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Remarks on logarithmic K-stability

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We introduce the logarithmic version of K-stability by generalizing Donaldson's algebraic formulation of K-stability.

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1. Introduction

1.1. *Log-K-stability*

Let (X, J) be a Fano manifold, that is, K_X^{-1} is ample. A basic problem in Kähler geometry is to determine whether (X, J) has a Kähler–Einstein metric (see [22]). The existence problem of Kähler–Einstein metric is a special case of the existence problem of constant scalar curvature Kähler (cscK) metric. For the latter, we fix an ample line bundle L on (X, J). We have the following folklore conjecture. For the definition of K-stability, (see [22, 5]) or Definition 4.

Conjecture 1 (Yau–Tian–Donaldson conjecture, [22, 5]). *There is a smooth constant scalar curvature Kähler metric in* $2\pi c_1(L)$ *on* (X, J) *if and only if* (X, J, L) *is K-polystable.*

The first purpose of this note is to introduce and discuss the logarithmic version of K-stability. Before introducing log-K-stability (Definition 4 in Sec. 4), we will recall the log-Futaki-invariant introduced by Donaldson [6] and then "integrate" it to get the log-K-energy. These generalize the important elements in theory of smooth Kähler–Einstein metrics to the conical setting and will play important roles in [10].

1.2. *Calculation of log-Futaki-invariant in the toric case*

One way to attack the Kähler–Einstein problem is to use continuity method. Fix a reference Kähler metric $\omega \in 2\pi c_1(X)$. Its Ricci curvature $\mathrm{Ric}(\omega)$ also lies in $2\pi c_1(X)$. Consider the following family of equations:

$$\mathrm{Ric}(\omega_\phi) = t\omega_\phi + (1-t)\omega. \tag{$(*)_t$}$$

By Yau's theorem [25], we can always solve $(*)_t$ for $t = 0$. If we could solve $(*)_t$ for $t = 1$, we would get Kähler–Einstein metric. However, it was first showed by Tian [19] that it is not possible to solve $(*)_t$ on certain Fano manifold for t sufficiently close to 1. Equivalently, for such a Fano manifold, there is some $t_0 < 1$, such that there is no Kähler metric ω in $2\pi c_1(X)$ which can have $\mathrm{Ric}(\omega) \geq t_0\omega$. Let $R(X) = \sup\{t : (*)_t \text{ is solvable}\}$. Székelyhidi proved the following.

Proposition 1 ([17]). *$R(X) = \sup\{t : \exists$ a Kähler metric $\omega \in 2\pi c_1(X)$ such that $\mathrm{Ric}(\omega) > t\omega\}$.*

In particular, $R(X)$ is independent of reference metric ω.

There is another continuity method we can use. Let $Y \in |-K_X|$ be a general element; then Y is a smooth Calabi–Yau hypersurface. The Kähler–Einstein metric with cone singularity along Y of cone angle $2\pi\beta$ is a solution to the following distributional equation

$$\mathrm{Ric}(\omega) = \beta\omega + 2\pi(1-\beta)\{Y\}. \tag{$(**)_\beta$}$$

Conjecture 2 (Donaldson [6]). *There is a cone-singularity solution ω_β to $(**)_\beta$ for any parameter $\beta \in (0, R(X))$. If $R(X) < 1$, there is no solution for parameter $\beta \in (R(X), 1)$.*

We will prove the following result.

Theorem 1. *Let $X_\triangle$ be a toric Fano variety with a $(\mathbb{C}^*)^n$-action. Let Y be a general hyperplane section of $(X_\triangle, -K_{X_\triangle})$. When $\beta < R(X_\triangle)$, $(X_\triangle, (1-\beta)Y)$ is log-K-stable along any 1-parameter subgroup in $(\mathbb{C}^*)^n$. When $\beta = R(X_\triangle)$, $(X_\triangle, (1-\beta)Y)$ is semi-log-K-stable along any 1-parameter subgroup in $(\mathbb{C}^*)^n$ and there is a 1-parameter subgroup in $(\mathbb{C}^*)^n$ which has vanishing log-Futaki invariant. When $\beta > R(X_\triangle), (X_\triangle, (1-\beta)Y)$ is not log-K-stable.*

This explains and generalizes slightly the calculation in [6] and gives some evidence for the Conjecture 2 (combined with Conjecture 3). We will prove the above result by calculating $R(X_\triangle)$ and log-Futaki invariant explicitly. The main formula for log-Futaki invariant is (18).

We end this introduction by stating the result in [9] where $R(X_\triangle)$ was calculated. A toric Fano manifold $X_\triangle$ is determined by a reflexive lattice polytope $\triangle$. (For details on toric manifolds, see [15].) Any such polytope $\triangle$ contains the origin $O \in \mathbb{R}^n$. We denote the barycenter of $\triangle$ by P_c. If $P_c \neq O$, the ray

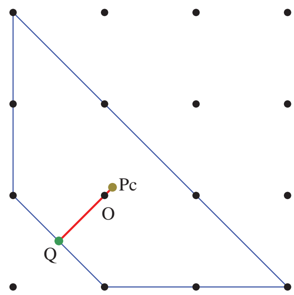


Fig. 1. $Bl_p\mathbb{P}^2$.

$P_c + \mathbb{R}_{\geq 0} \cdot \overrightarrow{P_cO}$ intersects the boundary $\partial\triangle$ at point Q. Then we have the following formula for $R(X_\triangle)$.

Theorem 2 ([9]). *If $P_c \neq O$, then*

$$R(X_\triangle) = \frac{|\overline{OQ}|}{|\overline{P_cQ}|}.$$

Here $|\overline{OQ}|$, $|\overline{P_cQ}|$ are lengths of line segments $\overline{OQ}$, $\overline{P_cQ}$. In other words, $Q = -\frac{R(X_\triangle)}{1-R(X_\triangle)}P_c \in \partial\triangle$.

Figure 1 is the example $Bl_p\mathbb{P}^2$, which is $\mathbb{P}^2$ blown up one point. $R(Bl_p\mathbb{P}^2) = 6/7$ (see [17, 9]). Note that if $P_c = O$, Wang–Zhu [24] already proved that there is Kähler–Einstein metric on $X_\triangle$ and so $R(X_\triangle) = 1$.

2. Log-Futaki Invariant

In this section, we recall Donaldson's definition of log-Futaki invariant (4). Let (X, L) be a polarized projective variety and D be a normal crossing divisor:

$$D = \sum_{i=1}^{r} \alpha_i D_i$$

with $\alpha_i \in (0, 1)$. Here D_i are different smooth irreducible divisors. From now on, we fix a Hermitian metric $\|\cdot\|_i = h_i$ and defining holomorphic section s_i of the line bundle $[D_i]$.

Assume $\omega \in 2\pi c_1(L)$ is a smooth Kähler form. We define

$$\overline{\mathcal{P}}(\omega) = \{\omega_\phi := \omega + \sqrt{-1}\partial\bar{\partial}\phi; \phi \in L^\infty(X) \cap C^\infty(X\backslash D) \text{ such that } \omega + \sqrt{-1}\partial\bar{\partial}\phi \geq 0\}.$$

Around any point $p \in X$, we can find local coordinate chart $\{U_p, \{z_j\}_{1\leq j\leq n}\}$, such that D is locally defined by

$$D \cap U_p = \bigcup_{i=1}^{r_p} \alpha_i\{z_i = 0\},$$

where $r_p = \sharp\{i; p \in D_i\}$.

Definition 1. We say that $\hat{\omega} \in \overline{\mathcal{P}}(\omega)$ is a conical Kähler metric on (X, D), if around p, $\hat{\omega}$ is quasi-isometric to the model metric

$$\sqrt{-1}\left(\sum_{i=1}^{r_p} \frac{dz_i \wedge d\bar{z}_i}{|z_i|^{2\alpha_i}} + \sum_{j=r_p+1}^{n} dz_j \wedge d\bar{z}_j\right)$$
$$= \sqrt{-1}\partial\bar{\partial}\left(\sum_{i=1}^{r_p} \frac{|z_i|^{2(1-\alpha_i)}}{(1-\alpha_i)^2} + \sum_{j=r_p+1}^{n} |z_j|^2\right). \tag{1}$$

We will simply say that $\hat{\omega}$ is a conical metric if it is clear what D is.

Geometrically, this means that the Riemannian metric determined by ω has conical singularity along each D_i of cone angle $2\pi(1-\alpha_i)$.

Remark 1. Construction of Kähler–Einstein metrics with conical singularities was proposed long time ago by Tian, see [21] in which he used such metrics to prove inequalities of Chern numbers in algebraic geometry.

One consequence of this definition is that globally the volume form has the form

$$\hat{\omega}^n = \frac{\Omega}{\prod_{i=1}^{r} \|s_i\|_i^{2\alpha_i}},$$

where Ω is a bounded volume form, which is smooth away from D. Note that any volume form Ω determines a Hermitian metric on K_X^{-1} in the following way. Choose any local holomorphic coordinates $z = \{z^i\}$ and denote $dz = dz^1 \wedge \cdots \wedge dz^n$. Then the Hermitian metric on K_X^{-1} is

$$\|\partial_z\|_\Omega^2 = \frac{\Omega}{\sqrt{-1}dz \wedge d\bar{z}}.$$

Let $\mathrm{Ric}(\Omega)$ denote the curvature of the L^∞-Hermitian metric on K_X^{-1} determined by Ω. Then, by abuse of notation,

$$\mathrm{Ric}(\hat{\omega}) = \mathrm{Ric}(\hat{\omega}^n) = \mathrm{Ric}(\Omega) + \sqrt{-1}\sum_{i=1}^{r} \alpha_i \partial\bar{\partial} \log \|s_i\|_i^2$$
$$= \mathrm{Ric}(\Omega) - 2\pi \sum_{i=1}^{r} \alpha_i c_1([D_i], h_i) + 2\pi \sum_{i=1}^{r} \alpha_i \{D_i\}$$
$$= \mathrm{Ric}(\Omega) - 2\pi c_1([D], h) + 2\pi\{D\}, \tag{2}$$

where $h = \bigotimes_{i=1}^r h_i^{\alpha_i}$ and $s = \bigotimes_{i=1}^r s_i^{\alpha_i}$ are Hermitian metric and defining holomorphic section of the $\mathbb{R}$-line bundle $[D] = \bigotimes_{i=1}^r [D_i]^{\alpha_i}$. Here we used the Poincáre–Lelong identity:

$$\frac{\sqrt{-1}}{2\pi}\partial\bar{\partial} \log \|s_i\|_i^2 = -c_1([D_i], h_i) + \{D_i\},$$

where $\{D_i\}$ is the current of integration along the divisor D_i.

The scalar curvature of $\hat{\omega}$ on its smooth locus $X \backslash D$ is

$$S(\hat{\omega}) = \hat{g}^{i\bar{j}}\hat{R}_{i\bar{j}} = \frac{n\mathrm{Ric}(\hat{\omega}) \wedge \hat{\omega}^{n-1}}{\hat{\omega}^n} = \frac{n(\mathrm{Ric}(\Omega) - 2\pi c_1([D], h)) \wedge \hat{\omega}^{n-1}}{\hat{\omega}^n}.$$

So if $S(\hat{\omega})$ is constant, then the constant only depends on cohomological classes by the identity:

$$\begin{aligned} n\mu_1 &:= \frac{n(c_1(X) - c_1([D])) \wedge [c_1(L)]^{n-1}}{c_1(L)^n} \\ &= \frac{-n(K_X + D) \cdot L^{n-1}}{L^n} = n\mu - 2\pi\frac{\mathrm{Vol}(D)}{\mathrm{Vol}(X)}. \end{aligned} \tag{3}$$

Here

$$n\mu = \frac{nc_1(X) \cdot c_1(L)^{n-1}}{c_1(L)^n} = \frac{-nK_X \cdot L^{n-1}}{L^n}$$

is the average scalar curvature for smooth Kähler form in $2\pi c_1(L)$, and

$$\mathrm{Vol}(D) = \int_D \frac{\omega^{n-1}}{(n-1)!} = (2\pi)^{n-1}\frac{L^{n-1} \cdot D}{(n-1)!}, \quad \mathrm{Vol}(X) = \int_X \frac{\omega^n}{n!} = (2\pi)^n\frac{L^n}{n!}.$$

Now assume $\mathbb{C}^*$ acts on (X, L) and v is the generating holomorphic vector field. Recall that the ordinary Futaki–Calabi invariant (see [7, 2]) is defined as

$$\mathrm{Fut}(2\pi c_1(L))(v) = -\int_X \theta_v(S(\omega) - n\mu)\frac{\omega^n}{n!},$$

where θ_v satisfies $\iota_v\omega = \sqrt{-1}\bar{\partial}\theta_v$.

From now on, assume $\hat{\omega}_\infty \in \overline{\mathcal{P}}(\omega)$ is a conical metric and satisfies $S(\hat{\omega}_\infty) = n\mu_1$. Also assume D is preserved by the $\mathbb{C}^*$-action. Let us calculate the ordinary Futaki invariant using the conical metric $\hat{\omega}_\infty$. Let $\hat{\theta}_v = \hat{\theta}(\hat{\omega}_\infty, v)$. Then near $p \in D$, $v \sim \sum_{i=1}^{r_p} c_i z_i \partial_{z_i} + \tilde{v}$ with $\tilde{v} = o(|z_1| + \cdots + |z_{r_p}|)$ holomorphic; $\hat{\theta}_v \sim O(\sum_{i=1}^{r_p} |z_i|^{2(1-\alpha_i)}) + g(z_{r_p+1}, \ldots, z_n)$.

We then make use of the distributional identity (2) to get

$$\begin{aligned} \mathrm{Fut}(2\pi c_1(L))(v) &= -\int_X \hat{\theta}_v(n\mathrm{Ric}(\hat{\omega}_\infty) - n\mu\hat{\omega}_\infty) \wedge \frac{\hat{\omega}_\infty^{n-1}}{n!} \\ &= -\int_X \hat{\theta}_v[(n\mathrm{Ric}(\Omega) - n \cdot 2\pi c_1([D], h) - n\mu_1\hat{\omega}_\infty) + n \cdot 2\pi\{D\} \\ &\quad - (n\mu - n\mu_1)\hat{\omega}_\infty] \wedge \frac{\hat{\omega}_\infty^{n-1}}{n!} \\ &= -\int_X \hat{\theta}_v(S(\hat{\omega}_\infty) - n\mu_1)\frac{\hat{\omega}_\infty^n}{n!} - 2\pi\int_X \{D\}\hat{\theta}_v\frac{\hat{\omega}_\infty^{n-1}}{(n-1)!} \\ &\quad + (n\mu - n\mu_1)\int_X \hat{\theta}_v\frac{\hat{\omega}_\infty^n}{n!} \\ &= -2\pi\left(\int_D \hat{\theta}_v\frac{\hat{\omega}_\infty^{n-1}}{(n-1)!} - \frac{\mathrm{Vol}(D)}{\mathrm{Vol}(X)}\int_X \hat{\theta}_v\frac{\hat{\omega}_\infty^n}{n!}\right). \end{aligned}$$

Remark 2. For technical reasons, in the above calculation, we can assume $\hat{\omega}$ is the $C^{2,\gamma,\beta}$ conical metric as defined in [6] (it is straightforward to generalize the definition in [6] to the case of several components), where $\beta = \{\beta_i = 1 - \alpha_i\}_{1\le i\le r}$ and $\gamma = \{\gamma_i; \gamma_i < \beta_i^{-1} - 1\}_{1\le i\le r}$.

So we get

$$0 = \mathrm{Fut}(2\pi c_1(L))(v) + 2\pi \left(\int_D \hat{\theta}_v \frac{\hat{\omega}_\infty^{n-1}}{(n-1)!} - \frac{\mathrm{Vol}(D)}{\mathrm{Vol}(X)} \int_X \hat{\theta}_v \frac{\hat{\omega}_\infty^n}{n!} \right).$$

The two integrals in the above formula are integrations of singular equivariant forms. They are independent of the chosen Kähler metric in $\overline{\mathcal{P}}(\omega)$ with at worst conical singularities. To see this, we can approximate $\hat{\omega}$ by a sequence of smooth Kähler forms ω_i and reduce the invariance to the case of smooth equivariant Kähler forms. Or we can use the theory of equivariant forms in its distributional version. The point is that, if we write $\hat{\omega} = \omega + \sqrt{-1}\partial\bar{\partial}\phi$, then

$$\hat{\omega} + \hat{\theta} = \omega + \theta - (\bar{\partial} - \iota_v)\partial\phi$$

in the sense of distributions and the integration by parts work in the setting of distributions.

Remark 3. If we assume the metric to be polyhomogeneous in the sense as in [8], then we can even get rid of the tools of approximation or distribution.

In particular, we can choose the smooth Kähler metric ω, then we just discover the log-Futaki invariant defined by Donaldson.

Definition 2 ([6]).

$$\mathrm{Fut}(2\pi c_1(L), D)(v) = \mathrm{Fut}(2\pi c_1(L))(v) + 2\pi \left(\int_D \theta_v \frac{\omega^{n-1}}{(n-1)!} - \frac{\mathrm{Vol}(D)}{\mathrm{Vol}(X)} \int_X \theta_v \frac{\omega^n}{n!} \right). \tag{4}$$

Remark 4. This differs from the formula in [6] by a sign. Here we think of D as an analytic cycle with real coefficients and we extend the integral and volume function linearly under linear combination of cycles. So if we replace D by $(1-\beta)\triangle$, we have the same formula as that in [6].

3. Log-K-Energy and Berman's Formulation

Similar to the smooth case in [13], we can integrate along paths of Kähler potentials to get log-K-energy

$$\begin{aligned}\nu_{\omega,D}(\phi) = &- \int_0^1 dt \int_X (S(\omega_t) - \underline{S})\dot{\phi}\frac{\omega_t^n}{n!} + 2\pi \int_0^1 dt \int_D \dot{\phi}\frac{\omega_t^{n-1}}{(n-1)!} \\ &- 2\pi \frac{\mathrm{Vol}(D)}{\mathrm{Vol}(X)} \int_0^1 dt \int_X \dot{\phi}\frac{\omega_t^n}{n!}\end{aligned}$$

$$
\begin{aligned}
&= \nu_\omega(\phi) + \int_0^1 \int_X (\sqrt{-1}\partial\bar{\partial}\log\|s\|^2 + 2\pi c_1([D],h))\dot{\phi}\frac{\omega_t^{n-1}}{(n-1)!} \\
&\quad + 2\pi\frac{\mathrm{Vol}(D)}{\mathrm{Vol}(X)}F_\omega^0(\phi) \\
&= \nu_\omega(\phi) + 2\pi\frac{\mathrm{Vol}(D)}{\mathrm{Vol}(X)}F_\omega^0(\phi) + \mathcal{J}_\omega^{\chi_D}(\phi) + \int_X \log\|s_D\|^2\frac{\omega_\phi^n - \omega^n}{n!}, \qquad (5)
\end{aligned}
$$

where $\chi_D = 2\pi c_1([D],h)$ is a Chern curvature form. The functionals $F_\omega^0(\phi)$ and $\mathcal{J}_\omega^\chi(\phi)$ are defined as:

$$
F_\omega^0(\phi) = -\int_0^1 dt \int_X \dot{\phi}\frac{\omega_{\phi_t}^n}{n!}, \quad \mathcal{J}_\omega^\chi(\phi) = \int_0^1 dt \int_X \dot{\phi}\chi \wedge \frac{\omega_{\phi_t}^{n-1}}{(n-1)!}.
$$

Let us now focus on the Fano case as in the beginning of this paper. Equation $(**)_\beta$ is equivalent to the following singular complex Monge–Ampère equation:

$$
(\omega + \sqrt{-1}\partial\bar{\partial}\phi)^n = e^{-\beta\phi}\frac{\Omega_1}{\|s\|^{2(1-\beta)}}, \qquad (6)
$$

with $\Omega_1 = e^{h_\omega}\omega^n$ satisfying $\mathrm{Ric}(\Omega_1) = \omega$. Here s is a defining holomorphic section of $[Y]$. Also we have chosen the Hermitian metric $\|\cdot\|$ on the line bundle $[Y] = K_X^{-1}$ whose Chern curvature is ω. We have $D = (1-\beta)Y$. Since $[Y] = K_X^{-1}$, we can assume $\chi_D = (1-\beta)\omega$, $2\pi\mathrm{Vol}(D) = n(1-\beta)\mathrm{Vol}(X)$. Then (5) becomes

$$
\begin{aligned}
\nu_{\omega,D}(\omega_\phi) &= \nu_\omega(\omega_\phi) + (1-\beta)(nF_\omega^0(\phi) + \mathcal{J}_\omega^\omega(\phi)) + (1-\beta)\int_X \log\|s\|^2\frac{\omega_\phi^n - \omega^n}{n!} \\
&= \nu_\omega(\phi) + (1-\beta)(I_\omega - J_\omega) + (1-\beta)\int_X \log\|s\|^2\frac{\omega_\phi^n - \omega^n}{n!} \\
&= \int_X \log\frac{\omega_\phi^n}{\omega^n}\frac{\omega_\phi^n}{n!} - \beta(I_\omega - J_\omega) + (1-\beta)\int_X (\log\|s\|^2 - h_\omega)\frac{\omega_\phi^n - \omega^n}{n!}.
\end{aligned}
$$

Here we have used the well-known formula for K-energy (see [20]):

$$
\nu_\omega(\phi) = \int_X \log\frac{\omega_\phi^n}{\omega^n}\frac{\omega_\phi^n}{n!} - (I_\omega - J_\omega)(\phi) + \int_X h_\omega(\omega^n - \omega_\phi^n)/n!,
$$

where

$$
I_\omega(\phi) = \int_X \phi(\omega^n - \omega_\phi^n)/n!, \quad J_\omega(\phi) = F_\omega^0(\phi) + \int_X \phi\frac{\omega^n}{n!}.
$$

Also it is easy to verify that

$$
nF_\omega^0(\phi) + \mathcal{J}_\omega^\omega(\phi) = (I_\omega - J_\omega)(\phi) = -\left(\int_X \phi\frac{\omega_\phi^n}{n!} + F_\omega^0(\phi)\right).
$$

From above formula, we see that, in Fano case, the log-K-energy coincides with Berman's free energy associated with (6) (see [1]) up to a constant depending on the reference metric ω:

$$\nu_{\omega,D}(\omega_\phi) = \int_X \log \frac{\omega_\phi^n}{\Omega_1/\|s\|^{2(1-\beta)}} \frac{\omega_\phi^n}{n!} + \beta\left(\int_X \phi \frac{\omega_\phi^n}{n!} + F_\omega^0(\phi)\right) + (1-\beta)\int_X (h_\omega - \log\|s\|^2)\frac{\omega^n}{n!}. \tag{7}$$

From the above calculations, we immediately get the following.

Lemma 1. *If $\sigma(t)$ is a 1-parameter group of holomorphic transformations generated by holomorphic vector field v, then for any Kähler metric $\omega_1, \omega_2 \in 2\pi c_1(L)$, we have*

$$\frac{d}{dt}\nu_{\omega_1,D}(\sigma(t)^*\omega_2) = \mathrm{Fut}(2\pi c_1(L), D)(v).$$

Remark 5. Note that the formula (7) also appears in the recent work of [8] and is studied in more detail in [10] of its relation with log-K-stability defined below.

4. Log-K-Stability

We imitate the definition of K-stability to define log-K-stability. First we recall the definition of test configuration [5] or special degeneration [22] of a polarized projective variety (X, L).

Definition 3 ([22, 5]). A test configuration of (X, L), consists of

(1) a scheme $\mathcal{X}$ with a $\mathbb{C}^*$-action;
(2) a $\mathbb{C}^*$-equivariant line bundle $\mathcal{L} \to \mathcal{X}$;
(3) a flat $\mathbb{C}^*$-equivariant map $\pi : \mathcal{X} \to \mathbb{C}$, where $\mathbb{C}^*$ acts on $\mathbb{C}$ by multiplication in the standard way;

such that any fiber $X_t = \pi^{-1}(t)$ for $t \neq 0$ is isomorphic to X and (X, L) is isomorphic to $(X_t, \mathcal{L}|_{X_t})$.

Any test configuration can be equivariantly embedded into $\mathbb{P}^N \times \mathbb{C}$ where the $\mathbb{C}^*$-action on $\mathbb{P}^N$ is given by a 1-parameter subgroup of $\mathrm{SL}(N+1, \mathbb{C})$. If Y is any subvariety of X, the 1-parameter subgroup of $\mathrm{SL}(N+1, \mathbb{C})$ associated to any test configuration of (X, L) also induces a test configuration $(\mathcal{Y}, \mathcal{L}|_{\mathcal{Y}})$ of $(Y, L|_Y)$.

Let d_k, $\tilde{d}_k$ be the dimensions of $H^0(X, L^k)$, $H^0(Y, L|_Y^k)$, and w_k, $\tilde{w}_k$ be the weights of $\mathbb{C}^*$-action on $H^0(\mathcal{X}_0, \mathcal{L}|_{\mathcal{X}_0}^k)$, $H^0(\mathcal{Y}_0, \mathcal{L}|_{\mathcal{Y}_0}^k)$, respectively. Then by equivariant Riemann–Roch we have expansions:

$$w_k = a_0 k^{n+1} + a_1 k^n + O(k^{n-1}), \quad d_k = b_0 k^n + b_1 k^{n-1} + O(k^{n-2}),$$
$$\tilde{w}_k = \tilde{a}_0 k^n + O(k^{n-1}), \quad \tilde{d}_k = \tilde{b}_0 k^{n-1} + O(k^{n-2}).$$

If the central fiber $\mathcal{X}_0$ is smooth, we can use equivariant differential forms to calculate the coefficients by [5]. Let ω be a smooth Kähler form in $2\pi c_1(L)$, and

$\theta_v = 2\pi(\mathcal{L}_v - \nabla_v)$; then

$$a_0 = -\frac{1}{(2\pi)^n}\int_{\mathcal{X}_0}\theta_v\frac{\omega^n}{n!}, \quad a_1 = -\frac{1}{(2\pi)^n}\frac{1}{2}\int_{\mathcal{X}_0}\theta_v S(\omega)\frac{\omega^n}{n!}, \tag{8}$$

$$b_0 = \frac{1}{(2\pi)^n}\int_{\mathcal{X}_0}\frac{\omega^n}{n!} = \frac{1}{(2\pi)^n}\mathrm{Vol}(X), \quad b_1 = \frac{1}{(2\pi)^n}\frac{1}{2}\int_{\mathcal{X}_0}S(\omega)\frac{\omega^n}{n!}. \tag{9}$$

Similarly, for central fiber $\mathcal{Y}_0$, we have

$$\tilde{a}_0 = -\frac{1}{(2\pi)^{n-1}}\int_{\mathcal{Y}_0}\theta_v\frac{\omega^{n-1}}{(n-1)!}, \quad \tilde{b}_0 = \frac{1}{(2\pi)^{n-1}}\int_{\mathcal{Y}_0}\frac{\omega^{n-1}}{(n-1)!} = \frac{1}{(2\pi)^{n-1}}\mathrm{Vol}(\mathcal{Y}_0). \tag{10}$$

Remark 6. Note that the above formula can be naturally generalized to the case when Y is a cycle of varieties. If $Y = \sum_i^n \alpha_i Y_i$ for different irreducible subvarieties Y_i, then formally $H^0(Y, L|_Y^k) = \bigoplus_{i=1}^r H^0(Y_i, L|_{Y_i}^k)^{\bigoplus \alpha_i}$, and the $\mathbb{C}^*$-weight of $H^0(\mathcal{Y}_0, \mathcal{L}|_{\mathcal{Y}_0}^k)$ is define to be the sum of the $\alpha_i \cdot \mathbb{C}^*$-weight of $H^0((\mathcal{Y}_0)_i, \mathcal{L}|_{(\mathcal{Y}_0)_i}^k)$.

Remark 7. To see our convention for signs of coefficients and give an example, consider the case where $X = \mathbb{P}^1$, $L = \mathcal{O}_{\mathbb{P}^1}(k)$. $\mathbb{C}^*$ acts on $\mathbb{P}^1$ by multiplication: $t \cdot z = tz$. A general $D \in |L|$ consists of k points. As $t \to 0$, $t \cdot D \to k\{0\}$. D is the zero set of a general degree k homogeneous polynomial $P_k(z_0, z_1)$ and $k\{0\}$ is the zero set of z_1^k. $\mathbb{C}^*$ acts on $H^0(\mathbb{P}^1, \mathcal{O}(k))$ by $t \cdot z_0^i z_1^j = t^{-j} z_0^i z_1^j$ so that $\lim_{t\to 0}[t \cdot P_k(z_0, z_1)] = [z_1^k]$, where $[P_k] \in \mathbb{P}(H^0(\mathbb{P}^1, \mathcal{O}(k)))$. Take the Fubini–Study metric $\omega_{\mathrm{FS}} = \sqrt{-1}\partial\bar{\partial}\log(1+|z|^2) = \sqrt{-1}\frac{dz\wedge d\bar{z}}{(1+|z|^2)^2}$. Then $\theta_v = \frac{\partial \log(1+|z|^2)}{\partial \log |z|^2} = \frac{|z|^2}{1+|z|^2}$. So

$$-a_0 = \frac{1}{2\pi}\int_{\mathbb{P}^1}\theta_v\omega_{\mathrm{FS}} = \int_0^{+\infty}\frac{r^2}{(1+r^2)^3}2rdr = \frac{1}{2},$$

$$-a_1 = \frac{1}{2\pi}\frac{1}{2}\int_{\mathbb{P}^1}S(\omega_{\mathrm{FS}})\theta_v\omega_{\mathrm{FS}} = \frac{1}{2\pi}\int_{\mathbb{P}^1}\theta_v\omega_{\mathrm{FS}} = \frac{1}{2}.$$

On the other hand,

$$w_k = -(1 + \cdots + k) = -\frac{1}{2}k^2 - \frac{1}{2}k,$$

which gives exactly $a_0 = a_1 = -\frac{1}{2}$.

Comparing (4), (8)–(10), we can define the algebraic log-Futaki invariant of the given test configuration to be

$$\mathrm{Fut}(\mathcal{X}, \mathcal{Y}, \mathcal{L}) = \frac{2(a_1 b_0 - a_0 b_1)}{b_0} + \left(-\tilde{a}_0 + \frac{\tilde{b}_0}{b_0}a_0\right)$$

$$= \frac{(2a_1 - \tilde{a}_0)b_0 - a_0(2b_1 - \tilde{b}_0)}{b_0}. \tag{11}$$

Note that here we use the convention of notation in Remark 6.

Definition 4. • (X, Y, L) is log-K-polystable along the test configuration $(\mathcal{X}, \mathcal{L})$ if $\mathrm{Fut}(\mathcal{X}, \mathcal{Y}, \mathcal{L}) \le 0$, and equality holds if and only if the normalization $(\mathcal{X}^\nu, \mathcal{Y}^\nu, \mathcal{L}^\nu)$ is a product configuration.

- (X, Y, L) is semi-log-K-stable along $(\mathcal{X}, \mathcal{L})$ if $\mathrm{Fut}(\mathcal{X}, \mathcal{Y}, \mathcal{L}) \le 0$. Otherwise, it is unstable.
- (X, Y, L) is log-K-polystable (semi-log-K-stable) if, for any integer $r > 0$, (X, Y, L^r) is log-K-stable (semi-log-K-stable) along any normal test configuration of (X, Y, L^r).

Remark 8. When Y is empty, then definition of log-K-stability becomes the definition of K-stability [22, 5]. For the assumption of being normal in the definition see [11].

Remark 9. In applications, we sometimes meet the following situation. Let $\lambda(t)$: $\mathbb{C}^* \to \mathrm{SL}(N+1, \mathbb{C})$ be a 1-parameter subgroup. As $t \to \infty$, $\lambda(t)$ will move $X, Y \subset \mathbb{P}^N$ to the limit scheme $\mathcal{X}_0$, $\mathcal{Y}_0$. Then stability condition is equivalent to the other opposite sign condition $\mathrm{Fut}(\mathcal{X}_0, \mathcal{Y}_0, v) \ge 0$. This is of course related to the above definition by transformation $t \to t^{-1}$.

Example 1 (Orbifold in codimension one). Assume X is smooth and $D = \sum_{i=1}^r (1 - \frac{1}{n_i}) D_i$ is a normal crossing divisor, where $n_i > 0$ are integers. The orbifold Kähler metric on the orbifold (X, D) is also a conical Kähler metric. Orbifold behaves similarly as smooth variety, but in the calculation, we need to use orbifold canonical bundle $K_{\mathrm{orb}} = K_X + D$. For example, thinking L as an orbifold line bundle on X, then the orbifold Riemann–Roch gives that

$$\begin{aligned}\dim H^0_{\mathrm{orb}}((X, D), L) &= \frac{L^n}{n!} k^n + \frac{1}{2} \frac{-(K_X + D) \cdot L^n}{(n-1)!} k^{n-1} + O(k^{n-2}) \\ &= b_0 k^n + \frac{1}{2}(2b_1 - \tilde{b}_0) k^{n-1} + O(k^{n-2}).\end{aligned}$$

For the $\mathbb{C}^*$-weight of $H^0_{\mathrm{orb}}((X, D), L)$, we have the expansion:

$$w_k^{\mathrm{orb}} = a_0^{\mathrm{orb}} k^{n+1} + a_1^{\mathrm{orb}} k^n + O(k^{n-1}).$$

By orbifold equivariant Riemann–Roch, we have the formula:

$$a_0^{\mathrm{orb}} = \frac{1}{(2\pi)^n} \int_X \hat{\theta}_v \frac{\hat{\omega}^n}{n!} = \frac{1}{(2\pi)^n} \int_X \theta_v \frac{\omega^n}{n!} = a_0, \quad a_1^{\mathrm{orb}} = \frac{1}{(2\pi)^n} \int_X \hat{\theta}_v S(\hat{\omega}) \frac{\hat{\omega}^n}{n!}.$$

To calculate the second coefficient a_1^{orb}, we choose an orbifold metric $\hat{\omega} \in 2\pi c_1(X)$. Then by (8):

$$\begin{aligned}a_1 &= -\frac{1}{(2\pi)^n} \frac{1}{2} \int_X \hat{\theta}_v n \mathrm{Ric}(\hat{\omega}) \wedge \frac{\hat{\omega}^{n-1}}{n!} \\ &= -\frac{1}{(2\pi)^n} \frac{1}{2} \int_X \hat{\theta}_v n (\mathrm{Ric}(\Omega) - 2\pi c_1([D], h) + 2\pi \{D\}) \wedge \frac{\hat{\omega}^{n-1}}{n!}\end{aligned}$$

$$= -\frac{1}{(2\pi)^n}\frac{1}{2}\int_X \hat{\theta}_v S(\hat{\omega})\frac{\hat{\omega}^n}{n!} - \frac{1}{(2\pi)^{n-1}}\frac{1}{2}\int_D \hat{\theta}_v \frac{\hat{\omega}^{n-1}}{(n-1)!}$$
$$= a_1^{\mathrm{orb}} - \frac{1}{(2\pi)^{n-1}}\frac{1}{2}\int_D \theta_v \frac{\omega^{n-1}}{(n-1)!} = a_1^{\mathrm{orb}} + \frac{1}{2}\tilde{a}_0.$$

So we have

$$a_1^{\mathrm{orb}} = \frac{1}{2}(2a_1 - \tilde{a}_0). \tag{12}$$

Comparing (11), we see that the log-Futaki invariant recovers the orbifold Futaki invariant, and similarly log-K-stability recovers orbifold K-stability. Orbifold Futaki and orbifold K-stability were studied in detail by Ross–Thomas [16].

Example 2. $X = \mathbb{P}^1$, $L = K_{\mathbb{P}^1}^{-1} = \mathcal{O}_{\mathbb{P}^1}(2)$, $Y = \sum_{i=1}^r \alpha_i p_i$. Fix $i \in \{1, \ldots, r\}$ and we choose the coordinate z on $\mathbb{P}^1$, such that $z(p_i) = 0$. Then consider the holomorphic vector field $v = z\partial_z$. v generates the 1-parameter subgroup $\lambda(t) : \lambda(t)\cdot z = t\cdot z$. As $t \to \infty$, $\lambda(t)$ degenerates (X, Y) into the pair $(\mathbb{P}^1, \alpha_i\{0\} + \sum_{j\neq i}\alpha_j\{\infty\})$. We take $\theta_v = \frac{-|z|^{-2}+|z|^2}{|z|^{-2}+1+|z|^2}$. Then it is easy to get the log-Futaki invariant of the degeneration determined by λ:

$$\mathrm{Fut}\left(\mathbb{P}^1, \sum_{i=1}^r \alpha_i p_i, \mathcal{O}_{\mathbb{P}^1}(2)\right)(\lambda) = \sum_{j\neq i}\alpha_j - \alpha_i.$$

If $(\mathbb{P}^1, \sum_{i=1}^r \alpha_i p_i)$ is log-K-stable, by Remark 9, we have

$$\sum_{j\neq i}\alpha_j - \alpha_i > 0. \tag{13}$$

Equivalently, if we let $t \to 0$, we get $\alpha_i - \sum_{j\neq i}\alpha_j < 0$ from log-K-stability. On the other hand, one considers the problem of constructing singular Riemannian metric g of constant scalar curvature on $\mathbb{P}^1$ which has conical angle $2\pi(1-\alpha_i)$ at p_i and is smooth elsewhere. Assume $p_i \neq \infty$ for any $i = 1, \ldots, r$. Under conformal coordinate z of $\mathbb{C} \subset \mathbb{P}^1$, $g = e^{2u}|dz|^2$. u is a smooth function in the punctured complex plane $\mathbb{C} - \{p_1, \ldots, p_r\}$ such that near each p_i, $u(z) = -2\alpha_i \log|z - p_i| +$ a continuous function, where $\alpha_i \in (0, 1)$ and $u = -2\log|z| +$ a continuous function near infinity. We call such function is of conical type. The condition of constant scalar curvature corresponds to the following Liouville equations: (1) $\Delta u = -e^{2u}$; (2) $\Delta u = 0$; (3) $\Delta u = e^{2u}$; which correspond to scalar curvature equals $1, 0, -1$ case respectively. For such equations, we have the following theorem due to many people's work.

Theorem 3 ([23, 14, 4, 3, 12]). *We have the following existence and uniqueness results.*

(i) *For Eq.* (1), *it has a solution of conical type if and only if* (a) $\sum_{i=1}^r \alpha_i < 2$, *and* (b) $\sum_{j\neq i}\alpha_j - \alpha_i > 0$, *for all* $i = 1, \ldots, n$.

(ii) *For Eq.* (2), *it has a solution of conical type if and only if* (a) $\sum_{i=1}^{r}\alpha_i = 2$. *In this case,* (a) *implies the condition* (b) $\sum_{j\neq i}\alpha_j - \alpha_i > 0$, *for all* $i = 1,\ldots,r$.

(iii) *For Eq.* (3), *it has a solution of conical type if and only if* (a) $\sum_{i=1}^{r}\alpha_i > 2$. *Again in this case,* (a) *implies the condition* (b) $\sum_{j\neq i}\alpha_j - \alpha_i > 0$, *for all* $i = 1,\ldots,r$.

Moreover, the above solutions are all unique (*up to biholomorphism*).

Note that $\deg(-(K_{\mathbb{P}^1} + \sum_{i=1}^{r}\alpha_i p_i)) = 2 - \sum_{i=1}^{r}\alpha_i$, *so by* (3), *conditions* (a) *in above theorem correspond to the cohomological conditions for the scalar curvature to be positive, zero, negative respectively, while the condition* (b) *is the same as* (13). *So by the above theorem, if* $(\mathbb{P}^1, \sum_{i=1}^{r}\alpha_i p_i)$ *is log*-K-*polystable, then there is a conical metric on* $(\mathbb{P}^1, \sum_{i=1}^{r}\alpha_i p_i)$ *with constant curvature whose sign is the same as that of* $2 - \sum_i \alpha_i$.

This example clearly suggests the following.

Conjecture 3 (Logarithmic version of Yau–Tian–Donaldson conjecture). *There is a constant scalar curvature conical Kähler metric on* (X, Y) *in* $2\pi c_1(L)$ *if and only if* $(X, Y; L)$ *is log*-K-*stable.*

5. Toric Fano Case

5.1. *Log-Futaki invariant for 1-parameter subgroup on toric Fano variety*

For a reflexive lattice polytope $\triangle$ in $\mathbb{R}^n = \Lambda \otimes_{\mathbb{Z}} \mathbb{R}$, we have a Fano toric manifold $(\mathbb{C}^*)^n \subset X_\triangle$ with a $(\mathbb{C}^*)^n$-action. In the following, we will sometimes just write X for $X_\triangle$ for simplicity. Let $(S^1)^n \subset (\mathbb{C}^*)^n$ be the standard real maximal torus. Let $\{z_i\}$ be the standard coordinate of the dense orbit $(\mathbb{C}^*)^n$, and $x_i = \log|z_i|^2$. We have the following lemma.

Lemma 2. *Any* $(S^1)^n$ *invariant Kähler metric* ω *on* X *has a potential* $u = u(x)$ *on* $(\mathbb{C}^*)^n$, *i.e.* $\omega = \sqrt{-1}\partial\bar{\partial}u$. u *is a proper convex function on* $\mathbb{R}^n$, *and satisfies the momentum map condition*:

$$Du(\mathbb{R}^n) = \triangle.$$

Note that

$$\frac{(\sqrt{-1}\partial\bar{\partial}u)^n/n!}{(\sqrt{-1})^n\frac{dz_1}{z_1}\wedge\frac{d\bar{z}_1}{\bar{z}_1}\wedge\cdots\wedge\frac{dz_n}{z_n}\wedge\frac{d\bar{z}_n}{\bar{z}_n}} = \det\left(\frac{\partial^2 u}{\partial x_i \partial x_j}\right). \tag{14}$$

Let $\{p_\alpha; \alpha = 1,\ldots,N\}$ be all the lattice points of $\triangle$. Each p_α corresponds to a holomorphic section $s_\alpha \in H^0(X_\triangle, K_{X_\triangle}^{-1})$. We can embed $X_\triangle$ into $\mathbb{P}^N$ using $\{s_\alpha\}$. Define u to be the potential on $(\mathbb{C}^*)^n$ for the pull back of Fubini–Study metric (i.e. $\sqrt{-1}\partial\bar{\partial}u = \omega_{\mathrm{FS}}$):

$$u = \log\left(\sum_{\alpha=1}^{N} e^{\langle p_\alpha, x\rangle}\right) + C. \tag{15}$$

C is some constant determined by normalization condition:

$$\int_{\mathbb{R}^n} e^{-u} dx = \mathrm{Vol}(\triangle) = \frac{1}{(2\pi)^n n!} \int_{X_\triangle} \omega^n = \frac{c_1(X_\triangle)^n}{n!}.$$

By the above normalization of u, it is easy to see that the Ricci potential h_ω satisfies:

$$e^{h_\omega} = \frac{|\cdot|^2_{FS}}{|\cdot|^2_{\omega^n}} = \frac{e^{-u}}{\dfrac{\omega^n}{\left((\sqrt{-1})^n \dfrac{dz_1}{z_1} \wedge \dfrac{d\bar{z}_1}{\bar{z}_1} \wedge \cdots \wedge \dfrac{dz_n}{z_n} \wedge \dfrac{d\bar{z}_n}{\bar{z}_n}\right)}}.$$

So we have

$$h_\omega = -\log\det(u_{ij}) - u. \tag{16}$$

Now let us calculate the log-Futaki invariant for any 1-parameter subgroup in $(\mathbb{C}^*)^n$. Each 1-parameter subgroup in $(\mathbb{C}^*)^n$ is determined by some $\lambda \in \mathbb{R}^n$ such that the generating holomorphic vector field is

$$v_\lambda = \sum_{i=1}^n \lambda_i z_i \frac{\partial}{\partial z_i}.$$

A general Calabi–Yau hypersurface $Y \in |-K_X|$ is a hyperplane section given by an equation:

$$s := \sum_{\alpha=1}^N b(p_\alpha) z^{p_\alpha} = 0.$$

By abuse of notation, we denote $\lambda(t)$ to be the 1-parameter subgroup generated by v_λ. Then

$$\lambda(t) \cdot s = \sum_{\alpha=1}^N b(p_\alpha) t^{-\langle p_\alpha, \lambda \rangle} z^{p_\alpha}. \tag{17}$$

Let $W(\lambda) = \max_{p\in\triangle} \langle p, \lambda \rangle$. Then $H_\lambda = \{p \in \mathbb{R}^n, \langle p, \lambda \rangle = W(\lambda)\}$ is a supporting plane of $\triangle$, and

$$\mathcal{F}_\lambda := \{p \in \triangle; \langle p, \lambda \rangle = W(\lambda)\} = H_\lambda \cap \triangle$$

is a face of $\triangle$. We have $\lim_{t\to 0}[s] = [s_0 := \sum_{p_\alpha \in \mathcal{F}_\lambda} b(p_\alpha) z^{p_\alpha}]$, and by (17), the $\mathbb{C}^*$-weight of s_0 is $-W(\lambda)$.

Proposition 2. *Let* $\mathrm{Fut}(X, (1-\beta)Y, K_X^{-1})(\lambda)$ *denote the log-Futaki invariant of the test configuration associated with the* 1*-parameter subgroup generated by* v_λ*. We have*

$$\mathrm{Fut}(X, (1-\beta)Y, K_X^{-1})(\lambda) = -(\beta\langle P_c, \lambda \rangle + (1-\beta)W(\lambda))\mathrm{Vol}(\triangle). \tag{18}$$

Proof. We will use the algebraic definition of log-Futaki invariant (11) to do the calculation. Note that (X, Y, K_X^{-1}) degenerates to (X, Y_0, K_X^{-1}) under λ. $Y_0 = \{s_0\}$

is a hyperplane section of X. Then

$$H^0(Y_0, K_X^{-1}|_{Y_0}^k) \cong \frac{H^0(X, K_X^{-k})}{(s_0 \otimes H^0(X, K_X^{-(k-1)}))}.$$

So we get $\tilde{w}_k = w_k - (w_{k-1} - W(\lambda)d_{k-1})$. Plugging the expansions, we get

$$\tilde{a}_0 = (n+1)a_0 + W(\lambda)b_0.$$

Note that $\tilde{b}_0 = nb_0 = n\mathrm{Vol}(\triangle)$, we have

$$-\tilde{a}_0 + \frac{\tilde{b}_0}{b_0}a_0 = -a_0 - W(\lambda)b_0,$$

where

$$-a_0 = \frac{1}{(2\pi)^n}\int_X \theta_v \frac{\omega^n}{n!} = \int_{\mathbb{R}^n} \sum_i \lambda_i u_i \det(u_{ij})dx = \int_\triangle \sum_i \lambda_i y_i dy = \mathrm{Vol}(\triangle)\langle P_c, \lambda\rangle.$$

By (16), the ordinary Futaki invariant is given by

$$\begin{aligned}\mathrm{Fut}(c_1(X))(v_\lambda) &= \frac{1}{(2\pi)^n}\int_X v(h_\omega)\frac{\omega^n}{n!} = -\int_{\mathbb{R}^n}\sum_{i=1}^n \lambda_i \frac{\partial u}{\partial x_i}\det(u_{ij})dx \\ &= -\int_\triangle \sum_i \lambda_i y_i dy = -\mathrm{Vol}(\triangle)\langle P_c, \lambda\rangle.\end{aligned}$$

Substituting these equations into (11), we get

$$\begin{aligned}&\mathrm{Fut}(X, (1-\beta)Y), K_X^{-1})(\lambda) \\ &\quad = -\mathrm{Vol}(\triangle)\langle P_c, \lambda\rangle + (1-\beta)(\mathrm{Vol}(\triangle)\langle P_c, \lambda\rangle - W(\lambda)\mathrm{Vol}(\triangle)) \\ &\quad = -(\beta\langle P_c, \lambda\rangle + (1-\beta)W(\lambda))\mathrm{Vol}(\triangle).\end{aligned}$$

□

Proof of Theorem 1. Note that for any $P_\lambda \in \mathcal{F}_\lambda \subset \partial\triangle$, $W(\lambda) = \langle P_\lambda, \lambda\rangle$. By Theorem 2, we have

$$\begin{aligned}\mathrm{Fut}(X, (1-\beta)Y, K_X^{-1})(\lambda) &= \left(\frac{\beta}{1-\beta}\frac{1-R(X)}{R(X)}\langle Q, \lambda\rangle - W(\lambda)\right)(1-\beta)\mathrm{Vol}(\triangle) \\ &= \langle Q_\beta - P_\lambda, \lambda\rangle,\end{aligned}$$

where $Q_\beta = \frac{\beta}{1-\beta}\frac{1-R(X)}{R(X)}Q$. Note that λ is an outward normal vector of H_λ. By convexity of $\triangle$, it is easy to see that (see the picture after Example 5.2)

- $\beta < R(X)$: $Q_\beta \in \triangle^\circ$. For any $\lambda \in \mathbb{R}^n$, $\langle Q_\beta - P_\lambda, \lambda\rangle < 0$.
- $\beta = R(X)$: $Q_\beta = Q \in \partial\triangle$. For any $\lambda \in \mathbb{R}^n$, $\langle Q_\beta - P_\lambda, \lambda\rangle \le 0$. Equality holds if and only if $\langle Q, \lambda\rangle = W(\lambda)$, i.e. H_λ is a supporting plane of $\triangle$ at point Q.
- $\beta > R(X)$: $Q_\beta \notin \overline{\triangle}$. There exists $\lambda \in \mathbb{R}^n$ such that $\langle Q_\beta - P_\lambda, \lambda\rangle > 0$. □

5.2. *Example*

(1) $X_\triangle = Bl_p\mathbb{P}^2$ (see Fig. 1 in Sec. 1), $P_c = \frac{1}{4}(\frac{1}{3}, \frac{1}{3})$, $Q = -6P_c \in \partial\triangle$, so $R(X) = \frac{6}{7}$. If we take $\lambda = \langle -1, -1\rangle$, then $W(\lambda) = 1$. So by (18)

$$\mathrm{Fut}(X, (1-\beta)Y, K_X^{-1})(\lambda) = \frac{2}{3}\beta - 4(1-\beta).$$

So $\mathrm{Fut}(X, (1-\beta)Y, K_X^{-1})(\lambda) \le 0$ if and only if $\beta \le \frac{6}{7}$, and equality holds exactly when $\beta = \frac{6}{7}$.

(2) $X_\triangle = Bl_{p,q}\mathbb{P}^2$, $P_c = \frac{2}{7}(-\frac{1}{3}, -\frac{1}{3})$, $Q = -\frac{21}{4}P_c \in \partial\triangle$, so $R(X_\triangle) = \frac{21}{25}$. If we take $\lambda_1 = \langle 1, 1\rangle$, then $W(\lambda_1) = 1$. By (18),

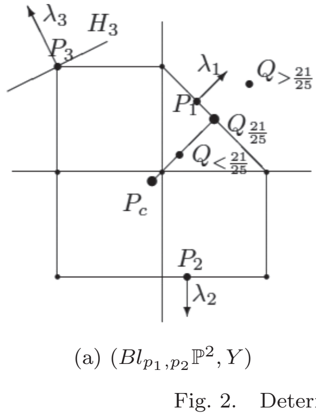
$$\mathrm{Fut}(X, (1-\beta)Y, K_X^{-1})(\lambda_1) = \frac{2}{3}\beta - \frac{7}{2}(1-\beta).$$

$\mathrm{Fut}(X, (1-\beta)Y, K_X^{-1})(\lambda_1) \le 0$ if and only if $\beta \le \frac{21}{25}$. This recovers Donaldson's calculation in [6]. If we take $\lambda_3 = \langle -1, 2\rangle$, then $W(\lambda_3) = \langle -1, 2\rangle \cdot \langle -1, 1\rangle = 3$. By (18)

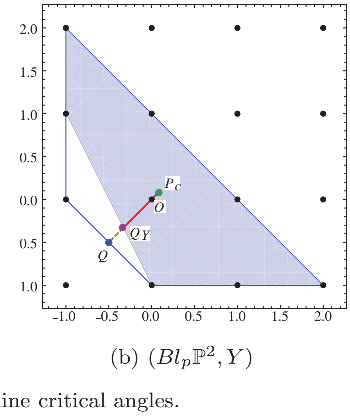
$$\mathrm{Fut}(X, (1-\beta)Y, K_X^{-1})(\lambda_3) = \frac{1}{3}\beta - \frac{21}{2}(1-\beta).$$

So $\mathrm{Fut}(X, (1-\beta)Y, K_X^{-1})(\lambda_3) \le 0$ if and only if $\beta \le \frac{63}{65}$ which means that $(X, (1-\beta)Y)$ is log-K-stable along λ_3 when $\beta \le \frac{21}{25} < \frac{63}{65}$. See Fig. 2(a) for picture.

(*The following paragraph is added in December* 2013 *to the original version arXiv*: 1104.0428.) From the above discussion, we see that the maximum value of β for which the torus action is log-K-stable can also be determined using two polytopes. One polytope is the polytope $\triangle_X$ corresponding to the toric Fano manifold.



(a) $(Bl_{p_1,p_2}\mathbb{P}^2, Y)$



(b) $(Bl_p\mathbb{P}^2, Y)$

Fig. 2. Determine critical angles.

The other polytope $\triangle_Y$ is the weight polytope of the divisor Y. Define

$$I(\triangle_X, \triangle_Y) = \left\{t \in (0,1); -\frac{t}{1-t}\overrightarrow{P_cO} \in \triangle_Y\right\}.$$

Then the pair $(X,(1-\beta)Y)$ is log-K-stable with respect to the given torus action if and only if $\beta \in I(\triangle_X, \triangle_Y)$. As before, let $\overrightarrow{P_cO}$ be the ray connecting P_c to O and assume $\partial\triangle_Y \cap \overrightarrow{P_cO} = Q_Y$. Then

$$\sup\{\beta; \beta \in I(\triangle_X, \triangle_Y)\} = \frac{|\overline{OQ_Y}|}{|\overline{P_cQ_Y}|} < \frac{|\overline{OQ}|}{|\overline{P_cQ}|} = R(X_\triangle).$$

Very recently Székelyhidi [18] disproved Donaldson's expectation in Conjecture 2. For example, for $Bl_p\mathbb{P}^2$, he observed that for any smooth divisor $D \in |-K_X|$, one can always choose a torus action such that $\triangle_Y \subsetneqq \triangle_X$ and so $I(\triangle_X, \triangle_Y) \subsetneqq (0, R(X))$ (see Fig. 2(b)).

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