

K-semistability is equivariant volume minimization

Chi Li

June 23, 2017

Abstract

This is a continuation to the paper [Li15a] in which a problem of minimizing normalized volumes over \mathbb{Q} -Gorenstein klt singularities was proposed. Here we consider its relation with K-semistability, which is an important concept in the study of Kähler-Einstein metrics on Fano varieties. In particular, we prove that for a \mathbb{Q} -Fano variety V , the K-semistability of $(V, -K_V)$ is equivalent to the condition that the normalized volume is minimized at the canonical valuation ord_V among all \mathbb{C}^* -invariant valuations on the cone associated to any positive Cartier multiple of $-K_V$. In this case, it's shown that ord_V is the unique minimizer among all \mathbb{C}^* -invariant quasi-monomial valuations. These results allow us to give characterizations of K-semistability by using equivariant volume minimization, and also by using inequalities involving divisorial valuations over V .

Contents

1	Introduction	2
2	Preliminaries	3
2.1	Normalized volumes of valuations	3
2.2	K-semistability and Ding-semistability	5
3	Statement of Main Results	7
4	K-semistable implies equivariant volume minimizing	9
4.1	A general volume formula	9
4.1.1	A summation formula	12
4.2	Case of \mathbb{C}^* -invariant valuations	14
4.2.1	Appendix: Interpolations in the \mathbb{C}^* -invariant case	16
5	Uniqueness among \mathbb{C}^*-invariant valuations	17
6	Equivariant volume minimization implies K-semistability	19
6.1	Valuation wt_β on \mathcal{X}_0	20
6.2	Valuation w_β on X	20
6.3	w_β versus wt_β	24
6.4	Completion of the proof of Theorem 6.1	29
6.5	An example of calculation	31
7	Proof of Theorem 3.7	32
8	Two examples of normalized volumes	34
9	Appendix: Convex geometric interpretation of volume formulas	36
9.1	Integration formula by Okounkov bodies and coconvex sets	36
9.2	Examples	42
10	Postscript Note:	44
11	Acknowledgment	45

1 Introduction

Valuation theory is a classical subject which has been studied in different fields of mathematics for more than a century. As a celebrated example of their applications in algebraic geometry, Zariski proved the resolution of surface singularities by proving a local uniformization theorem showing that every valuation of a surface can be resolved. Recently valuations have also become important in studying singularities of plurisubharmonic functions in complex analysis (see for example [BFJ08]). In this paper, we use the tool of valuation to study K-stability of Fano varieties.

The concept of K-stability was introduced by Tian [Tia97] in his study of Kähler-Einstein problem on any Fano manifold. It was introduced there to test the properness of so-called Mabuchi-energy. This energy is defined on the space of smooth Kähler potentials whose critical points are Kähler-Einstein metrics. Tian proved in [Tia97] that the properness of the Mabuchi-energy is equivalent to the existence of a Kähler-Einstein metric on a smooth Fano manifold without holomorphic vector field. The notion of K-stability was later generalized to a more algebraic formulation in [Don02]. By the work of many people on Yau-Tian-Donaldson conjecture spanning a long period of time (see in particular [Tia97], [Ber15] and [CDS15, Tia15]), we now know that, for a smooth Fano manifold V , the existence of Kähler-Einstein metric on V is equivalent to K-polystability of V . In [Li13], based on recent progresses on the subject, the author proved some equivalent characterizations of K-semistability. Roughly speaking, for smooth Fano manifolds, “K-semistable” is equivalent to “almost Kähler-Einstein” (see [Li13] for details). In particular, it was proved there that K-semistability is equivalent to the lower boundedness of the Mabuchi-energy.

Motivated by the work of [MSY08] in the study of Sasaki-Einstein metric, which is a metric structure closely related to the Kähler-Einstein metric, in [Li15a] the author proposed to understand K-semistability of Fano manifolds from the point view of volume minimizations. In this point of view, for any Fano manifold V , we consider the space of real valuations on the affine cone $X = C(V, -K_V)$ with center being the vertex o (we denote this space by $\text{Val}_{X,o}$) and study a normalized volume functional $\widehat{\text{vol}}$ on $\text{Val}_{X,o}$. Such space of valuations were studied in [JM10, BFFU13] with close relations to the theory of Berkovich spaces. The volume functional $\widehat{\text{vol}}(v)$ of a real valuation v is defined to be $A_X(v)^n \text{vol}(v)$, where $A_X(v)$ is the log discrepancy of v (see [JM10, BFFU13, Kol13]) and $\text{vol}(v)$ is the volume of v (see [ELS03, Laz96, BFJ12]). In [Li15a], we conjectured that the Fano manifold V is K-semistable if and only if $\widehat{\text{vol}}$ is minimized at the canonical valuations ord_V . Here we prove one direction of this conjecture, i.e. volume minimizing implies K-semistability. Indeed we prove that any \mathbb{Q} -Fano variety V is K-semistable if and only if $\widehat{\text{vol}}$ is minimized among \mathbb{C}^* -invariant valuations (see Theorem 3.1).¹

The key idea to prove our result is to extend the calculations in [Li15a] to show a generalization of a result in [MSY08] in which they showed that the derivative of a volume functional on the space of (normalized) Reeb vector fields is the classical Futaki invariant originally defined in [Fut83]. Here we will show that the derivative of the normalized volume at the canonical valuation ord_V is a variant of CM weight that generalizes the classical Futaki invariant. We achieve this by first deriving an integral formula for the normalized volumes for any real valuations centered at o . The derivation of this formula uses the tool of filtrations (see [BC11]) and has a very concrete convex geometric meaning which allows us to use the theory of Okounkov bodies ([Oko96, LM09]) and coconvex sets ([KK14]) to understand it.

Given this formula, then as in [Li15a] we can take its derivative and use the recent remarkable work of Fujita [Fuj15b] to show that Ding-semistability implies equivariant volume minimization, where Ding-semistability is a concept derived from Berman’s work in [Ber15] and is shown very recently to be equivalent to K-semistability in [BBJ15, Fuj16]. Moreover, the volume formula shows that any \mathbb{C}^* -invariant minimizer has a Dirac-type Duistermaat-Heckman measure. This together with some gap estimates allow us to conclude the uniqueness of \mathbb{C}^* -invariant minimizing valuation that is quasi-monomial (Theorem 3.4).

For the other direction of implication, we directly relate the volume function on X (i.e. on $\text{Val}_{X,o}$) to that on some degeneration of X which appears in the definition of K-stability.

¹Recently it has been clear that the general case can be reduced to the equivariant case considered in this paper, see [LX16].

The argument in this part also depends on the work of [BHJ15], which seems to be the first to systematically use the tool of valuations to study K-stability. On the other hand, we will also use our earlier work in [LX14] which essentially helps us to reduce the test of K-stability to Tian's original definition.

As a spin-off of our proof, we get the an equivalent characterization of K-semistability using divisorial valuations in Theorem 3.7. This generalizes a result about Fujita's divisorial stability in [Fuj15a] (see also [Li15a]).

We end this introduction by remarking that while the work here is a natural continuation of the idea initiated in [Li15a], the techniques and tools used here are mostly independent of those used in [Li15a] and so this paper can be read independent of [Li15a].

2 Preliminaries

2.1 Normalized volumes of valuations

In this section, we will briefly recall the concept of real valuations and their associated invariants (see [ZS60] and [ELS03] for details). We will also recall normalized volume functional defined in [Li15a].

Let $X = \text{Spec}(R)$ be an n -dimensional normal affine variety. A real valuation v on the function field $\mathbb{C}(X)$ is a map $v : \mathbb{C}(X) \rightarrow \mathbb{R}$, satisfying:

1. $v(fg) = v(f) + v(g)$;
2. $v(f + g) \geq \min\{v(f), v(g)\}$.

In this paper we also require $v(\mathbb{C}^*) = 0$, i.e. v is trivial on \mathbb{C} . Denote by $\mathcal{O}_v = \{f \in \mathbb{C}(X); v(f) \geq 0\}$ the valuation ring of v . Then \mathcal{O}_v is a local ring. The valuation v is said to be finite over R , or on X , if $\mathcal{O}_v \supseteq R$. Let Val_X denote the space of all real valuations that are trivial on \mathbb{C} and finite over R . For any $v \in \text{Val}_X$, the center of v over X , denoted by $\text{center}_X(v)$ or $c_X(v)$, is defined to be the image of the closed point of $\text{Spec}(\mathcal{O}_v)$ under the map $\text{Spec}(\mathcal{O}_v) \rightarrow \text{Spec}(R) = X$. For any $v \in \text{Val}_X$, let $\Gamma'_v = v(R)$ denote its valuation semigroup and Γ_v denote the abelian group generated by Γ'_v inside \mathbb{R} . Denote by $\text{rat.rank } v$ the rational rank of the abelian group Γ_v . Denote by $\text{tr.deg } v$ the transcendental degree of the field extension K_v/\mathbb{C} where K_v is the residue field of \mathcal{O}_v . Abhyankar proved the following inequality in [Abh56]:

$$\text{tr.deg } v + \text{rat.rank } v \leq n. \quad (1)$$

v is called Abhyankar if the equality in (1) holds. A valuation is called divisorial if $\text{tr.deg } v = n - 1$ (and hence $\text{rat.rank } v = 1$ by (1)). Divisorial valuations are Abhyankar. Since we are working in characteristic 0, it's well-known that Abhyankar valuations are quasi-monomial (see [ELS03, Proposition 2.8], [JM10, Proposition 3.7]). $v \in \text{Val}_{X,o}$ is quasi-monomial if there exists a birational morphism $\mu : Y \rightarrow X$, and a regular system of parameters $z = \{z_1, \dots, z_r\}$ for the local ring $\mathcal{O}_{Y,W}$ of v on Y (W is the center of v on Y) such that $v(z_1), \dots, v(z_r)$ freely generates the value group Γ_v . In this case, we can assume that there exist r \mathbb{Q} -linearly independent positive real numbers $\alpha_1, \dots, \alpha_r$ such that v is defined as follows. For any $f \in R$, we can expand $\mu^* f = \sum_{m \in \mathbb{Z}_{\geq 0}^r} c_m z^m$ with each c_m either 0 or a unit in $\widehat{\mathcal{O}_{Y,W}}$, and then:

$$v(f) = \min \left\{ \sum_{i=1}^r m_i \alpha_i; c_m \neq 0 \right\}.$$

Following [JM10], we will also use log smooth models in addition to algebraic coordinates for representing quasi-monomial valuations. A log smooth pair over X is a pair (Y, D) with Y regular and D a reduced effective simple normal crossing divisor, together with a proper birational morphism $\mu : Y \rightarrow X$. We will denote by $\text{QM}_\eta(Y, D)$ the set of all quasi-monomial valuations v that can be described in the above fashion at the point $\eta \in Y$ with respect to coordinates $\{z_1, \dots, z_r\}$ such that z_i defines at η an irreducible component D_i of D . We put $\text{QM}(Y, D) = \cup_\eta \text{QM}_\eta(Y, D)$.

From now on in this paper, we assume that X has \mathbb{Q} -Gorenstein klt singularity. Following [JM10] and [BFFU13], we can define the log discrepancy for any valuation $v \in \text{Val}_X$. This is achieved in three steps in [JM10] and [BFFU13]. Firstly, for a divisorial valuation ord_E associated to a prime divisor E over X , define $A_X(E) = \text{ord}_E(K_{Y/X}) + 1$, where $\pi : Y \rightarrow X$ is

a smooth model of X containing E . Next for any quasi-monomial valuation $v \in \text{QM}_\eta(Y, D)$ where $(Y, D = \sum_{k=1}^N D_k)$ is log smooth and η is a generic point of an irreducible component of $D_1 \cap \dots \cap D_N$, we define $A_X(v) = \sum_{k=1}^N v(D_k) A_X(D_k)$. Lastly for any valuation $v \in \text{Val}_X$, we define

$$A_X(v) = \sup_{(Y, D)} A_X(r_{Y, D}(v))$$

where (Y, D) ranges over all log smooth models over X , and $r_{(Y, D)} : \text{Val}_X \rightarrow \text{QM}(Y, D)$ are contraction maps that induce a homeomorphism $\text{Val}_X \rightarrow \varinjlim_{(Y, D)} \text{QM}(Y, D)$. For details, see

[JM10] and [BFFU13, Theorem 3.1]. We will need one basic property of A_X : for any proper birational morphism $Y \rightarrow X$, we have (see [JM10, Remark 5.6], [BFFU13, Proof 3.1]):

$$A_X(v) = A_Y(v) + v(K_{Y/X}). \quad (2)$$

From now on, we also fix a closed point $o \in X$ with the corresponding maximal ideal of R denoted by \mathfrak{m} . We will be interested in the space $\text{Val}_{X, o}$ of all valuations v with center $v = o$. If $v \in \text{Val}_{X, o}$, then v is centered on the local ring $R_{\mathfrak{m}}$ (see [ELS03]). In other words, v is nonnegative on $R_{\mathfrak{m}}$ and is strictly positive on the maximal ideal of the local ring $R_{\mathfrak{m}}$. For any $v \in \text{Val}_{X, o}$, we consider its valuative ideals:

$$\mathfrak{a}_p(v) = \{f \in R; v(f) \geq p\}.$$

Then by [ELS03, Proposition 1.5], $\mathfrak{a}_p(v)$ is \mathfrak{m} -primary, and hence is of finite codimension in R (cf. [AM69]). We define the volume of v as:

$$\text{vol}(v) = \lim_{p \rightarrow +\infty} \frac{\dim_{\mathbb{C}} R/\mathfrak{a}_p(v)}{p^n/n!}.$$

By [ELS03, Mus02, LM09, Cut12], the limit on the right hand side exists and is equal to the multiplicity of the graded family of ideals $\mathfrak{a}_\bullet = \{\mathfrak{a}_p\}$:

$$\text{vol}(v) = \lim_{p \rightarrow +\infty} \frac{e(\mathfrak{a}_p)}{p^n} =: e(\mathfrak{a}_\bullet), \quad (3)$$

where $e(\mathfrak{a}_p)$ is the Hilbert-Samuel multiplicity of \mathfrak{a}_p .

Now we can define the normalized volume for any $v \in \text{Val}_{X, o}$ (see [Li15a]):

$$\widehat{\text{vol}}(v) = \begin{cases} A_X(v)^n \text{vol}(v), & \text{if } A_X(v) < +\infty, \\ +\infty, & \text{if } A_X(v) = +\infty. \end{cases}$$

The following estimates were proved in [Li15a]. The second estimate motivates the definition of $\widehat{\text{vol}}(v) = +\infty$ when $A_X(v) = +\infty$.

Theorem 2.1 ([Li15a, Corollary 2.6, Theorem 3.3]). *Let (X, o) be a \mathbb{Q} -Gorenstein klt singularity. The following uniform estimates hold:*

1. *There exists a positive constant $K = K(X, o) > 0$ such that $\widehat{\text{vol}}(v) > K$ for any $v \in \text{Val}_{X, o}$.*
2. *There exists a positive constant $c = c(X, o) > 0$ that*

$$\widehat{\text{vol}}(v) \geq c \frac{A_X(v)}{v(\mathfrak{m})}, \text{ for all } v \in \text{Val}_{X, o} \text{ with } A_X(v) < +\infty.$$

From now on in this paper, we will always assume our valuation v satisfies $A_X(v) < +\infty$.

Notice that $\widehat{\text{vol}}(v)$ is rescaling invariant: $\widehat{\text{vol}}(\lambda v) = \widehat{\text{vol}}(v)$ for any $\lambda > 0$. By Izumi-type theorem proved in [Li15a, Proposition 2.3], we know that $\widehat{\text{vol}}$ is uniformly bounded from below by a positive number on $\text{Val}_{X, o}$. If $\widehat{\text{vol}}$ has a global minimum v_* on $\text{Val}_{X, o}$, i.e. $\widehat{\text{vol}}(v_*) = \inf_{v \in \text{Val}_{X, o}} \widehat{\text{vol}}(v)$, then we will say that $\widehat{\text{vol}}$ is globally minimized at v_* over (X, o) . In [Li15a], the author conjectured that this holds for all \mathbb{Q} -Gorenstein klt singularities and proved that this is the case under a continuity hypothesis.² In this paper, we will be interested in the following \mathbb{C}^* -invariant setting.

²H.Blum recently confirmed this conjecture in [Blum16]

Definition 2.2. Assume that there is a \mathbb{C}^* -action on (X, o) . We denote by $(\text{Val}_{X,o})^{\mathbb{C}^*}$ the space of \mathbb{C}^* -invariant valuations in $\text{Val}_{X,o}$. We say that $\widehat{\text{vol}}$ is \mathbb{C}^* -equivariantly minimized at a (\mathbb{C}^* -invariant) valuation $v_* \in (\text{Val}_{X,o})^{\mathbb{C}^*}$ if

$$\widehat{\text{vol}}(v_*) = \inf_{v \in (\text{Val}_{X,o})^{\mathbb{C}^*}} \widehat{\text{vol}}(v).$$

2.2 K-semistability and Ding-semistability

In this section we recall the definition of K-semistability, and its recent equivalence Ding-semistability.

Definition 2.3 ([Tia97, Don02], see also [LX14]). Let V be an $(n-1)$ -dimensional \mathbb{Q} -Fano variety.

1. For $r > 0 \in \mathbb{Q}$ such that $r^{-1}K_V^{-1} = L$ is Cartier. A test configuration (resp. a semi test configuration) of (X, L) consists of the following data

- A variety \mathcal{V} admitting a \mathbb{C}^* -action and a \mathbb{C}^* -equivariant morphism $\pi : \mathcal{V} \rightarrow \mathbb{C}$, where the action of \mathbb{C}^* on \mathbb{C} is given by the standard multiplication.
- A \mathbb{C}^* -equivariant π -ample (resp. π -semiample) line bundle \mathcal{L} on \mathcal{V} such that $(\mathcal{V}, \mathcal{L})|_{\pi^{-1}(\mathbb{C} \setminus \{0\})}$ is equivariantly isomorphic to $(V, r^{-1}K_V^{-1}) \times \mathbb{C}^*$ with the natural \mathbb{C}^* -action.

A test configuration is called a special test configuration, if the following are satisfied

- \mathcal{V} is normal, and \mathcal{V}_0 is an irreducible \mathbb{Q} -Fano variety;
- $\mathcal{L} = r^{-1}K_{\mathcal{V}/\mathbb{C}}^{-1}$.

2. Assume that $(\mathcal{V}, \mathcal{L}) \rightarrow \mathbb{C}$ is a normal test configuration. Let $\bar{\pi} : (\bar{\mathcal{V}}, \bar{\mathcal{L}}) \rightarrow \mathbb{P}^1$ be the natural equivariant compactification of $(\mathcal{V}, \mathcal{L}) \rightarrow \mathbb{C}$. The CM weight of $(\mathcal{V}, \mathcal{L})$ is defined by the intersection formula (see [Wan12, Oda13]):

$$\text{CM}(\mathcal{V}, \mathcal{L}) = \frac{1}{n(-K_V)^{n-1}} ((n-1)r^n \bar{\mathcal{L}}^n + nr^{n-1} \bar{\mathcal{L}}^{n-1} \cdot K_{\bar{\mathcal{V}}/\mathbb{P}^1}).$$

3. • V is called K-semistable if $\text{CM}(\mathcal{V}, \mathcal{L}) \geq 0$ for any normal test configuration $(\mathcal{V}, \mathcal{L})/\mathbb{C}^1$ of $(V, r^{-1}K_V^{-1})$.
- V is called K-polystable if $\text{CM}(\mathcal{V}, \mathcal{L}) \geq 0$ for any normal test configuration $(\mathcal{V}, \mathcal{L})/\mathbb{C}^1$ of $(V, r^{-1}K_V^{-1})$, and the equality holds if and only if $\mathcal{V} \cong V \times \mathbb{C}^1$.

Remark 2.4. For any special test configuration, the \mathbb{C}^* -action on \mathcal{V}_0 induces a natural action on $K_{\mathcal{V}_0}^{-1}$ by pushing forward the holomorphic $(n, 0)$ -vectors. Because $\mathcal{L}_0 = r^{-1}K_{\mathcal{V}_0}^{-1}$, we can always assume that the \mathbb{C}^* -action on \mathcal{L}_0 is induced from this natural \mathbb{C}^* -action. Moreover, for special test configurations, CM weight reduces to a much simpler form:

$$\text{CM}(\mathcal{V}, \mathcal{L}) = \frac{-r^n \bar{\mathcal{L}}^n}{n(-K_V)^{n-1}} = -\frac{(-K_{\mathcal{V}/\mathbb{P}^1})^n}{n(-K_V)^{n-1}}. \quad (4)$$

We will need the concept of Ding-semistability, which was derived from Berman's work in [Ber15].

Definition 2.5 ([Ber15, Fuj15b]). 1. Let $(\mathcal{V}, \mathcal{L})/\mathbb{C}^1$ be a normal semi-test configuration of $(V, r^{-1}K_V^{-1})$ and $(\bar{\mathcal{V}}, \bar{\mathcal{L}})/\mathbb{P}^1$ be its natural compactification. Let $D_{(\mathcal{V}, \mathcal{L})}$ be the \mathbb{Q} -divisor on \mathcal{V} satisfying the following conditions:

- The support $\text{Supp} D_{(\mathcal{V}, \mathcal{L})}$ is contained in \mathcal{V}_0 .
- The divisor $-r^{-1}D_{(\mathcal{V}, \mathcal{L})}$ is a \mathbb{Z} -divisor corresponding to the divisorial sheaf $\bar{\mathcal{L}}(r^{-1}K_{\bar{\mathcal{V}}/\mathbb{P}^1})$.

2. The Ding invariant $\text{Ding}(\mathcal{V}, \mathcal{L})$ of $(\mathcal{V}, \mathcal{L})/\mathbb{C}$ is defined as:

$$\text{Ding}(\mathcal{V}, \mathcal{L}) := \frac{-r^n \bar{\mathcal{L}}^n}{n(-K_V)^{n-1}} - (1 - \text{lct}(\mathcal{V}, D_{(\mathcal{V}, \mathcal{L})}; \mathcal{V}_0)).$$

3. V is called Ding semistable if $\text{Ding}(\mathcal{V}, \mathcal{L}) \geq 0$ for any normal test configuration $(\mathcal{V}, \mathcal{L})/\mathbb{C}$ of $(V, -r^{-1}K_V)$.

Notice that $\text{Ding}(\mathcal{V}, \mathcal{L}) = \text{CM}(\mathcal{V}, \mathcal{L})$ for special test configurations. It was proved in [LX14] that to test K-semistability (or K-polystability), one only needs to test on special test configurations. By the recent work in [BBJ15, Fuj16], the same is true for Ding-semistability. Moreover we have the following equivalence result:

Theorem 2.6 ([BBJ15, Fuj16]). *For a \mathbb{Q} -Fano variety V , V is K-semistable if and only if V is Ding-semistable.*

Remark 2.7. *In [BBJ15] Berman-Boucksom-Jonsson outlined a proof of the above result when V is a smooth Fano manifold. They showed that it's sufficient to check Ding semistability on special test configurations. This was achieved by showing that the (normalized) Ding invariant is decreasing along a process that transforms any test configuration into a special test configuration. This process was first used in [LX14] for proving results on K-(semi)stability of \mathbb{Q} -Fano varieties. The equivalence of Ding-semistability and K-semistability for the general case of \mathbb{Q} -Fano varieties has been proved in [Fuj16] by similar and more detailed arguments.*

We will need another result of Fujita, which was proved by applying a criterion for Ding semistability ([Fuj15b, Proposition 3.5]) to a sequence of semi-test configurations constructed from a filtration. We here briefly recall the relevant definitions about filtrations and refer the details to [BC11] (see also [BHJ15] and [Fuj15b, Section 4.1]).

Definition 2.8. *A good filtration of a graded \mathbb{C} -algebra $R = \bigoplus_{m=0}^{+\infty} R_m$ is a decreasing, multiplicative and linearly bounded \mathbb{R} -filtrations of R . In other words, for each $m \geq 0 \in \mathbb{Z}$, there is a family of subspaces $\{\mathcal{F}^x R_m\}_{x \in \mathbb{R}}$ of R_m such that:*

1. $\mathcal{F}^x R_m \subseteq \mathcal{F}^{x'} R_m$, if $x \geq x'$;
2. $\mathcal{F}^x R_m \cdot \mathcal{F}^{x'} R_{m'} \subseteq \mathcal{F}^{x+x'} R_{m+m'}$, for any $x, x' \in \mathbb{R}$ and $m, m' \in \mathbb{Z}_{\geq 0}$;
3. $e_{\min}(\mathcal{F}) > -\infty$ and $e_{\max}(\mathcal{F}) < +\infty$, where $e_{\min}(\mathcal{F})$ and $e_{\max}(\mathcal{F})$ are defined by the following operations:

$$\begin{aligned} e_{\min}(R_m, \mathcal{F}) &= \inf\{t \in \mathbb{R}; \mathcal{F}^t R_m \neq R_m\}; \\ e_{\max}(R_m, \mathcal{F}) &= \sup\{t \in \mathbb{R}; \mathcal{F}^t R_m \neq 0\}; \\ e_{\min}(\mathcal{F}) &= e_{\min}(R_{\bullet}, \mathcal{F}) = \liminf_{m \rightarrow +\infty} \frac{e_{\min}(R_m, \mathcal{F})}{m}; \\ e_{\max}(\mathcal{F}) &= e_{\max}(R_{\bullet}, \mathcal{F}) = \limsup_{i \rightarrow +\infty} \frac{e_{\max}(R_m, \mathcal{F})}{m}. \end{aligned} \tag{5}$$

Following [BHJ15], given any good filtration $\{\mathcal{F}^x R_m\}_{x \in \mathbb{R}}$ and $m \in \mathbb{Z}_{\geq 0}$, the successive minima is the decreasing sequence

$$\lambda_{\max}^{(m)} = \lambda_1^{(m)} \geq \dots \geq \lambda_{N_m}^{(m)} = \lambda_{\min}^{(m)}$$

where $N_m = \dim_{\mathbb{C}} R_m$, defined by:

$$\lambda_j^{(m)} = \max\{\lambda \in \mathbb{R}; \dim_{\mathbb{C}} \mathcal{F}^{\lambda} R_m \geq j\}.$$

Denote $R^{(t)} = \bigoplus_{k=0}^{+\infty} \mathcal{F}^{kt} R_k$. When we want to emphasize the dependence of $R^{(t)}$ on the filtration \mathcal{F} , we also denote $R^{(t)}$ by $\mathcal{F}R^{(t)}$.

From now on, we let $R_m = H^0(V, mL)$. The following concept of volume will be important for us:

$$\text{vol}\left(R^{(t)}\right) = \text{vol}\left(\mathcal{F}R^{(t)}\right) := \limsup_{k \rightarrow +\infty} \frac{\dim_{\mathbb{C}} \mathcal{F}^{kt} H^0(V, mL)}{m^{n-1}/(n-1)!}. \tag{6}$$

We need the following lemma by in [BHJ15] in our proof of uniqueness result.

Proposition 2.9 ([BC11], [BHJ15, Corollary 5.4]). *1. The probability measure*

$$\nu_m := \frac{1}{N_m} \sum_j \delta_{m^{-1} \lambda_j^{(m)}} = -\frac{d}{dt} \frac{\dim_{\mathbb{C}} \mathcal{F}^{mt} H^0(V, mL)}{N_m}$$

converges weakly as $m \rightarrow +\infty$ to the probability measure:

$$\nu := -V^{-1} d \text{vol}\left(R^{(t)}\right) = -V^{-1} \frac{d}{dt} \text{vol}\left(R^{(t)}\right) dt.$$

2. The support of the measure ν is given by $\text{supp } \nu = [\lambda_{\min}, \lambda_{\max}]$ with

$$\lambda_{\min} := \inf \left\{ t \in \mathbb{R} \mid \text{vol} \left(R^{(t)} \right) < L^{n-1} \right\}; \quad (7)$$

$$\lambda_{\max} = \lim_{m \rightarrow +\infty} \frac{\lambda_{\max}^{(m)}}{m} = \sup_{m \geq 1} \frac{\lambda_{\max}^{(m)}}{m}. \quad (8)$$

Moreover, ν is absolutely continuous with respect to the Lebesgue measure, except perhaps for a Dirac mass at λ_{\max} .

Following [Fuj15b], we also define a sequence of ideal sheaves on V :

$$I_{(m,x)}^{\mathcal{F}} = \text{Image} \left(\mathcal{F}^x R_m \otimes L^{-m} \rightarrow \mathcal{O}_V \right), \quad (9)$$

and define $\overline{\mathcal{F}}^x R_m := H^0(V, L^m \cdot I_{(m,x)}^{\mathcal{F}})$ to be the saturation of $\mathcal{F}^x R_m$ such that $\mathcal{F}^x R_m \subseteq \overline{\mathcal{F}}^x R_m$. \mathcal{F} is called saturated if $\overline{\mathcal{F}}^x R_m = \mathcal{F}^x R_m$ for any $x \in \mathbb{R}$ and $m \in \mathbb{Z}_{\geq 0}$. Notice that with our notations we have:

$$\text{vol} \left(\overline{\mathcal{F}} R^{(t)} \right) := \limsup_{k \rightarrow +\infty} \frac{\dim_{\mathbb{C}} \overline{\mathcal{F}}^{kt} H^0(V, kL)}{k^{n-1}/(n-1)!}.$$

The following result of Fujita will be crucial for us in proving that Ding-semistability implies volume minimization.

Theorem 2.10 ([Fuj15b, Theorem 4.9]). *Assume $(V, -K_V)$ is Ding-semistable. Let \mathcal{F} be a good \mathbb{R} -filtration of $R = \bigoplus_{m=0}^{+\infty} H^0(V, mL)$ where $L = r^{-1}K_V^{-1}$. Then the pair $(V \times \mathbb{C}, \mathcal{I}_{\bullet} \cdot (t)^{d_{\infty}})$ is sub log canonical, where*

$$\mathcal{I}_m = I_{(m, m e_+)}^{\mathcal{F}} + I_{(m, m e_+ - 1)}^{\mathcal{F}} t^1 + \cdots + I_{(m, m e_- + 1)}^{\mathcal{F}} t^{m(e_+ - e_-) - 1} + (t^{m(e_+ - e_-)}), \quad (10)$$

$$d_{\infty} = 1 - r(e_+ - e_-) + \frac{r^n}{(-K_V)^{n-1}} \int_{e_-}^{e_+} \text{vol} \left(\overline{\mathcal{F}} R^{(t)} \right) dt \quad (11)$$

and $e_+, e_- \in \mathbb{Z}$ with $e_+ \geq e_{\max}(R_{\bullet}, \mathcal{F})$ and $e_- \leq e_{\min}(R_{\bullet}, \mathcal{F})$.

Remark 2.11. *By Theorem 2.6, we can replace “Ding-semistable” by “K-semistable” in Fujita’s result.*

3 Statement of Main Results

Let V be a \mathbb{Q} -Fano variety. If $L = r^{-1}K_V^{-1}$ is an ample Cartier divisor for an $r \in \mathbb{Q}_{>0}$, we will denote the affine cone by $C_r := C(V, L) = \text{Spec} \bigoplus_{k=0}^{+\infty} H^0(V, kL)$. Notice that C_r has \mathbb{Q} -Gorenstein klt singularities at the vertex o (see [Kol13, Lemma 3.1]). (C_r, o) has a natural \mathbb{C}^* -action. Denote by v_0 the canonical \mathbb{C}^* -invariant divisorial valuation ord_V where V is considered as the exceptional divisor of the blow up $Bl_o X \rightarrow X$. The following is the main theorem of this paper, which in particular confirms one direction of part 4 of the conjecture in [Li15a].

Theorem 3.1. *Let V be a \mathbb{Q} -Fano variety. The following conditions are equivalent:*

1. $(V, -K_V)$ is K-semistable;
2. For an $r_* \in \mathbb{Q}_{>0}$ such that $L = -r_*^{-1}K_V$ is Cartier, $\widehat{\text{vol}}$ is \mathbb{C}^* -equivariantly minimized at ord_V over (C_{r_*}, o_*) ;
3. For any $r \in \mathbb{Q}_{>0}$ such that $L = -r^{-1}K_V$ is Cartier, $\widehat{\text{vol}}$ is \mathbb{C}^* -equivariantly minimized at ord_V over (C_r, o) .

Example 3.2. 1. Over $(\mathbb{C}^n, 0)$, $\widehat{\text{vol}}$ is globally minimized at the valuation $v_* = \text{ord}_E$, where $E \cong \mathbb{P}^{n-1}$ is the exceptional divisor of the blow up $Bl_o \mathbb{C}^n \rightarrow \mathbb{C}^n$. This follows from an inequality of de-Fernex-Ein-Mustařa as pointed out in [Li15a]. As a corollary we have a purely algebraic proof of the following statement:

Corollary 3.3. \mathbb{P}^{n-1} is K -semistable.

As pointed out in [PW16], an algebraic proof of this result could also be given by combining Kempf's result on Chow stabilities of rational homogeneous varieties, and the fact that asymptotic Chow-semistability implies K -semistability.

2. Over n -dimensional A_1 singularity $X^n = \{z_1^2 + \cdots + z_{n+1}^2 = 0\} \subset \mathbb{C}^{n+1}$ with $o = (0, \dots, 0)$, $\widehat{\text{vol}}$ is minimized among \mathbb{C}^* -invariant valuations at $v_* = \text{ord}_E$, where $E \cong Q$ (the smooth quadric hypersurface) is the exceptional divisor of the blowup $\text{Bl}_o X \rightarrow X$. This follows from the above theorem plus the fact that Q has Kähler-Einstein metric and hence is K -polystable. In [Li15a], v_* was shown to minimize $\widehat{\text{vol}}$ among valuations of the special form $v_{\mathbf{x}}$ determined by weight $\mathbf{x} \in \mathbb{R}_{>0}^{n+1}$. In the following paper [LL16], we will show that $v_* = \text{ord}_Q$ indeed globally minimizes $\widehat{\text{vol}}$.

Under any one of the equivalent conditions of Theorem 3.1, we can prove the uniqueness of minimizer among \mathbb{C}^* -invariant quasi-monomial valuations. Further existence and uniqueness results for general klt singularities will be dealt with in a forthcoming paper [LX16].

Theorem 3.4. Assume V is K -semistable. Then ord_V is the unique minimizer among all \mathbb{C}^* -invariant quasi-monomial valuations.

Through the proof of the above theorem, we get also a characterization of K -semistability using divisorial valuations of the field $\mathbb{C}(V)$.

Definition 3.5. For any divisorial valuation ord_F over V where F is a prime divisor on a birational model of V , we define the quantity:

$$\Theta_V(F) = A_V(F) - \frac{r^n}{(-K_V)^{n-1}} \int_0^{+\infty} \text{vol}(\mathcal{F}_F R^{(t)}) dt, \quad (12)$$

where $L = -r^{-1}K_V$ is an ample Cartier divisor for $r \in \mathbb{Q}_{>0}$ and $\mathcal{F}_F R_{\bullet} = \{\mathcal{F}_F R_k\}$ is the filtration defined in (81) by ord_F on $R = \bigoplus_{k=0}^{+\infty} H^0(V, kL)$.

Remark 3.6. 1. It's easy to verify that the $\Theta_V(F)$ does not depend on the choice of L . See Lemma 7.1. In particular when $-K_V$ is Cartier then we can choose $L = -K_V$ and $r = 1$.

2. If F is a prime divisor on V , then $A_V(F) = 1$ and $\Theta_V(F)$ is nothing but (a multiple of) Fujita's invariant $\eta(F)$ defined in [Fuj15a]. In particular, the following theorem can be seen as a generalization of result about Fujita's divisorial stability in [Fuj15a] (see also [Li15a]).

Theorem 3.7. Assume V is a \mathbb{Q} -Fano variety. Then V is K -semistable if and only if $\Theta_V(F) \geq 0$ for any divisorial valuation ord_F over V . Moreover, if $\Theta_V(F) > 0$ for any divisorial valuation ord_F over V , then V is K -stable.

The rest of this paper is devoted to the proof of Theorem 3.1, Theorem 3.4 and Theorem 3.7. Before we go into details, we make some general discussion on idea of the proof of Theorem 3.1, from which the proof of Theorem 3.4 and the proof Theorem 3.7 will be naturally derived.

In Section 4, we will prove that K -semistability, or equivalently Ding-semistability (by Theorem 2.6), implies equivariant volume minimization. More precisely we prove (1) \Rightarrow (3). To achieve this, we will generalize the calculations in [Li15a] where we compared the $\widehat{\text{vol}}(v_0)$, the normalized volume of the canonical valuation $v_0 = \text{ord}_V$, with $\widehat{\text{vol}}(v_1)$ for some special \mathbb{C}^* -invariant divisorial valuation v_1 . Here we will deal with general real valuations directly using the tool of filtrations. More precisely, we will associate an appropriate graded filtration to any real valuation v_1 with $A_X(v_1) < +\infty$, and will be able to calculate $\text{vol}(v_1)$ using the volumes of associated sublinear series. Izumi's theorem is a key to make this work, ensuring that we get linearly bounded filtrations. Motivated by the formulas in the case of \mathbb{C}^* -invariant valuations, we will consider a concrete convex interpolation between $\widehat{\text{vol}}(v_0)$ and $\widehat{\text{vol}}(v_1)$. By the convexity, we just need to show that the directional derivative of the interpolation at v_0 is nonnegative. The formulas up to this point works for any real valuation not necessarily \mathbb{C}^* -invariant.

In the \mathbb{C}^* -invariant case, same as for the case in [Li15a], it will turn out that the derivative matches Fujita's formula and his result in Theorem 2.10 ([Fuj15b]) gives exactly the nonnegativity. As mentioned earlier, we will deal with the non- \mathbb{C}^* -invariant case in [LL16].

In Section 6, we will prove the implication (2) \Rightarrow (1). Since (3) \Rightarrow (2) trivially, this completes the proof of Theorem 3.1. This direction of the implication depends on the work [BHJ15] and [LX14]. We learned from [BHJ15] that test configurations can be studied using the point of view of valuations: the irreducible components of the central fibre give rise to \mathbb{C}^* -invariant divisorial valuations, which however is not finite over the cone in general. In our new point of view, these \mathbb{C}^* -invariant valuations should be considered as tangent vectors at the canonical valuation. By [LX14], special test configurations are enough for testing K-semistability. For any special test configuration, there is just one tangent vector and so it's much easier to deal with. Then again the point is that the derivative of $\widehat{\text{vol}}$ along the tangent direction is exactly the Futaki invariant on the central fibre. So we are done.

4 K-semistable implies equivariant volume minimizing

Assume that V is a \mathbb{Q} -Fano variety and $L = -r^{-1}K_V$ is an ample Cartier divisor for a fixed $r \in \mathbb{Q}_{>0}$. Define $R = \bigoplus_{i=0}^{+\infty} H^0(V, iL) =: \bigoplus_{i=0}^{+\infty} R_i$ and denote by $X = C(V, L) = \text{Spec}(R)$ the affine cone over V with the polarization L . Let $v_0 = \text{ord}_V$ be the "canonical valuation" on R corresponding to the canonical \mathbb{C}^* -action on X .

Theorem 4.1. *If V is K-semistable, then $\widehat{\text{vol}}$ is \mathbb{C}^* -equivariantly minimized at the canonical valuation ord_V over $(X = C(V, L), o)$.*

The rest of this section is devoted to the proof of this theorem.

4.1 A general volume formula

Let v_1 be any real valuation centered at o with $A_X(v_1) < +\infty$. In this section, we don't assume v_1 is \mathbb{C}^* -invariant. We will derive a volume formula for $\text{vol}(v_1)$ with the help of an appropriately defined filtration associated to v_1 . To define this filtration, we decompose any $g \in R = \bigoplus_{k=0}^{+\infty} R_k$ into "homogeneous" components $g = g_{k_1} + \cdots + g_{k_p}$ with $g_{k_j} \neq 0$ and $k_1 < k_2 < \cdots < k_p$ and define $\mathbf{in}(g)$ to be the "initial component" g_{k_1} . Then the filtration we will consider is the following:

$$\mathcal{F}^x R_k = \{f \in R_k; \exists g \in R \text{ such that } v_1(g) \geq x \text{ and } \mathbf{in}(g) = f\} \cup \{0\}. \quad (13)$$

We notice that following properties of $\mathcal{F}^x R_\bullet$.

1. For fixed k , $\{\mathcal{F}^x R_k\}_{x \in \mathbb{R}}$ is a family of decreasing \mathbb{C} -subspace of R_k .
2. \mathcal{F} is multiplicative: $v_1(g_i) \geq x_i$ and $\mathbf{in}(g_i) = f_i \in R_{k_i}$ implies

$$v_1(g_1 g_2) \geq x_1 + x_2, \quad \mathbf{in}(g_1 g_2) = f_1 \cdot f_2 \in R_{k_1 + k_2}.$$

3. If $A(v_1) < +\infty$, then \mathcal{F} is linearly bounded. Indeed by Izumi's theorem in [Li13, Proposition 1.2] (see also [Izu85, Ree89, BFJ12]), there exist $c_1, c_2 \in (0, +\infty)$ such that

$$c_1 v_0 \leq v_1 \leq c_2 v_0.$$

If $v_1(g) \geq x$, then $k := v_0(g) = v_0(\mathbf{in}(g)) \geq c_2^{-1}x$. This implies that if $x > c_2 k$, then $\mathcal{F}^x R_k = 0$. So \mathcal{F} is linearly bounded from above. On the other hand, if $v_0(f) = k$, then $v_1(f) \geq c_1 k$. This implies that if $x \leq c_1 k$, then $\mathcal{F}^x R_k = R_k$. So \mathcal{F} is linearly bounded from below. Note that the argument in particular shows the following relation:

$$\inf_{\mathfrak{m}} \frac{v_1}{v_0} \leq e_{\min}(\mathcal{F}) \leq e_{\max}(\mathcal{F}) \leq \sup_{\mathfrak{m}} \frac{v_1}{v_0}. \quad (14)$$

For later convenience, from now on we will fix the following constant:

$$c_1 := \inf_{\mathfrak{m}} \frac{v_1}{v_0} > 0. \quad (15)$$

To state the following lemma, we first introduce a notation (see [DH09, Section 2]). If Z is a proper closed subscheme of Y and $v \in \text{Val}_Y$, let \mathcal{I}_Z be the ideal sheaf of Z and define:

$$v(Z) = v(\mathcal{I}_Z) := \min \{v(\phi); \phi \in \mathcal{I}_Z(U), U \cap \text{center}_Y(v) \neq \emptyset\}. \quad (16)$$

Lemma 4.2. For any $v_1 \in \text{Val}_{X,o}$, we have the following identity:

$$c_1 = v_1(V) = \inf_{i>0} \frac{v_1(R_i)}{i}, \quad (17)$$

where $v_1(V)$ is defined as in (16) and $v_1(R_i) = \inf\{v(f); f \neq 0 \in R_i\}$.

Proof. Let $\mu : Y := \text{Bl}_o X \rightarrow X$ be the blow up of o with exceptional divisor V . Then Y can be identified with the global space of the line bundle $L^{-1} \rightarrow V$ and we have the natural projection $\pi : Y \rightarrow V$. For any $f \in H^0(V, iL)$, we have $v_1(f) = v_1(\mu^* f) \geq iv_1(V)$. To see this, we write $f = h \cdot s^i$ on any affine open set U of X where $h \in \mathcal{O}_X(U)$ and s is a generator of $\mathcal{O}_V(L)(U)$. Then $\mu^* f = \mu^* h \cdot s^i$ can be considered as a regular function on $\pi^{-1}U \cong U \times \mathbb{C}$. Since $V \cap \pi^{-1}(U) = \{s = 0\}$, we have $v_1(s) = v_1(V)$ and $v_1(\mu^* f) = v_1(h \cdot s^i) = v_1(h) + iv_1(s) \geq iv_1(V)$. The identity holds if and only if $v_1(h) = 0$, i.e. if and only if f does not vanish on the center of v_1 over Y which is contained in V . When m is sufficiently large, mL is globally generated on V so that we can find $f \in H^0(V, mL)$ such that f does not vanish on the center of v on V . Then we have $\frac{v_1(f)}{v_0(f)} = v_1(f)/m = v_1(\mu^* f)/m = v_1(V)$. So from the above discussion we get:

$$c_1 = \inf_m \frac{v_1}{m} \leq \inf_{i>0} \frac{v_1(R_i)}{i} = v_1(V).$$

For any $t > c_1$, there exists $f \in R$ such that $v_1(f)/v_0(f) < t$. Decompose f into components: $f = f_{i_1} + \dots + f_{i_k}$, where $f_{i_j} \in R_{i_j} = H^0(V, i_j L)$ ($1 \leq j \leq k$) and $1 \leq i_1 < \dots < i_k$. Then there exists $i_* \in \{i_1, \dots, i_k\}$ such that

$$\begin{aligned} \frac{v_1(f)}{v_0(f)} &= \frac{v_1(f_{i_1} + \dots + f_{i_k})}{i_1} \geq \frac{\min\{v_1(f_{i_1}), \dots, v_1(f_{i_k})\}}{i_1} \\ &= \frac{v_1(f_{i_*})}{i_1} \geq \frac{v_1(f_{i_*})}{i_*} \geq v_1(V). \end{aligned}$$

So we get $t > v_1(V)$ for any $t > c_1$. As a consequence $c_1 \geq v_1(V)$. \square

The reason why we can use filtration in (13) to calculate volume comes from the following observation.

Proposition 4.3. For any $m \in \mathbb{R}$, we have the following identity:

$$\sum_{k=0}^{+\infty} \dim_{\mathbb{C}}(R_k/\mathcal{F}^m R_k) = \dim_{\mathbb{C}}(R/\mathfrak{a}_m(v_1)).$$

Notice that because of linear boundedness, the sum on the left hand side is a finite sum. More precisely, by the above discussion, when $k \geq m/c_1$, then $\dim_{\mathbb{C}}(R_k/\mathcal{F}^m R_k) = 0$.

Proof. For each fixed k , let $d_k = \dim_{\mathbb{C}}(R_k/\mathcal{F}^m R_k)$. Then we can choose a basis of $R_k/\mathcal{F}^m R_k$:

$$\left\{ [f_i^{(k)}]_k \mid f_i^{(k)} \in R_k, 1 \leq i \leq d_k \right\},$$

where $[\cdot]_k$ means taking quotient class in $R_k/\mathcal{F}^m R_k$. Notice that for $k \geq \lceil m/c_1 \rceil$, the set is empty. We want to show that the set

$$\mathfrak{B} := \left\{ [f_i^{(k)}] \mid 1 \leq i \leq d_k, 0 \leq k \leq \lceil m/c_1 \rceil - 1 \right\},$$

is a basis of $R/\mathfrak{a}_m(v_1)$, where $[\cdot]$ means taking quotient in $R/\mathfrak{a}_m(v_1)$.

- We first show that \mathfrak{B} is a linearly independent set. For any nontrivial linear combination of $[f_i^{(k)}]$:

$$\sum_{k=0}^N \sum_{i=1}^{d_k} c_i^{(k)} [f_i^{(k)}] = \left[\sum_{k=0}^N \sum_{i=1}^{d_k} c_i^{(k)} f_i^{(k)} \right] = [f^{(k_1)} + \dots + f^{(k_p)}] =: [F],$$

where $f^{(k_j)} \neq 0 \in R_{k_j} \setminus \mathcal{F}^m R_{k_j}$ and $k_1 < k_2 < \dots < k_p$. In particular $\mathbf{in}(F) = f^{(k_1)} \notin \mathcal{F}^m R_{k_1}$. By the definition of $\mathcal{F}^m R_{k_1}$, we know that

$$f^{(k_1)} + \dots + f^{(k_p)} \notin \mathfrak{a}_m(v_1),$$

which is equivalent to $[F] \neq 0 \in R/\mathfrak{a}_m(v_1)$.

- We still need to show that \mathfrak{B} spans $R/\mathfrak{a}_m(v_1)$. Suppose on the contrary \mathfrak{B} does not span $R/\mathfrak{a}_m(v_1)$. Then there is some $k \in \mathbb{Z}_{>0}$ and $f \in R_k - \mathfrak{a}_m(v_1)$ such that $[f] \neq 0 \in R/\mathfrak{a}_m(v_1)$ can not be written as a linear combination of $[f_i^{(k)}]$, i.e. not in the span of \mathfrak{B} . We claim that we can choose a maximal k such that this happens. Indeed, this follows from the fact that the set

$$\{v_0(g) \mid g \in R - \mathfrak{a}_m(v_1)\}$$

is finite (because $v_1(g) < m$ implies $v_0(g) \leq c_1^{-1}v_1(g) < m/c_1$).

So from now on we assume that k has been chosen such that for any $k' > k$ and $g \in R_{k'}$, $[g]$ is in the span of \mathfrak{B} . Then there are two cases to consider.

1. If $f \in R_k \setminus \mathcal{F}^m R_k$, then since $\{[f_i^{(k)}]_k\}$ is a basis of $R_k/\mathcal{F}^m R_k$, we can write $f = \sum_{j=1}^{d_k} c_j f_j^{(k)} + h_k$ where $h_k \in \mathcal{F}^m R_k$. So there exists $h \in \mathfrak{a}_m(v_1)$ such that $\mathbf{in}(h) = h_k$ and $f = \sum_{j=1}^{d_k} c_j f_j^{(k)} + h$. By the maximality of k , we know that $[h_k - h] = [h_k]$ is in the span of \mathfrak{B} . Then so is $[f] = \sum_{j=1}^{d_k} c_j [f_j^{(k)}] \in R/\mathfrak{a}_m(v_1)$. This contradicts the condition that $[f]$ is not in the span of \mathfrak{B} .
2. If $f \in \mathcal{F}^m R_k \subseteq R_k$, then by the definition of $\mathcal{F}^m R_k$, $f+h \in \mathfrak{a}_m(v_1)$ for some $h \in R$ such that $\mathbf{in}(f+h) = f$. Since we assumed that $[f] \neq 0 \in R/\mathfrak{a}_m(v_1)$, we have $h \neq 0$ and $k' := v_0(h) > v_0(f) = k$. Now we can decompose h into homogeneous components:

$$h = h^{(k_1)} + \dots + h^{(k_p)},$$

with $h^{(k_j)} \in R_{k_j}$ and $k' = k_1 < \dots < k_p$. Because $k' > k$ and the maximal property of k , we know that each $[h^{(k_j)}]$ is in the span of \mathfrak{B} . So we have $[f] = [(f+h) - h] = [-h]$ is in the span of \mathfrak{B} . This contradicts our assumption that $[f]$ is not in the span of \mathfrak{B} . □

The above proposition allows us to derive a general formula for $\widehat{\text{vol}}(v_1)$ with the help of \mathcal{F} defined in (13). Indeed, we have:

$$\begin{aligned} n! \dim_{\mathbb{C}} R/\mathfrak{a}_m(v_1) &= n! \sum_{k=0}^{+\infty} \dim_{\mathbb{C}} R_k/\mathcal{F}^m R_k \\ &= n! \sum_{k=0}^{\lfloor m/c_1 \rfloor} (\dim_{\mathbb{C}} R_k - \dim_{\mathbb{C}} \mathcal{F}^m R_k). \end{aligned} \quad (18)$$

For the first part of the sum, we have:

$$n! \sum_{k=1}^{\lfloor m/c_1 \rfloor} \dim_{\mathbb{C}} R_k = \frac{m^n}{c_1^n} L^{n-1} + O(m^{n-1}). \quad (19)$$

For the second part of the sum, we use Lemma 4.5 in Section 4.1.1 to get:

$$\lim_{m \rightarrow +\infty} \frac{n!}{m^n} \sum_{k=0}^{\lfloor m/c_1 \rfloor} \dim_{\mathbb{C}}(\mathcal{F}^m R_k) = n \int_{c_1}^{+\infty} \text{vol}(R^{(t)}) \frac{dt}{t^{n+1}}. \quad (20)$$

Combining (18), (19) and (20), we get the first version of volume formula:

$$\begin{aligned} \text{vol}(v_1) &= \lim_{p \rightarrow +\infty} \frac{n!}{m^n} \dim_{\mathbb{C}} R/\mathfrak{a}_m(v_1) \\ &= \frac{1}{c_1^n} L^{n-1} - n \int_{c_1}^{+\infty} \text{vol}(R^{(t)}) \frac{dt}{t^{n+1}}. \end{aligned} \quad (21)$$

We can use integration by parts to get a second version:

$$\begin{aligned}
\text{vol}(v_1) &= \frac{1}{c_1^n} L^{n-1} + \int_{c_1}^{+\infty} \text{vol}(R(t)) d\left(\frac{1}{t^n}\right) \\
&= \frac{1}{c_1^n} L^{n-1} + \left[\text{vol}(R(t)) \frac{1}{t^n} \Big|_{c_1}^{+\infty} \right] - \int_{c_1}^{+\infty} \frac{d\text{vol}(R(t))}{t^n} \\
&= - \int_{c_1}^{+\infty} \frac{d\text{vol}(R(t))}{t^n}.
\end{aligned} \tag{22}$$

Motivated by the case of \mathbb{C}^* -invariant valuations (see (37) in Section 4.2), we define a function of two parametric variables $(\lambda, s) \in (0, +\infty) \times [0, 1]$:

$$\begin{aligned}
\Phi(\lambda, s) &= \frac{1}{(\lambda c_1 s + (1-s))^n} L^{n-1} - n \int_{c_1}^{+\infty} \text{vol}(R(t)) \frac{\lambda s dt}{(1-s + \lambda s t)^{n+1}} \\
&= \int_{c_1}^{+\infty} \frac{-d\text{vol}(R(t))}{((1-s) + \lambda s t)^n}.
\end{aligned} \tag{23}$$

In Section 9, we will interpret and re-derive the above formulas (21)-(23) using the theory of Okounkov bodies and coconvex sets.

The usefulness of $\Phi(\lambda, s)$ can be seen from the following lemma.

Lemma 4.4. *$\Phi(\lambda, s)$ satisfies the following properties:*

1. For any $\lambda \in (0, +\infty)$, we have:

$$\Phi(\lambda, 1) = \text{vol}(\lambda v_1) = \lambda^{-n} \text{vol}(v_1), \quad \Phi(\lambda, 0) = \text{vol}(v_0) = L^{n-1}.$$

2. For fixed $\lambda \in (0, +\infty)$, $\Phi(\lambda, s)$ is continuous and convex with respect to $s \in [0, 1]$.

3. The derivative of $\Phi(\lambda, s)$ at $s = 0$ is equal to:

$$\Phi_s(\lambda, 0) = n\lambda L^{n-1} \left(\lambda^{-1} - c_1 - \frac{1}{L^{n-1}} \int_{c_1}^{+\infty} \text{vol}(R(t)) dt \right). \tag{24}$$

Proof. The first item is clear. To see the second item, first notice that $-d\text{vol}(R(t))$ has a positive density with respect to the Lebesgue measure dt since $\text{vol}(R(t))$ is decreasing with respect to $t \in \mathbb{R}$. Moreover it has finite total measure equal to $\text{vol}(R(x)) = L^{n-1}$ for any $x \leq c_1$. So the claim follows from the fact that the function $s \rightarrow 1/((1-s) + \lambda s t)^n$ is continuous and convex. The third item follows from direct calculation using the formula in (23). \square

Roughly speaking, the parameter λ is a rescaling parameter, and s is an interpolation parameter. To apply this lemma to our problem, we let $\lambda = \frac{r}{A_X(v_1)} =: \lambda_*$ such that

$$\Phi(\lambda_*, 1) = \frac{A_X(v_1)^n \text{vol}(v_1)}{r^n} = \frac{\widehat{\text{vol}}(v_1)}{r^n}.$$

Recall that $\widehat{\text{vol}}(v_0) = r^n L^{n-1}$. So our problem of showing $\widehat{\text{vol}}(v_1) \geq \widehat{\text{vol}}(v_0)$ is equivalent to showing that $\Phi(\lambda_*, 1) \geq \Phi(\lambda_*, 0)$. By item 2-3 of the above lemma, we just need to show that $\Phi_s(\lambda_*, 0)$ is non-negative. This will be proved in the \mathbb{C}^* -invariant case in the next section.

4.1.1 A summation formula

Lemma 4.5. *Let $\{\mathcal{F}^x R_i\}_{x \in \mathbb{R}}$ be a good filtration as in Definition 2.8. For any $\alpha \geq 0 \in \mathbb{R}$ and $\beta > -c_1$, we have the following identity:*

$$\begin{aligned}
\lim_{p \rightarrow +\infty} \frac{n!}{p^n} \sum_{i=0}^{\lfloor \alpha p / (\beta + c_1) \rfloor} \dim_{\mathbb{C}}(\mathcal{F}^{\alpha p - \beta i} R_i) \\
= n \int_{c_1}^{+\infty} \text{vol}(R(x)) \frac{\alpha^n dx}{(\beta + x)^{n+1}}.
\end{aligned} \tag{25}$$

Proof. Define $\phi(y) = \dim_{\mathbb{C}}(\mathcal{F}^{\alpha p - \beta y} R_{\lfloor y \rfloor})$. Then $\phi(y)$ is an increasing function on $[m, m+1]$ for any $m \in \mathbb{Z}_{\geq 0}$ and $\phi(y) \leq \dim_{\mathbb{C}} R_{\lfloor y \rfloor} \leq C y^{n-1}$. Moreover, because \mathcal{F}^x is decreasing in x , $\phi(y) \geq \dim_{\mathbb{C}}(\mathcal{F}^{\alpha p - \beta \lfloor y \rfloor} R_{\lfloor y \rfloor})$. So we have:

$$\begin{aligned} \sum_{i=0}^{\lfloor \alpha p / (\beta + c_1) \rfloor} \dim_{\mathbb{C}}(\mathcal{F}^{\alpha p - \beta i} R_i) &\leq \left(\sum_{i=0}^{\lfloor \alpha p / (\beta + c_1) \rfloor - 1} \dim_{\mathbb{C}}(\mathcal{F}^{\alpha p - \beta i} R_i) \right) + \dim_{\mathbb{C}} R_{\lfloor \alpha p / (\beta + c_1) \rfloor} \\ &\leq \left(\int_0^{\alpha p / (\beta + c_1)} \phi(y) dy \right) + O(p^{n-1}) \\ &= \left(p \int_{c_1}^{+\infty} \phi \left(\frac{\alpha p}{\beta + x} \right) \frac{\alpha dx}{(\beta + x)^2} \right) + O(p^{n-1}). \end{aligned}$$

We notice that:

$$\begin{aligned} \limsup_{p \rightarrow +\infty} \frac{\phi(\alpha p / (\beta + x))}{p^{n-1} / (n-1)!} &= \limsup_{p \rightarrow +\infty} \frac{\dim_{\mathbb{C}}(\mathcal{F}^{\alpha p x / (\beta + x)} R_{\lfloor \alpha p / (\beta + x) \rfloor})}{p^{n-1} / (n-1)!} \\ &\leq \limsup_{p \rightarrow +\infty} \frac{\dim_{\mathbb{C}}(\mathcal{F}^{x \lfloor \alpha p / (\beta + x) \rfloor} R_{\lfloor \alpha p / (\beta + x) \rfloor})}{\lfloor \alpha p / (\beta + x) \rfloor^{n-1} / (n-1)!} \frac{\lfloor \alpha p / (\beta + x) \rfloor^{n-1}}{(\alpha p / (\beta + x))^{n-1}} \frac{\alpha^{n-1}}{(\beta + x)^{n-1}} \\ &= \text{vol}(R^{(x)}) \frac{\alpha^{n-1}}{(\beta + x)^{n-1}}. \end{aligned}$$

The last identity holds by [LM09] and [BC11] (see also [BHJ15, Theorem 5.3]). So by Fatou's lemma, we have:

$$\begin{aligned} \limsup_{p \rightarrow +\infty} \frac{n!}{p^n} \sum_{i=0}^{\lfloor \alpha p / (\beta + c_1) \rfloor} \dim_{\mathbb{C}}(\mathcal{F}^{\alpha p - \beta i} R_i) &\leq n \limsup_{p \rightarrow +\infty} \left(\int_{c_1}^{+\infty} \frac{(n-1)!}{p^{n-1}} \phi \left(\frac{\alpha p}{\beta + x} \right) \frac{\alpha dx}{(\beta + x)^2} + O(p^{-1}) \right) \\ &\leq n \int_{c_1}^{+\infty} \limsup_{p \rightarrow +\infty} \frac{\phi(\alpha p / (\beta + x))}{p^{n-1} / (n-1)!} \frac{\alpha dx}{(\beta + x)^2} \\ &\leq n \int_{c_1}^{+\infty} \text{vol}(R^{(x)}) \frac{\alpha^n dx}{(\beta + x)^{n+1}}. \end{aligned} \tag{26}$$

Similarly we can prove the other direction. Define $\psi(y) = \dim_{\mathbb{C}}(\mathcal{F}^{\alpha p - \beta y} R_{\lceil y \rceil})$. Then $\psi(y)$ is an increasing function on $(m, m+1]$ for any $m \in \mathbb{Z}_{\geq 0}$ and satisfies $\psi(y) \leq \dim_{\mathbb{C}} R_{\lceil y \rceil} \leq C y^{n-1}$. Moreover, $\psi(y) \leq \dim_{\mathbb{C}}(\mathcal{F}^{\alpha p - \beta \lceil y \rceil} R_{\lceil y \rceil})$. So we have:

$$\begin{aligned} \sum_{i=0}^{\lfloor \alpha p / (\beta + c_1) \rfloor} \dim_{\mathbb{C}}(\mathcal{F}^{\alpha p - \beta i} R_i) &\geq \left(\sum_{i=1}^{\lceil \alpha p / (\beta + c_1) \rceil} \dim_{\mathbb{C}}(\mathcal{F}^{\alpha p - \beta i} R_i) \right) - \dim_{\mathbb{C}} R_{\lceil \alpha p / (\beta + c_1) \rceil} \\ &\geq \left(\int_0^{\alpha p / (\beta + c_1)} \psi(y) dy \right) + O(p^{n-1}) \\ &= \left(p \int_{c_1}^{+\infty} \psi \left(\frac{\alpha p}{\beta + x} \right) \frac{\alpha dx}{(\beta + x)^2} \right) + O(p^{n-1}). \end{aligned}$$

We can then estimate:

$$\begin{aligned} \liminf_{p \rightarrow +\infty} \frac{\psi(\alpha p / (\beta + x))}{p^{n-1} / (n-1)!} &= \liminf_{p \rightarrow +\infty} \frac{\dim_{\mathbb{C}}(\mathcal{F}^{\alpha p x / (\beta + x)} R_{\lceil \alpha p / (\beta + x) \rceil})}{p^{n-1} / (n-1)!} \\ &\geq \liminf_{p \rightarrow +\infty} \frac{\dim_{\mathbb{C}}(\mathcal{F}^{x \lceil \alpha p / (\beta + x) \rceil} R_{\lceil \alpha p / (\beta + x) \rceil})}{\lceil \alpha p / (\beta + x) \rceil^{n-1} / (n-1)!} \frac{\lceil \alpha p / (\beta + x) \rceil^{n-1}}{(\alpha p / (\beta + x))^{n-1}} \frac{\alpha^{n-1}}{(\beta + x)^{n-1}} \\ &= \text{vol}(R^{(x)}) \frac{\alpha^{n-1}}{(\beta + x)^{n-1}}. \end{aligned}$$

Similar as before, by using Fatou's lemma, we get the other direction of the inequality:

$$\begin{aligned}
& \liminf_{p \rightarrow +\infty} \frac{n!}{p^n} \sum_{i=0}^{\lfloor \alpha p / (\beta + c_1) \rfloor} \dim_{\mathbb{C}}(\mathcal{F}^{\alpha p - \beta i} R_i) \\
& \geq n \liminf_{p \rightarrow +\infty} \left(\int_{c_1}^{+\infty} \frac{(n-1)!}{p^{n-1}} \psi \left(\frac{\alpha p}{\beta + x} \right) \frac{\alpha dx}{(\beta + x)^2} + O(p^{-1}) \right) \\
& \geq n \int_{c_1}^{+\infty} \liminf_{p \rightarrow +\infty} \frac{\psi(\alpha p / (\beta + x))}{p^{n-1} / (n-1)!} \frac{\alpha dx}{(\beta + x)^2} \\
& \geq n \int_{c_1}^{+\infty} \text{vol} \left(R^{(x)} \right) \frac{\alpha^n dx}{(\beta + x)^{n+1}}. \tag{27}
\end{aligned}$$

Combining (26) and (27), we get the identity (25) we wanted:

$$\lim_{p \rightarrow +\infty} \frac{n!}{p^n} \sum_{i=0}^{\lfloor \alpha p / (\beta + c_1) \rfloor} \dim_{\mathbb{C}}(\mathcal{F}^{\alpha p - \beta i} R_i) = n \int_{c_1}^{+\infty} \text{vol} \left(R^{(x)} \right) \frac{\alpha^n dx}{(\beta + x)^{n+1}}.$$

□

4.2 Case of \mathbb{C}^* -invariant valuations

$v_0 = \text{ord}_V$ corresponds to the canonical \mathbb{C}^* -action on R given by $a \circ f_k = a^k f_k$ for $f_k \in R_k$ and $a \in \mathbb{C}^*$. From now on, v_1 is assumed to be \mathbb{C}^* -invariant: $v_1(a \circ f) = v_1(f)$ for any $f \in R$ and $a \in \mathbb{C}^*$. Let $w := v_1|_{\mathbb{C}(V)}$. Then it's easy to see that v_1 is a \mathbb{C}^* -invariant extension of w in the following way. First choose an affine neighborhood U of $\text{center}_V(w)$ and a local generator s of $\mathcal{O}_X(L)(U)$. Each $f_k \in R_k = H^0(V, kL)$ can be written as $f_k = h \cdot s^k$ on the open set U . Define $w(f_k) := w(h)$. Then for any $f_k \in R_k = H^0(V, kL)$, we have:

$$v_1(f_k) = w(f_k) + c_1 k$$

where $c_1 = v(V)$ is the constant appearing in Lemma 4.2. More generally if $f = \sum_k f_k \in R = \bigoplus_k R_k$, we have:

$$v_1(f) = \min \left\{ w(f_k) + c_1 k; f = \sum_k f_k \text{ with } f_k \neq 0 \right\}. \tag{28}$$

By (28) it's clear that the valuative ideals $\mathfrak{a}_m(v_1)$ are homogeneous with respect to the \mathbb{N} -grading of $R = \bigoplus_{k \in \mathbb{N}} R_k$.

Lemma 4.6. *Assume that v_1 is \mathbb{C}^* -invariant. $\mathcal{F} = \{\mathcal{F}^x\}_{x \in \mathbb{R}}$ defined in (13) is equal to:*

$$\mathcal{F}^x R_k = \mathfrak{a}_x(v_1) \cap R_k = \{f \in R_k \mid v_1(f) \geq x\}. \tag{29}$$

As a consequence, \mathcal{F} is left-continuous: $\mathcal{F}^x R_m = \bigcap_{x' < x} \mathcal{F}^{x'} R_m$ and saturated.

Proof. The identity (29) follows easily from the \mathbb{C}^* -invariance of v_1 and the left continuity is clear from (29). We just need to show the saturatedness. Let $\pi : Y = \text{Bl}_o X \rightarrow X$ be the blow up of o on X . For any $f \in H^0(V, mL)$, locally we can write $f = h \cdot s$ where s is a local trivializing section of L^m on an open neighborhood of $\text{center}_Y(v_1) \subset V$. Then we have

$$v_1(f) = v_1(\pi^*(hs)) = v_1(\pi^*h) + v_1(V)m.$$

So we have $v_1(f) \geq x$ if and only if $v_1(\pi^*h) = v_1(h) \geq x - c_1 m$. From this we see that $v_1 \left(I_{(m,x)}^{\mathcal{F}} \right) \geq x - c_1 m$. If $f \in H^0(V, mL \cdot I_{(m,x)}^{\mathcal{F}})$, then locally $f = h \cdot s$ with $h \in I_{(m,x)}^{\mathcal{F}}$. So we have: $v_1(f) = v_1(h) + c_1 m \geq x$ and hence $f \in \mathcal{F}^x H^0(V, mL)$. □

Remember that we want to prove $\Phi_s(\lambda_*, 0) \geq 0$. By (24) and $\lambda_* = r/A_X(v_1)$, we have

$$\Phi_s(\lambda_*, 0) = n \lambda_* L^{n-1} \left(\lambda_*^{-1} - c_1 - \frac{1}{L^{n-1}} \int_{c_1}^{+\infty} \text{vol} \left(R^{(t)} \right) dt \right) \tag{30}$$

$$= \frac{n L^{n-1}}{A_X(v_1)} \left(A_X(v_1) - c_1 r - \frac{r}{L^{n-1}} \int_{c_1}^{+\infty} \text{vol} \left(R^{(t)} \right) dt \right). \tag{31}$$

Under the blow-up $\pi : Y \rightarrow X$, we have $\pi^*K_X = K_Y - (r-1)V$ (see [Kol13, 3.1]). So by the property of log discrepancy in (2), we have

$$\begin{aligned} A_X(v_1) &= A_{(Y, (1-r)V)}(v_1) = A_Y(v_1) + (r-1)v_1(V) \\ &= A_Y(v_1) + (r-1)c_1. \quad (\text{by (17)}) \end{aligned}$$

This gives $A_X(v_1) - c_1 r = A_Y(v_1) - c_1$. So, using also $L = r^{-1}K_V^{-1}$, we get another expression for $\Phi_s(\lambda_*, 0)$ in (30):

$$\begin{aligned} \Phi_s(\lambda_*, 0) &= \frac{nL^{n-1}}{A_X(v_1)} \left(A_Y(v_1) - c_1 - \frac{r^n}{(-K_V)^{n-1}} \int_{c_1}^{+\infty} \text{vol}(\mathcal{F}R^{(t)}) dt \right) \\ &= \frac{nL^{n-1}}{A_X(v_1)} \left(A_Y(v_1) - c_1 \left(1 + \frac{r^n}{(-K_V)^{n-1}} \int_1^{+\infty} \text{vol}(\mathcal{F}R^{(c_1 t)}) dt \right) \right). \end{aligned} \quad (32)$$

Next we bring in Fujita's criterion for Ding-semistability in [Fuj15b]. For simplicity, we denote $\tilde{\mathcal{F}}^x = \mathcal{F}^{c_1 x}$. By (14) and (15) we can choose e_+ with

$$e_+ \geq \frac{\sup_m \frac{v_1}{v_0}}{\inf_m \frac{v_1}{v_0}} \geq \frac{e_{\max}(\mathcal{F})}{c_1} = e_{\max}(\tilde{\mathcal{F}})$$

and $e_- = 1$. Notice that by relation (14), $e_{\min}(\tilde{\mathcal{F}}) = e_{\min}(\mathcal{F})/c_1 \geq 1$. Denote by the \mathcal{I}_V the ideal sheaf of V as a subvariety of Y . Notice that Y is nothing but the total space of the line bundle $L^{-1} \rightarrow V$ so that we have a canonical projection denoted by $\rho : Y \rightarrow V$. Now similar to that in [Fuj15b], we define:

$$\begin{aligned} \mathcal{I}_{(m,x)}^{\tilde{\mathcal{F}}} &= \text{Image} \left(\tilde{\mathcal{F}}^x R_m \otimes L^{-m} \rightarrow \mathcal{O}_V \right) \\ \tilde{\mathcal{I}}_m &= \rho^* \mathcal{I}_{(m, m e_+)}^{\tilde{\mathcal{F}}} + \rho^* \mathcal{I}_{(m, m e_+ - 1)}^{\tilde{\mathcal{F}}} \cdot \mathcal{I}_V + \cdots \\ &\quad \cdots + \rho^* \mathcal{I}_{(m, m e_- + 1)}^{\tilde{\mathcal{F}}} \cdot \mathcal{I}_V^{m(e_+ - e_-) - 1} + \mathcal{I}_V^{m(e_+ - e_-)}. \end{aligned} \quad (33)$$

Then $\mathcal{I}_{(m,x)}^{\tilde{\mathcal{F}}}$ (resp. $\tilde{\mathcal{I}}_m$) is an ideal sheaf on V (resp. Y). Moreover, $\tilde{\mathcal{I}}_\bullet := \{\tilde{\mathcal{I}}_m\}_m$ is a graded family of coherent ideal sheaves by [Fuj15b, Proposition 4.3]. By Lemma 4.2, we have $v_1(\mathcal{I}_V) = c_1$. On the other hand, using Lemma 4.6, we get:

$$v_1(\mathcal{I}_{(m,x)}^{\tilde{\mathcal{F}}}) = v_1(\text{Image}(\mathcal{F}^{c_1 x} R_m \otimes L^{-m} \rightarrow \mathcal{O}_V)) \geq c_1 x - c_1 m. \quad (34)$$

Combining (34) and the inclusion $\mathcal{I}_V^{m(e_+ - e_-)} \subset \tilde{\mathcal{I}}_m$, we get $v_1(\tilde{\mathcal{I}}_m) = v_1(\mathcal{I}_V^{m(e_+ - e_-)}) = m(e_+ - e_-)v_1(\mathcal{I}_V) = mc_1(e_+ - 1)$ and hence

$$v_1(\tilde{\mathcal{I}}_\bullet) = \inf_m \frac{v_1(\tilde{\mathcal{I}}_m)}{m} = c_1(e_+ - e_-) = c_1(e_+ - 1).$$

We define d_∞ as in Fujita's Theorem 2.10:

$$d_\infty = 1 - r(e_+ - e_-) + \frac{r^n}{(-K_V)^{n-1}} \int_1^{e_+} \text{vol}(\tilde{\mathcal{F}}R^{(t)}) dt.$$

Then we get:

$$\begin{aligned} v_1(\tilde{\mathcal{I}}_\bullet \cdot \mathcal{I}_V^{d_\infty}) &= rc_1(e_+ - e_-) + c_1 d_\infty = c_1 \left(1 + \frac{r^n}{(-K_V)^{n-1}} \int_1^{e_+} \text{vol}(\tilde{\mathcal{F}}R^{(t)}) dt \right) \\ &= c_1 \left(1 + \frac{r^n}{(-K_V)^{n-1}} \int_1^{+\infty} \text{vol}(\tilde{\mathcal{F}}R^{(c_1 t)}) dt \right) \end{aligned}$$

Comparing the above expression with that in (32) we get the estimate:

$$\begin{aligned} \Phi_s(\lambda_*, 0) &= \frac{nL^{n-1}}{A_X(v_1)} \left(A_Y(v_1) - v_1(\tilde{\mathcal{I}}_\bullet \cdot \mathcal{I}_V^{d_\infty}) \right) \\ &\quad + \frac{nL^{n-1}}{A_X(v_1)} \left(\frac{r^n}{(-K_V)^{n-1}} \int_1^{+\infty} \left(\text{vol}(\tilde{\mathcal{F}}R^{(c_1 t)}) - \text{vol}(\mathcal{F}R^{(c_1 t)}) \right) dt \right) \\ &\geq \frac{nL^{n-1}}{A_X(v_1)} \left(A_Y(v_1) - v_1(\tilde{\mathcal{I}}_\bullet \cdot \mathcal{I}_V^{d_\infty}) \right). \end{aligned}$$

Note that the last inequality is actually an equality because \mathcal{F} is saturated by Lemma 4.6. So to show $\Phi_s(\lambda_*, 0) \geq 0$, we just need to show

$$A_Y(v_1) - v_1(\tilde{\mathcal{I}}_\bullet^r \cdot \mathcal{I}_V^{d_\infty}) \geq 0. \quad (35)$$

We prove this by deriving the following regularity from Fujita's result.

Lemma 4.7. *Assume that (V, K_V^{-1}) is K -semistable. Then $(Y, \tilde{\mathcal{I}}_\bullet^r \cdot \mathcal{I}_V^{d_\infty})$ is sub log canonical.*

Proof. By Fujita's Theorem 2.10 ([Fuj15b, Theorem 4.9]) and Remark 2.11, $(V \times \mathbb{C}, \mathcal{I}_\bullet^r \cdot (t)^{d_\infty})$ is sub log canonical where $\mathcal{I}_\bullet = \{\mathcal{I}_m\}$ is the ideal sheaf on $V \times \mathbb{C}$ defined in (10). Informally we get \mathcal{I}_\bullet on $V \times \mathbb{C}$ if in (33) ρ is replaced by the canonical fibration $\rho_0 : V \times \mathbb{C} \rightarrow V$. By choosing an affine cover of $\{U_i\}$ of V , we have $(\tilde{\mathcal{I}}_m^r \cdot \mathcal{I}_V^{d_\infty})(\rho^{-1}(U_i)) \cong (\mathcal{I}_m^r \cdot (t)^{d_\infty})(\rho_0^{-1}(U_i))$ for any $m \geq 0$. Since sub log canonicity can be tested by testing on all open sets of an affine cover, we get the conclusion. \square

By the above Lemma, we get (35) using the same approximation argument as in [BFFU13, Proof of Theorem 4.1]. Because the space of divisorial valuations is dense in $\text{Val}_{X,o}$ we want to use some semicontinuity properties to get the inequality (35) that already holds for divisorial valuations. More precisely, the sub log canonicity means that, for any divisorial valuation ord_F over Y , we have:

$$A_Y(F) - r \cdot \text{ord}_F(\tilde{\mathcal{I}}_\bullet) - d_\infty \cdot \text{ord}_F(\mathcal{I}_V) \geq 0.$$

Consider the function $\phi : \text{Val}_{X,o} \rightarrow \mathbb{R} \cup \{+\infty\}$ defined by:

$$\phi(v) = A_Y(v) - rv(\tilde{\mathcal{I}}_\bullet) - d_\infty \cdot v(\mathcal{I}_V) = A_X(v) - rv(\tilde{\mathcal{I}}_\bullet) - (r - 1 + d_\infty) \cdot v(\mathcal{I}_V).$$

If we endow $\text{Val}_{X,o}$ with the topology of pointwise convergence, then

- $v \mapsto v(\mathcal{I}_V)$ is continuous by [BFFU13, Proposition 2.4];
- $v \mapsto v(\tilde{\mathcal{I}}_\bullet)$ is upper semicontinuous by [BFFU13, Proposition 2.5];
- $v \mapsto A_Y(v)$ is lower semicontinuous by [BFFU13, Theorem 3.1].

Combining these properties we know that ϕ is a lower semi-continuous function on $\text{Val}_{X,o}$. Furthermore it was shown in [BFFU13] that ϕ is continuous on small faces of $\text{Val}_{X,o}$ and satisfies $\phi \geq \phi \circ r_\tau$ for any τ a good resolution dominating the blow up $\tau : Y \rightarrow X$ (we refer to [BFFU13] for the definition of the contraction r_τ and the notions of *small faces* and *good resolutions*). This allows us to show that $\phi \geq 0$ on the space of divisorial valuations implies $\phi(v) \geq 0$ for any valuation $v \in \text{Val}_{X,o}$.

4.2.1 Appendix: Interpolations in the \mathbb{C}^* -invariant case

Assume v_1 is \mathbb{C}^* -invariant. Let $w = v_1|_{\mathbb{C}(V)}$ be its restriction as considered at the beginning of Section 4.2. We can connect the two valuations v_0 and v_1 by a family of \mathbb{C}^* -invariant valuations v_s for $s \in [0, 1]$. v_s is defined as the following \mathbb{C}^* -invariant extension of sw : for any $f_k \in R_k$ define:

$$v_s(f_k) = sw(f_k) + (sc_1 + (1 - s))k = (1 - s)v_1(f_k) + sc_1v_0(f_k). \quad (36)$$

The second identity follows from (28). For $f \in R$, define

$$v_s(f) = \min \left\{ v_s(f_k); f = \sum_k f_k \text{ with } f_k \neq 0 \right\}.$$

Because $c_1 > 0$, it's easy to see that $\text{center}_X(v_s) = o$, i.e. $v_s \in \text{Val}_{X,o}$. From (36) v_s is indeed an interpolation between $v_0 = \text{ord}_V$ and the given valuation v_1 . The valuation v_s is \mathbb{C}^* -invariant and its valuative ideals $\mathfrak{a}_m(v_s)$ are homogeneous under the natural \mathbb{N} -grading of $R = \bigoplus_k R_k$.

Proposition 4.8. *Assume that v_1 is \mathbb{C}^* -invariant. The volume of v_s defined above is equal to $\Phi(1, s)$ defined in (23). In other words, we have the formula:*

$$\text{vol}(v_s) = \int_{c_1}^{+\infty} \frac{-d\text{vol}(R^{(t)})}{((1 - s) + st)^n}. \quad (37)$$

Proof. Because $\mathfrak{a}_m(v_s)$ is homogeneous and using (36) and Lemma 4.6, we have:

$$\begin{aligned} \dim_{\mathbb{C}} R/\mathfrak{a}_m(v_s) &= \sum_{i=0}^{+\infty} \dim_{\mathbb{C}} R_i / (\mathfrak{a}_m(v_s) \cap R_i) \\ &= \sum_{i=0}^{+\infty} (\dim_{\mathbb{C}} R_i - \dim_{\mathbb{C}} (\mathfrak{a}_{m-(1-s)i/s}(v_1) \cap R_i)) \\ &= \sum_{i=0}^{+\infty} \left(\dim_{\mathbb{C}} R_i - \dim_{\mathbb{C}} \left(\mathcal{F}^{\frac{m-(1-s)i}{s}} R_i \right) \right). \end{aligned}$$

Because $\mathcal{F}^x R_i = R_i$ if $x \leq c_1 i$ by the definition of c_1 (see the discussion before Lemma 4.2), we deduce that:

$$\mathcal{F}^{\frac{m-(1-s)i}{s}} R_i = R_i \text{ if } \frac{m-(1-s)i}{s} \leq c_1 i \text{ or equivalently } i \geq \frac{m}{c_1 s + (1-s)}.$$

So we get the following identity:

$$n! \dim_{\mathbb{C}} R/\mathfrak{a}_m(v_s) = n! \sum_{i=0}^{\lfloor m/(c_1 s + (1-s)) \rfloor} \dim_{\mathbb{C}} R_i - \dim_{\mathbb{C}} \mathcal{F}^{\frac{m-(1-s)i}{s}} R_i. \quad (38)$$

For the first part of the sum, we have:

$$n! \sum_{i=1}^{\lfloor m/(c_1 s + (1-s)) \rfloor} \dim_{\mathbb{C}} R_i = \frac{m^n}{(c_1 s + (1-s))^n} L^{n-1} + O(p^{n-1}). \quad (39)$$

For the second part of the sum, we use Lemma 4.5 to get:

$$\begin{aligned} \lim_{m \rightarrow +\infty} \frac{n!}{m^n} \sum_{i=0}^{\lfloor m/(c_1 s + (1-s)) \rfloor} \dim_{\mathbb{C}} (\mathcal{F}^{\frac{m-(1-s)i}{s}} R_i) \\ = n \int_{c_1}^{+\infty} \text{vol}(R^{(t)}) \frac{sdt}{((1-s) + st)^{n+1}}. \end{aligned} \quad (40)$$

Combining (38), (39) and (40), we have:

$$\begin{aligned} \text{vol}(v_s) &= \lim_{m \rightarrow +\infty} \frac{n!}{m^n} \dim_{\mathbb{C}} R/\mathfrak{a}_m(v_s) \\ &= \frac{1}{(c_1 s + (1-s))^n} L^{n-1} - n \int_{c_1}^{+\infty} \text{vol}(R^{(t)}) \frac{sdt}{((1-s) + st)^{n+1}}. \end{aligned} \quad (41)$$

We can use integration by parts to further simplify the formula:

$$\begin{aligned} \text{vol}(v_s) &= \frac{1}{(c_1 s + (1-s))^n} L^{n-1} + \int_{c_1}^{+\infty} \text{vol}(R^{(t)}) d \left(\frac{1}{((1-s) + st)^n} \right) \\ &= \frac{1}{(c_1 s + (1-s))^n} L^{n-1} + \\ &\quad \left[\text{vol}(R^{(t)}) \frac{1}{(st + (1-s))^n} \Big|_{t=c_1}^{+\infty} \right] - \int_{c_1}^{+\infty} \frac{d\text{vol}(R^{(t)})}{(st + (1-s))^n} \\ &= - \int_{c_1}^{+\infty} \frac{d\text{vol}(R^{(t)})}{(st + (1-s))^n}. \end{aligned} \quad (42)$$

□

5 Uniqueness among \mathbb{C}^* -invariant valuations

In this section, we prove Theorem 3.4. We first prove the result for divisorial valuations. Suppose E_2 is a prime divisor centered at o such that $\widehat{\text{vol}}(\text{ord}_{E_2}) = \widehat{\text{vol}}(\text{ord}_V)$. We want to

show that $E_2 = V$. The main observation is that the interpolation function $\Phi(\lambda_*, s)$ in (23) must be a constant function independent of $s \in [0, 1]$. Indeed in this case $\Phi(s) := \Phi(\lambda_*, s)$ is a convex function satisfying $\Phi(0) = \Phi(1) = \min_{s \in [0, 1]} \Phi(s)$. So $\Phi(s) \equiv \Phi(0) = \Phi(1)$. Recall the expression in (23):

$$\Phi(s) = \frac{1}{(\lambda_* c_1 s + (1-s))^n} L^{n-1} - n \int_{c_1}^{c_2} \text{vol}(R^{(t)}) \frac{\lambda_* s dt}{(1-s + \lambda_* s t)^{n+1}}. \quad (43)$$

Here we have changed the improper integral to a finite integral by choosing $c_2 \gg 1$ such that $\text{vol}(R^{(t)}) = 0$ for $t \geq c_2$. Because $\text{vol}(R^{(t)})$ is piecewise continuous on $[c_1, c_2]$, we easily verify that $\Phi(s)$ in (43) is a smooth function of $s \in [0, 1]$. We can calculate its second order derivative:

$$\begin{aligned} \Phi''(s) &= \frac{(n+1)n(\lambda_* c_1 - 1)^2}{(\lambda_* c_1 s + (1-s))^{n+2}} L^{n-1} \\ &\quad - n(n+1) \int_{c_1}^{c_2} \text{vol}(R^{(t)}) \frac{\lambda_*(\lambda_* t - 1)(-2 + ns(\lambda_* t - 1)) dt}{(1-s + \lambda_* s t)^{n+3}} \\ &= -n(n+1) \int_{c_1}^{c_2} (\lambda_* t - 1)^2 \frac{d\text{vol}(R^{(t)})}{(\lambda_* s t + (1-s))^{n+2}}. \end{aligned} \quad (44)$$

The second identity follows from integration by parts. By Proposition 2.9, $\nu = -\frac{1}{L^{n-1}} d\text{vol}(R^{(t)})$, supported on the interval $[\lambda_{\min}, \lambda_{\max}]$, is absolutely continuous on $[\lambda_{\min}, \lambda_{\max})$ with respect to the Lebesgue measure and possibly has a Dirac mass at λ_{\max} . We also know that $\Phi''(s) \equiv 0$ for $s \in [0, 1]$ since $\Phi(s)$ is a constant. Using the last expression in (44), we see that $\lambda_{\max} = \lambda_{\min}$ and $\nu = \delta_{\lambda_{\max}}$. Otherwise, $\Phi''(s)$ is not identically equal to zero on the nonempty open interval $(\lambda_{\min}, \lambda_{\max})$.

We will show that ν being a Dirac measure indeed implies $\text{ord}_{E_2} = \text{ord}_V$. The latter statement can be thought of a counterpart of the result in [BHJ15, Theorem 6.8]. Indeed, the argument given below is motivated the proof in [BHJ15, Lemma 5.13].

Let $\text{ord}_{E_2}^{(0)}$ denote the restriction of ord_{E_2} under the inclusion $\mathbb{C}(V) \subset \mathbb{C}(X)$. As before, $v_1 := \text{ord}_{E_2}$ is a \mathbb{C}^* -invariant extension of $\text{ord}_{E_2}^{(0)}$ in the following way. Choose an affine open neighborhood U of $\text{center}_V(\text{ord}_{E_2}^{(0)})$ and a trivializing section $s \in \mathcal{O}_V(L)(U)$, then any $f \in H^0(V, kL)$ can be locally written as $f = h \cdot s^k$. We define $\text{ord}_{E_2}^{(0)}(f) = \text{ord}_{E_2}(h)$ such that

$$\text{ord}_{E_2}(f) = \text{ord}_{E_2}(h_j) + k \cdot \text{ord}_{E_2}(s_j) = \text{ord}_{E_2}(h_j) + kc_1 = \text{ord}_{E_2}^{(0)}(f) + kc_1.$$

In particular we have,

$$\mathcal{F}^{kx} R_k = \mathcal{F}_{\text{ord}_{E_2}^{(0)}}^{k(x-c_1)} R_k = \{f \in R_k; \text{ord}_{E_2}^{(0)}(f) \geq k(x-c_1)\}.$$

Let Z be the center of $\text{ord}_{E_2}^{(0)}$ on V . Then by general Izumi's theorem (see [HS01]), we have:

$$\text{ord}_{E_2}^{(0)}(h) \leq C_2 w(h), \text{ for any } h \in \mathcal{O}_{V,Z}, \quad (45)$$

where w is a Rees valuation of Z . Let $V' \rightarrow V$ be the normalized blow up of Z inside V and $G = \mu^{-1}Z$. The Rees valuations are given up to scaling by vanishing order along irreducible components of G (see [Laz04, Example 9.6.3], [BHJ15, Definition 1.9]). By Izumi's theorem, the Rees valuations of Z are comparable to each other ([Ree89, HS01]). So for any $x > c_1$ and $0 < \delta \ll 1$, we have:

$$\mathcal{F}^{kx} R_k = \mathcal{F}_{\text{ord}_{E_2}^{(0)}}^{k(x-c_1)} R_k \subseteq H^0(\text{Bl}_Z V, k(L - \delta G)).$$

Since $\mu^*L - \delta G$ is ample on V' for $0 < \delta \ll 1$, this implies:

$$\text{vol}(\mathcal{F}R^{(x)}) \leq (\mu^*L - \delta G)^{n-1} < L^{n-1}$$

for any $x > c_1$. So $\lambda_{\min} = c_1 = \inf_m \frac{\text{ord}_{E_2}}{\text{ord}_V}$ by (7) in Lemma 2.9. For any $x < \lambda_{\max}$, $\text{vol}(R^{(x)}) > 0$ by [BC11, Lemma 1.6] (see also [BHJ15, Theorem 5.3]). It's easy to see that

$\lambda_{\max} = \sup_m \frac{\text{ord}_{E_2}}{\text{ord}_V} =: c_2$ by (8) and our definition of filtration \mathcal{FR} (in the \mathbb{C}^* -invariant case).

So if $d\text{vol}(R^{(x)})$ is the Dirac measure, then $c_1 = c_2$ and hence $\text{ord}_{E_2} = \text{ord}_V$.

It's clear that the above proof works well for a \mathbb{C}^* -invariant valuation v_1 as long as the general Izumi's inequality as in (45) holds for $w := v_1|_{\mathbb{C}(V)}$. In particular, this holds for any \mathbb{C}^* -invariant quasi-monomial valuation v_1 . Indeed, if v_1 is a \mathbb{C}^* -invariant quasi-monomial valuation. Then $w = v_1|_{\mathbb{C}(V)}$ is also a quasi-monomial valuation by Abhyankar-Zariski inequality (see [BHJ15, (1.3)]). Denote $Z = \text{center}_V(w)$. We can find a log smooth model $(W, \sum_i E_i) \rightarrow V$ such $\text{center}_V E_i = Z$ and w is a monomial valuation on the model $(W, \sum_i E_i)$. Since the general Izumi inequality holds for ord_{E_i} , it also holds for w (cf. [BFJ12]).

Remark 5.1. *We expect that the generalized Izumi inequality holds for any valuation $v^{(0)}$ of $\mathbb{C}(V)$ with $A_V(v^{(0)}) < +\infty$, i.e. there exists $C = C(v^{(0)})$ such that*

$$v^{(0)}(h) \leq Cw(h) \text{ for any } h \in \mathcal{O}_{V,Z}$$

where $Z = \text{center}_V(v^{(0)})$ and w is a Rees valuation of Z . If Z is a closed point, this is indeed true (see [Li15a]). The general case can probably be reduced to the case of closed point (cf. [JM10, Proof of Proposition 5.10]).

6 Equivariant volume minimization implies K-semistability

In this section, we will prove the other direction of implication.

Theorem 6.1. *Let V be a \mathbb{Q} -Fano variety and $L_* = r_*^{-1}K_V^{-1}$ an ample Cartier divisor for a fixed $r_* \in \mathbb{Q}_{>0}$. Assume that $\widehat{\text{vol}}$ is minimized among \mathbb{C}^* -invariant valuations at ord_V on $X_* := C(V, L_*)$. Then V is K-semistable.*

As explained in Section 3, this combined with Theorem 4.1 will complete the proof of Theorem 3.1. Indeed, Theorem 4.1 shows (1) \Rightarrow (3) and (3) \Rightarrow (2) trivially.

The rest of this section is devoted to the proof of this theorem. By [LX14], to prove the K-semistability, we only need to consider special test configurations. As we will see, this reduction simplifies the calculations and arguments in a significant way. So from now on, we assume that there is a special test configuration $(\mathcal{V}, \mathcal{L}) \rightarrow \mathbb{C}$ such that $\mathcal{L} = -r^{-1}K_{\mathcal{V}/\mathbb{C}}$ is a relatively very ample line bundle. Notice that r is in general not the same as r_* . However, for the sake of testing K-semistability which is of asymptotic nature (see [Tia97, Don02, LX14]), we can choose $0 < r = \frac{1}{N}$ for $N \gg 1$ sufficiently divisible such that L is an integral multiple of L_* . For later convenience we will define this constant:

$$\sigma := \frac{r^{-1}}{r_*^{-1}} = \frac{r_*}{r} \in \mathbb{Z}_{>0}, \text{ such that } L = \sigma L_*. \quad (46)$$

The central fibre of $\mathcal{V} \rightarrow \mathbb{C}$ is a \mathbb{Q} -Fano variety, denoted by \mathcal{V}_0 or E . Because the polarized family $(\mathcal{V}, \mathcal{L}) \rightarrow \mathbb{C}$ is flat, we can choose $r^{-1} = N \gg 1$ sufficiently divisible such that $\mathcal{L} = -r^{-1}K_{\mathcal{V}/\mathbb{C}} = -NK_{\mathcal{V}/\mathbb{C}}$ is a relatively very ample line bundle over \mathbb{C} and $H^0(V, mL) = H^0(\mathcal{V}_0, m\mathcal{L}_0)$ for every $m \in \mathbb{Z}_{>0}$. By taking affine cones, we then get a flat family $\mathcal{X} \rightarrow \mathbb{C}$ with general fibre $X = C(V, L)$ and the central fibre $\mathcal{X}_0 = C(\mathcal{V}_0, \mathcal{L}_0)$ (see [Kol13, Aside on page 98]). Notice that we can write:

$$\begin{aligned} X_* &= \text{Spec} \bigoplus_{i=0}^{+\infty} H^0(V, iL_*) &= \text{Spec } R_*; \\ X &= \text{Spec} \bigoplus_{i=0}^{+\infty} H^0(V, iL) &= \text{Spec } R; \\ \mathcal{X}_0 &= \text{Spec} \bigoplus_{i=0}^{+\infty} H^0(\mathcal{V}_0, i\mathcal{L}_0) &= \text{Spec } S; \\ \mathcal{X} &= \text{Spec} \bigoplus_{i=0}^{+\infty} H^0(\mathcal{V}, i\mathcal{L}) &= \text{Spec } \mathcal{R}. \end{aligned}$$

Our plan is to define families of valuations wt_β on \mathcal{X}_0 , w_β on X and $w_{*\beta}$ on X_* , and then study the relation between their normalized volumes.

6.1 Valuation wt_β on \mathcal{X}_0

By the definition of special test configuration, there is an equivariant $T_1 = \mathbb{C}^*$ action on $(\mathcal{V}, \mathcal{L}) \rightarrow \mathbb{C}$ which fixes the central fibre \mathcal{V}_0 . This naturally induces an equivariant $T_1 =: e^{\mathbb{C}\eta}$ action on $\mathcal{X} \rightarrow \mathbb{C}$ which fixes \mathcal{X}_0 . There is also another natural $T_0 := e^{\mathbb{C}\xi_0} \cong \mathbb{C}^*$ -action which fixes each point on the base \mathbb{C} and rescales the fibre. These two actions commute and generate a $\mathbb{T} = (\mathbb{C}^*)^2$ action on \mathcal{X}_0 . For any $\tau = (\tau_0, \tau_1) \in \mathbb{Z}^2$ and $t = (t_0, t_1) \in (\mathbb{C}^*)^2$, we will denote $t^\tau = t_0^{\tau_0} t_1^{\tau_1}$. Consider the weight decomposition of S under the \mathbb{T} action:

$$S = \bigoplus_{\tau \in \Gamma} S_\tau,$$

where $S_\tau = \{f \in S; t \circ f = t^\tau \cdot f \text{ for all } t \in (\mathbb{C}^*)^2\}$ for any $\tau \in \mathbb{Z}^2$ and $\Gamma = \{\tau \in \mathbb{Z}^2; S^\tau \neq \{0\}\} \subset \mathbb{Z}^2$.

Denote by $\mathfrak{t}_{\mathbb{C}}$ the Lie algebra of \mathbb{T} . The two generator $\{\xi_0, \eta\}$ allows us to identify $\mathfrak{t}_{\mathbb{C}} = N \otimes_{\mathbb{Z}} \mathbb{C}$ for a lattice $N \cong \mathbb{Z}^2$. Let $\mathfrak{t}_{\mathbb{R}} = N \otimes_{\mathbb{Z}} \mathbb{R}$. Any element $\xi \in \mathfrak{t}_{\mathbb{R}}$ is a holomorphic vector field of the form $\xi = a_0 \xi_0 + a_1 \eta$. Any $\xi = a_0 \xi_0 + a_1 \eta \in \mathfrak{t}_{\mathbb{R}}$ determines a valuation wt_ξ on $\mathbb{C}(\mathcal{X}_0)$:

$$\text{wt}_\xi(f) = \min \{\langle \tau, \xi \rangle; f^\tau \neq 0\} \text{ for } f = \sum_{\tau} f^\tau \in S.$$

This is clearly a \mathbb{T} -invariant valuation. In particular it is \mathbb{C}^* -invariant, i.e. T_0 -invariant.

Definition 6.2. *The cone of positive vectors is defined to be:*

$$\mathfrak{t}_{\mathbb{R}}^+ = \{\xi = a_0 \xi_0 + a_1 \eta \in \mathfrak{t}_{\mathbb{R}}; \langle \tau, \xi \rangle = a_0 \tau_0 + a_1 \tau_1 > 0 \text{ for any } \tau \in \Gamma \setminus (0, 0)\}. \quad (47)$$

$\mathfrak{t}_{\mathbb{R}}^+$ is essentially the same as the cone of Reeb vector fields considered in Sasaki geometry (see [MSY08]). The following is a standard fact:

Lemma 6.3. *The cone $\mathfrak{t}_{\mathbb{R}}^+$ contains an open neighborhood of ξ_0 .*

Proof. For a fixed $\tau_0 \in \mathbb{Z}_{>0}$,

$$H^0(\mathcal{V}_0, \tau_0 L_0) = S_{\tau_0} = \bigoplus_{\tau_1} S_{(\tau_0, \tau_1)} \text{ with } S_{(\tau_0, \tau_1)} \neq 0$$

is the weight space decomposition with respect to the action $T_1 = e^{\mathbb{C}\eta}$ on $(\mathcal{V}_0, \mathcal{L}_0)$. Because S is finitely generated, there exists $A > 0 \in \mathbb{R}$ such that $-A\tau_0 < \tau_1 < A\tau_0$ for any (τ_0, τ_1) appearing in the above decomposition (see [BHJ15, Proof of Corollary 3.4]). So for any $(a_0, a_1) \in \mathbb{R}^2$ with $a_0 > 0$ and $|a_1 a_0^{-1}| < A^{-1}$, we have:

$$a_0 \tau_0 + a_1 \tau_1 \geq a_0(\tau_0 + a_0^{-1} a_1 \tau_1) \geq a_0(\tau_0 - |a_0^{-1} a_1| A \tau_0) > 0.$$

The lemma follows immediately by choosing $a_0 = 1$. \square

One sees directly that $\xi \in \mathfrak{t}_{\mathbb{R}}^+$ if and only if the center of wt_ξ is the vertex $o' \in \mathcal{X}_0$, i.e. $\text{wt}_\xi \in \text{Val}_{\mathcal{X}_0, o'}$. For $\beta \in \mathbb{R}$, let $\xi_\beta = \xi_0 + \beta \eta \in \mathfrak{t}_{\mathbb{R}}$. The associated valuation wt_{ξ_β} will be denoted by wt_β . It's clear that $\xi_0 \in \mathfrak{t}_{\mathbb{R}}^+$. By Lemma 6.3, for $0 \leq \beta \ll 1$, $\xi_\beta \in \mathfrak{t}_{\mathbb{R}}^+$ and hence the valuation wt_β is in $\text{Val}_{\mathcal{X}_0, o'}$. Moreover, from the definition, we know that

$$\text{wt}_\beta(f) = k + \beta \cdot \text{wt}_\eta(f) \text{ for any } f \in S_k = H^0(\mathcal{V}_0, k\mathcal{L}_0). \quad (48)$$

6.2 Valuation w_β on X

For any $f \in H^0(V, mL)$, f determines a T_1 -invariant section $\bar{f} \in H^0(V \times \mathbb{C}^*, \pi_1^*(mL))$. Since we have a T_1 -equivariant isomorphism $(\mathcal{V} \setminus \mathcal{V}_0, \mathcal{L}) \cong (V \times \mathbb{C}^*, \pi_1^* L)$, \bar{f} can be seen as a meromorphic section of $H^0(\mathcal{V}, m\mathcal{L})$. Recall that \mathcal{V}_0 will also be denoted by E . We will define a valuation $v_E^{(1)}$ on $\mathbb{C}(X)$ satisfying $v_E^{(1)}(f) = \text{ord}_{\mathcal{V}_0}(\bar{f})$ for any $f \in H^0(V, mL)$ and another valuation $v_{*E}^{(1)}$ on $\mathbb{C}(X_*)$ such that $v_E^{(1)}$ is the restriction of $v_{*E}^{(1)}$ under inclusion $\mathbb{C}(X) \hookrightarrow \mathbb{C}(X_*)$ that is induced by the natural embedding:

$$R = \bigoplus_{k=0}^{+\infty} H^0(V, kL) = \bigoplus_{k=0}^{+\infty} H^0(V, \sigma kL_*) \hookrightarrow R_* = \bigoplus_{m=0}^{+\infty} H^0(V, mL_*).$$

To get a precise formula for $v_E^{(1)}$ and $v_{*E}^{(1)}$, we need some preparations. Denote by $v_{\mathcal{V}_0}$ or v_E the restriction of the valuation $\text{ord}_{\mathcal{V}_0}$ on $\mathbb{C}(\mathcal{V}) \cong \mathbb{C}(V \times \mathbb{C})$ to $\mathbb{C}(V)$. By [BHJ15, Lemma 4.1], if \mathcal{V}_0 is not the strict transform of $V \times \{0\}$, then

$$v_{\mathcal{V}_0} = v_E = \text{ord}_{\mathcal{V}_0}|_{\mathbb{C}(V)} = q \cdot \text{ord}_F \quad (49)$$

for a prime divisor F over V and a rational $q > 0$ (called multiplicity of v). Otherwise $v_{\mathcal{V}_0} = v_E$ is trivial.

Remark 6.4. *A product test configuration, i.e. a test configuration induced from a \mathbb{C}^* action on V , does not necessarily give rise to a trivial valuation. See [LX14, Example 3] (see also [BHJ15, Example 2.8]).*

v_E induces a valuation $v_E^{(0)}$ on $\mathbb{C}(X)$ and a valuation $v_{*E}^{(0)}$ on $\mathbb{C}(X_*)$ in the following way. Choose an affine open neighborhood U of $\text{center}_V(v_E)$ and a local trivializing section s of L on U . Any $f \in H^0(V, mL)$ can be locally written as $f = hs^m$ on U . Define:

$$v_E^{(0)}(f) = v_E(h).$$

It's clear that $v_E^{(0)}(f) \geq 0$ for any $f \in H^0(V, mL)$ and the equality holds if and only if f does not vanish on the center of $v_E^{(0)}$ over V . In particular, we see that $v_E^{(0)}$ is trivial if and only if v_E is trivial. If $v_E = q \cdot \text{ord}_F$ is not trivial and hence a divisorial valuation. We claim that

Lemma 6.5. *$v_E^{(0)}$ is a divisorial valuation.*

Proof. We can find a birational morphism $\mu : V' \rightarrow V$ such that F is a prime divisor on V' . Y' denotes the total space of the pull back line bundle $\pi' : (\mu')^*L^{-1} \rightarrow V'$. Define $\mathbb{F} = \pi'^{-1}(F)$, which is a prime divisor on Y' . Then we have a sequence of birational morphisms $Y' \xrightarrow{\pi_3} Y = \text{Bl}_o X \xrightarrow{\pi_2} X$, and by definition it's easy to see that $v_E^{(0)} = q \cdot \text{ord}_{\mathbb{F}}$. \square

The above construction can be carried out for $X_* = C(V, L_*)$ by using L_* instead of L and we get a valuation denoted by $v_{*E}^{(0)}$, a divisor \mathbb{F}_* over X_* such that $v_{*E}^{(0)} = q \cdot \text{ord}_{\mathbb{F}_*}$. One verifies that $v_E^{(0)}$ is the restriction of $v_{*E}^{(0)}$ under the natural embedding $R \hookrightarrow R_*$. Later we will need the following log discrepancies.

Lemma 6.6. *We have identities of log discrepancies:*

$$A_X(\mathbb{F}) = A_V(F) = A_{X_*}(\mathbb{F}_*). \quad (50)$$

Proof. Let $Y = \text{Bl}_o X \rightarrow X$ be the blow up with the exceptional divisor V . Then we have:

$$A_X(\text{ord}_{\mathbb{F}}) = A_{(Y, (1-r)V)}(\text{ord}_{\mathbb{F}}) = A_Y(\text{ord}_{\mathbb{F}}) + (r-1)\text{ord}_{\mathbb{F}}(V) = A_Y(\text{ord}_{\mathbb{F}}).$$

The last identity is because that $\text{ord}_{\mathbb{F}}(V) = 0$ by the definition of $\text{ord}_{\mathbb{F}} = v_E^{(0)}$. Now let $\mu : V' \rightarrow V$ and $\pi_2 : Y' \rightarrow Y$ be the same as in the above paragraph. Assume that the support of exceptional divisors of μ is decomposed into irreducible components as $F + D_1 + \dots + D_k =: F + D$. Then correspondingly the support of exceptional divisor of π_2 can be decomposed into $\mathbb{F} + \mathbb{D}$ where $\mathbb{D} = \pi'^{-1}D$. So we can write:

$$K_{Y'} + \mathbb{F} = \pi_2^* K_Y + A_Y(\mathbb{F})\mathbb{F} + \Delta, \quad \pi_2^* V = V',$$

where Δ has the support contained in D . By adjunction, we get $K_{V'} + F = \mu^* K_V + A_Y(\mathbb{F})F + \Delta|_{V'}$, and hence $A_V(F) = A_Y(\mathbb{F})$. The same argument can be applied to (X_*, \mathbb{F}_*) and we get the second identity. \square

Analogous to the discussion in [BHJ15], there are \mathbb{C}^* -invariant extensions of v_E from $\mathbb{C}(V)$ to $\mathbb{C}(X)$ defined as follows. For any $f \in \mathbb{C}(X)$, we can use T_0 -action to get Laurent typed series $f = \sum_{m \in \mathbb{Z}} f_m$ with f_m a meromorphic section of mL . Then we define a family of extensions of v_E with parameter λ :

$$\mathbb{V}_\lambda(f) := \min_{m \in \mathbb{Z}} \left\{ v_E^{(0)}(f_m) + \lambda m \right\}. \quad (51)$$

It's clear that \mathbb{V}_λ is \mathbb{C}^* -invariant, i.e. T_0 -invariant. In the same way, we define the extensions of v_E from $\mathbb{C}(V)$ to $\mathbb{C}(X_*)$. For any $g_* = \sum_{k \in \mathbb{Z}} g_{*k} \in \mathbb{C}(X_*)$ with g_{*k} a meromorphic section of kL_* , we define:

$$\mathbb{V}_{*\mu}(g_*) := \min_{k \in \mathbb{Z}} \left\{ v_{*E}^{(0)}(g_{*k}) + \mu k \right\}.$$

Because $H^0(V, mL) = H^0(V, m\sigma L_*)$, it's easy to verify that \mathbb{V}_λ is the restriction of $\mathbb{V}_{*\lambda/\sigma}$ under the embedding $R \subset R_*$ and $\mathbb{C}(X) \subset \mathbb{C}(X_*)$.

Let $A_{V \times \mathbb{C}}(\mathcal{V}_0)$ be the log discrepancy of \mathcal{V}_0 over $V \times \mathbb{C}$, and define a constant

$$a_1 := -\frac{A_{V \times \mathbb{C}}(\mathcal{V}_0) - 1}{r}. \quad (52)$$

Remark 6.7. By [BHJ15, Proposition 4.11], we have $a_1 = -q \cdot A_V(F)/r$.

Definition 6.8. Define a valuation on $\mathbb{C}(X)$ and a valuation on $\mathbb{C}(X_*)$ as follows:

$$v_E^{(1)} = \mathbb{V}_{a_1} \text{ on } R, \quad v_{*E}^{(1)} = \mathbb{V}_{*(a_1/\sigma)} \text{ on } R_*. \quad (53)$$

By the following lemma, this is exactly what we want.

Lemma 6.9. If v_0 denotes the divisorial valuation ord_V , then

1. For any $f \in H^0(V, mL)$, we have:

$$\text{ord}_{\mathcal{V}_0}(\bar{f}) = v_E^{(0)}(f) + a_1 m = v_E^{(0)}(f) + a_1 v_0(f) = \mathbb{V}_{a_1}(f).$$

2. We have a sharp lower bound:

$$v_E^{(1)}(V) = \inf_{\mathfrak{m}} \frac{v_E^{(1)}}{v_0} = a_1.$$

In the following, we will denote by v_{*0} the divisorial valuation ord_V on R_* . If we want to emphasize ord_V as a valuation on $\mathbb{C}(X_*)$, we will write ord_{V_*} . Notice that $\text{ord}_V = v_0 = \frac{v_{*0}}{\sigma} = \frac{\text{ord}_{V_*}}{\sigma}$ under the embedding $R \hookrightarrow R_*$. The following corollary is clearly a consequence of the above lemma.

Corollary 6.10. For any $f \in H^0(V, \sigma mL_*) = H^0(V, mL)$, we have:

$$\text{ord}_{\mathcal{V}_0}(\bar{f}) = v_E^{(0)}(f) + a_1 m = v_{*E}^{(0)}(f) + \frac{a_1}{\sigma} v_{*0}(f) = \mathbb{V}_{*(a_1/\sigma)}(f),$$

and there is a sharp lower bound:

$$v_{*E}^{(1)}(V) = \inf_{\mathfrak{m}} \frac{v_{*E}^{(1)}}{v_{*0}} = \frac{a_1}{\sigma}.$$

Moreover, \mathbb{V}_{a_1} is the restriction of $\mathbb{V}_{*(a_1/\sigma)}$ under the embedding $R \hookrightarrow R_*$.

Proof of Lemma 6.9. The proof is motivated by that of [BHJ15, Lemma 5.17, 5.18]. Let \mathcal{W} be a normal test configuration dominating both $V \times \mathbb{C}$ and \mathcal{V} by the morphisms $\rho_1 : \mathcal{W} \rightarrow V \times \mathbb{C}$ and $\rho_2 : \mathcal{W} \rightarrow \mathcal{V}$. We can write down the identities:

$$K_{\mathcal{W}} = \rho_1^* K_{V \times \mathbb{C}} + a \widehat{\mathcal{V}}_0 + \sum_{k=1}^N d_k F_k = \rho_2^* K_{\mathcal{V}} + b \widehat{\mathcal{V}} + \sum_{k=1}^N e_k F_k \quad (54)$$

By the definition of log discrepancy, we see that $A_{V \times \mathbb{C}}(\mathcal{V}_0) = a + 1$. By (54), we also get:

$$\rho_2^* K_{\mathcal{V}}^{-m/r} = \rho_1^* K_{V \times \mathbb{C}}^{-m/r} - \frac{am}{r} \widehat{\mathcal{V}}_0 + \frac{bm}{r} \widehat{\mathcal{V}} - \sum_{k=1}^N \frac{m(d_k - e_k)}{r} F_k. \quad (55)$$

So by the above identity, for any $f \in H^0(V, mL)$, we have (cf. [BHJ15, Proof of Lemma 5.17])

$$\text{ord}_{\mathcal{V}_0}(\bar{f}) = v_{\mathcal{V}_0}^{(0)}(f) - \frac{am}{r} = v_E^{(0)}(f) - \frac{A_{V \times \mathbb{C}}(\mathcal{V}_0) - 1}{r} m.$$

Because $v_E^{(0)}(f) = \text{ord}_F(f) \geq 0$, we have

$$\frac{v_E^{(1)}(f)}{\text{ord}_V(f)} = \frac{\text{ord}_{\mathcal{V}_0}(\bar{f})}{m} \geq -\frac{a}{r}. \quad (56)$$

Because mL is globally generated when $m \gg 1$, we can find $f' \in H^0(V, mL)$ which is nonzero on the center of $v_{\mathcal{V}_0} = v_E$, so that $\text{ord}_{\mathcal{V}_0}(\bar{f}') = -\frac{a}{r}m$. So we see that the lower bound in (56) can be obtained, and the second statement follows from the proof of Lemma 4.2. \square

Although the valuation $v_E^{(1)}$ is not necessarily positive over R , we can still consider the graded filtration of $v_E^{(1)}$ on R :

$$\begin{aligned} \mathcal{F}_{v_E^{(1)}}^\lambda H^0(V, kL) &= \{f \in H^0(V, kL); v_E^{(1)}(f) = \text{ord}_E(\bar{f}) \geq \lambda\} \\ &= \{f \in H^0(V, kL); t^{-\lambda} \bar{f} \in H^0(\mathcal{V}, k\mathcal{L})\}. \end{aligned}$$

Notice that this filtration was studied in [BHJ15] and [WN12]. This is a decreasing, left-continuous, multiplicative filtration. Moreover, it's linearly bounded from below, which means that:

$$e_{\min} = e_{\min}(R_\bullet, \mathcal{F}_{v_E^{(1)}}) = \liminf_{k \rightarrow +\infty} \frac{\inf \left\{ \lambda; \mathcal{F}_{v_E^{(1)}}^\lambda H^0(V, kL) \neq H^0(V, kL) \right\}}{k} > -\infty.$$

Note that this follows from Lemma 6.9, by which we have $e_{\min} \geq -\frac{A_{V \times \mathbb{C}}(\mathcal{V}_0) - 1}{r}$, and is clearly equivalent to the following statement.

Lemma 6.11. *There exists $M_0 \gg 1 \in \mathbb{Z}$, such that for any $f \in H^0(V, kL)$ and any $k \geq 0$, $t^{M_0 k} \bar{f}$ extends to become a holomorphic section of $H^0(\mathcal{V}, k\mathcal{L})$.*

As a consequence, for any $M \in \mathbb{Z}$ with $M \geq M_0$ we have an embedding of algebras $\Psi_M : R \rightarrow \mathcal{R}$, which maps $f \in H^0(V, kL)$ to $t^{Mk} \bar{f} \in H^0(\mathcal{V}, k\mathcal{L})$. Now we can define a family of valuations on $\mathbb{C}(X)$ which are finite over $R = \bigoplus_{k=0}^{+\infty} H^0(V, kL)$ with the center being the vertex o . For any $\beta = M^{-1}$ with $M \in \mathbb{Z}$ and $\beta^{-1} = M \geq M_0$ and any $f \in H^0(V, kL)$, define

$$\begin{aligned} w_\beta(f) &:= \beta \cdot \text{ord}_{\mathcal{V}_0}(\Psi_{\beta^{-1}}(f)) = \beta \cdot \text{ord}_{\mathcal{V}_0}(t^{\beta^{-1}k} \bar{f}) \\ &= k + \beta \cdot \text{ord}_{\mathcal{V}_0}(\bar{f}) = \text{ord}_V(f) + \beta \cdot v_E^{(1)}(f). \end{aligned} \quad (57)$$

In other words, w_β is a family of valuations starting from $v_0 = \text{ord}_V$ ‘‘in the direction of $v_E^{(1)}$ ’’. Notice that by Lemma 6.9, we have $v_E^{(1)}(f) = v_E^{(0)}(f) + a_1 v_0(f)$, where $a_1 = v_E^{(1)}(V) = -(A_{V \times \mathbb{C}}(\mathcal{V}_0) - 1)/r$. So we can rewrite $w_\beta(f)$ as:

$$w_\beta(f) = v_0(f) + \beta \cdot (v_E^{(0)}(f) + a_1 v_0(f)) = \left[(1 + \beta a_1) v_0 + \beta v_E^{(0)} \right] (f). \quad (58)$$

From the above formula, we see that w_β is nothing but the quasi-monomial valuation of $(Y', V + \mathbb{F})$ with weight $(1 + \beta a_1, \beta q)$, where $(Y', V + \mathbb{F})$ is the same pair as in the proof of Lemma 6.5. Note that $(1 + \beta a_1, \beta q)$ is an admissible weight (i.e. having non-negative components) when $0 \leq \beta \ll 1$. If v_E is trivial, then $w_\beta \equiv v_0 = \text{ord}_V$. Notice that the description of w_β as quasi-monomial valuations does not require $\beta^{-1} = M$ to be an integer. So from now on we define

Definition 6.12. *Using the above notation, for any $0 \leq \beta \ll 1$, the valuation w_β is defined to be the quasi-monomial valuation on the model $(Y', V + \mathbb{F})$ with the weight $(1 + \beta a_1, \beta q)$. Equivalently, for any $f \in R = \bigoplus_k H^0(V, kL)$, we define:*

$$w_\beta(f) = \min \left\{ k + \beta v_E^{(1)}(f); f = \sum_k f_k \text{ with } f_k \neq 0 \right\}.$$

Correspondingly, on X_* we define $w_{*\beta}$ to be the quasi-monomial valuation of $V_* + \mathbb{F}_*$ with the weight $(\frac{1}{\sigma} + \beta \frac{a_1}{\sigma}, \beta q)$ if v_E is not trivial, and otherwise $w_{*\beta} \equiv \frac{v_* o}{\sigma} = \frac{1}{\sigma} \text{ord}_{V_*}$, where $V_* = V$ is considered as a divisor over X_* . By the above discussion, w_β is the restriction of $w_{*\beta}$ under the embedding $R \hookrightarrow R_*$.

In the following two lemmas, we compare the volumes and log discrepancies of w_β and $w_{*\beta}$.

Lemma 6.13. *We have an identity of volumes:*

$$\text{vol}(w_\beta) = \frac{1}{\sigma} \text{vol}(w_{*\beta}). \quad (59)$$

Proof. For any fixed β , we use $\mathcal{F}R_k$ (resp. \mathcal{F}_*R_*) to denote the filtrations determined by w_β (resp. $w_{*\beta}$) on R (resp. R_*). For any $f \in H^0(kL)$, $f \in \mathcal{F}^m R_k$ if and only if $w_\beta(f) \geq m$. Because $w_\beta = w_{*\beta}$ when both are restricted to $H^0(kL) = H^0(\sigma kL_*)$, this holds if and only if $f \in \mathcal{F}_*^m R_{*\sigma k}$. So we see that $\mathcal{F}^m R_k = \mathcal{F}_*^m R_{*\sigma k}$. As a consequence we can calculate:

$$\begin{aligned} \text{vol}(\mathcal{F}R^{(t)}) &= \lim_{k \rightarrow +\infty} \frac{\dim_{\mathbb{C}} \mathcal{F}^{kt} H^0(kL)}{k^{n-1}/(n-1)!} \\ &= \lim_{k \rightarrow +\infty} \frac{\dim_{\mathbb{C}} \mathcal{F}_*^{\sigma k(t/\sigma)} H^0(\sigma kL_*)}{(\sigma k)^{n-1}/(n-1)!} \sigma^{n-1} \\ &= \sigma^{n-1} \text{vol}(\mathcal{F}_* R_*^{(t/\sigma)}). \end{aligned}$$

By the volume formula obtained in (22), we get the identity (59) by change of variables in the integral formula. Notice the change of integration limits is valid by the sharp lower bounds obtained in Lemma 6.9 and Corollary 6.10. \square

Lemma 6.14. *We have the identities of log discrepancies:*

$$A_X(w_\beta) \equiv r \equiv A_{X_*}(w_{*\beta}). \quad (60)$$

Proof. If v_E is trivial, then $w_\beta = v_0$ and $A_X(w_\beta) = A_X(v_0) = r$. Otherwise, by (58) w_β is a quasi monomial valuation of $(V + \mathbb{F})$ with weight $(1 + \beta a_1, \beta q)$. So we have:

$$\begin{aligned} A_X(w_\beta) &= (1 + \beta a_1) A_X(\text{ord}_V) + \beta q A_X(\text{ord}_{\mathbb{F}}) = (1 + \beta a_1) r + \beta q A_X(\mathbb{F}) \\ &= (1 + \beta a_1) r + \beta q A_V(F), \end{aligned} \quad (61)$$

where we used $A_X(\mathbb{F}) = A_V(F)$ by Lemma 6.6. By Remark 6.7, $q \cdot A_V(F) = -a_1 r$. So we see that $A_X(w_\beta) \equiv r$ by (61).

For $A_{X_*}(w_{*\beta})$, we notice that:

$$\begin{aligned} A_{X_*}(w_{*\beta}) &= \frac{1 + \beta a_1}{\sigma} A_{X_*}(\text{ord}_{V_*}) + \beta q A_{X_*}(\text{ord}_{\mathbb{F}_*}) \\ &= (1 + \beta a_1) r + \beta q A_V(F) = A_X(w_\beta). \end{aligned}$$

\square

As a corollary of above two lemmas, we have:

Corollary 6.15. *We have the identity of normalized volumes:*

$$\widehat{\text{vol}}(w_\beta) = \frac{1}{\sigma} \cdot \widehat{\text{vol}}(w_{*\beta}).$$

6.3 w_β versus wt_β

First recall the definition of wt_β from Section 6.1. If we denote $\text{wt}^{(1)} = \text{wt}_\eta$, then for any $f \in H^0(\mathcal{V}_0, k\mathcal{L}_0)$, we have (see (48))

$$\text{wt}_\beta(f) = \text{wt}_0(f) + \beta \cdot \text{wt}^{(1)}(f) := \text{ord}_{\mathcal{V}_0}(f) + \beta \cdot \text{wt}^{(1)}(f). \quad (62)$$

Note that this decomposition corresponds exactly to (57). Our main goal is to understand the following correspondence:

$$\text{wt}_\beta \longleftrightarrow w_\beta, \quad \text{wt}^{(1)} \longleftrightarrow v_E^{(1)}.$$

Proposition 6.16. *For any $0 \leq \beta \ll 1$, we have the following relation between w_β and wt_β .*

1. $\text{vol}(\text{wt}_\beta) = \text{vol}(w_\beta)$.
2. $A_{\mathcal{X}_0}(\text{wt}_\beta) \equiv r \equiv A_{\mathcal{X}_0}(\text{ord}_{\mathcal{V}_0})$.

Proof. Notice that if $\beta = 0$, then $w_0 = \text{ord}_V = v_0^X$ and $\text{wt}_0 = \text{ord}_{\mathcal{V}_0} =: v_0^{X_0}$. So we have:

$$\text{vol}(v_0^X) = L^{n-1} = \text{vol}(v_0^{X_0}), \quad A_X(v_0^X) = r = A_{X_0}(v_0^{X_0}).$$

For simplicity of notations, when it's clear, we will denote both ord_V and $\text{ord}_{\mathcal{V}_0}$ by v_0 . So Proposition 1 holds for $\beta = 0$. From now on we assume $\beta \neq 0$.

1. We first prove item 1. We will prove

$$\dim_{\mathbb{C}} \mathcal{F}_{\text{wt}_\beta}^\lambda H^0(\mathcal{V}_0, m\mathcal{L}_0) = \dim_{\mathbb{C}} \mathcal{F}_{w_\beta}^\lambda H^0(V, mL) \quad (63)$$

for any $\lambda \in \mathbb{R}$ and $m \gg 1$. Notice that, by the definition of w_β and wt_β , we have that, (when $\beta \neq 0$):

$$\begin{aligned} \mathcal{F}_{\text{wt}_\beta}^\lambda H^0(\mathcal{V}_0, m\mathcal{L}_0) &= \mathcal{F}_{\text{wt}^{(1)}}^{(\lambda-m)/\beta} H^0(\mathcal{V}_0, m\mathcal{L}_0); \\ \mathcal{F}_{w_\beta}^\lambda H^0(V, mL) &= \mathcal{F}_{v_E^{(1)}}^{(\lambda-m)/\beta} H^0(V, mL). \end{aligned}$$

So we just need to show

$$\dim_{\mathbb{C}} \mathcal{F}_{v_E^{(1)}}^\lambda H^0(X, mL) = \dim_{\mathbb{C}} \mathcal{F}_{\text{wt}^{(1)}}^\lambda H^0(\mathcal{V}_0, m\mathcal{L}_0) \quad (64)$$

for any $\lambda \in \mathbb{R}$ and $m \gg 1$. This actually follows from [BHJ15, Section 2.5]. We give a separate proof for the convenience of the reader.

When $m \gg 1$, by flatness, we have a T_1 -equivariant vector bundle \mathcal{E}_m over \mathbb{C}^1 , whose space of global sections is $H^0(\mathcal{V}, m\mathcal{L}) =: \mathbb{E}_m$. Notice that \mathbb{E}_m is a natural $\mathbb{C}[t]$ module satisfying $\mathbb{E}_m/(t)\mathbb{E}_m \cong H^0(\mathcal{V}_0, m\mathcal{L}_0) =: \mathbb{E}_m^{(0)}$. The T_1 -action is compatible under this quotient map. For any $g \neq 0 \in \mathbb{E}_m^{(0)}$ of weight λ , we can extend it to become $\mathcal{G} \neq 0 \in \mathbb{E}_m$ of weight λ . In this way, we get a T_1 -equivariant splitting morphism $\tau_m : \mathbb{E}_m^{(0)} \rightarrow \mathbb{E}_m$. We denote by $\Phi_m : \mathbb{E}_m^{(0)} \rightarrow \mathbb{E}_m^{(1)}$ the composition of τ_m with the quotient $\mathbb{E}_m \rightarrow \mathbb{E}_m/(t-1)\mathbb{E}_m \cong H^0(V, mL) =: \mathbb{E}_m^{(1)}$. From the construction, if g has weight λ , then $t^\lambda \mathcal{G} =: \tilde{f}$ is T_1 -invariant with its restriction on $\mathcal{V}_1 \cong V$ denoted by f . Then we have $v_{\mathcal{V}_0}(f) = \text{ord}_{\mathcal{V}_0}(f) = \text{ord}_{\mathcal{V}_0}(t^\lambda g) = \lambda$. So we see that $\Phi_m(\mathcal{F}_{\text{wt}^{(1)}}^\lambda H^0(\mathcal{V}_0, m\mathcal{L}_0)) \subseteq \mathcal{F}_{v_E^{(1)}}^\lambda H^0(V, mL)$. We claim that Φ_m is injective. For any $g = g_{\lambda_1} + \dots + g_{\lambda_k} \neq 0$ with each nonzero g_{λ_j} having weight λ_j and $\lambda_1 < \lambda_2 < \dots < \lambda_k$, we have that $\tilde{\mathcal{G}}_{\lambda_j} := t^{\lambda_j} \mathcal{G}_{\lambda_j}$ are T_1 -invariant sections of $\mathcal{L} \rightarrow \mathcal{V}$. So $\tilde{\mathcal{G}}_{\lambda_j}|_{\mathcal{V} \setminus \mathcal{V}_0} \cong: (\pi_1^* f_{\lambda_j})|_{V \times \mathbb{C}^*}$ where $\pi_1 : V \times \mathbb{C} \rightarrow V$ is the projection and we used the isomorphism $(\mathcal{V} \setminus \mathcal{V}_0, \mathcal{L}) \cong (V \times \mathbb{C}^*, \pi_1^* L)$. $\tilde{\mathcal{G}}$ are linearly independent because they have different vanishing orders along \mathcal{V}_0 . Hence f_{λ_j} are also linearly independent. So $\Phi_m(g) = f_{\lambda_1} + \dots + f_{\lambda_k} \neq 0$ if $g \neq 0$. So we get $\dim_{\mathbb{C}} \mathcal{F}_{\text{wt}^{(1)}}^\lambda H^0(\mathcal{V}_0, m\mathcal{L}_0) \leq \dim_{\mathbb{C}} \mathcal{F}_{v_E^{(1)}}^\lambda H^0(V, mL)$.

The above argument also shows that for any $\lambda \in \mathbb{R}$, we have:

$$\dim_{\mathbb{C}} \frac{\mathcal{F}_{\text{wt}^{(1)}}^\lambda H^0(\mathcal{V}_0, m\mathcal{L}_0)}{\mathcal{F}_{\text{wt}^{(1)}}^{>\lambda} H^0(\mathcal{V}_0, m\mathcal{L}_0)} \leq \dim_{\mathbb{C}} \frac{\mathcal{F}_{v^{(1)}}^\lambda H^0(V, mL)}{\mathcal{F}_{v^{(1)}}^{>\lambda} H^0(V, mL)}. \quad (65)$$

On the other hand, we have the equality:

$$\begin{aligned} \sum_{\lambda} \dim_{\mathbb{C}} \frac{\mathcal{F}_{\text{wt}^{(1)}}^\lambda H^0(\mathcal{V}_0, m\mathcal{L}_0)}{\mathcal{F}_{\text{wt}^{(1)}}^{>\lambda} H^0(\mathcal{V}_0, m\mathcal{L}_0)} &= \dim_{\mathbb{C}} H^0(\mathcal{V}_0, m\mathcal{L}_0) \\ &= \dim_{\mathbb{C}} H^0(V, mL) = \sum_{\lambda} \dim_{\mathbb{C}} \frac{\mathcal{F}_{v^{(1)}}^\lambda H^0(V, mL)}{\mathcal{F}_{v^{(1)}}^{>\lambda} H^0(V, mL)}. \end{aligned}$$

So in fact (65) is an equality and consequently (63) holds for any $\lambda \in \mathbb{R}$ and $m \gg 1$. Combining the above discussions, we see that (63) hold for any $\lambda \in \mathbb{R}$ and $m \gg 1$. So we can compare the volumes by calculating:

$$\begin{aligned} \dim_{\mathbb{C}} R/\mathfrak{a}_\lambda(w_\beta) &= \sum_{m=0}^{+\infty} \dim_{\mathbb{C}} H^0(V, mL)/\mathcal{F}_{w_\beta}^\lambda H^0(V, mL) \\ &= \sum_{m=0}^{+\infty} \dim_{\mathbb{C}} H^0(\mathcal{V}_0, m\mathcal{L}_0)/\mathcal{F}_{\text{wt}_\beta}^\lambda H^0(\mathcal{V}_0, m\mathcal{L}_0) + O(1) \\ &= \dim_{\mathbb{C}} S/\mathfrak{a}_\lambda(\text{wt}_\beta) + O(1). \end{aligned}$$

Now the identity on volumes follow by dividing both sides by $\lambda^n/n!$ and taking limits as $\lambda \rightarrow +\infty$.

2. Next we calculate the log discrepancies of wt_β . For this purpose we will use the interpretation of log discrepancy of wt_β as the weight on the corresponding \mathbb{C}^* -action on an equivariant nonzero pluricanonical-form (see [Li15a]). Choose m sufficiently divisible so that mr is an integer and $mK_{\mathcal{X}_0} = mr\mathcal{L}_0^{-1}$ is Cartier. Then on $\mathcal{X}_0 = C(\mathcal{V}_0, \mathcal{L}_0)$, there is a nowhere zero m -canonical section $\Omega \in H^0(\mathcal{X}_0, mK_{\mathcal{X}_0})$ (alternatively we could think of $\Omega^{1/m}$ as a multi-section of $K_{\mathcal{X}_0}$). This is a well known construction showing that \mathcal{X}_0 is \mathbb{Q} -Gorenstein in our setting (see [Kol13, Proposition 3.14.(4)]). For the convenience of the reader, we give an explicit description.

To define Ω first choose an affine covering $\{U_\alpha\}$ of \mathcal{V}_0 that induces an open covering $\{\pi^{-1}U_\alpha\}$ of $\mathcal{X}_0 \setminus \{o\}$. Notice that there is a natural identification $\mathcal{X}_0 \setminus \{o\} \cong \mathcal{L}_0^{-1} \setminus \mathcal{V}_0$ where \mathcal{V}_0 is considered as the zero section of the total space \mathcal{L}_0^{-1} of the line bundle $\mathcal{L}_0^{-1} \rightarrow \mathcal{V}_0$. So we have a canonical projection $\pi : \mathcal{X}_0 \setminus \{o\} \rightarrow \mathcal{V}_0$.

For any affine open set $U = U_\alpha \subset \mathcal{V}_0$, choose a local generator $s = s(p)$ of $\mathcal{O}_{\mathcal{V}_0}(\mathcal{L}_0^{-1})(U)$. Then

$$\pi^{-1}(U) = \{h \cdot s(p); p \in U, h \neq 0\} \subset \mathcal{X}_0 \setminus \{o\}.$$

In other words, any point $P \in \mathcal{X}_0 \setminus \{o\}$ can be represented as $h \cdot s$ over an affine open set U of \mathcal{V}_0 containing $p = \pi(P)$. Since $mK_{\mathcal{V}_0}$ is Cartier and $K_{\mathcal{V}_0} = r\mathcal{L}_0^{-1}$, $s^{\otimes mr}$ is a local generator of $\mathcal{O}_{\mathcal{V}_0}(mK_{\mathcal{V}_0})$. Moreover if $p = \pi(P)$ is a smooth point, we can choose local coordinates $z = \{z_1, \dots, z_{n-1}\}$ around p such that $s^{\otimes mr} = (dz_1 \wedge \dots \wedge dz_{n-1})^{\otimes m} =: (dz)^{\otimes m}$. Then $\{z_1, \dots, z_{n-1}, h\}$ are local holomorphic coordinates on a neighborhood U' of $P \in \pi^{-1}(U_{\text{reg}}) \subset \mathcal{X}_0 \setminus \{o\}$. We define

$$\Omega = (dh^r)^{\otimes m} \wedge (dz)^{\otimes m} := \pm r^m h^{rm-m} (dh \wedge dz)^{\otimes m} \in H^0(U', mK_{\mathcal{X}_0}). \quad (66)$$

Lemma 6.17. Ω defines a global section in $H^0(\mathcal{X}_0, mK_{\mathcal{X}_0})$. Moreover Ω is invariant under the canonical lifting (given by pull-back of pluri-canonical forms) of T_1 -action on $mK_{\mathcal{X}_0}$.

Proof. Because \mathcal{X}_0 is normal and $mK_{\mathcal{X}_0}$ is Cartier, we just need to verify the statement on the regular locus of \mathcal{X}_0 . To verify Ω is globally defined, we need to show that Ω does not depend on the choices of local holomorphic coordinates.

If $\tilde{h}^{mr} \cdot (d\tilde{z})^{\otimes m} = h^{mr} \cdot (dz)^{\otimes m}$, then $\tilde{h}^{mr} = h^{mr} \text{Jac}(\frac{\partial z}{\partial \tilde{z}})^m$. So

$$\begin{aligned} mr\tilde{h}^{mr-1}d\tilde{h} \wedge d\tilde{z} &= mrh^{mr-1}(dh)\text{Jac}(\partial z/\partial \tilde{z})^m \wedge \text{Jac}(\partial \tilde{z}/\partial z)dz \\ &= mrh^{mr-1}\tilde{h}^{mr}/h^{mr} \text{Jac}(\partial \tilde{z}/\partial z)dh \wedge dz. \end{aligned}$$

So we get $\tilde{h}^{-1}d\tilde{h} \wedge d\tilde{z} = \text{Jac}(\partial \tilde{z}/\partial z) \cdot h^{-1}dh \wedge dz$. From this identity we easily get:

$$\begin{aligned} \tilde{h}^{rm-m}(d\tilde{h} \wedge d\tilde{z})^{\otimes m} &= \tilde{h}^{rm}(\tilde{h}^{-1}d\tilde{h} \wedge d\tilde{z})^{\otimes m} \\ &= h^{rm}(h^{-1}dh \wedge dz)^{\otimes m} = h^{rm-m}(dh \wedge dz)^{\otimes m}. \end{aligned} \quad (67)$$

So Ω is indeed globally defined.

To see the invariance under T_1 -action, we just need to verify the invariance of Ω on the regular locus. Recall that T_1 action on \mathcal{X}_0 is induced by the canonical lifting of \mathbb{C}^* -action on $\mathcal{L}_0^{-mr} = K_{\mathcal{V}_0}^m$ that is given by the pull back of m -pluri-canonical-forms (see Remark 2.4). For any smooth point $p \in \mathcal{V}_0$ and $t \in T_1$, $t \circ p = \tilde{p}$ is also a smooth point of \mathcal{V}_0 . By the above change of variable formula in (67), we get $t^*\Omega = \Omega$ for any $t \in T_1 = e^{\mathbb{C}\eta}$. So Ω is indeed invariant under the action of T_1 on \mathcal{X}_0 . □

On the other hand, Ω has weight mr under the T_0 -action generated by $\xi_0 = h\partial_h$:

$$\mathfrak{L}_{\xi_0}\Omega = \mathfrak{L}_{h\partial_h}(\pm r^m h^{rm-m}(dh \wedge dz)^{\otimes m}) = mr \cdot \Omega,$$

where \mathfrak{L} denotes the Lie derivative with respect to the generating holomorphic vector field. So we get

$$\mathfrak{L}_{\xi_\beta}\Omega = \mathfrak{L}_{\xi_0}\Omega + \beta\mathfrak{L}_\eta\Omega = mr \cdot \Omega. \quad (68)$$

We claim that (68) implies $A_{\mathcal{X}_0}(\text{wt}_\beta) \equiv r$ which is independent of β . In the case where $\beta \in \mathbb{Q}$ and hence ξ_β generates a \mathbb{C}^* -action, this was observed in [Li15a], which in essence depends on Dolgachev-Pinkham-Demazure's description of normal graded rings as expounded and developed by Kollár. For general $\beta \in \mathbb{R}$ (with $0 < \beta \ll 1$), the claim follows from the case where $\beta \in \mathbb{Q}$ and the fact that we can realize wt_β as quasi-monomial valuations on a common log smooth model. One way to see this fact is to use the same construction as used for w_β in Section 6.2 which lead to the Definition 6.12. The other way, which works for general torus invariant valuations, is to use structure results of normal T -varieties from [AH06, AIP⁺11, LS13]. The latter papers indeed generalized Dolgachev-Pinkham-Demazure's construction to varieties with torus actions of higher ranks. For the reader's convenience, we briefly explain how the arguments fit together to work.

- (a) In the case where $\beta \in \mathbb{Q}$ and $0 \leq \beta \ll 1$, the vector field $\xi_\beta = \xi_0 + \beta\eta$ generates a \mathbb{C}^* -action $\lambda_\beta : \mathbb{C}^* \rightarrow \text{Aut}(\mathcal{X}_0)$. The GIT quotient $\mathcal{X}_0/\lambda_\beta$ is an orbifold (or a stack) (S_β, Δ_β) and $\mathcal{X}_0 \setminus \{\sigma'\}$ is a Seifert \mathbb{C}^* -bundle over (S_β, Δ_β) (see [Kol04]). There is a natural compactification $\tilde{\mathcal{X}}_0$ of $\mathcal{X}_0 \setminus \{\sigma'\}$ such that the following conditions hold.
- (i) $\tilde{\mathcal{X}}_0$ is the total space of an orbifold line bundle L_β over (S_β, Δ_β) . In other words, if we denote $A = \bigoplus_{m=0}^{+\infty} H^0(S_\beta, kL_\beta)$ where L_β is viewed as a \mathbb{Q} -Weil divisor, then $\mathcal{X}_0 = \text{Spec}_{\mathbb{C}}(A)$ and $\tilde{\mathcal{X}}_0 = \text{Spec}_{S_\beta}(A)$. Moreover $-(K_{S_\beta} + \Delta_\beta) = r_\beta L_\beta$ for some $r_\beta > 0 \in \mathbb{Q}$ (see [Wat81], [Kol04, 40-42] and [Kol13, 3.1]).
 - (ii) The vector field ξ_β lifts to become a vector field ξ_β on $\tilde{\mathcal{X}}_0$, which generates a \mathbb{C}^* -action and is a real positive multiple of the natural rescaling vector field $\xi'_\beta = u\partial_u$ along the fibre of $\tilde{\mathcal{X}}_0 \rightarrow (S_\beta, \Delta_\beta)$ where u is an affine coordinate along the fibre of $L_\beta \rightarrow S_\beta$. Notice that ξ'_β is well-defined, independent of the choice of u and invariant under finite stabilizers over the orbifold locus Δ_β .
 - (iii) There is a λ_β -equivariant nowhere vanishing holomorphic section $\Omega' \in H^0(\mathcal{X}_0, mK_{\mathcal{X}_0})$ which can be constructed by using the structure of the orbifold line bundle $L_\beta \rightarrow (S_\beta, \Delta_\beta)$ in the same way that Ω was constructed. By the same calculation as above, we know that $\mathcal{L}_{\xi'_\beta}\Omega' = mr_\beta\Omega'$.

From (i) and by using the adjunction formula, it's easy to get $A_{\mathcal{X}_0}(\text{wt}_{\xi'_\beta}) = A_{\mathcal{X}_0}(\text{ord}_{S_\beta}) = r_\beta$ (see [Kol13, 3.1]). Because both Ω' and Ω are equivariant under the \mathbb{C}^* -action λ_β , their ratio $f = \Omega'/\Omega$, which is a nowhere vanishing regular function over \mathcal{X}_0 , is also λ_β -equivariant. In other words, there exists $\kappa \in \mathbb{Z}$ such that $f(t \cdot p) = t^\kappa \cdot f(p)$ for any $p \in \mathcal{X}_0$ and $t \in \mathbb{C}^*$. Letting $t \rightarrow 0$ and using that $|f(\sigma')|$ is bounded away from 0 and $+\infty$, we see that $\kappa = 0$, i.e. Ω'/Ω is λ_β -invariant. So f is a pull back of a regular function that is globally defined on S_β . This implies that f has to be a constant. So we get from (iii) that $\mathcal{L}_{\xi'_\beta}\Omega = mr_\beta\Omega$. By (ii) above, $\xi_\beta = a\xi'_\beta$ for some $a \in \mathbb{R}$. So $\mathcal{L}_{\xi_\beta}\Omega = a \cdot \mathcal{L}_{\xi'_\beta}\Omega = amr_\beta\Omega$. Comparing this with (68), we see that $amr_\beta = mr$ and hence $A_{\mathcal{X}_0}(\text{wt}_{\xi_\beta}) = aA_{\mathcal{X}_0}(\text{wt}_{\xi'_\beta}) = ar_\beta = r$.

- (b) If $\beta \notin \mathbb{Q}$, then wt_{ξ_β} is quasi-monomial of rational rank 2. Because \mathcal{X}_0 is a normal variety with a $(\mathbb{C}^*)^2$ -action, by [AH06, Theorem 3.4] there exists a normal semi-projective variety Y , a birational morphism $\mu : \tilde{\mathcal{X}}_0 \rightarrow \mathcal{X}_0$ and a projection $\pi : \tilde{\mathcal{X}}_0 \rightarrow Y$ such that the generic fibre of π is a normal toric variety of dimension 2. This toric variety, denoted by Z , is associated to the polyhedral cone $\sigma \subset N \otimes_{\mathbb{Z}} \mathbb{R} \cong \mathbb{R}^2$ where $N = \text{Hom}(\mathbb{C}^*, (\mathbb{C}^*)^2) \cong \mathbb{Z}^2$. Each valuation wt_{ξ_β} (for $0 \leq \beta \ll 1$) then corresponds to a vector contained in the interior of σ (see [AIP⁺11, 11]). Let $\tilde{Z} \rightarrow Z$ be a toric resolution of singularities of Z and choose a Zariski open set U of Y such that the fibre $\pi^{-1}(p)$ of any point $p \in U$ is isomorphic to Z . Then there exists a smooth model $\mathfrak{X} \rightarrow \mathcal{X}_0$ that dominates both $\tilde{\mathcal{X}}_0$ and $U \times \tilde{Z}$. In fact one can choose \mathfrak{X} to be a toroidal desingularization of $\tilde{\mathcal{X}}_0$ as constructed in [LS13, Section 2]. On the model \mathfrak{X} we can realize wt_{ξ_β} , for all $0 \leq \beta \ll 1$, as monomial valuations around the same regular point $p \in Y$ but with different non-negative weights.³ By the linearity and hence the continuity of log discrepancy function with respect to the weights (see [JM10, 5.1]), we use (a) and the denseness of rational weights to finally conclude that $A_{\mathcal{X}_0}(\text{wt}_{\xi_\beta}) \equiv r$ for any $\beta \in \mathbb{R}$ with $0 \leq \beta \ll 1$.

³In the case where $Y = \{\text{pt}\}$ we are just using toric resolution of singularities. This kind of construction can be considered as a special case of the construction in [JM10, Proof of Lemma 3.6.(ii)].

□

We provide a simple example illustrating the proof of $A(\text{wt}_\beta) \equiv r$ in Proposition 6.16.

Example 6.18. Choose $V = \mathbb{C}\mathbb{P}^{n-1}$ and $L = n^{-1}K_V^{-1}$ the hyperplane bundle over $\mathbb{C}\mathbb{P}^{n-1}$. Then $X = C(V, H) = \mathbb{C}^n$. Consider a holomorphic vector field η on $\mathbb{C}\mathbb{P}^{n-1}$ that is induced by the holomorphic vector field $\sum_{i=0}^{n-1} \lambda_i Z_i \frac{\partial}{\partial Z_i}$ on \mathbb{C}^n under the quotient $\pi : \mathbb{C}^n - \{0\} \rightarrow \mathbb{C}\mathbb{P}^{n-1}$, where $\lambda_i \in \mathbb{Z}$ and $Z = [Z_0, \dots, Z_{n-1}]$ are the homogeneous coordinates of $\mathbb{C}\mathbb{P}^{n-1}$ which are also considered as coordinates on \mathbb{C}^n . Using the same notations as in the above proof of item 2 of Proposition 6.16 (in particular $m = 1$ and $r = n$), we are going to show:

Lemma 6.19. $\Omega = ndZ_0 \wedge \dots \wedge dZ_{n-1}$. The canonical lifting of η to \mathbb{C}^n is given by

$$\eta = \sum_{i=0}^{n-1} (\lambda_i - \bar{\lambda}) Z_i \frac{\partial}{\partial Z_i}$$

where $\bar{\lambda} = \frac{1}{n} \sum_{p=0}^{n-1} \lambda_p$.

Proof. We will do the calculation on the affine chart $U_0 = \{Z_0 \neq 0\}$. Exactly the same calculation can be done on other $U_i = \{Z_i \neq 0\}$ and the result will turn out to be symmetric with respect to the index i . First choose s_0 to be the canonical trivializing section of L^{-1} over $U_0 = \{Z_0 \neq 0\}$ given by $s_0(Z) = (1, Z_1/Z_0, \dots, Z_{n-1}/Z_0)$. Any point $Z = (Z_i) \in \mathbb{C}^n - \{0\}$ can be written as:

$$Z = Z_0(1, Z_1/Z_0, \dots, Z_{n-1}/Z_0) = Z_0 \cdot s_0.$$

So the globally defined holomorphic n -form from (66) is equal to

$$\Omega = d(Z_0^n) \wedge (dz_1 \wedge \dots \wedge dz_{n-1}) = ndZ_0 \wedge dZ_1 \wedge \dots \wedge dZ_{n-1}. \quad (69)$$

On the other hand, the natural trivializing section of the holomorphic line bundle K_V over U_0 is $dz = dz_1 \wedge \dots \wedge dz_{n-1}$. Under the isomorphism $K_V = nL^{-1}$, we have $s_0^{\otimes n} = dz$ on U_0 (by looking at the transition functions).

In the affine coordinate chart $U_0 = \{Z_0 \neq 0\}$, we have $\eta = \sum_{i=1}^{n-1} (\lambda_i - \lambda_0) z_i \frac{\partial}{\partial z_i}$ where $\{z_i = Z_i/Z_0\}$ are inhomogeneous coordinates on $U_0 \cong \mathbb{C}^{n-1}$. So we have $t^*dz = t(\sum_{i=1}^{n-1} (\lambda_i - \lambda_0)) dz$ and hence $t \circ s_0 = t(\sum_{i=1}^{n-1} (\lambda_i - \lambda_0)/n) s_0$ for any $t \in e^{\mathbb{C}\eta} \cong \mathbb{C}^*$. So the canonical lifting of η on $\pi^{-1}(U_0)$, still denoted by η , is equal to (under the coordinate $\{Z_0, z_1, \dots, z_{n-1}\}$):

$$\eta = -\frac{1}{n} \sum_{i=1}^{n-1} (\lambda_i - \lambda_0) Z_0 \frac{\partial}{\partial Z_0} + \sum_{i=1}^{n-1} (\lambda_i - \lambda_0) z_i \frac{\partial}{\partial z_i}. \quad (70)$$

Now the coordinate change from $\{Z_0, z_1, \dots, z_{n-1}\}$ to $\{Z_0, Z_1, \dots, Z_{n-1}\}$ is given by:

$$Z_0 = Z_0, \quad Z_i = z_i \cdot Z_0, \quad i = 1, \dots, n-1;$$

which implies the change of basis formula:

$$\frac{\partial}{\partial Z_0} = \frac{\partial}{\partial Z_0} + \sum_{j=1}^{n-1} \frac{Z_j}{Z_0} \frac{\partial}{\partial Z_j}, \quad \frac{\partial}{\partial z_i} = Z_0 \frac{\partial}{\partial Z_i}, \quad i = 1, \dots, n-1.$$

Substituting these into (70) we get:

$$\begin{aligned} \eta &= \frac{1}{n} \sum_{i=1}^{n-1} (\lambda_0 - \lambda_i) Z_0 \left(\frac{\partial}{\partial Z_0} + \sum_{j=1}^{n-1} \frac{Z_j}{Z_0} \frac{\partial}{\partial Z_j} \right) + \sum_{i=1}^{n-1} (\lambda_i - \lambda_0) \frac{Z_i}{Z_0} Z_0 \frac{\partial}{\partial Z_i} \\ &= \sum_{p=0}^{n-1} (\lambda_p - \bar{\lambda}) Z_p \frac{\partial}{\partial Z_p}. \end{aligned}$$

Indeed, this follows from easy manipulation:

$$\frac{1}{n} \sum_{i=1}^{n-1} (\lambda_0 - \lambda_i) = \lambda_0 - \bar{\lambda}, \quad \left(\frac{1}{n} \sum_{j=1}^{n-1} (\lambda_0 - \lambda_j) \right) + (\lambda_i - \lambda_0) = \lambda_i - \bar{\lambda}.$$

□

We let $\xi_0 = \sum_{i=0}^{n-1} Z_i \frac{\partial}{\partial Z_i}$ as in the proof of Proposition 6.16. Then $\xi_\beta = \sum_{i=0}^{n-1} (1 + \beta(\lambda_i - \bar{\lambda})) Z_i \frac{\partial}{\partial Z_i}$. It's clear that, for $0 \leq \beta \ll 1$, the valuation wt_β is the monomial valuation centered at $0 \in \mathbb{C}^n$ and its log discrepancy is given by:

$$A_{\mathbb{C}^n}(\text{wt}_\beta) = \sum_{i=0}^{n-1} (1 + \beta(\lambda_i - \bar{\lambda})) \equiv n.$$

6.4 Completion of the proof of Theorem 6.1

Now we can finish the proof of Theorem 6.1. Indeed, it's well known now that the derivative of $\widehat{\text{vol}}(\text{wt}_\beta) = r^n \text{vol}(\text{wt}_\beta)$ on \mathcal{X}_0 is given by a positive multiple of the Futaki invariant on \mathcal{V}_0 defined in [Fut83, DT92], or equivalently the CM weight in (4). This was first discovered in [MSY08] when V is smooth, and generalized to the mildly singular case in [DS15] (see also [CS12]). For the convenience of the reader, we give a different proof in Lemma 6.20 following the spirit of calculations in previous sections and using results on Duistermaat-Heckman measures in [BC11, BHJ15].⁴

Under the assumption that $\widehat{\text{vol}}$ obtains the minimum at ord_V on (X_*, o_*) , by the above Proposition and Corollary 6.15, we have

$$\widehat{\text{vol}}(\text{wt}_\beta) = \widehat{\text{vol}}(w_\beta) = \sigma^{-1} \widehat{\text{vol}}(w_{*\beta}) \geq \sigma^{-1} \widehat{\text{vol}}(w_{*0}) = \widehat{\text{vol}}(w_0) = \widehat{\text{vol}}(\text{wt}_0).$$

for any $0 \leq \beta \ll 1$. So we must have $\left. \frac{d}{d\beta} \widehat{\text{vol}}(\text{wt}_\beta) \right|_{\beta=0} \geq 0$ which is equivalent to $\text{CM}(\mathcal{V}, \mathcal{L}) \geq 0$. Since this non-negativity holds for any special test configuration $(\mathcal{V}, \mathcal{L})$, we are done.

Lemma 6.20. *The following properties hold.*

1. *The derivative of $\widehat{\text{vol}}(\text{wt}_\beta) = r^n \cdot \text{vol}(\text{wt}_\beta)$ at $\beta = 0$ is a positive multiple of the Futaki invariant on the central fibre \mathcal{V}_0 for the T_1 -action. More precisely, we have the equality:*

$$\left. \frac{d}{d\beta} \widehat{\text{vol}}(\text{wt}_\beta) \right|_{\beta=0} = -(-K_{\widehat{V}/\mathbb{P}^1})^n = (n(-K_V)^{n-1}) \cdot \text{CM}(\mathcal{V}, \mathcal{L}). \quad (71)$$

2. *For any $\xi_1 \in \mathfrak{t}_{\mathbb{R}}^+$ (see (47)) with associated weight function $\text{wt}_1 := \text{wt}_{\xi_1}$, $\Phi(t) = \text{vol}((1-t)\text{wt}_0 + t\text{wt}_1)$ is convex with respect to $t \in [0, 1]$.*

Proof. First notice that, by definition, $\text{wt}_\beta = \text{wt}_0 + \beta \text{wt}^{(1)}$ where $\text{wt}_0 = \text{ord}_{\mathcal{V}_0}$ is the canonical valuation. So we see that

$$\mathfrak{a}_\lambda(\text{wt}_\beta) \cap S_m = \mathcal{F}_{\text{wt}_\beta}^\lambda S_m = \mathcal{F}_{\text{wt}^{(1)}}^{(\lambda-m)/\beta} H^0(\mathcal{V}_0, m\mathcal{L}_0).$$

Similar as before, we define $c_3 := \inf_m \text{wt}^{(1)}/\text{wt}_0 > -\infty$ and calculate:

$$\begin{aligned} n! \dim_{\mathbb{C}} S / \mathfrak{a}_\lambda(\text{wt}_\beta) &= n! \sum_{m=0}^{\lfloor \lambda / (1+c_3\beta) \rfloor} (\dim_{\mathbb{C}} S_m - \dim_{\mathbb{C}} \mathcal{F}_{\text{wt}^{(1)}}^{(\lambda-m)/\beta} S_m) \\ &= \frac{\lambda^n \mathcal{L}_0^{n-1}}{(1+\beta c_3)^n} - n\lambda^n \int_{c_3}^{+\infty} \text{vol}(\mathcal{F}_{\text{wt}^{(1)}} S^{(x)}) \frac{\beta dx}{(1+\beta x)^{n+1}} + O(\lambda^{n-1}). \end{aligned}$$

For simplicity of notations, in the rest part of the argument we will denote $\mathcal{F}_{\text{wt}^{(1)}} S^{(x)}$ simply by $\mathcal{F}_{\text{wt}^{(1)}}^{(x)}$. This gives the following formula (compare (41) and (42)):

$$\text{vol}(\text{wt}_\beta) = \frac{\mathcal{L}_0^{n-1}}{(1+\beta c_3)^n} - n \int_{c_3}^{+\infty} \text{vol}(\mathcal{F}_{\text{wt}^{(1)}}^{(x)}) \frac{\beta dx}{(1+\beta x)^{n+1}} \quad (72)$$

$$= - \int_{c_3}^{+\infty} \frac{d \text{vol}(\mathcal{F}_{\text{wt}^{(1)}}^{(x)})}{(1+\beta x)^n}. \quad (73)$$

⁴The generalized convexity of the volume in Lemma 6.20 has also been noticed by Collins-Székelyhidi in [CS15, Page 6].

The second identity follows from integration by parts as in (42). So the normalized volume is equal to (compare (41)):

$$\widehat{\text{vol}}(\text{wt}_\beta) = r^n \left(\frac{\mathcal{L}_0^{n-1}}{(1 + \beta c_3)^n} - n\beta \int_{c_3}^{+\infty} \text{vol} \left(\mathcal{F}_{\text{wt}^{(1)}}^{(x)} \right) \frac{dx}{(1 + \beta x)^{n+1}} \right) =: \Phi(\beta).$$

Again $\Phi(0) = r^n \mathcal{L}_0^{n-1} = \widehat{\text{vol}}(\text{wt}_0)$ and its derivative at $\beta = 0$ is equal to:

$$\Phi'(0) = -nc_3 r^n \mathcal{L}_0^{n-1} - nr^n \int_{c_3}^{+\infty} \text{vol} \left(\mathcal{F}_{\text{wt}^{(1)}}^{(x)} \right) dx$$

Using integration by parts and noticing that $\text{vol}(\mathcal{F}_{\text{wt}^{(1)}}^{(x)}) = \mathcal{L}_0^{n-1}$ if $x \leq c_3$, we get:

$$\begin{aligned} \Phi'(0) &= -nc_3 r^n \mathcal{L}_0^{n-1} - nr^n \lim_{M \rightarrow +\infty} x \text{vol}(\mathcal{F}_{\text{wt}^{(1)}}^{(x)}) \Big|_{c_3}^M + nr^n \int_{c_3}^{+\infty} x d\text{vol}(\mathcal{F}_{\text{wt}^{(1)}}^{(x)}) \\ &= nr^n \int_{c_3}^{+\infty} x \cdot d\text{vol} \left(\mathcal{F}_{\text{wt}^{(1)}}^{(x)} \right). \end{aligned}$$

Now by [BC11] (see [BHJ15, Theorem 5.3]), we know that

$$\int_{c_3}^{+\infty} x \cdot d\text{vol} \left(\mathcal{F}_{\text{wt}^{(1)}}^{(x)} \right) = - \lim_{m \rightarrow +\infty} \frac{w_m}{mN_m} = - \frac{\bar{\mathcal{L}}^n}{n}, \quad (74)$$

where w_m is the weight of T_1 -action on $H^0(\mathcal{V}_0, m\mathcal{L}_0)$ and $N_m = \dim_{\mathbb{C}} H^0(\mathcal{V}_0, m\mathcal{L}_0)$. So $\frac{\Phi'(0)}{n(-K_V)^{n-1}} = -\frac{r^n \bar{\mathcal{L}}^n}{n(-K_V)^{n-1}}$ is nothing but the Futaki invariant on \mathcal{V}_0 or equivalently the CM weight in (4).

To see the second item, it's clear that the same calculation gives the formula as in (42):

$$\text{vol}((1-t)\text{wt}_0 + t\text{wt}_1) = \int_{c_1}^{+\infty} \frac{-d\text{vol}(\mathcal{F}_{\text{wt}_1} S^{(x)})}{((1-t) + tx)^n},$$

where $c_1 = \inf_m \text{wt}_1/\text{wt}_0$. Because $\xi_1 \in \mathfrak{t}_{\mathbb{R}}^+$, $\text{wt}_1 \in (\text{Val}_{\mathcal{X}_0, \sigma'})^{\mathbb{C}^*}$ (see Section 6.1). In particular $c_1 > 0$. The convexity with respect to t follows from the fact that $t \mapsto 1/((1-t) + tx)^n$ is convex for $t \in [0, 1]$ (cf. proof of Lemma 4.4). \square

Remark 6.21. *As pointed by a referee, one could simplify the proof of Theorem 6.1 by directly calculating the normalized volume w_β and its derivative at $\beta = 0$ by using the Duistermaat-Heckman measure of $\mathcal{F}_{v_E^{(1)}}^x R$ without the help of wt_β . Indeed, based on the results in Section 6.2 and by the same calculation as in the proof of Lemma 6.20 (see (82) and (83)), we can get the following formula of $\widehat{\text{vol}}(w_\beta)$:*

$$\widehat{\text{vol}}(w_\beta) = r^n \left(\frac{L^n}{(1 + \beta a_1)^n} - n \int_{a_1}^{+\infty} \text{vol} \left(\mathcal{F}_{v_E^{(1)}}^{(x)} \right) \frac{\beta dx}{(1 + \beta x)^{n+1}} \right)$$

where $a_1 = -\frac{A_{V \times \mathbb{C}(\mathcal{V}_0) - 1}}{r} = -q \cdot A_V(F)/r$ (see (52) and Remark 6.7). Taking derivative on both sides at $\beta = 0$ and then integrating by parts, we get:

$$\begin{aligned} \frac{d}{d\beta} \Big|_{\beta=0} \widehat{\text{vol}}(w_\beta) &= -nr^n a_1 L^n - nr^n \int_{a_1}^{+\infty} \text{vol} \left(\mathcal{F}_{v_E^{(1)}}^{(x)} R \right) dx \\ &= nr^n \int_{a_1}^{+\infty} x \cdot d\text{vol} \left(\mathcal{F}_{v_E^{(1)}}^{(x)} R \right). \end{aligned}$$

Now applying directly [BHJ15, Proposition 3.12] gives:

$$\frac{d}{d\beta} \Big|_{\beta=0} \widehat{\text{vol}}(w_\beta) = -r^n \bar{\mathcal{L}}^n = n(-K_V)^{n-1} \cdot \text{CM}(\mathcal{V}, \mathcal{L}).$$

Arguing in the same way as at the beginning of this subsection, we then complete the proof of Theorem 6.1. On the other hand, we keep the original argument because the correspondence between w_β and wt_β is natural and will be important for our future work.

6.5 An example of calculation

Here we provide an example to illustrate our calculations. This is related to the example discussed in [Li13, Section 5]. We let V be an $(n - 1)$ -dimensional Fano manifold and assume that D is a smooth prime divisor such that $D \sim_{\mathbb{Q}} -\lambda K_V$ with $0 < \lambda < 1$. Then we can construct a special degeneration of V by using deformation to the normal cone in the following way. First denote by $\Pi_1 : \mathcal{W} := Bl_{D \times \{0\}}(V \times \mathbb{C}) \rightarrow V \times \mathbb{C}$ the blow up of $V \times \mathbb{C}$ along the codimension 2 subvariety $D \times \{0\}$. Then $\mathcal{W} \rightarrow \mathbb{C}$ is a flat family and the central fibre \mathcal{W}_0 is the union of two components $\widehat{V} \cup \widehat{E}$. Here the \widehat{V} component is the strict transform of $V \times \{0\}$ which is isomorphic to V because $D \times \{0\}$ is of codimension one inside $V \times \{0\}$. \widehat{E} denotes the exceptional divisor, which in this case is nothing but $\mathbb{P}(N_D \oplus \mathbb{C})$, where N_D is the normal bundle of $D \subset V$. For any $c > 0 \in \mathbb{Q}$, there is a \mathbb{Q} -line bundle $\mathcal{M}_c := \Pi_1^*(-K_{V \times \mathbb{C}}) - c\widehat{E}$ on \mathcal{W} . \mathcal{M}_c is relatively ample over \mathbb{C} if and only if $c \in (0, \lambda^{-1})$. Moreover $\mathcal{M}_{\lambda^{-1}}$ is semi ample over \mathbb{C} and the linear system $|-p\mathcal{M}_{\lambda^{-1}}|$ for sufficiently divisible p gives a birational morphism $\Pi_2 : \mathcal{W} \rightarrow \mathcal{V}$ by contracting the component V in the central fibre, and we have $\mathcal{M}_{\lambda^{-1}}^p \sim_{\mathbb{Q}} \Pi_2^*(-pK_{\mathcal{V}/\mathbb{C}})$ over \mathbb{C} (see (76)). We choose a $p \in \mathbb{Z}_{>0}$ sufficiently divisible such that $\mathcal{L} := -pK_{\mathcal{V}/\mathbb{C}}$ is Cartier. Then the pair $(\mathcal{V}, \mathcal{L})$ is a special degeneration of (V, L) where $L = -pK_V$. Notice that the central fibre \mathcal{V}_0 , also denoted by E , is obtained from $\widehat{E} = \mathbb{P}(N_D \oplus \mathbb{C})$ by contracting the infinity section D_∞ of the \mathbb{P}^1 -bundle and hence in general has an isolated singularity.

The divisorial valuation v_E on $\mathbb{C}(V)$ coincides with the divisorial valuation ord_D . In other words, by using the notation of (49), we have $F = D$ and $q = 1$. The valuation $v_E^{(0)}$ on $X = C(V, L)$ is the divisorial valuation $\text{ord}_{\mathbb{D}}$ where \mathbb{D} is the closure of τ^*D where $\tau : L \setminus V \rightarrow V$ is the natural \mathbb{C}^* -bundle. By Lemma 6.6, $A_X(\mathbb{D}) = A_V(D) = 1$. Because $K_{\mathcal{W}} = \Pi_1^*K_{V \times \mathbb{C}} + \widehat{E}$ and $r = p^{-1}$, we have (see (52))

$$A_{V \times \mathbb{C}}(E) - 1 = 1 = A_V(v_E), \quad a_1 := -\frac{A_{V \times \mathbb{C}}(E) - 1}{r} = -p.$$

Lemma 6.22. *We have the identity of \mathbb{Q} -Cartier divisors:*

$$\Pi_2^*(-K_{\mathcal{V}}) = \Pi_1^*K_{V \times \mathbb{C}}^{-1} - \widehat{E} + (\lambda^{-1} - 1)\widehat{V}. \quad (75)$$

Proof. We have the identities:

$$K_{\mathcal{W}} = \Pi_2^*K_{\mathcal{V}} + b\widehat{V} = \Pi_1^*K_{V \times \mathbb{C}} + \widehat{E}.$$

To determine the coefficient b , we use the adjunction formula:

$$\begin{aligned} K_{\widehat{V}} &= (K_{\mathcal{W}} + \widehat{V})|_{\widehat{V}} = \Pi_2^*K_{\mathcal{V}}|_{\widehat{V}} + (b+1)\widehat{V}|_{\widehat{V}} = (b+1)\widehat{V}|_{\widehat{V}} \\ &= (b+1)(\widehat{V} + \widehat{E})|_V - (b+1)\widehat{E}|_{\widehat{V}} = -(b+1)\widehat{E}|_{\widehat{V}} \\ &= (b+1)\mathcal{O}_{\widehat{V}}(-D_\infty) = \lambda(b+1)K_{\widehat{V}}. \end{aligned}$$

In the above we used $\mathcal{O}_{\mathcal{V}}(\widehat{V} + \widehat{E}) = \mathcal{O}_{\mathcal{V}}(\mathcal{W}_0)$ is trivial over \widehat{V} . So $b = \lambda^{-1} - 1$ and we obtain the equality. \square

Remark 6.23. *Since $\mathcal{W}_0 = \widehat{V} + \widehat{E}$, the equality (75) can also be written in a different form:*

$$\begin{aligned} \Pi_2^*(-K_{\mathcal{V}}) &= \Pi_1^*K_{V \times \mathbb{C}}^{-1} - \lambda^{-1}\widehat{E} + (\lambda^{-1} - 1)(\widehat{V} + \widehat{E}) \\ &= \mathcal{M}_{\lambda^{-1}} + (\lambda^{-1} - 1)\mathcal{W}_0. \end{aligned} \quad (76)$$

As above, let w_β be the quasi-monomial valuation of $V + \mathbb{D}$ with weight $(1 + \beta a_1, \beta q) = (1 - \beta p, \beta)$. We can calculate the volume of w_β by applying the formula (21) to get $(c_1 = 1 + \beta a_1 = 1 - \beta p)$:

$$\text{vol}(w_\beta) = \frac{1}{(1 - \beta p)^n} L^{n-1} - n \int_{1 - \beta p}^{+\infty} \text{vol}(\mathcal{F}_{w_\beta} R^{(t)}) \frac{dt}{t^{n+1}}.$$

To calculate $\text{vol}(R^{(t)})$, we notice that:

$$\mathcal{F}_{w_\beta}^x R_k = \mathcal{F}_{v_E^{(0)}}^{(x - (1 - \beta p)k)/\beta} R_k,$$

so that we have the identity:

$$\text{vol}\left(\mathcal{F}_{w_\beta} R^{(t)}\right) = \text{vol}\left(\mathcal{F}_{v_E^{(0)}} R^{\left(\frac{t-(1-\beta p)}{\beta}\right)}\right). \quad (77)$$

So by changing of variables, we get another formula

$$\text{vol}(w_\beta) = \frac{L^{n-1}}{(1-\beta p)^n} - n \int_0^{+\infty} \text{vol}\left(\mathcal{F}_{v_E^{(0)}} R^{(s)}\right) \frac{\beta ds}{(1-\beta p + \beta s)^{n+1}}. \quad (78)$$

It's clear that the filtration defined by $v_E^{(0)} = \text{ord}_D$ is given by

$$\mathcal{F}_{v_E^{(0)}}^y R_k = H^0(V, \mathcal{O}_V(kL - [y]D)) = H^0(V, \mathcal{O}_V((k - \lambda p^{-1}[y])L)).$$

So it's straight-forward to calculate that:

$$\text{vol}\left(\mathcal{F}_{v_E^{(0)}} R^{(s)}\right) = \begin{cases} (1 - \lambda p^{-1}s)^{n-1} L^{n-1} & \text{if } s < p\lambda^{-1}; \\ 0 & \text{if } s \geq p\lambda^{-1}. \end{cases} \quad (79)$$

Substituting (77) into (78), we get the volume of w_β . We can also calculate the derivative:

$$\begin{aligned} \left. \frac{d}{d\beta} \right|_{\beta=0} \text{vol}(w_\beta) &= pnL^{n-1} - n \int_0^{+\infty} \text{vol}\left(\mathcal{F}_{v_E^{(0)}} R^{(s)}\right) ds \\ &= pnL^{n-1} - nL^{n-1} \int_0^{p\lambda^{-1}} (1 - \lambda p^{-1}s)^{n-1} ds \\ &= p(n - \lambda^{-1})L^{n-1} \geq 0. \end{aligned}$$

For the last inequality, we used the bound of Fano index. Indeed since $K_V^{-1} = \lambda^{-1}D$, it's well know that $\lambda^{-1} \leq \dim V + 1 = n$. So $\text{vol}(w_\beta)$ is increasing with respect to β in this example. On the other hand, noticing that $\widehat{V}|_{\widehat{V}} = (\widehat{V} + \widehat{E})|_{\widehat{V}} - \widehat{E}|_{\widehat{V}} = \mathcal{O}_{\widehat{V}}(-D) = \lambda K_V$, it's an exercise to obtain:

$$\begin{aligned} (-K_{\widehat{V}/\mathbb{P}^1})^n &= (\Pi_2^*(-K_{\widehat{V}/\mathbb{P}^1}))^n = \left(\Pi_1^*K_V^{-1} + \lambda^{-1}\widehat{V} - \mathcal{W}_0\right)^n \\ &= (\lambda^{-1} - n)\langle K_V^{-(n-1)}, [V] \rangle = p^{1-n}(\lambda^{-1} - n) \cdot L^{n-1}. \end{aligned}$$

So we obtain an identity illustrating the general formula (71):

$$\left. \frac{d}{d\beta} \right|_{\beta=0} \widehat{\text{vol}}(w_\beta) = p^{-n+1}(n - \lambda^{-1})L^{n-1} = -(-K_{\mathcal{V}/\mathbb{P}^1})^n. \quad (80)$$

7 Proof of Theorem 3.7

For any divisorial valuation ord_F of $\mathbb{C}(V)$, we consider the filtration:

$$\mathcal{F}_F^x R_k := \{f \in H^0(V, kL) \mid \text{ord}_F(f) \geq x\}. \quad (81)$$

We know that this is a multiplicative, left-continuous and linearly bounded filtration by [BKMS15] (see also [BHJ15, Section 5.4]). Similar to Lemma 6.13, we have the following observation.

Lemma 7.1. *If $L = \sigma L_*$ and we denote $R_* = \bigoplus_{k=0}^{+\infty} H^0(V, kL_*) =: \bigoplus_{k=0}^{+\infty} R_{*k}$, then $\mathcal{F}_F^x R_k = \mathcal{F}_F^x R_{*k}$. As a consequence*

$$\text{vol}\left(\mathcal{F}_F R^{(x)}\right) = \sigma^{n-1} \cdot \text{vol}\left(\mathcal{F}_F R_*^{(x/\sigma)}\right),$$

and by change of variables we have:

$$r^n \int_0^{+\infty} \text{vol}\left(\mathcal{F}_F R^{(x)}\right) dx = (r\sigma)^n \int_0^{+\infty} \text{vol}\left(\mathcal{F}_F R_*^{(x)}\right) dx.$$

To put Theorem 3.7 in the setting of our discussions, we adopt the notation in Section 6 and consider the quasi-monomial \mathbb{C}^* -invariant valuation v_α of (V, \mathbb{F}) with weight $(1, \alpha)$, similar to what we did in [Li15a]. Then we have:

- $\inf_{\mathfrak{m}} \frac{v_\alpha}{v_0} = 1$;
- $\mathcal{F}_{v_\alpha}^x R_k = \mathcal{F}_F^{(x-k)/\alpha} R_k$. As a consequence:

$$\text{vol}\left(\mathcal{F}_{v_\alpha} R^{(x)}\right) = \text{vol}\left(\mathcal{F}_F R^{\left(\frac{x-1}{\alpha}\right)}\right).$$

We can use the volume formula (21) to get:

$$\begin{aligned} \text{vol}(v_\alpha) &= L^{n-1} - n \int_1^{+\infty} \text{vol}\left(\mathcal{F}_F R^{\left(\frac{x-1}{\alpha}\right)}\right) \frac{dx}{x^{n+1}} \\ &= L^{n-1} - n \int_0^{+\infty} \text{vol}\left(\mathcal{F}_F R^{(t)}\right) \frac{\alpha dt}{(1+\alpha t)^{n+1}}. \end{aligned} \quad (82)$$

The log discrepancy of v_α is given by

$$A_X(v_\alpha) = r + \alpha A_X(\text{ord}_{\mathbb{F}}) = r + \alpha A_V(F),$$

where the last identity is by Lemma 6.6. So we get the normalized volume of v_α :

$$\widehat{\text{vol}}(v_\alpha) = (r + \alpha A_V(F))^n L^{n-1} - n A_X(v_\alpha)^n \int_0^{+\infty} \text{vol}\left(\mathcal{F}_F R^{(t)}\right) \frac{\alpha dt}{(1+\alpha t)^{n+1}}. \quad (83)$$

The derivative at $\alpha = 0$ is equal to:

$$\begin{aligned} \left. \frac{d}{d\alpha} \right|_{\alpha=0} \widehat{\text{vol}}(v_\alpha) &= nr^{n-1} A_V(F) L^{n-1} - nr^n \int_0^{+\infty} \text{vol}\left(\mathcal{F}_F R^{(t)}\right) dt \\ &= n(-K_V)^{n-1} \left(A_V(F) - \frac{r^n}{(-K_V)^{n-1}} \int_0^{+\infty} \text{vol}\left(\mathcal{F}_F R^{(t)}\right) dt \right) \\ &= n(-K_V)^{n-1} \Theta_V(F). \end{aligned} \quad (84)$$

We can apply these formula for F coming from any nontrivial special test configuration as considered in Section 6. Using the notations in Section 6, the valuation w_β is associated to the divisor:

$$(1 + \beta a_1)V + \beta q \mathbb{F} = \frac{1}{1 + \beta a_1} \left(V + \frac{\beta q}{1 + \beta a_1} \mathbb{F} \right).$$

So $w_\beta = v_\alpha / (1 + \beta a_1)$ with $\alpha = \frac{\beta q}{1 + \beta a_1}$. So we get $\widehat{\text{vol}}(w_\beta) = \widehat{\text{vol}}(v_\alpha)$ and

$$\left. \frac{d}{d\beta} \right|_{\beta=0} \widehat{\text{vol}}(w_\beta) = q \left. \frac{d}{d\alpha} \right|_{\alpha=0} \widehat{\text{vol}}(v_\alpha) = n(-K_V)^{n-1} \Theta_V(F).$$

Note that $q > 0$. By the proof in Section 6, we know that $\Theta_V(F)$ is the Futaki invariant (or CM weight) of the special test configuration.

Remark 7.2. For any special test configuration, one can also prove that $\Theta_V(F)$ is equal to the Futaki invariant by applying integration by parts to the formula in [BHJ15, Theorem 3.12.(iii)].

Proof of Theorem 3.7. If V is K -semistable or equivalently Ding-semistable, then by Theorem 4.1 $\widehat{\text{vol}}(v_\alpha) \geq \widehat{\text{vol}}(v_0)$ for $0 \leq \alpha \ll 1$. So $\Theta_V(F) \geq 0$ by (84). Actually the proof of Theorem 4.1 in Section 4.2 shows that one can apply Fujita's result directly for $\mathcal{F}_F R_\bullet$ to get $\Theta_V(F) \geq 0$ (see also [LL16]).

Reversely, if $\Theta_V(F) \geq 0$ (resp. > 0) for any divisorial valuation ord_F over V , then this holds for F coming from any non-trivial special test configuration. By the above discussion this exactly means that the Futaki invariant of any nontrivial special test configuration is nonnegative (resp. positive). So V is indeed K -semistable (resp. K -stable) in the sense of Tian ([Tia97]). By using [LX14] we conclude that V is K -semistable (resp. K -stable) as in definition 2.3. \square

8 Two examples of normalized volumes

1. Consider the quasi monomial valuation v on $\mathbb{C}(x, y)$ determined by $\gamma \in \mathbb{R}_{>0}^2$:

$$v \left(\sum_{e_1, e_2 \geq 0} p_{e_1, e_2} x^{e_1} y^{e_2} \right) = \min\{e_1 \gamma_1 + e_2 \gamma_2 \mid p_{e_1, e_2} \neq 0\}.$$

For any $m \in \mathbb{Z}_{>0}$, $\dim_{\mathbb{C}}(\mathbb{C}[x, y]/\mathfrak{a}_m(v))$ is the number of positive integral solutions to the inequalities:

$$\gamma_1 a + \gamma_2 b < m, a \geq 0, b \geq 0.$$

So it's easy to see that

$$\text{vol}(v) = \lim_{m \rightarrow +\infty} \frac{\dim_{\mathbb{C}}(\mathbb{C}[x, y]/\mathfrak{a}_m(v))}{m^2/2!} = \frac{1}{\gamma_1 \gamma_2}.$$

On the other hand, the log discrepancy $A_{\mathbb{C}^2}(v) = \gamma_1 + \gamma_2$ so that

$$\widehat{\text{vol}}(v) = \frac{(\gamma_1 + \gamma_2)^2}{\gamma_1 \gamma_2} \geq 2^2.$$

The last equality holds if and only if $\gamma_1 = \gamma_2$.

2. (**Non-Abhyankar examples which are not \mathbb{C}^* -invariant**) Here we calculate the normalized volumes of non-Abhyankar valuations in [ELS03, Example 3.15]. These valuations were constructed in [ZS60, pp. 102-104] and we follow the exposition in [ELS03, Example 3.15] to recall their constructions. Let $v(y) = 1$ and set $v(x) = \beta_1 > 1$ some rational number (compared to [ELS03], we shift the subindex by 1). Let c_1 be the smallest positive integer such that $c_1 \beta_1 \in \mathbb{Z}$. By [ZS60], there exists a valuation v such that the polynomial

$$q_2 = x^{c_1} + y^{\beta_1 c_1}$$

has value β_2 equal to any rational number greater than (or equal to) $\beta_1 c_1$. We choose this value so that $\beta_2 = \frac{b_2}{c_2} > \beta_1 c_1$, where b_2 and c_2 are relatively prime positive integers with c_2 relatively prime to c_1 .

Remark 8.1. Notice that this already implies that v is not \mathbb{C}^* -invariant because for general $t \in \mathbb{C}^*$, we have

$$\begin{aligned} v(t^{c_1} x^{c_1} + t^{\beta_1 c_1} y^{\beta_1 c_1}) &= v(t^{c_1} (x^{c_1} + y^{\beta_1 c_1}) + (t^{\beta_1 c_1} - t^{c_1}) y^{\beta_1 c_1}) \\ &= v(y^{\beta_1 c_1}) = \beta_1 c_1 \neq \beta_2 = v(x^{c_1} + y^{\beta_1 c_1}). \end{aligned}$$

This process can be repeated and in this way, we inductively construct a sequence of polynomials q_i , rational numbers $\beta_i = \frac{b_i}{c_i}$ such that:

- (a) $c_i > 1$ and b_i are relatively prime positive integers with c_i relatively prime to the product $c_1 c_2 \cdots c_{i-1}$;
- (b) $q_{i+1} = q_i^{c_i} + y^{\beta_i c_i}$;
- (c) $\beta_{i+1} > \beta_i c_i$.

As shown in [ZS60], this uniquely defines a valuation v on $\mathbb{C}(x, y)$ such that $v(q_i) = \beta_i$ in the following way. Using Euclidian algorithm with respect to the power of x , and setting $q_0 = y$ and $q_1 = x$, every polynomial has a unique expression as a sum of "monomials" in the q_i :

$$q^{\mathbf{a}} := q_0^{a_0} q_1^{a_1} q_2^{a_2} \cdots q_k^{a_k},$$

where a_0 can be arbitrary but the remaining exponents a_i satisfy $a_i < c_i$. By the pair wisely prime properties of c_i , it's easy to see that these "monomials" have distinct values, i.e. $v(q^{\mathbf{a}}) = a_0 + \sum_{i=1}^k a_i \beta_i$ are all distinct. So the valuation ideals on $\mathbb{C}[x, y]$ have the form:

$$\mathfrak{a}_m = \left(\left\{ q_0^{a_0} q_1^{a_1} q_2^{a_2} \cdots q_k^{a_k} \mid a_0 + \sum_{i=1}^k a_i \beta_i \geq m, a_j \leq c_j - 1 \text{ for } j \geq 1 \right\} \right).$$

In particular, the quotients $\mathbb{C}[x, y]/\mathfrak{a}_m$ have vector space basis consisting of “monomials”

$$q_0^{a_0} q_1^{a_1} q_2^{a_2} \cdots q_k^{a_k} \text{ where } a_0 + \sum_{i=1}^k a_i \beta_i < m \text{ and } a_j \leq c_j - 1 \text{ for } j \geq 1.$$

So computing the volume of v amounts to counting the number of “monomials” in this basis and the result was given in [ELS03]:

$$\text{vol}(v) = \lim_{k \rightarrow +\infty} \alpha_k^{-1} =: \alpha_*^{-1}, \quad (85)$$

where

$$\alpha_k^{-1} = \frac{c_1 c_2 \cdots c_{k-1}}{\beta_k} = \frac{1}{\beta_1} \left(\frac{c_1 \beta_1}{\beta_2} \right) \cdots \left(\frac{c_{k-1} \beta_{k-1}}{\beta_k} \right),$$

is a decreasing sequence of rational numbers strictly less than 1. As pointed out in [ELS03], a key to show this is that:

$$\alpha_*^{-1} = \lim_{k \rightarrow +\infty} \frac{\dim_{\mathbb{C}}(\mathbb{C}[x, y]/\mathfrak{a}_{c_k \beta_k})}{(c_k \beta_k)^2 / 2!} \leq \text{vol}(v). \quad (86)$$

Indeed, because $\beta_{k+1} > c_k \beta_k$, $\dim_{\mathbb{C}}(\mathbb{C}[x, y]/\mathfrak{a}_{c_k \beta_k})$ is equal to the number of integral solutions to the inequalities:

$$a_0 + \sum_{i=1}^k a_i \beta_i < c_k \beta_k, \text{ and } 0 \leq a_j \leq c_j - 1 \text{ for } j \geq 1. \quad (87)$$

We notice that

$$c_k \beta_k - \sum_{i=1}^k (c_i - 1) \beta_i = \sum_{i=1}^k \beta_i - \sum_{i=1}^{k-1} c_i \beta_i = \sum_{i=1}^{k-1} (\beta_{i+1} - c_i \beta_i) + \beta_1 > \beta_1 > 1,$$

so that in particular

$$c_k \beta_k > \sum_{i=1}^k (c_i - 1) \beta_i.$$

So the number of integral solutions to (87) is equal to:

$$\sum_{a_1=0}^{c_1-1} \sum_{a_2=0}^{c_2-1} \cdots \sum_{a_k=0}^{c_k-1} [c_k \beta_k - a_1 \beta_1 - a_2 \beta_2 - \cdots - a_k \beta_k] =: \tilde{\alpha}_k^{-1}.$$

A straight forward calculation shows that $\lim_{k \rightarrow +\infty} \frac{\alpha_k}{\tilde{\alpha}_k} = 1$, so that (86) holds.

On the other hand, there is a sequence of approximating divisorial valuations v_i satisfying $v_i(q_j) = \beta_j$ for any $j \leq i$ and $v_i \leq v$ (see [ELS03, FJ04]). One can similarly show that $\text{vol}(v_i) = \alpha_i^{-1}$. So we get $\alpha_i^{-1} = \text{vol}(v_i) \geq \text{vol}(v)$. Combining this with (86), we see that the equality (85) holds.

Next we calculate the log discrepancy of v by using the work of Favre-Jonsson ([FJ04]). In [FJ04, Chapter 2], Favre-Jonsson studied the representation of a valuation on $\mathbb{C}[x, y]$ by a countable *sequence of key-polynomials (SKP)* and real numbers. Using their notations, the valuation considered here is represented by $[(q_i, \beta_i)]_{i=0}^{+\infty}$. The representation via key-polynomials allowed Favre-Jonsson to approximate any valuation by a sequence of divisorial valuations ([FJ04, Section 3.5]) and calculate various numerical invariants. In particular, they obtained the formula for $\text{vol}(v)$ and $A(v) = A_{\mathbb{C}^2}(v)$. Their formula for log discrepancy reads that ([FJ04, (3.8)]):

$$A(v) = 2 + \sum_{i=1}^{+\infty} m_i (\alpha_i - \alpha_{i-1}), \quad (88)$$

where $\alpha_i = \alpha(v_i)$ is the *skewness* of v_i (see [FJ04, Definition 3.23]) defined as:

$$\alpha_i = \sup \left\{ \frac{v_i(f)}{\text{ord}_0(f)}; f \in \mathfrak{m} \right\}$$

and $m_i = m(v_i)$ is the *multiplicity* of v_i (see [FJ04, Section 3.4]). By [FJ04, Lemma 3.32] and [FJ04, Lemma 3.42] (the roles of x and y are reversed), we get for our valuation v that,

$$\alpha_i = \frac{\beta_i}{c_1 c_2 \cdots c_{i-1}}, \quad m_i = c_1 c_2 \cdots c_{i-1}, \quad \text{for any } i \geq 1.$$

So by Favre-Jonsson's formula (88), we get:

$$\begin{aligned} A(v) &= 2 + \sum_{i=1}^{+\infty} c_1 c_2 \cdots c_{i-1} \left(\frac{\beta_i}{c_1 c_2 \cdots c_{i-1}} - \frac{\beta_{i-1}}{c_1 c_2 \cdots c_{i-2}} \right) \\ &= 2 + \sum_{i=1}^{+\infty} (\beta_i - c_{i-1} \beta_{i-1}). \end{aligned}$$

For an explicit example, we follow [ELSO3] to take c_i to be the i -th prime number ($c_1 = 2, c_2 = 3, \dots$) and set $\beta_1 = \frac{3}{2}$ and $\beta_{i+1} = c_i \beta_i + \frac{1}{c_{i+1}}$ for $i \geq 1$. Then the volume is equal to

$$\begin{aligned} \text{vol}(v) = \alpha_*^{-1} &= \left(1 + \frac{1}{c_1} + \frac{1}{c_1 c_2} + \frac{1}{c_1 c_2 c_3} + \cdots \right)^{-1} \\ &\approx 0.58643. \end{aligned}$$

On the other hand,

$$A(v) = 2 + \sum_{i=1}^{+\infty} (\beta_i - c_{i-1} \beta_{i-1}) = 2 + \sum_{i=1}^{+\infty} \frac{1}{c_i} = +\infty.$$

The last identity is Euler's result: the sum of reciprocals of the primes diverges.

9 Appendix: Convex geometric interpretation of volume formulas

In this section, we use the tool of Okounkov bodies [Oko96, LM09] and coconvex sets [KK14] to provide a clear geometric interpretation of our volume formulas.

9.1 Integration formula by Okounkov bodies and coconvex sets

Consider a \mathbb{Z}^n -valuation defined on $R = \bigoplus_{k=0} H^0(V, kL)$ as follows. Choose a flag:

$$V_\bullet : V = V_0 \supset V_1 \supset V_2 \supset \cdots \supset V_{n-2} \supset V_{n-1} = \{\text{pt}\}$$

of irreducible subvarieties of V , where $\text{codim}_V(V_i) = i$ and each V_i is non-singular at the point V_{n-1} . Following [Oko96, LM09], this determines a \mathbb{Z}^n -valuation \mathbb{V} on R . This is a function satisfying the axiom of valuations (\mathbb{Z}^n is endowed with the lexicographic order):

$$\mathbb{V} = \mathbb{V}_{V_\bullet} = (\mathbb{V}_0, \mathbb{V}_1, \dots, \mathbb{V}_{n-1}) =: (\mathbb{V}_0, \mathbb{V}') : R \rightarrow \mathbb{Z}^n \cup \{\infty\}.$$

It's defined in the following way (see [Oko96, LM09] for more details)

- $g = g_{k_1} + \cdots + g_{k_p}$ with $g_{k_j} \neq 0 \in R_{k_j}$ and $k_1 < k_2 < \cdots < k_p$. Define $\mathbb{V}_0(g) = k_1$ and denote $g^{(0)} = g_{k_1} \in R_{k_1} = H^0(V, k_1 L)$;
- Define $\mathbb{V}_1(g) = \text{ord}_{V_1}(g^{(0)}) =: \nu_1$. After choosing a local equation for V_1 in V , $g^{(0)}$ determines a section $\tilde{g}^{(0)} \in H^0(V, \mathcal{O}_V(k_1 L - \nu_1 V_1))$ that does not vanish identically along V_1 , and so we get by restricting a non-zero section $g^{(1)} \in H^0(V_1, \mathcal{O}_{V_1}(k_1 L - \nu_1 V_1))$.
- Define $\mathbb{V}_2(g) = \text{ord}_{V_2}(g^{(1)}) =: \nu_2$. After choosing a local equation for V_2 in V_1 , $g^{(1)}$ determines a section $\tilde{g}^{(1)} \in H^0(V_1, \mathcal{O}_{V_1}(k_1 L - \nu_1 V_1 - \nu_2 V_2))$ that does not vanish identically along V_2 , and so we get by restricting a non-zero section $g^{(2)} \in H^0(V_2, \mathcal{O}_{V_2}(k_1 L - \nu_1 V_1 - \nu_2 V_2))$.
- This process continues to define \mathbb{V}' .

Denote by $\mathcal{S} = \mathbb{V}(R \setminus \{0\})$ the value semigroup of \mathbb{V} and by $\mathcal{C} = \mathcal{C}(\mathcal{S})$ the cone of \mathcal{S} , i.e. the closure of the convex hull of $\mathcal{S} \cup \{0\}$.

With the above valuation \mathbb{V} , we follow [KK14] to interpret the volumes (mixed multiplicities) as covolumes of convex sets. From now on, let v_1 be a real valuation in $\text{Val}_{X,o}$ satisfying $A_X(v_1) < +\infty$. The graded family of ideals $\mathfrak{a}_\bullet := \mathfrak{a}_\bullet(v_1) = \{\mathfrak{a}_m(v_1)\}$ give rise to a primary sequence of subsets $\mathcal{A}_\bullet = \{\mathcal{A}_m\}$ where

$$\mathcal{A}_m = \{\mathbb{V}(f); f \in \mathfrak{a}_m(v_1) \setminus \{0\}\}.$$

Define the convex set

$$\Gamma = \Gamma(\mathcal{A}_\bullet) = \text{closed convex hull} \left(\bigcup_{m>0} \frac{1}{m} \mathcal{A}_m \right).$$

By [KK14, Theorem 8.12], we have (notice that the multiplicity $\mathfrak{e}(\mathfrak{a}_\bullet(v_1))$ is $\text{vol}(v_1)$ in our notation, see (3)):

$$\text{vol}(v_1) = n! \lim_{m \rightarrow +\infty} \frac{\dim_{\mathbb{C}} R/\mathfrak{a}_m(v_1)}{m^n} = n! \text{covol}(\Gamma).$$

To get $\text{vol}(v_1)$, we will calculate the covolume $\text{covol}(\Gamma)$ by integrating the volumes of slices $(\mathcal{C} \setminus \Gamma) \cap \{\mathbb{V}_0 = \tau\}$. The volumes of these slices will be shown to be ‘‘covolume’’s of rescaled Okounkov bodies. From now on let $\mathcal{F}^x R_k$ be the filtration determined by v_1 . We recall that (see [Oko96, LM09]) the Okounkov body Δ^t of the graded sub algebra $R^{(t)} := \bigoplus_k \mathcal{F}^{kt} R_k$ is defined as:

$$\Delta^t := \text{closed convex hull} \left(\bigcup_{k \geq 1} \frac{1}{k} \mathcal{S}_k^t \right) \subset \mathbb{R}^{n-1},$$

where we denoted:

$$\mathcal{S}_k^t := \{\mathbb{V}'(f); f \in \mathcal{F}^{kt} R_k - \{0\}\}.$$

To see the relation between the Okounkov bodies $\{\Delta^t\}$ and convex set Γ , we consider the following set of displaced union of rescaled Okounkov bodies:

$$\tilde{\Gamma} := \bigcup_{t>0} t^{-1} (\{1\} \times \Delta^t). \quad (89)$$

Proposition 9.1. $\tilde{\Gamma}$ is convex. Moreover its closure is equal to the closed convex set Γ :

$$\Gamma = \overline{\tilde{\Gamma}} = \overline{\bigcup_{t>0} (\{t^{-1}\} \times t^{-1} \Delta^t)}.$$

Proof. We first show the easier inclusion:

$$\overline{\bigcup_{t>0} (\{t^{-1}\} \times t^{-1} \Delta^t)} \subseteq \Gamma. \quad (90)$$

Since Γ is closed, we just need to show that:

$$\bigcup_{t>0} (\{t^{-1}\} \times t^{-1} \Delta^t) \subseteq \Gamma. \quad (91)$$

For any $f \neq 0 \in \mathcal{F}^{kt} R_k \subseteq R_k$, $\mathbb{V}(f) = (k, \mathbb{V}'(f)) \in \{k\} \times \mathcal{S}_k^t$. By the definition of $\mathcal{F}^{kt} R_k$, there exists $g \in R$ with $v_1(g) \geq kt$ and $\mathbf{in}(g) = f$. By the definition of \mathbb{V} , we have $\mathbb{V}(g) = \mathbb{V}(f)$. So we have:

$$\left(1, \frac{\mathbb{V}'(f)}{k} \right) = \frac{\mathbb{V}(f)}{k} = \frac{\mathbb{V}(g)}{k}. \quad (92)$$

We claim the right-hand-side is contained in $t\Gamma$. To see this, first notice that, since $v_1(g) \geq kt \geq \lfloor kt \rfloor$, we have

$$\frac{\mathbb{V}(g)}{k} = \frac{\lfloor kt \rfloor}{kt} t \frac{\mathbb{V}(g)}{\lfloor kt \rfloor} \in \frac{\lfloor kt \rfloor}{kt} t \frac{\mathcal{A}_{\lfloor kt \rfloor}}{\lfloor kt \rfloor} \subset \frac{\lfloor kt \rfloor}{kt} t \Gamma.$$

Replacing f by f^p and g by g^p for $p \gg 1$, we also have:

$$\frac{\mathbb{V}(g)}{k} = \frac{\mathbb{V}(g^p)}{pk} = \frac{\lfloor pkt \rfloor}{pkt} \frac{\mathbb{V}(g^p)}{\lfloor pkt \rfloor} \in \frac{\lfloor pkt \rfloor}{pkt} t\Gamma.$$

Taking $p \rightarrow +\infty$ and using Γ is a closed set, we indeed get $\frac{\mathbb{V}(g)}{k} \in t\Gamma$. So by (92), we get:

$$\{1\} \times \frac{\mathcal{S}_k^t}{k} \subset t\Gamma \cap \{\mathbb{V}_0 = 1\}.$$

Since this holds for any $k \geq 1$ and $f \neq 0 \in \mathcal{F}^{kt}R_k$, we get an inclusion:

$$\{1\} \times \Delta^t = \text{closed convex hull} \left(\bigcup_k \frac{\mathcal{S}_k^t}{k} \right) \subseteq t\Gamma \cap \{\mathbb{V}_0 = 1\}.$$

Equivalently we have that,

$$\{t^{-1}\} \times t^{-1}\Delta^t \subseteq \Gamma \cap \{\mathbb{V}_0 = t^{-1}\}, \text{ for any } t > 0.$$

This gives the inclusion (91).

Next we show the other direction of inclusion. For any $g \neq 0 \in R$ with $v_1(g) \geq m$ and $\mathbf{in}(g) = f \neq 0 \in \mathcal{F}^m R_k$. By definition, we have $\mathbb{V}(g) = (k, \mathbb{V}'(f))$ so that:

$$\frac{\mathbb{V}(g)}{m} = \frac{1}{m}(k, \mathbb{V}'(f)) = \frac{k}{m} \left(1, \frac{\mathbb{V}'(f)}{k} \right).$$

Notice that

$$\frac{\mathbb{V}'(f)}{k} \in \frac{\mathbb{V}'(\mathcal{F}^{(m/k)k}R_k - \{0\})}{k} = \frac{\mathcal{S}_k^{m/k}}{k} \subset \Delta^{m/k}.$$

So we see that:

$$\frac{\mathbb{V}(g)}{m} \in \frac{k}{m} \left(\{1\} \times \Delta^{m/k} \right). \quad (93)$$

Because (93) holds for any $m \geq 1$ and any $g \neq 0 \in \mathbf{a}_m(v_1) - \{0\}$, we get:

$$\bigcup_{m \geq 1} \frac{\mathcal{A}_m}{m} = \bigcup_{m \geq 1} \frac{\mathbb{V}(\mathbf{a}_m(v_1) - \{0\})}{m} \subset \tilde{\Gamma}, \quad (94)$$

where $\tilde{\Gamma}$ denotes the displaced union in (89). Now we claim that $\tilde{\Gamma}$ is a convex set. Assuming the claim, we can take the closed convex hull on the left hand side of (94) to get the other direction of inclusion:

$$\Gamma \subseteq \overline{\tilde{\Gamma}} = \overline{\bigcup_{t>0} t^{-1}(\{1\} \times \Delta^t)} = \overline{\bigcup_{t>0} \{t^{-1}\} \times t^{-1}\Delta^t}.$$

Next we verify the claim on the convexity of $\tilde{\Gamma}$. First notice that we can rewrite:

$$\tilde{\Gamma} = \bigcup_{\tau > 0} \{\tau\} \times \tau\Delta^{\tau^{-1}}.$$

Because each slice $\tau\Delta^{\tau^{-1}}$ is a convex set, the convexity of $\tilde{\Gamma}$ is equivalent to the inclusion relation (compare to a similar but different relation in [BC11, (1.6)]):

$$(1-s)\tau_1\Delta^{\tau_1^{-1}} + s\tau_2\Delta^{\tau_2^{-1}} \subseteq ((1-s)\tau_1 + s\tau_2)\Delta^{((1-s)\tau_1 + s\tau_2)^{-1}}, \quad (95)$$

for any $\tau_1, \tau_2 \in \mathbb{R}_{>0}$ and $s \in [0, 1]$. We first reduce this to proving the inclusion:

$$(1-s)\tau_1 \frac{\mathcal{S}_{k_1}^{\tau_1^{-1}}}{k_1} + s\tau_2 \frac{\mathcal{S}_{k_2}^{\tau_2^{-1}}}{k_2} \subset ((1-s)\tau_1 + s\tau_2)\Delta^{((1-s)\tau_1 + s\tau_2)^{-1}}. \quad (96)$$

For any two points $u_i \in \text{int}(\Delta^{\tau_i^{-1}})$, $i = 1, 2$, we can find two sets $\{a_i, x_i\}_{i=1}^{N_1}$ and $\{b_j, y_j\}_{j=1}^{N_2}$ satisfying:

$$\sum_i a_i = 1 = \sum_j b_j, \quad 0 \leq a_i, b_j \leq 1, \quad x_i \in \frac{\mathcal{S}_{k_i}^{\tau_i^{-1}}}{k_i}, y_j \in \frac{\mathcal{S}_{l_j}^{\tau_j^{-1}}}{l_j},$$

such that $u_1 = \sum_i a_i x_i$ and $u_2 = \sum_j b_j y_j$. The linear combination $(1-s)\tau_1 u_1 + s\tau_2 u_2 = (1-s)\tau_1 \sum_i a_i x_i + s\tau_2 \sum_j b_j y_j$ can be written as a convex linear combination:

$$\sum_{i,j} c_{i,j} ((1-s)\tau_1 x_i + s\tau_2 y_j) \quad (97)$$

with $c_{i,j} \in [0, 1]$, $\sum_j c_{i,j} = a_i$ and $\sum_i c_{i,j} = b_j$. If each term in (97) is in the right-hand-side of (96), the convex combination is also in it as well. For any two points $u_i \in \Delta^{\tau_i^{-1}}$, $i = 1, 2$, we can find two sequence of points $\{u_i^{(k)}\}_{k=1}^{+\infty} \subset \text{int}(\Delta^{\tau_i^{-1}})$, $i = 1, 2$ such that $u_i^{(k)} \rightarrow u_i$ as $k \rightarrow +\infty$. By the above discussion, we have $(1-s)\tau_1 u_1^{(k)} + s\tau_2 u_2^{(k)}$ is contained in the right-hand-side of (96) that is a closed set. Letting $k \rightarrow +\infty$, we get $(1-s)\tau_1 u_1 + s\tau_2 u_2 \in$ is contained in it too.

So finally, we show (96) holds. For $i = 1, 2$, choose $k_i \in \mathbb{Z}_{>0}$ and $f_i \in \mathcal{F}^{k_i \tau_i^{-1}} R_{k_i}$. Then there exists $g_i \in R_{k_i}$ such that $v_1(g_i) \geq k_i \tau_i^{-1}$ and $\mathbf{in}(g_i) = f_i$. We fix any $s \in [0, 1]$, and for simplicity we will denote

$$u_i = \frac{\mathbb{V}'(f_i)}{k_i}, i = 1, 2; \quad \tau = \tau(s) = (1-s)\tau_1 + s\tau_2.$$

For any integer $l_1, l_2 \in \mathbb{Z}_{>0}$, we define

$$g = g(l_1, l_2) = g_1^{l_1} g_2^{l_2}, \quad f = f(l_1, l_2) = f_1^{l_1} f_2^{l_2} = \mathbf{in}(g).$$

Then we have

$$\begin{aligned} v_0(f) &= l_1 k_1 + l_2 k_2 =: k; \\ \mathbb{V}'(f) &= l_1 \mathbb{V}'(f_1) + l_2 \mathbb{V}'(f_2) = l_1 k_1 u_1 + l_2 k_2 u_2; \\ v_1(g) &\geq l_1 \frac{k_1}{\tau_1} + l_2 \frac{k_2}{\tau_2} =: m. \end{aligned} \quad (98)$$

So we get:

$$f = \mathbf{in}(g) \in \mathcal{F}^m R_k = \mathcal{F}^{(m/k)k} R_k. \quad (99)$$

We want to show that for any $\epsilon > 0$, there exists $l_1 = l_1(\epsilon)$ and $l_2 = l_2(\epsilon)$ such that the following properties are satisfied:

1. $m/k \geq \tau^{-1}$ such that $f \in \mathcal{F}^{k\tau^{-1}} R_k$ and $\frac{\mathbb{V}'(f)}{k} \in \Delta^{\tau^{-1}}$. By (98), this is equivalent to:

$$\frac{l_1 k_1}{l_1 k_1 + l_2 k_2} \frac{1}{\tau_1} + \frac{l_2 k_2}{l_1 k_1 + l_2 k_2} \frac{1}{\tau_2} \geq \frac{1}{\tau}. \quad (100)$$

- 2.

$$\left\| \tau \frac{\mathbb{V}'(f)}{k} - (1-s)\tau_1 u_1 - s\tau_2 u_2 \right\| \leq \epsilon. \quad (101)$$

By (98), this is equivalent to:

$$\left\| \left(\tau \frac{l_1 k_1}{l_1 k_1 + l_2 k_2} - (1-s)\tau_1 \right) u_1 + \left(\tau \frac{l_2 k_2}{l_1 k_1 + l_2 k_2} - s\tau_2 \right) u_2 \right\| \leq \epsilon. \quad (102)$$

Assuming that we can achieve these, then the proof would be completed. Indeed, by these two conditions, we would have that for any $\epsilon > 0$, $(1-s)\tau_1 u_1 + s\tau_2 u_2$ is approximated arbitrarily close by elements in $\tau \Delta^{\tau^{-1}}$. Because $\Delta^{\tau^{-1}}$ is a closed set, we indeed get $(1-s)\tau_1 u_1 + s\tau_2 u_2 \in \tau \Delta^{\tau^{-1}}$.

Notice that the equalities in (100) and (102) are both satisfied if and only if

$$\frac{l_2 k_2}{l_1 k_1} = \frac{s\tau_2}{(1-s)\tau_1}. \quad (103)$$

More precisely, if we define the ratios:

$$\delta = \delta(l_1, l_2) = \frac{l_2 k_2}{l_1 k_1}, \quad \delta_0 = \frac{s\tau_2}{(1-s)\tau_1},$$

then the left hand side minus the right hand side of (100) is equal to:

$$\frac{\delta - \delta_0}{(1 + \delta)(1 + \delta_0)} \frac{\tau_1 - \tau_2}{\tau_1 \tau_2}; \quad (104)$$

while the absolute difference on the left hand side of (102) is equal to:

$$\frac{\tau \|u_1 - u_2\|}{(1 + \delta)(1 + \delta_0)} |\delta - \delta_0|. \quad (105)$$

If the right-hand-side of (103) is a rational number then we can find a solution to (103). Otherwise, it's clear that we can find infinitely many pairs of $(l_1, l_2) \in \mathbb{Z}_{>0}^2$ such that (104) is strictly positive and (105) is as small as possible. \square

We can now calculate the covolume of Γ using \mathbb{V}_0 -slices. First there exists $b_1 > 0$ such that $\Gamma \cap \{\mathbb{V}_0 > b_1\} = \mathcal{C} \cap \{\mathbb{V}_0 > b_1\}$. So we have:

$$\begin{aligned} \text{covol}(\Gamma) &= \text{vol}(\mathcal{C} \cap \{\mathbb{V}_0 \leq b_1\}) - \text{vol}(\Gamma \cap \{\mathbb{V}_0 \leq b_1\}) \\ &= \frac{L^{n-1}}{(n-1)!} \int_0^{b_1} \tau^{n-1} d\tau - \text{vol}(\Gamma \cap \{\mathbb{V}_0 \leq b_1\}) \\ &= \frac{L^{n-1}}{n!} b_1^n - \text{vol}(\Gamma \cap \{\mathbb{V}_0 \leq b_1\}). \end{aligned}$$

For the second term, we first notice that

$$\text{vol}(\Gamma \cap \{\mathbb{V}_0 \leq b_1\}) = \text{vol}(\tilde{\Gamma} \cap \{\mathbb{V}_0 \leq b_1\}), \quad (106)$$

where $\tilde{\Gamma}$ was the set defined in (89):

$$\tilde{\Gamma} = \bigcup_{t>0} t^{-1} (\{1\} \times \Delta^t).$$

To see this, recall that by Proposition 9.1, $\tilde{\Gamma}$ is convex and $\Gamma = \overline{\tilde{\Gamma}}$. It's also clear that $\tilde{\Gamma} \cap \{\mathbb{V}_0 > b_1\} = \mathcal{C} \cap \{\mathbb{V}_0 > b_1\}$ for b_1 sufficiently large. So for $b_1 \gg 1$ we also have:

$$\Gamma \cap \{\mathbb{V}_0 \leq b_1\} = \overline{\tilde{\Gamma} \cap \{\mathbb{V}_0 \leq b_1\}}.$$

Since $\tilde{\Gamma} \cap \{\mathbb{V}_0 \leq b_1\}$ is again convex, its boundary has Lebesgue measure 0 (see [Lan86] for example) and hence (106) holds. So we get:

$$\begin{aligned} \text{vol}(\Gamma \cap \{\mathbb{V}_0 \leq b_1\}) &= \text{vol}(\tilde{\Gamma} \cap \{\mathbb{V}_0 \leq b_1\}) \\ &= \int_0^{b_1} \text{vol}(\tilde{\Gamma} \cap \{\mathbb{V}_0 = \tau\}) d\tau = \int_0^{b_1} \text{vol}(\tau \Delta^{\tau^{-1}}) d\tau \\ &= \int_0^{b_1} \tau^{n-1} \text{vol}(\Delta^{\tau^{-1}}) d\tau = \int_{b_1^{-1}}^{+\infty} \text{vol}(\Delta^t) \frac{dt}{t^{n+1}} \\ &= \frac{1}{(n-1)!} \int_{b_1^{-1}}^{+\infty} \text{vol}(R^{(t)}) \frac{dt}{t^{n+1}}. \end{aligned}$$

So we indeed recover the formula (21) for the volume of v_1 :

$$\text{vol}(v_1) = n! \text{covol}(\Gamma) = b_1^n L^{n-1} - n \int_{b_1^{-1}}^{+\infty} \text{vol}(R^{(t)}) \frac{dt}{t^{n+1}}. \quad (107)$$

As before, by integration by parts we get formula (22):

$$\begin{aligned} \text{vol}(v_1) &= b_1^n L^{n-1} + \int_{b_1^{-1}}^{+\infty} \text{vol}(R^{(t)}) d\left(\frac{1}{t^n}\right) \\ &= \int_{b_1^{-1}}^{+\infty} \frac{-d\text{vol}(R^{(t)})}{t^n}. \end{aligned} \quad (108)$$

To get another formula, we will consider the following function

$$H_{\mathcal{F}}(x) = \inf \left\{ \tau > 0; x \in \Delta^{\tau^{-1}} \right\} \quad \text{for any } x \in \Delta^0.$$

So we see that $H_{\mathcal{F}}(x)$ is related to the function considered by Boucksom-Chen in [BC11].

$$H_{\mathcal{F}}^{-1}(x) = G_{\mathcal{F}}(x) = \sup \{ t > 0; x \in \Delta^t \}.$$

By [BC11], $H_{\mathcal{F}}^{-1}(x)$ is a concave function and it's continuous on $\text{int}(\Delta^0)$. If we denote by λ the Lebesgue measure of \mathbb{R}^{n-1} , then we have (cf. [BC11, Proof of Theorem 1.11]):

$$d\text{vol}(\Delta^{\tau^{-1}}) = \left(\frac{d}{d\tau} \text{vol}(\{x \in \Delta^0; H_{\mathcal{F}}(x) \leq \tau\}) \right) d\tau = (H_{\mathcal{F}})_* \lambda.$$

We can then calculate covolume of Γ :

$$\text{covol}(\Gamma) = \text{covol}(\tilde{\Gamma}) \tag{109}$$

$$\begin{aligned} &= \int_0^{b_1} \tau^{n-1} \left(\text{vol}(\Delta^0) - \text{vol}(\Delta^{\tau^{-1}}) \right) d\tau \\ &= \int_0^{b_1} \tau^{n-1} \left(\int_{\Delta^0} \mathbf{1}_{\{\tau \leq H_{\mathcal{F}}(x)\}} dx \right) d\tau \\ &= \int_{\Delta^0} dx \int_0^{H_{\mathcal{F}}(x)} \tau^{n-1} d\tau = \frac{1}{n} \int_{\Delta^0} H_{\mathcal{F}}^n(x) dx. \end{aligned} \tag{110}$$

So we get another formula for $\text{vol}(v_1)$:

$$\text{vol}(v_1) = n! \text{covol}(\Gamma) = (n-1)! \int_{\Delta^0} H_{\mathcal{F}}^n(x) dx. \tag{111}$$

The relation between (108) and (111) is given by:

$$\begin{aligned} (n-1)! \int_{\Delta^0} H_{\mathcal{F}}^n(x) dx &= (n-1)! \int_0^{+\infty} \tau^n (H_{\mathcal{F}})_* \lambda \\ &= (n-1)! \int_0^{+\infty} \tau^n d\text{vol}(\Delta^{\tau^{-1}}) \\ &= -(n-1)! \int_0^{+\infty} t^{-n} d\text{vol}(\Delta^t) \\ &= \int_0^{+\infty} \frac{-d\text{vol}(R^{(t)})}{t^n}. \end{aligned}$$

Finally, let's point out the geometric meaning of the expression in (23). For simplicity we can assume $\lambda = 1$ by rescaling the coconvex set. Then

$$\begin{aligned} \Phi(1, s) &= \int_{c_1}^{+\infty} \frac{-d\text{vol}(R^{(t)})}{((1-s) + st)^n} \\ &= (n-1)! \int_0^{c_1^{-1}} \frac{d\text{vol}(\Delta^{\tau^{-1}})}{(1-s + s\tau^{-1})^n} \\ &= (n-1)! \int_0^{c_1^{-1}} \frac{\tau^n}{((1-s)\tau + s)^n} (H_{\mathcal{F}})_* \lambda \\ &= \frac{n!}{n} \int_{\Delta^0} \left(\frac{H_{\mathcal{F}}}{(1-s)H_{\mathcal{F}} + s} \right)^n dx. \end{aligned}$$

Comparing with (111), it's clear that, if we define the function

$$H_s(x) = \frac{H_{\mathcal{F}}}{(1-s)H_{\mathcal{F}} + s}.$$

Then $\Phi(1, s) = n! \text{covol}(\Gamma_s)$ where the convex set Γ_s for any $s \in [0, 1]$ is defined as:

$$\Gamma_s = \{ (t, tx); x \in \Delta^0, t > 0, H_s(x) \leq t \}.$$

9.2 Examples

We will give convex geometric interpretation of the volumes of valuations in Section 8.

1. Let $Y \rightarrow \mathbb{C}^2$ be the blow up at the origin with exceptional divisor \mathbb{P}^1 . We choose the point $\{0\} = [1, 0] \in \mathbb{P}^1$ and get a \mathbb{Z}^2 -valuation $\mathbb{V} = (\mathbb{V}_0, \mathbb{V}_1)$ on $\mathbb{C}[x, y]$ such that for each homogeneous polynomial \mathbb{V}_1 gives the lowest degree in y variable. Denote by (τ, μ) the rectangular coordinates on the $\mathbb{R}^2 = \mathbb{Z}^2 \otimes_{\mathbb{Z}} \mathbb{R}$. Then using the previous notations, it's easy to see that

$$\mathcal{S} = \{(a+b, b); (a, b) \in \mathbb{Z}_{\geq 0}^2\}; \quad \mathcal{C} = \mathcal{C}(\mathcal{S}) = \{(\tau, \mu) \in \mathbb{R}_{\geq 0}^2; \mu \leq \tau\}.$$

The convex set Γ associated to v is a closed \mathcal{C} -convex set. Because it contains the vectors:

$$\frac{(a+b, b)}{[\gamma_1 a + \gamma_2 b]} \text{ with } a > 0, b > 0,$$

it's easy to see that:

$$\Gamma = \mathcal{C} \setminus \triangle_{OAB},$$

where the triangle \triangle_{OAB} has vertices $O = (0, 0)$, $A = (\frac{1}{\gamma_1}, 0)$ and $B = (\frac{1}{\gamma_2}, \frac{1}{\gamma_2})$. Notice that on the blow up Y , the center of v is $[1, 0] \in \mathbb{P}^1$ (resp. $[0, 1] \in \mathbb{P}^1$) if and only if $\gamma_2 > \gamma_1$ (resp. $\gamma_2 < \gamma_1$) if and only if the slope of the line \overline{AB} is negative (resp. positive).

2. We will use the same \mathbb{Z}^2 -valuation \mathbb{V} as in the previous example. Denote by $v_0 = \text{ord}_o$ (where $o = (0, 0)$) the canonical valuation on $\mathbb{C}[x, y]$:

$$v_0 \left(\sum_{e_1, e_2 \in \mathbb{Z}_{\geq 0}} p_{\alpha_1, \alpha_2} x^{e_1} y^{e_2} \right) = \min\{e_1 + e_2; p_{e_1, e_2} \neq 0\}.$$

For convenience, we define the numbers:

$$d_1 = 1 \text{ and } d_i = c_1 c_2 \cdots c_{i-1} \text{ for } i \geq 2.$$

Notice that $\alpha_i = \frac{d_i}{\beta_i}$. Then we have:

Lemma 9.2. *For any $k \in \mathbb{Z}_{\geq 0}$, $a_0 \in \mathbb{Z}_{\geq 0}$ and $a_i \in \mathbb{Z}_{\geq 0} \cap [0, c_i - 1]$, we have*

$$v_0(q^{\mathbf{a}}) = a_0 + \sum_{i=1}^k a_i d_i$$

and the homogeneous component of lowest degree of $q^{\mathbf{a}}$, denoted by $\mathbf{in}(q^{\mathbf{a}})$, is $y^{a_0} x^{\sum_{i=1}^k a_i d_i}$. Moreover, for any linear combination of $q^{\mathbf{a}}$,

$$v_0 \left(\sum_{\mathbf{a}} b_{\mathbf{a}} q^{\mathbf{a}} \right) = \min\{v_0(q^{\mathbf{a}}), b_{\mathbf{a}} \neq 0\}.$$

Proof. For the first statement, it's clear that we just need to prove it for each q_i . We prove this by induction. Since $q_1 = x$, $\deg(q_1) = 1$ and $\mathbf{in}(q_1) = 1$. Since $q_2 = x^{c_1} + y^{c_1 \beta_1}$, $\deg(q_2) = c_1 < \beta_1 c_1$ and $\mathbf{in}(q_2) = x^{c_1}$. Assume we have proved $\deg(q_i) = d_i$ and $\mathbf{in}(q_i) = x^{d_i}$. To show that $q_{i+1} = q_i^{c_i} + y^{c_i \beta_i}$ satisfies the first statement, we just need to show that $c_i d_i < c_i \beta_i$. This is equivalent to $\alpha_i = \frac{d_i}{\beta_i} < 1$ which indeed holds true.

To prove the second statement, we choose the minimum among $v_0(q^{\mathbf{a}})$ with $b_{\mathbf{a}} \neq 0$. Then $\mathbf{in}(q^{\mathbf{a}}) = y^{a_0} x^{\sum_{i=1}^k a_i d_i}$. We claim that $\mathbf{in}(q^{\mathbf{a}})$ can not be cancelled in the standard monomial expansion of $\sum_{\mathbf{a}} b_{\mathbf{a}} q^{\mathbf{a}}$. Indeed, if it is cancelled, then there exists \mathbf{a}' with $p_{\mathbf{a}'} \neq 0$ and $a'_0 = a_0$ such that $\mathbf{in}(q^{\mathbf{b}}) = y^{a_0} x^{\sum_{j=1}^{k'} a'_j d_j}$ satisfies $\sum_{j=1}^{k'} a'_j d_j = \sum_{i=1}^k a_i d_i$.

By allowing zero coefficients and without loss of generality, we can assume that $k' = k$ and $c_j - 1 \geq a'_j > a_j \geq 0$. Then we can estimate that

$$\begin{aligned} 0 &= \sum_{j=1}^k a'_j d_j - \sum_{i=1}^k a_i d_i = (a'_k - a_k) d_k - \sum_{i=1}^{k-1} (a'_i - a_i) d_i \\ &\geq d_k - \sum_{i=1}^{k-1} (c_i - 1) d_i = \sum_{i=1}^{k-1} (d_{i+1} - c_i d_i) + d_1 = 1 > 0. \end{aligned}$$

So we get a contradiction. \square

Corollary 9.3. *We have the following formula:*

$$v(\mathbf{m}) = \inf_{\mathbf{m}} \frac{v}{v_0} = 1, \quad \sup_{\mathbf{m}} \frac{v}{v_0} = \alpha_*.$$

Proof. We first notice the estimates:

$$1 \leq \frac{v(q^{\mathbf{a}})}{v_0(q^{\mathbf{a}})} = \frac{a_0 + \sum_{i=1}^k a_i \beta_i}{a_0 + \sum_{i=1}^k a_i d_i} \leq \frac{\beta_i}{d_i} \leq \alpha_*.$$

The supreme can be proved by letting $k \rightarrow +\infty$. The infimum can be proved using the second statement of the above lemma. \square

Remark 9.4. *Notice that by (85), the above corollary is compatible with Favre-Jonsson's formula for $\text{vol}(v)$ in dimension 2 ([FJ04, Remark 3.33]):*

$$\text{vol}(v) = \left(\sup_{\mathbf{m}} \frac{v}{\text{ord}_0} \right)^{-1} \frac{1}{v(\mathbf{m})}.$$

As before, let $Y \rightarrow \mathbb{C}^2$ be the blow up at the origin with exceptional divisor \mathbb{P}^1 . We choose the point $\{0\} = [1, 0] \in \mathbb{P}^1$ then we get a \mathbb{Z}^2 -valuation $\mathbb{V} = (\mathbb{V}_0, \mathbb{V}_1)$ on $\mathbb{C}[x, y]$. Then by lemma 9.2, we get:

Corollary 9.5. *For any exponents $\mathbf{a} = (a_0, a_1, \dots, a_k)$ satisfying the assumption in the above lemma, we have $\mathbb{V}(q^{\mathbf{a}}) = (a_0 + \sum_{i=1}^k a_i d_i, a_0)$.*

Proposition 9.6. *The convex set Γ associated to v is equal to the region:*

$$\Gamma = \mathcal{C} \cap \left\{ \mu \leq \frac{\tau - \alpha_*^{-1}}{1 - \alpha_*^{-1}} \right\} = \mathcal{C} \setminus \Delta_{OAB},$$

where $O = (0, 0)$, $A = (\alpha_*^{-1}, 0)$ and $B = (1, 1)$. As a consequence, $\text{vol}(v) = 2! \cdot \text{covol}(\Gamma) = \alpha_*^{-1}$.

Notice that the set $\mathcal{H} := \left\{ \mu \leq \frac{\tau - \alpha_*^{-1}}{1 - \alpha_*^{-1}} \right\}$ is the half space on the right hand side of the line $\mu = \frac{\tau - \alpha_*^{-1}}{1 - \alpha_*^{-1}}$ which passes through $(1, 1)$ and $(\alpha_*^{-1}, 0)$.

Proof. By Corollary 9.2, we know that for any exponent \mathbf{a} satisfying $a_0 \geq 0$ and $0 \leq a_i \leq c_i - 1$, we have:

$$\Gamma \ni \frac{\mathbb{V}(q^{\mathbf{a}})}{[v(q^{\mathbf{a}})]} = \frac{(a_0 + \sum_{i=1}^k a_i d_i, a_0)}{[a_0 + \sum_{i=1}^k a_i \beta_i]}$$

Letting $a_i = 0$ and $a_0 \neq 0$, we see that $(1, 1) \in \Gamma$. Letting $a_0 = a_1 = \dots = a_{k-1} = 0$ and $a_i = c_i - 1$, we see that

$$\Gamma \ni \frac{((c_k - 1)d_k, 0)}{[(c_k - 1)\beta_k]} = \left(\frac{d_k}{\beta_k}, 0 \right) \xrightarrow{k \rightarrow +\infty} (\alpha_*^{-1}, 0).$$

So $(\alpha_*^{-1}, 0) \in \Gamma$ because Γ is closed. Because Γ is a \mathcal{C} -convex closed set, we deduce that $\Gamma \supseteq \mathcal{C} \cap \mathcal{H}$. To prove the other direction of inclusion, by the definition of Γ we just need to show that:

$$\frac{\mathcal{A}_m}{m} = \frac{\mathbb{V}(\mathbf{a}_m(v) - \{0\})}{m} \in \mathcal{C} \cap \mathcal{H}.$$

In other words, for any $f \in \mathfrak{a}_m(v)$, we want to show $\frac{\mathbb{V}(f)}{m} \in \mathcal{C} \cap \mathcal{H}$. To prove this, we expand f into “monomials” $f = \sum_{\mathbf{a}} b_{\mathbf{a}} q^{\mathbf{a}}$ with each $b_{\mathbf{a}} \neq 0$. Then by the proof of Lemma 9.2, we see that there is an exponent \mathbf{a}^* among the \mathbf{a} ’s such that:

$$\mathbb{V}(f) = v_0(f) = v_0(q^{\mathbf{a}^*}) = a_0^* + \sum_{i=1}^k a_i^* \beta_i, \text{ and } \mathbb{V}_1(f) = a_0^*.$$

Recall that we have $v(f) = \min_{\mathbf{a}} \{v(q^{\mathbf{a}})\}$, so that we can write the normalized valuation vector as:

$$\frac{\mathbb{V}(f)}{m} = \lambda \frac{\mathbb{V}(q^{\mathbf{a}^*})}{v(q^{\mathbf{a}^*})},$$

where

$$\lambda = \frac{v(q^{\mathbf{a}^*})}{m} = \frac{v(q^{\mathbf{a}^*})}{\min_{\mathbf{a}} \{v(q^{\mathbf{a}})\}} \frac{v(f)}{m} \geq 1.$$

Because $\mathcal{C} \cap \mathcal{H}$ is closed under rescalings, we just need to show

$$\frac{\mathbb{V}(q^{\mathbf{a}^*})}{v(q^{\mathbf{a}^*})} = \frac{(a_0 + \sum_i a_i^* d_i, a_0^*)}{a_0^* + \sum_i a_i^* \beta_i} \in \mathcal{C} \cap \mathcal{H}.$$

This is indeed true because it can be written as a convex combination:

$$\frac{\mathbb{V}(q^{\mathbf{a}^*})}{v(q^{\mathbf{a}^*})} = \frac{a_0^*}{a_0^* + \sum_i a_i^* \beta_i} (1, 1) + \sum_i \frac{a_i^*}{a_0^* + \sum_i a_i^* \beta_i} \left(\frac{d_i}{\beta_i}, 0 \right)$$

and the set of points $\{(1, 1)\} \cup \cup_i \{(\frac{d_i}{\beta_i}, 0)\}$ is contained in $\mathcal{C} \cap \mathcal{H}$. □

c_k	$\beta_k = v(q_k)$	q_k	$\#(\mathcal{S} \setminus \mathcal{A}_{c_k \beta_k})$	$\frac{\dim_{\mathbb{C}}(\mathbb{C}[x, y]/\mathfrak{a}_{c_k \beta_k})}{(c_k \beta_k)^2 / 2!}$
$c_1 = 2$	$\frac{3}{2}$	x	5	1.11111
$c_2 = 3$	$\frac{10}{3}$	$x^2 + y^3$	38	0.76
$c_3 = 5$	$\frac{51}{5}$	$(x^2 + y^3)^3 + y^{10}$	810	0.62284
$c_4 = 7$	$\frac{358}{7}$	$q_3^5 + y^{51}$	37923	0.59179
$c_5 = 11$	$\frac{3939}{11}$	$q_4^7 + y^{358}$	4553318	0.58693

Figure 1 shows $\frac{1}{c_k \beta_k} (\mathcal{S} \setminus \mathcal{A}_{c_k \beta_k})$ for $k = 2, 3, 4$. As an example of Proposition 4.3, we let $m = c_2 \beta_2 = 3 \cdot 10/3 = 10$ so that $\dim_{\mathbb{C}}(\mathbb{C}[x, y]/\mathfrak{a}_m(v)) = \#(\mathcal{S} \setminus \mathcal{A}_{10}) = 38$. On the other hand, we can write down the basis of $\mathcal{F}^m R_k$ for $k = 6, 7, 8, 9$ for which $0 \subsetneq \mathcal{F}^{10} R_k \subsetneq R_k$:

- (a) $k = 6$: $x^6 = \mathbf{in}((x^2 + y^3)^3) = \mathbf{in}(q_2^3)$.
- (b) $k = 7$: $x^7, x^6 y, x^5 y^2 = \mathbf{in}(y^2 x(x^2 + y^3)^2) = \mathbf{in}(y^2 q_1 q_2^2)$.
- (c) $k = 8$: $x^8, x^7 y, x^6 y^2, x^5 y^3, x^4 y^4$.
- (d) $k = 9$: $x^9, x^8 y, x^7 y^2, x^6 y^3, x^5 y^4, x^4 y^5, x^3 y^6, x^2 y^7$.

Notice that the elements $x^6, x^5 y^2$ show that the equality $\mathcal{F}^m R_k = R_k \cap \mathfrak{a}_m(v)$, which holds in the \mathbb{C}^* -invariant case by Lemma 4.6, does not hold in general.

10 Postscript Note:

After the second version of the preprint on arXiv was posted, we were informed by K. Fujita that he independently prove the results in Theorem 3.7 (see [Fuj16]). Moreover, he obtained a similar result for the uniform K-stability.

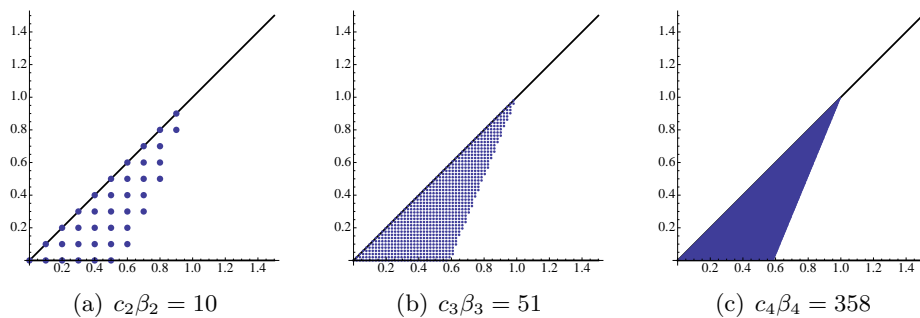


Figure 1: Complement of Primary Sequence

11 Acknowledgment

The author is partially supported by NSF DMS-1405936. I would like to thank the math department at Purdue University for providing a nice environment for my research. I would like to thank especially Laszlo Lempert, Kenji Matsuki, Bernd Ulrich and Sai-Kee Yeung for their interest and encouragement. I am grateful to Kento Fujita for pointing out a mistake in an earlier draft which helps me to get a correct statement of the result. I also would like to thank Chenyang Xu for helpful comments, Yuchen Liu for fruitful discussions, Tommaso de Fernex and Mircea Mustața for their interests in this work, and anonymous referees for very useful comments which help me to improve the presentation.

References

- [Abh56] S.S.Abhyankar, On the valuations centered in a local domain, Amer. Jour. Math., **78** (1956), pp 321-348.
- [AH06] K. Altmann, J. Hausen, Polyhedral divisors and algebraic torus actions, Math. Ann. **334** (2006) 557-607.
- [AIP⁺11] K. Altmann, N.O. Itten, L. Petersen, H. Süß, and R. Vollmert, The Geometry of T-varieties. In Piotr Pragacz, editor, *Contributions to Algebraic Geometry*, Imsganga Lect. Notes, 2011, arXiv:1102.5760.
- [AM69] M.F. Atiyah, and I.G. MacDonaldd, Introduction to commutative algebra, Addison-Wesley, Reading, Mass., 1969.
- [Ber15] R. Berman; K-polystability of \mathbb{Q} -Fano varieties admitting Kähler-Einstein metrics, Invent. Math., **203** (2015), no. 3, 973-1025. arXiv:1205.6214.
- [BBJ15] R. Berman, S. Boucksom and M. Jonsson. A variational approach to the Yau-Tian-Donaldson conjecture. arXiv:1509.04561
- [BC11] S. Boucksom, H. Chen, Okounkov bodies of filtered linear series, Compositio Math. **147** (2011) 1205-1229.
- [BFFU13] S. Boucksom, T. de Fernex, C. Favre and S. Urbinati. Valuation spaces and multiplier ideals on singular varieties, Recent advances in algebraic geometry, 29-51, London Math. Soc. Lecture Note Ser., 417, Cambridge Univ. Press, Cambridge, 2015. arXiv:1307.0227.
- [BFJ08] S. Boucksom, C. Favre and M. Jonsson. Valuations and plurisubharmonic singularities. Publ. RIMS. Kyoto Univ. **44** (2008), 449-494.
- [BFJ12] S. Boucksom, C. Favre and M. Jonsson, A refinement of Izumi's theorem, Valuation Theory in Interaction, 2014, pp 55-81. arXiv:1209.4104.
- [BHJ15] S. Boucksom, T. Hisamoto, and M. Jonsson. Uniform K-stability, Duistermaat-Heckman measures and singularities of pairs, to appear in Ann. Inst. Fourier, arXiv:1504.06568v3.
- [BKMS15] S. Boucksom, A. Küronya, C. Maclean and T. Szemberg. Vanishing sequences and Okounkov boides. Math. Ann., Volume 361, Issue 3, pp 811-834, 2015.

- [Blum16] H. Blum, Existence of valuations with smallest normalized volume, arXiv:1606.08894.
- [CS12] T. Collins, G. Székelyhidi, K-semistability for irregular Sasakian manifolds, arXiv:1204.2230 (2012).
- [CS15] T. Collins, G. Székelyhidi, Sasaki-Einstein metrics and K-stability, arXiv:1512.07213 (2015).
- [CDS15] X.X. Chen, S. K. Donaldson and S. Sun, Kähler-Einstein metrics on Fano manifolds, I-III, J. Amer. Math. Soc. **28** (2015), 183-197, 199-234, 235-278.
- [Cut12] S.D. Cutkosky, Multiplicities associated to graded families of ideals. Algebra & Number Theory, Vol. 7 (2013) No. 9, 2059-2083. arXiv:1206.4077.
- [CS16] G. Codogni, J. Stoppa, Torus equivariant K-stability, arXiv:1602.03451.
- [DaSz15] V. Datar, G. Székelyhidi, Kähler-Einstein metrics along smooth continuity method, to appear in Geom. Funct. Anal., arXiv:1506.07495.
- [DH09] T. de Fernex, C. Hacon, Singularities on normal varieties, Composition Math. **145** (2009), 393-414.
- [Don02] S. Donaldson, Scalar curvature and stability of toric varieties, J. Differential Geometry. 62 (2002), no. 2, 289-349.
- [DS15] S. Donaldson, S. Sun, Gromov-Hausdorff limits of Kahler manifolds and algebraic geometry, II, arXiv:1507.05082.
- [DT92] W. Ding, G. Tian. Kähler-Einstein metrics and the generalized Futaki invariant. Invent. Math. **110** (1992), no. 2, 315-335.
- [ELS03] L. Ein, R. Lazarsfeld, and K. E. Smith, Uniform approximation of Abhyankar valuation ideals in smooth function fields, American Journal of Mathematics 125 (2003), 409-440.
- [FEM04] T. de Fernex, L. Ein, M. Mustață, multiplicities and log canonical threshold, J. Algebraic Geometry, 13 (2004) 603-615.
- [Fut83] A. Futaki, An obstruction to the existence of Einstein Kähler metrics, Inventiones Mathematicae (1983), 437-443.
- [FJ04] C. Favre and M. Jonsson, The valuative tree. Lecture Notes in Mathematics, 1853, Springer-Verlag, Berlin, 2004
- [Fuj15a] K. Fujita, On K-stability and the volume functions of \mathbb{Q} -Fano varieties, Proc. London Math. Soc. (2016) 113 (5): 541-582, arXiv:1508.04052.
- [Fuj15b] K. Fujita, Optimal bounds for the volumes of Kähler-Einstein Fano manifolds, arXiv:1508.04578.
- [Fuj16] K. Fujita, A valuative criterion for uniform K-stability of \mathbb{Q} -Fano varieties, to appear in J. Reine Angew. Math., arXiv:1602.00901.
- [HS01] R. Hübl and I. Swanson. Divisorial valuations centered on local domains. J. Pure Appl. Algebra, **161** (2001) 145-166.
- [Izu85] S. Izumi. A measure of integrity for local analytic algebras. Publ. RIMS Kyoto Univ. 21 (1985), 719-735.
- [JM10] M. Jonsson and M. Mustață, Valuations and asymptotic invariants for sequences of ideals, Ann. Inst. Fourier (Grenoble) 62 (2012), no. 6, 2145-2209. arXiv:1011.3699.
- [Kol04] J. Kollár, Seifert G_m -bundles, arXiv:0404386.
- [Kol13] J. Kollár, Singularities of the Minimal Model Program, Cambridge Tracts in Mathematics, 200, Cambridge University Press, 2013.
- [KK14] K. Kaveh, A. Khovanskii. Convex bodies and multiplicities of ideals. Proceedings of the Steklov Institute of Mathematics, 2014, Vol. 286, pp. 268-284.
- [Lan86] R. Lang, A note on the measurability of convex sets, Arch. Math., Vol. 47, 90-92 (1986).
- [Laz96] R. Lazarsfeld, Positivity in algebraic geometry, I: Classical setting: line bundles and linear series, Erger Mathematik und ihrer Grenzgebiete. (3) **48**, Springer-Verlag, Berlin, 1996.

- [LS13] A. Liendo, H. Süß, Normal singularities with torus actions, *Tohoku Math J.* (2) volume 65, Number 1 (2013), 105-130.
- [LM09] R. Lazarsfeld and M. Mustașă. Convex bodies associated to linear series, *Ann. Sci. Éc. Norm. Supér.* **42** (2009), no. 5, 783-835.
- [Li13] C. Li. Yau-Tian-Donaldson correspondence for K-semistable Fano varieties, to appear in *J. Reine Angew. Math.*, arXiv:1302.6681.
- [Li15a] C. Li. Minimizing normalized volumes of valuations. arXiv:1511.08164.
- [LL16] C. Li, and Y. Liu, Kähler-Einstein metrics and volume minimization, arXiv:1602.05094.
- [LX14] C. Li, and C. Xu, Special test configurations and K-stability of Fano varieties, *Ann. of Math.* (2) 180 (2014), no.1, 197-232.
- [LX16] C. Li and C. Xu, Stability of valuations and Kollár components, arXiv:1604.05398.
- [Oda13] Y. Odaka, A generalization of the Ross-Thomas slope theory, *Osaka J. Math.* **50** (2013), 171-185.
- [Laz04] R. Lazarsfeld, Positivity in algebraic geometry, II, *Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Survey in Mathematics*, **49**. Berlin, Springer-Verlag, 2004.
- [MSY08] D. Martelli, J. Sparks, S.-T. Yau, Sasaki-Einstein manifolds and volume minimisation, *Commu. Math. Phys.* 280, 611-673 (2008).
- [Mus02] M. Mustașă. On multiplicities of graded sequences of ideals, *Journal of Algebra* 256 (2002) 229-249.
- [Oko96] A. Okounkov. Brunn-Minkowski inequality for multiplicities, *Invent. Math.* **125** (1996), 405-411.
- [PW16] J. Park, J. Won, K-stability of smooth del Pezzo surfaces, arXiv:1608.06053.
- [Ree89] D. Rees. Izumi's theorem. In *Commutative algebra* (Berkeley, CA, 1987), 407-416. *Math. Sci. Res. Inst. Publ.*, 15. Springer, New York. 1989.
- [Tia97] G. Tian, Kahler-Einstein metrics with positive scalar curvature, *Invent. Math.* 130 (1997), no. 1, 1-37.
- [Tia15] G. Tian, K-stability and Kähler-Einstein metrics, *Communications on Pure and Applied Mathematics*, Volume 8, Issue 7, pages 1085-1156, July 2015.
- [Wan12] X. Wang, Heights and GIT weight, *Math. Res. Lett.* **19** (2012), 909-926.
- [Wat81] K. Watanabe, Some remarks concerning Demazure's construction of normal graded rings, *Nagoya Math. J.*, Volume 83 (1981), 203-211.
- [WN12] D. Witt Nyström. Test configurations and Okounkov bodies. *Compositio Math.* **148** Issue 06 (2012), 1736-1756.
- [ZS60] O. Zariski and P. Samuel. *Commutative Algebra II, Grad. Texts in Math.*, Vol. 29, Springer-Verlag, New York, 1960.

li2285@purdue.edu

Department of Mathematics, Purdue University, West Lafayette, IN 47907-2067 USA