Zoll Manifolds of Type $\mathbb{CP}^n$ with Entire Grauert Tubes

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Abstract

We show that a Zoll manifold of type $\mathbb{CP}^n$ with an entire Grauert tube is isometric to $\mathbb{CP}^n$ with the canonical Fubini-Study metric, up to constant multiplication.

Contents

1 Introduction and main results

A Riemannian manifold (M, g) is called a *Zoll manifold* (or sometimes an SC manifold) if there exists a positive real number l satisfying:

- Every geodesic $\gamma : \mathbb{R} \to M$ is periodic of period l, i.e. $\gamma(t + l) = \gamma(t)$ for all $t \in \mathbb{R}$,
- Each geodesic is injective on the interval $(0, l)$.

A fundamental example of a Zoll manifold is S^2 with its round metric. Other known examples are given by the so-called *CROSS* (Compact, Rank-One Symmetric Spaces), consisting of the following five cases:

1. S^n with the round metric,
2. $\mathbb{RP}^n$ with the induced metric from the round S^n,
3. $\mathbb{CP}^n$ with the Fubini–Study metric,
4. $\mathbb{HP}^n$ with its canonical metric,
5. $\mathbb{OP}^2$ with its canonical metric.

By the Bott–Samelson Theorem, any Zoll manifold must either be simply connected or satisfy $\pi_1(M) = \mathbb{Z}_2$. In the former case, the cohomology of M coincides with that of one among S^n, $\mathbb{CP}^n$, $\mathbb{HP}^n$, $\mathbb{OP}^2$, whereas in the latter, it coincides with that of $\mathbb{RP}^n$. Consequently, an n-dimensional Zoll manifold can exhibit at most one of these five cohomological types. We refer to them as the *types* of Zoll manifolds, calling each of the five spaces above the *canonical model* of its respective type.

A natural question then arises: for those Zoll manifolds of the five types described, can there exist a Zoll metric different from that of the canonical model? This was first answered in 1903 by Zoll, who constructed a two-parameter family of Zoll metrics on S^2. A. Weinstein later extended this line of reasoning to show that such *exotic* Zoll metrics can exist on S^n for $n \geq 2$. L. Green proved that there are no exotic Zoll metric on $\mathbb{RP}^2$. R. Michel proved that, at least infinitesimally, no such exotic Zoll metric can arise on $\mathbb{RP}^n$. Very little is currently known for the other three types. We refer to the classical book [3] for extensive study on related topics.

In that context, one naturally wonders under what additional conditions a Zoll metric might coincide with its canonical model. One of the most intriguing geometric features of a Zoll manifold lies in its tangent bundle. Since all geodesics of a Zoll manifold M share the same period, the geodesic flow on TM is itself periodic. Also recall that on any Riemannian manifold (M, g), one may define the Liouville 1-form

$$\alpha_v(-) \;=\; g\big(\pi_*(-), v\big),$$

and hence the symplectic form $\omega = d\alpha$ on TM. Since ω remains invariant under the geodesic flow, the tangent bundle of a Zoll manifold is a symplectic manifold equipped with an equivariant S^1-action. It is thus natural to ask if imposing an additional symmetry condition, compatible with this symplectic structure—like, Kähler—, might force a Zoll metric to be the canonical one.

In fact, each of the above five CROSS models has a tangent bundle with a Kähler structure which is compatible with the above symplectic structure. According to Patrizio and Wong [8], each of the tangent bundles is a Stein manifold, where the function $u(v) = \|v\|$ measuring the norm of the tangent vector, is plurisubharmonic on $TM \setminus M$ and satisfies the Homogeneous Complex Monge–Ampère equation (HCMA)

$$\big(i\,\partial\,\overline{\partial}\,u\big)^n \;=\; 0.$$

If a Riemannian manifold's entire tangent bundle admits such a complex structure—so that u is a solution of the above HCMA on all of $TM \setminus M$—we say the manifold has an *entire Grauert tube*. In this paper, our main result is as follows.

Theorem 1.1. *Let (M, g) be a Zoll manifold of type $\mathbb{CP}^n$. Assume that (M, g) has the entire Grauert tube condition. Then, M is isometric to $\mathbb{CP}^n$ with its Fubini-Study Metric, up to constant multiplication.*

The uniqueness theorem for Zoll manifolds of type S^n with an entire Grauert tube was proved by Burns and Leung in [4] and they asked similar questions for Zoll manifolds of other types. The above result answers their question for the case of type $\mathbb{CP}^n$. Our proof follows a similar line of argument from [4] and there are some new ingredients. We first use the result of [4] to get an analytic compactification X of the tangent bundle of M as a complex manifold, and consider the added point set D as an ample divisor. After that, we will calculate the cohomology of X and D via algebraic topological techniques, and consider the generators of their second cohomology classes. We can show that the complex manifold X is $2n$-dimensional Fano manifold with its index equal to $n+1$. In the case considered in [4] this step is relatively easy because the second Betti number of compactified space is equal to 1, while in our case $b_2(X) = 2$ and more arguments are needed. In particular, we will calculate the Fano index of X with the help of the Bott-Samelson Theorem and by relating the Morse index of closed geodesics to the Fano index of X. Finally, by a classification theorem of Wiśniewski [13], we can show that X is biholomorphic to the standard model $\mathbb{CP}^n \times \mathbb{CP}^n$ ([8]). Some algebro-geometric argument allows us to establish that the biholomorphism preserves key structures in X, which shows that the metric on M should be isometric to the Fubini-Study Metric, up to constant multiplication.

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2 Complexification and Compactification of Tangent Bundle

2.1 Homogeneous Complex Monge-Ampère Equation on the Tangent Bundle

To investigate the condition under which there is a natural complexification of the entire tangent bundle TM of a manifold M (i.e., an entire Grauert tube), we first need to recall its definition. The basic references are [5, 6].

Let (M, g) be an n-dimensional compact Riemannian manifold. Define the energy function $E : TM \to \mathbb{R}$ by

$$E(v) \;=\; \frac{1}{2}\|v\|^2.$$

For $T^rM = \{\, v \in TM : \|v\| < r \,\}$, suppose there exists a complex structure J and let $u = \sqrt{2E}$ be a function on T^rM which serves as a plurisubharmonic exhaustion satisfying the homogeneous complex Monge-Ampère equation (HCMA)

$$(i\partial\bar{\partial}u)^n = 0$$

on $T^rM\backslash M$. We call (T^rM, M, u) a *Monge-Ampère Model.* If there exists such a complex structure on T^rM, we call T^rM a *Grauert tube.* Moreover, if $r = \infty$ is possible, then M is said to admit an *Entire Grauert tube.* Since $Rank(\partial\bar{\partial}u) = n-1$, the kernel of $\partial\bar{\partial}u$ is a rank 1 subbundle of the tangent bundle of $T^rM\backslash M$, implying that $T^rM\backslash M$ admits a foliation by Riemann surfaces [2], which is called the *Monge-Ampère Foliation.*

This foliation admits a more concrete description, often referred to as the *Riemann foliation.* For a real number τ, define the dilation operator

$$N_\tau : TM \to TM, \quad N_\tau(v) \;=\; \tau\, v.$$

For a geodesic γ, define the immersion $\phi_\gamma : \mathbb{C} \to TM$ by

$$\phi_\gamma(\sigma + i\tau) \;=\; N_\tau\big(\gamma'(\sigma)\big).$$

If two geodesics γ_1 and γ_2 have different images, then under this construction, their images of $\mathbb{C}\backslash\mathbb{R}$ do not intersect. Consequently, these immersions define a well-defined foliation on $T^rM\backslash M$. Moreover, each leaf of this foliation inherits a well-defined complex structure since it is the image of $\mathbb{C}$. According to Theorem 3.1 in [6], the two foliations described above coincide, and each leaf is a complex submanifold under the complex structure J. A complex structure J that endows each leaf with a complex submanifold structure in this manner is called an *adapted complex structure*.

Furthermore, by [5, 6] and [12], if (T^rM, M, u) admits such a complex structure, it is uniquely determined by the metric g. In particular, Theorem 2.2 of [12] states that any compact real-analytic Riemannian manifold admits an $\epsilon > 0$ such that $T^\epsilon M$ carries a unique adapted complex structure. Moreover, the Riemanniann metric g is recovered as the restriction of the Kähler metric $\sqrt{-1}\partial\bar{\partial}u^2$ on M.

An adapted complex structure can be concretely constructed from the metric g via *parallel vector fields* on each leaf. Before defining it, recall that for $v \in TM$, a vector $\zeta \in T_vTM$ can be written in terms of its horizontal part $h \in TM$ and vertical part $w \in TM$. That is, if $V : \mathbb{R} \to TM$ is a vector field along a path in M such that $V(0) = v$, $(\pi V)'(0) = h$, and $\frac{DV}{dt}(0) = w$, we can express $\zeta = V'(0) \in T_vTM$ as $\zeta = (h, w)$.

For a unit speed geodesic $\gamma : \mathbb{R} \to M$, consider the leaf $\phi_\gamma : \mathbb{C} \to TM$ of the Riemann foliation. A vector field $V : \mathbb{C} \to TTM$ on this leaf is called a *parallel vector field* if it is invariant under the geodesic flow Φ_t and the dilation N_τ on TM. That is, for $\Gamma : (-\epsilon, \epsilon) \to TM$ be a defining curve of a vector $V(i) = \zeta = (h, w) \in T_vTM$, i.e. $\Gamma'(0) = \zeta$, $(\pi \circ \Gamma)'(0) = h$, $\frac{D}{dt}\Gamma(0) = w$, $V(\sigma + i\tau)$ should be defined as $(\Phi_\sigma \circ N_\tau \circ \Gamma)'(0)$. Therefore, a parallel vector field can be determined by its value at $i \in \mathbb{C}$. let $\iota : \mathbb{R} \to TM$ be the Jacobi field along $\gamma(\sigma)$ with $\iota(0) = h$ and $\iota'(0) = w$. By definition, in horizontal/vertical notation we then have

$$V(\sigma + i\tau) = \left(\iota(\sigma), \tau\, \tfrac{D\iota}{dt}(\sigma)\right)$$

Now suppose $J : TT^rM \to TT^rM$ is a well-defined adapted complex structure on T^rM, meaning that each leaf of the Riemann foliation is a complex submanifold. By [12] and [6], such a complex structure J is uniquely determined by the metric g. Concretely, let $p \in M$ and a unit vector $v \in T_pM$. Consider the geodesic γ with $\gamma(0) = p$ and $\gamma'(0) = v$. Take an orthonormal basis $v_1, \ldots, v_{n-1}, v_n = v$ of T_pM. Define parallel vector fields ζ_i, η_i by

$$\zeta_i(i) = (v_i, 0), \quad \eta_i(i) = (0, v_i)$$

Then ζ_i and η_i give rise to holomorphic sections $\zeta_i^{1,0} = \zeta_i - \sqrt{-1}J\zeta_i$ and $\eta_i^{1,0} = \eta_i - \sqrt{-1}J\eta_i$ on $\mathbb{C}\backslash\mathbb{R}$. The vector fields $\{\xi_i^{1,0}; i = 1, \ldots, n\}$ are linearly independent, thus forming a holomorphic basis. For holomorphic vector ζ, η such that

$$\zeta^{1,0} = [\zeta_1^{1,0}, \ldots, \zeta_{n-1}^{1,0}]^T, \quad \eta^{1,0} = [\eta_1^{1,0}, \ldots, \eta_{n-1}^{1,0}]^T$$

we have a holomorphic matrix $\Psi : \mathbb{C}\backslash\mathbb{R} \to M_{n-1}(\mathbb{C})$ such that

$$\eta^{1,0} = \Psi\zeta^{1,0}$$

On the real axis (except for some discrete set $S \subset \mathbb{R}$), ζ_i and η_i also do not vanish identically, because they are non-trivial Jacobi fields. For real vectors

$$\zeta = [\zeta_1, \ldots, \zeta_{n-1}]^T, \quad \eta = [\eta_1, \ldots, \eta_{n-1}]^T$$

we have a real matrix $\psi : \mathbb{R}\backslash S \to M_{n-1}(\mathbb{R})$ such that

$$\eta \;=\; \psi\,\zeta.$$

On the entire complex plane $\mathbb{C}$, we can write

$$\eta \;=\; \mathrm{Re}\,\Psi\,\zeta \;+\; \mathrm{Im}\,\Psi\,J\zeta,$$

which implies

$$J\,\zeta \;=\; (\mathrm{Im}\,\Psi)^{-1}\big(\eta \;-\; \mathrm{Re}\,\Psi\,\zeta\big).$$

Note that ψ is defined even without the existence of a Grauert tube. Therefore, the existence of Grauert tube T^rM is equivalent to the condition that the analytic continuation Ψ must be well-defined and $\mathrm{Im}\Psi$ invertible. Indeed, for a compact real-analytic manifold M, there exists some $\epsilon > 0$ such that ψ can be analytically continued on $T^\epsilon M$.

2.2 Compactification of the Tangent Bundle of Zoll Manifolds

We now describe the algebraization of the tangent bundle of a Zoll manifold with an Entire Grauert tube, following Burns and Leung [4] and Leung [7].

On a Zoll manifold, all geodesics γ are periodic with a fixed period l. We view the Riemann foliation as a map

$$\phi_\gamma \;:\; \mathbb{C}/\mathbb{Z} \;\longrightarrow\; TM,$$

so each leaf is topologically a long cylinder. By adding two points

$$\infty_\gamma \;=\; \lim_{\tau\to\infty} \phi_\gamma(\sigma + i\tau), \quad 0_\gamma \;=\; \lim_{\tau\to-\infty} \phi_\gamma(\sigma + i\tau),$$

one can compactify each leaf to a sphere. Denote by D the set of all such added points:

$$D \;=\; \{\,0_\gamma,\; \infty_\gamma :\; \gamma\,\text{is a geodesic on}\; M\}.$$

Therefore, we have a compactification of TM, $X = TM \cup D$.

Remark 2.1 (**Manifold structure on** D)**.** *One may regard D as the set of all oriented closed geodesics on M. Equivalently, D can be realized as the quotient of the unit tangent bundle UM by the geodesic flow, i.e. the equivalence classes of UM under this flow. Consequently, D admits the structure of a manifold via the the free quotient*

$$D \;\cong\; UM/S^1.$$

Accordingly, elements of the tangent bundle TD can be described by the following one-to-one correspondence. Let $v \in UM$, and let γ be the geodesic with $\gamma'(0) = v$. Then:

- *A Jacobi field ι on γ such that $\iota(0) \perp v$ and $\iota'(0) \perp v$,*
- *A parallel vector field ζ on the leaf ϕ_γ given by*

$$\zeta(\sigma + i\tau) \;=\; \big(\iota(\sigma),\, \tau\,\iota'(\sigma)\big),$$

- *The vector* $[\zeta(i)] \in T_{\infty_\gamma} D$*, where* $\zeta(i) \in T_v UM$*.*

From this, we deduce two important facts:

1. *A parallel vector field on the leaf* ϕ_γ *can extend to the boundary point* $\infty_\gamma \in D$*. Indeed, recall that for any parallel vector field* ζ *on* ϕ_γ*, each value* $\zeta(i\tau)$ *arises by applying the dilation* N_τ *to* $\zeta(i)$*. Thus, for the defining curve* $\Gamma : (-\epsilon, \epsilon) \to TM \backslash M$ *of* $\zeta(i)$*,* $\lim_{\tau\to\infty} N_\tau \circ \Gamma$ *should be the defining curve of the parallel vector at* ∞_γ*. Consequently, in the correspondence outlined above, if* $\zeta(i) = (\iota(0), \iota'(0))$ *extends as a parallel vector field, then at* ∞_γ *it corresponds precisely to the vector determined by* $[\zeta] \in T_{\infty_\gamma} D$*. In particular, note that if* ι *is a nonzero normal Jacobi field, then the induced parallel vector field remains nonzero on* D*.*

2. *Via this extension, the complex structure on* TM *induce a complex structure on* D*. In other words,* D *is a complex manifold.*

According to [4] and [7], we have the following:

Proposition 2.2. *For a Zoll manifold* M *with period* l *admitting an Entire Grauert tube, let* $X = TM \cup D$ *be the manifold obtained by compactifying all leaves.*

(i) X *is a complex manifold, and* D *is a (complex) codimension 1 submanifold of* X*.*

(ii) $\rho = \log\big(1 + \cosh\big(\frac{4\pi\sqrt{2E}}{l}\big)\big)$ *is a Kähler potential of* TM*, and the Kähler structure derived from this potential extends to* X*.*

(iii) $\mathcal{O}(D)$ *is a positive line bundle.*

Therefore, by the Kodaira Embedding Theorem, D is an ample divisor and X is a projective manifold.

Moreover, we must pay attention to the dilation N_τ with $\tau = -1$, namely $N_{-1} : X \to X$. According to [6], N_{-1} is an anti-holomorphic involution on X. In particular, N_{-1} fixes exactly the points of M and acts as an orientation-reversing map on X, on D, and on each leaf ϕ_γ. The existence of this map will be crucial in the proofs that follow.

2.3 Concrete Description on the Model Case

Our goal is to show that the compactification X of TM, as described in 2.2, is biholomorphic to the compactification of the tangent bundle of the canonical model which is $\mathbb{CP}^n$ equipped with the Fubini–Study metric. We aim to demonstrate that this biholomorphism preserves the various structures attached to X: M, D, N_{-1}, and the exhaustion u. If achieved, it in particular implies that M is diffeomorphic to $\mathbb{CP}^n$, and furthermore, since u is defined as the vector norm, preserving u ensures that the diffeomorphism is in fact an isometry.

To carry out this comparison, we need to analyze precisely how the aforementioned structures are realized in the canonical model of $\mathbb{CP}^n$. Patrizio and Wong [8] showed that the compactification of $T\mathbb{CP}^n$ can be realized as $\mathbb{CP}^n \times \mathbb{CP}^n$. Building on that, we will give a concrete description of how the leaves, D, u, N_{-1}, and the Kähler potential appear in $\mathbb{CP}^n \times \mathbb{CP}^n$.

First, within $\mathbb{CP}^n \times \mathbb{CP}^n$, one identifies $\mathbb{CP}^n$ via the following totally real embedding:

$$\begin{aligned} \iota : \quad \mathbb{CP}^n &\longrightarrow \mathbb{CP}^n \times \mathbb{CP}^n, \\ [z] &\longmapsto [\,z\,] \times [\overline{z}\,]. \end{aligned}$$

To understand how $T\mathbb{CP}^n$ embeds into $\mathbb{CP}^n \times \mathbb{CP}^n$, let us examine the geodesics on $\mathbb{CP}^n$ and their corresponding leaves. Observe that any geodesic γ on $\mathbb{CP}^n$ can be represented using two complex vectors $z, w \in \mathbb{C}^{n+1}$ with $\|z\| = \|w\| = 1$ and $\langle z, w\rangle = \sum_i z_i \bar{w}_i = 0$, so that

$$\gamma(t) \;=\; \big[\cos(\tfrac{t}{2})\, z \;+\; \sin(\tfrac{t}{2})\, w\big], \quad 0 \le t \le 2\pi.$$

Thus,

$$(\iota \circ \gamma)(t) \;=\; \Big[\cos(\tfrac{t}{2})\, z \;+\; \sin(\tfrac{t}{2})\, w\Big] \times \Big[\cos(\tfrac{t}{2})\, \overline{z} \;+\; \sin(\tfrac{t}{2})\, \overline{w}\Big].$$

A natural holomorphic extension of this map is given by

$$\phi_\gamma(\sigma + i\tau) \;=\; \Big[\cos\frac{\sigma + i\tau}{2}\, z \;+\; \sin\frac{\sigma + i\tau}{2}\, w\Big] \times \Big[\cos\frac{\sigma + i\tau}{2}\, \overline{z} \;+\; \sin\frac{\sigma + i\tau}{2}\, \overline{w}\Big].$$

Hence any nonzero vector $\tau \cdot \gamma'(0) \in T\mathbb{CP}^n$ can be represented in $\mathbb{CP}^n \times \mathbb{CP}^n$ via

$$\phi_\gamma(i\tau) \;=\; \big[\cosh(\tfrac{\tau}{2})\, z \;+\; i\,\sinh(\tfrac{\tau}{2})\, w\big] \times \big[\cosh(\tfrac{\tau}{2})\, \overline{z} \;+\; i\,\sinh(\tfrac{\tau}{2})\, \overline{w}\big].$$

The two points added by compactification (those in D that correspond to the endpoints of each leaf) are

$$\lim_{\tau\to\infty} \phi_\gamma(i\tau) \;=\; \infty_\gamma \;=\; [z + i\,w] \times [\overline{z} + i\,\overline{w}], \qquad \lim_{\tau\to-\infty} \phi_\gamma(i\tau) \;=\; 0_\gamma \;=\; [z - i\,w] \times [\overline{z} - i\,\overline{w}],$$

corresponding to points $[Z] \times [W] \in \mathbb{CP}^n \times \mathbb{CP}^n$ satisfying $\sum_{0\le\alpha\le n} Z_\alpha W_\alpha = 0$.

Next, let $N = (n+1)^2 - 1$ and consider the Segre embedding

$$\begin{aligned} S : \quad \mathbb{CP}^n \times \mathbb{CP}^n &\longrightarrow \mathbb{CP}^N, \\ [z] \times [w] &\longmapsto [\,z w^T\,]. \end{aligned}$$

Define a plurisubharmonic exhaustion $\mathcal{N} : \mathbb{CP}^N \to [1, \infty]$ by

$$\mathcal{N}(\zeta) \;=\; \frac{\sum_{0\le i,j\le n} |\zeta_{ij}|^2}{\big|\sum_{0\le\alpha\le n} \zeta_{\alpha\alpha}\big|^2}.$$

Restricting to $\mathbb{CP}^n \times \mathbb{CP}^n \subset \mathbb{CP}^N$, we have

$$\mathcal{N}(z w^T) \;=\; \frac{\sum_{0\le i,j\le n} |z_i w_j|^2}{|\sum_{0\le\alpha\le n} z_\alpha w_\alpha|^2} \;=\; \frac{\sum_{0\le i\le n} |z_i|^2 \, \sum_{0\le j\le n} |w_j|^2}{\big|\sum_{0\le\alpha\le n} z_\alpha w_\alpha\big|^2} \ge 1.$$

A straightforward computation shows that

$$\big(\mathcal{N} \circ \phi_\gamma\big)(i\tau) \;=\; \cosh^2(\tau) \;=\; \frac{\cosh(2\tau) + 1}{2},$$

so one obtains

$$u_0 \;=\; \sqrt{2E} = |\tau| \;=\; \tfrac{1}{2}\,\cosh^{-1}\!\big(2\,\mathcal{N} - 1\big). \tag{1}$$

WLOG, for $z = [z_0 : \cdots : z_n]$ and $w = [w_0 : \cdots : w_n]$, consider the local coordinate with $z_0 = w_0 = 1$. Then the Kähler potential $\rho = \log\big(1 + \cosh\big(\frac{4\pi\sqrt{2E}}{l}\big)\big)$ should be, with $l = 2\pi$ and $\sqrt{2E} = u_0$,

$$\log\big(2\mathcal{N}(zw^t)\big) = \log\big(1 + \sum_{1\le i\le n} |z_i|^2\big) + \log\big(1 + \sum_{1\le i\le n} |w_i|^2\big) - 2\log\big|\big(\sum_{0\le\alpha\le n} z_\alpha w_\alpha\big)\big| + \log 2$$

Since the last two terms are pluriharmonic, the associated Kähler metrics is exactly the canonical product metric on $\mathbb{CP}^n \times \mathbb{CP}^n$.

Remark 2.3. *The aforementioned embedding of $T\mathbb{CP}^n$ into $\mathbb{CP}^n \times \mathbb{CP}^n$ can also be viewed in the reverse direction as follows. Take an arbitrary point $\big[z\big] \times \big[w\big] \in \mathbb{CP}^n \times \mathbb{CP}^n$.*

- *If $\sum_{0\le\alpha\le n} z_\alpha\, w_\alpha \neq 0$, then we can choose representatives z, w such that*

$$\sum_{0\le\alpha\le n} z_\alpha\, w_\alpha \;=\; 1, \quad \textit{and} \quad \|z\| \;=\; \|w\|.$$

  Then $\big[z\big] \times \big[w\big]$ corresponds to a tangent vector at the point $\big[\frac{z+\overline{w}}{2}\big] \in \mathbb{CP}^n$. Let $\rho = \frac{\|z+\overline{w}\|}{2}$. Then $2\cosh^{-1}(\rho)$ should be the length of the vector and the direction of the vector should be the direction of the following geodesic.

$$\Big[\cos(\frac{t}{2})\frac{z+\overline{w}}{2\rho} \;+\; \sin(\frac{t}{2})\,\frac{z-\overline{w}}{2i\sqrt{\rho^2-1}}\Big]$$

- *If $\sum_{0\le\alpha\le n} z_\alpha\, w_\alpha = 0$, then pick any representatives z, w with $\|z\| = \|w\| = \sqrt{2}$. In this case, $\big[z\big] \times \big[w\big]$ can be interpreted as the point in D which represents the geodesic*

$$\Big[\cos(\frac{t}{2})\,\frac{z+\overline{w}}{2} \;+\; \sin(\frac{t}{2})\,\frac{z-\overline{w}}{2i}\Big]$$

  in $\mathbb{CP}^n$.

Remark 2.4. ***(Description of N_{-1})*** *By definition, for each leaf,*

$$N_{-1} \circ \phi_\gamma(\sigma + i\tau) \;=\; \phi_\gamma(\sigma - i\tau).$$

A direct calculation in this model shows that, in $\mathbb{CP}^n \times \mathbb{CP}^n$,

$$N_{-1} : \quad [z] \times [w] \;\longmapsto\; [\overline{w}] \times [\overline{z}],$$

and in $\mathbb{CP}^N$, with the conjugate transpose *, we have

$$N_{-1} : \quad [\zeta] \;\longmapsto\; [\zeta^*].$$

Notably, all structures we have placed on $\mathbb{CP}^n \times \mathbb{CP}^n$ are invariant under N_{-1}. First, the base manifold $\mathbb{CP}^n$ is the fixed locus of N_{-1} acting on $\mathbb{CP}^n \times \mathbb{CP}^n \subset \mathbb{CP}^N$. Furthermore, N_{-1} sends points of $\mathbb{CP}^n \times \mathbb{CP}^n$ to $\mathbb{CP}^n \times \mathbb{CP}^n$, points of $D = \big\{\, [z] \times [w] : \sum_{0\le\alpha\le n} z_\alpha\, w_\alpha = 0\big\}$ to D, and each leaf L to itself.

Remark 2.5. ***(Harmonicity of u on each leaf)*** *In the general case, consider a geodesic γ corresponding to a point $p \in D$ and its compactified complexified leaf ϕ_γ. Since this leaf is biholomorphic to $\mathbb{CP}^1$, we may choose a local coordinate $z = e^{\,i(\sigma+i\tau)}$ centered at p. Note that this setup implies $p = \phi_\gamma(i\infty)$, where $z = e^{-\infty} = 0$; thus it matches the coordinate naturally defined by the defining section of p. On $\phi_\gamma(\sigma + i\tau)$, the value of u_1 is $|\tau|$. Hence*

$$u \;=\; |\tau| \;=\; -\log|z|,$$

which shows that u (restricted to each leaf) is harmonic.

3 Cohomology of X and D

Assume M is a Zoll manifold of type $\mathbb{CP}^n$ as before. First, we compute the cohomology of X and D, and analyze the behavior of their cocycles. This computation is influenced by [10] and [14]; see also [1].

3.1 Computation of Cohomology

Note that $TM\backslash M$ deformation retracts to and hence is homotopic to the unit tangent bundle UM. Recall that the real dimensions: $\dim M = 2n$, $\dim UM = 4n-1$ and $\dim D = 4n-2$.

Proposition 3.1. *The following isomorphisms are true.*

1. $H^j(UM,\mathbb{Z}) \cong \begin{cases} \mathbb{Z} & j=2k,\ 0\le k\le n-1, \text{ or } j=2k-1,\ n+1\le k\le 2n \\ \mathbb{Z}/(n+1)\mathbb{Z} & j=2n, \\ 0 & \text{otherwise}, \end{cases}$

2. $H^j(D,\mathbb{Z}) \cong \begin{cases} \mathbb{Z}^{\frac{j}{2}+1} & \text{if } j \text{ is even and } 0\le j\le 2n-2, \\ \mathbb{Z}^{2n-\frac{j}{2}} & \text{if } j \text{ is even and } 2n\le j\le 4n-2, \\ 0 & \text{otherwise}, \end{cases}$

3. $H^j(X,\mathbb{Z}) \cong H^j\big(\mathbb{CP}^n\times\mathbb{CP}^n,\mathbb{Z}\big)$ *for any* $j\in\mathbb{Z}$.

Proof. Since M is type $\mathbb{CP}^n$, by the Bott-Samelson Theorem, $\pi_1(M)=0$. Therefore, for $n=1$, M is a simply connected 2-dimensional manifold - a sphere, which is resolved in [4]. Hence $UM\cong\mathbb{RP}^3$, $D\cong S^2$, $X\cong S^2\times S^2$. Now we assume that $n\ge 2$. First, observe that UM can be viewed as a bundle in two different ways:

$$S^{2n-1} \hookrightarrow UM \overset{\tau}{\twoheadrightarrow} M \qquad \text{and} \qquad S^1 \hookrightarrow UM \overset{\pi}{\twoheadrightarrow} D.$$

By the long exact sequence of homotopy groups for each fibration, we obtain

$$\pi_1(S^{2n-1})=0 \longrightarrow \pi_1(UM) \longrightarrow \pi_1(M)=0$$

Hence $\pi_1(UM)=0$. Also,

$$\pi_1(UM)=0 \longrightarrow \pi_1(D) \longrightarrow \pi_0(S^1) \longrightarrow \pi_0(UM),$$

which implies $\pi_1(D)=0$.

From the Gysin exact sequence associated to the first fibration, we have

$$H^{j-2n}(M) \longrightarrow H^j(M) \overset{\tau^*}{\longrightarrow} H^j(UM) \longrightarrow H^{j-2n+1}(M) \longrightarrow H^{j+1}(M).$$

Note that we know that $H^j(M)\cong H^j(\mathbb{CP}^n)$ for any $j\in\mathbb{Z}$. Thus, for $0\le j\le 2n-2$, it follows that

$$H^j(UM) \cong H^j(M)\cong H^j(\mathbb{CP}^n),$$

and for $2n+1\le j\le 4n-1$,

$$H^j(UM) \cong H^{j-2n+1}(M)\cong H^{j-2n+1}(\mathbb{CP}^n).$$

Next, consider the case $j = 2n-1$. The part of the Gysin sequence becomes

$$\underbrace{H^{2n-1}(M)}_{\cong 0} \longrightarrow H^{2n-1}(UM) \longrightarrow H^0(M) \xrightarrow{\cup\chi} H^{2n}(M) \longrightarrow H^{2n}(UM) \longrightarrow \underbrace{H^1(M)}_{\cong 0}.$$

Here χ is the Euler class of TM which satisfies $\chi([M]) = \chi(\mathbb{CP}^n) = n+1$. Consequently,

$$H^{2n-1}(UM) \cong \operatorname{Ker}(\cup\chi) \cong 0,$$

$$H^{2n}(UM) \cong \operatorname{Coker}(\cup\chi) \cong \mathbb{Z}/(n+1)\mathbb{Z}.$$

This completes the computation of $H^j(UM)$.

Next, we use the Gysin exact sequence for the fibration $S^1 \hookrightarrow UM \overset{\pi}{\twoheadrightarrow} D$. From

$$H^{2j}(UM) \longrightarrow H^{2j-1}(D) \longrightarrow H^{2j+1}(D) \longrightarrow H^{2j+1}(UM),$$

and the facts $H^{-1}(D) \cong 0$ and $H^{4n-1}(D) \cong 0$, we deduce $H^{2j+1}(D) = 0$ for $0 \leq j \leq n-1$ by an induction from $j = 0$ to $j = n-1$ via the last three terms, and for $j \geq n$ by an induction from $j = 2n-1$ to $j = n+1$ via the first three terms.

Since D is a connected real $(4n-2)$-dimensional manifold, we have $H^0(D) \cong H^{4n-2}(D) \cong \mathbb{Z}$. Consider then the Gysin exact sequence

$$\underbrace{H^{2j-1}(D)}_{\cong 0} \longrightarrow H^{2j-1}(UM) \longrightarrow H^{2j-2}(D) \longrightarrow H^{2j}(D) \xrightarrow{\pi^*} H^{2j}(UM) \longrightarrow \underbrace{H^{2j-1}(D)}_{\cong 0}.$$

For $0 \leq j \leq n-1$, we have $H^{2j}(D) = \mathbb{Z}^{j+1}$ by an inductive argument on the right five terms (from $j = 1$ to $j = n-1$). For $j \geq n$, we have $H^{2j}(D) = \mathbb{Z}^{2n-j}$ by an inductive argument on the left five terms (from $j = 2n-1$ to $j = n+1$). This completes the computation of $H^j(D)$.

Finally, since X can be decomposed into the subset of vectors of norm at most 1 and those of norm at least 1, we apply the Mayer–Vietoris sequence:

$$H^{j-1}(UM) \xrightarrow{\partial} H^j(X) \longrightarrow H^j(D) \oplus H^j(M) \xrightarrow{\iota^*} H^j(UM) \xrightarrow{\partial} H^{j+1}(X).$$

For $j \leq 2n$, the map ι^* satisfies $\iota|_{H^j(D)} = 0$ and that $\iota|_{H^j(M)}$ is surjective onto $H^j(UM)$. So the boundary maps ∂ are zero, and we obtain a short exact sequence from the middle three terms. Moreover,

$$H^j(D) \oplus H^j(M) \xrightarrow{\iota^*} H^j(UM) \xrightarrow{\partial} H^{j+1}(X) \longrightarrow H^{j+1}(D) \oplus H^{j+1}(M) \xrightarrow{\iota^*} H^{j+1}(UM)$$

shows that for $j \geq 2n+1$, the map ι^* is the zero map, and hence again the middle three terms form a short exact sequence. Since $\operatorname{Ext}^1_{\mathbb{Z}}(\mathbb{Z}^k, \mathbb{Z}) = 0$, these sequences split. Lastly, for $j = 2n$, the map ι^* on the left of the above sequence is surjective, so the ∂ is a zero map. Therefore, we have $H^{2n+1}(X) = 0$, so the computation is completed. $\square$

3.2 Behavior of Cocycles

A crucial step in our proof is to show that X is a Fano manifold and that $-K_X = \mathcal{O}\big((n+1)D\big)$. To this end, we must first prove that for some integer r,

$$-K_X \;=\; \mathcal{O}(rD) \quad \text{or equivalently} \quad c_1(X) = r\,[D].$$

By Proposition 3.1, the inclusion $\iota : D \to X$ induces an isomorphism $\iota^* : H^2(X,\mathbb{Z}) \to H^2(D,\mathbb{Z})$. Hence, it suffices to show that

$$c_1(D) \;=\; r\,[D]\big|_D.$$

In addition, since $\mathcal{O}(D)\big|_D \cong \mathcal{N}_{D/X}$ and $e\big(\mathcal{N}_{D/X}\big) = c_1\big(\mathcal{N}_{D/X}\big)$, it follows that

$$[D]\big|_D = \; c_1\big(\mathcal{O}(D)\big|_D\big) \;=\; c_1\big(\mathcal{N}_{D/X}\big) \;=\; [\omega].$$

Where, $[\omega]$ is the Euler class in $H^2(D,\mathbb{Z})$ induced by the fibration $UM \overset{\pi}{\twoheadrightarrow} D$. Therefore, we seek explicit expressions for the generators of $H^2(D,\mathbb{Z}) \cong \mathbb{Z}\oplus\mathbb{Z}$. Observe the following short exact sequence:

$$0 \;\longrightarrow\; H^0(D) \xrightarrow{\;\cup\,\omega\;} H^2(D) \xrightarrow{\;\pi^*\;} H^2(UM) \;\longrightarrow\; 0.$$

Hence, for the generator $y \in H^2(UM) \cong \mathbb{Z}$, we see that $H^2(D) \cong \mathbb{Z}\oplus\mathbb{Z}$ is generated by the class $[\omega]$ and $x \in H^2(D)$ such that $\pi^*(x) = y$.

Next, recall the antiholomorphic involution $N_{-1} : X \to X$,

$$N_{-1}(v) \;=\; -v, \quad N_{-1}(\infty_\gamma) \;=\; 0_\gamma, \quad N_{-1}(0_\gamma) \;=\; \infty_\gamma.$$

We note the following properties of N_{-1}:

1. N_{-1} sends points of D to points of D and reverses the orientation on D. Therefore,

$$N_{-1}^*\,c_1(D) \;=\; N_{-1}^*\,c_1(-K_D) \;=\; -\,c_1(D).$$

2. Recall the correspondence $[\omega] = c_1(\mathcal{O}(D)|_D)$. Again, since N_{-1} reverses the orientation on D, we have

$$N_{-1}^*[\omega] \;=\; N_{-1}^*c_1(\mathcal{O}(D)|_D) \;=\; -c_1(\mathcal{O}(D)|_D) \;=\; -[\omega]$$

3. N_{-1} fixes the base manifold M. Since $y \in H^2(UM)$ is induced by a generator $z \in H^2(M)$ i.e. $\pi^*x = y = \tau^*z$, it follows that $N_{-1}^*(y) = y$. Thus,

$$\pi^*\big(x - N_{-1}^*x\big) \;=\; y - N_{-1}^*(y) \;=\; 0.$$

So for some integer d, we obtain $x - N_{-1}^*x = d\,\omega$ and the matrix of N_{-1}^* with respect to the basis $\{x,\omega\}$ is given by $\begin{pmatrix} 1 & -d \\ 0 & -1 \end{pmatrix}$.

Consequently, $c_1(D)$ and ω both belong to the (-1)-eigenspace, which is one-dimensional. Since ω is one of the generators, there exists an integer r such that $c_1(D) = r[\omega]$. Therefore, we conclude the following:

Lemma 3.2. *We have the identity:* $-K_X \;=\; \mathcal{O}(rD)$ *for some integer* r.

4 Evaluation of Fano Index

One of the powerful features of having an entire Grauert tube is that all compactified leaves of the Riemann foliation become complex submanifolds. This property can be used effectively in computing the Fano index by applying the adjunction formula to a compactified leaf C. Let us consider a geodesic γ on M, the corresponding leaf ϕ_γ of the Riemann foliation, and its compactification C. Denote by $\mathcal{N}$ the holomorphic normal bundle of C in X. By the adjunction formula, we have

$$K_C \;=\; K_X\big|_C + \det\mathcal{N}.$$

To exploit this identity, we first prove the following lemma regarding $\det\mathcal{N}$. Let $2n$ be the complex dimension of X.

Lemma 4.1. $\det\mathcal{N} \cong \mathcal{O}_{\mathbb{CP}^1}(2n)$.

Proof. Fix a unit-speed closed geodesic γ. Since M has a type of $\mathbb{CP}^n$, the Bott–Samelson theorem implies that the index k of γ is 1. In other words, there exist unit vectors $v_1, v_2, \ldots, v_{2n-1} \in T_{\gamma(0)}M$ all orthogonal to $\gamma'(0)$, and corresponding Jacobi fields η_i on γ such that

$$\eta_i(0) \;=\; 0, \qquad \frac{D\eta_i}{dt}(0) \;=\; v_i,$$

and among these, η_1 has exactly one additional first-order vanishing point along γ (besides $t = 0$), while $\eta_2, \ldots, \eta_{2n-1}$ vanish to first order only at $t = 0$.

From these Jacobi fields, we obtain parallel vector fields over C. Since each v_i is orthogonal to $\gamma'(0)$, the associated parallel vector fields define holomorphic sections $\eta_i^{1,0}$ of $\mathcal{N}$. Consequently, their wedge product $\bigwedge \eta_i^{1,0}$ gives a holomorphic section of $\det\mathcal{N}$. As noted above, η_1 vanishes to first order at two points on γ, and each $\eta_2, \ldots, \eta_{2n-1}$ vanishes to first order at exactly one point. Moreover, by the construction of complex structures from [4], $\{\eta_i^{1,0}, i = 1, \ldots, n\}$ extend over $D \subset X$ to holomorphic sections of $T^{(1,0)}X$ and are linearly independent at points of D. In fact, at points of D, they form a basis of the tangent space $T^{1,0}D$. Thus, $\bigwedge \eta_i^{1,0}$ has total vanishing order $2n$. Hence, $\det\mathcal{N} = \mathcal{O}_{\mathbb{CP}^1}(2n)$. □

Using this, we obtain the following result (cf. [4], Lemma 8):

Lemma 4.2. *We have the identity: $-K_X \;\cong\; \mathcal{O}\big((n+1)D\big)$. In particular, X is a Fano manifold, i.e. $-K_X$ is an ample line bundle.*

Proof. By the adjunction formula,

$$K_C \;=\; K_X\big|_C + \det\mathcal{N}.$$

Since C is biholomorphic to $\mathbb{CP}^1$, we have $K_C = \mathcal{O}(-2)$. From Lemma 4.1, $\det\mathcal{N} = \mathcal{O}_{\mathbb{CP}^1}(2n)$, implying

$$K_X\big|_C \cong \; \mathcal{O}\big(-2n-2\big).$$

Moreover, the divisor D meets C transversally at two points $\{\infty_\gamma, 0_\gamma\}$, so

$$\mathcal{O}(D)\big|_C \cong \; \mathcal{O}(2).$$

By Lemma 3.2, we already know that $-K_X \cong \mathcal{O}(rD)$ for some integer r. Restricting to C, we get

$$\mathcal{O}\big(-(2n+2)\big) \;\cong\; K_X\big|_C \;\cong\; -\mathcal{O}(rD)\big|_C \;=\; \mathcal{O}(-2r).$$

Hence $\mathcal{O}(-2n-2) \cong \mathcal{O}(-2r)$, implying $r = n+1$. Thus the Fano index is $n+1$. □

5 Characterization of the Complex Structure and Metric

According to Theorem B of Wiśniewski [13], any (complex) $2n$-dimensional Fano manifold with Fano index $n+1$ and Picard number 2 is biholomorphic to $\mathbb{CP}^n \times \mathbb{CP}^n$. By the Kodaira vanishing theorem, the map

$$c_1 : H^1\big(X, \mathcal{O}^*\big) \longrightarrow H^2(X, \mathbb{Z})$$

is an isomorphism, and from Proposition 3.1 we have $H^2(X,\mathbb{Z}) \cong \mathbb{Z} \oplus \mathbb{Z}$. Thus the Picard number of X is 2. It follows that:

Proposition 5.1. *There is an biholomorphism $X \cong \mathbb{CP}^n \times \mathbb{CP}^n$. and D is identified with a smooth divisor in $\mathcal{O}_X(1,1) = p_1^*\mathcal{O}_{\mathbb{CP}^n}(1) \otimes p_2^*\mathcal{O}_{\mathbb{CP}^n}(1)$ where p_i, $i = 1, 2$ are projections to the two factors of $\mathbb{CP}^n$.*

Proof of the Main Theorem. By Theorem 5.1, we obtained $X \cong \mathbb{CP}^n \times \mathbb{CP}^n$, together with projections $p_i : X \to \mathbb{CP}^n$ $(i = 1, 2)$ such that

$$\mathcal{O}(D) \;\cong\; p_1^*\mathcal{O}(1) \,\otimes\, p_2^*\mathcal{O}(1).$$

Consider $\phi = \nu \circ p_1 \circ N_{-1}$ where $\nu : \mathbb{CP}^n \to \mathbb{CP}^n$ is the canonical conjugation involution on $\mathbb{CP}^n$. Then ϕ is a surjective holomorphic morphism $X \to \mathbb{CP}^n$. By Mori's theory, there are only two such holomorphic morphisms p_1 and p_2 up to isomorphism which correspond to the contraction of two extremal rays. In other words, there exists a $\sigma \in \mathrm{Aut}(\mathbb{CP}^n)$ such that either $\phi = \sigma \circ p_2$ or $\phi = \sigma \circ p_1$.

We consider the first case: $\phi = \nu \circ p_1 \circ N_{-1} = \sigma \circ p_2$ with $\sigma \in \mathrm{Aut}(\mathbb{CP}^n)$. Replacing p_2 by $\sigma \circ p_2$, we can assume that $\nu \circ p_1 \circ N_{-1} = p_2$ which implies $\nu \circ p_2 \circ N_{-1} = p_1$. Then we have $p_1 \circ N_{-1}([Z],[W]) = \nu \circ p_2([Z],[W]) = \nu([W]) = [\overline{W}]$ and similarly $p_2 \circ N_{-1}([Z],[W]) = [\overline{Z}]$. So we get the expression for N_{-1}:

$$N_{-1}([Z],[W]) = ([\overline{W}],[\overline{Z}]).$$

To proceed, we want to exclude the second case. Suppose $\nu \circ p_1 \circ N_{-1} = \sigma_1 \circ p_1$ for some $\sigma_1 \in \mathrm{Aut}(\mathbb{CP}^n)$. By the same reasoning as above, we must have $\nu \circ p_2 \circ N_{-1} = \sigma_2 \circ p_2$ for some $\sigma_2 \in \mathrm{Aut}(\mathbb{CP}^n)$. So we get:

$$N_{-1}([Z],[W]) = (\overline{\sigma_1([Z])},\ \overline{\sigma_2([W])}).$$

Since N_{-1} is an anti-holomorphic involution, both $\nu \circ \sigma_i$ are anti-holomorphic involution of $\mathbb{CP}^n$. The fixed point set M of N_{-1} is equal to $K_1 \times K_2$ where K_i, $i = 1, 2$ is the fixed point set of the anti-holomorphic involution $\nu \circ \sigma_i$ which are smooth submanifolds. The anti-holomorphic involutions of $\mathbb{CP}^n$ are well understood and we know that K_i is either diffeomorphic to $\mathbb{RP}^n$ or is empty. This would contradict that assumption that M has the same cohomology ring as $\mathbb{CP}^n$. The latter claim follows from [11, Lemma 5.29], which states that any anti-holomorphic involution is projectively equivalent to the standard complex conjugation or when $n + 1 = 2m$ is even to the involution given by

$$[Z_1, \cdots, Z_{2m}] \mapsto [-\overline{Z_{m+1}}, \cdots, -\overline{Z_{2m}}, \overline{Z_1}, \cdots, \overline{Z_m}].$$

To explain this further, note that since $\mathrm{Aut}(\mathbb{CP}^n) = PSL(n+1,\mathbb{C})$, $\sigma = \sigma_A$ for an invertible $(n+1) \times (n+1)$-matrix A and the anti-holomorphic involution $\nu \circ \sigma$ is given by $[Z] \mapsto [\overline{Av}] = [\bar{A}\bar{v}]$. Then there exists a invertible $(n+1) \times (n+1)$-matrix U such that either $A = \bar{U}^{-1}U$ or $A = \bar{U}^{-1}JU$ with $J = \begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix}$. Note that the latter case happens only when $(n+1) = 2m$ is even. With this normal form, we see that $[v]$ with $v \in \mathbb{C}^{n+1} \setminus \{0\}$ is a fixed point of $\nu \circ \sigma_A$ if and only if $\overline{Av} = \lambda v$ with $\lambda \neq 0$, which is equivalent to either $Uv = \bar{\lambda}\overline{Uv}$ or $JUv = \bar{\lambda}\overline{Uv}$. We easily see that the fixed point set of $\nu \circ \sigma$ is either diffeomorphic to $\mathbb{RP}^n$ or is empty.

From now on, we can assume the equality $p_2 = \overline{p_1 \circ N_{-1}}$. Assume $D \in H^0(\mathbb{CP}^n \times \mathbb{CP}^n, \mathcal{O}(1,1))$ is given by the equation $\sum_{i,j} c_{ij} Z_i W_j = 0$. Because D is invariant under the conjugation $N_{-1} : ([Z],[W]) \mapsto ([\bar{W}],[\bar{Z}])$, we see easily that $C = (c_{ij})$ can be chosen to be a Hermitian matrix $C^* = C$. So we can diagonalize the matrix C by a unitary matrix U: $UCU^* = I$ where $U^* = \bar{U}^t$. By a simultaneous change of variables $Z \mapsto U^t Z$, $W \mapsto \bar{U}^t W$, we can assume D is given by the equation $\sum_{i,j} Z_i W_i = 0$. Hence, under the new coordinate for the Segre embedding $X \to \mathbb{P}\left(H^0\big(X, \mathcal{O}(D)\big)^*\right)$, the data (N_{-1}, D, M) coincide with the data of standard model $\mathbb{CP}^n$.

Finally we can complete the proof as in [4]. Consider the leaf $\phi_\gamma \cong \mathbb{C}\backslash\{0\}$ in $\mathbb{CP}^n \times \mathbb{CP}^n$ arising from $u_1 := u = \sqrt{2E}$ as in section 2.1, which meets D transversally at ∞_γ and 0_γ. We know that (see 2.5) that $u_1 = -\log|z| + O(1)$ as $z \to +\infty$ and, by a change of coordinate on $\mathbb{C}\backslash 0$, $u_1 = \log|z| + O(1)$ as $z \to 0$. From the explicit description of u_0 in (1), we also know that $u_0 = -\log|z| + O(1)$ as $z \to +\infty$ and $u_0 = \log|z| + O(1)$ as $z \to 0$. Restricted to ϕ_γ, the function u_1 is harmonic, whereas u_0 is subharmonic, and both vanish on $M = \mathbb{CP}^n \subset \mathbb{CP}^n \times \mathbb{CP}^n$. Therefore, $u_0 - u_1$ is subharmonic on $\phi_\gamma \cong \mathbb{C}\backslash 0$, remains bounded near infinity and 0, and attains the value 0. By Hadamard Three-Cicles Theorem we know that $F(r) := \max\{(u_0 - u_1)(z); |z| = r\}$ is convex in $\log r$ for $0 < r < +\infty$ (see [9, Chapter 2, Theorem 28]). Since $F(r)$ is bounded as $r \to 0$ and $r \to +\infty$, $F(r)$ must be constant. As a consequence, $u_0 - u_1$ must also be constant on $\phi_\gamma \cong \mathbb{C}\backslash\{0\}$ (again by [9, Chapter 2, Theorem 28]) and hence be identically equal 0. Since the Zoll metric on M corresponding to u_i is the restriction of the Kähler metric $\sqrt{-1}\partial\bar{\partial}u_i^2$ on M, the Zoll metric corresponding to u_1 is the same as the Zoll metric corresponding to u_0 which is the canonical Fubini-Study metric on $M = \mathbb{CP}^n$. □

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