Thus a subsequence converges (as explained above for the continuity method) to an element ψ_∞ of $\mathcal{D}^{0,\gamma}_w \cap \mathcal{C}^\infty(M\backslash D)$. Since each step in the iteration follows a continuity path of the form (30) with ω replaced by $\omega_{k\tau}$, Lemma 6.8 implies that $E^\beta_0(\omega_{(k-1)\tau}, \omega_{k\tau}) < 0$ (unless ω was already Kähler–Einstein). Since E^β_0 is an exact energy functional, i.e., satisfies a cocyle condition [42], then $E^\beta_0(\omega, \omega_{k\tau}) = \sum_{j=1}^k E^\beta_0(\omega_{(j-1)\tau}, \omega_{j\tau}) < 0$. Therefore, ψ_∞ is a fixed point of E^β_0, hence a Kähler–Einstein edge metric. By Lemma 5.2 such Kähler–Einstein metrics are unique; we conclude that the original iteration converges to ψ_∞ both in $\mathcal{A}_0$ and in $\mathcal{D}^{0,\gamma'}_w$ for each $\gamma' \in (0, \gamma)$.

Next, consider the case $\mu > 0$, and take $\mu = 1$ for simplicity. By the properness assumption, Corollary 6.9 implies the iteration exists (uniquely by Lemma 6.6) for each $\tau \in (0, \infty)$ and then the monotonicity of E^β_0 implies that $J(\omega, \omega_{k\tau}) \leq C$. To obtain a uniform estimate on $\operatorname{osc}\psi_{k\tau}$ we will employ the argument of [10] as explained to us by Berman. By Lemma 6.10, have $\int_M e^{-p(\psi_{k\tau} - \sup\psi_{k\tau})}\omega^n \leq C$, where $p/3 = \max\{1 - \frac{1}{\tau}, \frac{1}{\tau}\}$. Now rewrite (33) as

$$\omega^n_{\psi_{k\tau}} = \omega^n e^{f_\omega - (1-\frac{1}{\tau})\psi_{k\tau} - \frac{1}{\tau}\psi_{(k-1)\tau}}. \tag{96}$$

Using Kołodziej's estimate and the Hölder inequality this yields the uniform estimate $\operatorname{osc}\psi_{k\tau} \leq C$. Unlike for solutions of (30), the functions $\psi_{k\tau}$ need not be changing signs. Therefore we let $\tilde{\psi}_{k\tau} := \psi_{k\tau} - \frac{1}{V}\int_M \psi_{k\tau}\omega^n$. As in the previous paragraph we obtain a uniform estimate $\operatorname{tr}_{\omega_{k\tau}}\omega \leq C$. However, to conclude that $\operatorname{tr}_\omega \omega_{k\tau} \leq C$ from (96) we must show that $|(1 - \frac{1}{\tau})\psi_{k\tau} - \frac{1}{\tau}\psi_{(k-1)\tau}| \leq C$. This is shown in [52, p. 1543]. Thus, as before, we conclude that $\{\tilde{\psi}_{k\tau}\}$ subconverges to the potential of a Kähler–Einsteinedge metric. Whenever it is unique, the iteration itself necessarily converges. Berndtsson's generalized Bando–Mabuchi Theorem [7], [12] shows uniqueness of Kähler–Einstein edge metrics up to an automorphism (which must preserve D by (7) or Lemma 6.1). This concludes the proof of Theorem 2.5.

Appendix A. Upper bound on the bisectional curvature of the reference metric

By Chi Li[1] and Yanir A. Rubinstein

Proposition A.1. *Let $\beta \in (0, 1]$, and let $\omega = \omega_0 + \sqrt{-1}\partial\bar{\partial}|s|^{2\beta}_h$ be given by (26). The bisectional curvature of ω is bounded from above on $M\backslash D$.*

We denote throughout by $\hat{g}, g$ the Kähler metrics associated to ω_0, ω, respectively. As in [66], to simplify the calculation and estimates we need

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a lemma to choose an appropriate local holomorphic frame and coordinate system, whose elementary proof we include for the reader's convenience. We thank Gang Tian for pointing out to us the calculations in [66] that were helpful in writing this appendix.

LEMMA A.2 ([66, p. 599]). *There exists* $\varepsilon_0 > 0$ *such that if* $0 < \mathrm{dist}_{\hat{g}}(p, D) \le \varepsilon_0$, *then we can choose a local holomorphic frame* e *of* L_D *and local holomorphic coordinates* $\{z_i\}_{i=1}^n$ *valid in a neighborhood of* p, *such that*

(i) $s = z_1 e$, *and* $a := |e|_h^2$ *satisfies* $a(p) = 1$, $da(p) = 0$, $\frac{\partial^2 a}{\partial z_i \partial z_j} a(p) = 0$; *and*

(ii) $\hat{g}_{i\bar{j},k}(p) = \frac{\partial}{\partial z^k}\omega_0(\frac{\partial}{\partial z^i}, \overline{\frac{\partial}{\partial z^j}})|_p = 0$, *whenever* $j \ne 1$.

Proof. (i) Fix any point $q \in D$, and choose a local holomorphic frame e' and holomorphic coordinates $\{w_i\}_{i=1}^n$ in $B_{\hat{g}}(q, \varepsilon(q))$ for $0 < \varepsilon(q) \ll 1$. Let $s = f'e'$ with f' a holomorphic function and $|e'|_h^2 = c$. Let $e = Fe'$ for some nonvanishing holomorphic function F to be specified later. Then $a = |Fe'|_h^2 = |F|^2 c$. Now fix any point $p \in B_{\hat{g}}(q, \varepsilon(q)) \setminus \{q\}$. In order for a to satisfy the vanishing properties with respect to the variables $\{w_i\}_{i=1}^n$ at a point p, we can just choose F such that $F(p) = c(p)^{-1/2}$, and

$$\begin{aligned}\partial_{w^i} F(p) &= -c^{-1} F \partial_{w^i} c(p) = -c^{-3/2}\partial_{w^i} c(p)\\ \partial_{w^i}\partial_{w^j} F(p) &= -c^{-1}(F\partial_{w^i}\partial_{w^j} c + \partial_{w^j} c \partial_{w^i} F + \partial_{w^i} c \partial_{w^j} F)(p)\\ &= -c^{-3/2}\partial_{w^i}\partial_{w^j} c(p) + 2c^{-5/2}\partial_{w^i} c \partial_{w^j} c(p).\end{aligned}$$

Since $c = |e'|_h^2$ is never zero, when $\varepsilon(q)$ is small, which implies $|w - w(p)|$ is small, we can assume $F \ne 0$ in $B_{\hat{g}}(q, \varepsilon(q))$. Now $s = fe = f'e'$ with $f = f'F^{-1}$ a holomorphic function. Since $D = \{s = 0\}$ is a smooth divisor, we can assume $\partial_{w^1} f(q) \ne 0$, and choosing $\varepsilon(q)$ sufficiently small, we can assume that $\partial_{w^1} f \ne 0$ in $B_{\hat{g}}(q, \varepsilon(q))$. Thus by the inverse function theorem, $z_1 = f(w_1, \ldots, w_n), z_2 = w_2, \ldots, z_n = w_n$ are holomorphic coordinates in $B_{\hat{g}}(q, \varepsilon(q)/2)$ and now $s = f(w)e = z_1 e$. By the chain rule, it then follows that a satisfies $a(p) = 1$, $\partial_{z^i} a(p) = \partial_{z^i}\partial_{z^j} a(p) = 0$.

Now cover D by $\cup_{q\in D} B_{\hat{g}}(q, \varepsilon(q)/2)$. By compactness of D the conclusion follows.

(ii) Denote by $\{w^i\}_{i=1}^n$ the coordinates obtained in (i). Following [29, p. 108], let $\tilde{z}^k := w^k - w^k(p) + \frac{1}{2} b^k_{st}(w^s - w^s(p))(w^t - w^t(p))$, with $b^k_{st} = b^k_{ts}$, define a new coordinate system. Then, $\omega_0(\frac{\partial}{\partial w^i}, \overline{\frac{\partial}{\partial w^j}}) = \omega_0(\frac{\partial}{\partial \tilde{z}^i}, \overline{\frac{\partial}{\partial \tilde{z}^j}}) + \hat{g}_{t\bar{j}} b^t_{ip} w^p + \hat{g}_{i\bar{t}} \overline{b^t_{jp} w^p} + O(\sum_{i=1}^n |w^i - w^i(p)|^2)$, and

$$d_{i\bar{j}k} := \frac{\partial}{\partial w^k}\omega_0(\frac{\partial}{\partial w^i}, \overline{\frac{\partial}{\partial w^j}})|_p = \frac{\partial}{\partial \tilde{z}^k}\omega_0(\frac{\partial}{\partial \tilde{z}^i}, \overline{\frac{\partial}{\partial \tilde{z}^j}})|_p + \hat{g}_{t\bar{j}}(p) b^t_{ik} =: e_{i\bar{j}k} + \hat{g}_{t\bar{j}}(p) b^t_{ik}.$$

Let $\hat{g}'_{r\bar{s}} := \hat{g}_{r\bar{s}}$ for each $r, s > 1$, and denote the inverse of the $(n-1)\times(n-1)$ matrix $[\hat{g}'_{r\bar{s}}]$ by $[\hat{g}'^{r\bar{s}}]$. Let $b^1_{ik} = 0$. Then, for each $j > 1$, the equations can be rewritten as $d_{i\bar{j}k} - \sum_{t>1} \hat{g}'_{t\bar{j}}(p) b^t_{ik} = e_{i\bar{j}k}$. Hence, $\sum_{j>1} \hat{g}'^{s\bar{j}} e_{i\bar{j}k} = \sum_{j>1} \hat{g}'^{s\bar{j}} d_{i\bar{j}k} -$

b^s_{ik}, $s > 1$. For each $s > 1$, define b^s_{ik} so that the right-hand side vanishes. Multiplying the equations by $[\hat{g}'_{s\bar{t}}]$, we obtain $e_{i\bar{t}k} = 0$ for each $t > 1$. Finally, set $z^i := \tilde{z}^i + w^i(p)$, $i = 1, \ldots, n$. Since $b^1_{ik} = 0$, we have $z^1 = w^1$, and therefore these coordinates satisfy both properties (i) and (ii) of the statement, as desired. □

Let $H := a^\beta$, then $|s|^{2\beta}_h = |z_1 e|^{2\beta}_h = H|z_1|^{2\beta}$. Note that both a and H are locally defined smooth positive functions. Let $\omega = \frac{\sqrt{-1}}{2} g_{i\bar{j}} dz^i \wedge \overline{dz^j}$, $\omega_0 = \frac{\sqrt{-1}}{2} \hat{g}_{i\bar{j}} dz^i \wedge \overline{dz^j}$, and write $z \equiv z_1$ and $\rho := |z|$. Using the symmetry for subindices, we can calculate in a straightforward manner:

$$\begin{aligned} g_{i\bar{j}} &= \hat{g}_{i\bar{j}} + H_{i\bar{j}}|z|^{2\beta} + \beta H_i \delta_{1\bar{j}} |z|^{2\beta-2} z + \beta H_{\bar{j}} \delta_{1i} |z|^{2\beta-2} \bar{z} + \beta^2 H |z|^{2\beta-2} \delta_{1i} \delta_{1\bar{j}}, \\ g_{i\bar{j},k} &= \hat{g}_{i\bar{j},k} + H_{i\bar{j}k}|z|^{2\beta} + \beta H_{ik} \delta_{1\bar{j}} |z|^{2\beta-2} z + \beta (H_{k\bar{j}} \delta_{1i} + H_{i\bar{j}} \delta_{1k}) |z|^{2\beta-2} \bar{z} \\ &\quad + \beta^2 (H_i \delta_{1\bar{j}} \delta_{1k} + H_k \delta_{1i} \delta_{1\bar{j}} + H_{\bar{j}} \delta_{1i} \delta_{1k}) |z|^{2\beta-2} \\ &\quad + \beta^2 (\beta - 1) H |z|^{2\beta-4} \bar{z} \delta_{1i} \delta_{1\bar{j}} \delta_{1k}, \end{aligned}$$

$$\begin{aligned} g_{i\bar{j},k\bar{\ell}} &= \hat{g}_{i\bar{j},k\bar{\ell}} + H_{i\bar{j}k\bar{\ell}} |z|^{2\beta} \\ &\quad + \beta \left[(H_{ik\bar{\ell}} \delta_{1\bar{j}} + H_{ik\bar{j}} \delta_{1\bar{\ell}}) |z|^{2\beta-2} z + (H_{\bar{j}\bar{\ell}i} \delta_{1k} + H_{\bar{j}\bar{\ell}k} \delta_{1i}) |z|^{2\beta-2} \bar{z} \right] \\ &\quad + \beta^2 (H_{k\bar{j}} \delta_{1i} \delta_{1\bar{\ell}} + H_{i\bar{j}} \delta_{1k} \delta_{1\bar{\ell}} + H_{k\bar{\ell}} \delta_{1i} \delta_{1\bar{j}} + H_{i\bar{\ell}} \delta_{1\bar{j}} \delta_{1k}) |z|^{2\beta-2} \\ &\quad + \beta (\beta - 1) \left[H_{ik} \delta_{1\bar{j}} \delta_{1\bar{\ell}} |z|^{2\beta-4} z^2 + H_{\bar{j}\bar{\ell}} \delta_{1i} \delta_{1k} |z|^{2\beta-4} \bar{z}^2 \right] \\ &\quad + \beta^2 (\beta - 1) \left[(H_i \delta_{1k} + H_k \delta_{1i}) \delta_{1\bar{j}} \delta_{1\bar{\ell}} |z|^{2\beta-4} z \right. \\ &\qquad\qquad \left. + (H_{\bar{j}} \delta_{1\bar{\ell}} + H_{\bar{\ell}} \delta_{1\bar{j}}) \delta_{1i} \delta_{1k} |z|^{2\beta-4} \bar{z} \right] \\ &\quad + \beta^2 (\beta - 1)^2 H |z|^{2\beta-4} \delta_{1i} \delta_{1\bar{j}} \delta_{1k} \delta_{1\bar{\ell}}. \end{aligned}$$

Let $p \in M \setminus D$ satisfy $\mathrm{dist}_{\hat{g}}(p, D) \leq \varepsilon_0$. The lemma implies, in particular, $H(p) = 1$, $H_i(p) = H_{ij}(p) = 0$, and the expressions above simplify to

$$\begin{aligned} g_{i\bar{j}}(p) &= \hat{g}_{i\bar{j}} + H_{i\bar{j}} |z|^{2\beta} + \beta^2 |z|^{2\beta-2} \delta_{i1} \delta_{1\bar{j}}, \\ g_{i\bar{j},k}(p) &= \hat{g}_{i\bar{j},k} + H_{i\bar{j}k} |z|^{2\beta} + \beta (\delta_{i1} H_{k\bar{j}} + \delta_{k1} H_{i\bar{j}}) |z|^{2\beta-2} \bar{z} \\ &\quad + \beta^2 (\beta - 1) \delta_{i1} \delta_{\bar{j}1} \delta_{k1} |z|^{2\beta-4} \bar{z}, \\ g_{i\bar{j},k\bar{\ell}}(p) &= \hat{g}_{i\bar{j},k\bar{\ell}} + H_{i\bar{j}k\bar{\ell}} |z|^{2\beta} + \beta (\delta_{i1} H_{\bar{j}k\bar{\ell}} + \delta_{k1} H_{i\bar{j}\bar{\ell}}) |z|^{2\beta-2} \bar{z} \\ &\quad + \beta (\delta_{\bar{j}1} H_{ik\bar{\ell}} + \delta_{\bar{\ell}1} H_{i\bar{j}k}) |z|^{2\beta-2} z + \beta^2 (\delta_{i1} \delta_{\bar{j}1} H_{k\bar{\ell}} + \delta_{i1} \delta_{\bar{\ell}1} H_{k\bar{j}} \\ &\quad + \delta_{k1} \delta_{\bar{j}1} H_{i\bar{\ell}} + \delta_{k1} \delta_{\bar{\ell}1} H_{i\bar{j}}) |z|^{2\beta-2} + \beta^2 (\beta - 1)^2 \delta_{i1} \delta_{\bar{j}1} \delta_{k1} \delta_{\bar{\ell}1} |z|^{2\beta-4}. \end{aligned}$$

It follows that

$$g^{r\bar{s}}(p) = O(1), \quad g^{1\bar{s}}(p) = O(\rho^{2-2\beta}) \quad \text{for } r, s > 1 \tag{97}$$

and

$$g^{1\bar{1}}(p) = \beta^{-2} \rho^{2-2\beta} \frac{1}{1 + b(p) \rho^{2-2\beta}} + O(\rho^2), \tag{98}$$

where $O(\rho^2) < C_3\rho^2$ and $b(p) := \beta^{-2}\det[\hat{g}_{i\bar{j}}]/\det[\hat{g}_{r\bar{s}}]_{r,\bar{s}>1}|_p$ with $0 < C_1 < b(p) < C_2$, and C_1, C_2, C_3 independent of $p \in M\backslash D$.

Take two unit vectors $\eta = \eta^i\frac{\partial}{\partial z^i}, \nu = \nu^i\frac{\partial}{\partial z^i} \in T_p^{1,0}M$, so that $g(\eta,\eta)|_p = g(\nu,\nu)|_p = 1$. Then from the expression of $g_{i\bar{j}}$, we have

$$\eta^1, \nu^1 = O(\rho^{1-\beta}) \quad \eta^r, \nu^r = O(1) \text{ for } r > 1. \tag{99}$$

Set

$$\mathrm{Bisec}_\omega(\eta,\nu) = R(\eta,\bar{\eta},\nu,\bar{\nu}) = R_{i\bar{j}k\bar{\ell}}\eta^i\overline{\eta^j}\nu^k\overline{\nu^\ell} = \sum_{i,j,k,l}\Lambda_{i\bar{j}k\bar{\ell}} + \Pi_{i\bar{j}k\bar{\ell}},$$

with $\Lambda_{i\bar{j}k\bar{\ell}} := -g_{i\bar{j},k\bar{\ell}}\eta^i\overline{\eta^j}\nu^k\overline{\nu^\ell}$ and $\Pi_{i\bar{j}k\bar{\ell}} := g^{s\bar{t}}\, g_{i\bar{t},k}\, g_{s\bar{j},\bar{\ell}}\eta^i\overline{\eta^j}\nu^k\overline{\nu^\ell}$ (no summations). By (97)–(99), we have $|\Lambda_{i\bar{j}k\bar{\ell}}| \le C$ except for

$$\Lambda_{1\bar{1}1\bar{1}} = -\beta^2(\beta-1)^2|z|^{2\beta-4}|\eta^1|^2|\nu^1|^2,$$

hence

$$\sum_{i,j,k,l}\Lambda_{i\bar{j}k\bar{\ell}}(p) = O(1) + \Lambda_{1\bar{1}1\bar{1}}(p) = O(1) - \beta^2(\beta-1)^2|z|^{2\beta-4}|\eta^1|^2|\nu^1|^2. \tag{100}$$

The proposition follows immediately by combining (100) and the following estimate.

Lemma A.3. *There exists a uniform constant $C > 0$ such that for every $p \in M\backslash D$,*

$$\sum_{i,j,k,l}\Pi_{i\bar{j}k\bar{\ell}}(p) \le C + \beta^2(\beta-1)^2|z|^{2\beta-4}|\eta^1|^2|\nu^1|^2.$$

Proof. Define a bilinear Hermitian form of two tensors $a = [a_{i\bar{j}k}]$, $b = [b_{p\bar{q}r}] \in (\mathbb{C}^n)^3$ satisfying $a_{i\bar{j}k} = a_{k\bar{j}i}$ and $b_{p\bar{q}r} = b_{r\bar{q}p}$ by setting

$$\langle [a_{i\bar{j}k}], [b_{p\bar{q}r}]\rangle := \sum_{i,j,k,p,q,r} g^{q\bar{j}}(\eta^i a_{i\bar{j}k}\nu^k)(\overline{\eta^p b_{p\bar{q}r}\nu^r}).$$

It is easy to see that this is a nonnegative bilinear form. We denote by $\|\cdot\|$ the associated norm. Then $\sum_{i,j,k,l}\Pi_{i\bar{j}k\bar{\ell}} = \|[g_{i\bar{j},k}]\|^2$. Write

$$g_{i\bar{j},k} = A_{i\bar{j}k} + B_{i\bar{j}k} + D_{i\bar{j}k} + E_{i\bar{j}k},$$

with $A_{i\bar{j}k} := \hat{g}_{i\bar{j},k}$, $B_{i\bar{j}k} := H_{i\bar{j}k}|z|^{2\beta}$, $D_{i\bar{j}k} := \beta(\delta_{i1}H_{k\bar{j}} + \delta_{k1}H_{i\bar{j}})|z|^{2\beta-2}\bar{z}$ and $E_{i\bar{j}k} := \beta^2(\beta-1)\delta_{i1}\delta_{\bar{j}1}\delta_{k1}|z|^{2\beta-4}\bar{z}$. Denote $A = [A_{i\bar{j}k}]$ and similarly B, D, E. Using (97),

$$\langle D, E\rangle \le C\left|\sum_j g^{1\bar{j}}|\eta^1|^2\overline{\nu^1}\rho^{2\beta-1}\rho^{2\beta-3}\right| \le C\rho^{1-\beta},$$

and similarly we conclude that $\|[g_{i\bar{j},k}]\|^2 \le C + \|A+E\|^2$. Now, since $\|\frac{1}{\sqrt{\varepsilon}}A - \sqrt{\varepsilon}E\|^2 \ge 0$, we obtain $\|A+E\|^2 \le (1+\frac{1}{\varepsilon})\|A\|^2 + (1+\varepsilon)\|E\|^2$. Note now that by (98),

$$\|E\|^2 = g^{1\bar{1}}|E_{1\bar{1}1}|^2|\eta^1|^2|\nu^1|^2 \leq C + \frac{\beta^2(1-\beta)^2}{1+b(p)\rho^{2-2\beta}}\rho^{2\beta-4}|\eta^1|^2|\nu^1|^2.$$

Thus, letting $\varepsilon = \varepsilon(p) = b(p)\rho^{2-2\beta}$, we will have proved the lemma provided we can bound $(1+\rho^{2\beta-2})\|A\|^2$. Now, by (98) and Lemma A.2(ii),

$$\rho^{2\beta-2}\|A\|^2 = \sum_{i,k,p,r} \rho^{2\beta-2}\hat{g}_{i\bar{1},k}\hat{g}_{1\bar{j},\bar{\ell}}g^{1\bar{1}}\eta^i\overline{\eta^j}\nu^k\overline{\nu^\ell} \leq C.$$

This concludes the proof of Lemma A.3. □

Appendix B. A local third derivative estimate (after Tian)

A general result due to Tian [57], proved in his M.Sc. thesis, gives a local *a priori* estimate in $W^{3,2}$ for solutions of both real and complex Monge–Ampère equations under the assumption that the solution has bounded real or complex Hessian and the right-hand side is at least Hölder. By the classical integral characterization of Hölder spaces this implies a uniform Hölder estimate on the Laplacian. This result can be seen as an alternative to the Evans–Krylov theorem (and, in fact, appeared independently around the same time).

We present a very special case of this here that applies, in particular, to $\varphi(s)$ along the Ricci continuity path (30). Unlike Calabi's estimates, this local estimate does not require curvature bounds on the reference geometry (which works only when $\beta < 1/2$ [15]). The argument here is an immediate adaptation of [57] to the complex edge setting and is based entirely on the presentation in [57] and Tian's unpublished notes [64]. He understood the applicability of this method in the edge setting for some time and had described this in various courses and lectures over the years.

Theorem B.1 (Tian [57], [64]). *Let $\varphi(s) \in \mathcal{D}^{0,0}_w \cap \mathcal{C}^4(M\setminus D)\cap \mathrm{PSH}(M,\omega)$ be a solution to (30), with $s > S$ and $0<\beta<1$. For any $\gamma \in (0,\beta^{-1}-1)\cap(0,1)$, there are constants $r_0\in(0,1)$ and $C>0$ such that for any $x\in M$ and $0<a<r_0$,*

$$\int_{B_a(x)} |\nabla\omega_\varphi|^2\,\omega^n \;\leq\; C\,a^{2n-2+2\gamma}, \tag{101}$$

where $B_a(x)$ denotes the geodesic ball with center x and radius a, ∇ the covariant derivative and $|\cdot|$ the norm, all taken with respect to ω_β (3). The constant C depend only on $\gamma,\beta,\omega,n,||\Delta_\omega\varphi||_{L^\infty(M)}$ and $||\varphi||_{L^\infty(M)}$.

For the proof, we may assume that $x\in D$ and fix some neighborhood U of x in M. We will also always assume $1/2<\beta<1$ purely for simplicity of notation. Setting $t=1$ in (87),

$$\log\det[u_{i\bar{j}}] = f_\omega - s\varphi + \log\det[\psi_{i\bar{j}}] =: \log h, \tag{102}$$