

# A Pohožaev identity and critical exponents of some complex Hessian equations

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In this paper, we prove some sharp non-existence results for Dirichlet problems of complex Hessian equations. In particular, we consider a complex Monge-Ampère equation which is a local version of the equation of Kähler-Einstein metric. The non-existence results are proved using the Pohožaev method. We also prove existence results for radially symmetric solutions. The main difference of the complex case with the real case is that we don't know if a priori radially symmetric property holds in the complex case.

## 1 Introduction

In [19], Tso considered the following real  $k$ -Hessian equation:

$$S_k(u_{\alpha\beta}) = (-u)^p \text{ on } \Omega, \quad u = 0 \text{ on } \partial\Omega. \quad (1)$$

$\Omega$  denotes a domain inside  $\mathbb{R}^d$ .  $k$  is an integer satisfying  $1 \leq k \leq d$ .  $p$  is a positive real number.  $S_k(u_{\alpha\beta})$  denotes the  $k$ -th symmetric polynomial of eigenvalues of the Hessian matrix  $(u_{\alpha\beta}) = \left(\frac{\partial^2 u}{\partial x^\alpha \partial x^\beta}\right)$ . The following formula is well known:

$$S_k(u_{\alpha\beta}) = \frac{1}{k!} \sum_{1 \leq i_1, \dots, i_k \leq n} \delta_{j_1 \dots j_k}^{i_1 \dots i_k} u_{i_1 j_1} \dots u_{i_k j_k}.$$

Here we used the generalized Kronecker symbol  $\delta_{j_1 \dots j_k}^{i_1 \dots i_k}$ , which is equal to the sign of permutation from  $\{i_1 \dots i_k\}$  to  $\{j_1 \dots j_k\}$  if the two sets of indices are the same or is equal to 0 otherwise. Tso ([19]) proved following result.

**Theorem 1** ([19]). *Let  $\Omega$  be a ball and  $\tilde{\gamma}(k, d) = \begin{cases} \frac{(d+2)k}{d-2k} & 1 \leq k < \frac{d}{2} \\ \infty & \frac{d}{2} \leq k < d \end{cases}$ . Then (i) (1) has no negative solution in  $C^1(\bar{\Omega}) \cap C^4(\Omega)$  when  $p \geq \tilde{\gamma}(k, d)$ ; (ii) It admits a negative solution which is radially symmetric and is in  $C^2(\bar{\Omega})$  when  $0 < p < \tilde{\gamma}(k, d)$ ,  $p$  is not equal to  $k$ .*

The non-existence result above was proved by the Pohožaev method. In this article, we first generalize Tso's result to case of complex  $k$ -Hessian equation. From now on, let  $B_R$  denote the ball of radius  $R$  in  $\mathbb{C}^n$ . We consider the following equation

$$S_k(u_{i\bar{j}}) = (-u)^p \text{ on } B_R, \quad u = 0 \text{ on } \partial B_R. \quad (2)$$

where the complex  $k$ -Hessian operator  $S_k(u_{i\bar{j}})$  is the  $k$ -th symmetric polynomial of eigenvalues of the complex Hessian matrix  $(u_{i\bar{j}}) = \left(\frac{\partial^2 u}{\partial z^i \partial \bar{z}^j}\right)$ , or equivalently we have the following formula:

$$S_k(u_{i\bar{j}}) = \frac{1}{k!} \sum_{1 \leq i_1, \dots, i_k \leq n} \delta_{j_1 \dots j_k}^{i_1 \dots i_k} u_{i_1 \bar{j}_1} \dots u_{i_k \bar{j}_k}.$$

Our first result is

**Theorem 2.** Define  $\gamma(k, n) = \frac{(n+1)k}{n-k} = \tilde{\gamma}(k, 2n)$ . Then (i) (2) has no nontrivial nonpositive solution in  $C^2(\bar{B}_R) \cap C^4(B_R)$  when  $p \geq \gamma(k, n)$ ; (ii) It admits a negative solution which is radially symmetric and is in  $C^2(\bar{B}_R)$  when  $0 < p < \gamma(k, n)$  and  $p$  is not equal to  $k$ .

**Remark 1.** By scaling, we get a solution to  $S_k(u_{i\bar{j}}) = \lambda(-u)^p$  for any  $\lambda > 0$  if  $p$  satisfies the restrictions. When  $p = k$ , we are with the eigenvalue problem, as in the real Hessian case ([18]), one should be able to show that there exists a  $\lambda_1 > 0$  such that there is a nontrivial nonpositive solution to the equation:  $S_k(u_{i\bar{j}}) = \lambda_1(-u)^k$ . Moreover, the solution is unique up to scaling.

**Remark 2.** By the work of [8] and [7], the solution to (1) is a priori radially symmetric. However, it's not known if all the solutions to (2) are radially symmetric. The classical moving plane method for proving radial symmetry works for many classes of real elliptic equations but doesn't seem to work in the complex case (cf. [7]). For the recent study of complex Hessian equations, see [4], [12], [21] and the reference therein.

Next we use Pohožaev method to prove a non-existence result for the following equation:

$$S_k(u_{l\bar{m}}) = a \frac{e^{-u}}{\int_{B_1} e^{-u} dV} \text{ on } B_1, \quad u = 0 \text{ on } \partial B_1. \quad (3)$$

**Remark 3.** We have the following scaling property. Suppose  $u = u(x)$  is a solution to (3). Defining  $v(x) = u(x/R)$ , then  $v$  is a solution to the following equation:

$$S_k(v_{l\bar{m}}) = aR^{2(n-k)} \frac{e^{-v}}{\int_{B_R} e^{-v} dV} \text{ on } B_R, \quad v = 0 \text{ on } \partial B_R. \quad (4)$$

Note that when  $k = n$ , we have a (scaling-invariant) Monge-Ampère equation:

$$\det(u_{l\bar{m}}) = a \frac{e^{-u}}{\int_{B_1} e^{-u} dV} \text{ on } B_1, \quad u = 0 \text{ on } \partial B_1. \quad (5)$$

Note that equation (5) is a local version of the Kähler-Einstein metric equation. Indeed, by taking log and  $\sqrt{-1}\partial\bar{\partial}$  on both sides of (5), we get:

$$\text{Ric}(\sqrt{-1}\partial\bar{\partial}u) = \sqrt{-1}\partial\bar{\partial}u,$$

which says the Kähler metric  $\sqrt{-1}\partial\bar{\partial}u$  is a Kähler-Einstein metric with positive Einstein constant. (Here we assume  $u$  is a strictly plurisubharmonic function so that  $\sqrt{-1}\partial\bar{\partial}u$  is a Kähler metric on  $B_1$ .) Since the domain we consider is the unit ball, there are natural solutions to (5) coming from potentials of Fubini-Study metrics on  $\mathbb{P}^n$ :

$$u_\epsilon = (n+1)[\log(|z|^2 + \epsilon^2) - \log(1 + \epsilon^2)], \quad (6)$$

with the parameter  $a$  in (5) being

$$a_\epsilon = (n+1)^n \epsilon^2 \int_0^1 \frac{r^{2n-1} dr}{(r^2 + \epsilon^2)^{n+1}} \omega_{2n-1} = (n+1)^n \frac{\omega_{2n-1}}{2n(1 + \epsilon^2)^n}.$$

The 2nd identity can be seen verified using the substitution  $x = r^2/(r^2 + \epsilon^2)$ . Note that we will use  $\omega_{d-1} = \frac{2\pi^{d/2}}{\Gamma(d/2)}$  to denote the volume of the  $(d-1)$ -dimensional unit sphere  $S^{d-1}$ . In particular  $\omega_{2n-1} = \frac{2\pi^n}{(n-1)!}$ . So we get that when

$$0 < a < a_0 = (n+1)^n \frac{\pi^n}{n!}, \quad (7)$$

there exists a radially symmetric solution for (5). Again it's an open question ([7], [1]) whether all solutions to (5) are a priori radially symmetric, which would imply (6) gives all the solutions to (5). Without a priori radially symmetric properties, we can still use Pohožaev method to get

**Theorem 3.** *For the Dirichlet problem (3), there exists  $\alpha(k, n) > 0$  such that there exists no solution to (3) in  $C^2(\bar{B}_1) \cap C^4(B_1)$  when  $a > \alpha(k, n)$ . Moreover, when  $k = n$ , we can make  $\alpha(n, n) = a_0 = (n+1)^n \frac{\pi^n}{n!}$  and (5) has no solution in  $C^2(\bar{B}_1) \cap C^4(B_1)$  if  $a \geq a_0$ . In other words, the  $a_0$  in (7) is sharp and can not be obtained, at least for solutions with enough regularities.*

We briefly compare with some previous results. Equation (5) was extensively studied in [1] on general hyperconvex domains. Note that, the normalization here differs from that in [1] by a factor of  $\pi^n/n!$ . Berman-Berndtsson [1] proved that equation (5) has a solution when  $a < a_0$  on any hyperconvex domain. The solution actually is a global maximizer of the Moser-Trudinger-Onofri functional. They also showed that when  $a > a_0$  the Moser-Trudinger-Onofri functional is not bounded from above, which makes them expect that there is no solution to (5) when  $a \geq a_0$ . Again, if the a priori radially symmetric property holds, then it would be true. Unfortunately this property is not known yet in the complex case. In [2, Theorem 1.6] Berman-Berndtsson further proved that there is no  $S^1$ -invariant solution to equation (5) if  $a \geq a_0$ , where the  $S^1$ -action is taken as  $e^{i\theta}(z^1, \dots, z^n) = (e^{i\theta}z^1, \dots, e^{i\theta}z^n)$ . Guedj-Kolev-Yeganefar ([9]) used Bishop-Gromov comparison theorem (for the Kähler-Einstein metric associated to any solution) to prove a (non-sharp) nonexistence result: there is no smooth solution to equation (5) if  $a > 2^n(2n-1)^n \frac{(n-1)! \pi^n}{(2n-1)!}$  (by [9, Section 6.2] and taking the difference of normalization into account).

Here Theorem 3 in particular gives a sharp (without symmetry assumption) non-existence result for equation (5) using the Pohožaev method, and thus confirms an expectation of Berman-Berndtsson. We emphasize that the Pohožaev method actually gives non-existence results for general star-shaped and (strongly) k-pseudoconvex domains. Here we restrict to the case of balls to state our result because in this case we get the sharp exponent. Note that, when  $n = 1$ ,  $a_0 = 2\pi$ . Since  $\Delta = 4u_{z\bar{z}}$  on the complex plane, this is the well known result for Laplace equation of type (3) ([6]).

In the last part, we will restrict ourselves to radially symmetric solutions. Radial symmetry reduces the equation (3) to the following equation.

$$(u_s^k s^n)_s s^{1-n} = A(k, n)^{-1} \frac{ae^{-u}}{\int_0^1 e^{-u(s)} s^{n-1} ds}, \quad u = 0 \text{ on } \partial B_1. \quad A(k, n) = \frac{\omega_{2n-1}}{2k} \binom{n-1}{k-1}. \quad (8)$$

See equation (33). Using the phase plane method, we will prove the following result.

**Theorem 4.** *Define  $\beta(k, n) = k^{k-1} \binom{n-1}{k-1} \frac{\pi^n}{(n-1)!}$ . We have the following description of solutions of (8), or equivalently the radially symmetric solutions of (3).*

1. *When  $n - k \geq 4$ , (8) admits at most one solution and it admits a solution if and only if  $0 < a \leq \beta(k, n)$ .*
2. *When  $0 < n - k < 4$ , there exists  $\alpha^*(k, n)$  with  $\alpha^*(k, n) > \beta(k, n)$  such that (8) admits a solution if and only if  $0 < a \leq \alpha^*(k, n)$ . Furthermore, the solution to (8) is unique for small  $a > 0$ . When  $a = \beta(k, n)$ , there exist infinitely many solutions for (8). When  $\alpha^*(k, n) \geq a \neq \beta(k, n)$ , there exist finitely many solutions to (8). Moreover, the number of solutions tends to infinity as  $a$  approaches  $\beta(k, n)$ .*
3. *When  $k = n$ , (8) admits at most one solution and it admits a solution if and only if  $0 < a < (n+1)^n \frac{\pi^n}{n!}$ .*

Similar radially symmetric problems for real equations were considered before by several people ([11], [3], [10]). They all used the phase plane method initiated in [11]. The above theorem generalizes [3, Theorem 1] to the complex Hessian case. This is achieved by generalizing and modifying the argument used in [3]. See also Remark 9.

The paper is organized as follows. In Section 2, we derive a Pohožaev identity for general complex Hessian equations and prove the nonexistence part of Theorem 2. In Section 3, we prove

Theorem 3. In Section 4.1, we prove the existence part of Theorem 2. Finally, in Section 4.2, we prove Theorem 4.

We conclude the introduction by remarking that, although our proofs follow similar methods in the real case, there are several new technical difficulties (see e.g. Remark 9) to be resolved. Most notably, in the proof of Theorem 2 and Theorem 3, the introduction of the  $CR$ -type invariant  $\tilde{S}_{k-1}(\partial\Omega)$  in (20) is essential for the complex case.

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## 2 A Pohožaev identity for complex Hessian equations

In [13] Pohožaev established an identity for solutions of the Dirichlet problem

$$\Delta u = f(u) \text{ in } \Omega, \quad u = 0 \text{ on } \partial\Omega. \quad (9)$$

He used this identity to show that the problem (9) has no nontrivial solutions when  $\Omega$  is a bounded star-shaped domain in  $\mathbb{R}^d$  and  $f = f(u)$  is a continuous function on  $\mathbb{R}$  satisfying the condition

$$(d-2)uf(u) - 2dF(u) < 0 \quad \text{for } u \neq 0,$$

where  $F$  denotes the primitive  $F(u) = \int_0^u f(t)dt$  of  $f$ . Later, Pucci-Serrin [14] generalized Pohožaev identity to identities for much more general variational equations, and they obtained non-existence results using these type of identities. We will follow Pucci-Serrin to derive a Pohožaev identity in the complex case. We will consider the general variational problem associated to the functional

$$\mathbb{F} = \int_{\Omega} \mathcal{F}(z, u(z), u_{i\bar{j}}(z)) dV.$$

Here and henceforth, we assume  $\mathcal{F} = \mathcal{F}(z, u, r_{i\bar{j}})$  is a smooth function on  $\mathbb{C} \times \mathbb{R} \times \mathbb{C}^{(n(n+1)/2)}$ , where we denote by  $\{r_{i\bar{j}}\}_{1 \leq i, j \leq n}$  a Hermitian matrix. It's easy to verify that the Euler-Lagrange equation for  $\mathbb{F}$  is

$$\frac{\partial^2}{\partial z^i \partial \bar{z}^j} \mathcal{F}_{r_{i\bar{j}}} + \mathcal{F}_u = 0. \quad (10)$$

We can now state the Pohožaev type identity we need. Note that the coefficient for the last term is slightly different with the formula in [14, (29)] in the real case. Also see remark 4.

**Proposition 1.** *Suppose  $u \in C^2(\bar{\Omega}) \cap C^4(\Omega)$  is a solution to the equation (10). For any constant  $c$ , then following identity holds:*

$$\begin{aligned} & \frac{\partial}{\partial z^i} \left( z^i \mathcal{F} + (cu + z^q u_q) \frac{\partial}{\partial \bar{z}^j} \mathcal{F}_{r_{i\bar{j}}} \right) - \frac{\partial}{\partial \bar{z}^j} \left( \frac{\partial}{\partial z^i} (cu + z^q u_q) \mathcal{F}_{r_{i\bar{j}}} \right) \\ &= n\mathcal{F} + z^i \mathcal{F}_{z^i} - cu \mathcal{F}_u - (c+1)u_{i\bar{j}} \mathcal{F}_{r_{i\bar{j}}}. \end{aligned} \quad (11)$$

*Proof.* This follows from direct computation. We give some key steps in the calculation.

- Multiply  $u$  on both sides of equation (10) and use the product rule for differentiation we get:

$$\frac{\partial}{\partial z^i} \left( u \frac{\partial}{\partial \bar{z}^j} \mathcal{F}_{r_{i\bar{j}}} \right) - \frac{\partial}{\partial \bar{z}^j} \left( \frac{\partial u}{\partial z^i} \mathcal{F}_{r_{i\bar{j}}} \right) + u_{i\bar{j}} \mathcal{F}_{r_{i\bar{j}}} + u \mathcal{F}_u = 0. \quad (12)$$

- Multiply  $z^q u_q$  on both sides of equation (10) and use product rule twice, we get

$$\frac{\partial}{\partial z^i} \left( z^q u_q \frac{\partial}{\partial \bar{z}^j} \mathcal{F}_{r_{i\bar{j}}} \right) - \frac{\partial}{\partial \bar{z}^j} \left( \frac{\partial}{\partial z^i} (z^q u_q) \mathcal{F}_{r_{i\bar{j}}} \right) + u_{i\bar{j}} \mathcal{F}_{r_{i\bar{j}}} + z^k u_{i\bar{j}k} \mathcal{F}_{r_{i\bar{j}}} + z^q u_q \mathcal{F}_u = 0. \quad (13)$$

- Use product rule and chain rule, we get

$$\frac{\partial}{\partial z^i}(z^i \mathcal{F}) - n \mathcal{F} = z^q \frac{\partial}{\partial z^q} \mathcal{F} = z^q \mathcal{F}_{z^q} + z^q u_q \mathcal{F}_u + z^q u_{i\bar{j}q} \mathcal{F}_{r_{i\bar{j}}}. \quad (14)$$

- Multiplying (12) by constant  $c$  and combining it with (13) and (14), we immediately get (11). □

**Remark 4.** By the similar calculations, we can derive a more general formula for any holomorphic vector field  $V = V^i \partial_i$ :

$$\begin{aligned} & \frac{\partial}{\partial z^i} \left( V^i \mathcal{F} + (cu + V^q u_q) \frac{\partial}{\partial z^j} \mathcal{F}_{r_{i\bar{j}}} \right) - \frac{\partial}{\partial z^j} \left( \frac{\partial}{\partial z^i} (cu + V^q u_q) \mathcal{F}_{r_{i\bar{j}}} \right) \\ &= \frac{\partial V^i}{\partial z^i} \mathcal{F} + V^i \mathcal{F}_{z^i} - cu \mathcal{F}_u - \left( cu_{i\bar{j}} + \frac{\partial V^k}{\partial z^i} u_{k\bar{j}} \right) \mathcal{F}_{r_{i\bar{j}}}. \end{aligned} \quad (15)$$

When  $V = z^i \partial_{z^i}$ , we just recover (11).

The relevant example to us is when

$$\mathcal{F} = -\frac{u S_k(u_{i\bar{j}})}{k+1} + F(z, u), \quad \text{and } \mathbb{F} = \mathbb{F}_k = \mathbb{H}_k + \int_{\Omega} F(z, u) dV. \quad (16)$$

where we define

$$\mathbb{H}_k = -\frac{1}{k+1} \int_{\Omega} u S_k(u_{l\bar{m}}) dV; \quad F(z, u) = \int_0^u f(z, t) dt.$$

The following lemma is well-known for the real  $k$ -Hessian operator ([16]). We give the complex version to see that (10) in this case becomes the general complex  $k$ -Hessian equation

$$\begin{cases} S_k(u_{l\bar{m}}) = f(z, u), & \text{on } \Omega, \\ u = 0, & \text{on } \partial\Omega. \end{cases} \quad (17)$$

**Lemma 1.** Define the Newton tensor

$$T_{k-1}(u_{l\bar{m}})^{i\bar{j}} = \frac{1}{k!} \sum \delta_{j_1 \dots j_{k-1} \bar{j}}^{i_1 \dots i_{k-1} i} u_{i_1 \bar{j}_1} \dots u_{i_{k-1} \bar{j}_{k-1}}.$$

Then we have

1. The tensor  $(T_{k-1}(u_{l\bar{m}})^{i\bar{j}})$  is divergence free, i.e.

$$\frac{\partial}{\partial z^i} T_{k-1}(u_{l\bar{m}})^{i\bar{j}} = 0 = \frac{\partial}{\partial z^j} T_{k-1}(u_{l\bar{m}})^{i\bar{j}}$$

- 2.

$$S_k(u_{l\bar{m}}) = \frac{1}{k} T_{k-1}(u_{l\bar{m}})^{i\bar{j}} u_{i\bar{j}}.$$

- 3.

$$\frac{\partial S_k(u_{l\bar{m}})}{\partial u_{i\bar{j}}} = T_{k-1}(u_{l\bar{m}})^{i\bar{j}}.$$

For the complex Hessian equation, we substitute (16) into (11) and use lemma (1) to get

$$\begin{aligned}
& \frac{\partial}{\partial z^i} \left( z^i \left( \frac{-uS_k(u_{l\bar{m}})}{k+1} + F(z, u) \right) + (cu + z^q u_q) \frac{-u_{\bar{j}} T_{k-1}(u_{l\bar{m}})^{i\bar{j}}}{k+1} \right) \\
& + \frac{\partial}{\partial \bar{z}^j} \left( \frac{\partial}{\partial z^i} (cu + z^q u_q) \frac{u T_{k-1}(u_{l\bar{m}})^{i\bar{j}}}{k+1} \right) \\
& = [k(c+1) + c - n] \frac{uS_k(u_{l\bar{m}})}{k+1} + nF - cuf + z^i F_{z^i}.
\end{aligned} \tag{18}$$

If we make the coefficient of the first term vanish, we get the important constant which will be useful later:

$$c_0 = \frac{n-k}{k+1}.$$

The following lemma is just the divergence theorem in complex coordinate. Note that we use the following standard normalizations.

$$\frac{\partial}{\partial z^i} = \frac{1}{2} \left( \frac{\partial}{\partial x^{2i-1}} - \sqrt{-1} \frac{\partial}{\partial x^{2i}} \right), \quad g_{i\bar{j}} = \frac{1}{2} \delta_{ij}, \quad \nu_i = g_{i\bar{j}} \nu^{\bar{j}} = \frac{1}{2} \nu^{\bar{i}}, \quad z^i \nu_i + \bar{z}^i \nu_{\bar{i}} = x^\alpha \nu_\alpha. \tag{19}$$

**Lemma 2.**  $\Omega$  is a bounded domain in  $\mathbb{C}^n$  with  $C^2$  boundary. Let  $X = X^i \frac{\partial}{\partial z^i}$  be a  $C^1$  vector field on  $\bar{B}_1$  of type  $(1,0)$ . Let  $\nu$  denote the outward unit normal vector of  $\partial\Omega$ . Decompose  $\nu = \nu^{(1,0)} + \nu^{(0,1)}$  such that  $\nu^{(1,0)} = \nu^i \frac{\partial}{\partial z^i}$  and  $\nu^{(0,1)} = \nu^{\bar{j}} \frac{\partial}{\partial \bar{z}^j}$ . Then we have

$$\int_{\Omega} \frac{\partial X^i}{\partial z^i} dV = \oint_{\partial\Omega} X^i \nu_i d\sigma,$$

where  $d\sigma$  is the induced volume form on  $\partial\Omega$  from the Euclidean volume form on  $\mathbb{C}^n = \mathbb{R}^{2n}$ .

Assume  $\Omega$  is a bounded domain with  $C^2$ -boundary. For any  $p \in \partial\Omega$ , choose a small ball  $B_\epsilon(p)$  such that  $\Omega \cap B_\epsilon = \{\rho \leq 0\}$ , where  $\rho$  is a  $C^2$ -function satisfying  $|\nabla\rho|(p) = 1$ . Recall that the Levi form can be defined as

$$\mathbb{L} = \sqrt{-1} \frac{\partial^2 \rho}{\partial z^i \partial \bar{z}^j} dz^i \otimes d\bar{z}^j.$$

$\mathbb{L}$  is a symmetric Hermitian form on the space  $\mathcal{T} = T^{(1,0)}\mathbb{C}^n \cap T(\partial\Omega) \otimes_{\mathbb{R}} \mathbb{C} = \{\xi \in \mathbb{C}^n; \xi^i \rho_i = 0\} \cong (T(\partial\Omega) \cap J T(\partial\Omega), J)$ , where  $J$  is the standard complex structure on  $\mathbb{C}^n \cong \mathbb{R}^{2n}$ . Assume  $\nu$  is the outer unit normal vector to  $\partial\Omega$  then at point  $p$ , we have  $\nu_i = \rho_i$ . Denote

$$\tilde{S}_{k-1}(\partial\Omega) = \frac{1}{(k-1)!} \sum_{1 \leq i_1, \dots, i_k \leq n} \delta_{j_1 \dots j_k}^{i_1 \dots i_k} \rho_{i_1 \bar{j}_1} \dots \rho_{i_k \bar{j}_k} \nu_{i_k} \nu_{\bar{j}_k} = T_{k-1}(\rho_{l\bar{m}})^{i\bar{j}} \nu_i \nu_{\bar{j}}. \tag{20}$$

Locally around  $p$  we can choose coordinates such that  $\nu = \partial_{z^n} + \partial_{\bar{z}^n}$  and so  $\nu_i = \frac{1}{2} \delta_{in} = \frac{1}{2} \nu^{\bar{i}}$ . Then we see that, up to a constant,  $\tilde{S}_{k-1}(\partial\Omega)$  is equal to  $S_{k-1}(\mathbb{L}|_{\mathcal{T}})$ , the later being the  $(k-1)$ -th symmetric function of the eigenvalues of the restricted operator  $\mathbb{L}|_{\mathcal{T}}$ .

Note that  $\tilde{S}_{k-1}(\partial\Omega)$  is a well defined local invariant for  $\partial\Omega$ , i.e. it is independent of the defining function  $\rho$ .  $\Omega$  is called to be strongly  $k$ -pseudoconvex, if  $\tilde{S}_{k-1}(\partial\Omega) > 0$ . Note that the real version of  $\tilde{S}_{k-1}(\partial\Omega)$  appeared in [20, formula (6)].

For example, when  $\Omega$  is a ball  $B_R(0)$ ,  $\nu_i = \frac{z^i}{2R}$  and we can choose  $\rho = \frac{1}{2R}(|z|^2 - R^2)$ . By

symmetry, we can calculate at point  $(0, \dots, 0, 1)$  to get:

$$\begin{aligned}
\tilde{S}_{k-1}(\partial B_R) &= \frac{1}{(k-1)!} \sum_{i_1, \dots, j_{k-1}} \delta_{j_1 \dots j_{k-1} n}^{i_1 \dots i_{k-1} n} \rho_{i_1 \bar{j}_1} \dots \rho_{i_{k-1} \bar{j}_{k-1}} \nu_i \nu_{\bar{j}} \\
&= \frac{1}{4R^2} \frac{1}{(2R)^{k-1} (k-1)!} \sum_{1 \leq i_1, \dots, j_{k-1} \leq n-1} \delta_{j_1 \dots j_{k-1}}^{i_1 \dots i_{k-1}} \delta_{i_1 \bar{j}_1} \dots \delta_{i_{k-1} \bar{j}_{k-1}} \\
&= \frac{1}{2^{k+1} R^{k+1}} \binom{n-1}{k-1}. \tag{21}
\end{aligned}$$

We can now derive the important integral formula for us.

**Proposition 2.** *Let  $\Omega$  be a  $C^2$ -domain. Suppose  $f$  belongs  $C(\bar{\Omega} \times (-\infty, 0]) \cap C^1(\Omega \times (-\infty, 0))$  and is positive in  $\Omega \times (-\infty, 0)$ . Assume  $u \in C^2(\bar{\Omega}) \cap C^4(\Omega)$  is a solution to (17). Then we have the identity*

$$\oint_{\partial\Omega} z^i \nu_i \tilde{S}_{k-1}(\partial\Omega) |\nabla u|^{k+1} d\sigma = -(k+1) \int_{\Omega} \left( nF - \frac{n-k}{k+1} uf + z^i F_{z^i} \right) dV. \tag{22}$$

$$\oint_{\partial\Omega} \langle x, \nu \rangle \tilde{S}_{k-1}(\partial\Omega) |\nabla u|^{k+1} d\sigma = -(k+1) \int_{\Omega} \left( 2(nF - \frac{n-k}{k+1} uf) + x^\alpha F_{x^\alpha} \right) dV. \tag{23}$$

*Proof.* The second identity follows from the first easily. So we only prove the first identity. When  $\nabla u \neq 0$ , letting  $\rho = \frac{u}{|\nabla u|}$  in (20), we get

$$\begin{aligned}
z^k u_k u_{\bar{j}} T_{k-1}(u_{p\bar{q}})^{i\bar{j}} \nu_i &= z^k \nu_k (\nu_{\bar{j}} T_{k-1}(u_{p\bar{q}})^{i\bar{j}} \nu_i |\nabla u|^{-(k-1)}) |\nabla u|^{k+1} \\
&= z^k \nu_k \tilde{S}_{k-1} |\nabla u|^{k+1}.
\end{aligned}$$

When  $\nabla u = 0$ , then both sides are equal to zero. Now we can integrate (18) on  $B_1$  using divergence theorem (Lemma 2) and the boundary condition  $u = 0$  on  $\partial B_1$  to get the first identity.  $\square$

*Proof of the nonexistence part of Theorem 2.* When  $\Omega = B_R$ ,  $\langle x, \nu \rangle = R > 0$ .  $\tilde{S}_{k-1}(\partial B_R) \geq 0$  is a positive constant. So the left hand side of (23) is positive. When  $f(u) = (-u)^p$ ,  $F(u) = -\frac{1}{p+1}(-u)^{p+1}$ . So if  $\frac{n}{p+1} - \frac{n-k}{k+1} \leq 0$ , i.e.  $p \geq \frac{(n+1)k}{n-k} = \gamma(k, n)$ , there is no nontrivial nonpositive solution in  $C^2(\bar{B}_1) \cap C^4(B_1)$  to (2).  $\square$

**Remark 5.** *The same argument actually gives a non-existence result for star-shaped and strongly  $k$ -pseudoconvex domains.*

### 3 Non-local problem with exponential nonlinearities

In this section, we prove Theorem 3 using Pohožaev method. When  $k = 1$  the argument was used in [6]. The argument can be generalized to higher  $k$  by the introduction of  $\tilde{S}_{k-1}(\partial\Omega)$  in (20). Recall that we consider the following non-local equation:

$$S_k(u_{l\bar{m}}) = a \frac{e^{-u}}{\int_{B_1} e^{-u} dV}, \quad u = 0 \text{ on } \partial B_1. \tag{24}$$

*Proof.* In identity (23), if  $f$  does not depend on  $z$ , then it becomes:

$$-2 \int_{\Omega} (n(k+1)F(u) - (n-k)uf(u)) dV = \int_{\partial\Omega} \langle x, \nu \rangle \tilde{S}_{k-1}(\partial\Omega) |\nabla u|^{k+1} d\sigma. \tag{25}$$

To estimate the right hand side, note that we can integrate both sides of (3) and use divergence theorem to get

$$\begin{aligned} a &= \int_{\Omega} S_k(u_{l\bar{m}}) = \frac{1}{k} \int_{\Omega} T_{k-1}(u_{l\bar{m}})^{i\bar{j}} u_{i\bar{j}} dV \\ &= \frac{1}{k} \oint_{\partial\Omega} u_i T_{k-1}(u_{l\bar{m}})^{i\bar{j}} (\nu^p g_{p\bar{j}}) = \frac{1}{k} \oint_{\partial\Omega} \tilde{S}_{k-1}(\partial\Omega) |\nabla u|^k. \end{aligned}$$

For simplicity, we let  $\tilde{S}_{k-1}$  denote the quantity  $\tilde{S}_{k-1}(\partial\Omega)$  defined in (20). Now by Hölder's inequality, we have

$$\begin{aligned} ka &= \oint_{\partial\Omega} \tilde{S}_{k-1} |\nabla u|^k = \oint_{\partial\Omega} (\langle x, \nu \rangle \tilde{S}_{k-1})^{k/(k+1)} |\nabla u|^k (\langle x, \nu \rangle)^{-k/(k+1)} \tilde{S}_{k-1}^{1/(k+1)} \\ &\leq \left( \oint_{\partial\Omega} \langle x, \nu \rangle \tilde{S}_{k-1} |\nabla u|^{k+1} \right)^{k/(k+1)} \left( \oint_{\partial\Omega} \langle x, \nu \rangle^{-k} \tilde{S}_{k-1} \right)^{1/(k+1)}. \end{aligned}$$

So we get

$$\oint_{\partial\Omega} \langle x, \nu \rangle \tilde{S}_{k-1} |\nabla u|^{k+1} \geq \frac{(ka)^{(k+1)/k}}{\left( \oint_{\partial\Omega} \langle x, \nu \rangle^{-k} \tilde{S}_{k-1} \right)^{1/k}}. \quad (26)$$

Now we specialize to equation (24). When  $\Omega$  is the unit ball,  $\langle x, \nu \rangle \equiv 1$  and, by (21) and for simplicity, we denote

$$\tilde{S}_{k-1} = \tilde{S}_{k-1}(\partial B_1) \equiv \frac{1}{2^{k+1}} \binom{n-1}{k-1}.$$

Also we have

$$f(u) = a \frac{e^{-u}}{\int_{B_1} e^{-u} dV}, \quad F(u) = a \frac{1 - e^{-u}}{\int_{B_1} e^{-u} dV}.$$

Combining (25) and (26), we get

$$2a \left( \tilde{S}_{k-1} |\partial B_1| \right)^{1/k} \int_{B_1} [(n-k)u e^{-u} + n(k+1)(e^{-u} - 1)] dV \geq (ka)^{(k+1)/k} \int_{B_1} e^{-u} dV, \quad (27)$$

or

$$-2(\tilde{S}_{k-1} |\partial B_1|)^{1/k} \int_{B_1} (n-k)(-u)e^{-u} + n(k+1) \geq (k^{(k+1)/k} a^{1/k} - 2n(k+1)(\tilde{S}_{k-1} |\partial B_1|)^{1/k}) \int_{B_1} e^{-u} dV.$$

So there is no nontrivial non-positive solution if  $a$  satisfies

$$a \geq \frac{(2n(k+1))^k \omega_{2n-1} \tilde{S}_{k-1}}{k^{k+1}} = \left( \frac{n(k+1)}{k} \right)^k \binom{n}{k} \frac{\pi^n}{n!} =: \alpha_1(k, n). \quad (28)$$

When  $k = n$ , the righthand is equal to  $(n+1)^n \pi^n / n!$  which is sharp.  $\square$

**Remark 6.** When  $k < n$ , we can get better estimate for  $a$ . For this, by (27), consider the function

$$\mu(x) = c_1(e^x - 1) - c_2 x e^x - c_3 e^x.$$

with  $c_1 = n(k+1)$ ,  $c_2 = (n-k)$  and  $c_3 = k^{(k+1)/k} \alpha_2(k, n)^{1/k} \frac{1}{2} (\tilde{S}_{k-1} \omega_{2n-1})^{-1/k}$ . The condition  $\max\{\mu(x); x \geq 0\} = 0$  gives a better upper bound  $\alpha_2(k, n)$  for  $a$ , although it's still not sharp:

$$0 < \alpha_2(k, n) = \alpha_1(k, n) \left[ 1 - \frac{n-k}{n(k+1)} + \frac{n-k}{n(k+1)} \log \frac{n-k}{n(k+1)} \right]^k \leq \alpha_1(k, n).$$



**Remark 7.** If we consider the similar real Hessian equation on  $B_1 \subset \mathbb{R}^d$ :

$$S_k(u_{\alpha\beta}) = \tilde{a} \frac{e^{-u}}{\int_{B_1} e^{-u} dV} \text{ on } B_1, \quad u = 0 \text{ on } \partial B_1, \quad (29)$$

then we can use the real version of above calculation to get the following necessary condition for  $\tilde{a}$  in order for (29) to have a solution in  $C^2(\bar{B}_1) \cap C^4(B_1)$ .

$$\tilde{a} < \tilde{\alpha}(k, d) := \frac{((k+1)d)^k \binom{d-1}{k-1} \omega_{d-1}}{k^{k+1}}.$$

The case when this bound is sharp is when the real dimension is even  $d = 2n$  and  $k = d/2 = n$ . Indeed, we have

**Proposition 3.** When  $k = \frac{d}{2}$ , then there exists a solution in  $C^2(\bar{B}_1) \cap C^4(B_1)$  to (29) if and only if  $\tilde{a} < \tilde{\alpha}(d/2, d)$ .

*Proof.* We just need to show that, for  $k = d/2$ , there exists a radially symmetric solution for (29) when  $a < \tilde{\alpha}(k, d)$ . First it's easy to verify that the radial symmetry reduces the equation (29) to the following equation:

$$\frac{d-2k}{d} \binom{d}{k} (u_r r)^k + \frac{1}{k} \binom{d-1}{k-1} (u_r r)^{k-1} (u_r r)_{,r} r = \frac{\tilde{a}}{\omega_{d-1}} \frac{r^{2k} e^{-u}}{\int_0^1 e^{-u(r)} r^{2n-1} dr}. \quad u = 0 \text{ on } \partial B_1.$$

Now assume  $k = \frac{d}{2} = n$  and we introduce the variable  $s = r^2$ . Then the above equation becomes:

$$((u_s s)^n)_{,s} s = \frac{n^2 \tilde{a}}{\omega_{d-1} \binom{d-1}{n-1} 2^n} \frac{s^n e^{-u}}{\int_0^1 e^{-u(s)} s^{n-1} ds}. \quad (30)$$

This equation is integrable since it's the same as the radial reduction of complex Monge-Ampère equation. See (33), (6) and (7). So it has solution

$$u_\epsilon = (n+1)[\log(|x|^2 + \epsilon^2) - \log(1 + \epsilon^2)],$$

with the parameter

$$\tilde{a}_\epsilon = \frac{1}{n} \binom{d-1}{n-1} 2^{n+1} a_\epsilon = \frac{1}{n} \binom{d-1}{n-1} (2n+2)^n \frac{\omega_{d-1}}{(1+\epsilon^2)^n}.$$

So  $\tilde{a}_\epsilon \in (0, \tilde{\alpha}(d/2, d) = \frac{2}{d} \binom{d-1}{d/2-1} (d+2)^{d/2} \omega_{d-1})$ .  $\square$

From another point of view, in [17], Tian-Wang proved the following Moser-Trudinger inequality for  $k = d/2$ :

$$\int_{\Omega} \exp \left( D \left( \frac{u}{\|u\|_{\Phi_0^k}} \right)^{p_0} \right) \leq C.$$

with

$$\|u\|_{\Phi_0^k} = \left( \int_{\Omega} -u S_k(u_{\alpha\beta}) \right)^{1/(k+1)},$$

$$D = d \left[ \frac{\omega_{d-1}}{k} \binom{d-1}{k-1} \right]^{2/d}, \quad p_0 = \frac{d+2}{d}.$$

If we let  $x = u/\|u\|_{\Phi_0^k}$  and  $y = \|u\|_{\Phi_0^k}$  and use the inequality

$$xy \leq Dx^{p_0} + Ey^{q_0}, \text{ with } q_0 = \frac{d}{2} + 1, E = (Dp_0)^{-q_0/p_0} q_0^{-1} = \left[ (d+2)^{d/2} \frac{\omega_{d-1}}{k} \binom{d-1}{k-1} \frac{d+2}{2} \right]^{-1}.$$

we get the Moser-Trudinger-Onofri inequality:

$$-(E(d/2 + 1))^{-1} \log \left( \int_{\Omega} \exp(-u) dV \right) \leq \frac{1}{k+1} \int_{\Omega} -u S_{d/2}(u_{\alpha\beta}) dV + C.$$

This implies that when  $0 < a < E(k+1)^{-1}$ , there exists a solution to (29). Now note that we indeed have: ( $k=d/2$ )

$$\tilde{\alpha}(d/2, d) = (E(k+1))^{-1} = (d+2)^{d/2} \frac{2}{d} \binom{d-1}{k-1} \omega_{d-1}.$$

## 4 Radially symmetric solutions

### 4.1 Reduction in the radially symmetric case

In this section, we assume  $\Omega = B_R$  and  $u(z) = u(s)$  is radially symmetric, where  $s = r^2 = |z|^2$ . Then we can calculate that

$$u_{i\bar{j}} = u_s \delta_{ij} + u_{ss} \bar{z}^i z^j.$$

By the unitary invariance of operator  $S_k$ , we get

$$\begin{aligned} S_k(u_{l\bar{m}}) &= \binom{n-1}{k} u_s^k + \binom{n-1}{k-1} u_s^{k-1} (u_s + u_{ss}s) \\ &= \frac{1}{k} \binom{n-1}{k-1} (u_s^k s^n)_s s^{1-n}. \end{aligned}$$

So the radially symmetric solution to (2) satisfies the equation:

$$\frac{1}{k} \binom{n-1}{k-1} (u_s^k s^n)_s s^{1-n} = (-u)^p, \quad u(R) = 0. \quad (31)$$

The Hessian energy becomes

$$\mathbb{H}_k = -\frac{1}{k+1} \int_{\Omega} u S_k(u_{l\bar{m}}) dV = \frac{\omega_{2n-1}}{2k(k+1)} \binom{n-1}{k-1} \int_0^R u_s^{k+1} s^n ds.$$

so the functional whose Euler-Lagrange equation is (31) becomes

$$\mathbb{F}_k = \frac{A}{k+1} \int_0^R |u_s|^{k+1} s^n ds - \frac{B}{p+1} \int_0^R |u|^{p+1} s^{n-1} ds$$

where

$$A = A(k, n) = \frac{\omega_{2n-1}}{2k} \binom{n-1}{k-1}, \quad B = B(k, n) = \frac{\omega_{2n-1}}{2}. \quad (32)$$

As in [19], denote  $\mathcal{E} = \{u \in C^1([0, R]); u(R) = 0\}$ . For any  $1 \leq k \leq n$  and  $0 < \delta < \gamma(k, n) = \frac{(n+1)k}{n-k}$ , and let  $\mathcal{W}_k$  be the completion of  $\mathcal{E}$  under the norm

$$\|u\| = \left( \int_0^R u_s^{k+1} s^n ds \right)^{1/(k+1)}.$$

**Lemma 3.** *There exists a constant  $C = C(\delta, k, R, n)$  such that, for all  $u \in E$ ,*

$$\left( \int_0^R |u|^{\delta+1} s^{n-1} ds \right)^{1/(\delta+1)} \leq C \left( \int_0^R |u_s|^{k+1} s^n ds \right)^{1/(k+1)}.$$

*Proof.* By applying Hölder's inequality to  $u(s) = \int_R^s u_s(s) ds$ , we have

$$|u(s)| \leq C s^{-(n-k)/(k+1)} \left( \int_0^R |u_s|^{k+1} s^n \right)^{1/(k+1)}.$$

Then raising the  $(\delta + 1)$ -th power, multiplying  $s^{n-1}$  and integrating from 0 to  $R$  we get the inequality. The range for  $\delta$  is determined by the inequality:

$$-\frac{n-k}{k+1}(\delta+1) + n - 1 > -1.$$

□

**Remark 8.** By [5], when  $k < n$ , we actually have the sharp Sobolev inequalities of complex Hessian operator for radial functions,

$$\left( \int_0^R |u|^{\gamma(k,n)+1} s^{n-1} \right)^{1/(\gamma(k,n)+1)} \leq C \left( \int_0^R |u_s|^{k+1} s^n \right)^{1/(k+1)}.$$

Since we don't have symmetrization process as in the real case, the sharp Sobolev inequalities for general  $k$ -plurisubharmonic functions are still open ([21]).

As in [19], we define the notion of weak solution. We use the constants in (32).

**Definition 1.** We say  $u \in \mathcal{W}_k$  is a weak solution to equation (31), if for every  $\phi \in C^1([0, R])$  with  $\phi(R) = 0$ , the following identity is satisfied.

$$A \int_0^R |u_s|^k u_s \phi'(s) s^n ds = B \int_0^R |u|^p \phi(s) s^{n-1} ds.$$

Arguing as in [19, Lemma 4], we get the following regularity result which reduces the problem to finding critical point of  $\mathbb{F}_k$  on  $\mathcal{W}_k$ .

**Lemma 4** ([19]). Any generalized solution of (31) is in  $C^2([0, R])$ , and solves (31) in the classical sense. Moreover, it is negative in  $[0, R)$  unless it vanishes identically.

*Proof of the existence part of Theorem 2.* When  $p < k$ , we are in the sub-linear (with respect to complex  $k$ -Hessian operator) case, by the Sobolev inequality, we have we have

$$\int_0^R |u|^{p+1} s^{n-1} ds \leq C(p) \left( \int_0^R |u_s|^{k+1} s^n ds \right)^{(p+1)/(k+1)} \leq \epsilon \int_0^R |u_s|^{k+1} s^n ds + C(\epsilon, p).$$

Then by taking  $\epsilon$  sufficiently small, we get

$$\mathbb{F}_k \geq \epsilon \int_0^R |u_s|^{k+1} s^n ds - C(\epsilon, p).$$

So the functional  $\mathbb{F}_k$  is a coercive functional on  $\mathcal{W}_k$  and one can use the direct method in variational calculus to find an absolute minimizer. On the other hand, it's easy to see that

$$\mathbb{F}_k(tu) = O(t^{k+1}) - O(t^{p+1}) < 0, \text{ if } t \ll 1.$$

So the absolute minimizer is not 0.

In the super-linear case, i.e. when  $k < p < \gamma(k)$ , we have

1.  $\mathbb{F}_k(0) = 0$ , and  $\mathbb{F}_k(tu) = O(t^{k+1}) - O(t^{p+1}) \rightarrow -\infty$  as  $t \rightarrow +\infty$ .

2. Choose  $\alpha$  to be sufficiently small. Then when  $\|u\| = \alpha$ , we have

$$\mathbb{F}_k(u) \geq \|u\| - C(p)\|u\|^{(p+1)/(k+1)} = \|u\| \left(1 - C(p)\|u\|^{\frac{p-k}{k+1}}\right) = \alpha \left(1 - C(p)\alpha^{\frac{p-k}{k+1}}\right) > 0.$$

So  $\mathbb{F}_k$  satisfies the Montain Pass condition. Now as in the semi-linear case, it's known that under the assumption,  $\mathbb{F}_k$  is in  $C^1(\mathcal{W}_k, \mathbb{R})$  and satisfies the Palais-Smale condition. So the minimax method proves the existence of critical point of  $\mathbb{F}_k$  on  $\mathcal{W}_k$ . For details, see [15].  $\square$

## 4.2 Nonlocal problem with exponential nonlinearity

Denote  $s = |z|^2$ . Assume  $u = u(s)$  is any radial symmetric solution of (3). Then by (31), we see that (3) is reduced to the following equation for  $u$ :

$$(u_s^k s^n)_s s^{1-n} = \lambda e^{-u}, \quad \lambda = \frac{2k}{\binom{n-1}{k-1} \omega_{2n-1}} \frac{a}{\int_0^1 e^{-u(s)} s^{n-1} ds} = A(k, n)^{-1} \frac{a}{\int_0^1 e^{-u(s)} s^{n-1} ds}. \quad (33)$$

We use the phase plane method to study this equation. Define

$$v = \left(\frac{1}{k} u_s s\right)^k, \quad w = \lambda k^{-k} s^k e^{-u}.$$

Introduce a new variable  $t = \log s$ . Then it's easy to verify that (33) is equivalent to the following system of equations:

$$v_t = -(n-k)v + w, \quad w_t = kw(1 - v^{1/k}). \quad (34)$$

For the boundary condition, when  $t = -\infty$ , or equivalently  $s = 0$ .

$$v(t = -\infty) = 0 = w(t = -\infty).$$

To find the boundary condition when  $t = 0$ , or equivalently  $s = 1$ , we note that

$$\int_{B_{\sqrt{s}}} \det(u_{l\bar{m}}) dV = \frac{1}{k} \binom{n-1}{k-1} \frac{\omega_{2n-1}}{2} \int_0^s (u_s^k s^n)_s ds = A(k, n) u_s^k s^n.$$

So

$$v(t = 0) = k^{-k} A(k, n)^{-1} \int_{B_1} \det(u_{l\bar{m}}) dV = k^{-k} A(k, n)^{-1} a.$$

while  $w(t = 0) = \lambda k^{-k}$ . So we are looking for the trajectory from  $(0, 0)$  to the point  $(k^{-k} A(k, n)^{-1} a, \lambda k^{-k})$ , as  $t$  changes from  $-\infty$  to 0. The critical point of system (34) is  $(1, (n-k))$ . The Hessian matrix is

$$\begin{pmatrix} -(n-k) & 1 \\ -wv^{(1-k)/k} & k(1 - v^{1/k}) \end{pmatrix} \Big|_{(1, (n-k))} = \begin{pmatrix} k-n & 1 \\ -(n-k) & 0 \end{pmatrix}.$$

whose trace and determinant are

$$\text{tr} = k - n, \quad \det = n - k.$$

So the two eigenvalue is

$$\beta_1 = \frac{k-n + \sqrt{(n-k)^2 - 4(n-k)}}{2}, \quad \beta_2 = \frac{k-n - \sqrt{(n-k)^2 - 4(n-k)}}{2}.$$

There are two complex eigenvalue with negative real part if and only if

$$0 < n - k < 4.$$

Now we can prove Theorem 4 using similar analysis as in [3] (see also [11] and [10]).

*Proof of Theorem 4.* When  $n = k$ , the equation is integrable.  $u = (n + 1)[\log(s + \epsilon^2) - \log(1 + \epsilon^2)]$ .

$$v(s) = \left(\frac{1}{n}u_s s\right)^k = \left(\frac{n+1}{n}\right)^n \left(\frac{s}{s+\epsilon^2}\right)^n, \quad w(s) = \frac{(n+1)^n}{n^{n-1}} \frac{\epsilon^2 s^n}{(s+\epsilon^2)^{n+1}}.$$

So there is a trajectory  $\mathcal{O}$  connecting  $(0, 0)$  to the point  $\left(\left(\frac{n+1}{n}\right)^n, 0\right)$  and  $a_\epsilon = n^n v(t=0)A(n, n) = (n+1)^n \frac{\pi^n}{n!(1+\epsilon^2)^n}$  lies in  $(0, a_0 = (n+1)^n \frac{\pi^n}{n!})$ .

When  $k < n$ , consider the function defined by

$$L(v, w) = k \left( \frac{k}{k+1} v^{(k+1)/k} - v + \frac{1}{k+1} \right) + (w - (n-k)) - (n-k) \log \frac{w}{(n-k)}.$$

Then it's easy to verify that  $L(1, n-k) = 0$  and  $L(v, w) > 0$  for  $\mathbb{R}_+^2 \ni (v, w) \neq (1, n-k)$ . Moreover, if  $(v(t), w(t))$  is a trajectory for the system (34), then

$$\frac{d}{dt} L(v(t), w(t)) = -(n-k)k(v^{1/k} - 1)(v-1) \leq 0, \text{ and } < 0 \text{ when } v \neq 1.$$

So  $L(v, w)$  is a Lyapunov function for the system (34). So we conclude that the basin of attraction of  $(1, n-k)$  contains the whole positive quadrant. The solution to (33) corresponds to a trajectory  $\tilde{\mathcal{O}}$  connecting  $(0, 0)$  to  $(v(t=0), w(t=0))$ .

1. When  $n - k < 4$ ,  $\text{Im}(\beta_{1,2}) \neq 0$  and  $\text{Re}(\beta_{1,2}) < 0$ . There is a trajectory  $\mathcal{O}$  connecting  $(0, 0)$  and  $(1, n-k)$ , which turns around  $(1, n-k)$  infinitely many times. In particular, the line  $v = 1$  intersects with  $\mathcal{O}$  at infinitely many points. This behavior of  $\mathcal{O}$  clearly implies part 2 of Theorem 4.
2. When  $n - k \geq 4$ , we consider the region  $\mathcal{D}$  bounded by the curves  $\mathcal{C} = \{w = (n-k)v^b\}$  and  $w = (n-k)v$ .

**Claim:** When  $(-\beta_2)^{-1} \leq b \leq (-\beta_1)^{-1}$ , the region is invariant under the system (34).

**Proof of the claim:** We just need to show the vector field on the boundary of the region points to the interior of the region. For the boundary  $w = (n-k)v$  this is clear since the vector field has direction  $\langle 0, 1 \rangle$ . For the boundary  $w = (n-k)v^b$ , we parametrize it by  $\{v = \tau, w = (n-k)\tau^b; 0 \leq \tau \leq 1\}$ . For  $0 < \tau < 1$ , the vector field points to the interior if and only if

$$\begin{aligned} \frac{k w (1 - v^{1/k})}{-(n-k)v + w} &= \frac{k(n-k)t^b(1 - t^{1/k})}{-(n-k)\tau + (n-k)\tau^b} < b(n-k)\tau^{b-1} \\ \iff h(\tau) &:= k(1 - \tau^{1/k}) - b(n-k)(\tau^{b-1} - 1) < 0. \end{aligned}$$

$$h(0) = -\infty, h(1) = 0, h'(\tau) = \tau^{b-2}((n-k)b(1-b) - \tau^{\frac{1}{k}+1-b}).$$

So if  $h(\tau)$  is increasing, i.e.  $h'(\tau) > 0$  when  $\tau \in (0, 1)$ , then (35) holds. Now  $h'(\tau) > h'(1) = (n-k)b(1-b) - 1$ . It's easy to see that

$$h'(1) \geq 0 \iff (-\beta_2)^{-1} \leq b \leq (-\beta_1)^{-1}.$$

So we can just choose the curve  $\mathcal{C} = \{w = (n-k)v^{-1/\beta_1}\}$ . Now it's easy to see that  $\mathcal{O}$  lies in the region  $\mathcal{D}$ . Since  $\mathcal{D}$  is above the curve  $w = (n-k)v$ , so  $v'(t) \geq 0$  along  $\mathcal{O}$ . This implies for any  $0 < v(t=0) \leq 1$ , or equivalently, when  $0 < a \leq k^k A(k, n) = k^{k-1} \binom{n-1}{k-1} \frac{\pi^n}{(n-1)!} = \beta(k, n)$ , there exists a unique solution to (33). □

In figure 1, we give phase diagrams in three cases of the above proof when  $n = 6$ .

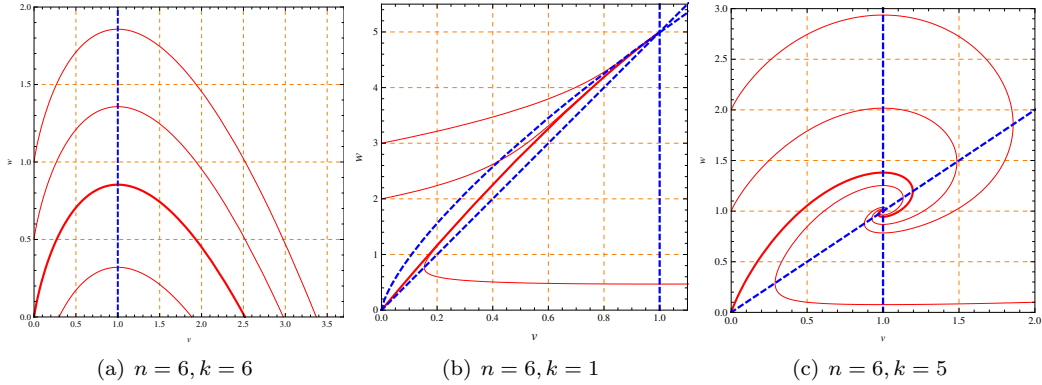


Figure 1: Phase diagrams for system (34) when  $n = 6$

**Remark 9.** In the case where  $n - k \geq 4$ , define the line  $\mathcal{L}$  to be one characteristic line of the system:  $(n - k)(v - 1) + \beta_1(w - (n - k)) = 0$ . Note that the curve  $\mathcal{C} = \{w = (n - k)v^{-1/\beta_1}\}$  is tangent to  $\mathcal{L}$  at point  $(1, (n - k))$ . In [3], the region was chosen to be a triangle bounded by  $\mathcal{L}$ ,  $v = 0$  and  $w = (n - k)v$ . But one can verify that, for some choices of  $(n, k)$  for complex Hessian equation this triangle is not invariant under the flow. So it's more natural to consider the above invariant region  $\mathcal{D}$  when one deals with general Hessian case.

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