, \\ z_\alpha^r \cdot T - t \cdot Z_\alpha^r = 0; \text{ for } r, s = 1, \ldots, m; p = m+1, \ldots, n \big\}, \end{aligned}$$

where $[Z_\alpha^r, T]$ are homogenous coordinates on $\mathbb{P}^m$. Near S_0, the coordinate $T \neq 0$, and so we can define new coordinate charts $\{w_\alpha, t\}$ such that the map π is given by

$$z_\alpha^1 = t w_\alpha^1, \ldots, z_\alpha^m = t w_\alpha^m; \qquad z_\alpha^{m+1} = w_\alpha^{m+1}, \ldots, z_\alpha^n = w_\alpha^n; \qquad t = t.$$

Without loss of generality, we can assume that $V_\alpha = \{z_\alpha; |z_\alpha| < \epsilon\}$ for sufficiently small $\epsilon > 0$. Then if we denote the polydisk on the total space

$$\mathcal{U}_\alpha = \big\{(t, w_\alpha); |t| < 2, |w_\alpha| < \epsilon\big\},$$

then $\pi(\mathcal{U}_\alpha) \subset V_\alpha \times \mathbb{C}$, and when $t \neq 0$ satisfies $|t| < 2$,

$$\pi(\mathcal{U}_\alpha) \cap X_t \cong \big\{z_\alpha; |z_\alpha^r| < 2\epsilon t, |z_\alpha^p| < \epsilon; \text{ for } r = 1, \ldots, m; p = m+1, \ldots, n\big\}.$$

Denote by $\mathcal{S}$ the strict transform of $S \times \mathbb{C}$ on $\tilde{\mathcal{X}}$. Let π_1 be the composition $\tilde{\mathcal{X}} \to X \times \mathbb{C} \to \mathbb{C}$. For any $a > 0 \in \mathbb{R}$, denote $\mathcal{S}_{|t|<a} = \pi^{-1}(\{t; |t| < a\})$. Then the collection of open sets $\{\mathcal{U}_\alpha\}$ is a covering of $\mathcal{S}_{|t|<1}$ inside the total space $\tilde{\mathcal{X}}$, and on $\mathcal{U}_\alpha$ the ideal sheaf $\mathcal{I}_{\mathcal{S}}$ is generated by $w_\alpha^1, \ldots, w_\alpha^m$. Denote $\mathcal{U} = \bigcup_\alpha \mathcal{U}_\alpha$. We can find a small neighborhood $\mathcal{W}$ of $\mathcal{S}_{|t|<1} \subset \tilde{\mathcal{X}}$ such that $\mathcal{W} \subset\subset \mathcal{U}$. Denote $w_\alpha' = (w_\alpha^1, \ldots, w_\alpha^m)$, $w_\alpha'' = (w_\alpha^{m+1}, \ldots, w_\alpha^n)$, and define

$$\tilde{R}_k^r(t; w'_\alpha, w''_\alpha) = t^{-k} R_k^r(t w'_\alpha, w''_\alpha),$$
$$\tilde{R}_k^p(t; w'_\alpha, w''_\alpha) = t^{-k} R_k^p(t w'_\alpha, w''_\alpha).$$

Then $\tilde{R}_k^r \in \mathcal{I}_{\mathcal{S}}^k$, $\tilde{R}_k^p \in \mathcal{I}_{\mathcal{S}}^k$. Note that $\{\mathcal{U}_\alpha \cap \mathcal{W}_t, w_\alpha\}_\alpha$ form an atlas covering $\mathcal{W}_t := \pi^{-1}(X_t) \cap \mathcal{W}$ for $|t| < 1$. The transition function on $(\mathcal{U}_\alpha \cap \mathcal{W}_t) \cap (\mathcal{U}_\beta \cap \mathcal{W}_t)$ is given by

$$\begin{cases} w_\beta^r = \sum_{s=1}^m (a_{\beta\alpha})_s^r (w''_\alpha) w_\alpha^s + t^{k-1} \tilde{R}_k^r & \text{for } r = 1, \ldots, m, \\ w_\beta^p = \phi_{\beta\alpha}^p (w''_\alpha) + t^k \tilde{R}_k^p & \text{for } p = m+1, \ldots, n. \end{cases} \tag{3.4}$$

So we get a $(k-2)$-trivial atlas covering $\mathcal{W}_0$ in the sense of Definition 2.4. Next we can construct the diffeomorphism that we want. Choose a partition of unity $\{\rho_\alpha, \tilde{\rho}\}$ subordinate to the covering $\{\mathcal{U}_\alpha, \tilde{\mathcal{X}} \backslash \overline{\mathcal{W}}\}$. In particular, $\mathrm{Supp}(\rho_\alpha) \subset \mathcal{U}_\alpha$, $\mathrm{Supp}(\tilde{\rho}) \cap \mathcal{W} = \emptyset$. As in Section 2, define the differentiable vector field in the small neighborhood $\mathcal{W}$ of $\mathcal{S}_{|t|<1} \subset \tilde{\mathcal{X}}$:

$$\begin{aligned} \mathbb{V} &= \sum_\alpha \rho_\alpha \Big(\frac{\partial}{\partial t}\Big)_\alpha \\ &= \sum_{i=1}^n \Big(\sum_\alpha \rho_\alpha \frac{\partial f_{\beta\alpha}^i (w_\alpha, t)}{\partial t} \Big) \frac{\partial}{\partial w_\beta^i} + \Big(\frac{\partial}{\partial t}\Big)_\beta \\ &= \sum_{r=1}^m \sum_\alpha \rho_\alpha \partial_t (t^{k-1} \tilde{R}_k^r) \frac{\partial}{\partial w_\beta^r} + \sum_{p=m+1}^n \sum_\alpha \rho_\alpha \partial_t (t^k \tilde{R}_k^p) \frac{\partial}{\partial w_\beta^p} + \Big(\frac{\partial}{\partial t}\Big)_\beta . \end{aligned}$$

Let $\{\sigma(s); s \in (-\epsilon, \epsilon)\}$ be the one-parameter subgroup generated by $\mathrm{Re}(\mathbb{V})$, which exists when ϵ is sufficiently small. Then we get a map $\sigma(s) : \mathcal{W} \cap \tilde{\mathcal{X}}_0 \to \mathcal{U} \cap \tilde{\mathcal{X}}_s$ which gives a diffeomorphism to its image.

Note that the vector field $\mathbb{V}$ is tangent to $\mathcal{S}$ so that $\sigma(s)$ preserves $\mathcal{S}$. Denote by $\mathcal{J}$ the complex structure on the total space $\tilde{\mathcal{X}}$ of the blowup. Denote

$$\Phi(s) = \sigma(s)^* \mathcal{J} - \mathcal{J}.$$

Then we can calculate

$$\begin{aligned} \dot{\Phi}(s) &= \frac{d}{ds}\big(\sigma(s)^* \mathcal{J}\big) = \mathcal{L}_{\mathrm{Re}(\mathbb{V})} \mathcal{J} = \bar{\partial} \mathbb{V} + \overline{\bar{\partial} \mathbb{V}} \\ &= \sum_{r=1}^m \sum_\alpha \big[\partial_t (t^{k-1} \tilde{R}_k^r)\big] (\bar{\partial} \rho_\alpha) \otimes \frac{\partial}{\partial w_\beta^r} \\ &\quad + \sum_{p=m+1}^n \sum_\alpha \big[\partial_t (t^k \tilde{R}_k^p)\big] (\bar{\partial} \rho_\alpha) \otimes \frac{\partial}{\partial w_\beta^p} \Big|_{t=s} \\ &\quad + \text{complex conjugates}. \end{aligned}$$

Assume that $\tilde{\omega}_0$ is a smooth Kähler metric on the open set $\mathcal{W}$. Because both $\tilde{R}^r_k, \tilde{R}^p_k \in \mathcal{I}^k_{\mathcal{S}}$, we get

$$|\dot{\Phi}|_{\tilde{\omega}_0} \leq C s^{\max\{0,k-2\}} |w'|^k.$$

So we can integrate to get

$$\big|\Phi(s)\big|_{\tilde{\omega}_0} = \big|\sigma(s)^*\mathcal{J} - \mathcal{J}\big|_{\tilde{\omega}_0} = \Big|\int_0^s \sigma(s)^*(\mathcal{L}_{\mathbb{V}}\mathcal{J})\, ds\Big|_{\tilde{\omega}_0} \leq C s^{k-1}|w'|^k. \tag{3.5}$$

If $\tilde{r}$ is the distance function to S_0 with respect to $\tilde{g}_0$, then $\tilde{r}$ is comparable to the norm $|w'|$. So the above estimate proves the inequality (1.1) for $j = 0$. The higher-order estimates of Φ can be proved in the same way by taking the higher-order Lie derivative of $\mathbb{V}$.

Next we show that if $S \hookrightarrow X$ is $(k-1)$-comfortable, the estimates of some components of Φ can be improved. In this case, by Theorem A.8, we can choose the coordinate charts such that the following holds:

$$\begin{cases} z^r_\beta = \sum_{s=1}^m (a_{\beta\alpha})^r_s(z''_\alpha) z^s_\alpha + R^r_{k+1} & \text{for } r = 1, \dots, m, \\ z^p_\beta = \phi^p_{\beta\alpha}(z''_\alpha) + R^p_k & \text{for } p = m+1, \dots, n, \end{cases} \tag{3.6}$$

where $R^r_{k+1} \in \mathcal{I}^{k+1}_S$, $R^p_k \in \mathcal{I}^k_S$. Similarly as before, denote $\tilde{R}^r_{k+1}(t; w'_\alpha, w''_\alpha) = t^{-(k+1)} R^r_{k+1}(t w'_\alpha, w''_\alpha)$ and $\tilde{R}^p_k(t; w'_\alpha, w''_\alpha) = t^{-k} R^p_k(t w'_\alpha, w''_\alpha)$. Then $\tilde{R}^r_{k+1} \in \mathcal{I}^{k+1}_{\mathcal{S}}$ and $\tilde{R}^p_k \in \mathcal{I}^k_{\mathcal{S}}$. On the total space of the deformation to the normal cone, we have

$$\begin{cases} w^r_\beta = \sum_{s=1}^m (a_{\beta\alpha})^r_s(w''_\alpha) w^s_\alpha + t^k \tilde{R}^r_{k+1} & \text{for } r = 1, \dots, m, \\ w^p_\beta = \phi^p_{\beta\alpha}(w''_\alpha) + t^k \tilde{R}^p_k & \text{for } p = m+1, \dots, n. \end{cases} \tag{3.7}$$

Notice that this is a $(k-1)$-trivial atlas covering $\mathcal{W}_0$ in the sense of Definition 2.4.

Similarly as before, the differentiable vector field $\mathbb{V}$ (see Section 2) becomes

$$\begin{aligned} \mathbb{V} &= \sum_{i=1}^n \Big(\sum_\alpha \rho_\alpha \frac{\partial f^i_{\beta\alpha}(w_\alpha, t)}{\partial t}\Big) \frac{\partial}{\partial w^i_\beta} + \Big(\frac{\partial}{\partial t}\Big)_\beta \\ &= \sum_{r=1}^m \sum_\alpha \rho_\alpha \big[\partial_t(t^k \tilde{R}^r_{k+1})\big] \otimes \frac{\partial}{\partial w^r_\beta} + \sum_{p=m+1}^n \sum_\alpha \rho_\alpha \big[\partial_t(t^k \tilde{R}^p_k)\big] \otimes \frac{\partial}{\partial w^p_\beta} \\ &\quad + \Big(\frac{\partial}{\partial t}\Big)_\beta. \end{aligned} \tag{3.8}$$

Use the same notation $\sigma(s)$, $\mathcal{J}$, $\Phi(s)$, and $\dot{\Phi}(s)$, as before. We have

$$
\begin{aligned}
\dot{\Phi}(s) &= \frac{d}{ds}\big(\sigma(s)^*\mathcal{J}\big) = \mathcal{L}_{\mathrm{Re}(\mathbb{V})}\mathcal{J} = \bar{\partial}\mathbb{V} + \overline{\bar{\partial}\mathbb{V}} \\
&= \sum_{r=1}^{m}\sum_{\alpha}\big[\partial_t(t^k \tilde{R}^r_{k+1})\big](\bar{\partial}\rho_\alpha) \otimes \frac{\partial}{\partial w^r_\beta} \\
&\quad + \sum_{p=m+1}^{n}\sum_{\alpha}\big[\partial_t(t^k \tilde{R}^p_{k})\big](\bar{\partial}\rho_\alpha) \otimes \frac{\partial}{\partial w^p_\beta}\Big|_{t=s} \\
&\quad + \text{complex conjugates.}
\end{aligned}
$$

We assume the index $v \in \{1,\dots,m,\overline{1},\dots,\overline{m}\}$, $h \in \{m+1,\dots,n,\overline{m+1},\dots,\overline{n}\}$, and we decompose Φ into four types of components:

$$
\begin{aligned}
\Phi &= \Phi^h_v + \Phi^v_h + \Phi^v_v + \Phi^h_h \\
&:= \phi^h_v\, dw^v \otimes \partial_{w^h} + \phi^v_h\, dw^h \otimes \partial_{w^v} \\
&\quad + \phi^v_v\, dw^v \otimes \partial_{w^v} + \phi^h_h\, dw^h \otimes \partial_{w^h}.
\end{aligned} \tag{3.9}
$$

Again we assume that $\tilde{\omega}_0$ is a smooth Kähler metric on $\mathcal{W}$.

Since $\tilde{R}^r_{k+1} \in \mathcal{I}^{k+1}_{\mathcal{S}}$, $\tilde{R}^p_k \in \mathcal{I}^k_{\mathcal{S}}$, it is easy to see that

$$
\begin{aligned}
&|\dot{\phi}^v_h| \le C s^{k-1}|w'|^{k+1}, \qquad |\dot{\phi}^v_v| \le C s^{k-1}|w'|^{k+1}, \\
&|\dot{\phi}^h_h| \le C s^{k-1}|w'|^{k}, \qquad |\dot{\phi}^h_v| \le C s^{k-1}|w'|^{k}.
\end{aligned}
$$

Integrating these, we get

$$
\begin{aligned}
&|\Phi^v_h|_{\tilde{\omega}_0} \le C s^{k}|w'|^{k+1}, \qquad |\Phi^v_v|_{\tilde{\omega}_0} \le C s^{k}|w'|^{k+1}, \\
&|\Phi^h_h|_{\tilde{\omega}_0} \le C s^{k}|w'|^{k}, \qquad |\Phi^h_v|_{\tilde{\omega}_0} \le C s^{k}|w'|^{k}.
\end{aligned} \tag{3.10}
$$

When $|s| < \epsilon$ with ϵ sufficiently small, since $\tilde{r}$ is comparable to $|w'|$, we get the estimates that improve the estimates in (3.2) for $j = 0$. The higher-order estimates can be proved similarly by taking higher-order Lie derivatives of J with respect to $\mathbb{V}$. □

3.2. Order of embedding via deformation to the normal cone

Let S be a smooth submanifold of a complex manifold X. We will denote by $\pi_S : N_S \to S$ the normal bundle of S inside X and by Θ_{N_S} the tangent sheaf on the total space of N_S. The natural $\mathbb{C}^*$-action on N_S induces $\mathbb{C}^*$-actions on various cohomology groups. Since we will use various Čech cohomology groups frequently, we choose a Stein covering $\{\hat{U}_\alpha\}$ of N_S by first choosing a Stein covering $\{U_\alpha\}$ of S and then defining $\hat{U}_S = \pi_S^{-1}(U_\alpha)$. In particular, $\hat{U}_\alpha$ is invariant under the natural

$\mathbb{C}^*$-action. On each $\hat{U}_\alpha$, choose a coordinate system $w_\alpha = \{w'_\alpha, w''_\alpha\} = \{w^r_\alpha, w^p_\alpha | r = 1, \ldots, m; p = m+1, \ldots, n\}$ such that w^r_α are fiber variables and w^p_α are base variables. Then the $\mathbb{C}^*$-action is given by

$$t \cdot \{w'_\alpha, w''_\alpha\} = \{t^{-1} w'_\alpha, w''_\alpha\}.$$

The transition function on $\hat{U}_\alpha \cap \hat{U}_\beta$ is of the form

$$\begin{cases} w^r_\beta = \sum_{s=1}^m (a_{\beta\alpha})^r_s (w''_\alpha) w^s_\alpha & \text{for } r = 1, \ldots, m, \\ w^p_\beta = \phi^p_{\beta\alpha}(w''_\alpha) & \text{for } p = m+1, \ldots, n. \end{cases} \tag{3.11}$$

Let $\mathbf{V}$ be a Čech cohomology space $H^i(X, \mathscr{F})$, where X is an analytic space with a $\mathbb{C}^*$-action and $\mathscr{F}$ is the coherent sheaf associated to a $\mathbb{C}^*$-equivariant vector bundle $F \to X$. The space of cocycles of $\mathscr{F}$ with respect to a $\mathbb{C}^*$-invariant Stein covering has a continuous S^1-action. By the result from [20, Proposition 3.4], this space can be written as the closure of the algebraic direct sum of eigenspaces. This induces a weight decomposition of the cohomology space $\mathbf{V} = H^i(X, \mathscr{F})$. We will denote by $\mathbf{V}(-k)$ the subspace of elements of weight $-k$.

LEMMA 3.2
For $k \geq 0$, we have the commutative diagram of exact sequences

$$\begin{tikzcd} H^1(N_S, \Theta_{N_S} \otimes \mathscr{I}_S^{k+1})(-k) \arrow[r, "\mathfrak{N}'_k"] \arrow[d, "\mathfrak{R}_k"', "\cong"] & H^1(N_S, \Theta_{N_S} \otimes \mathscr{I}_S^{k})(-k) \arrow[r, "\mathfrak{T}'_k"] \arrow[d, "\mathfrak{I}_k"', "\cong"] & H^1(S, \Theta_S \otimes \mathscr{I}_S^{k}/\mathscr{I}_S^{k+1}) \arrow[d, equal] \\ H^1(S, N_S \otimes \mathscr{I}_S^{k+1}/\mathscr{I}_S^{k+2}) \arrow[r, "\mathfrak{N}_k"] & H^1(N_S, \Theta_{N_S})(-k) \arrow[r, "\mathfrak{T}_k"] & H^1(S, \Theta_S \otimes \mathscr{I}_S^{k}/\mathscr{I}_S^{k+1}) \end{tikzcd} \tag{3.12}$$

where the morphisms are given as follows:
(1) $\mathfrak{I}_k$, $\mathfrak{N}'_k$ *are induced by inclusion of sheaves;*
(2) $\mathfrak{T}'_k$, $\mathfrak{R}_k$ *will be defined in the proof, and* $\mathfrak{R}_k$ *is an isomorphism;*
(3) $\mathfrak{N}_k = \mathfrak{I}_k \circ \mathfrak{N}'_k \circ \mathfrak{R}_k^{-1}$ *and* $\mathfrak{T}_k = \mathfrak{T}'_k \circ \mathfrak{I}_k^{-1}$ *are defined by using the commutativity of the diagram.*

Note that in the above diagram the sheaf $\mathscr{I}_S^k/\mathscr{I}_S^{k+1}$ is a sheaf supported on S. The Kodaira–Spencer class θ_k of the atlas constructed in the proof of Proposition 3.1 lives in $H^1(N_S, \Theta_{N_S})(-k)$ and the bottom exact sequence will serve to compare θ_k to Abate, Bracci, and Tovena's obstruction in Proposition 3.3.

Proof
We first notice that $\mathfrak{T}'_k$ is well defined as the composition of maps

$$H^1(N_S, \Theta_{N_S} \otimes \mathcal{I}_S^k) \to H^1(S, \Theta_{N_S}|_S \otimes \mathcal{I}_S^k/\mathcal{I}_S^{k+1}) \to H^1(S, \Theta_S \otimes \mathcal{I}_S^k/\mathcal{I}_S^{k+1}).$$

In the last map, we used the holomorphic splitting $\Theta_{N_S}|_S = \Theta_S \oplus N_S$. Similarly, $\mathfrak{R}_k$ is well defined as the composition of maps

$$\begin{aligned} H^1(N_S, \Theta_{N_S} \otimes \mathcal{I}_S^{k+1}) &\to H^1(S, \Theta_{N_S}|_S \otimes \mathcal{I}_S^{k+1}/\mathcal{I}_S^{k+2}) \\ &\to H^1(S, N_S \otimes \mathcal{I}_S^{k+1}/\mathcal{I}_S^{k+2}). \end{aligned}$$

Let us first show that the first row of the sequence is exact. Assume that $\theta_k \in H^1(N_S, \Theta_{N_S} \otimes \mathcal{I}_S^k)(-k)$. Let θ_k be represented by a cocycle $\{\theta_{\alpha\beta}\}$ with respect to a $\mathbb{C}^*$-invariant covering of N_S. Then by [20], we can write $\theta_{\alpha\beta}$ as a convergent series $\theta_{\alpha\beta} = \sum_\ell \theta_{\alpha\beta,\ell}$, where $\theta_{\alpha\beta,\ell}$ has weight ℓ. Because δ commutes with the $\mathbb{C}^*$-action, we know that $\theta_{\alpha\beta,\ell}$ is also a cocycle. Because $\theta_k = [\{\theta_{\alpha\beta}\}]$ has weight $-k$ and the weight decomposition of cohomology is induced by the weight decomposition on the space of cocycles, we know that $[\{\theta_{\alpha\beta,\ell}\}] = 0$ if $\ell \neq k$. So we can assume that θ_k is represented by a weight $(-k)$ cocycle

$$(\theta_k)_{\beta\alpha} = \sum_{r=1}^{m} b^r_{\beta\alpha}(w)\frac{\partial}{\partial w^r_\beta} + \sum_{p=m+1}^{n} c^p_{\beta\alpha}(w)\frac{\partial}{\partial w^p_\beta},$$

where $b^r_{\beta\alpha}, c^p_{\beta\alpha} \in \mathcal{I}_S^k$. Since $\frac{\partial}{\partial w^r_\beta}$ (resp., $\frac{\partial}{\partial w^p_\beta}$) has weight 1 (resp., 0), we know that $b^r_{\beta\alpha}$ (resp., $c^p_{\beta\alpha}$) is homogeneous of degree $(k+1)$ (resp., k) in $w' = \{w^r_\beta\}$. Then

$$\big(\mathfrak{T}'_k(\theta_k)\big)_{\beta\alpha} = \sum_{p=m+1}^{n} \big[c^p_{\beta\alpha}(w)\big]_{k+1}\frac{\partial}{\partial w^p_\beta}.$$

If $\mathfrak{T}'_k(\theta_k) = 0$, then we can write

$$\begin{aligned} &\sum_{p=m+1}^{n} \big[c^p_{\beta\alpha}(w)\big]_{k+1}\frac{\partial}{\partial w^p_\beta} \\ &\quad = \sum_{p=m+1}^{n} [d^p_\beta]_{k+1}\frac{\partial}{\partial w^p_\beta} - \sum_{p=m+1}^{n} [d^q_\alpha]_{k+1}\frac{\partial}{\partial w^q_\alpha} \quad \text{over } \hat{U}_\alpha \cap \hat{U}_\beta. \end{aligned}$$

We can assume that d^p_β and d^q_β are homogeneous of degree k. Then it is easy to see that $c^p_{\beta\alpha} = d^p_\beta - d^q_\alpha \frac{\partial w^p_\beta}{\partial w^q_\alpha}$. So if we define

$$(\tilde{\theta}_k)_{\beta\alpha} = (\theta_k)_{\beta\alpha} - \sum_{p=m+1}^{n} d_\beta^p \frac{\partial}{\partial w_\beta^p} + \sum_{q=m+1}^{n} d_\alpha^q \frac{\partial}{\partial w_\alpha^q},$$

then it is easy to see that $(\tilde{\theta}_k)_{\beta\alpha} \in H^0(\hat{U}_\alpha \cap \hat{U}_\beta, \Theta_{N_S} \otimes \mathcal{I}_S^{k+1})(-k)$ and we have $\theta_k = \mathfrak{N}'_k(\tilde{\theta}_k)$.

To show that $\mathfrak{R}_k$ is an isomorphism, we will construct its inverse. Assume that $\mathfrak{h} \in H^1(S, N_S \otimes \mathcal{I}_S^{k+1}/\mathcal{I}_S^{k+2})$; we can represent it as a cocycle

$$\mathfrak{h}_{\beta\alpha} = \sum_{r=1}^{m} [b_{\beta\alpha}^r]_{k+2} \frac{\partial}{\partial w_\beta^r}. \tag{3.13}$$

We can assume that $b_{\beta\alpha}^r$ is homogeneous of degree $k+1$ in $w'_\beta = \{w_\beta^r\}$. Then because of homogeneity, the cocycle condition of $\{\mathfrak{h}_{\beta\alpha}\}$ becomes

$$\sum_{r=1}^{m} \Big(b_{\beta\alpha}^r(w_\beta) \frac{\partial}{\partial w_\beta^r} + b_{\alpha\gamma}^r \frac{\partial}{\partial w_\alpha^r} + b_{\gamma\beta}^r \frac{\partial}{\partial w_\gamma^r} \Big) = 0. \tag{3.14}$$

So if we define

$$\mathfrak{h}'_{\beta\alpha} := \mathfrak{R}_k^{-1}(\mathfrak{h}_{\beta\alpha}) = \sum_{r=1}^{m} b_{\beta\alpha}^r \frac{\partial}{\partial w_\beta^r} \in H^0(\hat{U}_\alpha \cap \hat{U}_\beta, \Theta_{N_S} \otimes \mathcal{I}_S^{k+1})(-k),$$

where $\frac{\partial}{\partial w_\beta^r}$ and so on are considered as tangent vectors along the fibers of $N_S \to S$, then by (3.14) $\{\mathfrak{h}'_{\beta\alpha}\}$ satisfies the cocycle condition and hence represents a cohomology class in $H^1(N_S, \Theta_{N_S} \otimes \mathcal{I}_S^{k+1})$ of weight $-k$. Now we can define $\mathfrak{N}_k$. Choose $\mathfrak{h} \in H^1(S, N_S \otimes \mathcal{I}_S^{k+1}/\mathcal{I}_S^{k+2})$ represented by the cocycle as in (3.13) such that $b_{\beta\alpha}^p$ is homogeneous of degree $k+1$ in $w'_\beta = \{w_\beta^r\}$. Then we define

$$\mathfrak{N}_k(\mathfrak{h}_{\beta\alpha}) = \mathfrak{I}_k \circ \mathfrak{N}'_k \circ \mathfrak{R}_k^{-1}(\mathfrak{h}_{\beta\alpha}) = \sum_{r=1}^{m} b_{\beta\alpha}^r \frac{\partial}{\partial w_\beta^r} \in H^0(\hat{U}_\alpha \cap \hat{U}_\beta, \Theta_{N_S})(-k).$$

Using a similar homogeneity argument, one can also construct an inverse of $\mathfrak{I}_k$ showing that it is an isomorphism. Indeed, for any $\theta \in H^1(N_S, \Theta_{N_S})$ of weight $(-k)$, we can choose a $\mathbb{C}^*$-equivariant Čech cocycle $\{\theta_{\beta\alpha}\}$ of weight $(-k)$ representing θ. On $\hat{U}_\alpha \cap \hat{U}_\beta$, we can write

$$\theta_{\beta\alpha} = \sum_{r=1}^{m} a_{\beta\alpha}^r(w_\alpha) \frac{\partial}{\partial w_\beta^r} + \sum_{p=m+1}^{n} b_{\beta\alpha}^p(w_\alpha) \frac{\partial}{\partial w_\beta^p}.$$

Since $\frac{\partial}{\partial w_\beta^r}$ (resp., $\frac{\partial}{\partial w_\beta^p}$) has weight 1 (resp., 0), we see that $a_\alpha^r(w_\alpha)$ (resp., b_α^p) is homogeneous of degree $(k+1)$ (resp., k) in w_α^r. In particular, $a_{\beta\alpha}^r \in \mathcal{I}_D^{k+1}$ and $b_{\beta\alpha}^p \in$

$\mathcal{J}_D^k$. So $\theta_{\beta\alpha} \in H^0(\hat{U}_\alpha \cap \hat{U}_\beta, \Theta_{N_S} \otimes \mathcal{J}_S^k)$ and $\{\theta_{\beta\alpha}\}$ represents a cohomology class in $H^1(N_S, \Theta_{N_S} \otimes \mathcal{J}_S^k)$ of weight $(-k)$. □

Our main result in this subsection is the following technical proposition which, under appropriate assumptions, reinterprets the obstructions to splitting and comfortable embeddings via the deformation to normal cone construction.

PROPOSITION 3.3

Assume that S is a $(k-1)$-comfortably embedded submanifold of X for some $k \geq 1$, and let $(\rho_{k-1}, \boldsymbol{\nu}_{k-1})$ be a $(k-1)$-comfortable pair. Then for the holomorphic family of complex manifolds $\mathcal{W}$ from Proposition 3.1, the associated k-order Kodaira–Spencer class $\theta_k \in H^1(\mathcal{W}_0, \Theta_{\mathcal{W}_0})$ extends uniquely to a class in $H^1(N_S, \Theta_{N_S})$. This extension lies in the $(-k)$-weight space and will still be denoted by θ_k. Moreover, θ_k satisfies the following properties under the exact sequence from Lemma 3.2:

(1) *$\mathfrak{T}_k(\theta_k) = \mathfrak{g}_k^{\rho_{k-1}} \in H^1(S, \Theta_S \otimes \mathcal{J}_S^k/\mathcal{J}_S^{k+1})$ is the obstruction to k-splitting relative to ρ_{k-1}. As a consequence, if S is not k-splitting relative to ρ_{k-1}, then $\theta_k \in H^1(N_S, \Theta_{N_S})(-k)$ is nonzero.*

(2) *If S is k-splitting relative to ρ_{k-1} (i.e., we have a kth-order lifting ρ_k such that $\phi_{k,k-1} \circ \rho_k = \rho_{k-1}$), then $\theta_k = \mathfrak{N}_k(\mathfrak{h}_k^{\rho_k})$, where $\mathfrak{h}_k^{\rho_k} \in H^1(S, N_S \otimes \mathcal{J}_S^{k+1}/\mathcal{J}_S^{k+2})$ is the obstruction to k-comfortable embedding with respect to ρ_k.*

Proof

Suppose that the embedding $S \hookrightarrow X$ is $(k-1)$-comfortably embedded. As shown in (3.7), we can choose a $(k-1)$-comfortable atlas adapted to (ρ_{k-1}, ν_{k-1}) such that we get an induced atlas on the blowup with coordinate changes given by

$$\begin{cases} w_\beta^r = \sum_{s=1}^m (a_{\beta\alpha})_s^r (w_\alpha'') w_\alpha^s + t^k \tilde{R}_{k+1}^r & \text{for } r = 1, \ldots, m, \\ w_\beta^p = \phi_{\beta\alpha}^p (w_\alpha'') + t^k \tilde{R}_k^p & \text{for } p = m+1, \ldots, n. \end{cases} \tag{3.15}$$

We can substitute the transition function in (3.15) into (2.5) above to get

$$\begin{aligned} (\theta_k)_{\beta\alpha} &= \frac{1}{k!} \sum_{i=1}^n \frac{\partial^k f_{\beta\alpha}^i (w_\alpha, t)}{\partial t^k}\Big|_{t=0} \frac{\partial}{\partial w_\beta^i} \\ &= \sum_{r=1}^m \tilde{R}_{k+1}^r (0; w_\alpha) \frac{\partial}{\partial w_\beta^r} + \sum_{p=m+1}^n \tilde{R}_k^p (0; w_\alpha) \frac{\partial}{\partial w_\beta^p}, \end{aligned} \tag{3.16}$$

where in the last expression, w_α and w_β are related by the following relation on $\tilde{\mathcal{X}}_0$ near $S_0 \cong S$:

$$\begin{cases} w_\beta^r = \sum_{s=1}^m (a_{\beta\alpha})_s^r (w_\alpha'') w_\alpha^s & \text{for } r = 1, \dots, m, \\ w_\beta^p = \phi_{\beta\alpha}^p (w_\alpha'') & \text{for } p = m+1, \dots, n, \end{cases} \tag{3.17}$$

which is nothing but the transition function on N_S. Recall that $\tilde{R}_{k+1}^r(t; w_\alpha', w_\alpha'') = t^{-(k+1)} R_{k+1}(t w_\alpha', w_\alpha'')$ and $\tilde{R}_k^p(t; w_\alpha', w_\alpha'') = t^{-k} R_k^p(t w_\alpha', w_\alpha'')$. So $\tilde{R}_{k+1}^r(0; w_\alpha)$ (resp., $\tilde{R}_k^p(0; w_\alpha)$) is nothing but the $(k+1)$th-order (resp., kth-order) leading term of $R_{k+1}^r(w_\alpha)$ (resp., $R_k^p(w_\alpha)$) in its Taylor expansion with respect to w_α'.

Since w_α' are global coordinates on the whole $\hat{U}_\alpha \subset N_S$, we see that $(\theta_k)_{\beta\alpha}$ is actually defined over $\hat{U}_\alpha \cap \hat{U}_\beta \subset N_S$. This shows the statement that $\theta_k \in H^1(\mathcal{W}_0, \Theta_{\mathcal{W}_0})$ extends uniquely to a class in $H^1(N_S, \Theta_{N_S})$ which will still be denoted by θ_k.

So if we denote by $\pi_S : N_S \to S$ the natural projection of the normal bundle to its base, and by $\hat{U}_\alpha = \pi_S^{-1}(\mathcal{U}_\alpha \cap \tilde{\mathcal{X}}_0 \cap S_0)$ the $\mathbb{C}^*$-invariant open set on N_S, then we have

$$(\theta_k)_{\beta\alpha} \in H^0(\hat{U}_\alpha \cap \hat{U}_\beta, \Theta_{N_S} \otimes \mathcal{J}_S^k).$$

So we get a Čech cohomology class:

$$\theta_k' := \{(\theta_k)_{\beta\alpha}\} \in \check{H}^1(N_S, \Theta_{N_S} \otimes \mathcal{J}_S^k).$$

From (3.16) and the homogeneity of $\tilde{R}_{k+1}^r$, $\tilde{R}_k^p$ in w_α', we see that θ_k' has weight $(-k)$ under the natural $\mathbb{C}^*$-action on N_S. When we restrict to $S_0 = S \subset N_S$ and mod out by $\mathcal{J}_{S_0}^{k+1}$, we get

$$\begin{aligned} (\mathfrak{g}_k)_{\beta\alpha} &:= (\theta_k)_{\beta\alpha}|_{S_0} \\ &= \sum_{r=1}^m \big[\tilde{R}_{k+1}^r(0; w_\alpha', w_\alpha'')\big]_{(k)} \frac{\partial}{\partial w_\beta^r} + \sum_{p=m+1}^n \big[\tilde{R}_k^p(0; w_\alpha', w_\alpha'')\big]_{(k)} \frac{\partial}{\partial w_\beta^p} \\ &= \sum_{p=m+1}^n \big[\tilde{R}_k^p(0; w_\alpha', w_\alpha'')\big]_{(k)} \frac{\partial}{\partial w_\beta^p}, \end{aligned} \tag{3.18}$$

which form a cocycle

$$\begin{aligned} \{(\mathfrak{g}_k)_{\beta\alpha}\} &\in \check{H}^1\big(\{U_\alpha\}, \Theta_{N_S}|_{S_0} \otimes \mathcal{J}_{S_0}^k / \mathcal{J}_{S_0}^{k+1}\big) \\ &= \check{H}^1\big(\{U_\alpha\}, N_{S_0} \otimes \mathcal{J}_{S_0}^k / \mathcal{J}_{S_0}^{k+1}\big) \oplus \check{H}^1\big(\{U_\alpha\}, \Theta_{S_0} \otimes \mathcal{J}_{S_0}^k / \mathcal{J}_{S_0}^{k+1}\big). \end{aligned}$$

In the last equality, we used the holomorphic splitting $\Theta_{N_S}|_{S_0} = \Theta_{S_0} \oplus N_{S_0}$. Because we assumed that S is $(k-1)$-comfortably embedded, the component in the first summand is 0 as seen in (3.18). So using the notation in Lemma 3.2, we can write $\mathfrak{g}_k = \mathfrak{T}_k(\theta_k)$. By Proposition A.2, we see that $\mathfrak{g}_k = \{(\mathfrak{g}_k)_{\beta\alpha}\}$ is the obstruction to the existence of ρ_k satisfying $\phi_{k,k-1} \circ \rho_k = \rho_{k-1}$. In other words, $\mathfrak{g}_k^{\rho_{k-1}} := \mathfrak{g}_k$ is the obstruction to k-splitting relative to ρ_{k-1}. So we get the first part of Proposition 3.3.

Now if we assume that the obstruction to k-splitting vanishes (i.e., that the above $\mathfrak{g}_k^{\rho_{k-1}}$ vanishes), then by Theorem A.9 the transition functions in (3.15) can be improved to

$$\begin{cases} w_\beta^r = \sum_{s=1}^m (a_{\beta\alpha})_s^r (w''_\alpha) w_\alpha^s + t^k \tilde{R}_{k+1}^r & \text{for } r = 1, \dots, m, \\ w_\beta^p = \phi_{\beta\alpha}^p (w''_\alpha) + t^{k+1} \tilde{R}_{k+1}^p & \text{for } p = m+1, \dots, n. \end{cases} \tag{3.19}$$

Substituting this into (3.16), $(\theta_k)_{\beta\alpha}$ now becomes

$$(\theta_k)_{\beta\alpha} = \frac{1}{k!} \sum_{i=1}^n \frac{\partial^k f_{\beta\alpha}^i (w_\alpha, t)}{\partial t^k} \Big|_{t=0} \frac{\partial}{\partial w_\beta^i} = \sum_{r=1}^m \tilde{R}_{k+1}^r (0; w_\alpha) \frac{\partial}{\partial w_\beta^r}. \tag{3.20}$$

So we see that in this case $(\theta_k)_{\beta\alpha} \in H^0(\hat{U}_\alpha \cap \hat{U}_\beta, \Theta_{N_S} \otimes \mathcal{I}_S^{k+1})$. Again we get a weight $(-k)$ Čech cohomology class

$$\theta''_k := \{(\theta_k)_{\beta\alpha}\} \in \check{H}^1(\{\hat{U}_\alpha\}, \Theta_{N_S} \otimes \mathcal{I}_S^{k+1})(-k),$$

which satisfies $\mathfrak{N}'_k(\theta''_k) = \theta_k$. When we restrict to S_0 and mod out by $\mathcal{I}_{S_0}^{k+2}$, we get

$$\begin{aligned} (\mathfrak{h}_k)_{\beta\alpha} &:= (\theta_k)_{\beta\alpha}|_{S_0} \\ &= \sum_{r=1}^m \big[\tilde{R}_{k+1}^r (0; w'_\alpha, w''_\alpha)\big]_{(k+1)} \frac{\partial}{\partial w_\beta^r} \\ &\in H^0(\hat{U}_\alpha \cap \hat{U}_\beta \cap S_0, N_{S_0} \otimes \mathcal{I}_{S_0}^{k+1}/\mathcal{I}_{S_0}^{k+2}). \end{aligned} \tag{3.21}$$

Comparing with (A.3), we see that $\mathfrak{h}_k := \{(\mathfrak{h}_k)_{\beta\alpha}\}$ is nothing but the obstruction $\mathfrak{h}_k^{\rho_k}$ to k-comfortable embedding with respect to the k-splitting ρ_k. By Lemma 3.2, we can write $\theta''_k = \mathfrak{R}_k^{-1}(\mathfrak{h}_k)$. □

4. Special case: $S = D$ is an ample divisor

One of the main goals of this section is to prove Theorem 1.5. The proof is essentially based on the construction in Section 3.1 and Proposition 3.3. Roughly speaking, under the assumption that $D \to X$ is $(m-1)$-comfortable, we get an $(m-1)$-trivial atlas by the construction in Section 3.1 and hence a reduced Kodaira–Spencer class defined as a class in $H^1(U, \Theta_U)$. Then Proposition 3.3 is also used to show that this reduced Kodaira–Spencer class is nontrivial if the embedding $D \to X$ is not m-comfortable (and $n \geq 3$). Finally, by Proposition 2.15, the reduced Kodaira–Spencer class near the "infinity" divisor via coordinate changes coincides with the reduced Kodaira–Spencer class for the deformation of the cone defined in Definition 2.12. This allows us to complete the proof.

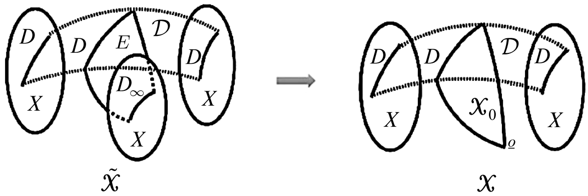

Figure 1. $\mu : \tilde{\mathcal{X}} \longrightarrow \mathcal{X}$.

4.1. Degeneration to the projective cone

From now on, we assume that $S = D$ is a smooth ample divisor in X. Then we can further modify the deformation to the normal cone construction. Recall from the above section that $\tilde{\mathcal{X}} = \mathrm{Bl}_{D\times\{0\}}(X \times \mathbb{C})$ and that $\tilde{\mathcal{X}}_0 = (\mathrm{Bl}_D X) \cup E = X \cup E$, where $E = \mathbb{P}(N_D \oplus \mathbb{C})$. Denote by $L := L_D$ the holomorphic line bundle associated to the divisor D. Since D is an ample divisor, one can verify that the line bundle $\tilde{\mathcal{L}} = \pi_1^* L - E$ is π_2-relatively semiample, where π_1 is the composition $\tilde{\mathcal{X}} \to X \times \mathbb{C} \to X$ and π_2 is the composition $\tilde{\mathcal{X}} \to X \times \mathbb{C} \to \mathbb{C}$. Moreover, the strict transform of X under the blowup becomes exceptional and can be blown down so that we get $\mathcal{X}$ under the morphism associated to $\tilde{\mathcal{L}}$. Then the canonical morphism $\pi : \mathcal{X} \to \mathbb{C} = \mathrm{Spec}(\mathbb{C}[t])$ gives a flat family of projective varieties, satisfying that $\mathcal{X}_t \cong X$ for $t \neq 0$ and $\mathcal{X}_0$ is obtained from E by contracting the infinity section D_∞ (see Figure 1 for an illustration). The central fiber $\mathcal{X}_0$ thus obtained is very close to being the projective cone $\overline{C}(D, L)$. One delicate point here is that $\mathcal{X}_0$ may not be normal.

LEMMA 4.1

The central fiber $\mathcal{X}_0$ coincides with $\bar{C}(D, L)$ if the restriction map $\psi_m : H^0(X, mL) \to H^0(D, mL|_D)$ is surjective for any $m \geq 1$.

Proof

We first describe the above construction of $\mathcal{X}$ in the algebraic category (see [13, Chapter 5]). Let $\mathcal{I}_D$ denote the ideal sheaf of D as a subvariety of X. Then $\tilde{\mathcal{X}}$ is the blowup of the ideal sheaf $\mathcal{I}_D + (t)$ on $X \times \mathbb{C}$:

$$\tilde{\mathcal{X}} = \mathrm{Proj}_{X\times\mathbb{C}}\Big(\bigoplus_{k=0}^{+\infty} \big(\mathcal{I}_D + (t)\big)^k\Big).$$

Moreover $\mathcal{X} = \mathrm{Proj}_{\mathbb{C}[t]} \mathcal{R}$, where $\mathcal{R}$ is the following finitely generated graded algebra over $\mathbb{C}[t]$:

$$\mathcal{R} = \bigoplus_{k=0}^{+\infty} H^0\big(\mathbb{C}, (\pi_2)_*(k\tilde{\mathcal{L}})\big) = \bigoplus_{k=0}^{+\infty} H^0(\tilde{\mathcal{X}}, k\tilde{\mathcal{L}}) = \bigoplus_{k=0}^{+\infty} \mathcal{R}_k, \tag{4.1}$$

where $\tilde{\mathcal{L}} = \pi_1^* L - E$. The graded pieces $\mathcal{R}_k$ can be calculated in the same way as in [29, Section 4]:

$$\begin{aligned}\mathcal{R}_k &= H^0\big(\tilde{X}, k(\pi_1^* L - E)\big) = H^0\big(X \times \mathbb{C}, L^k \otimes \big(\mathcal{I}_D + (t)\big)^k\big) \\ &= \bigoplus_{j=0}^{k-1} t^j H^0(X, L^k \otimes \mathcal{I}_D^{k-j}) \oplus t^k \mathbb{C}[t] H^0(X, L^k) \\ &= \bigoplus_{j=0}^{k-1} t^j H^0(X, L^j) \oplus t^k \mathbb{C}[t] H^0(X, L^k). \end{aligned} \tag{4.2}$$

In the last identity, we used $\mathcal{I}_D = \mathcal{O}_X(-D) \cong L^{-1}$. The central fiber is thus equal to

$$\mathcal{X}_0 = \mathrm{Proj}_{\mathbb{C}}\big(\mathcal{R}/(t)\mathcal{R}\big) = \mathrm{Proj}_{\mathbb{C}}\Big(\bigoplus_{k=0}^{+\infty} \mathcal{R}_k/(t)\mathcal{R}_k\Big). \tag{4.3}$$

From (4.2), we see directly that

$$\mathcal{R}_k/(t)\mathcal{R}_k = \mathbb{C} \oplus \bigoplus_{j=1}^{k} t^j \frac{H^0(X, L^j)}{H^0(X, L^{j-1})} = \bigoplus_{j=0}^{k} t^j H^0(X, L^j)|_D. \tag{4.4}$$

Here $H^0(X, L^j)|_D$ denotes the image of the restriction map $H^0(X, L^r) \to H^0(D, L^j|_D)$ for any $j \geq 0$. To see the last identity, we consider the exact sequence of ideal sheaves

$$0 \longrightarrow L^{j-1} = L^j \otimes \mathcal{O}(-D) \longrightarrow L^j \longrightarrow L^j|_D \longrightarrow 0, \tag{4.5}$$

and the corresponding long exact sequence

$$0 \longrightarrow H^0(X, L^{j-1}) \longrightarrow H^0(X, L^j) \longrightarrow H^0(D, L^j|_D) \longrightarrow H^1(X, L^{j-1}). \tag{4.6}$$

So indeed $H^0(X, L^j)/H^0(X, L^{j-1}) = H^0(X, L^j)|_D$ for $j \geq 1$.

On the other hand, we have

$$\bar{C}(D, L) = \mathrm{Proj} \bigoplus_{k=0}^{\infty} \Big(\bigoplus_{j=0}^{k} H^0(D, L^j|_D) \cdot t^{k-j}\Big).$$

Combining this with (4.3) and (4.4), we see that $\mathcal{X}_0 \cong \bar{C}(D, L)$ if $H^0(X, L^j)|_D = H^0(D, L^j|_D)$ for any $j \geq 1$ (the $j = 0$ case is automatic). □

For example, let X be any Riemann surface of genus at least 1, and let $D = \{p\}$ be any point. Then D is ample. In this special case, the central fiber $\mathcal{X}_0$ is a singular curve whose normalization is $\mathbb{P}^1$. Here the map $\psi_0 = \mathrm{Id} : H^0(X, \mathcal{O}_X) \to H^0(p, \mathcal{O}_p)$. But $\psi_1 = 0 : H^0(X, L_p) = \mathbb{C} \to H^0(\{p\}, L_p|_{\{p\}}) = \mathbb{C}$ because ψ_1 factors through the inverse of isomorphism $H^0(X, \mathcal{O}_X) = \mathbb{C} \xrightarrow{\cdot s_{\{p\}}} H^0(X, L_p)$ by the assumption that $g(X) \geq 1$. In particular, ψ_1 is not surjective.

Remark 4.2
The above lemma was communicated to me by H.-J. Hein. One referee provided an even more explicit example to me: if X is an elliptic curve and p is a Weierstrass point, then by using the Weierstrass form, one can verify that the total space has a singularity of type $\tilde{E}_8$.

On the other hand, from the exact sequence

$$0 \to H^0\big(X, (m-1)L_D\big) \to H^0(X, mL_D) \to H^0(D, mL|_D) \\ \to H^1\big(X, (m-1)L\big) \to \cdots,$$

we see that ψ_m is surjective if $H^1(X, (m-1)L) = 0$ for all $m \geq 1$. In particular, this is satisfied in the Tian–Yau setting. Indeed, if X is Fano and $m \geq 1$, then $H^1(X, (m-1)L) = H^1(X, \Omega_X^n \otimes \mathcal{O}_X(-K_X + (m-1)L)) = 0$ by the Kodaira–Nakano vanishing theorem.

4.2. Proof of Theorem 1.5
From now on, we assume that we are in the situation that the above central fiber $\mathcal{X}_0$, that is, the strict transform of the exceptional divisor of the blowup, is normal and hence coincides with $\bar{C}(D, L) =: \bar{C}$. Let $\mathcal{D}$ be the strict transform of $D \times \mathbb{C}$, and let $\mathcal{X}^\circ = \mathcal{X} \setminus \mathcal{D}$. Because $\mathcal{D}$ is a relatively ample divisor over $\mathbb{C}$, we know that $\mathcal{X}^\circ$ is a flat family of affine varieties. In particular, we can define $\mathrm{Ord}_{\mathcal{X}^\circ}$ as in Definition 2.11. Notice that $\mathcal{X}_0^\circ = C(D, L) =: C$ and we can define the reduced Kodaira–Spencer class $\mathbf{KS}_{\mathcal{X}^\circ}^{\mathrm{red}} \in \mathbf{T}_C^1$. Since there is a natural $\mathbb{C}^*$-action on $\mathbf{T}_C^1$, we can talk about the weight of $\mathbf{KS}_{\mathcal{X}^\circ}^{\mathrm{red}} \in \mathbf{T}_C^1$ and denote it by $w(\mathcal{X}^\circ)$ (see Section A.2.2). With this notation and combining the calculations from the previous subsection, we can derive the following.

PROPOSITION 4.3
Let $\mathcal{X} \to \mathbb{B}$ be the flat family constructed in the above section, and assume that $\mathcal{X}_0 = \bar{C}(D, N_D)$. Let $D \hookrightarrow X$ be $(k-1)$-comfortably embedded, and let $(\rho_{k-1}, \boldsymbol{\nu}_{k-1})$ be a $(k-1)$-comfortable pair. If D is not k-splitting relative to ρ_{k-1}, then $\mathrm{Ord}(\mathcal{X}^\circ) = k =$

$-w(\mathcal{X}^\circ)$. *In particular, if* D *is* $(k-1)$*-comfortably embedded and not* k*-splitting, then* $\mathrm{Ord}(\mathcal{X}^\circ) = k = -w(\mathcal{X}^\circ)$.

Proof

Let $(\rho_{k-1}, \boldsymbol{\nu}_{k-1})$ be a $(k-1)$-comfortable pair. Then by the proof of Proposition 3.3, we have a $(k-1)$-trivial atlas covering $\mathcal{W}_0$. Without loss of generality, we can assume that $\mathcal{W}_0 = \bar{C} \setminus K$ where K is a strongly pseudoconvex neighborhood of the vertex $o \in \bar{C}$. Then we also have a $(k-1)$-trivial atlas covering $\mathcal{W}_0 \setminus D = C \setminus K$. In particular, this atlas covers the annulus $Y = (C \setminus \overline{K}_{c_1}) \cap \mathring{K}_{c_2}$. By Proposition 2.15, we get $\mathrm{Ord}(\mathcal{X}^\circ) \geq k$. Moreover from Proposition 3.3, we get a cohomology class $\theta_k \in H^1(L, \Theta_L)$ with weight $-k$, which is represented by a cocycle $\{(\theta_k)_{\beta\alpha}\}$. Proposition 2.15 yields that

$$(\mu_1 \circ \tau_U)(\mathbf{KS}^{(k)}_{\mathcal{X}^\circ}) = \theta_k|_Y,$$

where Y again denotes the annulus and μ_1, τ_U are from the diagram (2.34). But $\theta_k|_Y = \mu_1(\theta_k|_U)$. So thanks to the injectivity of both τ_U and μ_1, we may reduce to proving that the class $\vartheta_k := \theta_k|_U \in H^1(U, \Theta_U)$ is not zero.

By Proposition 3.3, we know that $\mathfrak{T}_k(\theta_k) = \mathfrak{g}_k$ is the obstruction to k-splitting relative to ρ_{k-1}. So if the embedding is not k-splitting with respect to ρ_{k-1}, then θ_k is nonzero. Now the claim follows from Lemma 4.6. $\square$

COROLLARY 4.4

Assume that $\dim D = n-1 \geq 2$. *If* D *is* $(k-1)$*-comfortably embedded and not* k*-comfortably embedded, then the following holds.*

(1) $\mathrm{Ord}(\mathcal{X}^\circ) = k = -w(\mathcal{X}^\circ)$*; that is, Theorem 1.5 is true.*

(2) *For any* $l < k$ *and any* $(l-1)$*th-order lifting* $\rho_{l-1} : \mathcal{O}_D \to \mathcal{O}_X/\mathcal{I}_D^l$*, there exists a* $(k-1)$*th-order lifting* $\rho_{k-1} : \mathcal{O}_D \to \mathcal{O}_X/\mathcal{I}_D^k$ *such that* $\phi_{k-1.l-1}(\rho_{k-1}) = \rho_{l-1}$*, where* $\phi_{k-1,l-1} : \mathcal{O}_X/\mathcal{I}_D^k \to \mathcal{O}_X/\mathcal{I}_D^l$ *is the natural map.*

Proof

We first recall Remark A.7. If $\dim D \geq 2$ and D is ample, then $H^1(D, N_D \otimes \mathcal{I}_D^{k+1}/\mathcal{I}_D^{k+2}) = H^1(D, L_D^{-k}) = 0$ for any $k \geq 1$ by the Kodaira-Nakano vanishing theorem. So there is no obstruction to k-comfortably embedded relative to any kth-order lifting. As a consequence, k-comfortable is equivalent to k-splitting for any $k \geq 0$, and is also equivalent to k-linearizable for all $k \geq 0$.

By the assumption, we know that (X, D) is $(k-1)$-splitting but not k-splitting, and hence there exists a comfortable pair $(\rho_{k-1}, \boldsymbol{\nu}_{k-1})$ such that there is no kth-order lifting relative to ρ_{k-1}. So the first statement holds by Proposition 4.3.

Suppose that for some $l < k$, there exists an $(l-1)$th-order lifting ρ_{l-1} that cannot be lifted to a $(k-1)$-order lifting. By choosing the maximal l and using Remark A.7, we can assume that there is a comfortable pair $(\rho_{l-1}, \boldsymbol{\nu}_{l-1})$ such that ρ_{l-1} cannot be lifted to an lth-order lifting. By Proposition 4.3, we get $w(X,D) = -l > -k$, which contradicts Corollary 4.4(1). ☐

Remark 4.5
We will see in Proposition 4.9 that Corollary 4.4(2) is not necessarily true if $n = 2$.

LEMMA 4.6
For $k \geq 1$, the natural restriction map induces an isomorphism $H^1(L, \Theta_L)(-k) \xrightarrow{\cong} H^1(U, \Theta_U)(-k)$.

Proof
This is already clear by the homogeneity argument as in the proof of Lemma 3.2. Indeed, we just need to construct an inverse of the natural morphism. Let $\theta_k \in H^1(U, \Theta_U)(-k)$. Then by the same argument as in the proof of Lemma 3.2, we can assume that θ_k is represented by a weight $(-k)$ cocycle

$$(\theta_k)_{\beta\alpha} = \sum_{r=1}^{m} b^r_{\beta\alpha}(w) \frac{\partial}{\partial w^r_\beta} + \sum_{p=m+1}^{n} c^p_{\beta\alpha}(w) \frac{\partial}{\partial w^p_\beta}.$$

Since $\frac{\partial}{\partial w^r_\beta}$ (resp., $\frac{\partial}{\partial w^p_\beta}$) has weight 1 (resp., 0), we know that $b^r_{\beta\alpha}$ (resp., $c^p_{\beta\alpha}$) is homogeneous of degree $(k+1)$ (resp., k) in $w' = \{w^r_\beta\}$. Because $k \geq 1$, θ_k can be extended to become a cocycle $H^1(L, \Theta_L)(-k)$. This defines the inverse of the restriction morphism. ☐

Remark 4.7
We sketch a slightly more conceptual proof by using the Dolbeault cohomology. On the total space L, we have the exact sequence

$$0 \to \pi_L^* L \to \Theta_L \to \pi_L^* \Theta_D \to 0. \tag{4.7}$$

By restricting this exact sequence to $U = L \backslash D$, we have a similar exact sequence on U. So we get a commutative diagram of long exact sequences:

$$\begin{array}{ccccccccc}
H^0(L, \pi_L^*\Theta_D) & \to & H^1(L, \pi_L^* L) & \to & H^1(L, \Theta_L) & \to & H^1(L, \pi_L^*\Theta_D) & \to & H^2(L, \pi_L^* L) \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
H^0(U, \pi_U^*\Theta_D) & \to & H^1(U, \pi_U^* L) & \to & H^1(U, \Theta_L) & \to & H^1(L, \pi_U^*\Theta_D) & \to & H^2(U, \pi_U^*)
\end{array}$$

(4.8)

For any $k \geq 0$, we have the weight-$(-k)$ pieces of the cohomology groups under the natural $\mathbb{C}^*$-action:

$$\begin{aligned} H^p(L, \pi_L^* L)(-k) &= H^p(D, L^{-k}), \\ H^p(L, \pi_L^* \Theta_D)(-k) &= H^p(D, \Theta_D \otimes L^{-k}); \\ H^p(U, \pi_U^* L)(-k) &= H^p(D, L^{-k}), \\ H^p(U, \pi_U^* \Theta_D)(-k) &= H^p(D, \Theta_D \otimes L^{-k}). \end{aligned}$$

If we were to work in the algebraic category, the weight decomposition is directly obtained by using a projection formula as in [4, Section 11]. Since we are working in the analytic category, we need to be more careful as we now explain. Since the arguments to get the decompositions are the same, we just explain the first identity. Using the isomorphism between Dolbeault cohomology and sheaf cohomology, any cohomology class $\alpha \in H^p(L, \pi_L^* L)$ is represented by a $\bar\partial$-closed $\pi_L^* L$-valued $(0, p)$-form denoted by η. For any point $p \in D$, we first choose local holomorphic coordinates $\{z^i, \xi\}$, where the $\{z^i\}$'s are holomorphic coordinates on D and ξ is a linear coordinate along the fiber associated to a local trivializing holomorphic section s. By using the Fourier expansion along the circle $|\xi| =$ constant and extending to the whole U, one can show that η can be expressed as a convergent sum:

$$\begin{aligned} \eta = &\sum_{m \in \mathbb{N}_0, |I|=p} \big[A'_{m,I}\big(z, |\xi|^2\big)\xi^m + A'_{\bar m,I}\big(z, |\xi|^2\big)\bar\xi^m\big] s\, d\bar z^I \\ &+ \bar\partial\Big(\sum_{m \in \mathbb{N}_0, |J|=p-1} \big[B'_{m,J}\big(z, |\xi|^2\big)\xi^{m-1} + B'_{\bar m,J}\big(z, |\xi|^2\big)\bar\xi^{m+1}\big] s\, d\bar z^J\Big) \\ =: &\ \eta' + \bar\partial\zeta. \end{aligned}$$

Furthermore, by using the fact that $\bar\partial\eta' = 0$, one can see that $A'_{m,I}(z, |\xi|^2)$ and $A'_{\bar m,I}(z, |\xi|^2)|\xi|^{2m}$ are constants in ξ. In particular, by smoothness, $A'_{\bar m,I} = 0$ for all $m \in \mathbb{N}_0$. So we see that η is $\bar\partial$-cohomologous to a $(0, p)$-form of the form

$$\eta' = \sum_{m \in \mathbb{N}_0} \sum_{|I|=p} A'_{m,I}(z)\xi^m s\, d\bar z^I =: \sum_m \eta_m.$$

$\mathbb{C}^*$ acts on η' by $t \circ \eta' = \sum_m t^{m-1}\eta'_m$. Using the $\bar\partial$-closedness of η', it is easy to see that each component η'_m is $\bar\partial$-closed. We have that η' is of weight $-k$ if and only if $\eta' = \eta'_{k+1} = \sum_{|I|=p} A'_{k+1,I}\xi^{k+1} s\, d\bar z^I$ which represents a cohomology class $\alpha \in H^p(D, L^{-k})$. We can now extract the weight $(-k)$-part from (4.8) to get exact sequences:

$$\begin{array}{ccccccccc} H^0(D,\Theta_D\otimes L^{-k}) & \to & H^1(D,L^{-k}) & \xrightarrow{\mathfrak{N}_k} & H^1(L,\Theta_L)(-k) & \xrightarrow{\mathfrak{T}_k} & H^1(D,\Theta_D\otimes L^{-k}) & \to & H^2(D,L^{-k}) \\ \| & & \| & & \downarrow & & \| & & \| \\ H^0(D,\Theta_D\otimes L^{-k}) & \to & H^1(D,L^{-k}) & \xrightarrow{\mathfrak{N}_k^\circ} & H^1(U,\Theta_U)(-k) & \xrightarrow{\mathfrak{T}_k^\circ} & H^1(D,\Theta_D\otimes L^{-k}) & \to & H^2(D,L^{-k}) \end{array} \tag{4.9}$$

The statement then follows from the five lemma.

Remark 4.8

If we rewrite the statement of Proposition 3.3 by using the isomorphism of Lemma 4.6, then we get the exact sequence

$$H^1(D,L^{-k}) \xrightarrow{\mathfrak{N}_k^\circ} H^1(U,\Theta_U)(-k) \xrightarrow{\mathfrak{T}_k^\circ} H^1(D,\Theta_D\otimes L^{-k}) \tag{4.10}$$

such that (1) $\mathfrak{T}_k^\circ(\vartheta^{(k)}) = \mathfrak{g}_k$ is the obstruction to k-splitting, and (2) if $\mathfrak{T}_k^\circ(\vartheta^{(k)}) = 0$, then there is a kth-order lifting ρ_k and $\vartheta^{(k)} = \mathfrak{N}_k^\circ(\mathfrak{h}^{(k)})$, where $\mathfrak{h}^{(k)}$ is the obstruction to k-comfortably embedding with respect to ρ_k.

4.3. 2-dimensional examples and a remark on comfortable embedding

As mentioned in the Introduction and recalled in Section A.1 of the Appendix, Abate, Bracci, and Tovena in [1] gave a detailed study of various conditions of embedding: k-linearizable, k-splitting, and k-comfortable embedding. In order to talk about k-comfortable embedding, we need to assume k-splitting (see Definition A.4). Under this assumption, we can study whether the embedding is comfortable with respect to any kth-order lifting. In [1, Remark 3.4], the authors asked whether k-comfortable embedding with respect to one kth-order lifting implies k-comfortable embedding with respect to any other kth-order lifting. Here we give a simple example showing that the answer to this question is in general negative.

PROPOSITION 4.9

For the diagonal embedding $D = \Delta(\mathbb{P}^1) \hookrightarrow X = \mathbb{P}^1 \times \mathbb{P}^1$, the following conditions hold.

(i) *It is k-splitting for any $k \geq 1$.*
(ii) *The set of all first-order liftings is parameterized by $\mathbb{C}$. So we can denote by ρ_1^a the first-order lifting corresponding to any $a \in \mathbb{C}$.*
(iii) *There exists a second-order lifting ρ_2 satisfying $\phi_{2,1} \circ \rho_2 = \rho_1^a$ if and only if $a = 0$.*
(iv) *The embedding is 1-comfortable with respect to ρ_1^a if and only if $a = -1/2$.*
(v) *The embedding is 1-linearizable but not 2-linearizable.*

Remark 4.10

This diagonal embedding is 2-splitting and 1-comfortable, but the embedding is only 1-linearizable. This does not contradict Theorem A.9, since the 1-comfortable embedding is with respect to $\rho_1^{-1/2}$ which cannot be lifted to a second-order lifting.

Proof of Proposition 4.9

Because there is a projection morphism onto the first factor $p_1 : X = \mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^1$, we see that there is a natural kth-order lifting $\rho_k : \mathcal{O}_D \to \mathcal{O}_X/\mathcal{I}_D^{k+1}$ given by $\phi_{\infty,k} \circ p_1^* \circ \Delta^*$, where $p_1^* : \mathcal{O}_D \to \mathcal{O}_X$ is the pullback and $\phi_{\infty,k} : \mathcal{O}_X \to \mathcal{O}_X/\mathcal{I}_D^{k+1}$ is the natural quotient map. So the embedding is k-splitting for any $k \geq 1$. Since any embedding is 0-comfortable, we know that the embedding is 1-linearizable by Theorem A.9. So we get (i) and the first half of (v).

We will quickly show that the embedding is not comfortable with respect to the natural first-order lifting ρ_1. We first construct an atlas near D. Choose the open covering of $\mathbb{P}^1 \times \mathbb{P}^1$,

$$\mathfrak{V} = \{U_i \times U_j ; 1 \leq i, j \leq 2\},$$

with (we denote $\mathbb{P}^1 = \mathbb{C} \cup \{\infty\}$ with $|\infty| = +\infty$)

$$U_1 = \big\{z \in \mathbb{P}^1; |z| < 2\big\}, \qquad U_2 = \big\{z \in \mathbb{P}^1; |z| > 1/2\big\}.$$

Then $S = \Delta(\mathbb{P}^1)$ is covered by two open sets $\{V_i := U_i \times U_i ; i = 1, 2\}$. We define the new coordinate functions by

$$\begin{aligned} V_1 = \big\{(z,z') \in \mathbb{P}^1 \times \mathbb{P}^1; |z| < 2, |z'| < 2\big\} &\to \mathbb{C}^2, \\ (z,z') &\mapsto (y_1 = z - z', z_1 = z), \\ V_2 = \big\{(z,z') \in \mathbb{P}^1 \times \mathbb{P}^1; |z| > 1/2, |z'| > 1/2\big\} &\to \mathbb{C}^2, \\ (z,z') &\mapsto (y_2 = z^{-1} - z'^{-1}, z_2 = z^{-1}). \end{aligned}$$

So we have $D \cap V_i = \{y_i = 0\}$. If V' is a small neighborhood of $S = \Delta(\mathbb{P}^1)$, then on the intersection $V_1 \cap V_2 \cap V'$, the transition functions are given by

$$y_2 = -\frac{y_1}{z_1(z_1 - y_1)} = -\frac{y_1}{z_1^2} - \frac{y_1^2}{z_1^3} + R_3, \qquad z_2 = z_1^{-1}. \tag{4.11}$$

In the above expansion, we assume that y_1 is sufficiently small, and we denote by R_3 a term belonging to $\mathcal{I}_D^3$. It is immediate to see that this atlas is adapted to the natural first-order lifting ρ_1 where we have

$$\rho_1(z_1) = [z_1]_2 \quad \text{on } V_1 \cap V', \qquad \rho_1(z_2) = [z_2]_2 \quad \text{on } V_2 \cap V'.$$

The obstruction to 1-comfortable embedding is given by

$$(\mathfrak{h}_1^{\rho_1})_{21} = -\frac{[y_1^2]_3}{z_1^3}\frac{\partial}{\partial y_2} \in H^0(U_1 \cap U_2, N_D \otimes \mathcal{I}_D^2/\mathcal{I}_D^3). \tag{4.12}$$

Here we consider $\frac{\partial}{\partial y_2}$ and $\frac{\partial}{\partial y_1}$ as local generators of N_D, so that we have $\frac{\partial}{\partial y_2} = -z_1^2\frac{\partial}{\partial y_1}$ on $U_1 \cap U_2$. We claim that $\mathfrak{h}_1^{\rho_1}$ represents a nonzero cohomology class in $H^1(D, N_D \otimes \mathcal{I}_D^2/\mathcal{I}_D^3) \cong H^1(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(-2)) = \mathbb{C}$. Otherwise, we can write

$$-\frac{[y_1^2]_3}{z_1^3}\frac{\partial}{\partial y_2} = a[y_1^2]_3\frac{\partial}{\partial y_1} - b[y_2^2]_3\frac{\partial}{\partial y_2} \quad \text{on } U_1 \cap U_2,$$

where $a = a(z_1)$ is analytic in z_1 and $b = b(z_1^{-1})$ is analytic in $z_2 = z_1^{-1}$. Using the change of coordinates, we arrive at an equation

$$-\frac{1}{z_1} = a(z_1) - \frac{b(z_1^{-1})}{z_1^2},$$

which obviously has no solutions by looking at the Laurent expansion. So we get that $D \hookrightarrow X$ is not 1-comfortably embedded with respect to ρ_1.

Let us find all possible first-order liftings, that is, homomorphisms of sheaves of rings $\rho : \mathcal{O}_D = \mathcal{O}_X/\mathcal{I}_D \to \mathcal{O}_X/\mathcal{I}_D^2$ with $\phi_{1,0} \circ \rho = \mathrm{Id}$. On U_1, we can write $\rho(z_1) = [z_1 + a(z_1)y_1]_2$ with $a(z_1)$ analytic in z_1 and $\rho(z_2) = [z_2 + b(z_2)y_2]_2$ with $b(z_2)$ analytic in $z_2 = z_1^{-1}$. Since ρ is a homomorphism of sheaves of rings, we must have

$$\begin{aligned} 1 &= \rho(z_1z_2) \\ &= \big[z_1z_2 + a(z_1)z_1y_1 + b(z_2)z_2y_2\big]_2 \\ &= 1 + a(z_1)z_1[y_1]_2 + b(z_2)z_2[y_2]_2 \quad \text{over } U_1 \cap U_2. \end{aligned}$$

Since we have $[y_2]_2 = -[y_1]_2z_1^{-2}$ by (4.11), we get $(a(z_1) - b(z_2))z_1[y_1]_2 = 0$. So we must have that $a(z_1) = b(z_2) = a =$ constant. Thus we get (ii). We will denote the corresponding first-order lifting by ρ_1^a.

Now for any fixed first-order lifting ρ^a, it is easy to find an atlas adapted to it. We simply need to make a coordinate change:

$$\begin{aligned} \hat{z}_1 &= z_1 + ay_1, \qquad \hat{y}_1 = y_1 \quad \text{on } V_1; \\ \hat{z}_2 &= z_2 + ay_2, \qquad \hat{y}_2 = y_2 \quad \text{on } V_2. \end{aligned} \tag{4.13}$$

We can calculate the new transition function

$$\hat{y}_2 = -\frac{\hat{y}_1}{\hat{z}_1^2} - (2a+1)\frac{\hat{y}_1^2}{\hat{z}_1^3} + R_3, \qquad \hat{z}_2 = \hat{z}_1^{-1} - (a^2+a)\frac{\hat{y}_1^2}{\hat{z}_1^3} + R_3, \tag{4.14}$$

where R_3 denotes terms in $\mathcal{I}_D^3$. So we see that the obstruction to 1-comfortable embedding with respect to ρ_1^a is equal to $(2a+1)\mathfrak{h}_1^{\rho_1}$ (see (4.12)). From the above we have seen that $H^1(D, N_D \otimes \mathcal{I}_D^2/\mathcal{I}_D^3) \cong \mathbb{C}$ is generated by $\mathfrak{h}_1^{\rho_1}$. So the embedding is comfortable with respect to ρ_1^a if and only if $a = -1/2$. So we get (iii).

Furthermore, we can calculate the obstruction to existence of second-order lifting ρ_2^a such that $\phi_{2,1} \circ \rho_2^a = \rho_1^a$:

$$(\mathfrak{g}_2^{\rho_1^a})_{21} = -a^2 \frac{[y_1^2]_3}{z_1^3} \frac{\partial}{\partial z_2} \in H^0(U_1 \cap U_2, \Theta_D \otimes \mathcal{I}_D^2/\mathcal{I}_D^3).$$

By similar reasoning as before, we can see that $H^1(D, \Theta_D \otimes \mathcal{I}_D^2/\mathcal{I}_D^3) = H^1(\mathbb{P}^1, \Theta_{\mathbb{P}^1} \otimes \mathcal{O}_{\mathbb{P}^1}(-4)) \cong \mathbb{C}$ is generated by the cohomology $\mathfrak{g}_2^{\rho_1^a}$ if and only if $a \neq 0$. So we get (iv).

If the embedding is 2-linearizable, then it is 2-splitting and 1-comfortable with respect to the induced 1-splitting (see Theorem A.9). But from (ii)–(iv), we see that no such kind of 1-splitting exists. So we get the second half of (v). □

Remark 4.11

By (4.13), it is clear that the special value $a = -1/2$ corresponds to the (most) "symmetric" coordinate atlas

$$V_1 \ni (z, z') \mapsto \Big(z - z', \frac{1}{2}(z + z')\Big) = (\hat{y}_1, \hat{z}_1),$$
$$V_2 \ni (z, z') \mapsto \Big(z^{-1} - z'^{-1}, \frac{1}{2}(z^{-1} + z'^{-1})\Big) = (\hat{y}_2, \hat{z}_2),$$

for which (see (4.14)) the transition functions are given by

$$\hat{y}_2 = -\frac{\hat{y}_1}{\hat{z}_1^2 - \frac{1}{4}\hat{y}_1^2} = -\frac{\hat{y}_1}{\hat{z}_1^2} - \frac{1}{4}\frac{\hat{y}_1^3}{\hat{z}_1^4} + R_5, \qquad \hat{z}_2 = \frac{\hat{z}_1}{\hat{z}_1^2 - \frac{1}{4}\hat{y}_1^2} = \frac{1}{\hat{z}_1} + \frac{1}{4}\frac{\hat{y}_1^2}{\hat{z}_1^3} + R_4.$$

So this is indeed a 1-comfortable atlas (see Theorem A.8).

Remark 4.12

By Theorem A.3, 1-comfortable embedding is equivalent to the splitting of the exact sequence

$$0 \to \mathcal{I}_D^2/\mathcal{I}_D^3 \to \mathcal{I}_D/\mathcal{I}_D^3 \to \mathcal{I}_D/\mathcal{I}_D^2 \to 0. \tag{4.15}$$

This is an a priori sequence of sheaves of $\mathcal{O}_X/\mathcal{I}_D^2$-modules. $\mathcal{I}_D^2/\mathcal{I}_D^3$ and $\mathcal{I}_D/\mathcal{I}_D^2$ are natural $\mathcal{O}_D$-modules. $\mathcal{I}_D/\mathcal{I}_D^3$ becomes an $\mathcal{O}_D$-module depending on the first-order lifting (ring homomorphism) $\rho_1^a : \mathcal{O}_D \to \mathcal{O}_X/\mathcal{I}_D^2$. Proposition 4.9(iv) is equivalent to saying that (4.15) splits as an exact sequence of $\mathcal{O}_D$-modules thus obtained if and only if $a=-1/2$. This can also be verified directly using the expressions $\rho_1^a(z_1)=[z_1+ay_1]_2$ on V_1 and $\rho_1^a(z_2)=[z_2+ay_2]_2$ on V_2.

Remark 4.13
If we denote by w_i the fiber variables of N_D satisfying $w_2=-z_1^{-2}w_1$, then using the notation in Lemma 3.2, we have $\theta_1^a=\mathfrak{N}_1(\mathfrak{h}_1^{\rho_1^a})=0$ and $\mathfrak{T}_2(\theta_2^{-1/2})=\mathfrak{g}_2^{\rho_1^{-1/2}}\neq 0$, where

$$\begin{aligned}(\theta_1^a)_{21}&=-(2a+1)\frac{w_1^2}{z_1^3}\frac{\partial}{\partial w_2}\\&=(2a+1)\Big(\frac{1}{2}w_2\frac{\partial}{\partial z_2}-\frac{1}{2}w_1\frac{\partial}{\partial z_1}\Big)\in H^0(\hat{U}_1\cap\hat{U}_2,\Theta_{N_D})(-1)\end{aligned}$$

and

$$(\theta_2^{-1/2})_{21}=-\frac{1}{4}\frac{w_1^2}{z_1^3}\frac{\partial}{\partial z_2}\in H^0(\hat{U}_1\cap\hat{U}_2,\Theta_{N_D})(-2).$$

Notice that the central fiber of $\mathcal{X}$ from the contracted deformation to the normal cone is $\bar{C}(\mathbb{P}^1,\mathcal{O}_{\mathbb{P}^1}(2))\cong\mathbb{P}(1,1,2)$. So by Proposition 4.3, we get the following corollary (see Example 5.2).

COROLLARY 4.14
The contracted deformation to the normal cone associated with $(\mathbb{P}^1\times\mathbb{P}^1,\Delta(\mathbb{P}^1))$ *degenerates* $\mathbb{P}^1\times\mathbb{P}^1$ *to* $\mathbb{P}(1,1,2)$. *The weight of this deformation is* -2.

Similarly, we can deal with the case $D_2=\{Z_0^2+Z_1^2+Z_2^2=0\}\hookrightarrow X_2=\mathbb{P}^2$. For this, we notice that there is a two-fold branched covering:

$$\begin{aligned}p_2:\mathbb{P}^1\times\mathbb{P}^1&\longrightarrow\mathbb{P}^2,\\\big([X_0,X_1],[Y_0,Y_1]\big)&\mapsto\big[X_0Y_0+X_1Y_1,\sqrt{-1}(X_0Y_0-X_1Y_1),\\&\qquad\sqrt{-1}(X_0Y_1+X_1Y_0)\big].\end{aligned}$$

The branch locus is exactly $\Delta(\mathbb{P}^1)$ with $p_2(\Delta(\mathbb{P}^1))=D_2$. Using this covering structure, it is easy to obtain two open sets $\{V_1,V_2\}$ covering D_2:

$$\begin{aligned}V_1&=(U_1\times U_1)/\mathbb{Z}^2\to\mathbb{C}^2,\\&\qquad(z,z')\mapsto\Big(y_1=\frac{1}{4}(z-z')^2,z_1=\frac{1}{2}(z+z')\Big),\\V_2&=(U_2\times U_2)/\mathbb{Z}^2\to\mathbb{C}^2,\\&\qquad(z,z')\mapsto\Big(y_2=\frac{1}{4}(z^{-1}-z'^{-1})^2,z_2=\frac{1}{2}(z^{-1}+z'^{-1})\Big).\end{aligned}$$

The transition function over $V_1 \cap V_2$ is given by

$$y_2 = \frac{y_1}{(z_1^2 - y_1)^2} = \frac{y_1}{z_1^4} + \frac{2y_1^2}{z_1^6} + R_3, \qquad z_2 = \frac{z_1}{z_1^2 - y_1} = \frac{1}{z_1} + \frac{y_1}{z_1^3} + R_2.$$

So this atlas is a 0-comfortable one. The associated $\theta_1 \in H^1(D_2, N_{D_2})(-1)$ is represented by

$$(\theta_1)_{21} = \frac{2w_1^2}{z_1^6}\frac{\partial}{\partial w_2} + \frac{w_1}{z_1^3}\frac{\partial}{\partial z_2} \in H^0(\hat{U}_1 \cap \hat{U}_2, \Theta_{N_{D_2}}),$$

where the w_i's are fiber variables of $N_{D_2} \cong \mathcal{O}_{\mathbb{P}^1}(4)$ satisfying $w_2 = z_1^{-4} w_1$. So we have

$$(\mathfrak{g}_1)_{21} = \big(\mathfrak{T}_1(\theta_1)\big)_{21} = \frac{[w_1]_2}{z_1^3}\frac{\partial}{\partial z_2} \in H^0(U_1 \cap U_2, \Theta_{D_2} \otimes \mathcal{I}_{D_2}/\mathcal{I}_{D_2}^2).$$

In the Čech cohomology $\check{H}^1(\{U_1, U_2\}, \Theta_{D_2} \otimes \mathcal{I}_{D_2}/\mathcal{I}_{D_2}^2)$, any coboundary can be represented by

$$a(z_1)[w_1]_2\frac{\partial}{\partial z_1} - b(z_2)[w_2]_2\frac{\partial}{\partial z_2} = \Big(\frac{-a(z_1)}{z_1^2} - \frac{b(z_1^{-1})}{z_1^4}\Big)[w_1]_2\frac{\partial}{\partial z_2}.$$

Since $a(z_1)$ (resp., $b(z_1^{-1})$) is analytic in z_1 (resp., z_1^{-1}), the term in the bracket of the right-hand side cannot contain any z_1^{-3}-term. So we see that $H^1(D_2, \Theta_{D_2} \otimes \mathcal{I}_{D_2}/\mathcal{I}_{D_2}^2) \cong H^1(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(-2)) \cong \mathbb{C}$ is generated by $\mathfrak{g}_1 \neq 0$. Because $\mathfrak{g}_1$ is the obstruction to 1-splitting (see Proposition A.2), we obtain that the embedding is not even 1-splitting and hence not 1-linearizable. In this case, $\mathcal{X}_0 = \bar{C}(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(4)) \cong \mathbb{P}(1,1,4)$. So by Proposition 4.3, we obtain the following result.

PROPOSITION 4.15

We have that $D_2 = \{Z_0^2 + Z_1^2 + Z_2^2 = 0\} \hookrightarrow \mathbb{P}^2$ is 0-linearizable. The contracted deformation to normal cone associated to $(\mathbb{P}^2, D_2)$ degenerates $\mathbb{P}^2$ to $\mathbb{P}(1,1,4)$. The deformation weight $w(X, D)$ is equal to -1.

5. Applications to AC Kähler metrics

In the first subsection, we explicitly compute the data of rotationally symmetric Kähler cone metrics on the affine cone. We also compare the norms with respect to a smooth metric (living on the projective cone) and norms with respect to the cone metric near the infinity divisor. This allows us to get the estimate in Proposition 1.3. In the second subsection, we combine this estimate with Conlon and Hein's estimates in (5.9) to get Corollary 1.4. We then calculate several examples to illustrate our results. In particular, we can indeed recover numerical quantities in examples of [10].

5.1. Proof of Proposition 1.3

First we review the Kähler cone metric on $C(D,L)$ given by the special Calabi ansatz $\omega_0 = \sqrt{-1}\partial\bar{\partial}h^\delta$. Then ω_0 is a Riemannian cone metric on $C(D,L)$:

$$g = dr^2 + r^2 g_Y,$$

where Y is the associated circle bundle over D. To see this, we consider the coordinate chart on $\mathbb{P}(L^{-1}\oplus\mathbb{C})$. Away from the infinity section D_∞, we have the coordinate chart given by $(z,[\zeta_\alpha e_\alpha,1]) = (z,[e_\alpha,\zeta_\alpha^{-1}]) = (z,[e_\alpha,\xi_\alpha])$. Let $h = |e_\alpha|_h^2|\zeta_\alpha|^2 = a_{\alpha-}(z)|\zeta_\alpha|^2 = (a_{\alpha+}(z)|\xi_\alpha|^2)^{-1}$. For simplicity, we will denote $\zeta=\zeta_\alpha$, $\xi=\xi_\alpha$, $a = a_{\alpha-} = a_{\alpha+}^{-1}$. Then we can calculate

$$\begin{aligned}\omega_0 = \sqrt{-1}\partial\bar{\partial}h^\delta &= \delta h^\delta\omega_D + \delta^2 h^\delta\frac{\nabla\zeta\wedge\overline{\nabla\zeta}}{|\zeta|^2}\\ &= \delta h^\delta\omega_D + \delta^2 h^\delta\frac{\nabla\xi\wedge\overline{\nabla\xi}}{|\xi|^2},\end{aligned}\tag{5.1}$$

where $\omega_D = \sqrt{-1}\partial\bar{\partial}\log h$ is a smooth Kähler metric on D, and we have used vertical and horizontal frames:

$$dz^i, \nabla\zeta = d\zeta + \zeta a^{-1}\partial a \quad \overset{\text{dual}}{\Longleftrightarrow} \quad \nabla_{z^i} = \frac{\partial}{\partial z^i} - a^{-1}\frac{\partial a}{\partial z^i}\zeta\frac{\partial}{\partial\zeta}, \frac{\partial}{\partial\zeta}.$$

Under the $\{z,\xi\}$ coordinate, we have similarly

$$\begin{aligned}&dz^i, \nabla\xi = d\xi - \xi a^{-1}\partial a = -\zeta^{-2}\nabla\zeta\\ &\qquad\overset{\text{dual}}{\Longleftrightarrow} \quad \nabla_{z^i} = \frac{\partial}{\partial z^i} + a^{-1}\frac{\partial a}{\partial z^i}\xi\frac{\partial}{\partial\xi}, \qquad \frac{\partial}{\partial\xi} = -\zeta^2\frac{\partial}{\partial\zeta}.\end{aligned}$$

To write the metric into a metric cone, we write $\zeta = \tilde{\rho}e^{i\theta}$. Then

$$\nabla\zeta = d\zeta + \zeta a^{-1}\partial a = e^{i\theta}(d\tilde{\rho} + i\tilde{\rho}\,d\theta + \tilde{\rho}a^{-1}\partial a) = e^{i\theta}\big(d\tilde{\rho} + i\tilde{\rho}(d\theta - ia^{-1}\partial a)\big).$$

So if we let $r = h^{\delta/2} = (a(z)|\zeta|^2)^{\delta/2}$ and $\nabla\theta = d\theta - Ja^{-1}\,da$, then it is easy to verify that the corresponding metric tensor is given by

$$g_{\omega_0} = dr^2 + r^2(\delta g_{\omega_D} + \delta^2\nabla\theta\otimes\nabla\theta).$$

Note that $\nabla\theta$ is nothing but the connection form on the unit S^1-bundle in L^{-1}. Now we compare the norm of tensors on $U = L\backslash D$ with respect to two metrics ω_0 and $\tilde{\omega}_0$, where $\tilde{\omega}_0$ is any smooth Kähler metric on a neighborhood of D in L. For example, we can take

$$\tilde{\omega}_0 = \pi_L^* \omega_D + \epsilon \sqrt{-1} \partial \bar{\partial} \big(a_+(z) |\xi|^2 \big)$$

for small $\epsilon > 0$. Suppose that Φ is a tensor of type $(p = p_h + p_v, q = q_h + q_v)$, that is,

$$\Phi \in (T_h^* X)^{\otimes p_h} \otimes (T_v^* X)^{\otimes p_v} \otimes (T_h X)^{\otimes q_h} \otimes (T_v X)^{\otimes q_v}.$$

Then, by noticing that $h^{\delta/2} \sim |\xi|^{-\delta}$, we have

$$\frac{|\Phi|_{\omega_0}}{|\Phi|_{\tilde{\omega}_0}} \sim |\xi|^{\delta p_h + (\delta+1) p_v - \delta q_h - (\delta+1) q_v}. \tag{5.2}$$

In particular, we get the following lemma.

LEMMA 5.1
If Φ is tensor of type $(1,1)$, then

$$|\Phi_v^h|_{\omega_0} \sim |\Phi_v^h|_{\tilde{\omega}_0} |\xi|, \qquad |\Phi_h^v|_{\omega_0} \sim |\Phi_h^v|_{\tilde{\omega}_0} |\xi|^{-1},$$
$$|\Phi_v^v|_{\omega_0} \sim |\Phi_v^v|_{\tilde{\omega}_0}, \qquad |\Phi_h^h|_{\omega_0} = |\Phi_h^h|_{\tilde{\omega}_0}.$$

As a consequence, under the assumption that the embedding $D \hookrightarrow X$ is $(k-1)$-comfortable, we combine Lemma 5.1 with estimates (3.10) to get

$$|\Phi|_{\omega_0} \leq C_0 |\xi|^k \sim C_0 r^{-\frac{k}{\delta}}. \tag{5.3}$$

Next we compare the Christoffel symbols of the two metrics, which will be useful for converting the estimate of covariant derivatives with respect to ω_0 to that with respect to $\tilde{\omega}_0$. See (6.2) and (6.3). To simplify the calculation, we can choose the coordinate $\{z_\alpha^i\}$ on D and holomorphic frame such that

$$g_{i\bar{j}}^D(0) = \omega_D(\partial_{z_\alpha^i}, \partial_{z_\alpha^j})(0) = \delta_{ij}, \qquad (\partial_{z_\alpha^k} g_{i\bar{j}}^D)(0) = 0;$$
$$(\partial_{z_\alpha^i} a)(0) = 0, \qquad (\partial_{z_\alpha^i} \partial_{z_\alpha^j} a)(0) = 0.$$

Denote by index 0 the coordinate corresponding to $\xi = \xi_\alpha$. Then the components of the metric tensor associated with ω_0 are given by

$$g_{i\bar{j}} = \delta a^\delta |\xi|^{-2\delta} \delta_{ij}, \qquad g_{0\bar{0}} = \delta^2 a^\delta |\xi|^{-2(\delta+1)}, \qquad g_{0\bar{j}} = g_{j\bar{0}} = 0.$$

So it is easy to calculate that

$$|dz_\alpha^i|_{\omega_0} = \delta^{-1/2} a^{-\delta/2} |\xi|^\delta \sim \frac{1}{|\xi|^{-\delta}}, \qquad |d\xi|_{\omega_0} = \delta^{-1} a^{-\delta/2} |\xi|^{(\delta+1)} \sim \frac{|\xi|}{|\xi|^{-\delta}},$$
$$\Gamma_{ij}^k = \Gamma_{ij}^0 = \Gamma_{i0}^0 = \Gamma_{00}^i = 0, \qquad \Gamma_{i0}^j = -\frac{\delta}{\xi} \delta_{ij}, \qquad \Gamma_{00}^0 = -\frac{\delta+1}{\xi}.$$

In other words,

$$\nabla\partial_{z_\alpha^i} = -\frac{\delta}{\xi}\,d\xi\otimes\partial_{z_\alpha^i}, \qquad \nabla\partial_\xi = -\frac{\delta+1}{\xi}\,dz_\alpha^i\otimes\partial_{z_\alpha^i} - \frac{\delta+1}{\xi}\,d\xi\otimes\partial_\xi,$$
$$\nabla\,dz_\alpha^i = -\frac{\delta}{\xi}(d\xi\otimes dz_\alpha^i + dz_\alpha^i\otimes d\xi), \qquad \nabla d\xi = -\frac{\delta+1}{\xi}\,d\xi\otimes d\xi.$$

So we see that

$$\begin{aligned} |\nabla_{\omega_0}\partial_{z_\alpha^i}|_{\omega_0} &\le C \sim \frac{|\partial_{z_\alpha^i}|_{\omega_0}}{|\xi|^{-\delta}} \sim \frac{|\partial_{z_\alpha^i}|_{\omega_0}}{r}, \\ |\nabla_{\omega_0}\partial_\xi|_{\omega_0} &\le C|\xi|^{-1} \sim \frac{|\partial_\xi|_{\omega_0}}{|\xi|^{-\delta}} \sim \frac{|\partial_\xi|_{\omega_0}}{r}, \end{aligned} \tag{5.4}$$

$$\begin{aligned} |\nabla_{\omega_0}\,dz_\alpha^i|_{\omega_0} &\le C|\xi|^{2\delta} \sim \frac{|dz_\alpha^i|_{\omega_0}}{|\xi|^{-\delta}} \sim \frac{|dz_\alpha^i|_{\omega_0}}{r}, \\ |\nabla_{\omega_0}\,d\xi|_{\omega_0} &\le C|\xi|^{1+2\delta} \sim \frac{|d\xi|_{\omega_0}}{|\xi|^{-\delta}} \sim \frac{|d\xi|_{\omega_0}}{r}. \end{aligned} \tag{5.5}$$

The above estimates imply that each time we take a covariant derivative with respect to ω_0, we get an extra decay factor $|\xi|^\delta \sim r^{-1}$. So by induction which starts from (5.3), we get the wanted estimate in Proposition 1.3:

$$|\nabla^j_{\omega_0}\Phi|_{\omega_0} \le C_j|\xi|^{k+j\delta} \sim C_j r^{-\frac{k}{\delta}-j} \quad \text{for any } j\ge 0. \tag{5.6}$$

5.2. Asymptotic rates of Tian and Yau's Ricci-flat metrics

In this subsection, we explain how to get Corollary 1.4. First we recall the Calabi–Yau cone metric on $C := C(D,L)$ in the case in which $K_D^{-1} = \mu L$ for $\mu > 0$ and D has a Kähler–Einstein metric $\omega_D = \omega_D^{\mathrm{KE}}$ such that $\mathrm{Ric}(\omega_D^{\mathrm{KE}}) = \mu\cdot\omega_D^{\mathrm{KE}}$. In this case, note that the Hermitian metric h satisfies $\sqrt{-1}\partial\bar\partial\log h = \omega_D^{\mathrm{KE}}$. To find the Calabi–Yau cone metric, it is straightforward to calculate that

$$\mathrm{Ric}(\omega_0) = -\sqrt{-1}\partial\bar\partial\log\omega_0^n = (-n\delta+\mu)\pi_L^*\omega_D^{\mathrm{KE}},$$

where $n = \dim D + 1$. So we get the exponent for the Calabi–Yau cone metric:

$$-K_D = \mu N_D \Longrightarrow \delta = \frac{\mu}{\dim D + 1}. \tag{5.7}$$

Now assume that X is a Fano manifold of dimension n and that D is a smooth divisor such that $-K_X \sim \alpha D$ with $\mathbb{Q}\ni\alpha > 1$. By the adjunction formula, we get that $-K_D = -K_X|_D - [D] = (\alpha-1)[D] = (1-\alpha^{-1})K_X^{-1}$ is still ample, and so D is also a Fano manifold. Assuming that D has a Kähler–Einstein metric, Tian and Yau [32] constructed an asymptotically conical (AC) Calabi–Yau Kähler metric ω_{TY} on $X\backslash D$ whose metric tangent cone at infinity is the conical Calabi–Yau metric on $C(D,N_D)$ discussed above with the exponent $\delta=\frac{\alpha-1}{n}$. See also [5]. More precisely, there is a diffeomorphism $\phi_K : C(D,N_D)\backslash B_R(\underline{o})\to (X\backslash D)\backslash K$ such that

$$\left\|\nabla^j_{\omega_0}\big(\phi_K^*(\omega_{\mathrm{TY}})-\omega_0\big)\right\|_{C^0}\le Cr^{-\lambda-j}\quad \text{for } j\ge 0. \tag{5.8}$$

Here K is a compact set in the noncompact manifold $M:=X\backslash D$ and $B_R(\underline{o})$ is the ball of radius R around the vertex $\underline{o}$ of the metric cone.

A natural problem is to determine the optimal order (i.e., the number λ in (5.8)) of such an AC Calabi–Yau metric. This issue was studied in detail by Cheeger and Tian in [8] and by Conlon and Hein in [10] and [11]. In [10] Conlon and Hein studied the estimates on solutions to the corresponding complex Monge–Ampère equation for Calabi–Yau metrics. If we denote by $\mathfrak{k}$ its Kähler class represented by ω_{TY}, then their estimate of the optimal rate is as follows (see [10] and [12, Remark 1.2]):

$$\lambda_{\max}\ge\begin{cases}\min(2n,\lambda_1) & \text{if } \mathfrak{k}\in H^2_c(M),\\ \min(2,\lambda_1) & \text{if } \mathfrak{k}\in H^2(M).\end{cases} \tag{5.9}$$

Here λ_1 is any number satisfying the following condition: there exists a diffeomorphism $F_K : C(D,N_D)\backslash B_R(\underline{o})\to M\backslash K$ such that

$$\left\|\nabla^j_{\omega_0}(F_K^*\Omega-\Omega_0)\right\|_{\omega_0}\le Cr^{-\lambda_1-j}\quad \text{for any } j\ge 0, \tag{5.10}$$

where Ω (resp., Ω_0) is the multivalued meromorphic volume form on X (resp., $\overline{C}(D,N_D)$) that is nonvanishing holomorphic on $M=X\backslash D$ (resp., $C(D,N_D)$) and has pole of order α along D. Conlon and Hein [10] also showed that the condition (5.10) is equivalent to the following condition:

$$\left\|\nabla^j_{\omega_0}(F_K^*J-J_0)\right\|_{\omega_0}\le Cr^{-\lambda_1-j}\quad \text{for any } j\ge 0, \tag{5.11}$$

where J (resp., J_0) is the complex structure on M (resp., $C(D,N_D)$). So we see that λ_1 essentially measures the difference between the complex structure of $M\backslash K$ and $C(D,N_D)\backslash B_R(\underline{o})$. Equivalently, we are indeed comparing the complex structure on the (punctured) neighborhood of D inside X and the complex structure of the (punctured) neighborhood of D inside N_D.

Now assuming that D is $(k-1)$-comfortably embedded, the diffeomorphism from Proposition 1.3 (constructed in Section 3.1) satisfies (5.11) with $\lambda_1=\frac{k}{\delta}$. By the above discussion, we indeed get Corollary 1.4 by using the estimates of Conlon and Hein.

Example 5.2

(1) $(X, D) \cong (\mathbb{P}^1 \times \mathbb{P}^1, \Delta(\mathbb{P}^1))$. In this case, ω_{TY} coincides with the Eguchi–Hanson metric. We have $\alpha = 2$, $n = 2$, $\delta = (\alpha - 1)/n = 1/2$. By Proposition 4.9, D is 1-comfortably embedded (and 1-linearizable) so that $k = 2$. So $\lambda = \frac{k}{\delta} = 4$.

(2) $(X, D) \cong (\mathbb{P}^2, \{Z_0^2 + Z_1^2 + Z_2^2 = 0\}) \cong (\mathbb{P}^1 \times \mathbb{P}^1, \Delta(\mathbb{P}^1))/\mathbb{Z}_2$. In this case, ω_{TY} is the Eguchi–Hanson metric$/\mathbb{Z}_2$. We have $\alpha = \frac{3}{2}$, $n = 2$, $\delta = (\alpha - 1)/n = 1/4$. By Proposition 4.15, D is 0-comfortably embedded (and 0-linearizable) so that $k = 1$. So $\lambda = \frac{k}{\delta} = 4$.

Example 5.3

We consider Pinkham's construction of sweeping out the cone (see [27, p. 46]). Assume that $D^{n-1} \subset \mathbb{P}^{N-1}$ is a smooth complete intersection

$$D = \bigcap_{i=1}^{m} \big\{F_i(Z_1, \ldots, Z_N) = 0\big\} \subset \mathbb{P}^{N-1},$$

where $m = N - n$ and $F_i(Z_1, \ldots, Z_N)$ is a (generic) homogeneous polynomial of degree $d_i > 0$. Denote the affine cone over D in $\mathbb{C}^N$ and projective cone over D inside $\mathbb{P}^N$ by

$$C(D, H) = \bigcap_{i=1}^{m} \big\{F_i(z_1, \ldots, z_N) = 0\big\} \subset \mathbb{C}^N,$$

$$\overline{C}(D, H) = \bigcap_{i=1}^{m} \big\{F_i(Z_1, \ldots, Z_N)\big\} \subset \mathbb{P}^N.$$

Notice that since we have assumed that D is a complete intersection, it is then known that D is projectively normal in $\mathbb{P}^{N-1}$ which implies that its projective cone inside $\mathbb{P}^N$ is normal and hence coincides with its normalization $\bar{C}(D, H)$.

Now assume that $G_i(Z_0, Z_1, \ldots, Z_N)$ is a generic homogeneous polynomial of degree e_i with $e_i < d_i$ for each $i = 1, \ldots, m$. In particular, $G_i(1, z_1, \ldots, z_N)$ is a polynomial of degree e_i. We construct a degeneration

$$\mathcal{X} = \bigcap_{i=1}^{m} \big\{F_i(Z_1, \ldots, Z_N) + (tZ_0)^{d_i - \deg G_i} G_i(tZ_0, Z_1, \ldots, Z_N) = 0\big\}$$
$$\subset \mathbb{P}^N \times \mathbb{C}. \tag{5.12}$$

By the "generic" assumption, $X = \mathcal{X}_1$ is smooth. This degenerates the variety $X = \mathcal{X}_1 \subset \mathbb{P}^N$ to $\overline{C}(D, H)$. In fact, $\mathcal{X}$ is a degeneration of $\mathcal{X}_1$ generated by the one-parameter subgroup of projective transformations

$$[Z_0, Z_1, \dots, Z_N] \to [t^{-1}Z_0, Z_1, \dots, Z_N].$$

Away from $\{Z_0 = 0\}$, we have the deformation of $C(D, H)$:

$$\mathcal{X}^\circ = \bigcap_{i=1}^{m} \bigl\{F_i(z_1, \dots, z_N) + t^{d_i - \deg G_i} G_i(t, z_1, \dots, z_N) = 0\bigr\} \subset \mathbb{C}^N \times \mathbb{C}. \quad (5.13)$$

From Digression 5.4, the degeneration $\mathcal{X}$ coincides with the family obtained by first blowing up $D \times \{0\}$ inside $X \times \mathbb{C}$ and then blowing down the strict transform of $X \times \{0\}$ as in the Introduction. Now using the representation of $\mathcal{X}$ in (5.12), we see that $\mathcal{X}$ can be obtained by applying the above construction to the case $X = \mathcal{X}_1$, $Y' = \bar{C}(X, H) \subset \bar{C}(\mathbb{P}^N, H) = \mathbb{P}^{N+1}$, and $D = \{Z_0 = 0\} \cap X$. The coincidence of $\bar{C}(D, H)$ with the central fiber from the contracted deformation to the normal cone can also be verified directly by using Lemma 4.1 and the projective normality of D.

By the adjunction formula, we know that $-K_{\mathcal{X}_1} = (N + 1 - \sum_{i=1}^m d_i)H$ and $-K_D = (N - \sum_{i=1}^m d_i)H$. Consider the hyperplane section $D = \mathcal{D}_1 = \mathcal{X}_1 \cap \{Z_0 = 0\} \subset \mathcal{X}_1$. Then if we assume that $\sum_{i=1}^m d_i \leq N - 1$, we are in Tian and Yau's setting above with $\alpha := N + 1 - \sum_{i=1}^m d_i \geq 2$.

By Section A.2, $\mathbf{T}^1_C$ can be calculated as a quotient ring. As in Example A.11, consider the class

$$[\mathcal{G}] := \Bigl[\sum_i G_i(1, z_1, \dots, z_n)\Bigr] \in \bigoplus_{i=1}^m \mathbf{T}^1_C\bigl(-(d_i - e_i)\bigr),$$

where $[\cdot]$ denotes the quotient morphism (see (A.11))

$$H^0(U, N_U) \to \mathbf{T}^1_C = \frac{H^0(U, N_U)}{H^0(U, \Theta_{\mathbb{C}^N}|_U)} = \bigoplus_{j=-\infty}^{+\infty} \frac{\bigoplus_{i=1}^m H^0(D, (d_i + j)H)}{\mathrm{Jac}(H^0(D, (j+1)H)^{\oplus N})}.$$

Notice the right-hand side is actually finite-dimensional (see [4], [31]). Now if we assume that $[\mathcal{G}]$ in $\mathbf{T}^1_C$ is nonzero, then the reduced Kodaira–Spencer class $\mathbf{KS}^{\mathrm{red}}_{\mathcal{X}^\circ}$ is the maximal weight piece of $[\mathcal{G}]$ and the weight of deformation $w(\mathcal{X}^\circ)$ of $\mathbf{KS}^{\mathrm{red}}_{\mathcal{X}^\circ/\mathbb{B}}$ is equal to the weight of $[\mathcal{G}]$.

Without loss of generality, we can assume that $e_1 > e_2 > \cdots > e_m$ so that $\min_{i=1}^m \{d_i - e_i\} = d_1 - e_1$. Then in general, $w := w(\mathcal{X}^\circ) \leq -(d_1 - e_1)$ which could be a strict inequality (see example item (3) of ordinary double point below). The equality holds if $[G_1] \neq 0 \in \mathbf{T}^1_C(-(d_1 - e_1))$. If we assume furthermore that $n \geq 3$, then by Theorem 1.5, we know that the divisor D is $(|w| - 1)$-comfortably embedded into X (but not $|w|$-comfortably embedded into X).

So by the above calculation, we see that the asymptotic rate of holomorphic form is given by

$$\lambda = \frac{|w|}{\delta} = \frac{n|w|}{\alpha - 1}.$$

If, furthermore, $e_i \leq d_i - 2$, then

$$\lambda = \frac{|w|}{\delta} = \frac{n|w|}{\alpha - 1} = \frac{n \cdot \min_{i=1}^{m}\{d_i - e_i\}}{N - \sum_{i=1}^{m} d_i}.$$

In this way, we can indeed give an algebraic interpretation of the corresponding calculations in [10].

(1) (See [10, Example 1]). This is the *smoothing of the cubic cone*:

$$C = \Big\{z \in \mathbb{C}^4; \sum_{i=1}^{4} z_i^3 = 0\Big\}$$
$$\rightsquigarrow M = \Big\{z \in \mathbb{C}^4; \sum_{i=1}^{4} z_i^3 = \sum_{i,j} a_{ij} z_i z_j + \sum_k a_k z_k + \epsilon\Big\},$$

where a_{ij}, a_i, ϵ are small (generic) constants. We have

$$\mathbf{T}_C^1 = \frac{\mathbb{C}[z_1, \ldots, z_4]}{\langle z_1^2, \ldots, z_4^2\rangle} = \bigoplus_{\nu=-3}^{1} \mathbf{T}_C^1(\nu).$$

With the earlier notation, $G(Z_0, \ldots, Z_4) = \sum_{i,j} a_{ij} Z_i Z_j + \sum_k a_k Z_k Z_0 + \epsilon Z_0^2$ with

$$[\mathscr{G}] = \Big[\sum_{ij} a_{ij} z_i z_j + \sum_k a_k z_k + \epsilon\Big] \in \mathbf{T}_C^1(-1) + \mathbf{T}_C^1(-2) + \mathbf{T}_C^1(-3).$$

Note that we assume that a_{ij}, a_k are generic if they are not zero. So we get

	$\mathbf{KS}^{\mathrm{red}}_{\mathcal{X}^\circ}$	$w(\mathcal{X}^\circ)$	λ
$a_{ij} = a_k = 0$	$[\sum_{i,j} a_{ij} z_i z_j]$	-3	$\frac{3\cdot 3}{4-3} = 9$
$a_{ij} = 0, a_k \neq 0$	$[\sum_k a_k z_k]$	-2	$\frac{3\cdot 2}{4-3} = 6$
$a_{ij} \neq 0$	$[\epsilon]$	-1	$\frac{3\cdot 1}{4-3} = 3$

(2) (See [10, Example 2]). This is the *smoothing of the complete intersection*:

$$C = \Big\{ z \in \mathbb{C}^5; f_1 = \sum_{i=1}^{5} z_i^2 = 0, f_2 = \sum_{i=1}^{5} \eta_i z_i^2 = 0 \Big\}$$
$$\rightsquigarrow M = \big\{ z \in \mathbb{C}^5; f_1(z) = f_2(z) = \epsilon \big\}.$$

Here η_i are distinct complex numbers. We have

$$\mathbf{T}^1_C = \frac{\mathbb{C}[z_1, \dots, z_5]^{\oplus 2}}{\operatorname{Im}\big(\begin{smallmatrix} z_1 & \dots & z_5 \\ \eta_1 z_1 & \dots & \eta_5 z_5 \end{smallmatrix}\big)} = \mathbf{T}^1_C(-2).$$

Because the images of $\mathcal{G} = (-\epsilon, -\epsilon)$ are not zero inside $\mathbf{T}^1_C$, we have $\lambda = \frac{3\cdot 2}{5-2-2} = 6$.

(3) (See [10, Example 3]). This is the *smoothing of the ordinary double point*:

$$C = \Big\{ z \in \mathbb{C}^{n+1}; \sum_{i=1}^{n+1} z_i^2 = 0 \Big\} \rightsquigarrow M = \Big\{ z \in \mathbb{C}^{n+1}; \sum_{i=1}^{n+1} z_i^2 = \sum_{i=1}^{n+1} a_i z_i + \epsilon \Big\},$$

$$\mathbf{T}^1_C = \frac{\mathbb{C}[z_1, \dots, z_{n+1}]}{\langle z_1, \dots, z_{n+1} \rangle} = \mathbf{T}^1_C(-2).$$

As a result, we get $G(Z_0, \dots, Z_{n+1}) = \sum_{i=1}^{n+1} a_i Z_i + \epsilon Z_0$. Therefore, we have that $[G(1, z_1, \dots, z_n)] = [\sum_{i=1}^{n+1} a_i z_i + \epsilon] = [\epsilon]$ is of weight -2. So we have $\lambda = \frac{n\cdot 2}{n+1-2} = \frac{2n}{n-1}$. Note that if $n = 2$, then $D \hookrightarrow X$ is isomorphic to $\Delta(\mathbb{P}^1) \hookrightarrow \mathbb{P}^1 \times \mathbb{P}^1$, where $\Delta : \mathbb{P}^1 \to \mathbb{P}^1 \times \mathbb{P}^1$ is the diagonal embedding which was studied in Section 4.3. The identification is easily constructed:

$$\begin{aligned} \big(\mathbb{P}^1 \times \mathbb{P}^1, \Delta(\mathbb{P}^1)\big) \quad &\longrightarrow \quad (X, D) \\ &\qquad = \big(\{Z_0^2 + Z_1^2 + Z_2^2 + Z_3^2 = 0\}, \{Z_0 = 0\} \cap X\big), \\ \big([X_0, X_1], [Y_0, Y_1]\big) \quad &\mapsto \quad \big[X_0 Y_1 - X_1 Y_0, \sqrt{-1}(X_0 Y_1 + X_1 Y_0), \\ &\qquad (X_0 Y_0 + X_1 Y_1), \sqrt{-1}(X_0 Y_0 - X_1 Y_1)\big]. \end{aligned}$$

Digression 5.4

Here we recall an equivalent description of the deformation to the normal cone by using MacPherson's graph construction. Let s_D denote the canonical holomorphic section of $L = L_D$ with $D = \{s_D = 0\}$. We can identify X with the graph of s_D as a subvariety of $Y = \mathbb{P}(L \oplus \mathbb{C})$: $\mathcal{X}_1 = \{(p, [s_D(p), 1]); p \in X\}$. We then use the natural $\mathbb{C}^*$-action on Y to get a family of subvarieties of Y: $\mathcal{X}_t = \{p, [t^{-1} s_D(p), 1]; p \in X\}$. For $t \neq 0$, $\mathcal{X}_t \cong X$. As $t \to 0$, $\mathcal{X}_t$ converges to a subscheme $\tilde{\mathcal{X}_0}$ of Y which is nothing but the union of X with E. Alternatively, there is a rational map

$$\Psi : X \times \mathbb{C} \dashrightarrow \mathbb{P}(L \oplus \mathbb{C}), \qquad (p, t) \mapsto \big(p, \big[t^{-1} s(p), 1\big]\big) = \big(p, \big[s(p), t\big]\big).$$

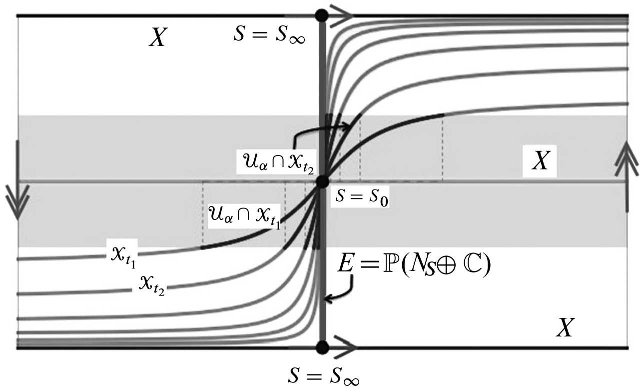

Figure 2. (Color online) Deformation to the normal cone: graph construction.

Notice that the indeterminacy locus of Ψ is exactly $D \times \{0\} = \{s = 0\} \times \{0\}$. So $\tilde{\mathcal{X}} = \mathrm{Bl}_{S\times\{0\}}(X \times \mathbb{C})$ is the graph Γ_Ψ of Ψ, that is, the closure of the graph of $\Psi :$ $(X \times \mathbb{C}) \setminus (D \times \{0\}) \to \mathbb{P}(L \oplus \mathbb{C})$.

Figure 2 is an illustration of deformation to the normal cone using the graph construction ($S = D$). Notice that the two pairs of opposite sides of the boundary in the figure are glued according to the direction of arrows and the total space $\tilde{\mathcal{X}}$ should be taken as the disjoint union of $\mathcal{X}_t$ in the figure. (See also [13, Remark 5.1.1, Section 5.1].) To get $\mathcal{X}$ from $\tilde{\mathcal{X}}$, we can use the similar construction, just by replacing $Y = \mathbb{P}(L \oplus \mathbb{C})$ by the projective cone $\bar{C}(X, L) =: Y'$ which is obtained from Y by contracting the infinity divisor X_∞.

Remark 5.5
As pointed out by the referee, for the above examples of complete intersections the result of Theorem 1.5 may not be surprising since we have explicit expressions:

$$\mathcal{X}_1 = \bigcap_{i=1}^{m} \big\{F_i(Z_1, \dots, Z_N) + Z_0^{d_i - e_i} G_i(Z_0, Z_1, \dots, Z_N) = 0\big\} \subset \mathbb{P}^N.$$

Noting that $|w| = \min_{i=1}^{m}\{d_i - e_i\}$, it is immediate that

$$\mathcal{O}_{\mathcal{X}_1}/\mathcal{I}_D^{|w|} \cong \mathcal{O}_{\mathcal{X}_0}/\mathcal{I}_D^{|w|}$$

using the fact that $\mathcal{I}_D(U_{\{Z_i \neq 0\}\cap \mathcal{X}_1}) = (\langle \frac{Z_0}{Z_i} \rangle + \mathcal{I}_{\mathcal{X}_1})/\mathcal{I}_{\mathcal{X}_1}$. In other words, $(\mathcal{X}_1, D)$ is $(|w|-1)$-linearizable. Then by Remark A.7, when $n \geq 3$, we know that D is $(|w|-1)$-comfortably embedded. So we get $m(X, D) \geq |w|$. Note however that the conclusion in Theorem 1.5 is stronger, saying that this is an equality for the more general case without using such explicit defining equations.

6. Analytic compactification

In this section, we will prove Theorem 1.6. We will first sketch a proof following the strategy of the classical work of Newlander and Nirenberg in [24] that is modified to adapt to the setting of weighted spaces. Then we will write down the detailed estimates by imitating the corresponding estimates in [24].

6.1. Reduction of Theorem 1.6 to Proposition 6.1

We refer to Section 5.1 for background. Denote $U = L\backslash D$. Denote the standard complex structure on U by J_0. Assume that we have a complex structure J on some neighborhood U_ϵ of D. Denote $\Phi = J - J_0$. We assume that the index $v \in \{1, \overline{1}\}$ associates to the fiber variable $\xi = z_\alpha^1$ and that $h \in \{2, \dots, n, \overline{2}, \dots, \overline{n}\}$ associates to the base variables $\{z_\alpha^2, \dots, z_\alpha^n\}$. By abuse of notation, we decompose Φ into four types of components:

$$\begin{aligned}\Phi &= \Phi_v^h + \Phi_h^v + \Phi_v^v + \Phi_h^h\\ &= \phi_v^h\, dz^v \otimes \partial_{z^h} + \phi_h^v\, dz^h \otimes \partial_{z^v} + \phi_v^v\, dz^v \otimes \partial_{z^v} + \phi_h^h\, dz^h \otimes \partial_{z^h}.\end{aligned} \tag{6.1}$$

We assume that Φ satisfies $|\nabla^j \Phi|_{\omega_0} \le C|r|^{-\lambda-j} \sim |\xi|^{\delta(\lambda+j)}$. We first need to transform this estimate to the corresponding estimate with respect to $\tilde{\omega}_0$. For this, since we know that the basic tensors satisfy (5.4) and (5.5), we can equivalently assume that Φ satisfies

$$\big|(\partial_{z^v}^{j_1} \partial_{z^h}^{j_2} \Phi) \otimes (dz^v)^{\otimes j_1} \otimes (dz^h)^{\otimes j_2}\big|_{\omega_0} \le C|r|^{-\lambda-j} = C|\xi|^{\delta(\lambda+j)}. \tag{6.2}$$

Recall the norm in Section 5.1:

$$\begin{aligned}&|dz^v|_{\omega_0} \le C|\xi|^{\delta+1}, \qquad |dz^h|_{\omega_0} \le C|\xi|^{\delta}\\ &\qquad\qquad \Longrightarrow |(dz^v)^{\otimes j_1} \otimes dz^h)^{\otimes j_2}|_{\omega_0} \le |\xi|^{j_1(\delta+1)+j_2\delta} = |\xi|^{\delta j + j_1}.\end{aligned}$$

Also, we have

$$\begin{aligned}&|dz^v \otimes \partial_{z^h}|_{\omega_0} \le C|\xi|, \qquad |dz^h \otimes \partial_{z^v}|_{\omega_0} \le C|\xi|^{-1},\\ &|dz^v \otimes \partial_{z^v}|_{\omega_0} \le C, \qquad |dz^h \otimes \partial_{z^h}|_{\omega_0} \le C.\end{aligned}$$

By these inequalities, it is easy to see that

$$\begin{aligned}&|\partial_{z^v}^{j_1} \partial_{z^h}^{j_2} \phi_v^h| \lesssim |\xi|^{\lambda\delta-1-j_1}, \qquad |\partial_{z^v}^{j_1} \partial_{z^h}^{j_2} \phi_h^v| \lesssim |\xi|^{\lambda\delta+1-j_1},\\ &|\partial_{z^v}^{j_1} \partial_{z^h}^{j_2} \phi_v^v| \lesssim |\xi|^{\lambda\delta-j_1}, \qquad |\partial_{z^v}^{j_1} \partial_{z^h}^{j_2} \phi_h^h| \lesssim |\xi|^{\lambda\delta-j_1}.\end{aligned} \tag{6.3}$$

PROPOSITION 6.1

Fix $\eta \in \mathbb{R}_{>0} \setminus \mathbb{N}$. Let J_0 denote the standard complex structure on $\mathbb{B}^* \times \mathbb{B}^{n-1}$. Assume that J is an integrable almost complex structure on $\mathbb{B}^* \times \mathbb{B}^{n-1}$ and that the tensor $\Phi = J - J_0$ is decomposed into four types of components:

$$\begin{aligned}\Phi &= J - J_0 \\ &= \Phi_v^h + \Phi_h^v + \Phi_v^v + \Phi_h^h \\ &= \phi_v^h\, dz^v \otimes \partial_{z^h} + \phi_h^v\, dz^h \otimes \partial_{z^v} + \phi_v^v\, dz^v \otimes \partial_{z^v} + \phi_h^h\, dz^h \otimes \partial_{z^h},\end{aligned} \tag{6.4}$$

where the index $v \in \{1, \overline{1}\}$ is associated to the first variable z^1, and $h \in \{2, \ldots, n, \overline{2}, \ldots, \overline{n}\}$ is associated to the variables $\{z^2, \ldots, z^n\}$. Assume that there exists a constant C such that for any $j_1 + j_2 \le 2n+1$ and all $(z_1, z_2, \ldots, z_n) \in \mathbb{B}^* \times \mathbb{B}^{n-1}$ it holds that

$$\begin{aligned}&|\partial_{z^v}^{j_1} \partial_{z^h}^{j_2} \phi_v^h| \le C|\xi|^{\eta-1-j_1}, \qquad |\partial_{z^v}^{j_1} \partial_{z^h}^{j_2} \phi_h^v| \le C|\xi|^{\eta+1-j_1}, \\ &|\partial_{z^v}^{j_1} \partial_{z^h}^{j_2} \phi_v^v| \le C|\xi|^{\eta-j_1}, \qquad |\partial_{z^v}^{j_1} \partial_{z^h}^{j_2} \phi_h^h| \le C|\xi|^{\eta-j_1}.\end{aligned} \tag{6.5}$$

Denote $m = \lceil \eta \rceil$. Then for sufficiently small $R > 0$, there exist J-holomorphic coordinates $\zeta = (\zeta_1, \zeta_2, \ldots, \zeta_n) : \mathbb{B}_R^* \times \mathbb{B}_R^{n-1} \to \mathbb{B}_{2R}^* \times \mathbb{B}_{2R}^{n-1}$ and a constant C' such that for any $j_1 + j_2 \le 2n+1$ it holds that

$$\begin{aligned}&\left|\partial_{z^v}^{j_1} \partial_{z^h}^{j_2} \left(\zeta^1 - z^1(\zeta)\right)\right| \le C'|\zeta^1|^{m+1-j_1}; \\ &\left|\partial_{z^v}^{j_1} \partial_{z^h}^{j_2} \left(\zeta^k - z^k(\zeta)\right)\right| \le C'|\zeta^1|^{m-j_1}, \quad 2 \le k \le n.\end{aligned}$$

Remark 6.2

The result obtained here is a counterpart of [18, Theorem 3.1] in our different asymptotically conical setting. In the proof of Theorem 3.1 there, the authors used gauge fixing and a result of Nijenhuis and Woolf [25, Theorem II]. (See [12] for a different proof following a similar argument as in [18].) We aim to give a more direct proof by following the fundamental work of Newlander and Nirenberg. One should also be able to adapt the work of Nijenhuis and Woolf [25] and Malgrange [23] to the current setting to prove the compactification (extension) of the complex structures considered here.

Remark 6.3

If we assume that $\eta > 1$, then the existence of such coordinates follows from the work of [19]. However, even in this case, Proposition 6.1 provides more information (weighted estimates), which is needed to read out the embedding order of the divisor at infinity.

In the remainder of Section 6.1, we will sketch the proof of Proposition 6.1 and show how Theorem 1.6 follows from it. Section 6.2 contains the technical details of the proof of Proposition 6.1.

The $(0,1)$-vector under the new complex structure J is given by

$$\frac{1}{2}(1+\sqrt{-1}J)\frac{\partial}{\partial \bar{z}^i}=\frac{\partial}{\partial \bar{z}^i}+\frac{\sqrt{-1}}{2}\phi_{\bar{i}}^{\bar{j}}\frac{\partial}{\partial \bar{z}^j}+\frac{\sqrt{-1}}{2}\phi_{\bar{i}}^{k}\frac{\partial}{\partial z^k}.$$

Denote $\eta=\lambda\delta$ and $\rho=|\xi|=|z^1|$. Then from (6.3), we can write

$$(\phi_{\bar{i}}^{\bar{j}})=\begin{pmatrix} O(\rho^{\eta})_{1\times 1} & O(\rho^{\eta+1})_{1\times(n-1)}\\ O(\rho^{\eta-1})_{(n-1)\times 1} & O(\rho^{\eta})_{(n-1)\times(n-1)}\end{pmatrix}. \tag{6.6}$$

We have the same order estimates for $(\phi_{\bar{i}}^{k})$. When ρ is sufficiently small, the matrix $(\delta_{\bar{i}}^{\bar{j}}+\frac{\sqrt{-1}}{2}\phi_{\bar{i}}^{\bar{j}})$ is invertible. It is easy to get the order estimates

$$\begin{aligned}(a_{\bar{i}}^{k}) &:= -\Big(\delta_{\bar{i}}^{\bar{j}}+\frac{\sqrt{-1}}{2}\phi_{\bar{i}}^{\bar{j}}\Big)^{-1}\Big(\frac{\sqrt{-1}}{2}\phi_{\bar{j}}^{k}\Big)\\ &= \begin{pmatrix} O(\rho^{\eta})_{1\times 1} & O(\rho^{\eta+1})_{1\times(n-1)}\\ O(\rho^{\eta-1})_{(n-1)\times 1} & O(\rho^{\eta})_{(n-1)\times(n-1)}\end{pmatrix}.\end{aligned} \tag{6.7}$$

To get an analytic compactification of the complex structure J, we want to solve for a map $z:\mathbb{B}_R^n\to\mathbb{B}_{2R}^n\subset\mathbb{C}^n$, where $\mathbb{B}_R^n=\{(\zeta^1,\dots,\zeta^n)\in\mathbb{C}^n;|\zeta^j|\le R\}$, such that z is a homeomorphism onto the image and is holomorphic with respect to J_0 and J. For the map z to be holomorphic, $dz(\partial/\partial\bar{\zeta}^l)$ should be a $(0,1)$-vector for any $l\ge 1$. It is easily seen that $z^i=z^i(\zeta)$ must solve the following equations:

$$\frac{\partial z^i}{\partial \bar{\zeta}^l}+\sum_{p=1}^{n}a_{\bar{p}}^{i}(z)\frac{\partial \bar{z}^p}{\partial\bar{\zeta}^l}=0,\quad i,l=1,\dots,n. \tag{6.8}$$

We first recall the important homotopy operator in [24]. For a vector of n complex-valued functions $F=(f_{\bar{1}},\dots,f_{\bar{n}})$, denote (see [24, (2.5)])

$$\mathbb{T}F=\sum_{s=0}^{n-1}\frac{(-1)^s}{(s+1)!}\sum{}'T^{j_1}\bar{\partial}_{j_1}\cdots T^{j_s}\bar{\partial}_{j_s}\cdot T^k f_{\bar{k}},$$

where $\sum'$ denotes the summation over all $(s+1)$-tuples with $j_1,\dots,j_s,k$ distinct, and

$$T^1 f(\zeta)=\frac{1}{2\pi i}\iint_{0<|\tau|<R}\frac{f(\tau,\zeta^2,\dots,\zeta^n)}{\zeta^1-\tau}\,d\tau\,d\bar{\tau},$$

$$T^j f(\zeta)=\frac{1}{2\pi i}\iint_{|\tau|<R}\frac{f(\zeta^1,\dots,\zeta^{j-1},\tau,\zeta^j,\dots,\zeta^n)}{\zeta^j-\tau}\,d\tau\,d\bar{\tau},\quad \text{for } j\ge 2.$$

To adapt this to our setting, we need to modify T^1. First choose $N = \lceil \eta \rceil$. Then we define (see (6.25) and Lemma 6.5)

$$\begin{aligned}\tilde{T}^1 f(\zeta) &= T^1 f(\zeta^1, \zeta^2, \dots, \zeta^n) - T^1 f(0, \zeta^2, \dots, \zeta^n)\\ &\quad - \sum_{k=1}^{N-1} (T^1 f)^{(k)}(0, \zeta^2, \dots, \zeta^n) \frac{\zeta^k}{k!},\\ \tilde{T}^j f(\zeta) &= T^j f(\zeta), \quad \text{if } j \geq 2.\end{aligned}$$

Then by Lemmas 6.5 and 6.6, these operators are well defined for functions f such that $f \sim O(|\zeta^1|^{\eta-1})$ and satisfy (see [9, p. 775]) the following identities on $\mathbb{B}_R^* \times \mathbb{B}_R^{n-1}$:

$$\overline{\partial}_j \tilde{T}^j f = f, \quad j = 1, \dots, n, \qquad \text{and} \qquad \overline{\partial}_j \tilde{T}^k f = \tilde{T}^k \overline{\partial}_j f, \quad \text{for } j \neq k. \tag{6.9}$$

Then we define

$$\widetilde{\mathbb{T}} F(\zeta) = \sum_{s=0}^{n-1} \frac{(-1)^s}{(s+1)!} \sum{}' \tilde{T}^{j_1} \overline{\partial}_{j_1} \cdots \tilde{T}^{j_s} \overline{\partial}_{j_s} \cdot \tilde{T}^k f_{\overline{k}}.$$

Then using relation (6.9) to manipulate, we can easily get the following formula which is a variation of the one in [24, (2.6)] by replacing the operator T^j by $\tilde{T}^j$:

$$\overline{\partial}_j \tilde{\mathbb{T}} F - f_{\overline{j}} = \sum_{s=0}^{n-2} \frac{(-1)^s}{(s+2)!} \sum{}^j \tilde{T}^{j_1} \overline{\partial}_{j_1} \cdots \tilde{T}^{j_s} \overline{\partial}_{j_s} \cdot \tilde{T}^k (\overline{\partial}_j f_{\overline{k}} - \overline{\partial}_k f_{\overline{j}}), \tag{6.10}$$

where $\sum^j$ denotes the summation over all $(s+1)$-tuples with $j_1, \dots, j_s, k$ distinct and different from j. From (6.8), we will denote

$$f_{\overline{l}}^i = -\sum_{p=1}^n a_{\overline{p}}^i(z) \frac{\partial \overline{z}^p}{\partial \overline{\zeta}^l}, \qquad F^i = (f_{\overline{1}}^i, f_{\overline{2}}^i, \dots, f_{\overline{n}}^i) = \sum_{l=1}^n f_{\overline{l}}^i \, d\overline{\zeta}^l. \tag{6.11}$$

Denote also $\mathfrak{z}^i(\zeta) = z^i(\zeta) - \zeta^i$. We then want to transform (6.8) into

$$z^i = \zeta^i + \widetilde{\mathbb{T}}\big(F^i(z)\big) \Longleftrightarrow \mathfrak{z}^i = \widetilde{\mathbb{T}}\big(F^i(\zeta + \mathfrak{z})\big) \Longleftrightarrow \mathfrak{z} = \mathfrak{J}[\mathfrak{z}]. \tag{6.12}$$

We will show in Lemma 6.13 that the solution to this equation with the appropriate control is indeed the solution to (6.8). To get solutions to the system (6.8) with the required order estimates, we prescribe the following asymptotic behaviors:

$$\begin{aligned}&z^1 = \zeta^1 + O(\rho^{1+\eta}), \qquad z^j = \zeta^j + O(\rho^\eta)\\ &\qquad\qquad \Longleftrightarrow \quad \mathfrak{z}^1 = O(\rho^{1+\eta}), \qquad \mathfrak{z}^k = O(\rho^\eta).\end{aligned} \tag{6.13}$$

Here and in the following, we still denote $\rho = |\zeta^1|$ since $|\zeta^1|$ and $|z^1|$ are comparable with this prescription. If we denote by h the index $\{2, \dots, n\}$, then the precise meaning of (6.13) is the following:

$$\begin{aligned} \left|\partial_{\zeta^1}^{l_1} \partial_{\zeta^h}^{l_2} (z^1 - \zeta^1)\right| &\le C(l_1, l_2) |\zeta^1|^{1+\eta-l_1}, \\ \left|\partial_{\zeta^1}^{l_1} \partial_{\zeta^h}^{l_2} (z^h - \zeta^h)\right| &\le C(l_1, l_2) |\zeta^1|^{\eta-l_1}, \quad \text{for all } l_1, l_2 \ge 0. \end{aligned} \tag{6.14}$$

However, to carry out the argument in [24], we first need to define the space of functions which have only "mixed" higher-order derivatives. So we will first consider the functions $\{z^i; i = 1, \dots, n\}$ satisfying

$$\begin{aligned} z^1 = \zeta^1 + \tilde{O}(\rho^{1+\eta}), \qquad z^j = \zeta^j + \tilde{O}(\rho^\eta) \\ \iff \quad \mathfrak{z}^1 = \tilde{O}(\rho^{1+\eta}), \qquad \mathfrak{z}^k = \tilde{O}(\rho^\eta), \end{aligned} \tag{6.15}$$

which implies that the following estimates hold:

$$\begin{aligned} \left|\partial_{\zeta^1}^{\prime l_1} \partial_{\zeta^h}^{\prime l_2} (z^1 - \zeta^1)\right| &\le C(l_1, l_2) |\zeta^1|^{1+\eta-l_1}, \\ \left|\partial_{\zeta^1}^{\prime l_1} \partial_{\zeta^h}^{\prime l_2} (z^h - \zeta^h)\right| &\le C(l_1, l_2) |\zeta^1|^{\eta-l_1}, \end{aligned} \tag{6.16}$$

where ∂' means that we do not allow repeated derivatives with respect to any single variable (see Section 6.2.2).

Under this prescription, by using (6.11) and the asymptotic behavior of $a^i_{\bar{p}}$, we first show (see Lemma 6.11) that

$$\begin{aligned} (f^1_{\bar{1}}, f^1_{\bar{m}}) &= \big(\tilde{O}(\rho^\eta + \rho^{2\eta}), \tilde{O}(\rho^{2\eta+1} + \rho^{\eta+1})\big) = (\tilde{O}(\rho^\eta), \tilde{O}(\rho^{\eta+1}), \\ (f^j_{\bar{1}}, f^j_{\bar{m}}) &= \big(\tilde{O}(\rho^{\eta-1} + \rho^{2\eta-1}), \tilde{O}(\rho^{2\eta} + \rho^\eta)\big) = \big(\tilde{O}(\rho^{\eta-1}), \tilde{O}(\rho^\eta)\big). \end{aligned} \tag{6.17}$$

Then we show (see Lemma 6.9) that

$$\widetilde{\mathbb{T}}[F^1] = \tilde{O}(\rho^{\eta+1}), \qquad \widetilde{\mathbb{T}}[F^k] = \tilde{O}(\rho^\eta) \quad \text{for } k \ge 2.$$

This is compatible with the prescription in (6.15) and should allow us to use the arguments in [24] to solve the system (6.12). However, to use the contraction-iteration principle (see Lemma 6.10), we have to relax the asymptotic behaviors in (6.15) a little bit by replacing η by a ν satisfying

$$0 < \nu < \eta, \qquad \lceil \nu \rceil = \lceil \eta \rceil \qquad \text{and} \qquad \nu \notin \mathbb{N}. \tag{6.18}$$

Although replacing η by ν might seem like a loss of derivative, we will gain this ϵ back using the analyticity of transition functions.

More precisely, in the next subsection, we will introduce the weighted multiple Hölder norm $\|\cdot\|_{n+n\alpha,(\nu+1,\nu)}$ and show in Theorem 6.12 that, for any $\mathfrak{z}$, $\tilde{\mathfrak{z}}$ satisfying that when R is sufficiently small and $\|\mathfrak{z}\|_{n+n\alpha,(\nu+1,\nu)} \le R$, $\|\tilde{\mathfrak{z}}\|_{n+n\alpha,(\nu+1,\nu)} \le R$, then the following estimates hold:

(1)

$$\left\|\mathfrak{I}[\mathfrak{z}]\right\|_{n+n\alpha,(\nu+1,\nu)} \le R, \tag{6.19}$$

(2)

$$\left\|\mathfrak{I}[\mathfrak{z}]-\mathfrak{I}[\tilde{\mathfrak{z}}]\right\|_{n+n\alpha,(\nu+1,\nu)} \le \frac{1}{2}\|\mathfrak{z}-\tilde{\mathfrak{z}}\|_{n+n\alpha,(\nu+1,\nu)}. \tag{6.20}$$

By standard iteration, there is a unique solution to the system (6.12) such that

$$\begin{aligned} &\mathfrak{z}^1 = \tilde{O}(\rho^{1+\nu}), \qquad \mathfrak{z}^j = \tilde{O}(\rho^{\nu}), \quad \text{or equivalently} \\ &z^1 = \zeta^1 + \tilde{O}(\rho^{1+\nu}), \qquad z^j = \zeta^j + \tilde{O}(\rho^{\nu}). \end{aligned} \tag{6.21}$$

In the following, $\mathbb{B}_R = \{\zeta \in \mathbb{C}; |\zeta| \le R\}$ denotes the closed disk of radius R with center 0, and $\mathbb{B}_R^* = \{\zeta \in \mathbb{C}; 0 < |\zeta| \le R\}$ denotes the punctured closed disk. We need to show that the map $\zeta \mapsto z$ gives a coordinate chart for $\zeta \in \mathbb{B}_R^n$ when R is sufficiently small. First note that $\{z^i(\zeta)\}$ is the identity for $\zeta^1 = 0$ and is Hölder continuous on $\{\zeta^1 = 0\}$. Second, on $\mathbb{U}_R = \mathbb{B}_R^* \times \mathbb{B}_R^{n-1}$, consider the Jacobian

$$\mathbb{J} = \left(\frac{\partial(z^i, \overline{z}^i)}{\partial(\zeta^j, \overline{\zeta}^j)}\right).$$

By an argument similar to the one used to obtain (6.7), it is easy to see that $\mathbb{J}$ is invertible if R is very small. So on $\mathbb{U}_R$, $\zeta \mapsto z$ is a local diffeomorphism to its image. We just need to show that it is an injective map and hence a homeomorphism.

To do this, we decompose the coordinate change in (6.13) into two steps. First we let

$$y^1 = z^1(\zeta) = \zeta^1 + \tilde{O}\big(|\zeta^1|^{1+\nu}\big), \qquad y^k = \zeta^k \quad \text{for } k \ge 2. \tag{6.22}$$

Since the Jacobian matrix is invertible and C^{ν}, the map is a $C^{1,\nu}$-diffeomorphism and is clearly a change of coordinates. We can express ζ in terms of y to get

$$\zeta^1 = y^1 + \tilde{O}\big(|y^1|^{1+\nu}\big), \qquad \zeta^k = y^k \quad \text{for } k \ge 2.$$

Now we can write the map in (6.13) as

$$z^1 = y^1, \qquad z^k = y^k + \tilde{O}\big(|y^1|^{\nu}\big) \quad \text{for } k \ge 2.$$

We just need to show that this is injective. We assume that $z(y) = z(\tilde{y})$. Then $y^1 = \tilde{y}^1$ and $z^j(y) = z^j(\tilde{y})$. On the slice $y^1 = \tilde{y}^1$, we connect y and $\tilde{y}$ by $y_t = (1-t)y + t\tilde{y}$. Then we have

$$\begin{aligned} 0 &= \big\| z(\tilde{y}) - z(y) \big\| \\ &= \sum_{j=1}^{n} \Big| \int_0^1 \sum_{k=1}^{n} (\partial_{y^k} z^j)(y_t) \cdot (\tilde{y}^k - y^k)\, dt \Big| \\ &= \sum_{j=2}^{n} \Big| \int_0^1 \sum_{k=2}^{n} \big(\delta^j_k + \tilde{O}(|y^1|^\nu)\big)(\tilde{y}^k - y^k)\, dt \Big| \geq C(1 - R^\nu)\|\tilde{y} - y\|. \end{aligned}$$

So if R is sufficiently small, then we indeed have $\tilde{y} = y$.

To get all the higher-order estimates for the functions as stated in Proposition 6.1, that is,

$$\begin{aligned} &\mathfrak{z}^1 = O(\rho^{1+\nu}), \qquad \mathfrak{z}^j = O(\rho^\nu), \quad \text{or equivalently} \\ &z^1 = \zeta^1 + O(\rho^{1+\nu}), \qquad z^j = \zeta^j + O(\rho^\nu), \end{aligned} \tag{6.23}$$

we need to apply similar arguments as in [24, Section 6] involving regularity theorems for elliptic equations (6.8). Since this part of argument is now standard, we will be brief and refer to [24, Section 6] for references on differentiability theorems. By (6.21), we know that $z^1, \dots, z^n$ are $C^{1+\alpha}_{1+\nu,\nu}$ functions of ζ^j, $\bar{\zeta}^j$ under the weighted Hölder norm. Because (6.8) is first-order elliptic, we infer that the z^k's are $C^{2+\alpha}_{1+\nu,\nu}$ with respect to the variables ζ^j, $\bar{\zeta}^j$. Combining this with the "mixed" second derivatives from (6.21), we see that the z^k's are of class $C^{2+\alpha}_{1+\nu,\nu}$ whose norm is defined using all derivatives (including repeated derivatives) with respect to $\zeta^1, \dots, \zeta^n$. The higher-order regularity follows from differentiating equations (6.8) and improving the derivatives by a standard bootstrapping argument. See [12] for a different proof of the higher-order estimates using the gauge-fixing method and regularity theorems.

This completes the sketch of the proof of Proposition 6.1. We will now explain the comfortable order of the divisor in the last statement in Theorem 1.6. Note that the transition functions on the bundle $N_D \to D$ in terms of $\{z^i_\alpha\}$ are standard ones:

$$z^1_\beta = a_{\beta\alpha}(z'')z^1_\alpha, \qquad z^k_\beta = \phi^k_{\beta\alpha}(z''_\alpha) \quad \text{for } k \geq 2.$$

By the asymptotic behavior (6.23) and its inverse, we see that the transition functions in the ζ-coordinates have the shape

$$\zeta^1_\beta = a_{\beta\alpha}(\zeta''_\alpha)\zeta^1_\alpha + O\big(|\zeta^1_\alpha|^{\nu+1}\big), \qquad \zeta^k_\beta = \phi^k_{\beta\alpha}(\zeta''_\alpha) + O\big(|\zeta^1_\alpha|^\nu\big).$$

We know that ζ^i_β, for any $1 \leq i \leq n$, is a holomorphic function of ζ_α outside D, and from the above expressions it is Hölder continuous across $D = \{\zeta^1_\alpha = 0\}$. So we see that ζ^i_β is holomorphic across D and hence is a holomorphic function of ζ_α. Denote $m = \lceil \nu \rceil = \lceil \eta \rceil = \lceil \lambda\delta \rceil$ (recall that $\eta = \lambda\delta$ and $\nu = \eta - \epsilon$ for small ϵ). Then the analyticity of holomorphic functions clearly implies that we must have the following improved transition:

$$\zeta^1_\beta = a_{\beta\alpha}(\zeta''_\alpha)\zeta^1_\alpha + R^1_{m+1}, \qquad \zeta^k_\beta = \phi^k_{\beta\alpha}(\zeta''_\alpha) + R^k_m,$$

where $R^1_{m+1} \in \mathcal{I}^{m+1}_D$, $R^k_m \in \mathcal{I}^m_D$, where $\mathcal{I}_D$ is the ideal sheaf of D generated by $\{\zeta^1_\alpha\}$. By Theorem A.8 (see also (3.6)), we see that in the compactification, the divisor D is indeed $(m-1)$-comfortably embedded. In this way, we prove Theorem 1.6.

6.2. *Estimates for the proof of Proposition 6.1*

Suppose that f is a complex-valued function defined on $\mathbb{B}^*_R \times \mathbb{B}^{n-1}_R$. Denote by D_j either of the differential operators $\frac{\partial}{\partial\zeta^j}, \frac{\partial}{\partial\bar{\zeta}^j}$. We denote by D^k a general kth-order derivative $D^k = D_{i_1}\cdots D_{i_k}$ with $i_1,\ldots,i_k$ distinct (i.e., we only consider "mixed" derivatives), and we denote by $D^{k,j} = D_{i_1}\cdots D_{i_k}$ (resp., $D^{k,\{1,j\}}$) such a derivative with the $i_1,\ldots,i_k$ distinct and different from j (resp., $\{1,j\}$). For a fixed positive $\alpha < 1$, we denote the difference quotient operators

$$\begin{aligned}\delta_1 f &= \frac{f(\tilde{\zeta}^1,\zeta^2,\ldots,\zeta^n) - f(\zeta^1,\zeta^2,\ldots,\zeta^n)}{|\tilde{\zeta}^1 - \zeta^1|^\alpha}\\ &\qquad \text{for } 0 < |\zeta^1| \le R, 0 < |\tilde{\zeta}^1| \le R, \zeta^1 \ne \tilde{\zeta}^1,\\ \delta_i f &= \frac{f(\zeta^1,\ldots,\tilde{\zeta}^i,\ldots,\zeta^n) - f(\zeta^1,\ldots,\zeta^i,\ldots,\zeta^n)}{|\tilde{\zeta}^i - \zeta^i|^\alpha}\\ &\qquad \text{for } i > 1, |\zeta^i| < R, |\tilde{\zeta}^i| < R, \zeta^i \ne \tilde{\zeta}^i.\end{aligned}$$

Denote $\delta^m = \delta_{j_1}\cdots\delta_{j_m}$ for $0 \le m \le n$ and $j_1,\ldots,j_m$ distinct; δ^0 will denote the identity operator; $\delta^{m,1}$ will denote such a difference quotient with $j_1,\ldots,j_m$ distinct and different from 1.

6.2.1. *Single-variable estimates*

The following is the standard Schauder estimate for the elliptic operator $\bar{\partial}$ for a single variable.

LEMMA 6.4
Assume that $\alpha \in (0,1)$ is fixed. There exists a constant $c > 0$ such that, if $w \in C^{1,\alpha}(\mathbb{B}_1(0))$ satisfies $\frac{\partial w}{\partial\bar{\zeta}} = f$ in $\mathbb{B}_1$ and if $f \in C^{0,\alpha}(\mathbb{B}_1(0))$, then

$$\|w\|_{C^{1,\alpha}(\mathbb{B}_{1/2})} \le c\big(\|w\|_{L^\infty(\mathbb{B}_1)} + \|f\|_{C^{0,\alpha}(\mathbb{B}_1)}\big). \tag{6.24}$$

Proof
In the following proof, the constant c may change but does not depend on $f \in C^{0,\alpha}(\mathbb{B}(0))$. Denote operators

$$Tf(\zeta) = \frac{1}{2\pi i} \iint_{\mathbb{B}_1} \frac{f(\tau)}{\tau - \zeta}\, d\tau \wedge d\bar{\tau}, \qquad Sw(\zeta) = \frac{1}{2\pi i} \int_C \frac{w(\tau)}{\tau - \zeta}\, d\tau.$$

Then $w \in C^{1,\alpha}(\mathbb{B}_1)$ satisfies

$$w = T\partial_{\bar{\zeta}} w + Sw = Tf + Sw.$$

By Chern [9, Main Lemma], we have

$$\|Tf\|_{C^{1,\alpha}(\mathbb{B}_1)} \leq c\|f\|_{C^{0,\alpha}(\mathbb{B}_1)}.$$

On the other hand, because $Sw = w - Tf$ is holomorphic, we have

$$\begin{aligned}
\|Sw\|_{C^{1,\alpha}(\mathbb{B}_{1/2})} &\leq c\|Sw\|_{L^\infty(\mathbb{B}_1)} \\
&\leq c\big(\|w\|_{L^\infty(\mathbb{B}_1)} + \|Tf\|_{L^\infty(\mathbb{B}_1)}\big) \\
&\leq c\|w\|_{L^\infty(\mathbb{B}_1)} + c\|f\|_{L^\infty(\mathbb{B}_1)}.
\end{aligned}$$

□

We need to extend the above Schauder estimate to the weighted Hölder space. We follow [26, Chapter 2] to define the weighted Hölder norm for functions on the punctured disks. For any $s > 0$, denote the annulus $\{\zeta^1 \in \mathbb{C}; s < |\zeta^1| < 2s\}$ by $A(s, 2s)$. First we define the norm on the annulus:

$$\begin{aligned}
[w]_{1,\alpha,s} := \sup_{A(s,2s)} |w| + s \sup_{A(s,2s)} |D_1 w| + s^\alpha \sup_{x,y\in A(s,2s)} \frac{|w(x) - w(y)|}{|x-y|^\alpha} \\
+ s^{1+\alpha} \sup_{x,y\in A(s,2s)} \frac{|D_1 w(x) - D_1 w(y)|}{|x-y|^\alpha}.
\end{aligned}$$

The following is the scaling-invariant weighted Hölder norm for functions on the punctured disk of radius R:

$$\|w\|_{C^{1,\alpha}_\nu(\mathbb{B}_R(0))} = \sup_{s\in(0,R/2]} s^{-\nu}[w]_{1,\alpha,s}.$$

As pointed out in [26, Corollary 2.1], the following lemma is important for deriving the rescaled Schauder estimate in Lemma 6.6.

Denote $m = \lceil \nu \rceil = \lceil \eta \rceil$, and denote the area form $\frac{d\tau\wedge d\bar{\tau}}{2\pi\sqrt{-1}}$ by dV, or $dV(\tau)$ if we want to emphasize the integration variable. For any $f \in C^{1,\alpha}_{\nu-1}(\mathbb{B}_R)$, define

$$\begin{aligned}\tilde{T}f(\zeta) &= Tf(\zeta) - Tf(0) - \sum_{k=1}^{m-1} (Tf)^{(k)}(0)\frac{\zeta^k}{k!} \\ &= \frac{1}{2\pi i}\Big(\iint_{\mathbb{B}_R} \frac{f(\tau)}{\tau-\zeta}\, d\tau \wedge d\bar{\tau} - \sum_{k=0}^{m-1}\iint_{\mathbb{B}_R} \frac{f(\tau)\zeta^k}{\tau^{k+1}}\, d\tau\wedge d\bar{\tau}\Big) \\ &= \frac{1}{2\pi i}\iint_{\mathbb{B}_R} \frac{f(\tau)\zeta^m}{(\tau-\zeta)\tau^m}\, d\tau\wedge d\bar{\tau}. \end{aligned} \tag{6.25}$$

LEMMA 6.5
Denote $\rho = |\zeta|$ for any $\zeta \in \mathbb{B}_R^*$. Then there exists a positive constant C independent of R such that for any $f \in C^{1,\alpha}_{\nu-1}(\mathbb{B}_R)$, we have

$$\|\rho^{-\nu}\tilde{T}f\|_{L^\infty(\mathbb{B}_R)} \le C\|\rho^{1-\nu}f\|_{L^\infty(\mathbb{B}_R)}. \tag{6.26}$$

Proof
We can first estimate

$$\begin{aligned}|\rho^{-\nu}\tilde{T}f| &= |\zeta|^{-\nu}\Big|\iint_{\mathbb{B}_R}\frac{f(\tau)\zeta^m}{(\tau-\zeta)\tau^m}\, dV\Big| \\ &\le \big\||\rho|^{1-\nu}f\big\|_{L^\infty}|\zeta|^{m-\nu}\iint_{\mathbb{B}_R(0)}\frac{dV}{|\tau-\zeta||\tau|^{m+1-\nu}}.\end{aligned}$$

We split the integral into three parts:

$$\iint_{\mathbb{B}_R(0)} = \iint_{\mathbb{B}_{\rho/2}(0)} + \iint_{\mathbb{B}_{\rho/2}(\zeta)} + \iint_{\mathbb{B}_R(0)\setminus(\mathbb{B}_{\rho/2}(0)\cup\mathbb{B}_{\rho/2}(\zeta))} = \mathbf{I} + \mathbf{II} + \mathbf{III}.$$

The inequality (6.26) follows from the following estimates:

$$\begin{aligned}\mathbf{I} &\le C\int_0^{\rho/2}\frac{ds}{s^{m-\nu}\rho/2} \le C\rho^{\nu-m}, \\ \mathbf{II} &\le C\int_0^{\rho/2}\frac{ds}{\rho^{m+1-\nu}} \le C\rho^{\nu-m}.\end{aligned}$$

To estimate part $\mathbf{III}$, it is easy to see that $|\tau-\zeta| \ge \frac{|\tau|}{4}$ for $\tau \in \mathbb{B}_R(0)\setminus\mathbb{B}_{\rho/2}(\zeta)$. So we can estimate for any $\nu < m$:

$$\mathbf{III} \le C\int_{\rho/2}^{R}\frac{ds}{s^{m+1-\nu}} \le \frac{C}{m-\nu}\Big(\Big(\frac{\rho}{2}\Big)^{\nu-m} - R^{\nu-m}\Big) \le C\rho^{\nu-m}.$$

It is clear that (6.26) follows by combining the above estimates. ☐

LEMMA 6.6
If $f \in C^{0,\alpha}_{\nu-1}(\mathbb{B}_R)$, then $\tilde{T} f \in C^{1,\alpha}_{\nu}(\mathbb{B}_R)$ and satisfies

$$\|\tilde{T} f\|_{C^{1,\alpha}_{\nu}(\mathbb{B}_R)} \le C \|f\|_{C^{0,\alpha}_{\nu-1}(\mathbb{B}_R)}.$$

Proof
Let $F(\zeta) = \tilde{T} f(\zeta)$. Let $\rho = |\zeta|$. By Lemmas 6.4 and 6.5 and a standard rescaling argument as in [26, Corollary 2.1], we have

$$\|\tilde{T} f\|_{C^{1,\alpha}_{\nu}(\mathbb{B}_{R/2})} \le C \|f\|_{C^{0,\alpha}_{\nu-1}(\mathbb{B}_R)}.$$

To get the estimate on $\mathbb{B}_R \backslash \mathbb{B}_{R/2}$, we use the explicit formula of $\tilde{T}$. As in [9, (18), (26)], we have

$$\begin{aligned} F_{\bar{\zeta}} &= f(\zeta), \\ F_{\zeta} &= \frac{1}{2\pi\sqrt{-1}} \iint_{\mathbb{B}_R(0)} \frac{f(\tau) - f(\zeta)}{(\tau - \zeta)^2}\, d\tau\, d\bar{\tau} \\ &\quad - \frac{1}{2\pi\sqrt{-1}} \sum_{k=1}^{m-1} k\zeta^{k-1} \iint_{\mathbb{B}_R(0)} \frac{f(\tau)}{\tau^{k+1}}\, d\tau \wedge d\bar{\tau}. \end{aligned}$$

So

$$\begin{aligned} \Big|\frac{|F_\zeta|}{|\zeta|^{\nu-1}}\Big| &\le \frac{1}{2\pi} \frac{1}{|\zeta|^{\nu-1}} \iint_{\mathbb{B}_R(0)} \frac{|f(\tau) - f(\zeta)|}{|\tau - \zeta|^2}\, dV(\tau) \\ &\quad + \sum_{k=1}^{m-1} kR^{k-\nu} \|\rho^{1-\nu} f\|_\infty \int_0^R \frac{ds}{s^{k-\nu+1}}. \end{aligned}$$

The second term on the right-hand side of the above identity is uniformly bounded by $C\|\rho^{1-\nu} f\|_\infty$. To estimate the first integral term, we split it into two parts:

$$\iint_{\mathbb{B}_R(0)} = \iint_{\mathbb{B}_{\rho/2}(0)} + \iint_{\mathbb{B}_R(0)\backslash\mathbb{B}_{\rho/2}(0)} = \mathbf{I} + \mathbf{II}.$$

Here we need to separate the integral over $\mathbb{B}_{\rho/2}(0)$ from each estimate since we only have Hölder estimates for x and y of comparable lengths. Notice that we can assume that $R/8 \le |\zeta| \le R$ and estimate

$$\begin{aligned} \mathbf{I} &\le \frac{1}{2\pi} \iint_{\mathbb{B}_{\rho/2}(0)} \frac{1}{(|\zeta| - |\tau|)^2} \Big(|\tau|^{1-\nu} \big|f(\tau)\big| \frac{|\zeta|^{1-\nu}}{|\tau|^{1-\nu}} + |\zeta|^{1-\nu} \big|f(\zeta)\big| \Big)\, dV(\tau) \\ &\le C\|\rho^{1-\nu} f\|_{L^\infty(\mathbb{B}_R)} \frac{1}{R^2} \int_0^{R/2} (R^{1-\nu} s^{\nu-1} + 1) s\, ds \end{aligned}$$

$$
\begin{aligned}
&\leq C\|\rho^{1-\nu} f\|_{L^\infty(\mathbb{B}_R(0))},\\
\mathbf{II} &\leq C \iint_{\mathbb{B}_R(0)\setminus\mathbb{B}_{\rho/2}(0)} \frac{\|f\|_{C^{0,\alpha}_{\nu-1}} |\tau-\zeta|^\alpha R^{-\alpha}}{|\tau-\zeta|^2}\, dV\\
&\leq C\|f\|_{C^{0,\alpha}_{\nu-1}} R^{-\alpha} \int_0^{2R} s^{\alpha-2+1}\, ds \leq C\|f\|_{C^{0,\alpha}_{\nu-1}}.
\end{aligned}
$$

So we get $\|\rho^{1-\nu} D_1 \tilde{T} f\|_{L^\infty} \leq C\|f\|_{C^{0,\alpha}_{\nu-1}}$, that is, the C^1-estimate. This implies the $C^{0,\alpha}$-estimate

$$
R^\alpha \sup_{x,y\in A(R/8,R)} \frac{|w(x)-w(y)|}{|x-y|^\alpha} \leq C\|f\|_{C^{0,\alpha}_{\nu-1}}.
$$

Similarly, one can prove that

$$
R^{1+\alpha} \sup_{x,y\in A(R/8,R)} \frac{|D_1 w(x) - D_1 w(y)|}{|x-y|^\alpha} \leq C\|f\|_{C^{0,\alpha}_{\nu-1}},
$$

with $w = \tilde{T}(f)$. In fact, we can prove the inequality as in [25, Section 6.1e], again the only difference is that we need to separate the integral over $\mathbb{B}_{\rho/2}(0)$ from each estimate since we only have Hölder estimates for x and y of comparable lengths. □

6.2.2. Multivariable estimates

Similarly to [24, (3.1)–(3.3)], we introduce the weighted multiple Hölder space by incorporating the weighted first-order Hölder space for ζ^1 and the usual first-order Hölder spaces for the other variables. Formally, we introduce various norms:

(1) (Integral part)

$$
\begin{aligned}
\|u\|_{n,\nu} = \sum_{k=0}^{n-1} \Big(&\frac{R^k}{k!} \sup_{\mathbb{B}_R(0)^*\times\mathbb{B}_R(0)^{n-1}} \Big(\frac{|D^{k,1}u|}{|\zeta^1|^\nu}\Big)\\
&+ \frac{R^{k+1}}{(k+1)!} \sup_{\mathbb{B}_R(0)^*\times\mathbb{B}_R(0)^{n-1}} \Big(\frac{|D_1 D^{k,1}u|}{|\zeta^1|^{\nu-1}}\Big)\Big).
\end{aligned}
$$

(2) (Fractional part, i.e., difference quotient part)

$$
\begin{aligned}
[u]_{n\alpha,\nu} = \sum_{m=1}^{n-1} \Big(&\frac{R^{m\alpha}}{m!} \sup_{\mathbb{B}_R(0)^*\times\mathbb{B}_R(0)^{n-1}} \Big(\frac{|\delta^{m,1}u|}{|\zeta^1|^\nu}\Big)\\
&+ \frac{R^{(m+1)\alpha}}{(m+1)!} \sup_{s\in(0,R/2)} s^{\alpha-\nu} \sup_{\{\zeta^1,\tilde{\zeta}^1\in A(s,2s)\}} |\delta_1 \delta^{m,1} u|\Big).
\end{aligned}
$$

(3) (0th-order weighted multiple Hölder norm)

$$\|u\|_{n\alpha,\nu} = \tilde{H}_{\alpha,\nu}[u] = \sup_{\mathbb{B}_R(0)^*\times\mathbb{B}_R(0)^{n-1}} \frac{|u|}{|\zeta^1|^\nu} + [u]_{n\alpha,\nu}.$$

(4) (First-order weighted multiple Hölder norm)

$$\begin{aligned}\|u\|_{n+n\alpha,\nu} &= \|u\|_{n,\nu} \\ &\quad + \sum_{k=0}^{n-1}\Big(\frac{R^k}{k!}[D^{k,1}u]_{n\alpha,\nu} + \frac{R^{k+1}}{(k+1)!}[D_1D^{k,1}u]_{n\alpha,\nu-1}\Big) \\ &= \sum_{k=0}^{n-1}\Big(\frac{R^k}{k!}\tilde{H}_{\alpha,\nu}[D^{k,1}u] + \frac{R^{k+1}}{(k+1)!}\tilde{H}_{\alpha,\nu-1}[D_1D^{k,1}u]\Big).\end{aligned}$$

(5) (Partial first-order weighted multiple Hölder norm)

$$\|u\|^1_{n-1+n\alpha,\nu} = \sum_{k=0}^{n-1}\frac{R^k}{k!}\sup_{\mathbb{B}_R(0)^*\times\mathbb{B}_R(0)^{n-1}}\tilde{H}_{\alpha,\nu}[D^{k,1}u],$$

$$\begin{aligned}\|u\|^j_{n-1+n\alpha,\nu} = \sum_{k=0}^{n-2}\Big(&\frac{R^k}{k!}\sup_{\mathbb{B}_R(0)^*\times\mathbb{B}_R(0)^{n-1}}\tilde{H}_{\alpha,\nu}[D^{k,\{1,j\}}u] \\ &+ \frac{R^{k+1}}{(k+1)!}\tilde{H}_{\alpha,\nu-1}[D_1D^{k,\{1,j\}}u]\Big) \quad \text{for } j\ge 2.\end{aligned}$$

(6) (Anisotropically weighted norm for vector of functions) Denote $\mathfrak{z} = (\mathfrak{z}^1(\zeta),\dots,\mathfrak{z}^n(\zeta))$, $F=(f_{\bar{1}},\dots,f_{\bar{n}})$. Denote

$$\|\mathfrak{z}\|_{n+n\alpha,(\nu+1,\nu)} = \|\mathfrak{z}^1\|_{n+n\alpha,\nu+1} + \sum_{j=2}^{n}\|\mathfrak{z}^j\|_{n+n\alpha,\nu},$$

$$\|F\|_{n-1+n\alpha,(\nu,\nu+1)} = \|f_{\bar{1}}\|^1_{n-1+n\alpha,\nu} + \sum_{j=2}^{n}\|f_{\bar{j}}\|^j_{n-1+n\alpha,\nu+1}.$$

Now we come back to solve the system (6.12) which is equivalent to

$$\mathfrak{z}^i = \widetilde{\mathbb{T}}\big(F^i(\zeta+\mathfrak{z})\big) = \mathfrak{I}^i[\mathfrak{z}], \quad \text{where } F^i = (f^i_{\bar{l}}) = \Big(-\sum_{p=1}^{n} a^i_{\bar{p}}\frac{\partial\bar{z}^p}{\partial\bar{\zeta}^l}\Big). \tag{6.27}$$

Arguing as in [24], the following lemma is a consequence of the definitions of the above norms and Lemma 6.6.

LEMMA 6.7 (cf. [24, (3.4), Lemma 4.1, Lemma 4.3])
We have the following estimates:

$$\begin{aligned} \|D_j f\|^j_{n-1+n\alpha,\nu} &\le \frac{c}{R}\|f\|_{n+n\alpha,\nu}, \quad j=1,\dots,n,\\ \|\tilde{T}^j D_j f\|^l_{n-1+n\alpha,\nu} &\le c\|f\|^l_{n-1+n\alpha,\nu}, \quad j,l=1,\dots,n, j\neq l,\\ \|\tilde{T}^1 f\|_{n+n\alpha,\nu+1} &\le cR\|f\|^1_{n-1+n\alpha,\nu},\\ \|\tilde{T}^j f\|_{n+n\alpha,\nu} &\le cR\|f\|^j_{n-1+n\alpha,\nu} \quad \textit{for } j\ge 2. \end{aligned} \tag{6.28}$$

Remark 6.8
Note that the idea for the above estimates are the following.

(1) Differentiation with respect to z^j for $j\neq 1$ keeps the weight unchanged and produces an R^{-1} factor under appropriate norms. $\tilde{T}^j$ for $j\neq 1$ keeps the weight unchanged and produces an R factor.
(2) Differentiation with respect to z^1 decreases the weight and produces an R^{-1} factor. $\tilde{T}^1$ improves the weight by 1 and produces an extra R factor.

Packing these estimates for components of F^1, F^j, the above lemma implies the following one.

LEMMA 6.9 (cf. [24, Theorem 4.1])
We have

$$\begin{aligned} \big\|\widetilde{\mathbb{T}}(F^1)\big\|_{n+n\alpha,\nu+1} &\le cR\|F^1\|_{n-1+n\alpha,(\nu,\nu+1)};\\ \big\|\widetilde{\mathbb{T}}(F^j)\big\|_{n+n\alpha,\nu} &\le cR\|F^j\|_{n-1+n\alpha,(\nu-1,\nu)}, \quad \textit{for } j\ge 2. \end{aligned}$$

The next lemma follows from the decay rate of $(a^i_{\bar{j}})$ in identity (6.7) and the definition of norms listed above. It shows the reason for relaxing the asymptotics by replacing η by ν.

LEMMA 6.10 (cf. [24, Lemma 3.1])
Suppose that $\|\mathfrak{z}\|_{n+n\alpha,(\nu+1,\nu)}\le 1$. *Then*

$$\begin{aligned} \big\|a^1_{\bar{1}}(\zeta+\mathfrak{z})\big\|_{n-1+n\alpha,\nu} &\le KR^{\eta-\nu}\big(1+R^\nu\|\mathfrak{z}\|_{n+n\alpha,(\nu+1,\nu)}\big),\\ \big\|a^1_{\bar{k}}(\zeta+\mathfrak{z})\big\|_{n-1+n\alpha,\nu+1} &\le KR^{\eta-\nu}\big(1+R^\nu\|\mathfrak{z}\|_{n+n\alpha,(\nu+1,\nu)}\big),\\ \big\|a^k_{\bar{1}}(\zeta+\mathfrak{z})\big\|_{n-1+n\alpha,\nu-1} &\le KR^{\eta-\nu}\big(1+R^\nu\|\mathfrak{z}\|_{n+n\alpha,(\nu+1,\nu)}\big),\\ \big\|a^j_{\bar{k}}(\zeta+\mathfrak{z})\big\|_{n-1+n\alpha,\nu} &\le KR^{\eta-\nu}\big(1+R^\nu\|\mathfrak{z}\|_{n+n\alpha,(\nu+1,\nu)}\big). \end{aligned}$$

The following lemma is the precise formulation of the estimates in (6.17). Notice that if $\eta = 1$, then we get back the estimate in [24, Lemma 5.1].

LEMMA 6.11 (cf. [24, Lemma 5.1])
If $\|\mathfrak{z}\|_{n+n\alpha,(\nu+1,\nu)} \le R$, *then*

$$\begin{aligned} \|F^1\|_{n-1+n\alpha,(\nu,\nu+1)} &\le CR^{\eta-\nu}\big(1 + R^\nu\|\mathfrak{z}\|_{n+n\alpha,(\nu+1,\nu)}\big), \\ \big\|F^1[\mathfrak{z}] - F^1[\tilde{\mathfrak{z}}]\big\|_{n+n\alpha,(\nu,\nu+1)} &\le CR^{\eta-1}\|\mathfrak{z} - \tilde{\mathfrak{z}}\|_{n+n\alpha,(\nu+1,\nu)}. \end{aligned} \tag{6.29}$$

For $j \ge 2$, *we have*

$$\begin{aligned} \|F^j\|_{n-1+n\alpha,(\nu-1,\nu)} &\le CR^{\eta-\nu}\big(1 + R^\nu\|\mathfrak{z}\|_{n+n\alpha,(\nu+1,\nu)}\big), \\ \big\|F^j[\mathfrak{z}] - F^j[\tilde{\mathfrak{z}}]\big\|_{n+n\alpha,(\nu-1,\nu)} &\le CR^{\eta-1}\|\mathfrak{z} - \tilde{\mathfrak{z}}\|_{n+n\alpha,(\nu+1,\nu)}. \end{aligned} \tag{6.30}$$

Proof
We prove the first two estimates for $F^1 = (f^1_{\bar{1}}, f^1_{\bar{m}})$. We first deal with $f^1_{\bar{1}}$:

$$f^1_{\bar{1}} = a^1_{\bar{1}}\frac{\partial \bar{z}^1}{\partial \bar{\zeta}^1} + \sum_{m>1} a^1_{\bar{m}}\frac{\partial \bar{z}^m}{\partial \bar{\zeta}^1}. \tag{6.31}$$

For the first term on the right-hand side of (6.31), we have the following estimate:

$$\begin{aligned} \Big\| a^1_{\bar{1}}\frac{\partial \bar{z}^1}{\partial \bar{\zeta}^1}\Big\|^1_{n-1+n\alpha,\nu} &\le \|a^1_{\bar{1}}\|^1_{n-1+n\alpha,\nu}\bigg(1 + \Big\|\frac{\partial \bar{\mathfrak{z}}^1}{\partial \bar{\zeta}^1}\Big\|^1_{n-1+n\alpha,0}\bigg) \\ &\lesssim KR^{\eta-\nu}\big(1 + R^\nu\|\mathfrak{z}\|_{n+n\alpha,(\nu+1,\nu)}\big) \\ &\qquad \times \big(1 + \|\mathfrak{z}^1\|_{n+n\alpha,\nu+1}R^{\nu-1}\big) \\ &\lesssim R^{\eta-\nu}\big(1 + R^\nu\|\mathfrak{z}\|_{n+n\alpha,(\nu+1,\nu)}\big), \end{aligned}$$

where we estimated $\|a^1_{\bar{1}}\|^1_{n-1+n\alpha,\nu}$ using Lemma 6.10. Using Remark 6.8, we can estimate

$$\begin{aligned} &\Big\| a^1_{\bar{1}}(\zeta + \mathfrak{z})\frac{\partial \bar{z}^1}{\partial \bar{\zeta}^1} - a^1_{\bar{1}}(\zeta + \tilde{\mathfrak{z}})\frac{\partial \bar{\tilde{z}}^1}{\partial \bar{\zeta}^1}\Big\|^1_{n-1+n\alpha,\nu} \\ &\quad \le \big\| a^1_{\bar{1}}(\zeta + \mathfrak{z}) - a^1_{\bar{1}}(\zeta + \tilde{\mathfrak{z}})\big\|^1_{n-1+n\alpha,\nu}\Big\|\frac{\partial \bar{z}^1}{\partial \bar{\zeta}^1}\Big\|^1_{n-1+n\alpha,0} \\ &\qquad + \big\| a^1_{\bar{1}}(\zeta + \tilde{\mathfrak{z}})\big\|_{n-1+n\alpha,\nu}\Big\|\frac{\partial(\mathfrak{z}^1 - \tilde{\mathfrak{z}}^1)}{\partial \bar{\zeta}^1}\Big\|^1_{n-1+n\alpha,0} \\ &\quad \le KR^{\eta-1}\|\mathfrak{z} - \tilde{\mathfrak{z}}\|_{n+n\alpha,(\nu+1,\nu)}. \end{aligned}$$

In the above estimates, similar to the method in our proof that $\zeta \mapsto z$ gives coordinate charts, we have estimated the difference of $a^1_{\bar{1}}(z) - a^1_{\bar{1}}(\tilde{z})$ by decomposing into two parts and then using the mean value theorem to get the above estimate (cf. [24, p. 401]):

$$\begin{aligned}
&\left\| a^1_{\bar{1}}(\zeta + \mathfrak{z}) - a^1_{\bar{1}}(\zeta + \tilde{\mathfrak{z}}) \right\|^1_{n-1+n\alpha,\nu} \\
&\quad = \left\| a^1_{\bar{1}}(\zeta + \mathfrak{z}) - a^1_{\bar{1}}(\zeta^1 + \tilde{\mathfrak{z}}^1, \zeta'' + \mathfrak{z}'') \right\|_{n-1+n\alpha,\nu} \\
&\qquad + \left\| a^1_{\bar{1}}(\zeta^1 + \tilde{\mathfrak{z}}^1, \zeta'' + \mathfrak{z}'') - a^1_{\bar{1}}(\zeta^1 + \tilde{\mathfrak{z}}^1, \zeta'' + \tilde{\mathfrak{z}}'') \right\|_{n-1+n\alpha,\nu} \\
&\quad \lesssim R^{\eta-1} \|\mathfrak{z}^1 - \tilde{\mathfrak{z}}^1\|_{n+n\alpha,\nu+1} + R^{\eta-1} \|\mathfrak{z}'' - \tilde{\mathfrak{z}}''\|_{n+n\alpha,\nu}.
\end{aligned}$$

The following estimates deal with the second part on the right-hand side of (6.31):

$$\begin{aligned}
\left\| a^1_{\bar{m}} \frac{\partial \bar{z}^m}{\partial \bar{\zeta}^1} \right\|^1_{n-1+n\alpha,\nu} &\le \|a^1_{\bar{m}}\|^1_{n-1+n\alpha,\nu+1} \left\| \frac{\partial \bar{\mathfrak{z}}^m}{\partial \bar{\zeta}^1} \right\|^1_{n-1+n\alpha,-1} \\
&\lesssim K R^{\eta-\nu} \big(1 + R^\nu \|\mathfrak{z}\|_{n+n\alpha,(\nu+1,\nu)}\big) \|\mathfrak{z}^m\|_{n+n\alpha,\nu} R^{\nu-1} \\
&\lesssim R^{\eta-\nu} \big(1 + R^\nu \|\mathfrak{z}\|_{n+n\alpha,(\nu+1,\nu)}\big).
\end{aligned}$$

In the last inequality, we used $\|\mathfrak{z}^m\|_{n+n\alpha,\nu} \le R$:

$$\begin{aligned}
&\left\| a^1_{\bar{m}}(\zeta + \mathfrak{z}) \frac{\partial \bar{z}^m}{\partial \bar{\zeta}^1} - a^1_{\bar{m}}(\zeta + \tilde{\mathfrak{z}}) \frac{\partial \bar{\tilde{z}}^m}{\partial \bar{\zeta}^1} \right\|^1_{n-1+n\alpha,\nu} \\
&\quad \le \left\| a^1_{\bar{m}}(\zeta + \mathfrak{z}) - a^1_{\bar{m}}(\zeta + \tilde{\mathfrak{z}}) \right\|^1_{n-1+n\alpha,\nu+1} \left\| \frac{\partial \bar{z}^m}{\partial \bar{\zeta}^1} \right\|^1_{n-1+n\alpha,-1} \\
&\qquad + \left\| a^1_{\bar{m}}(\zeta + \tilde{\mathfrak{z}}) \right\|_{n-1+n\alpha,\nu+1} \left\| \frac{\partial (\mathfrak{z}^m - \tilde{\mathfrak{z}}^m)}{\partial \bar{\zeta}^1} \right\|^1_{n-1+n\alpha,-1} \\
&\quad \le K R^{\eta-1} \|\mathfrak{z} - \tilde{\mathfrak{z}}\|_{n+n\alpha,(\nu+1,\nu)}.
\end{aligned}$$

We used the estimate

$$\begin{aligned}
&\left\| a^1_{\bar{m}}(\zeta + \mathfrak{z}) - a^1_{\bar{m}}(\zeta + \tilde{\mathfrak{z}}) \right\|^1_{n-1+n\alpha,\nu+1} \\
&\quad = \left\| a^1_{\bar{m}}(\zeta + \mathfrak{z}) - a^1_{\bar{m}}(\zeta^1 + \tilde{\mathfrak{z}}^1, \zeta'' + \mathfrak{z}'') \right\|_{n-1+n\alpha,\nu+1} \\
&\qquad + \left\| a^1_{\bar{m}}(\zeta^1 + \tilde{\mathfrak{z}}^1, \zeta'' + \mathfrak{z}'') - a^1_{\bar{m}}(\zeta^1 + \tilde{\mathfrak{z}}^1, \zeta'' + \tilde{\mathfrak{z}}'') \right\|_{n-1+n\alpha,\nu+1} \\
&\quad \lesssim R^{\eta-1} \|\mathfrak{z}^1 - \tilde{\mathfrak{z}}^1\|_{n+n\alpha,\nu+1} + R^{\eta-1} \|\mathfrak{z}'' - \tilde{\mathfrak{z}}''\|_{n+n\alpha,\nu}.
\end{aligned}$$

In the same way, one can verify the other estimates. ☐

6.2.3. Completion of the proof of Proposition 6.1
Combining Lemmas 6.9 and 6.11, we get the following result.

THEOREM 6.12
For any $\mathfrak{z}$, $\tilde{\mathfrak{z}}$ satisfying $\|\mathfrak{z}\|_{n+n\alpha,(\nu+1,\nu)} \le R$, $\|\tilde{\mathfrak{z}}\|_{n+n\alpha,(\nu+1,\nu)} \le R$ with R sufficiently small, we have

$$\begin{aligned}\big\|\mathfrak{J}(\mathfrak{z})\big\|_{n+n\alpha,(\nu+1,\nu)} &\le cR^{\eta-\nu}\big(1+R^{\nu}\|\mathfrak{z}\|_{n+n\alpha,(\nu+1,\nu)}\big),\\ \big\|\mathfrak{J}(\tilde{\mathfrak{z}})-\mathfrak{J}(\mathfrak{z})\big\|_{n+n\alpha,(\nu+1,\nu)} &\le cR^{\eta}\|\tilde{\mathfrak{z}}-\mathfrak{z}\|_{n+n\alpha,(\nu+1,\nu)}.\end{aligned}$$

So for R sufficiently small, we indeed get the desired inequalities (6.19) and (6.20) needed to apply the contraction-iteration principle to get a solution to the system (6.27).

LEMMA 6.13
If $\mathfrak{z}$ is a solution to the system (6.27), then $\mathfrak{z}$ is a solution to (6.8), that is,

$$g^{i}_{\overline{j}} = \frac{\partial z^i}{\partial \overline{\zeta}^j} + \sum_{p=1}^{n} a^{i}_{\overline{p}}(z)\frac{\partial \overline{z}^p}{\partial \overline{\zeta}^j} = 0, \quad i,j=1,\dots,n. \tag{6.32}$$

Proof
We follow the argument in [24, p. 403]. Using the formula (6.10) and calculating as in [24, (2.11)–(2.12)] (see also [25, (4.1.2)]), we get the identity

$$\begin{aligned}g^{i}_{\overline{j}} = \sum_{s=0}^{n-2}\frac{(-1)^s}{(s+2)!}&\sum\nolimits^{j}\tilde{T}^{j_1}\overline{\partial}_{j_1}\cdots\tilde{T}^{j_s}\overline{\partial}_{j_s}\\ &\cdot\tilde{T}^k\big[(\partial_p a^{i}_{\overline{m}})(\zeta)(\overline{\partial}_j\overline{z}^m\cdot g^{p}_{\overline{k}} - \overline{\partial}_k\overline{z}^m\cdot g^{p}_{\overline{j}})\big],\end{aligned} \tag{6.33}$$

where $\sum^j$ denotes the summation over all $(s+1)$-tuples with $j_1,\dots,j_s,k$ distinct and different from j. We claim that from (6.33) the following holds:

$$\begin{aligned}&\|G^1\|_{n-1+n\alpha,(\nu,\nu+1)} + \|G^j\|_{n-1+n\alpha,(\nu-1,\nu)}\\ &\quad\le CR^{\eta+\nu}\big(\|G^1\|_{n-1+n\alpha,(\nu,\nu+1)} + \|G^j\|_{n-1+n\alpha,(\nu-1,\nu)}\big),\end{aligned} \tag{6.34}$$

where we denote $G^i = (g^{i}_{\overline{1}},\dots,g^{i}_{\overline{n}})$. Assuming that (6.34) holds, then when R is sufficiently small we have $G^i = 0$ and so we indeed get the solution to (6.32). To verify the claim, we need to estimate the term in the bracket:

$$\mathfrak{G}^{i}_{\overline{j}\overline{k}} := \sum_{p,m}(\partial_p a^{i}_{\overline{m}})(\zeta)(\overline{\partial}_j\overline{z}^m\cdot g^{p}_{\overline{k}} - \overline{\partial}_k\overline{z}^m\cdot g^{p}_{\overline{j}}) =: \sum_{m,p}\mathfrak{G}^{i}_{\overline{j}\overline{k}mp}.$$

We will estimate it for different cases of indices.

(1) ($i=1,\ j=1$) In this case $k\ge 2$ (since $k\neq j$ in $\sum^j$).

(a) ($p=1,\ m=1$) Note that $\partial_1 a^1_{\bar 1}=\tilde O(R^{\eta-1})$, $\bar\partial_1\bar z^1=\tilde O(1+R^\nu)$, $\|g^1_{\bar k}\|_{n-1+n\alpha,\nu+1}=\tilde O(R^{\nu+1})$, $\bar\partial_k\bar z^1=\bar\partial_k\bar{\mathfrak{z}}^1=\tilde O(R^{\nu+1})$, $\|g^1_{\bar j}\|_{n-1+n\alpha,\nu}=\tilde O(R^\nu)$. So we can estimate the summand as:

$$
\begin{aligned}
&\|\mathfrak{G}^1_{\bar 1\bar k mp}\|_{n-1+n\alpha,(\nu,\nu+1)}\\
&\quad\le R^{\eta-1}\big((1+R^\nu)\cdot R^{\nu+1}\|G^1\|_{n-1+n\alpha,(\nu,\nu+1)}\\
&\qquad + R^{\nu+1}\|\bar\partial_k\bar{\mathfrak{z}}^1\|_{n-1+n\alpha,\nu+1}\cdot R^\nu\|G^1\|_{n-1+n\alpha,(\nu,\nu+1)}\big)\\
&\quad\le R^{\eta+\nu}\big(\|G^1\|_{n-1+n\alpha,(\nu,\nu+1)}+\|G^j\|_{n-1+n\alpha,(\nu-1,\nu)}\big).
\end{aligned}
$$

For convenience, we will just write formally that the following holds:

$$\mathfrak{G}^1_{\bar 1\bar k mp}=\tilde O\big(\rho^{\eta-1}(\rho^{0+\nu+1}+\rho^{\nu+1+\nu})\big)=\tilde O(\rho^\eta\rho^\nu).$$

By similar reasoning, we can estimate the other summands:

(b) ($p\ge 2, m=1$) $\mathfrak{G}^1_{\bar 1\bar k mp}=\tilde O(\rho^\eta(\rho^{0+\nu}+\rho^{\nu+1+\nu-1}))=\tilde O(\rho^\eta\rho^\nu)$.

(c) ($p=1, m\ge 2$) $\mathfrak{G}^1_{\bar 1\bar k mp}=\tilde O(\rho^\eta(\rho^{\nu-1+\nu+1}+\rho^{0+\nu}))=\tilde O(\rho^\eta\rho^\nu)$.

(d) ($p\ge 2, m\ge 2$) $\mathfrak{G}^1_{\bar 1\bar k mp}=\tilde O(\rho^{\eta+1}(\rho^{\nu-1+\nu}+\rho^{0+\nu-1}))=\tilde O(\rho^\eta\rho^\nu)$.

Combining estimates (a)–(d) above gives

$$\|\mathfrak{G}^1_{\bar 1\bar k}\|_{n-1+n\alpha,\nu}\lesssim R^{\eta+\nu}\big(\|G^1\|_{n-1+n\alpha,(\nu,\nu+1)}+\|G^j\|_{n-1+n\alpha,(\nu-1,\nu)}\big).$$

The same remark applies to the notation in the following estimates:

(2) ($i=1,\ j\ge 2$) In this case k can be 1.

(a) ($k=1$) We estimate norm $\|\mathfrak{G}^1_{\bar j\bar 1}\|_{n-1+n\alpha,\nu}$.

(i) ($p=1,\ m=1$) $\mathfrak{G}^1_{\bar j\bar 1mp}=\tilde O(\rho^{\eta-1}(\rho^{\nu+1+\nu}+\rho^{0+\nu+1}))=\tilde O(\rho^\eta\rho^\nu)$.

(ii) ($p\ge 2,\ m=1$) $\mathfrak{G}^1_{\bar j\bar 1mp}=\tilde O(\rho^\eta(\rho^{\nu+1+\nu-1}+\rho^{0+\nu}))=\tilde O(\rho^\eta\rho^\nu)$.

(iii) ($p=1,\ m\ge 2$) $\mathfrak{G}^1_{\bar j\bar 1mp}=\tilde O(\rho^\eta(\rho^{0+\nu}+\rho^{\nu-1+\nu+1}))=\tilde O(\rho^\eta\rho^\nu)$.

(iv) ($p\ge 2,\ m\ge 2$) $\mathfrak{G}^1_{\bar j\bar 1mp}=\tilde O(\rho^{\eta+1}(\rho^{0+\nu-1}+\rho^{\nu-1+\nu}))=\tilde O(\rho^\eta\rho^\nu)$.

(b) ($k\ge 2$) We use the norm $\|\mathfrak{G}^1_{\bar j\bar k}\|_{n-1+n\alpha,\nu+1}$.

(i) ($p=1, m=1$) $\mathfrak{G}^1_{\bar j\bar k mp}=\tilde O(\rho^{\eta-1}(\rho^{\nu+1+\nu+1}+\rho^{\nu+1+\nu+1}))=\tilde O(\rho^{\eta+\nu}\rho^{\nu+1})$.

(ii) $(p \geq 2,\ m = 1)$ $\mathfrak{G}^1_{\bar{j}kmp} = \tilde{O}(\rho^\eta(\rho^{\nu+1+\nu} + \rho^{\nu+1+\nu})) = \tilde{O}(\rho^{\eta+\nu}\rho^{\nu+1})$.

(iii) $(p = 1,\ m \geq 2)$ $\mathfrak{G}^1_{\bar{j}kmp} = \tilde{O}(\rho^\eta(\rho^{0+\nu+1} + \rho^{0+\nu+1})) = \tilde{O}(\rho^\eta\rho^{\nu+1})$.

(iv) $(p \geq 2,\ m \geq 2)$ $\mathfrak{G}^1_{\bar{j}kmp} = \tilde{O}(\rho^{\eta+1}(\rho^{0+\nu} + \rho^{0+\nu})) = \tilde{O}(\rho^\eta\rho^{\nu+1})$.

(3) $(i \geq 2,\ j = 1)$ In this case $k \geq 2$. From the expression of $\mathfrak{G}^i_{\bar{j}k}$, we see that the only difference between this and the case $i = 1,\ j = 1$ lies in the term $\partial_p a^i_{\bar{m}}$. We just need to decrease each order by 1 to get

$$\mathfrak{G}^i_{\bar{1}k} = \tilde{O}(\rho^\eta\rho^{\nu-1}),$$

or equivalently,

$$\|\mathfrak{G}^i_{\bar{1}k}\|_{n-1+n\alpha,\nu} \leq R^{\eta+\nu-1}\big(\|G^1\|_{n-1+n\alpha,(\nu,\nu+1)} + \|G^j\|_{n-1+n\alpha,(\nu-1,\nu)}\big).$$

(4) $(i \geq 2,\ j \geq 2)$ In this case, k can be 1. Again, we see that the only difference between this and the case $i = 1,\ j \geq 2$ lies in the term $\partial_p a^i_{\bar{m}}$. So we just need to decrease each order by 1 to get

$$\mathfrak{G}^i_{\bar{j}1} = \tilde{O}(\rho^\eta\rho^{\nu-1}) \qquad \text{and} \qquad \mathfrak{G}^i_{\bar{j}k} = \tilde{O}(\rho^\eta\rho^\nu).$$

Now from (1), we have that

$$\begin{aligned}\|g^1_{\bar{1}}\|^1_{n-1+n\alpha,\nu} &\leq C\sum_{k\geq 2}\|\tilde{T}^k\mathfrak{G}^1_{\bar{1}k}\|_{n-1+n\alpha,\nu}\\ &\leq CR^{\eta+\nu}\big(\|G^1\|_{n-1+n\alpha,(\nu,\nu+1)} + \|G^j\|_{n-1+n\alpha,(\nu-1,\nu)}\big).\end{aligned}$$

From (2), we have for $j \geq 2$,

$$\begin{aligned}\|g^1_{\bar{j}}\|^j_{n-1+n\alpha,\nu} &\leq C\Big(\|\tilde{T}^1\mathfrak{G}^1_{\bar{j}1}\|^j_{n-1+n\alpha,\nu+1} + \sum_{k\geq 2}\|\tilde{T}^k\mathfrak{G}^1_{\bar{j}k}\|^j_{n-1+n\alpha,\nu+1}\Big)\\ &\leq CR^{\eta+\nu}\big(\|G^1\|_{n-1+n\alpha,(\nu,\nu+1)} + \|G^j\|_{n-1+n\alpha,(\nu-1,\nu)}\big).\end{aligned}$$

Note that we have used the fact from (6.7) that the operator $\tilde{T}^1$ improves the weight from ν to $\nu + 1$. The same argument applies to items (3) and (4) as well. So we indeed get the estimate (6.34). □

Appendix

A.1. Neighborhoods of complex submanifolds after Grauert–Abate–Bracci–Tovena
Assume that S is a smooth complex submanifold of X. In the Introduction, we recalled the definition of $S(k)$ and the concept of linearizability. Grauert [14] showed

that the obstruction for extending an isomorphism $S(k-1) \to S_N(k-1)$ to an isomorphism $S(k) \to S_N(k)$ lies in the cohomology group $H^1(S, \Theta_X|_S \otimes \mathcal{I}_S^k/\mathcal{I}_S^{k+1})$. He also pointed out that this obstruction consists of two parts. To see this, consider the exact sequence

$$0 \to \Theta_S \otimes \mathcal{I}_S^k/\mathcal{I}_S^{k+1} \to \Theta_X|_S \otimes \mathcal{I}_S^k/\mathcal{I}_S^{k+1} \to N_S \otimes \mathcal{I}_S^k/\mathcal{I}_S^{k+1} \to 0$$

from which we get the long exact sequence

$$\begin{aligned}\cdots &\to H^1(S, \Theta_S \otimes \mathcal{I}_S^k/\mathcal{I}_S^{k+1}) \\ &\to H^1(S, \Theta_X|_S \otimes \mathcal{I}_S^k/\mathcal{I}_S^{k+1}) \to H^1(S, N_S \otimes \mathcal{I}_S^k/\mathcal{I}_S^{k+1}) \to \cdots .\end{aligned}$$

So, roughly speaking, the obstruction comes from two parts: one from $H^1(S, N_S \otimes \mathcal{I}_S^k/\mathcal{I}_S^{k+1})$, and the other from $H^1(S, \Theta_S \otimes \mathcal{I}_S^k/\mathcal{I}_S^{k+1})$. In [1], Abate, Bracci, and Tovena explicitly described these two cohomological obstruction classes, and introduced the notion of *k-splitting* and *k-comfortably embedded* such that k-linearizable equals k-splitting plus $(k-1)$-comfortably embedded with respect to the induced $(k-1)$th-order lifting. (This section includes results from Abate, Bracci, and Tovena's work referenced in the main body of the paper.)

Definition A.1 ([1, Definitions 2.1, 2.2])

(1) S is k-splitting in X (for some $k \geq 1$) if the exact sequence

$$0 \longrightarrow \mathcal{I}_S/\mathcal{I}_S^{k+1} \longrightarrow \mathcal{O}_X/\mathcal{I}_S^{k+1} \longrightarrow \mathcal{O}_S \to 0$$

splits as a sequence of sheaves of rings.

(2) A k-splitting atlas for $S \subset X$ is an atlas $\{(V_\alpha, z_\alpha)\}$ of X adapted to S (i.e., $V_\alpha \cap S \neq \emptyset$ implies $V_\alpha \cap S = \{z_\alpha^1 = \cdots = z_\alpha^m = 0\}$) such that

$$\frac{\partial^k z_\beta^p}{\partial z^{r_1} \cdots \partial z_\alpha^{r_k}}\bigg|_S \equiv 0,$$

for all $r_1, \ldots, r_k = 1, \ldots, m$, all $p = m+1, \ldots, n$, and all indices α, β such that $V_\alpha \cap V_\beta \cap S \neq \emptyset$.

(3) An atlas $\{(V_\alpha, z_\alpha)\}$ adapted to S is adapted to a kth-order lifting $\rho : \mathcal{O}_S \to \mathcal{O}_M/\mathcal{I}_S^{k+1}$ if

$$\rho[f]_1 = \sum_{l=0}^{k} (-1)^l \Big[\frac{\partial^l f}{\partial z_\alpha^{r_1} \cdots \partial z_\alpha^{r_l}} z_\alpha^{r_1} \cdots z_\alpha^{r_l}\Big]_{k+1}, \tag{A.1}$$

for every $f \in \mathcal{O}(V_\alpha)$ and all indices α such that $V_\alpha \cap S \neq \emptyset$.

In the following, if S is k-splitting, we will fix a lifting: $\rho_k : \mathcal{O}_S \to \mathcal{O}_X/\mathcal{J}_S^{k+1}$. We also denote by $\phi_{h,k}$ the natural map

$$\phi_{h,k} : \mathcal{O}_X/\mathcal{J}_S^{h+1} \to \mathcal{O}_X/\mathcal{J}_S^{k+1}, \quad \text{for } h \geq k. \tag{A.2}$$

PROPOSITION A.2 ([1, Proposition 2.2])
Assume that S is $(k-1)$-splitting in X, let $\rho_{k-1} : \mathcal{O}_S \to \mathcal{O}_X/\mathcal{J}_S^k$ be a $(k-1)$th-order lifting, and let $\mathfrak{V} = \{(V_\alpha, \phi_\alpha)\}$ be a $(k-1)$-splitting atlas adapted to ρ_{k-1}. Let $\mathfrak{g}_k^{\rho_{k-1}} \in H^1(S, \mathrm{Hom}(\Omega_S, \mathcal{J}_S^k/\mathcal{J}_S^{k+1}))$ be the Čech cohomology class represented by a 1-cocycle $\{(\mathfrak{g}_k^{\rho_{k-1}})_{\beta\alpha}\} \in H^1(\mathfrak{V}_S, \mathrm{Hom}(\Omega_S, \mathcal{J}_S^k/\mathcal{J}_S^{k+1}))$ given by

$$\begin{aligned}(\mathfrak{g}_k^{\rho_{k-1}})_{\beta\alpha} &= -\frac{1}{k!}\frac{\partial^k z_\alpha^p}{\partial z_\beta^{r_1}\cdots\partial z_\beta^{r_k}}\bigg|_S \frac{\partial}{\partial z_\alpha^p} \otimes [z_\beta^{r_1}\cdots z_\beta^{r_k}]_{k+1} \\ &\in H^0(V_\alpha \cap V_\beta \cap S, \Theta_S \otimes \mathcal{J}_S^k/\mathcal{J}_S^{k+1}).\end{aligned} \tag{A.3}$$

Then there exists a kth-order lifting $\rho_k : \mathcal{O}_S \to \mathcal{O}_X/\mathcal{J}_S^{k+1}$ such that $\rho_{k-1} = \phi_{k,k-1} \circ \rho_k$ if and only if $\mathfrak{g}_k^{\rho_{k-1}} = 0$. We call this $\mathfrak{g}_k^{\rho_{k-1}}$ the obstruction to k-splitting relative to ρ_{k-1}.

PROPOSITION A.3 ([1, Proposition 3.2])
Assume that S is k-splitting in X, and let $\rho : \mathcal{O}_S \to \mathcal{O}_X/\mathcal{J}_S^{k+1}$ be a kth-order lifting, with $k \geq 0$. Then for any $1 \leq h \leq k+1$, the lifting ρ induces a structure of locally $\mathcal{O}_S$-free modules on $\mathcal{J}_S/\mathcal{J}_S^{h+1}$ for $1 \leq h \leq k+1$ in such a way that the sequence

$$0 \longrightarrow \mathcal{J}_S^h/\mathcal{J}_S^{h+1} \longrightarrow \mathcal{J}_S/\mathcal{J}_S^{h+1} \longrightarrow \mathcal{J}_S/\mathcal{J}_S^h \longrightarrow 0 \tag{A.4}$$

becomes an exact sequence of locally $\mathcal{O}_S$-free modules.

Definition A.4 ([1, Definitions 3.1, 3.2])

(1) If S is k-splitting in X and the sequence (A.4) splits for $1 \leq h \leq k+1$, then S is called *k-comfortably embedded* in X. Denote by $\nu_{h-1,h} : \mathcal{J}_S/\mathcal{J}_S^h \to \mathcal{J}_S/\mathcal{J}_S^{h+1}$ the splitting $\mathcal{O}_S$-morphism of the sequence (A.4) and the comfortable splitting sequence $\boldsymbol{\nu}_k = (\nu_{0,1}, \ldots, \nu_{k,k+1})$.
(2) A k-comfortable atlas is an atlas $\{(V_\alpha, z_\alpha)\}$ adapted to S such that

$$\begin{aligned}&\frac{\partial z_\beta^p}{\partial z_\alpha^r} \in \mathcal{J}_S^k, \qquad \text{and} \qquad \frac{\partial^2 z_\beta^r}{\partial z_\alpha^{s_1}\partial z_\alpha^{s_2}} \in \mathcal{J}_S^k \\ &\qquad\iff \quad \frac{\partial^k z_\beta^p}{\partial z_\alpha^{r_1}\cdots\partial z_\alpha^{r_k}}\bigg|_S \equiv 0, \qquad \text{and} \qquad \frac{\partial^{k+1} z_\beta^s}{\partial z_\alpha^{r_1}\cdots\partial z_\alpha^{r_{k+1}}}\bigg|_S \equiv 0,\end{aligned}$$

for all $r_1, \dots, r_k = 1, \dots, m$, all $p = m+1, \dots, n$, and all indices α, β such that $V_\alpha \cap V_\beta \cap S \neq \emptyset$.

Remark A.5
Any submanifold S is always 0-comfortably embedded and 0-linearizable, but is not always 1-linearizable (which is equivalent to having a splitting tangent sequence). If S is k-comfortably embedded, then S is also k-splitting.

THEOREM A.6 ([1, Corollary 3.6])
Assume that there exists a kth-order lifting $\rho_k : \mathcal{O}_S \to \mathcal{O}_X/\mathcal{I}_S^{k+1}$ such that S is $(k-1)$-comfortably embedded in X with respect to $\rho_{k-1} = \phi_{k,k-1} \circ \rho_k$. Fix a $(k-1)$-comfortable pair $(\rho_{k-1}, \boldsymbol{\nu}_{k-1})$, and let $\mathfrak{V} = \{(V_\alpha, z_\alpha)\}$ be a projectable atlas adapted to ρ_k and $(\rho_{k-1}, \boldsymbol{\nu}_{k-1})$. Then the cohomology class $\mathfrak{h}^{\rho_k}$ associated to the exact sequence (A.4) is represented by 1-cocycle $\{\mathfrak{h}^{\rho_k}_{\beta\alpha}\} \in H^1(\mathfrak{V}_S, \mathcal{N}_S \otimes \mathcal{I}_S^{k+1}/\mathcal{I}_S^{k+2})$ given by

$$\mathfrak{h}^{\rho_k}_{\beta\alpha} = -\frac{1}{(k+1)!} \frac{\partial z_\beta^{s_1}}{\partial z_\alpha^{r_1}} \cdots \frac{\partial z_\beta^{s_{k+1}}}{\partial z_\alpha^{r_{k+1}}} \frac{\partial^{k+1} z_\alpha^t}{\partial z_\beta^{s_1} \cdots \partial z_\beta^{s_{k+1}}}\bigg|_S \partial_{z_\alpha^t} \otimes [z_\alpha^{r_1} \cdots z_\alpha^{r_{k+1}}]_{k+2}.$$

Remark A.7
If D is a smooth divisor, then the obstruction to k-comfortable embedding lies in $H^1(D, N_D \otimes \mathcal{I}_D^{k+1}/\mathcal{I}_D^{k+2}) = H^1(D, (N_D)^{-k})$. If we assume that the normal bundle N_D is ample on D and $n-1 = \dim D \geq 2$, then the Kodaira–Nakano vanishing theorem gives $H^1(D, (N_D)^{-k}) = 0$ for any $k \geq 1$. So in this case, there is no obstruction to passing from $(k-1)$-comfortable embedding to k-comfortable embedding (with respect to any k-splitting). Note that D is always 0-comfortably embedded. So we obtain that if N_D is ample on D and $\dim X \geq 3$, then D is k-comfortably embedded if and only if D is k-splitting, and if and only if D is k-linearizable (see Theorem A.9).

THEOREM A.8 ([1, Theorems 2.1, 3.5])
S is k-splitting in X if and only if there is a k-splitting atlas $\mathfrak{V} = \{(V_\alpha, z_\alpha)\}$ of X, that is an atlas adapted to S such that

$$\begin{cases} z_\beta^r = \sum_{s=1}^m (a_{\beta\alpha})_s^r (z_\alpha) z_\alpha^s, & \textit{for } r = 1, \dots, m, \\ z_\beta^p = \phi_{\beta\alpha}^p (z_\alpha'') + R_{k+1}^p, & \textit{for } p = m+1, \dots, n, \end{cases}$$

where $z_\alpha'' = (z_\alpha^{m+1}, \dots, z_\alpha^n)$ are local coordinates on S, and R_{k+1}^p denotes a term belonging to $\mathcal{I}_S^{k+1}$. Furthermore, S is k-comfortably embedded in X if and only if there is a k-comfortable atlas $\mathfrak{V} = \{(V_\alpha, z_\alpha)\}$, that is an atlas adapted to S such that

$$\begin{cases} z_\beta^r = \sum_{s=1}^m (a_{\beta\alpha})_s^r (z''_\alpha) z_\alpha^s + R_{k+2}^r, & \textit{for } r = 1, \dots, m, \\ z_\beta^p = \phi_{\beta\alpha}^p (z''_\alpha) + R_{k+1}^p, & \textit{for } p = m+1, \dots, n, \end{cases}$$

where $R_{k+2}^r \in \mathcal{I}_S^{k+2}$ *and* $R_{k+1}^p \in \mathcal{I}_S^{k+1}$.

THEOREM A.9 ([1, Theorem 4.1])
S is k-linearizable if and only if S is k-splitting in X and $(k-1)$-comfortably embedded with respect to the $(k-1)$th-order lifting induced by the k-splitting, if and only if there is an atlas $\mathfrak{V}$ such that the changes of coordinates are of the form

$$\begin{cases} z_\beta^r = \sum_{s=1}^m (a_{\beta\alpha})_s^r (z''_\alpha) z_\alpha^s + R_{k+1}^r & \textit{for } r = 1, \dots, m, \\ z_\beta^p = \phi_{\beta\alpha}^p (z''_\alpha) + R_{k+1}^p & \textit{for } p = m+1, \dots, n, \end{cases}$$

where $R_{k+1}^r, R_p^{k+1} \in \mathcal{I}_S^{k+1}$.

A.2. Deformation of normal algebraic varieties

A.2.1. First-order deformations

Assume that Z is a complex analytic variety in $\mathbb{C}^N$. Choose any analytically open set W of $\mathbb{C}^N$, and assume that $\mathcal{I}_Z(W)$ is generated by $\{f_1, \dots, f_d\}$. Let $\mathcal{Z} \to \mathbb{B}$ be a flat deformation of Z with $\mathcal{Z}_0 = Z$, which is realized as an embedding deformation of $Z \to \mathbb{C}^N$. Assume that $\mathcal{I}_{\mathcal{Z}_t}(W)$ is generated by $\{f_1 + tg_1, \dots, f_d + tg_d\}$. Then by the flatness condition, $\{g_i\}$ induces a morphism

$$\bar{g} : \mathcal{I}_Z / \mathcal{I}_Z^2 \to \mathcal{O}_{\mathbb{C}^N} / \mathcal{I}_Z = \mathcal{O}_Z, \qquad \sum_i [f_i h_i] \mapsto \sum_i g_i h_i |_Z.$$

So we get $\bar{g} \in \operatorname{Hom}_{\mathcal{O}_Z}(\mathcal{I}_Z/\mathcal{I}_Z^2, \mathcal{O}_Z) = H^0(Z, N_Z)$. To get the space of first-order infinitesimal deformations of Z, one considers the conormal exact sequence

$$\mathcal{I}_Z / \mathcal{I}_Z^2 \to \Omega_{\mathbb{C}^N}|_Z \to \Omega_Z \to 0,$$

whose dual defines the sheaf $\mathcal{T}_Z^1$ (see [31, Section 1.2] and [15, Proposition II1.25]):

$$0 \to \Theta_Z \to \Theta_{\mathbb{C}^N}|_Z \to N_Z \to \mathcal{T}_Z^1 \to 0.$$

Since we assumed that Z is an embedding in $\mathbb{C}^N$, we get the exact sequence

$$0 \to H^0(Z, \Theta_Z) \to H^0(Z, \Theta_{\mathbb{C}^N}|_Z) \to H^0(Z, N_Z) \xrightarrow{\psi_Z} \mathbf{T}_Z^1 \to 0. \tag{A.5}$$

In particular, $\mathbf{T}_Z^1$ is defined so that (A.5) becomes exact and is not equal to $H^1(Z, \Theta_Z)$ in general. The image of $\bar{g}$ in $\mathbf{T}_Z^1$ is the first-order information of the deformation $\mathcal{Z} \to \mathbb{B}$.

PROPOSITION A.10 ([31, Theorem 1])
Assume that Z has an isolated normal singularity o, and denote $U = Z \setminus \{o\}$. Then there are exact sequences

$$H^0(U, \Theta_{\mathbb{C}^N}|_U) \to H^0(U, N_U) \xrightarrow{\psi_U} \mathbf{T}^1_Z \to 0, \tag{A.6}$$

$$0 \longrightarrow \mathbf{T}^1_Z \xrightarrow{\tau_U} H^1(U, \Theta_U) \to H^1(U, \Theta_{\mathbb{C}^N}|_U). \tag{A.7}$$

Proof
For the reader's convenience, we sketch the proof here. Because Z is normal, by Serre's criterion for normality, Z has depth $\mathrm{depth}_{\underline{o}} Z \geq 2$ at its vertex. Because the first three sheaves in (A.5) are reflexive, by [30, Lemma 1] the depth of each is at least 2. So in (A.5) we can replace $H^0(Z, \cdot)$ by $H^0(U, \cdot)$ to get

$$0 \to H^0(U, \Theta_U) \to H^0(U, \Theta_{\mathbb{C}^N}|_U) \to H^0(U, N_U) \to \mathbf{T}^1_Z \to 0. \tag{A.8}$$

On the other hand, because U is smooth and embedded into $\mathbb{C}^N$, we have

$$0 \to \Theta_U \to \Theta_{\mathbb{C}^N}|_U \to N_U \to 0,$$

which gives us the exact sequence

$$\begin{aligned} 0 &\to H^0(U, \Theta_U) \to H^0(U, \Theta_{\mathbb{C}^N}|_U) \\ &\to H^0(U, N_U) \xrightarrow{\delta} H^1(U, \Theta_U) \to H^1(U, \Theta_{\mathbb{C}^N}|_U). \end{aligned} \tag{A.9}$$

Combining (A.8) and (A.9), we get (A.6) and (A.7). □

A.2.2. Deformation of affine cones
As an example of the above general theory, consider a projective manifold $D \subset \mathbb{P}^{N-1}$. We assume that D is projectively normal in $\mathbb{P}^{N-1}$ so that the affine cone over D is normal and is equal to $C = C(D, H)$, where H is the hyperplane bundle of $\mathbb{P}^{N-1}$. Then it is easily verified (see [4], [31]) that

$$H^0(U, \Theta_{\mathbb{C}^N}|_U) = \sum_{j=-\infty}^{+\infty} H^0\big(D, \mathcal{O}_D(j+1)\big),$$

$$H^0(U, N_U) = \sum_{j=-\infty}^{+\infty} H^0\big(D, N_D(j)\big).$$

Decompose $\mathbf{T}^1_C = \sum_{j=-\infty}^{+\infty} \mathbf{T}^1_C(j)$ into weight spaces. Then by (A.6) we have the exact sequence

$$H^0\big(D, \mathcal{O}_D(j+1)\big)^N \xrightarrow{\text{Jac}} H^0\big(D, N_D(j)\big) \longrightarrow \mathbf{T}^1_C(j) \to 0. \tag{A.10}$$

Example A.11 (cf. [4, Section 4], [21])
Assume that $D^{n-1} \subset \mathbb{P}^{N-1}$ is a complete intersection

$$D = \bigcap_{i=1}^{N-n} \{F_i = 0\} \subset \mathbb{P}^{N-1},$$

where F_i is a homogeneous polynomial of degree d_i. We assume that $\{Z_1, \dots, Z_N\}$ are homogeneous coordinates of $\mathbb{P}^{N-1}$ and denote

$$R(D,H) = \bigoplus_{m=0}^{+\infty} H^0(D, mH) \cong \mathbb{C}[Z_1, \dots, Z_N]/\langle F_1, \dots, F_{N-n}\rangle.$$

Note that this is nothing but the affine coordinate ring of $C(D,H)$. Then

$$\begin{aligned} H^0\big(D, \mathcal{O}_D(j+1)\big) &= H^0\big(D, (j+1)H\big) = R(D,H)(j+1), \\ H^0\big(D, N_D(j)\big) &= \bigoplus_{i=1}^{N-n} H^0\big(D, (d_i+j)H\big) \\ &= \bigoplus_{i=1}^{N-n} R(D,H)(d_i+j). \end{aligned}$$

The map

$$\mathrm{Jac} : R(D,H)(j+1)^N \to \bigoplus_{i=1}^{N-n} R(D,H)(d_i+j)$$

is given by the Jacobian matrix $(\partial F_k / \partial Z^l)_{k=1,\dots,N-n}^{l=1,\dots,N}$, with the quotient

$$\mathbf{T}^1_C(j) = \frac{\bigoplus_{i=1}^{N-n} R(D,H)(d_i+j)}{\mathrm{Jac}(R(D,H)(j+1)^{\oplus N})}. \tag{A.11}$$

Now assume that $\mathcal{G} = \{g_i = g_i(z_1, \dots, z_N), i = 1, \dots, N-n\}$ consists of (not necessarily homogeneous) polynomials. We can consider the deformation of $C(D,H) \subset \mathbb{C}^N$ given by

$$\mathcal{C}_t = \bigcap_{i=1}^{N-n} \big\{F_i(z_1, \dots, z_N) + t g_i = 0\big\} \subset \mathbb{C}^N.$$

If we assume that the image $[\mathcal{G}]$ in $\mathbf{T}^1_C$ is not zero, then, by (A.10), we see that the weight of this deformation is the weight of $[\mathcal{G}]$. Note that the polynomials in the image of Jac have degree at least $d_i - 1$. So if g_i is of degree $e_i \le d_i - 2$, then it is easy to see that $[\mathcal{G}]$ is indeed not zero and the weight is equal to $\max\{e_i - d_i\} = -\min\{d_i - e_i\}$.

Remark A.12
The reason that we assume the nonvanishing of $[\mathscr{G}]$ is to guarantee that the induced map $\mathbb{C} \to \mathbf{T}^1_C$ does not have a vanishing first-order derivative. Otherwise, we can consider the reduced Kodaira–Spencer class, as the following example shows:

$$\{z_1^2 + z_2^2 + z_3^2 = 0\} \rightsquigarrow \{z_1^2 + z_2^2 + z_3^2 + tz_3 = 0\}.$$

We have $\mathbf{T}^1_C = \mathbb{C}[z_1, z_2, z_3]/\langle z_1, z_2, z_3\rangle$. So $\mathscr{G} = (g = z_3)$ gives the vanishing image $[\mathscr{G}] = 0$. However, we have

$$\begin{aligned}\{z_1^2 + z_2^2 + z_3^2 + tz_3 = 0\} &= \left\{z_1^2 + z_2^2 + (z_3 + t/2)^2 - \frac{t^2}{4} = 0\right\}\\ &\cong \left\{z_1^2 + z_2^2 + \tilde{z}_3^2 - \frac{t^2}{4} = 0\right\}.\end{aligned}$$

So by Definition 2.11 and (2.32), we see that the order of the deformation is equal to 2 and the weight of the deformation is equal to -2.

Finally, we briefly recall Pinkham's results on deformation of isolated singularities with $\mathbb{C}^*$-actions. We state the results in our setting of affine cones.

THEOREM A.13 (see [27], [28])
(1) *There exists a formal versal $\mathbb{C}^*$-equivariant deformation $\mathscr{C} \to V$ of C.*
(2) *Let $\mathscr{Y} \to T$ be any formal $\mathbb{C}^*$-equivariant deformation of X. Then there exists a $\mathbb{C}^*$-equivariant morphism $\phi : T \to V$ and a $\mathbb{C}^*$-equivariant isomorphism of the deformation $\mathscr{Y} \to T$ with the pullback $\mathscr{X} \times_V T \to T$.*

Let t_j be homogeneous generators of the maximal ideal of weigh $d(t_j)$. Let J^- be the ideal in $\mathscr{O}_V$ generated by $\{t_j; d(t_j) < 0\}$. Let V^- be the subvariety defined by J^-.

THEOREM A.14 ([28, Theorem 2.9])
The deformation $\mathscr{C}^- \to V^-$ of C extends to a proper flat family $\overline{\mathscr{C}}^- \to V^-$ of deformations of $\bar{C}$. Moreover, there is an isomorphism $\overline{\mathscr{C}} - \mathscr{C} \cong D_\infty \times V^-$, and $\overline{\mathscr{C}}^- \to V^-$ is a locally trivial deformation near D_∞.

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References

[1] M. ABATE, F. BRACCI, and F. TOVENA, *Embeddings of submanifolds and normal bundles*, Adv. Math. **220** (2009), no. 2, 620–656. MR 2466428. DOI 10.1016/j.aim.2008.10.001. *(1398, 1399, 1401, 1423, 1440, 1474, 1475, 1476, 1477)*

[2] A. ANDREOTTI and H. GRAUERT, *Théorème de finitude pour la cohomologie des espaces complexes*, Bull. Soc. Math. France **90** (1962), 193–259. MR 0150342. *(1420)*

[3] C. AREZZO, A. DELLA VEDOVA, and G. LA NAVE, *On the curvature of conic Kähler-Einstein metrics*, J. Geom. Anal. **28** (2018), no. 1, 265–283. MR 3745858. DOI 10.1007/s12220-017-9819-y. *(1398)*

[4] M. ARTIN, *Lectures on deformation of singularities*, preprint, 1976, http://www.math.tifr.res.in/publ/ln/tifr54.pdf. *(1416, 1419, 1439, 1451, 1478, 1479, 1481)*

[5] S. BANDO and R. KOBAYASHI, "Ricci-flat Kähler metrics on affine algebraic manifolds" in *Geometry and Analysis on Manifolds (Katata/Kyoto, 1987)*, Lecture Notes in Math. **1339**, Springer, Berlin, 1988, 20–31. MR 0961470. DOI 10.1007/BFb0083044. *(1449)*

[6] C. CAMACHO and H. MOVASATI, *Neighborhoods of Analytic Varieties*, Monogr. IMCA **35**, Instituto de Matemática y Ciencias Afines IMCA, Lima, 2003. MR 2010707. *(1398)*

[7] C. CAMACHO, H. MOVASATI, and P. SAD, *Fibered neighborhoods of curves in surfaces*, J. Geom. Anal. **13** (2003), no. 1, 55–66. MR 1967036. DOI 10.1007/BF02930996. *(1398)*

[8] J. CHEEGER and G. TIAN, *On the cone structure at infinity of Ricci flat manifolds with Euclidean volume growth and quadratic curvature decay*, Invent. Math. **118** (1994), no. 3, 493–571. MR 1296356. DOI 10.1007/BF01231543. *(1449)*

[9] S.-S. CHERN, *An elementary proof of the existence of isothermal parameters on a surface*, Proc. Amer. Math. Soc. **6** (1955), no. 5, 771–782. MR 0074856. DOI 10.2307/2032933. *(1458, 1463, 1465)*

[10] R. J. CONLON and H.-J. HEIN, *Asymptotically conical Calabi–Yau manifolds, I*, Duke Math. J. **162** (2013), no. 15, 2855–2902. MR 3161306. DOI 10.1215/00127094-2382452. *(1400, 1401, 1445, 1449, 1452, 1453)*

[11] ———, *Asymptotically conical Calabi–Yau metrics on quasi-projective varieties*, preprint, arXiv:1301.5312v3 [math.DG]. *(1398, 1400, 1449)*

[12] ———, *Asymptotically conical Calabi–Yau manifolds, III*, preprint, arXiv:1405.7140v3 [math.DG]. *(1400, 1402, 1449, 1456, 1461)*

[13] W. FULTON, *Intersection Theory*, 2nd ed., Ergeb. Math. Grenzgeb. (3) **2**, Springer, Berlin, 1998. MR 1644323. DOI 10.1007/978-1-4612-1700-8. *(1423, 1434, 1454)*

[14] H. GRAUERT, *Über Modifikationen und exzeptionelle analytische Mengen*, Math. Ann. **146** (1962), 331–368. MR 0137127. DOI 10.1007/BF01441136. *(1398, 1399, 1473)*

[15] G.-M. GREUEL, C. LOSSEN, and E. SHUSTIN, *Introduction to Singularities and Deformations*, Springer Monogr. Math., Springer, Berlin, 2007. MR 2290112. *(1416, 1419, 1477, 1481)*

[16] P. A. GRIFFITHS, *The extension problem in complex analysis, II: Embeddings with positive normal bundle*, Amer. J. Math. **88** (1966), no. 2, 366–446. MR 0206980. DOI 10.2307/2373200. *(1398)*

[17] R. HARTSHORNE, *Deformation Theory*, Grad. Texts in Math. **257**, Springer, New York, 2010. MR 2583634. DOI 10.1007/978-1-4419-1596-2. *(1416)*

[18] M. HASKINS, H.-J. HEIN, and J. NORDSTRÖM, *Asymptotically cylindrical Calabi–Yau manifolds*, J. Differential Geometry **101** (2015), no. 2, 213–265. MR 3399097. *(1398, 1401, 1456)*

[19] C. D. HILL and M. TAYLOR, *Integrability of rough almost complex structures*, J. Geom. Anal. **13** (2003), no. 1, 163–172. MR 1967042. DOI 10.1007/BF02931002. *(1456)*

[20] R. A. JOHNSON, *Representations of compact groups on topological vector spaces: Some remarks*, Proc. Amer. Math. Soc. **61** (1976), no. 1, 131–136. MR 0430144. DOI 10.2307/2041678. *(1420, 1428, 1429)*

[21] A. KAS and M. SCHLESSINGER, *On the versal deformation of a complex space with an isolated singularity*, Math. Ann. **196** (1972), no. 1, 23–29. MR 0294701. DOI 10.1007/BF01419428. *(1479, 1480)*

[22] K. KODAIRA, *Complex Manifolds and Deformation of Complex Structures*, with appendix "Elliptic partial differential operators on a manifold" by D. Fujiwara, Grundlehren Math. Wiss. **283**, Springer, New York, 1986. MR 0815922. DOI 10.1007/978-1-4613-8590-5. *(1403, 1404)*

[23] B. MALGRANGE, "Sur l'intégrabilité des structures presque-complexes" in *Symposia Mathematica, Vol. II (INDAM, Rome, 1968)*, Academic Press, London, 289–296, 1969. MR 0253383. *(1456)*

[24] A. NEWLANDER and L. NIRENBERG, *Complex analytic coordinates in almost complex manifolds*, Ann. of Math. (2) **65** (1957), no. 3, 391–404. MR 0088770. DOI 10.2307/1970051. *(1455, 1457, 1458, 1459, 1461, 1466, 1467, 1468, 1469, 1470, 1471)*

[25] A. NIJENHUIS and W. B. WOOLF, *Some integration problems in almost-complex and complex manifolds*, Ann. of Math. (2) **77** (1963), no. 3, 424–489. MR 0149505. DOI 10.2307/1970126. *(1456, 1466, 1471)*

[26] F. PACARD and T. RIVIÈRE, *Linear and Nonlinear Aspects of Vortices: The Ginzburg-Landau Model*, Progr. Nonlinear Differential Equations Appl. **39**, Birkhäuser Boston, Boston, 2000. MR 1763040. DOI 10.1007/978-1-4612-1386-4. *(1463, 1465)*

[27] H. PINKHAM, *Deformations of algebraic varieties with G_m action*, Astérisque **20**, Soc. Math. France, Paris, 1974. MR 0376672. *(1450, 1480)*

[28] ———, *Deformations of normal surface singularities with* $\mathbf{C}^*$ *action*, Math. Ann. **232** (1978), no. 1, 65–84. MR 0498543. DOI 10.1007/BF01420623. *(1480)*

[29] J. ROSS and R. THOMAS, *An obstruction to the existence of constant scalar curvature Kähler metrics*, J. Differential Geom. **72** (2006), no. 3, 429–466. MR 2219940. *(1435)*

[30] M. SCHLESSINGER, *Rigidity of quotient singularities*, Invent. Math. **14** (1971), no. 1, 17–26. MR 0292830. DOI 10.1007/BF01418741. *(1478)*

[31] ———, *On rigid singularities*, Rice Univ. Stud. **59** (1973), no. 1, 147–162. MR 0344519. *(1419, 1451, 1477, 1478)*

[32] G. TIAN and S.-T. YAU, *Complete Kähler manifolds with zero Ricci curvature, II*, Invent. Math. **106** (1991), no. 1, 27–60. MR 1123371. DOI 10.1007/BF01243902. *(1398, 1448)*

[33] C. VAN COEVERING, *A construction of complete Ricci-flat Kähler manifolds*, preprint, arXiv:0803.0112v5 [math.DG]. *(1398)*

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