Li, C. (2011) "Constant Scalar Curvature Kähler Metric Obtains the Minimum of K-energy,"
International Mathematics Research Notices, Vol. 2011, No. 9, pp. 2161–2175
Advance Access publication September 1, 2010
doi:10.1093/imrn/rnq152

Constant Scalar Curvature Kähler Metric Obtains the Minimum of K-energy

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Based on Donaldson's method, we prove that, for an integral Kähler class, when there is a Kähler metric of constant scalar curvature, then it minimizes the K-energy. We do not assume that the automorphism group is discrete.

1 Introduction

Let X be a compact Kähler manifold of dimension n and fix a Kähler class $[\omega]$ on X. Define the Kähler potential space

$$\mathcal{K}(\omega) := \{\phi \in C^{\infty}(X, \mathbb{R})\ ;\ \omega + \frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}\phi > 0\} \tag{1}$$

We will write $\mathcal{K}$ for $\mathcal{K}(\omega)$ if the reference metric ω is clear. $\overline{\mathcal{K}} := \mathcal{K}/\mathbb{R}$ is the space of Kähler metrics in $[\omega]$. The K-energy functional was defined by Mabuchi [10].

Definition 1.1. For any $\phi \in \mathcal{K}$, let $\omega_\phi = \omega + (\sqrt{-1}/2\pi)\partial\bar{\partial}\phi \in \overline{\mathcal{K}}$, define

$$\nu_\omega(\omega_\phi) = -\frac{1}{V}\int_0^1 \mathrm{d}t \int_X (S(\omega_{\phi_t}) - \underline{S})\frac{\mathrm{d}\phi_t}{\mathrm{d}t}\ \omega_{\phi_t}^n$$

Received March 10, 2010; Revised June 24, 2010; Accepted July 7, 2010
Communicated by Prof. Simon Donaldson

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ϕ_t is any path connecting 0 and ϕ in $\mathcal{K}$. $S(\omega_\phi)$ denotes the scalar curvature of Kähler metric ω_ϕ, and

$$\underline{S} = \frac{1}{V}\int_X S(\omega)\omega^n = \frac{nc_1(X)\cdot[\omega]^{n-1}}{[\omega]^n}, \quad V = \int_X \omega^n$$

is the average of scalar curvature, which is independent of chosen Kähler metric in $[\omega]$. □

The K-energy is well defined, that is, it does not depend on the path connecting 0 and ϕ. In particular, we can take $\phi(t) = t\phi$. A Kähler metric of constant scalar curvature is a critical point of K-energy and it is a local minimizer.

Now assume the Kähler class is $c_1(L) \in H^2(X,\mathbb{Z}) \cap H^{1,1}(X,\mathbb{R})$ for some ample line bundle L over X.

In this note, we prove the following theorem:

Theorem 1.2. Suppose that there is a metric ω_∞ of constant scalar curvature in the Kähler class $c_1(L)$. Then ω_∞ minimizes the K-energy in this Kähler class. □

In the case of Kähler–Einstein metrics, this result was proved by Bando and Mabuchi in [1] and [2]. In [7], Donaldson proved the above theorem under the assumption that the automorphism group $\mathrm{Aut}(X, L)$ is discrete. Here $\mathrm{Aut}(X, L)$ denotes the group of automorphisms of the pair of (X, L) modulo the trivial automorphism $\mathbb{C}^*$ (acting by constant scalar multiplication on the fibres). The theorem was proved for more general extremal Kähler metrics on any compact Kähler manifolds by Chen–Tian in [5], where the authors used geodesics in infinite dimensional space of Kähler metrics in $[\omega]$. It can also be proved by recent result of Chen–Sun [4], where they use geodesic approximation to prove weak convexity of K-energy.

In this note, we use Donaldson's method to prove the theorem for every integral class case.

The strategy to prove the theorem is finite dimensional approximation. We sketch the idea here.

Tian's approximation theorem (Proposition 2.2 and Corollary 2.3) says that $\mathcal{K}$ can be approximated by a sequence of finite dimensional symmetric spaces $\mathcal{H}_k$. Here $\mathcal{H}_k \cong GL(N_k,\mathbb{C})/U(N_k)$ is the space of Hermitian metrics on the complex vector space $H^0(X, L^k)$.

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In [7], Donaldson defined a sequence of functional $\mathcal{L}_k$ on $\mathcal{K}$, which approximate K-energy as $k \to \infty$. When restricted to $\mathcal{H}_k$, $\mathcal{L}_k$ is bounded below by the logarithmic of Chow norm. It was known that balanced metric obtains the minimum of Chow norm [16, 23]. In [6], Donaldson already proved, in the case of discrete automorphism group, the existence of balanced metrics, which approximate Kähler metric of constant scalar curvature. Putting these together, he can prove the theorem.

Mabuchi [12–14] extended many results of [6] to the case where the varieties have infinitesimal automorphisms. As Mabuchi [11] showed, if the automorphism group is not discrete, in general there will be no balanced metrics. Instead, Mabuchi defined T-balanced metrics and T-stability with respect to some torus group contained in Aut(X). Donaldson claimed [7] one can use these new techniques to prove the above theorem without assuming that the automorphism group is discrete.

In this note, we use the same quantization strategy. But we don't need existence of balanced metrics nor T-balanced metrics. Instead we use simpler Bergman metrics constructed directly from ω_∞. Using asymptotic expansion of Bergman kernel, we show that Bergman metrics are almost balanced in an asymptotical sense (Proposition 2.4), and they can help us to prove the theorem. In this way, we don't need the restriction on the automorphism group; moreover, our argument is more direct.

As can be seen from the following, the argument follows [7] closely. The idea of using almost balanced metrics is inspired by works of Mabuchi. In particular, the proof of Lemma 3.3 is inspired by the argument of [15], page 13. See Remark 3.1. As the above work shows, the convexity of various functionals is the essential property behind the argument.

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2 Notations and Preliminaries

2.1 Maps between $\mathcal{K}_k$ and $\mathcal{H}_k$

We will use some definitions and notations from [7]. The set $\mathcal{K}$ defined in (1) depends on reference Kähler metric ω. However in the following, we will omit writing down this dependence, because it's clear that $\mathcal{K}$ is also the set of metrics h on L whose curvature form

$$c_1(L, h) := -\frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}\log h$$

is a positive (1,1) form on X. Let $\mathcal{K}_k$ denote the set of Hermitian metrics on L^k with positive curvature form, then $\mathcal{K}_k \simeq \mathcal{K} = \mathcal{K}_1$. Let $N_k = \dim H^0(X, L^k)$, $V = \int_X c_1(L)^n$. We have maps between $\mathcal{K}_k$ and $\mathcal{H}_k$.

Definition 2.1.

$$\begin{aligned}
\mathrm{Hilb} : \mathcal{K}_k &\longrightarrow \mathcal{H}_k \\
h_k &\mapsto \|s\|^2_{\mathrm{Hilb}(h_k)} = \frac{N_k}{Vk^n}\int_X |s|^2_{h_k} c_1(L^k, h_k)^n, \quad \forall s \in H^0(X, L^k) \\
\mathrm{FS} : \mathcal{H}_k &\longrightarrow \mathcal{K}_k \\
H_k &\mapsto |s|^2_{\mathrm{FS}(H_k)} = \frac{|s|^2}{\sum_{\alpha=1}^{N_k} |s_\alpha^{(k)}|^2}, \quad \forall s \in L^k.
\end{aligned}$$

In the above definition, $\{s_\alpha^{(k)}; 1 \le \alpha \le N_k\}$ is an orthonormal basis of the Hermitian complex vector space $(H^0(X, L^k), H_k)$. □

2.2 Bergman metrics, expansions of Bergman kernels

For any fixed Kähler metric $\omega \in c_1(L)$, take a Hermitian metric h on L such that $c_1(L, h) = \omega$, the kth Bergman metric of h is

$$h_k = \mathrm{FS}(\mathrm{Hilb}(h^{\otimes k})) \in \mathcal{K}_k.$$

Let $\{s_\alpha^{(k)}, 1 \le \alpha \le N_k\}$ be an orthonormal basis of $\mathrm{Hilb}(h^{\otimes k})$. Define the kth (suitably normalized) Bergman kernel of ω

$$\rho_k(\omega) = \frac{N_k n!}{Vk^n}\sum_{\alpha=1}^{N_k} |s_\alpha^{(k)}|^2_{h^{\otimes k}}.$$

Note that h is determined by ω up to a constant, but $\rho_k(\omega)$ doesn't depend on the chosen h.

The following proposition is now well known.

Proposition 2.2 ([3, 9, 19, 20, 22]).

(i) For fixed ω, there is an asymptotic expansion as $k \to +\infty$

$$\rho_k(\omega) = A_0(\omega) + A_1(\omega)k^{-1} + \dots$$

where $A_i(\omega)$ are smooth functions on X defined locally by ω.

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(ii) In particular

$$A_0(\omega) = 1, \quad A_1(\omega) = \frac{1}{2}S(\omega).$$

(iii) The expansion holds in C^∞ in that for any r, $N \geq 0$

$$\left\| \rho_k(\omega) - \sum_{i=0}^{N} A_i(\omega) k^{-i} \right\|_{C^r(X)} \leq K_{r,N,\omega} k^{-N-1}$$

for some constants $K_{r,N,\omega}$. Moreover the expansion is uniform in that for any r, N, there is an integer s such that if ω runs over a set of metrics, which are bounded in C^s, and with ω bounded below, the constants $K_{r,N,\omega}$ are bounded by some $K_{r,N}$ independent of ω. □

Remark 2.1. We choose a particular normalization of Bergman kernel, so that the expansion starts with order 0, other than order n as it appeared in [6, Proposition 6]. □

The following approximation result is a corollary of Proposition 2.2.(i)–(ii).

Corollary 2.3 ([20]). Using the notation at the beginning of this subsection, we have, as $k \to +\infty$, $(h_k)^{1/k} \to h$, and $(1/k)c_1(L^k, h_k) \to \omega$, the convergence in C^∞ sense. More precisely, for any $r > 0$, there exists a constant $C_{r,\omega}$ such that

$$\left\| \log \frac{h_k^{\frac{1}{k}}}{h} \right\|_{C^r} \leq C_{r,\omega} k^{-2}, \qquad \left\| \frac{1}{k} c_1(L^k, h_k) - \omega \right\|_{C^r} \leq C_{r,\omega} k^{-2}. \tag{2}$$

□

Proof. It's easy to see that

$$(h_k)^{\frac{1}{k}} = h \cdot \left(\sum_\alpha |s_\alpha^{(k)}|^2_{h^{\otimes k}} \right)^{-\frac{1}{k}} =: he^{-\phi_k}.$$

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Note that by the expansion in Proposition 2.2.(i)–(ii), we have

$$\sum_\alpha |s_\alpha|^2_{h^{\otimes k}} = \frac{(N_k n!/Vk^n)\sum_{\alpha=1}^{N_k}|s_\alpha^{(k)}|^2_{h^{\otimes k}}}{N_k n!/Vk^n} = \frac{1+\frac{1}{2}S(\omega)k^{-1}+O(k^{-2})}{1+\frac{1}{2}\underline{S}k^{-1}+O(k^{-2})}$$
$$= 1+O(k^{-1}).$$

So

$$\phi_k = \frac{1}{k}\log\left(\sum_\alpha |s_\alpha^{(k)}|^2_{h^{\otimes k}}\right) = O(k^{-2}).$$

The error term is in C^∞ sense. So the first inequality in (2) holds. The second inequality in (2) follows because

$$\frac{1}{k}c_1(L^k, h_k) - \omega = \frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}\phi_k.$$

■

Now assume we have a Kähler metric of constant scalar curvature ω_∞ in the Kähler class $c_1(L)$. Take a $h_\infty \in \mathcal{K}_1$ such that

$$\omega_\infty = c_1(L, h_\infty).$$

We will make extensive use of the kth Bergman metric of h_∞ and its associated objects, so for the rest of this note, we denote

$$H_k^* = \mathrm{Hilb}(h_\infty^{\otimes k}), \quad h_k^* = \mathrm{FS}(H_k^*) = \mathrm{FS}(\mathrm{Hilb}(h_\infty^{\otimes k})), \omega_k^* = c_1(L^k, h_k^*) = \frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}\log\left(\sum_{\alpha=1}^{N_k}|\tau_\alpha^{(k)}|^2\right). \tag{$*$}$$

Hereafter, we fix an orthonormal basis $\{\tau_\alpha^{(k)}, 1 \le \alpha \le N_k\}$ of $H_k^* = \mathrm{Hilb}(h_\infty^{\otimes k})$.

The next proposition says we can improve the convergence rate in corollary 2.3 for h_∞. This will be important for us. (Compare [15], (3.8)–(3.10))

Proposition 2.4. For any $r > 0$, there exists some constant C_{r,ω_∞} such that

$$\left\|\sum_{\alpha=1}^{N_k}|\tau_\alpha^{(k)}|^2_{h_\infty^{\otimes k}} - 1\right\|_{C^r} \le C_{r,\omega_\infty}k^{-2} \tag{3}$$

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So in particular,

$$\frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}\log\left(\sum_{\alpha=1}^{N_k}|\tau_\alpha^{(k)}|^2\right) - k\,\omega_\infty = O(k^{-2}) \tag{4}$$

□

Proof. Since $S(\omega_\infty) \equiv \underline{S}$, by proposition 2.2, we have

$$\sum_{\alpha=1}^{N_k}|\tau_\alpha^{(k)}|^2_{h_\infty^{\otimes}} - 1 = \frac{(N_k n!/Vk^n)\sum_{\alpha=1}^{N_k}|\tau_\alpha^{(k)}|^2_{h_\infty^{\otimes k}}}{N_k n!/Vk^n} - 1 = \frac{1+\frac{1}{2}S(\omega_\infty)k^{-1}+O(k^{-2})}{1+\frac{1}{2}\underline{S}k^{-1}+O(k^{-2})} - 1$$
$$= O(k^{-2})$$

(4) follows because the left hand side of it is equal to $\frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}\log\left(\sum_{\alpha=1}^{N_k}|\tau_\alpha^{(k)}|^2_{h_\infty^{\otimes k}}\right)$ ■

2.3 Aubin–Yau functional and Chow norm

We define the Aubin–Yau functional with respect to (L^k, h_k^*) by

$$I_k(h_k^* e^{-\phi}) = -\int_0^1 \mathrm{d}t \int_X \frac{\mathrm{d}\phi(t)}{\mathrm{d}t}\, c_1(L^k, h_k^* e^{-\phi(t)})^n.$$

Here $\phi \in \mathcal{K}_k$, that is, $(1/k)\phi \in \mathcal{K}$. $\phi(t)$ is a path connecting 0 and ϕ in $\mathcal{K}_k$.

Under the orthonormal basis $\{\tau_\alpha^{(k)}, 1 \le \alpha \le N_k\}$ of H_k^*, $H^0(X, L^k) \cong \mathbb{C}^{N_k}$ and $\mathbb{P}(H^0(X, L^k)^*) \cong \mathbb{CP}^{N_k-1}$.

For any $H_k \in \mathcal{H}_k$, take an orthonormal basis $\{s_\alpha, 1 \le \alpha \le N_k\}$ of H_k. Let $\det H_k$ denote the determinant of matrix $(H_k)_{\alpha\beta} = (H_k^*(s_\alpha, s_\beta))$. $\{s_\alpha\}$ determines a projective embedding into $\mathbb{CP}^{N_k-1}$. (Note that the fixed isomorphism $\mathbb{P}(H^0(X, L^k)^*) \cong \mathbb{CP}^{N_k-1}$ is determined by the basis $\{\tau_\alpha^{(k)}\}$.) The image of this embedding is denoted by $X_k(H_k) \subset \mathbb{CP}^{N_k-1}$ and has degree $d_k = Vk^n$. $X_k(H_k)$ has a Chow point [16, 23]

$$\hat{X}_k(H_k) \in \mathcal{W}_k := H^0(Gr(N_k - n - 2, \mathbb{P}^{N_k-1}), \mathcal{O}(d_k))$$

such that the corresponding divisor

$$Zero(\hat{X}_k(H_k)) = \{L \in Gr(N_k - n - 2, \mathbb{P}^{N_k-1});\, L \cap X_k(H_k) \neq \emptyset\}.$$

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Proposition 2.5 ([16, 23]). $\mathcal{W}_k$ has a Chow norm $\|\cdot\|_{\mathrm{CH}(H_k^*)}$, such that for all $H_k \in \mathcal{H}_k$ we have

$$\frac{1}{N_k}\log\det H_k - \frac{1}{Vk^n}I_k(\mathrm{FS}(H_k)) = \frac{1}{Vk^n}\log\|\hat{X}_k(H_k)\|^2_{\mathrm{CH}(H_k^*)} \qquad \square$$

$SL(N_k,\mathbb{C})$ acts on $\mathcal{H}_k$ and $\mathcal{W}_k$. Note that $X_k(\sigma\cdot H_k^*) = \sigma\cdot X_k(H_k^*)$. Define

$$f_k(\sigma) = \log\left(\|\hat{X}_k(\sigma\cdot H_k^*)\|^2_{\mathrm{CH}(H_k^*)}\right) \quad \forall\sigma\in SL(N_k,\mathbb{C})$$

It's easy to see that $f_k(\sigma\cdot\sigma_1) = f_k(\sigma)$ for any $\sigma_1\in SU(N_k)$, so f_k is a function on the symmetric space $SL(N_k,\mathbb{C})/SU(N_k)$. We have

Proposition 2.6 ([7, 8, 18, 23]). $f_k(\sigma)$ is convex on $SL(N_k,\mathbb{C})/SU(N_k)$. $\square$

To relate $\mathcal{K}_k$ and $\mathcal{H}_k$, following Donaldson [7], we change $FS(H_k)$ in the above formula into general $h_k\in\mathcal{K}_k$ and define:

Definition 2.7. For all $h_k\in\mathcal{K}_k$ and $H_k\in\mathcal{H}_k$,

$$\tilde{P}_k(h_k, H_k) = \frac{1}{N_k}\log\det H_k - \frac{1}{Vk^n}I_k(h_k). \qquad \square$$

Note that, for any $c\in\mathbb{R}$, $I_k(e^c h_k) = cVk^n + I_k(h_k)$, so

$$\tilde{P}_k(e^c h_k, e^c H_k) = \tilde{P}_k(h_k, H_k). \tag{5}$$

Remark 2.2. This definition differs from Donaldson's definition by omitting two extra terms, since we find no use for these terms in the following argument. $\square$

3 Proof of Theorem 1.2

Lemma 3.1. For any $h_k, h_k'\in\mathcal{K}_k$, with $h_k' = h_k e^{-\phi}$, we have

$$-\int_X \phi\, c_1(L^k, h_k)^n \le I_k(h_k') - I_k(h_k) \le -\int_X \phi\, c_1(L^k, h_k')^n. \qquad \square$$

This is [7, Lemma 1].

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Proof. This lemma just says I_k is a convex function on $\mathcal{K}_k$, regarded as an open subset of $C^\infty(X)$. We only need to calculate its second derivative along the path $h_k(t) = h_k e^{-t\phi}$:

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2} I_k(h_k) = -\int_X \phi \triangle_t \phi c_1(L^k, h_k(t))^n = \int_X |\nabla_t \phi|^2 c_1(L^k, h_k(t))^n \geq 0$$

$\triangle_t$ and ∇_t are the Laplace and gradient operators of Kähler metric $c_1(L^k, h_k(t))$. ∎

From now on, fix a $\omega \in c_1(L)$, take a Hermitian metric $h \in \mathcal{K}$ such that $\omega = c_1(L, h)$. We have the k-th Bergman metric $h_k = \mathrm{FS}(\mathrm{Hilb}(h^{\otimes k}))$ and corresponding Kähler metric $\omega_k = c_1(L^k, h_k)$. By Corollary 2.3

$$(h_k)^{\frac{1}{k}} \to h, \quad \frac{1}{k}\omega_k \to \omega, \quad \text{in} \quad C^\infty.$$

Lemma 3.2.

$$\tilde{P}_k(h_k, \mathrm{Hilb}(h_k)) \geq \tilde{P}_k(\mathrm{FS}(\mathrm{Hilb}(h_k)), \mathrm{Hilb}(h_k)).$$ □

This is a corollary of [7, Lemma 4]. Since the definition of $\tilde{P}$ is a little different from that in [7], we give a direct proof here.

Proof. Let $h_k' = \mathrm{FS}(\mathrm{Hilb}(h_k))$. Then

$$\tilde{P}_k(h_k, \mathrm{Hilb}(h_k)) - \tilde{P}_k(\mathrm{FS}(\mathrm{Hilb}(h_k)), \mathrm{Hilb}(h_k)) = \frac{1}{Vk^n}(I(h_k') - I(h_k)).$$

Let $\{s_\alpha^{(k)}, 1 \leq \alpha \leq N_k\}$ be an orthonormal basis of $\mathrm{Hilb}(h_k)$. Then $\log(h_k'/h_k) = -\log(\sum_{\alpha=1}^{N_k} |s_\alpha^{(k)}|_{h_k}^2)$. By Lemma 3.1 and concavity of the function log,

$$\begin{aligned}
\frac{1}{Vk^n}(I(h_k') - I(h_k)) &\geq -\frac{1}{Vk^n}\int_X \log\left(\sum_\alpha |s_\alpha^{(k)}|_{h_k}^2\right) c_1(L^k, h_k)^n \\
&\geq -\log\left(\frac{1}{N_k}\frac{N_k}{Vk^n}\int_X \sum_\alpha |s_\alpha^{(k)}|_{h_k}^2 c_1(L^k, h_k)^n\right) \\
&= -\log\left(\frac{1}{N_k}\sum_\alpha \|s_\alpha^{(k)}\|_{\mathrm{Hilb}(h_k)}^2\right) = 0.
\end{aligned}$$

∎

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Lemma 3.3. There exists a constant $C > 0$, depending only on h and h_∞, such that

$$\tilde{P}_k(\mathrm{FS}(\mathrm{Hilb}(h_k)), \mathrm{Hilb}(h_k)) - \tilde{P}_k(\mathrm{FS}(H_k^*), H_k^*) \geq -Ck^{-1}. \qquad \square$$

Proof. Recall that $H_k^* = \mathrm{Hilb}(h_\infty^{\otimes k})$ and $\{\tau_\alpha^{(k)}; 1 \leq \alpha \leq N_k\}$ is an orthonormal basis of H_k^* (see $(*)$). Let $H_k = \mathrm{Hilb}(h_k)$ and $\{s_\alpha^{(k)}; 1 \leq \alpha \leq N_k\}$ be an orthonormal basis of H_k. Transforming by a matrix in $SU(N_k)$, we can assume

$$s_\alpha^{(k)} = e^{\lambda_\alpha^{(k)}} \tau_\alpha^{(k)}$$

Evaluating the norm $\mathrm{Hilb}(h_k)$ on both sides, we see that

$$e^{-2\lambda_\alpha^{(k)}} = \frac{N_k}{Vk^n} \int_X |\tau_\alpha^{(k)}|_{h_k}^2 \omega_k^n. \tag{6}$$

Since by Corollary 2.3 we have the following uniform convergence in C^∞: $(h_k)^{\frac{1}{k}} \to h$, $\frac{1}{k}\omega_k \to \omega$. There exists a constant $C_1 > 0$, $C_2 > 0$, depending only on h and h_∞, such that $C_1^{-k} \leq \frac{h_k}{h_\infty^{\otimes k}} \leq C_1^k$, $C_2^{-1}\omega_\infty \leq \frac{1}{k}\omega_k \leq C_2\omega_\infty$, so we see from (6) that $|\lambda_\alpha^{(k)}| \leq Ck$.

Let $\underline{\lambda} = (1/N_k)\sum_{\beta=1}^{N_k} \lambda_\beta^{(k)}$, $H_k' = e^{2\underline{\lambda}} H_k$, $\hat{\lambda}_\alpha^{(k)} = \lambda_\alpha^{(k)} - \underline{\lambda}$. Then $\{\hat{s}_\alpha^{(k)} = e^{\hat{\lambda}_\alpha^{(k)}} \tau_\alpha^{(k)}\}$ is an orthonormal basis of H_k'. Note that $\hat{\lambda}_\alpha^{(k)}$ satisfies the same estimate as $\lambda_\alpha^{(k)}$:

$$|\hat{\lambda}_\alpha^{(k)}| \leq Ck. \tag{7}$$

$(e^{\hat{\Lambda}})_{\alpha\beta} = e^{\hat{\lambda}_\alpha^{(k)}} \delta_{\alpha\beta}$ is a diagonal matrix in $SL(N_k, \mathbb{C})$. By scaling invariance of $\tilde{P}_k$ (5) and Proposition 2.5, we have

$$\tilde{P}_k(\mathrm{FS}(H_k), H_k) = \tilde{P}_k(\mathrm{FS}(H_k'), H_k') = \frac{1}{Vk^n} \log \|\hat{X}_k(H_k')\|^2_{\mathrm{CH}(H_k^*)} \tag{8}$$

$$\tilde{P}_k(\mathrm{FS}(H_k^*), H_k^*) = \frac{1}{Vk^n} \log \|\hat{X}_k(H_k^*)\|^2_{\mathrm{CH}(H_k^*)}. \tag{9}$$

As in Section 2.3, let

$$X_k(s) = e^{s\hat{\Lambda}} \cdot X_k(H_k^*)$$

$$f_k(s) = \log \|\hat{X}_k(s)\|^2_{\mathrm{CH}(H_k^*)}.$$

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Then $X_k(0) = X_k(H_k^*)$ and $X_k(1) = X_k(H_k') = X_k(H_k)$. By Proposition 2.6, $f_k(s)$ is a convex function of s, so

$$f_k(1) - f_k(0) \geq f_k'(0).$$

We can estimate $f_k'(0)$ by the estimates in Proposition 2.4:

$$\begin{aligned} f_k'(0) &= \int_X \frac{\sum_\alpha \hat{\lambda}_\alpha^{(k)} |\tau_\alpha^{(k)}|^2_{h_\infty^{\otimes k}}}{\sum_\alpha |\tau_\alpha^{(k)}|^2_{h_\infty^{\otimes k}}} \left(\frac{\sqrt{-1}}{2\pi} \partial\bar{\partial} \log \sum_{\alpha=1}^{N_k} |\tau_\alpha^{(k)}|^2 \right)^n \\ &= \int_X \frac{\sum_\alpha \hat{\lambda}_\alpha^{(k)} |\tau_\alpha^{(k)}|^2_{h_\infty^{\otimes k}}}{1 + O(k^{-2})} (1 + O(k^{-2}))(k\omega_\infty)^n \\ &= \int_X O(k^{-2}) (\sum_{\alpha=1}^{N_k} \hat{\lambda}_\alpha^{(k)} |\tau_\alpha^{(k)}|^2_{h_\infty^{\otimes k}}) (k\omega_\infty)^n \end{aligned}$$

where the last equality is because of

$$\int_X \sum_{\alpha=1}^{N_k} \hat{\lambda}_\alpha^{(k)} |\tau_\alpha^{(k)}|^2_{h_\infty^{\otimes k}} (k\omega_\infty)^n = \frac{Vk^n}{N_k} \sum_{\alpha=1}^{N_k} \hat{\lambda}_\alpha^{(k)} = 0$$

By the estimate for $\hat{\lambda}_\alpha^{(k)}$ (7), we get

$$|f_k'(0)| \leq Ck^{-2} k N_k \leq Ck^{n-1}.$$

So $f_k(1) - f_k(0) \geq f_k'(0) \geq -Ck^{n-1}$, and

$$\frac{1}{Vk^n} (\log \|\hat{X}_k(H_k')\|^2_{\mathrm{CH}} - \frac{1}{Vk^n} \log \|\hat{X}_k(H_k^*)\|^2_{\mathrm{CH}}) = \frac{1}{Vk^n} (f_k(1) - f_k(0)) \geq -C \frac{1}{Vk^n} k^{n-1} \geq -Ck^{-1}.$$

So the lemma follows from identities (8) and (9). ■

Remark 3.1. The proof of this lemma is similar to the argument in the beginning part of [15, Section 5] where Mabuchi proved K-semistability of varieties with constant scalar curvature metrics. Roughly speaking, here we consider geodesic segment connecting H_k^* and H_k in $\mathcal{H}_k$, while Mabuchi [15, Section 5] considered geodesic ray in $\mathcal{H}_k$ defined by a test configuration. The estimates in Proposition 2.4 show that, to prove the K-semistability as in Mabuchi's argument [15, Section 5], we only need Bergman metrics of h_∞ instead of Mabuchi's T-balanced metrics. □

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Remark 3.2. The referee pointed out that the similar argument also appears in the proof of Theorem 2 in [17]. □

Remark 3.3. In [7, Corollary 2], H_k^* is taken to be balance metric, that is, H_k^* is a fixed point of the mapping $\mathrm{Hilb}(\mathrm{FS}(\cdot))$. Then the difference in Lemma 3.3 is nonnegative, instead of bounded below by error term $-Ck^{-1}$. □

Lemma 3.4. There exists a constant $C > 0$, which only depends on h_∞, such that

$$\left|\tilde{P}_k(\mathrm{FS}(H_k^*), H_k^*) - \tilde{P}_k(h_\infty^{\otimes k}, \mathrm{Hilb}(h_\infty^{\otimes k}))\right| \leq Ck^{-2}$$ □

Proof. Recall from $(*)$ that: $\mathrm{Hilb}(h_\infty^{\otimes k}) = H_k^*$, $h_k^* = \mathrm{FS}(H_k^*) = \mathrm{FS}(\mathrm{Hilb}(h_\infty^{\otimes k}))$, so it's easy to see that

$$\tilde{P}_k(\mathrm{FS}(H_k^*), H_k^*) - \tilde{P}_k(h_\infty^{\otimes k}, \mathrm{Hilb}(h_\infty^{\otimes k})) = \frac{1}{Vk^n}(I_k(h_\infty^{\otimes k}) - I_k(h_k^*))$$

For any section s of L^k, $|s|^2_{h_k^*} = \frac{|s|^2_{h_\infty^{\otimes k}}}{\sum_\alpha |\tau_\alpha^{(k)}|^2_{h_\infty^{\otimes k}}}$. So

$$\frac{h_\infty^{\otimes k}}{h_k^*} = \sum_{\alpha=1}^{N_k} |\tau_\alpha^{(k)}|^2_{h_\infty^{\otimes k}}$$

By proposition 2.4. $\left|\log \frac{h_k^*}{h_\infty^{\otimes k}}\right| = |\log(1 + O(k^{-2}))| = O(k^{-2})$. So by Lemma 3.1, we get

$$\left|\frac{1}{Vk^n}(I_k(h_\infty^{\otimes k}) - I_k(h_k^*))\right| \leq Ck^{-2}$$ ■

Definition 3.5. For any Kähler metric $\omega_\phi = c_1(L, h) + (\sqrt{-1}/2\pi)\partial\bar{\partial}\phi \in [\omega]$, let $h_k(\phi) = h^{\otimes k}e^{-k\phi}$. Define

$$\mathcal{L}_k(\omega_\phi) = \tilde{P}_k(h_k(\phi), \mathrm{Hilb}(h_k(\phi)))$$ □

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Lemma 3.6 ([7]). There exist constants μ_k, such that

$$\mathcal{L}_k(\omega_\phi) + \mu_k = \frac{1}{2}\nu_\omega(\omega_\phi) + O(k^{-1}).$$

Here $O(k^{-1})$ depends on ω and ω_ϕ. □

Proof. Let $\psi(t) = t\phi \in \mathcal{K}$ connecting 0 and ϕ, $h_k(t) = h_k e^{-tk\phi}$, $\omega_t = \omega + t(\sqrt{-1}/2\pi)\partial\bar{\partial}\phi$, $\triangle_t$ be the Laplace operator of metric ω_t. Plugging in expansions for Bergman kernels ρ_k in Proposition 2.2, we get

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\tilde{P}_k(h_k(t), \mathrm{Hilb}(h_k(t))) &= \frac{1}{N_k n!}\int_X \frac{N_k n!}{V k^n}\sum_\alpha |s_\alpha^{(k)}|^2_{h_k(t)}(-k\phi + \triangle_t\phi)k^n\omega_t^n + \frac{1}{Vk^n}\int_X k\phi k^n\omega_t^n \\ &= \frac{1}{V}\frac{k^n}{k^n + \frac{1}{2}\underline{S}k^{n-1} + \cdots}\int_X(-k\rho_k(\omega_t) + \triangle_t\rho_k(\omega_t))\phi\omega_t^n + \frac{k}{V}\int_X \phi\omega_t^n \\ &= -\frac{1}{2V}\int_X (S(\omega_t) - \underline{S})\phi\omega_t^n + O(k^{-1})\end{aligned}$$

$\{\omega_t, 0 \le t \le 1\}$ have uniformly bounded geometry, so by Proposition 2.2.(3), the expansions above are uniform. So the lemma follows after integrating the above equation. ■

Proof of Theorem 1.2. Let $\omega_\phi = c_1(L, h) + \frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}\phi \in [\omega]$, $h_k = h_k(\phi) = h^{\otimes k}e^{-k\phi}$

By Lemma 3.2, Lemma 3.3, Lemma 3.4

$$\begin{aligned}\tilde{P}_k(h_k, \mathrm{Hilb}(h_k)) &\ge \tilde{P}_k(\mathrm{FS}(\mathrm{Hilb}(h_k)), \mathrm{Hilb}(h_k)) \\ &\ge \tilde{P}_k(\mathrm{FS}(H_k^*), H_k^*) + O(k^{-1}) \\ &= \tilde{P}_k(h_\infty^{\otimes k}, \mathrm{Hilb}(h_\infty^{\otimes k})) + O(k^{-1})\end{aligned}$$

So by Lemma 3.6

$$\begin{aligned}\nu_\omega(\omega_\phi) &= 2\mathcal{L}_k(\omega_\phi) + 2\mu_k + O(k^{-1}) = 2\tilde{P}_k(h_k, \mathrm{Hilb}(h_k)) + 2\mu_k + O(k^{-1}) \\ &\ge 2\tilde{P}_k(h_\infty^{\otimes k}, \mathrm{Hilb}(h_\infty^{\otimes k})) + 2\mu_k + O(k^{-1}) = 2\mathcal{L}_k(\omega_\infty) + 2\mu_k + O(k^{-1}) \\ &= \nu_\omega(\omega_\infty) + O(k^{-1})\end{aligned}$$

The Theorem follows by letting $k \to +\infty$. ■

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Acknowledgements

The author would like to thank: Professor Gang Tian for his constant encouragement and help; Yanir Rubinstein for pointing out some inaccuracies in the first version of the paper, and for his friendship and encouragement. The author would also like to thank all the participants, particularly Yalong Shi, in the geometry seminar in Peking University. Their patience and encouragement all help the author to understand the work of Donaldson. The author also thanks the referees for careful reading of the paper.

References

[1] Bando, S. "The K-energy map, almost Einstein Kähler metrics and an inequality of the Miyaoka–Yau type." *Tohuku Mathematical Journal* 39 (1987): 231–5.

[2] Bando, S., and T. Mabuchi. "Uniqueness of Einstein Kähler Metrics Modulo Connected Group Actions." In *Algebraic Geometry, Sendai 1985*, 11–40. Advanced Studies in Pure Mathematics 10. Tokyo: Kinokuniya; Amsterdam: North-Holland, 1987.

[3] Catlin, D. "The Bergman Kernel and a Theorem of Tian." In *Analysis and Geometry in Several Complex Variables, Katata, 1997*, 1–23. Boston: Birkhauser, 1999.

[4] Chen, X., and S. Sun. "Space of Kähler metrics (V)-Kähler quantization." preprint arxiv:0902.4149.

[5] Chen, X., and G. Tian. "Geometry of Kähler metrics and holomorphic foliation by discs." *Publications Mathématiques de l'IHÉS* 107 (2008): 1–107.

[6] Donaldson, S. K. "Scalar curvature and projective embeddings I." *Journal of Differential Geometry* 59 (2001): 479–522.

[7] Donaldson, S. K. "Scalar curvature and projective embeddings, II." *Quarterly Journal of Mathematics* 56 (2005): 345–56.

[8] Kempf, G., and L. Ness. "The Length of Vectors in Representation Spaces." In *Algebraic Geometry*, 233–43. Lecture Notes in Mathematics 732. Springer, 1979.

[9] Lu, Z. "On the lower order terms of the asymptotic expansion of Tian–Yau–Zelditch." *American Journal of Mathematics* 122, no. 2 (2000): 235–73.

[10] Mabuchi, T. "K-energy maps integrating Futaki invariants." *Tohuku Mathematical Journal* 38 (1986): 245–57.

[11] Mabuchi, T. "An obstruction to asymptotic semistability and approximate critical metrics." *Osaka Journal of Mathematics* 41 (2004): 463–72.

[12] Mabuchi, T. "Stability of extremal Kähler manifolds." *Osaka Journal of Mathematics* 41 (2004): 563–82.

[13] Mabuchi, T. "An energy-theoretic approach to the Hitchin–Kobayashi correspondence for manifolds I." *Inventiones Mathematicae* 159 (2005): 225–43.

[14] Mabuchi, T. "An energy-theoretic approach to the Hitchin–Kobayashi correspondence for manifolds II." *Osaka Journal of Mathematics* 46 (2009): 115–39.

[15] Mabuchi, T. "K-stability of constant scalar curvature polarization." preprint arxiv:0812.4093.

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[16] Paul, S. T. “Geometric analysis of Chow Mumford stability.” *Advances in Mathematics* 182 (2004): 333–56.

[17] Phong, D. H., and J. Sturm. “The Monge–Ampère operator and geodesics in the space of Kähler potentials.” *Inventiones Mathematicae* 166 (2006): 125–49.

[18] Phong, D. H., and J. Sturm. “Stability, energy functionals, and Kähler–Einstein metrics.” preprint arxiv:0203254.

[19] Ruan, W. “Canonical coordinates and Bergman metrics.” *Communications in Analysis and Geometry* 6, no. 3 (1998): 589–631.

[20] Tian, G. “On a set of polarized Kähler metrics on algebraic manifolds.” *Journal of Differential Geometry* 32 (1990): 99–130.

[21] Tian, G. “The K-energy on hypersurfaces and stability.” *Communications in Analysis and Geometry* 2, no. 2 (1994): 239–65.

[22] Zelditch, S. “Szegö kernel and a theorem of Tian.” *International Mathematics Research Notices* 6 (1998): 317–31.

[23] Zhang, S. “Heights and reductions of semi-stable varieties.” *Compositio Mathematica* 104 (1996): 77–105.

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