ON THE PROPER MODULI SPACES OF SMOOTHABLE KÄHLER–EINSTEIN FANO VARIETIES

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Abstract

In this paper we investigate the geometry of the orbit space of the closure of the subscheme parameterizing smooth Kähler–Einstein Fano manifolds inside an appropriate Hilbert scheme. In particular, we prove that being K-semistable is a Zariski-open condition, and we establish the uniqueness of the Gromov–Hausdorff limit for a punctured flat family of Kähler–Einstein Fano manifolds. Based on these, we construct a proper scheme parameterizing the S-equivalent classes of $\mathbb{Q}$*-Gorenstein smoothable, K-semistable* $\mathbb{Q}$*-Fano varieties, and we verify various necessary properties to guarantee that it is a* good *moduli space.*

Contents

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1. Introduction

Constructing moduli spaces for higher-dimensional algebraic varieties is a fundamental problem in algebraic geometry. For the dimension 1 case, the moduli space parameterizing stable Deligne–Mumford curves was constructed via various kinds of methods, for example, by geometric theory (GIT), Teichmüller quotient space modulo the mapping class group action, and so on. For higher-dimensional cases, one of the natural classes to consider is that of all canonically polarized manifolds, for which the GIT machinery has been quite successful (see, e.g., [3], [13], [54], [56]). However, because the GIT method in its classical form fails to produce a geometrically natural compactification for these moduli spaces (see [55]), researchers have had to develop suitable substitutes. In fact, it has taken quite some time for people to realize what kind of varieties should be included in order to construct a *proper* moduli space (see [30]). Thanks to a recent breakthrough in the theory underlying the minimal model program (see, e.g., [8]), it has become possible to obtain a rather satisfactory theory on proper projective moduli spaces parameterizing *KSBA-stable varieties*, or *KSBA theory* (derived from *Kollár*, *Shepherd-Barron*, *and Alexeev*; see [27] for a concise survey of this theory). We also remark that it was only realized later that this compactification should coincide with the compactification from Kähler–Einstein metric/K-stability (see, e.g., [6], [36], [55]).

As for Fano varieties, the story is much more subtle. Apart from some local properties, for example, having only *Kawamata log terminal* (*klt*) singularities when a Fano variety is assumed to be K-semistable (see [36]) and admitting klt Fano degenerations as long as the general fiber is a klt Fano variety in a 1-parameter family (see [34]), it is still not clear what kind of general Fano varieties we should parameterize in order to obtain a nicely behaved moduli space, especially if we aim to find a compact Hausdorff space, or how to construct it. The recent breakthrough in the Kähler–Einstein problem—namely, the solution to the Yau–Tian–Donaldson conjecture (see, e.g., [9]–[11], [53])—is a major step forward, especially for understanding those Fano manifolds with Kähler–Einstein metrics. Furthermore, it implies that the right limits of smooth Kähler–Einstein manifolds form a bounded family. In this paper, we aim to use the analytic results established by these authors to investigate the geometry of the compact space of orbits, which is the closure of the space parameterizing smooth Fano varieties.

1.1. Main results

Our first main result of this paper is the following.

THEOREM 1.1

Let $\mathcal{X} \to C$ be a flat family of projective varieties over a pointed smooth curve $(C, 0)$ with $0 \in C$. Suppose that

(1) *$K_{\mathcal{X}}$ is $\mathbb{Q}$-Cartier and $-K_{\mathcal{X}/C}$ is relatively ample over C;*
(2) *for any $t \in C^\circ := C \setminus \{0\}$, $\mathcal{X}_t$ is smooth, and $\mathcal{X}_0$ is klt;*
(3) *$\mathcal{X}_0$ is K-polystable.*

Then

(i) *there is a Zariski-open neighborhood U of $0 \in C$ on which $\mathcal{X}_t$ is K-semistable for all $t \in U$, and K-stable if we further assume that $\mathcal{X}_0$ has a discrete automorphism group;*
(ii) *for any other flat projective family $\mathcal{X}' \to C$ satisfying (1)–(3) as above and*
$$\mathcal{X}' \times_C C^\circ \cong \mathcal{X} \times_C C^\circ,$$
we can conclude that $\mathcal{X}'_0 \cong \mathcal{X}_0$;
(iii) *$\mathcal{X}_0$ admits a weak Kähler–Einstein metric $\omega_{\mathrm{KE}}(\mathcal{X}_0)$, and moreover, if we further assume that $\mathcal{X}_t$ is K-polystable for all $t \in C^\circ$, then $(\mathcal{X}_0, \omega_{\mathrm{KE}}(\mathcal{X}_0))$ is the Gromov–Hausdorff limit of a family $\{(\mathcal{X}_t, \omega_{\mathrm{KE}}(\mathcal{X}_t))\}_{t \in C^\circ}$ as $t \to 0$, where $\omega_{\mathrm{KE}}(\mathcal{X}_t)$ is a Kähler–Einstein metric on $\mathcal{X}_t$ for each $t \in C^\circ$.*

If both $\mathcal{X}_0$ and $\mathcal{X}'_0$ are assumed to be *smooth* Kähler–Einstein manifolds, then part of Theorem 1.1 is a consequence of Székelyhidi's work in [48], where the more general case for arbitrary polarization is established. When the fiber is of dimension 2, this is also implied by the work of [39] and [50], in which explicit compactifications of Kähler–Einstein del Pezzo surfaces are constructed. We remark that Zariski openness has already been established in [18] and [38] when the fibers are Kähler–Einstein Fano *manifolds* with *discrete* automorphism group. Finally, we note that an independent work [47] obtains similar results along this line (see Remark 1.4).

Now let us give a brief description of our approach to Theorem 1.1. First, we note that, although part of our theorem is stated in algebro-geometric terms, the proof indeed relies heavily on known analytic results, which is especially true in some recent work (see, e.g., [10], [11], [53]). On the other hand, we note that no further analytic tools are developed beyond their work in the present paper. So our argument is actually more algebro-geometric in nature.

The first main tool for us is a continuity method very similar to the one proposed by Donaldson in [16]. Indeed, by throwing in an auxiliary divisor $\mathcal{D} \in |-mK_{\mathcal{X}}|$, we consider the following log extension of Theorem 1.1.

THEOREM 1.2

For a fixed $\beta \in [0, 1]$, let $\mathcal{X} \to C$ be a flat family over a pointed smooth curve $(C, 0)$ with a relative codimension 1 cycle $\mathcal{D}$ over C. Suppose that

(1) *$-K_{\mathcal{X}/C}$ is ample and $\mathcal{D} \sim_C -mK_{\mathcal{X}/C}$ for some positive integer $m > 1$;*
(2) *for any $t \in C^\circ := C \setminus \{0\}$, $\mathcal{X}_t$ and $\mathcal{D}_t$ are smooth, $(\mathcal{X}_0, \frac{1}{m}\mathcal{D}_0)$ is klt;*
(3) *$(\mathcal{X}_0, \mathcal{D}_0)$ is β-K-polystable (see Definition 2.3).*

Then

(i) *there is a Zariski neighborhood U of $0 \in C$ on which $(\mathcal{X}_t, \mathcal{D}_t)$ is β-K-semistable for all $t \in U$;*

(ii) *for any other flat projective family $(\mathcal{X}', \mathcal{D}') \to C$ with a relative codimension 1 cycle $\mathcal{D}'$ satisfying (1)–(3) as above and*

$$(\mathcal{X}', \mathcal{D}') \times_C C^\circ \cong (\mathcal{X}, \mathcal{D}) \times_C C^\circ,$$

we can conclude that $(\mathcal{X}'_0, \mathcal{D}'_0) \cong (\mathcal{X}_0, \mathcal{D}_0)$;

(iii) *$(\mathcal{X}_0, \mathcal{D}_0)$ admits a weak conical Kähler–Einstein metric with cone angle $2\pi(1-(1-\beta)/m)$ along $\mathcal{D}_0$, which is the Gromov–Hausdorff limit of $(\mathcal{X}_{t_i}, \mathcal{D}_{t_i})$ endowed with the conical Kähler–Einstein metric with cone angle $2\pi(1-(1-\beta_i)/m)$ along $\mathcal{D}_{t_i} \subset \mathcal{X}_{t_i}$ for any sequence $t_i \to 0$ and $\beta_i \nearrow \beta$.*

To prove Theorem 1.2, we begin by noting that the uniqueness is well understood when the angle is small. We give an account of this fact using completely algebro-geometric means. To be precise, we use the result that the set of log canonical thresholds satisfies the *ascending chain condition (ACC)* (see [23]) to show that when the angle β is smaller than a positive number $\beta_0 > 0$, there is only one extension with at worst klt singularities. Fix ϵ such that $0 < \epsilon < \beta_0$. We define a set $\mathbf{B} \subset [\epsilon, 1]$ (see Section 6 for the precise definition) for which the conclusions of Theorem 1.2 hold for the angles belonging to the set $\mathbf{B}$. The result in the small-angle case implies that $\mathbf{B} \supset [\epsilon, \beta_0]$.

Now to prove Theorem 1.1, let us first assume that all the nearby fibers $\mathcal{X}_t$ are K-semistable. Then it suffices to show that $\mathbf{B}$ is open and closed in $[\epsilon, 1)$. We establish closedness and openess using two facts. First we prove a simple but very useful fact (see Lemma 3.1), which says that for a point p on the *limiting orbit* with *reductive stabilizer*, there is a Zariski-open neighborhood $p \in U$ such that the closure of the $\mathrm{SL}(N+1)$-orbit of any point in the limiting orbit near p actually contains $g \cdot p$ for some $g \in \mathrm{SL}(N+1)$. In particular, it guarantees that there are no nearby nonequivalent K-polystable points on the limiting orbit. With this, using a crucial *intermediate value theorem* type of results (see Lemma 6.9), we show that if there is a different limit, which a priori could be far away from the given central fiber in the parameterizing Hilbert scheme, then we can indeed always find another limit which either specializes to (X_0, D_0) in a test configuration or becomes the central fiber of a test configuration of (X_0, D_0), violating the K-stability assumption. Similarly, this argument can also be applied to study the case when $\beta \nearrow 1$.

To finish the proof, we need to verify the assumption that all the nearby fibers $\mathcal{X}_t$ are *K-semistable*. For this, one needs two observations. First, it follows from the work of [10], [11], and [53] that to check for K-semistability of $\mathcal{X}_t$, $t \neq 0$, it suffices to test for all 1*-parameter subgroup (*1*-PS)* degenerations in a fixed $\mathbb{P}^{\mathbb{N}}$. Second, it follows from a straightforward GIT argument that the *K-semistable threshold* (*kst*) (see Section 7.2) is a constructible function. So what remains to show is that it is also lower semicontinuous (which is also observed in [47]), but this is a consequence of the upper semicontinuity of the dimension of the automorphism groups and the continuity method deployed in the proof of Theorem 1.2. With all this knowledge in hand, we are able to achieve our main goal here—that is, to construct a proper (or *good*) moduli space for all $\mathbb{Q}$-Gorenstein smoothable, K-semistable Fano varieties.

THEOREM 1.3
For $N \gg 0$, let Z^* be the seminormalization of the locus inside $\mathrm{Hilb}_\chi(\mathbb{P}^N)$ *parameterizing all $\mathbb{Q}$-Gorenstein smoothable, K-semistable Fano varieties in $\mathbb{P}^N$ with fixed Hilbert polynomial χ (see Section 8 for the precise definition of Z^*). Then the algebraic stack* $[Z^*/\mathrm{SL}(N+1)]$ *admits a proper seminormal* scheme $\mathcal{KF}_N$ *as its* good moduli space (in the sense of [1]). Furthermore, for sufficiently large N, $\mathcal{KF}_N$ does not depend on N.

Recall from [1, Section 1.2] that a quasicompact morphism $\phi : \mathcal{Z} \longrightarrow M$ from an Artin stack $\mathcal{Z}$ to an algebraic space M is a *good moduli space* if
(1) the pushforward functor on quasicoherent sheaves is exact, and
(2) the induced morphism on sheaves $\mathcal{O}_M \to \phi_*\mathcal{O}_{\mathcal{Z}}$ is an isomorphism.
This concept is a generalization of *good quotient* in classical GIT. In more concrete terms, Theorem 1.3 says that each $\mathrm{SL}(N+1)$-orbit inside Z^* corresponds to a $\mathbb{Q}$-Gorenstein smoothable, K-semistable $\mathbb{Q}$-Fano variety, and Z^* admits a categorical quotient $\mathcal{KF}_N$, whose points correspond to the S-equivalence (i.e., the equivalence relation generated by the orbital closure inclusion) classes of $\mathrm{SL}(N+1)$-action on Z^*. In particular, the set of $\mathbb{C}$-points in $\mathcal{KF}_N$ precisely corresponds to set of *closed minimal* $\mathrm{SL}(N+1)$-orbits in Z^*, that is, the set of $\mathbb{Q}$-Gorenstein smoothable, K-polystable $\mathbb{Q}$-Fano varieties over $\mathbb{C}$.

The existence of a moduli space for Kähler–Einstein Fano manifolds was anticipated by [50]. A local quotient picture was suggested in [15, Section 5.3] and [48], and was explicitly conjectured in [46, Sections 1.3, 1.4] and [39, Conjecture 6.2]. Furthermore, the moduli space is speculated to be projective by the existence of the descending CM (Chow–Mumford) line bundle (see, e.g., [39], [42]). We also note that smooth Kähler–Einstein Fano manifolds with discrete automorphism group (which are known to be asymptotically Chow stable by [13]) admit (possibly nonproper) algebraic moduli spaces thanks to the work of [18] and [38].

Now let us explain our approach to Theorem 1.3. Due to the lack of a *global GIT* interpretation of the K-stability, our strategy is to replace GIT by the work of [2]. So to obtain a good quotient, one needs to verify all the assumptions of [2, Theorem 1.2]. In particular, among other things one needs to establish the following two key properties:

(1) the stabilizer-preserving condition for the local presentation of the moduli stack;

(2) the affineness of the quotient morphism.

Intuitively, the first property implies that the local Zariski-open charts of the moduli space can be glued together, while the second property guarantees that the local charts constructed above are actually *affine*. Moreover, the second property guarantees the *goodness* of the quotient $[Z^*/\mathrm{SL}(N+1)] \to \mathcal{KF}_N$, and as a consequence the restriction of the CM line bundle to $Z^* \subset \mathrm{Hilb}$ descends to the good moduli space. This was crucial in our recent study of the projectivity of the moduli space $\mathcal{KF}$ in [32]. We singled out these two properties that depend in an essential way on the existence of a *global proper* topological (equipped with Gromov–Hausdorff topology) moduli space.

We remark that both properties follow from Luna's famous étale slice theorem for a reductive group G-acting on an *affine* variety Z; that is, if $z \in Z$ and the G-orbit $G \cdot z \subset Z$ is *closed*, then there is a nice slice containing z satisfying the above two properties. Unfortunately, we are unable to verify the *affineness* assumption of Luna's theorem since there is no global GIT interpretation of K-stability, but the closedness of $G \cdot x$ in an affine variety will be a *consequence* of our proof instead, which is based on the existence of a nice *continuous proper slice* (although nonalgebraic) lying over the stack. The slice is obtained via a family version of Tian's embeddings of Kähler–Einstein Fano varieties and its properness follows from Theorem 1.1. The slice can be regarded as an alternative to the zero set of the moment map in the classical Kempf–Ness–Kirwan picture.

Finally, we close this Introduction by outlining the plan of this article. In Section 2, we give some basic definitions. In Section 3, we review some facts on the linear action of a reductive group on a projective space. In Section 4, we list the main analytic results we need in this article. First we recall the recent results appearing in [10], [11], and [53]. Then we also state the Gromov–Hausdorff continuity for conical Kähler–Einstein metrics on a smooth family of Fano pairs (see [10], [11], [53]). In Section 5, we prove that when the angle is small enough, the filling is always unique. In Section 6, we establish the main technical tool of our argument—a continuity theorem. We remark that, with it, we can already show Theorem 1.2 under the assumption that the nearby fibers are all β-K-polystable. In Section 7, we prove the K-semistability of the nearby points by applying the continuity method. First, in Section 7.1 we prove Theorem 7.2, which says that any orbit closure of a K-semistable Fano manifold contains only one isomorphic class of K-polystable $\mathbb{Q}$-Fano variety (this is an extension of the result of [12] for the Fano case). In Section 7.2, we show that a smoothing of a K-semistable $\mathbb{Q}$-Fano variety is always K-semistable. In Section 7.3, by putting all the results together, we finish the proofs of Theorems 1.1 and 1.2. In Section 8, we apply our results and prove a Luna slice-type theorem for K-stability, which is used to establish Theorem 1.3. In the Appendix, we discuss several technical results that are needed on the general theory of linear action of a reductive group on projective space.

Remark 1.4 (Remarks on the history)
The present article was originally titled “Degeneration of Fano Kähler–Einstein manifolds” (see [33]). In the first version, we established the separateness of the moduli space and proved the uniqueness of K-polystable degeneration for K-semistable Fano manifolds. After it was posted on the arXiv, the authors of [47] notified us that they had independently investigated similar questions with a circle of parallel ideas (but in a more analytic fashion) and obtained closely related results. In particular, in [47], the authors *first* obtained: the existence of weak Kähler–Einstein metrics on $\mathbb{Q}$-Gorenstein smoothable, K-polystable $\mathbb{Q}$-Fano varieties; the *analytic* openness of K-stability under the assumption of *finite* automorphism group; and the lower semicontinuity of the cone angle for conical Kähler–Einstein metrics. Those statements were not included in the first version of our preprint. As a consequence, the uniqueness of K-stable filling with *finite* automorphism group was also independently obtained in [47]. After the appearance of [47] on the arXiv, we realized that the approach in the first version of our paper could be naturally extended, which we have done in the current version. After we posted the second version of our paper on the arXiv, we were contacted by Odaka, who claimed (see [37]) to have independently obtained some parts of Theorem 1.3 based on the work of [33] and [47].

2. Preliminaries

In this section, we establish the conventions used in our paper. The definitions of K-stability (resp., β-K-stability) given below are recalled from [14] and [51] (resp., [16]). We also refer the reader to the lecture notes [43] and [49] for both an analytic and an algebro-geometric point of view.

Definition 2.1
Let $(X, D; L)$ be an n-dimensional projective variety polarized by an ample line bundle L together with an effective divisor $D \subset X$. A *log test configuration* of $(X, D; L)$ consists of the following:

(1) a projective flat morphism $\pi : (\mathcal{X}, \mathcal{D}; \mathcal{L}) \to \mathbb{A}^1$ and an effective divisor $\mathcal{D}$ on $\mathcal{X}$ such that $\mathrm{Supp}(\mathcal{D})$ does not contain any component of the central fiber $\mathcal{X}_0$;

(2) a $\mathbb{G}_m$-action on $(\mathcal{X}, \mathcal{D}; \mathcal{L})$, such that π is $\mathbb{G}_m$-equivariant with respect to the standard $\mathbb{G}_m$-action on $\mathbb{A}^1$ via multiplication;

(3) $\mathcal{L}$ is relatively ample, and we have a $\mathbb{G}_m$-equivariant isomorphism

$$(\mathcal{X}^\circ, \mathcal{D}^\circ; \mathcal{L}|_{\mathcal{X}^\circ}) \cong (X \times \mathbb{G}_m, D \times \mathbb{G}_m; \pi_X^* L), \tag{1}$$

where $(\mathcal{X}^\circ, \mathcal{D}^\circ) = (\mathcal{X}, \mathcal{D}) \times_{\mathbb{A}^1} \mathbb{G}_m$ and $\pi_X : X \times \mathbb{G}_m \to X$.

A log test configuration is called a *product test configuration* if $(\mathcal{X}, \mathcal{D}; \mathcal{L}) \cong (X \times \mathbb{A}^1, D \times \mathbb{A}^1; \pi_X^* L)$ where $\pi_X : X \times \mathbb{A}^1 \to X$, and a *trivial test configuration* if $\pi : (\mathcal{X}, \mathcal{D}; \mathcal{L}) \to \mathbb{A}^1$ is a product test configuration with $\mathbb{G}_m$ acting trivially on X.

Assume X to be normal, and let χ denote the Hilbert polynomial. We introduce $a_i, \tilde{a}_i, b_i, \tilde{b}_i \in \mathbb{Q}$ via the following expansions:

- $\chi(X, L^{\otimes k}) := \dim H^0(X, L^k) = a_0 k^n + a_1 k^{n-1} + O(k^{n-2})$;
- $\chi(D, (L|_D)^{\otimes k}) := \dim H^0(D, L^k|_D) = \tilde{a}_0 k^{n-1} + O(k^{n-2})$;
- $w(k) :=$ weight of $\mathbb{G}_m$-action on $\wedge^{\mathrm{top}} H^0(\mathcal{X}_0, \mathcal{L}^{\otimes k}|_{\mathcal{X}_0}) = b_0 k^{n+1} + b_1 k^n + O(k^{n-1})$;
- $\tilde{w}(k) :=$ weight of $\mathbb{G}_m$-action on $\wedge^{\mathrm{top}} H^0(\mathcal{D}_0, \mathcal{L}^{\otimes k}|_{\mathcal{D}_0}) = \tilde{b}_0 k^n + O(k^{n-1})$.

In this article, we will focus on the projective pairs (X, D) introduced in Definition 2.1 satisfying that the divisor D is prime and $(X, \frac{1}{m} D)$ is a projective pair with klt singularities (see [29, Definition 2.34]) for a given positive integer m.

Definition 2.2

Let (X, D) be a projective pair with klt singularities. Then (X, D) to said to be a *log Fano* pair if $-(K_X + D)$ is an ample $\mathbb{Q}$-Cartier divisor, and a $\mathbb{Q}$-*Fano variety* if $D = 0$.

Now we are ready to state the algebro-geometric criterion for the existence of conical Kähler-Einstein metrics on a log Fano manifold (X, D) with cone angle $2\pi(1 - (1-\beta)/m)$ along the divisor $D \in |-mK_X|$.

Definition 2.3

For a $\mathbb{Q}$-Fano variety X with $D \in |-mK_X|$ and a real number $\beta \in [0, 1]$, we define the *log generalized Futaki invariant with the angle* β as

$$\mathrm{DF}_{1-\beta}(\mathcal{X}, \mathcal{D}; \mathcal{L}) = \mathrm{DF}(\mathcal{X}; \mathcal{L}) + (1-\beta) \cdot \mathrm{CH}(\mathcal{X}, \mathcal{D}; \mathcal{L})$$

with (see [31, Definition 3.3])

$$\mathrm{DF}(\mathcal{X}; \mathcal{L}) := \frac{a_1 b_0 - a_0 b_1}{a_0^2} \qquad \text{and} \qquad \mathrm{CH}(\mathcal{X}, \mathcal{D}; \mathcal{L}) := \frac{1}{m} \cdot \frac{a_0 \tilde{b}_0 - b_0 \tilde{a}_0}{2a_0^2}.$$

Then

$$\mathrm{DF}_{1-\beta}(\mathcal{X},\mathcal{D};\mathcal{L}^{\otimes r}) = \mathrm{DF}_{1-\beta}(\mathcal{X},\mathcal{D};\mathcal{L}).$$

We say that $(X, D; L)$ is *β-K-semistable* if $\mathrm{DF}_{1-\beta}(\mathcal{X},\mathcal{D};\mathcal{L}) \geq 0$ for any normal test configuration $(\mathcal{X},\mathcal{D};\mathcal{L})$, and *$\beta$-K-polystable* (resp., β-K-stable) if it is β-K-semistable with $\mathrm{DF}_{1-\beta}(\mathcal{X},\mathcal{D};\mathcal{L}) = 0$ if and only if $(\mathcal{X},\mathcal{D};\mathcal{L})$ is a product test configuration (resp., trivial test configuration).

Thanks to the linear dependence of $\mathrm{DF}_{1-\beta}(\mathcal{X},\mathcal{D};\mathcal{L})$ on β, we immediately obtain the following interpolation property.

LEMMA 2.4
If $(X, D; L)$ is both β_1-K-semistable and β_2-K-polystable with $\beta_1 < \beta_2$ (resp., $\beta_2 < \beta_1$), then $(X, D; L)$ is β-K-polystable for any $\beta \in (\beta_1, \beta_2]$ (resp., $\beta \in [\beta_2, \beta_1)$).

Remark 2.5
Note that if for $(X, D; K_X^{\otimes(-r)})$, where X is a $\mathbb{Q}$-Fano variety with $D \in |-mK_X|$,

$$\lambda : \mathbb{G}_m \to \mathrm{SL}(N_r + 1) \quad \text{with } N_r + 1 := \dim H^0(X, K_X^{\otimes(-r)})$$

induces a test configuration $(\mathcal{X},\mathcal{D};\mathcal{L})$, then

$$\mathrm{CH}(\mathcal{X},\mathcal{D};\mathcal{L}) = \frac{1}{2mr^n(-K_X)^n} \cdot \Big(\mathrm{CH}(\mathcal{D}_0) - \frac{nm}{(n+1)} r\, \mathrm{CH}(\mathcal{X}_0)\Big) \tag{2}$$

with $\mathrm{CH}(\mathcal{D}_0)$ and $\mathrm{CH}(\mathcal{X}_0)$ being precisely the λ-weights for the Chow points of $\mathcal{D}_0, \mathcal{X}_0 \subset \mathbb{P}^{N_r}$.

3. Linear action of reductive groups on projective spaces

In this section, we prove a basic fact on a reductive group acting on $\mathbb{P}^M$, which will be crucial for the later argument. Let G be a reductive algebraic group acting on $\mathbb{P}^M$ via a rational representation $\rho : G \to \mathrm{SL}(M+1)$, and let $z : C \to \mathbb{P}^M$ be an algebraic morphism satisfying $z(0) = z_0 \in \mathbb{P}^M$, where $(0 \in C)$ is a smooth pointed curve germ. Let

$$\overline{BO} := \lim_{t\to 0} \overline{O_{z(t)}}$$

with $O_{z(t)} := G \cdot z(t)$, and let $\overline{O_{z(t)}} \subset \mathbb{P}^M$ be its closure; that is, $\overline{BO}$ is a union of (broken) orbits to which $\overline{O_{z(t)}}$ is specialized.

LEMMA 3.1
Suppose that $G_{z_0} < G$, the stabilizer of $z_0 \in \mathbb{P}^M$ for the G-action on $\mathbb{P}^M$, is reductive. Then there is a G-invariant Zariski-open neighborhood of $z_0 \in U \subset \mathbb{P}^M$ satisfying

$$\overline{BO} \cap U = \bigcup_{\substack{O_p \subset \overline{BO} \\ O_{z_0} \cap \overline{O_p} \neq \varnothing}} O_p \cap U \quad \text{where } O_p := G \cdot p \subset \overline{BO}; \tag{3}$$

that is, the closure of the G-orbit of any point in $\overline{BO}$ near z_0 contains $g \cdot z_0$ for some (hence for all) $g \in G$. We will call O_{z_0} a minimal *orbit.*

Proof
We divide the proof into two steps.

Step 1: $G = G_{z_0}$. The representation $\rho : G \to \mathrm{SL}(M+1)$ induces a G-linearization of $\mathcal{O}_{\mathbb{P}^M}(1) \to \mathbb{P}^M$. Let $\rho_0 : G \to \mathbb{G}_m$ be the character of the resulting G-action on $\mathcal{O}_{\mathbb{P}^M}(1)|_{z_0}$, since z_0 is fixed by G. Then z_0 is GIT-polystable with respect to the linearization of $\mathcal{O}_{\mathbb{P}^M}(1)$ induced by the representation $\rho \otimes \rho_0^{-1} : G \to \mathrm{SL}(M+1)$. It follows from the construction in classical GIT that the semistable locus $z_0 \in U := (\mathbb{P}^M)^{\mathrm{ss}} \subset \mathbb{P}^M$ is G-invariant and Zariski-open. To see that U serves our purpose, it suffices to note that $G \cdot z_0$ is the *unique* polystable orbit in $(\mathbb{P}^M)^{\mathrm{ss}} \cap \overline{BO}$ and for any $z \in \overline{BO} \cap U$, $O_{z_0} \subset \overline{G \cdot z}$, which follows from the classical result of Kempf and Ness (see [35, proof of Theorem 8.3]).

Step 2: $G > G_{z_0}$. Since G_{z_0} is reductive, we have a decomposition of its Lie algebra

$$\mathrm{Lie}(G) = \mathfrak{g} = \mathfrak{g}_{z_0} \oplus \mathfrak{p}$$

as representations of G_{z_0}. The infinitesimal action of G at $0 \neq \hat{z}_0 \in \mathbb{C}^{M+1}$, a lifting of $z_0 \in \mathbb{P}^M$, induces a G_{z_0}-invariant decomposition $\mathbb{C}^{M+1} = \mathbb{C} \cdot \hat{z}_0 \oplus W' \oplus \mathfrak{p}$. By the proof of [17, Proposition 1],

$$\mathbb{P}W = \mathbb{P}(W' \oplus \mathbb{C}\hat{z}_0) \subset \mathbb{P}^M$$

satisfies the following properties:

(1) $z_0 \in \mathbb{P}W$ and is preserved by G_{z_0};
(2) $\mathbb{P}W$ is transversal to the G-orbit of z_0 at z_0;
(3) for $w \in \mathbb{P}W$ near z_0 and $\xi \in \mathfrak{g} := \mathrm{Lie}(G)$, if we let $\sigma_w : \mathfrak{g} \to T_w\mathbb{P}^M$ denote the infinitesimal action of G, then

$$\sigma_w(\xi) \in T_w\mathbb{P}W \iff \xi \in \mathfrak{g}_{z_0} := \mathrm{Lie}(G_{z_0}).$$

In particular, (3) implies that there exists a Zariski-open neighborhood $U_0 \subset \mathbb{P}W$ of z_0 such that the infinitesimal action induced by $\mathfrak{p}^{\perp}$ on $\mathbb{P}W$ is *transversal* for all points in U_0 (see Lemma A.7).

CLAIM 3.2

Let $S := G \cdot \operatorname{Im} z \subset \mathbb{P}^N$, and let H be the identity component of G_{z_0}. Then there is a Zariski-open subset $U_W \subset U_0 \subset \mathbb{P}W$ and a finite *collection of pointed arcs $\{z^i : (C_i, 0) \to (U_0, z_0)\}_{i=0}^d$ with $z^0 = z : C \to \mathbb{P}W$ such that*

$$\overline{S} \cap U_W = \bigcup_{i=0}^{d} \overline{O(H, z^i)} \cap U_W \quad \text{with } O(H, z^i) := H \cdot \operatorname{Im} z^i \subset \mathbb{P}W.$$

Assuming Claim 3.2 for the moment, let us define

$$\overline{BO_i^W} := \lim_{t\to 0} \overline{O^W_{z^i(t)}} \quad \text{with } O^W_{z^i(t)} := H \cdot z^i(t) \subset \overline{O(H, z^i)} \subset \mathbb{P}W.$$

Next, for each $0 \leq i \leq d$, by applying Step 1 to the H-action on $\mathbb{P}W$ and $\overline{BO_i^W} \subset \mathbb{P}W$, we obtain an H-invariant Zariski-open neighborhood $z_0 \in U_i' \subset \mathbb{P}W$ such that

$$\forall p \in U_i' \cap \overline{BO_i^W} \Longrightarrow z_0 \in \overline{G \cdot p}.$$

Then $U = G \cdot (\bigcap_{i=0}^d U_i')$ is the G-invariant Zariski-open set we want. In fact, to see that U is Zariski-open, one first notes that $\bigcap_{i=0}^d U_i'$ is Zariski-open as each of $U_i' \subset \mathbb{P}W$ is so for all i; hence U is constructible by Chevalley's lemma (see [25, Chapter II, Exercise 3.19]). On the other hand, U is also open in $\mathbb{P}^M$ with respect to the analytic topology. This follows from the fact that the $\mathfrak{g}_{z_0}^{\perp}$-action on $\mathbb{P}W$ is transversal (see Lemma A.7) and $\forall g \in G < \mathrm{SL}(M+1)$ is an automorphism of $\mathbb{P}^M$. Being constructible and analytically open implies that U is Zariski-open in $\mathbb{P}^M$.

Now let us proceed to the proof of Claim 3.2. To better illustrate the picture, let us treat the case $\dim G_{z_0} = 0$ first.

Case 1: $\dim G_{z_0} = 0$. Let us consider the variety $S := G \cdot \operatorname{Im} z \subset \mathbb{P}^M$, and let $\partial S := \overline{S} \setminus S$. Then there is an open neighborhood $U_W \subset U_0$ such that $\overline{S} \cap U_W$ has only finitely many irreducible components. Let us write

$$\overline{S} \cap U_W = \bigcup_{i=0}^{d} C_i$$

with $C_0 = \operatorname{Im} z(C)$ and where the C_i's are irreducible components passing through z_0.

Since $\partial\overline{S} \cap C_i$ is constructible, after a possible shrinking of C_i we have two possibilities:

(1) $\partial\overline{S} \cap C_i = C_i$,

(2) $\partial\overline{S} \cap C_i = \varnothing$ or z_0.

We claim that the first case does not happen. Then by choosing the arc $z^i : (C_i, 0) \to (U_0, z_0)$, we establish Claim 3.2. To prove our claim, we observe that there are two kinds of points on the boundary $\partial\overline{S}$:

- *first kind*: a boundary point of $\overline{G \cdot z(t)}$ for a fixed t,
- *second kind*: all the remaining points on $\partial\overline{S}$.

Note that the sets of both kinds of points form constructible sets. Any boundary point of the first kind can indeed be written as a limit of points in $G \cdot z(t) \cap U_W$ for a fixed t, but this is absurd as G acts on U_0 transversally. So we may assume that all the points on C_i are of the *second kind*, which implies that

$$\operatorname{Im} z \not\subset \overline{G \cdot z(t)} \quad \text{for a fixed } t \in C.$$

In particular, we have $\dim G + 1 = \dim \overline{S}$ as $\dim G_{z_0} = 0$. Since $\partial\overline{S}$ is G-invariant, we have $G \cdot C_i \subset \partial\overline{S}$. Now let us consider the G-action on $z \in C_i$, which implies that

$$\dim \partial\overline{S} \geq \dim G + \dim C_i = \dim G + 1 = \dim \overline{S},$$

which is a contradiction. Thus our claim is verified.

Case 2: *The general case*. Let us consider the variety $S := G \cdot \operatorname{Im} z \subset \mathbb{P}^M$, and let $\partial S := \overline{S} \setminus S$. Then there is an H-invariant open neighborhood $U_W \subset U_0$ such that $\overline{S} \cap U_W$ has only finitely many *irreducible* components, which are denoted by

$$\overline{S} \cap U_W = \bigcup_{i=0}^{d} V_i$$

with $V_0 = \overline{O(H, z)}$ and $z_0 \in V_i$, $0 \leq i \leq d$. Moreover, V_i is H-invariant for each i since $\overline{S}$ is.

Then Claim 3.2 amounts to saying that for each i there is an arc $z^i : C_i \to U_0$ such that

$$V_i = \overline{O(H, z^i)} \cap U_W.$$

To find such an arc, all we need is a *general* $v \in V_i$ satisfying

$$\dim H \cdot v + 1 \geq \dim V_i, \tag{4}$$

since that implies two situations: either $\dim H \cdot v < \dim V_i$, in which case we choose $z^i : C_i \to V_i$ to be an arc joining z_0 and v so that $\operatorname{Im} z^i \not\subset \overline{H \cdot v}$; or $\dim H \cdot v = \dim V_i$, in which case we choose any nonconstant arc $z^i : C_i \to V_i$ satisfying $z^i(0) = z_0$. Then $\dim V_i = \dim O(H, z^i)$ and our claim is justified.

To find such $v \in V_i$, we only need it to satisfy

$$\dim H \cdot v \geq \dim H \cdot z(t) \quad \text{for all } t \in C,$$

which again follows from the transversality. Indeed, there is a Zariski-open set U_C of C such that for any $t_0 \in U_C$,

$$\dim H \cdot z(t_0) = \max_{t \in C} \dim H \cdot z(t).$$

By the definition of V_i, for a fixed general $v \in V_i$ there are $g_i \in G$ and $t_0 \in U_C$ such that $g_i \cdot z(t_0) \in B(v, \epsilon) \in \mathbb{P}^M$. By the transversality of $\mathfrak{p}$-action on U_0, for $\epsilon \ll 1$ there is an $h \in G$ close to identity such that $h \cdot g_i \cdot z(t_0) \in V_i$. By the genericity of v, we obtain

$$\dim H \cdot v \geq \dim H \cdot h \cdot g_i \cdot z(t_0) = \dim H \cdot z(t_0) \geq \dim H \cdot z(t) \quad \text{for all } t \in C,$$

and hence $\dim O(H, z^i) \geq \dim O(H, z)$ by our choice of $z^i : C_i \to V_i$.

Now we prove (4). Suppose that (4) does not hold, which is equivalent to $\dim V_i > \dim O(H, z^i)$. Then we have

$$\begin{aligned}
\dim \overline{S} &\geq \dim G \cdot V_i \\
&\geq \dim G/H + \dim V_i \quad (\mathfrak{p}\text{-acting transversely on } U_0) \\
&> \dim G/H + \dim O(H, z^i) \\
&\geq \dim G/H + \dim O(H, z) = \dim \overline{S},
\end{aligned}$$

which is a contradiction. In this way we complete the proof of Claim 3.2 and hence of Lemma 3.1. □

The necessity of the assumption that G_{z_0} is reductive can be illustrated by the following example.

Example 3.3
Let $M_2(\mathbb{C}) = \{[v, w] \mid v, w \in \mathbb{C}^2\}$ be the linear space of 2×2 matrices on which $G := \mathrm{GL}(2)$ is acting via multiplication on the left. Let $V := M_2(\mathbb{C}) \oplus \mathbb{C} \oplus \mathbb{C}$. Then G acts on $\mathbb{P}V$ via the representation

$$\begin{aligned}
\rho : \mathrm{GL}(2) &\longrightarrow \mathrm{SL}(V) \\
g &\longmapsto \rho(g)
\end{aligned} \quad \text{with } \rho(g) \cdot \begin{bmatrix} A \\ x_5 \\ x_6 \end{bmatrix} := \begin{bmatrix} g \cdot A \\ \det(g^{-1}) x_5 \\ \det(g^{-1}) x_6 \end{bmatrix}.$$

Let

$$z_0 = \begin{bmatrix} 0_{2\times 2} \\ 1 \\ 0 \end{bmatrix} \quad \text{and} \quad z_0' = \begin{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \\ 0 \\ 0 \end{bmatrix} \in \mathbb{P}V.$$

Then their stabilizers are $G_{z_0} = G$ and $G_{z_0'} = \left[\begin{smallmatrix} * & * \\ 0 & * \end{smallmatrix}\right] < \mathrm{GL}(2)$. In particular, G_{z_0} is reductive while $G_{z_0'}$ is not. Now let

$$z(t) = \begin{bmatrix} \begin{bmatrix} t & 0 \\ 0 & t^2 \end{bmatrix} \\ 1 \\ t \end{bmatrix} \qquad \text{and} \qquad z'(t) = \begin{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & t \end{bmatrix} \\ t \\ t^2 \end{bmatrix} \in \mathbb{P}V$$

be two curves in $\mathbb{P}V$. Then we have

$$\lim_{t \to 0} \overline{O_{z(t)}} = \lim_{t \to 0} \mathbb{P}V_{[1,t]} = \lim_{t \to 0} \overline{O_{z'(t)}} = \mathbb{P}V_{[1,0]},$$

where $V_{[1,t]} := \{tx_5 = x_6\} \subset V$. Clearly, $z_0 := z(0)$ satisfies (3) while $z_0' := z'(0)$ does not, since

$$z_0' \notin \mathbb{P}^1 \cong \overline{G \cdot z_\epsilon''} \subset \mathbb{P}V_{[1,0]} \quad \text{for } 0 < |\epsilon| \ll 1 \text{ where } z_\epsilon'' := \begin{bmatrix} \begin{bmatrix} 1 & \epsilon \\ 0 & 0 \end{bmatrix} \\ 0 \\ 0 \end{bmatrix}.$$

4. Gromov–Hausdorff continuity of conical Kähler–Einstein metrics on smooth Fano pairs

In this section, we list the important analytic results that will be needed in our main argument.

4.1. *Gromov–Hausdorff limit of Kähler–Einstein Fano manifolds*

In this section, let us recall the main technical results obtained in the solution of the Yau–Tian–Donaldson conjecture (see, e.g., [4], [10], [11], [53]).

THEOREM 4.1

Let X_i be a sequence of n-dimensional Fano manifolds with a fixed Hilbert polynomial χ, and let $D_i \subset X_i$ be smooth divisors in $|-mK_{X_i}|$ for a fixed $m > 0$. Let $\beta_i \in (0,1)$ be a sequence converging to β_∞ with $0 < \epsilon_0 \le \beta_\infty \le 1$. Suppose that each X_i admits a Kähler metric $\omega_i(\beta_i)$ solving

$$\mathrm{Ric}\big(\omega(\beta_i)\big) = \beta_i \omega(\beta_i) + \frac{1-\beta_i}{m}[D] \quad \text{on } X_i; \tag{5}$$

that is, $\omega_i(\beta_i)$ is a conical Kähler–Einstein metric with cone angle $2\pi(1-(1-\beta_i)/m)$ along the divisor $D_i \subset X_i$. Then the Gromov–Hausdorff limit of any subsequence of $\{(X_i, \omega_i(\beta_i))\}_i$ is homeomorphic *to a $\mathbb{Q}$-Fano variety Y. Furthermore, there is a unique Weil divisor $E \subset Y$ such that*

(1) *$(Y, \frac{1-\beta_\infty}{m} E)$ is klt;*

(2) *Y admits a weak conical Kähler–Einstein metric solving*

$$\mathrm{Ric}\big(\omega(\beta_\infty)\big) = \beta_\infty \omega(\beta_\infty) + \frac{1-\beta_\infty}{m}[E] \quad \textit{on } Y,$$

where in particular, $\mathrm{Aut}(Y, E)$ *is reductive and the pair (Y, E) is β_∞-K-polystable;*

(3) *possibly after passing to a subsequence, there are embeddings $T_i : X_i \to \mathbb{P}^N$ and $T_\infty : Y \to \mathbb{P}^N$, defined by the complete linear systems $|-rK_{X_i}|$ and $|-rK_Y|$, respectively, for $r = r(m, \epsilon_0, \chi)$ and $N + 1 = \chi(X_i, K_{X_i}^{-\otimes r})$, such that the $T_i(X_i)$'s converge to $T_\infty(Y)$ as projective varieties and the $T_i(D_i)$'s converge to $T_\infty(E)$ as algebraic cycles.*

In the following corollary, we denote by $C^{,\alpha,\beta}$ the space of conical Kähler metrics defined in [16].

COROLLARY 4.2

Let (X, D) be a smooth Fano pair with $D \in |-mK_X|$. Then the following hold.

(1) *We have that (X, D) is β-K-stable if and only if it admits a conical Kähler–Einstein metric $\omega(\beta) \in C^{,\alpha,\beta}$ solving (5).*

(2) *Let $\gamma \in (0, 1]$. Then (X, D) is γ-K-semistable if and only if it admits a conical Kähler–Einstein metric $\omega(\beta) \in C^{,\alpha,\beta}$ solving (5) for any $\beta \in (0, \gamma)$.*

Remark 4.3

Note that the limiting divisor $E \subset Y$ is actually $\mathbb{Q}$-Cartier. To see this, one notes that on the smooth locus of Y

$$E|_{Y^{\mathrm{reg}}} \sim -mK_{Y^{\mathrm{reg}}}, \tag{6}$$

which implies that $E|_Y \sim -mK_Y$ as Y is normal. On the other hand, Y being $\mathbb{Q}$-Fano implies that K_Y is $\mathbb{Q}$-Cartier. This, together with (6), implies that E is $\mathbb{Q}$-Cartier. It was also pointed out in [19, Section 4.3] and [11, Section 5] that if the sequence $\{(X_i, D_i)\} = \{(\mathcal{X}_{t_i}, \mathcal{D}_{t_i})\}$ is a subsequence of a projective flat family $(\mathcal{X}^\circ, \mathcal{D}^\circ) \to C^\circ$ of smooth log Fano pairs over a smooth punctured (not necessarily complete) curve $C^\circ = C \setminus \{0\}$, that is, $\{t_i\} \subset C^\circ$ and $t_i \xrightarrow{i\to\infty} 0$, then the Gromov–Hausdorff limit (Y, E) can be realized as the central fiber of a *flat degeneration*

$$\begin{tikzcd}
(\mathcal{X}^\circ, \mathcal{D}^\circ) \arrow[r, hook] \arrow[d] & (\mathcal{X}, \mathcal{D}) \arrow[d] \\
C^\circ \arrow[r, hook] & C
\end{tikzcd}$$

that is, $(Y, E) = (\mathcal{X}_0, \mathcal{D}_0)$. This important consequence is used in [11] and [53] to construct the destabilizing test configurations. In particular, the flatness of $\mathcal{X} \to C$ is established in [19, Section 4.3], and the flatness of $\mathcal{D} \to C$ can be deduced (see [25, Chapter III, Exercise 10.9]) from the fact that $\mathcal{D}$ is Cohen–Macaulay since we have already shown that it is $\mathbb{Q}$-Cartier (see [29, Corollary 5.25]), and the morphism $\mathcal{D} \to C$ is equidimensional.

4.2. *Gromov–Hausdorff continuity of conical Kähler–Einstein metrics on smooth Fano families*

Definition 4.4
Let

$$\mathbb{H}^{\chi;N} := \mathrm{Hilb}_\chi(\mathbb{P}^N) \tag{7}$$

denote the Hilbert scheme of closed subschemes of $\mathbb{P}^N$ with Hilbert polynomial χ. For a closed subscheme $X \subset \mathbb{P}^N$ with Hilbert polynomial $\chi(X, \mathcal{O}_{\mathbb{P}^N}(k)|_X) = \chi(k)$, let $\mathrm{Hilb}(X) \in \mathbb{H}^{\chi;N}$ denote its *Hilbert point.*

To set the scene, let

$$\begin{array}{ccc} (\mathcal{X}, \mathcal{D}) & \xrightarrow{\ i\ } & \mathbb{P}^N \times \mathbb{P}^N \times \Delta \\ \Big\downarrow{\scriptstyle \pi} & & \Big\downarrow \\ \Delta & = & \Delta \end{array}$$

be a projective flat family of Fano varieties over the disk $\Delta = \{|t| < 1\} \subset \mathbb{C}$ such that

(1) $\mathcal{X}$ is *smooth* and $\mathcal{D} \in |-mK_{\mathcal{X}/\Delta}|$ is a *smooth* divisor defined by a *smooth* section $s_{\mathcal{D}} \in \Gamma(\Delta, \omega_{\mathcal{X}/\Delta}^{\otimes -m})$;

(2) both π and $\pi|_{\mathcal{D}}$ are holomorphic *submersions* over Δ.

To get rid of the $\mathrm{U}(N+1)$-ambiguity for the later argument, let us assume that $\omega_{\mathcal{X}}^{\otimes -r}$ is relatively very ample, and let i be the embedding induced by a *prefixed basis*

$$\{s_i(t)\}_{i=0}^N \subset \Gamma\big(\Delta, \pi_* \mathcal{O}_{\mathcal{X}}(-rK_{\mathcal{X}/\Delta})\big).$$

Then $i^*\mathcal{O}_{\mathbb{P}^N}(1) \cong \mathcal{O}_{\mathcal{X}}(-rK_{\mathcal{X}/\Delta})$. Now let $(r\omega_{\mathrm{FS}}(t), h_{\mathrm{FS}}^{\otimes r}(t))$ denote the metric on $(\mathcal{X}_t, \mathcal{O}_{\mathcal{X}}(-rK_{\mathcal{X}/C})|_{\mathcal{X}_t})$ induced from the embedding i via the basis $\{s_i\}$. Suppose that for each $t \in \Delta$, $\mathcal{X}_t$ is K-semistable. Then by Lemma 2.4, $(\mathcal{X}_t, \mathcal{D}_t)$ is β-K-polystable for any $\beta \in (0,1)$. So by Corollary 4.2, for any $\beta \in (0,1)$ there exists the conical Kähler–Einstein metric $\omega(t,\beta)$ on the pair $(\mathcal{X}_t, \frac{1-\beta}{m}\mathcal{D}_t)$ which satisfies

$$\mathrm{Ric}\big(\omega(t,\beta)\big) = \beta\omega(t,\beta) + \frac{1-\beta}{m}[\mathcal{D}_t].$$

In the following, by abuse of notation, sometimes we will abbreviate $\omega(t,\beta)$ as a conical Kähler–Einstein metric with *cone angle* β (*instead of* $2\pi(1-(1-\beta)/m)$) *along* D, since the integer m is fixed once and for all throughout the paper. Now assume that $\omega(t,\beta)=\omega_{\mathrm{KE}}(t,\beta)=\omega_{\mathrm{FS}}(t)+\sqrt{-1}\partial\bar\partial\varphi(t,\beta)$, where $r\cdot\omega_{\mathrm{FS}}(t)$ is equal to the Fubini–Study metric induced from the embedding of $\mathcal{X}_t\to\mathbb{P}^N$ using the basis $\{s_i(t)\}_{i=0}^N$. Then $\varphi(t,\beta)$ is the unique solution (see [7, Theorem 7.3]) to the equation

$$\big(\omega_{\mathrm{FS}}(t)+\sqrt{-1}\partial\bar\partial\varphi(t,\beta)\big)^n=e^{f(t)-\beta\varphi(t,\beta)}\frac{\omega^n_{\mathrm{FS}}(t)}{(|s_{\mathcal{D}_t}|^2_{h_{\mathrm{FS}}^{\otimes m}(t)})^{\frac{1-\beta}{m}}},\tag{8}$$

where $f(t)$ satisfies

$$\begin{aligned}\operatorname{Ric}\big(\omega_{\mathrm{FS}}(t)\big)=\omega_{\mathrm{FS}}(t)+\sqrt{-1}\partial\bar\partial f(t)\qquad\text{and}\\ \int_{\mathcal{X}_t}e^{f(t)}\cdot\omega^n_{\mathrm{FS}}(t)=\int_{\mathcal{X}_t}\omega^n_{\mathrm{FS}}(t).\end{aligned}\tag{9}$$

Remark 4.5
It is easy to check that an equivalent form of (8) is

$$\big(\omega_{\mathrm{FS}}(t)+\sqrt{-1}\partial\bar\partial\varphi(t,\beta)\big)^n\cdot|s_{\mathcal{D}_t}|^{-\frac{2\beta}{m}}_{h_{\mathrm{FS}}e^{-\varphi}}(s_{\mathcal{D}_t}\otimes\overline{s_{\mathcal{D}_t}})^{\frac{1}{m}}=1.\tag{10}$$

We define a positive definite Hermitian matrix

$$A_{\mathrm{KE}}(t,\beta)=\big[(s_i,s_j)_{\mathrm{KE},\beta}(t)\big]$$

with

$$(s_i,s_j)_{\mathrm{KE},\beta}(t)=\int_{\mathcal{X}_t}\big\langle s_i(t),s_j(t)\big\rangle_{h_{\mathrm{KE}}^{\otimes r}(t,\beta)}\omega^n(t,\beta),$$

where $h_{\mathrm{KE}}(t,\beta):=h_{\mathrm{FS}}(t)\cdot e^{-\varphi(t,\beta)}$. Now we introduce the *rth Tian's embedding*

$$T:\big(\mathcal{X}_t,\mathcal{D}_t;\omega(t,\beta)\big)\longrightarrow\mathbb{P}^N\tag{11}$$

to be the one given by the basis $\{g(t,\beta)\circ s_j(t)\}_{j=0}^N$ with $g(t,\beta)=A_{\mathrm{KE}}^{-1/2}(t,\beta)$.

Definition 4.6
We denote by

$$\operatorname{Hilb}\big(\mathcal{X}_t,(1-\beta)\mathcal{D}_t\big)\in\mathbb{H}^{\boldsymbol{\chi};N}:=\mathbb{H}^{\chi;N}\times\mathbb{H}^{\tilde\chi;N}\tag{12}$$

the *Hilbert point* of the pair $(\mathcal{X}_t,\mathcal{D}_t\subset\mathcal{X}_t)\subset\mathbb{P}^N$ using Tian's embedding for the basis $\{s_i\}$ with respect to a Kähler form $\omega(t,\beta)$, where $(\chi,\tilde\chi)$ are the Hilbert polynomials of $X\subset\mathbb{P}^N$ and $D\subset\mathbb{P}^N$, respectively. We note that when $\beta=1$, the second factor $\mathbb{H}^{\tilde{\chi};N}$ is not trivial as we still remember $\mathcal{D}_t$; that is, $\mathrm{Hilb}(X, 0 \cdot D)$ is not the same as $\mathrm{Hilb}(X)$. See Remark 4.7(1) below.

Remark 4.7
We make the following observations.

(1) It is by definition that

$$\mathrm{Hilb}\big(\mathcal{X}_t, (1-\beta)\mathcal{D}_t\big) = \big(\mathrm{Hilb}(\mathcal{X}_t), \mathrm{Hilb}(\mathcal{D}_t); \omega(t,\beta)\big).$$

In the following, we will always use the coefficient $(1-\beta)$ to stress that the cycle is obtained via Tian's embedding with respect to the metric $\omega(t,\beta)$.

(2) Tian's embedding is well defined for any klt $\mathbb{Q}$-Fano log pair with weak conical Kähler–Einstein metric $(X, (1-\beta)D; \omega_{\mathrm{KE}}(\beta))$. Note that for any weak conical Kähler–Einstein metric $\omega_{\mathrm{KE}}(\beta)$, we always assume that the local potential is bounded (see [5]).

(3) The advantage of fixing a basis $\{s_i(t)\}_{i=0}^N \subset \Gamma(\Delta, \pi_*\mathcal{O}_{\mathcal{X}}(-rK_{\mathcal{X}/\Delta}))$ lies in the fact that the image of Tian's embedding and hence the Hilbert point $\mathrm{Hilb}(\mathcal{X}_t, (1-\beta)\mathcal{D}_t)$ is completely determined by the isometric class of $\omega(t,\beta)$. See Lemma 4.9.

PROPOSITION 4.8
We have that $\mathrm{Hilb}(\mathcal{X}_t, (1-\beta)\mathcal{D}_t)$ *varies continuously in* $\mathbb{H}^{\chi,\tilde{\chi};N}$ *with respect to the pair* $(\beta, t) \in (0,1) \times \Delta$.

Proof
Using the above notation, we claim that $\varphi_{\mathrm{KE}}(t,\beta)$ is continuous with respect to t for any $\beta < 1$. Assuming the claim, $A_{\mathrm{KE}}(t,\beta)$ is then continuous with respect to t, and hence the images of Tian's embedding given by orthonormal basis change continuously.

Now we verify the claim by applying the implicit function theorem. First, we note that the complex manifold $(\mathcal{X}_t, \mathcal{D}_t)$ is diffeomorphic to a fixed pair (X, D) endowed with the integrable complex structure J_t thanks to the assumption that π is a submersion. Let $C^{2,\alpha;\beta}(\mathcal{X}_t, \mathcal{D}_t; J_t)$ and $C^{,\alpha;\beta}(\mathcal{X}_t, \mathcal{D}_t; J_t)$ denote the function spaces on $(\mathcal{X}_t, \mathcal{D}_t; J_t)$ defined in [16]. For each fixed $t \in \Delta$, we consider the map

$$\begin{aligned} F(t,\beta,\cdot) : C^{2,\alpha;\beta}(\mathcal{X}_t, \mathcal{D}_t; J_t) &\longrightarrow C^{,\alpha;\beta}(\mathcal{X}_t, \mathcal{D}_t; J_t) \\ \varphi &\longmapsto \log \frac{(\omega_t + \sqrt{-1}\partial_{J_t}\bar{\partial}_{J_t}\varphi)^n |s_{\mathcal{D}_t}|_{h_t}^{2(1-\beta)/m}}{\omega^n} - f_t + \beta\varphi \end{aligned} \tag{13}$$

where for simplicity we write $f_t = f(t)$, $\omega_t = \omega_{\mathrm{FS}}(t)$, and $h_t = h_{\mathrm{FS}}^{\otimes m}(t)$, and $s_{\mathcal{D}_t}$ is the defining section for $\mathcal{D}_t$ as before. Note that $\varphi_{\mathrm{KE}}(t,\beta)$ is exactly the solution to the equation $F(t,\beta,\varphi) = 0$. We would like to apply the implicit function theorem to obtain the continuity of $\varphi_{\mathrm{KE}}(t,\beta)$ with respect to t. In order to do that, we need to work with a *fixed* function space, whereas the spaces $C^{2,\alpha;\beta}(\mathcal{X}_t, \mathcal{D}_t; J_t)$ depend on the parameter t. To get around this, we note that the metrics $\{\omega_t(\cdot, J_t\cdot)\}_t$ vary smoothly and hence $C^{,\alpha;\beta}(X, D; J_t) = C^{,\alpha;\beta}(X, D; J_0)$. This key observation allows us to identify the spaces $C^{2,\alpha;\beta}(X, D; J_0)$ and $C^{2,\alpha;\beta}(X, D; J_t)$ via the following simple way. Let us fix a family of background conical Kähler metrics

$$\hat{\omega}_t = \omega_t + \epsilon\sqrt{-1}\partial_{J_t}\bar{\partial}_{J_t}|s_{\mathcal{D}_t}|_{h_t}^{2\gamma},$$

with $\gamma = 1 - \frac{1-\beta}{m} \in (0,1)$ being fixed and $0 < \epsilon \ll 1$. Then we define a linear map

$$\begin{aligned} Q_{t,\beta} := Q(t,\beta,\cdot) : C^{2,\alpha;\beta}(X, D; J_0) &\longrightarrow C^{2,\alpha;\beta}(X, D; J_t) \\ \tilde{\varphi} &\longmapsto (-\triangle_{\hat{\omega}_t} + 1)^{-1} \circ (-\triangle_{\hat{\omega}_0} + 1)\tilde{\varphi}. \end{aligned} \tag{14}$$

Since $\ker(-\triangle_{\hat{\omega}_t} + 1) = \{0\}$ by the proof of [16, Proposition 8], it follows from Donaldson's Schauder estimate in [16, Section 4.3] that $Q_{t,\beta'}$ is an isomorphism for $|t| \ll 1$ and $\beta' \in (\beta-\epsilon, \beta+\epsilon)$ with $0 < \epsilon \ll 1$. Also, by using the explicit parametrix constructed in [16, Section 3], Q_t gives rise to a *continuous* local linear trivialization of the family of subspaces $C^{2,\alpha;\beta}(X, D; J_t) \subset C^{,\alpha;\beta}(X, D; J_t) = C^{,\alpha;\beta}(X, D; J_0)$. Denoting $\tilde{\varphi}(t,\beta) = Q_{t,\beta}^{-1}(\varphi(t,\beta))$, we can calculate

$$\frac{\partial F(t,\beta,Q(t,\beta,\tilde{\varphi}))}{\partial\tilde{\varphi}}\Big|_{(0,\beta,\tilde{\varphi}_{\mathrm{KE}})}(\phi) = (\triangle_{\omega_{\mathrm{KE}}} + \beta) \circ Q_0\phi = (\triangle_{\omega_{\mathrm{KE}}} + \beta)\phi$$

which is invertible by [16, Theorem 2] since there exists no holomorphic vector field on the pair $(\mathcal{X}_0, \mathcal{D}_0)$ (see [45, Theorem 2.1] or Lemma 5.4). Now we can apply the effective version of the implicit function theorem as in [16, Section 4.4] to the map $F(t,\beta,Q(t,\beta,\cdot)) : C^{2,\alpha;\beta}(X, D; J_0) \to C^{,\alpha;\beta}(X, D; J_0)$ to get a continuous family of solutions $\tilde{\varphi}_{\mathrm{KE}}(t,\beta')$ to the equation $F(t,\beta',Q(t,\beta',\tilde{\varphi})) = 0$ for all $|t| \ll 1$ and $\beta' \in (\beta-\epsilon, \beta+\epsilon)$ with $0 < \epsilon \ll 1$. Since the argument for this last statement is standard, we will only sketch its proof. For a fixed β, by the usual implicit function theorem we first get a family of solutions $\tilde{\varphi}_{\mathrm{KE}}^{(1)}(t,\beta)$ to the equation $F(t,\beta,Q(t,\beta,\tilde{\varphi}_{\mathrm{KE}}^{(1)})) = 0$ for $|t| \ll 1$. Then we can apply Donaldson's argument of deforming cone angles in [16, Section 4.4] in a family version to further get $\tilde{\varphi}_{\mathrm{KE}}(t,\beta')$ for any $|t| \ll 1$ and $\beta' \in (\beta-\epsilon, \beta+\epsilon)$.

More precisely, let $\omega_{\mathrm{KE}}(t,\beta) = \omega_{\mathrm{FS}} + \sqrt{-1}\partial\bar{\partial}\tilde{\varphi}_{\mathrm{KE}}^{(1)}(t,\beta)$ be the continuous family of $C^{,\alpha;\beta}$ conical Kähler–Einstein metrics obtained earlier. For each $\beta' \in (\beta - \epsilon, \beta+\epsilon)$ and t near zero, we define the new reference metric $\omega(t,\beta') := \omega_{\mathrm{KE}}(t,\beta) +$

$\sqrt{-1}\partial\bar{\partial}(\|s_{\mathcal{D}_t}\|^{2\beta'/m} - \|s_{\mathcal{D}_t}\|^{2\beta/m})$, where $\|\cdot\|^2$ is a smooth extension of the Hermitian metric determined by $h_{\mathrm{FS}}\exp(-\tilde{\varphi}^{(1)}_{\mathrm{KE}}(t,\beta))$ on $(K^{-1}_{\mathcal{X}_t})^{\otimes m}|_{\mathcal{D}_t}$ (using the fact that $\tilde{\varphi}^{(1)}_{\mathrm{KE}}(t,\beta)$ is *smooth in tangential directions* by [16, Section 4.3]). Then as in the proof of [16, Proposition 7], one can show that

(1) $k_{\beta'} := |s_{\mathcal{D}}|^{-2\beta'/m}_{\beta'}(s_{\mathcal{D}} \otimes \overline{s_{\mathcal{D}}})^{1/m}\ \omega(t,\beta')^n$ (see (10)) satisfies $\|k_{\beta'} - 1\|_{C^{\cdot,\alpha;\beta'}} \to 0$ as $\beta' \to \beta$; here we used $|\cdot|^2_{\beta'}$ to denote the Hermitian metric on $\mathcal{O}_{\mathcal{X}_t}(-mK_{\mathcal{X}_t})$ whose curvature is equal to $m\cdot\omega(t,\beta')$;

(2) if we let $\Delta_{\beta'}$ denote the Laplace operator associated to $\omega(t,\beta')$, then $\Delta_{\beta'}+\beta'$ is invertible and the operator norm of its inverse is bounded by a fixed constant independent of β' and for t near zero.

So the effective version of the implicit function theorem allows us to get a continuous family of solutions $\tilde{\varphi}_{\mathrm{KE}}(t,\beta')$. In fact, one notes that $\omega(t,\beta') = \omega_{\mathrm{FS}} + \sqrt{-1}\partial\bar{\partial}\psi(t,\beta')$ with $\psi(t,\beta') = \tilde{\varphi}^{(1)}_{\mathrm{KE}}(t,\beta) + \|s_{\mathcal{D}_t}\|^{2\beta'/m} - \|s_{\mathcal{D}_t}\|^{2\beta/m}$ is an *approximate solution* to the conical Kähler–Einstein equation by item (1) above and is continuous with respect to both t and β'. By item (2) and the effective version of the implicit function theorem, we then know that the difference between the actual solution $\tilde{\varphi}_{\mathrm{KE}}(t,\beta')$ and $\psi(t,\beta')$ approaches zero in $C^{2,\alpha;\beta'}$-norm as $\beta' \to \beta$. As a consequence, $\tilde{\varphi}_{\mathrm{KE}}(t,\beta')$ is continuous at $\beta' = \beta$ in C^0-norm with respect to both β' and t. We note that the argument above does not depend on the origin 0 and β is chosen; hence $\varphi_{\mathrm{KE}}(t,\beta') = Q(t,\beta',\tilde{\varphi}_{\mathrm{KE}}(t,\beta'))$ is continuous with respect to all $t \in \Delta$ and $\beta' \in (\beta-\epsilon,\beta+\epsilon)$.

By using the complex Monge–Ampère equation in (8) or (10), we see that the family of volume forms $\omega_{\mathrm{KE}}(t,\beta')^n$ on the fixed *smooth* manifold X is continuous in $L^p(X)$, $\forall p \in [1, 1/(1-\beta'))$ with respect to β' and t, which implies that the family of matrices of L^2-inner products $A_{\mathrm{KE}}(t,\beta') = [(s_i,s_j)_{\mathrm{KE},\beta'}(t)]$ is also continuous with respect to t and β'. So Tian's embeddings $T(\mathcal{X}_t,\mathcal{D}_t;\omega(t,\beta'))$ determined by $\{A^{-1/2}_{\mathrm{KE}}(t,\beta')\circ s_j(t)\}_{j=0}^N$ indeed produce a continuous family of Hilbert points inside $\mathbb{H}^{\chi;N}$. $\square$

Let $\{(X_i,D_i)\}$ be a sequence of smooth Fano pairs with *a fixed* Hilbert polynomial χ and $D_i \in |-mK_{X_i}|$. Suppose that each X_i admits a unique conical Kähler–Einstein form $\omega(i,\beta_i)$ solving

$$\mathrm{Ric}\big(\omega(i,\beta_i)\big) = \beta_i\omega(i,\beta_i) + \frac{1-\beta_i}{m}[D_i] \quad \text{on } X_i$$

with $\inf \beta_i \geq \epsilon > 0$. Then we define

$$T_i : \big(X_i, D_i; \omega(i,\beta_i)\big) \longrightarrow \mathbb{P}^N$$

to be Tian's embedding with respect to $\omega(i,\beta_i)$ for sufficiently large N depending only on ϵ, m and the fixed Hilbert polynomial χ, and we let $\mathrm{Hilb}(X_i,(1-\beta_i)D_i) \in$

$\mathbb{H}^{\chi;N} \times \mathbb{H}^{\tilde{\chi};N}$ denote the Hilbert point corresponding to Tian's embedding of X_i with respect to $\omega(i, \beta_i)$. Then we have the following.

LEMMA 4.9
Let $(X, D) \subset \mathbb{P}^N$ be a log $\mathbb{Q}$-Fano pair with the same Hilbert polynomial χ, and let $D \in |-mK_X|$. Suppose that (X, D) admits a weak conical Kähler–Einstein form $\omega(\beta)$ with $\beta = \lim_{i\to\infty} \beta_i$ solving

$$\operatorname{Ric}\bigl(\omega(\beta)\bigr) = \beta\omega(\beta) + \frac{1-\beta}{m}[D] \quad \text{on } X.$$

Then

$$\bigl(X_i, D_i; \omega(i, \beta_i)\bigr) \xrightarrow{\text{GH}} \bigl(X, D; \omega'(\beta)\bigr) \quad \text{as } i \to \infty$$

for a conical Kähler–Einstein metric $\omega'(\beta)$ is equivalent to the following statement: there is a sequence of $\{g_i\} \subset \mathrm{U}(N+1)$ such that

$$g_i \cdot \operatorname{Hilb}\bigl(X_i, (1-\beta_i)D_i\bigr) \longrightarrow \operatorname{Hilb}\bigl(X, (1-\beta)D\bigr) \in \mathbb{H}^{\chi;N} \quad \text{as } i \to \infty,$$

where $\operatorname{Hilb}(X, (1-\beta)D)$ denotes the Hilbert point of Tian's embedding $T : (X, D; \omega(\beta)) \to \mathbb{P}^N$ for a fixed basis $\{s_i\}$.

Proof
This follows directly from Theorem 4.1 which in turn follows from work in [10], [11], [52], and [53]. Indeed, let us assume that $(X_i, D_i; \omega(i, \beta_i)) \xrightarrow{\text{GH}} (X, D; \omega'(\beta))$, where we can assume that the limit exists by Theorem 4.1. Then by Theorem 4.1(3), $(T_i(X_i), T_i(D_i)) \to (T'_\infty(X), T'_\infty(D))$, where T_i (resp., T'_∞) is given by Tian's embedding determined by an orthonormal basis of $H^0(X_i, -mK_{X_i})$ (resp., $H^0(X, -mK_X)$) with respect to $\omega(i, \beta_i)$ (resp., $\omega'(\beta)$). Assume that $\omega(\beta)$ is also a conical Kähler–Einstein metric on (X, D). By the uniqueness of conical Kähler–Einstein metrics proved in [5], there exists a holomorphic automorphism $\sigma \in \operatorname{Aut}(X, D)$ such that $\sigma^*\omega(\beta) = \omega'(\beta)$. Moreover, because σ lifts to $\operatorname{Aut}(X, D, -mK_X)$, there is a unitary isomorphism between $(H^0(X, -mK_X), \|\cdot\|^2_{\omega'(\beta)})$ and $(H^0(X, -mK_X), \|\cdot\|^2_{\omega(\beta)})$, where $\|\cdot\|^2_{\omega(\beta)}$ ($\|\cdot\|^2_{\omega'(\beta)}$) is the L^2-inner product induced by $\omega(\beta)$ (resp., $\omega'(\beta)$). Via this isomorphism, we have $(T_i(X_i), T_i(D_i)) \to (T_\infty(X), T_\infty(D))$, where T_∞ is given by Tian's embedding determined by an orthonormal basis of $H^0(X, -mK_X)$ with respect to $\omega(\beta)$. Now the statement of the lemma holds because the orthonormal basis of a unitary vector space is defined only up to a $\mathrm{U}(N+1)$-ambiguity. □

5. Strong uniqueness for $0 < \beta \ll 1$

In this section, we will give a purely algebro-geometric proof of the fact that when the angle $\beta > 0$ is sufficiently small, then there is a unique filling.

PROPOSITION 5.1
For a fixed finite set $I \subset [0,1]$, there exists a number $\beta_I > 0$ such that if $(X, (1-\beta_I)D)$ is a klt pair, D is $\mathbb{R}$-Cartier, and the coefficients of $\mathbb{R}$-divisor D are contained in I, then (X, D) is log canonical.

Proof
By [23, Theorem 1.1], we know that for the set of all n-dimensional log pairs (X, D) satisfying the property that D is an $\mathbb{R}$-divisor and its coefficients are contained in I, the set of log canonical thresholds

$$\left\{\mathrm{lct}(X, D) \;\middle|\; X \text{ is } n \text{ dimensional, the coefficients of } D \text{ are in } I\right\}$$

satisfies the ACC. In particular, there exists a maximum β_I among all log canonical thresholds which are strictly less than 1.

Then we know that if $(X, (1-\beta_I)D)$ is klt and D is $\mathbb{Q}$-Cartier, (X, D) is log canonical, since otherwise, we have a pair whose log canonical threshold is in $(1-\beta_I, 1)$, which is a contradiction. ☐

Let $\mathcal{X} \to C$ be a flat family of $\mathbb{Q}$-Fano varieties over a smooth pointed curve $0 \in C$, and assume that $\mathcal{X}$ is $\mathbb{Q}$-Gorenstein. Fix $m > 1$, and let $\mathcal{D} \sim_C -mK_{\mathcal{X}}$ be a divisor such that after a possible shrinking of the pointed curve C, for *every* $t \in C$ the fiber $(\mathcal{X}_t, \frac{1}{m}\mathcal{D}_t)$ is *klt*. For instance, we can choose m sufficiently divisible such that $|-mK_{\mathcal{X}}|$ is relatively basepoint-free over C, and let $\mathcal{D} \sim_C -mK_{\mathcal{X}}$ be a general divisor in $|-mK_{\mathcal{X}}|$. In particular, after a possible shrinking of C, if $\mathcal{X} \to C$ is smooth over C°, then one might choose $\mathcal{D}$ so that $\mathcal{D}_t$ is *smooth* for $t \in C^\circ$ provided that $\mathcal{X}_t$ is so for $t \in C^\circ$.

THEOREM 5.2
Let $(\mathcal{X}, \mathcal{D}) \to C$ be a flat family introduced as above. Let

$$\beta_0 := \min\left\{\beta_I, \frac{1}{m+1}\right\},$$

with β_I being given in Proposition 5.1 for the set $I = \{\frac{q}{m} | q = 1, 2, \ldots, m\}$. For any fixed $\beta \in (0, \beta_0]$, suppose that $(\mathcal{X}', \mathcal{D}') \to C$ is another flat family with $K_{\mathcal{X}'} + \frac{1}{m}\mathcal{D}'$ being $\mathbb{Q}$-Cartier satisfying

$$(\mathcal{X}', \mathcal{D}') \times_C C^\circ \cong (\mathcal{X}, \mathcal{D}) \times_C C^\circ \tag{15}$$

with $(Y_\beta, \frac{1-\beta}{m} E_\beta) := (\mathcal{X}'_0, \frac{1-\beta}{m}\mathcal{D}'_0)$ being $\mathbb{Q}$-Fano. Then the above isomorphism can be extended to an isomorphism

$$(\mathcal{X}', \mathcal{D}') \cong (\mathcal{X}, \mathcal{D}).$$

Proof
Since $\mathcal{D} \times_C C^\circ$ is integral by our choice, the coefficients of $\frac{1}{m}E_\beta$ lie in the set $\{\frac{q}{m} \mid q \in \mathbb{N}\}$. By our assumption that $(Y_\beta, \frac{1-\beta}{m}E_\beta)$ is klt and $\beta \le \beta_0 \le \frac{1}{m+1}$, we have

$$\frac{1}{m+1} \le \frac{1-\beta}{m} \qquad \text{and} \qquad \frac{1-\beta}{m}c_i < 1 \quad \text{hence } c_i < m+1,$$

where $E_\beta = \sum_i c_i E_{\beta,i}$ with $E_{\beta,i}$ being a prime divisor for each i. Hence the coefficients of $\frac{1}{m}E_\beta$ must lie in $I = \{\frac{q}{m} \mid q = 1, 2, \ldots, m\}$. By our assumption of $\beta \in (0, \beta_0] \subset (0, \beta_I]$, we know that $(Y_\beta, \frac{1}{m}E_\beta)$ is *log canonical* by Proposition 5.1. Furthermore, since Y_β is irreducible, we know that

$$K_{\mathcal{X}'} + \frac{1}{m}\mathcal{D}' \sim_{\mathbb{Q},C} 0$$

as this holds over C°.

Let W be a common resolution

$$\begin{tikzcd} & W \arrow[dl, "p"'] \arrow[dr, "q"] & \\ \mathcal{X} \arrow[rr, dashed, "\phi"] & & \mathcal{X}' \end{tikzcd}$$

that is an isomorphism over C°. If the birational map ϕ extends to a birational map $\mathcal{X}_0 \overset{\phi|_{\mathcal{X}_0}}{\dashrightarrow} Y_\beta$, then

$$q^* K_{\mathcal{X}'} \sim_{C,\mathbb{Q}} p^* K_{\mathcal{X}}$$

as ϕ is an isomorphism in codimension 1, which implies that

$$\mathcal{X} = \operatorname{Proj} \bigoplus_{r=0}^{\infty} \mathcal{O}_W(-rp^* K_{\mathcal{X}/C}) = \operatorname{Proj} \bigoplus_{r=0}^{\infty} \mathcal{O}_W(-rq^* K_{\mathcal{X}'/C}) = \mathcal{X}'$$

since both $\mathcal{X}_0$ and Y_β are $\mathbb{Q}$-Fano, and we are already done. So from now on we assume that $\mathcal{X}_0 \neq Y_\beta$ on W.

Now let us write

$$p^*\Big(K_{\mathcal{X}} + \frac{1}{m}\mathcal{D}\Big) + a_0 Y_\beta + \sum a_i E_i = K_W + \frac{1}{m}p_*^{-1}\mathcal{D}. \tag{16}$$

Since $(\mathcal{X}_0, \frac{1}{m}\mathcal{D}_0)$ is klt, this implies that $(\mathcal{X}, \frac{1}{m}\mathcal{D} + \mathcal{X}_0)$ is *purely log terminal (plt)* near $\mathcal{X}_0$ by inversion of adjunction (see [29, Theorem 5.50]). Hence for any divisor F whose center is contained in $\mathcal{X}_0$, we have

$$-1 < a\Big(F, \mathcal{X}, \frac{1}{m}\mathcal{D} + \mathcal{X}_0\Big) = a\Big(F, \mathcal{X}, \frac{1}{m}\mathcal{D}\Big) - \mathrm{ord}_F(\mathcal{X}_0) \le a\Big(F, \mathcal{X}, \frac{1}{m}\mathcal{D}\Big) - 1,$$

where ord_F denotes the vanishing order along the divisor F. Therefore, $(\mathcal{X}, \frac{1}{m}\mathcal{D})$ is terminal along $\mathcal{X}_0$ and $a_0 > 0$, $a_i > 0$. Similarly, by writing

$$q^*\Big(K_{\mathcal{X}'} + \frac{1}{m}\mathcal{D}'\Big) + b_0\mathcal{X}_0 + \sum b_i E_i = K_W + \frac{1}{m}q_*^{-1}\mathcal{D}', \tag{17}$$

we obtain $b_0, b_i \ge 0$ because $(Y, \frac{1}{m}E)$ is log canonical thanks to our choice of β and Proposition 5.1. Since the right-hand sides of (16) and (17) are equal to each other by (15), $\mathcal{X}_0 \ne Y_\beta$, and both $K_{\mathcal{X}} + \frac{1}{m}\mathcal{D}$ and $K_{\mathcal{X}'} + \frac{1}{m}\mathcal{D}'$ are $\mathbb{Q}$-linearly equivalent to a relatively *trivial* divisor over C, the implication is that there is a constant $c \le 0$ such that

$$a_0 Y_\beta + \sum a_i E_i = b_0\mathcal{X}_0 + \sum b_i E_i + c \cdot W_0.$$

By comparing the coefficients of Y_β on both sides, we see that $c > 0$; but by comparing the coefficients of $\mathcal{X}_0$ on both sides, we see that $c \le 0$. This contradiction implies that $\mathcal{X}' = \mathcal{X}$. ☐

Remark 5.3

If $m = 1$, then the pair we get is plt instead of klt. The above argument also applies to this case. A similar uniqueness statement is observed in [38, Corollary 4.3], and the above argument indeed gives a straightforward proof of it.

We also note that the automorphism group $\mathrm{Aut}(X, D)$ is always finite by the following well-known fact.

LEMMA 5.4

Let (X, D) be a klt pair such that $-K_X$ is ample and $D \sim_{\mathbb{Q}} -K_X$. Then $\mathrm{Aut}(X, D)$ *is finite.*

Proof

We can choose a sufficiently small $\epsilon > 0$ such that $(X, (1+\epsilon)D)$ is klt, and we know that $K_X + (1+\epsilon)D$ is ample. As $\mathrm{Aut}(X, D)$ preserves $K_X + (1+\epsilon)D$, it gives polarized automorphisms. Therefore, to prove that it is finite, we only need to show that it does not contain $\mathbb{G}_m$ or $\mathbb{G}_a$ as a subgroup. For $\mathbb{G}_m$ this follows from [24, Lemma 3.4]. As mentioned there, the same argument also works verbatim for $\mathbb{G}_a$. ☐

6. Continuity method

In this section, we will develop our continuity method which serves as the main technique in the proof of our main result. Let $(C, 0)$ be a smooth pointed curve. We define

$C^\circ := C \setminus \{0\}$ as before. To begin with, let us fix $\mathfrak{B} \in (0,1]$, and we will assume that the nearby smooth fibers are all $\mathfrak{B}$-*K-polystable* for the rest of this section. We fix an $\epsilon \in (0, \beta_0)$, with β_0 given as in Theorem 5.2. By Lemma 2.4, for any $\beta \in [\epsilon, \mathfrak{B}]$, $(\mathcal{X}_t, \mathcal{D}_t)$ is β-K-polystable. Applying [9]–[11], and [52] (see Corollary 4.2), we conclude that $(\mathcal{X}_t, \mathcal{D}_t)$ admits a (unique when $\beta < 1$ thanks to [7, Theorem 7.3]) conical Kähler–Einstein metric with cone angle $2\pi(1-(1-\beta)/m)$ along $\mathcal{D}_t$ for all $t \in C^\circ$ near zero. This leads us to introduce the following notion.

Definition 6.1
We say that

$$\begin{tikzcd} (\mathcal{X}, \mathcal{D}; \mathcal{L}) \arrow[r] \arrow[d, "\pi"] & \big(\mathbb{P}\mathcal{E}; \mathcal{O}_{\mathbb{P}\mathcal{E}}(1)\big) \arrow[d] \\ C \arrow[r, equal] & C \end{tikzcd}$$

is a *Kähler–Einstein degeneration of index* $(r, \mathfrak{B})$ if, for any $\beta \in [\epsilon, \mathfrak{B}]$,

(1) $\mathcal{D} \in |-mK_{\mathcal{X}}|$;

(2) $\mathcal{L} = K_{\mathcal{X}}^{\otimes -r}$ is relatively *very ample* and $\mathcal{E} = \pi_*\mathcal{L}$ is locally free of rank $N+1$;

(3) $\forall t \in C$, $(\mathcal{X}_t, \frac{1}{m}\mathcal{D}_t)$ is klt and $(\mathcal{X}_t, \mathcal{D}_t)$ is a *smooth* Fano pair for $\forall t \in C^\circ$;

(4) for $\beta < 1$ and $\forall t \in C^\circ$, $(\mathcal{X}_t, \mathcal{D}_t)$ admits a unique Kähler form $\omega(t,\beta) \in C^{,\alpha,\beta}$ in the sense of [16] solving

$$\operatorname{Ric}\big(\omega(t,\beta)\big) = \beta\omega(t,\beta) + \frac{1-\beta}{m}[\mathcal{D}_t] \quad \text{on } \mathcal{X}_t; \tag{18}$$

moreover, $\omega(t,\beta)$ gives rise to Tian's rth embedding

$$T : \big(\mathcal{X}_t, \mathcal{D}_t; \omega(t,\beta)\big) \longrightarrow \mathbb{P}^N.$$

By Theorem 4.1, there is a uniform $r = r(\mathcal{X}, \mathcal{D})$ independent of $\beta \in [\epsilon, \mathfrak{B}]$ such that all Gromov–Hausdorff limits of subsequences of the family $\{(\mathcal{X}_t, \mathcal{D}_t; \omega(t,\beta))\}_{t \in C, \beta \in [\epsilon, \mathfrak{B}]}$ can be embedded into $\mathbb{P}^N$.

Definition 6.2
Continuing with the above notation, we define

$$\mathbf{B}_r(\mathcal{X}, \mathcal{D}) := \left\{ \beta \in [\epsilon, \mathfrak{B}] \;\middle|\; \begin{array}{l} (X, D) \text{ admits a conical Kähler–Einstein metric } \omega(\beta) \\ \text{solving } \operatorname{Ric}\big(\omega(\beta)\big) = \beta\omega(\beta) + \dfrac{1-\beta}{m}[D] \text{ on } X; \\ \text{moreover, } \big(\mathcal{X}_t, \mathcal{D}_t; \omega(t,\beta)\big) \xrightarrow{\mathrm{GH}} \big(X, D; \omega(\beta)\big) \text{ as } t \to 0, \end{array} \right\}$$

and we fix $\mathbf{T}$ such that $\epsilon \le \mathbf{T} \le \sup\{\sigma \in [\epsilon, \mathfrak{B}] \mid [\epsilon, \sigma] \subset \mathbf{B}_r(\mathcal{X}, \mathcal{D})\}$.

By Theorem 4.1, the Gromov–Hausdorff limit of any subsequence of $(\mathcal{X}_{t_i}, \mathcal{D}_{t_i}, \omega(t_i, \beta))$ is a $\mathbb{Q}$-Fano Y together with a $\mathbb{Q}$-Cartier divisor E such that $(Y, \frac{1-\beta}{m}E)$ is log Fano.

LEMMA 6.3
We have

$$\mathbf{B}_r(\mathcal{X}, \mathcal{D}) \supset [\epsilon, \beta_0].$$

Proof
After shrinking C if necessary, we may choose a holomorphic basis

$$\{s_i(t)\}_{i=0}^{N} \subset \Gamma\big(\Delta, \pi_* \mathcal{O}_{\mathcal{X}}(-rK_{\mathcal{X}/\Delta})\big)$$

for the family $\mathcal{X} \to C$ as in Section 4.2, which gives rise to an *algebraic arc*

$$\begin{aligned} z : C &\longrightarrow \mathbb{H}^{\boldsymbol{\chi};N} \times C \\ t &\longmapsto \big(\mathrm{Hilb}(\mathcal{X}_t, \mathcal{D}_t), t\big). \end{aligned} \tag{19}$$

For this arc, we know that $\mathrm{Hilb}(Y, E)$ for the Gromov–Hausdorff limit $(Y, E; \omega_Y)$ of any subsequence $\{(\mathcal{X}_{t_i}, \mathcal{D}_{t_i}; \omega_{\mathrm{KE}}(t_i, \beta))\}_{t_i \to 0}$ lies in the fiber over $0 \in C$ of the morphism

$$\begin{tikzcd} \overline{\mathrm{SL}(N+1)\cdot \mathrm{Im}\, z} \arrow[r] \arrow[d, "\pi_C"] & \mathbb{H}^{\boldsymbol{\chi};N} \times C \arrow[d] \\ C \arrow[r, equal] & C \end{tikzcd}$$

By choosing an arc $\tilde{z} : \tilde{C} \to \overline{\mathrm{SL}(N+1)\cdot \mathrm{Im}\, z}$ that passes through $\mathrm{Hilb}(Y, E)$ and dominates C, and comparing the universal family over $\mathrm{Im}\, \tilde{z} \subset \mathbb{H}^{\boldsymbol{\chi};N} \times C$ with the pullback family induced by the map $\pi_C \circ \tilde{z} : \tilde{C} \to C$, we conclude that $(Y, E) = (X, D)$ as long as $\beta \le \beta_0$ thanks to Theorem 5.2. Our proof is thus completed. $\square$

Remark 6.4
Note that Lemma 6.3 implies that for $\beta \in [0, \beta_0]$, (X, D) is actually *β-K-stable* (see Lemma 5.4), which can also be proved by using Theorem 5.2 and a verbatim extension of the theory of special test configuration developed in [34] to the log setting. In fact, using the latter approach, we can indeed conclude that a pair (X_0, D_0) is β-K-stable if $D_0 \sim -mK_{X_0}$, $(X_0, \frac{1}{m}D_0)$ is klt, and $\beta \in [0, \beta_0]$, *without* assuming that X_0 is smoothable. However, this stronger fact is not needed for the rest of this paper.

From now on, let us assume that $(\mathcal{X}_0, \mathcal{D}_0)$ is $\mathfrak{B}$-*K-polystable*. We will now show that $\mathbf{B}_r(\mathcal{X}, \mathcal{D})$ is both open and closed in the set $[\epsilon, \mathfrak{B}]$; equivalently, we can choose

$$\mathbf{T} = \mathfrak{B} = \max_{[\epsilon,\sigma]\subset \mathbf{B}_r(\mathcal{X},\mathcal{D})} \{\sigma\}.$$

To do this, we first define a map (see Definition 4.6)

$$\begin{aligned} \tau : [\epsilon, \mathfrak{B}] \times C^\circ &\longrightarrow \mathbb{H}^{\boldsymbol{\chi};N}, \\ (\beta, t) &\longmapsto \mathrm{Hilb}\big(\mathcal{X}_t, (1-\beta)\mathcal{D}_t\big). \end{aligned} \tag{20}$$

Then we have the following.

LEMMA 6.5
We have that $\tau|_{[\epsilon,\mathfrak{B}]\times C^\circ}$ *is continuous.*

Proof
By Proposition 4.8, $\tau(\cdot,\cdot)$ is continuous with respect to (β, t) on $[\epsilon, \mathfrak{B}) \times C^\circ$. By Theorem 4.1, the Gromov–Hausdorff limit of $(\mathcal{X}_t, \mathcal{D}_t; \omega(t, \beta_i))$ for any sequence $\beta_i \nearrow \mathfrak{B}$ is $\mathfrak{B}$-K-polystable and lies in $\overline{\mathrm{SL}(N+1)\cdot \mathcal{X}_t}$. On the other hand, since $(\mathcal{X}_t, \mathcal{D}_t)$ is $\mathfrak{B}$-K-polystable, this implies that the limit must lie in $\mathrm{U}(N+1)\cdot \mathrm{Hilb}(\mathcal{X}_t, (1-\mathfrak{B})\mathcal{D}_t)$; hence the metrics $\{h_{\mathrm{KE}}(t,\beta)\}_{(t,\beta)\in[\epsilon,\mathfrak{B}]\times C^\circ}$ (see Section 4.2) vary continuously for $(\beta, t) \in [\epsilon, \mathfrak{B}] \times C^\circ$. So $\tau(\cdot, t)$ is also continuous at $\beta = \mathfrak{B}$ with respect to the basis $\{s_i\}$ in Definition 4.6. Thus the proof is completed. $\square$

By Lemma 6.3, we know that the continuity of $q \circ \tau$ can be extended to $[\epsilon, \beta_0] \times \{0\}$, where $q\colon \mathbb{H}^{\boldsymbol{\chi};N} \to \mathbb{H}^{\boldsymbol{\chi};N}/\mathrm{U}(N+1)$ is the natural quotient morphism, which is continuous with respect to the quotient topology on $\mathbb{H}^{\boldsymbol{\chi};N}/\mathrm{U}(N+1)$. Next, we show that the β-continuity of $q \circ \tau$ can indeed be extended to $[\epsilon, \mathbf{T}] \times \{0\}$ (i.e., including the central fiber) as long as $q \circ \tau$ can be continuously extended to $[\epsilon, \mathbf{T}) \times C$ based on the fact that (X, D) is a degeneration of smooth pairs $(\mathcal{X}_t, \mathcal{D}_t)$ admitting conical Kähler–Einstein metrics $\omega(t, \beta)$ for any $\beta \in [\epsilon, \mathbf{T})$. To do this, let us prefix a *continuous distance* function on $\mathbb{H}^{\boldsymbol{\chi};N}$:

$$\mathrm{dist}_{\mathbb{H}^{\boldsymbol{\chi};N}} : \mathbb{H}^{\boldsymbol{\chi};N} \times \mathbb{H}^{\boldsymbol{\chi};N} \longrightarrow \mathbb{R}_{\geq 0}. \tag{21}$$

LEMMA 6.6
Let us continue with the above setting. In particular, $(X, D) = (\mathcal{X}_0, \mathcal{D}_0)$ *is* $\mathfrak{B}$*-K-polystable. Then* (X, D) *admits a conical Kähler–Einstein metric* $\omega_X(\mathbf{T})$ *with cone angle* $2\pi(1-(1-\mathbf{T})/m)$ *along the divisor* D*. Furthermore, for any sequence* $\{\beta_i\} \subset (\epsilon, \mathbf{T})$ *satisfying* $\beta_i \nearrow \mathbf{T}$*, we have*

$$\mathrm{dist}_{\mathbb{H}^{\boldsymbol{\chi};N}}(\mathrm{Hilb}\big(X, (1-\beta_i)D\big), \mathrm{U}(N+1)\cdot \mathrm{Hilb}\big(X, (1-\mathbf{T})D\big) \longrightarrow 0,$$

where $\mathrm{Hilb}(X,(1-\mathbf{T})D)$ *is the Hilbert point corresponding to the cycle obtained via Tian's embedding of* $(X,D;\omega_X(\mathbf{T}))$.

Proof
By Theorem 4.1 and the definition of $\mathbf{T}$, for any $\beta<\mathbf{T}$, the Gromov–Hausdorff limit as $t\to 0$ of $(\mathcal{X}_t,\mathcal{D}_t;\omega(t,\beta))$ converges to a weak conical Kähler–Einstein metric on $(X,D;\omega(\beta))=(\mathcal{X}_0,\mathcal{D}_0;\omega(0,\beta))$. This implies that for each fixed $\beta_i<\mathbf{T}$, there is a $C^\circ\ni t_i\to 0$ so that

$$\mathrm{dist}_{\mathbb{H}^{\mathcal{X};N}}\big(\mathrm{Hilb}\big(\mathcal{X}_{t_i},(1-\beta_i)\mathcal{D}_{t_i}\big),\mathrm{U}(N+1)\cdot\mathrm{Hilb}\big(X,(1-\beta_i)D\big)\big)<1/i. \tag{22}$$

It follows from Theorem 4.1 that for any subsequence of $\{(\mathcal{X}_{t_i},\mathcal{D}_{t_i};\omega(t_i,\beta_i))\}$, there is a Gromov–Hausdorff convergent subsequence. Now suppose that there is a subsequence

$$\big(\mathcal{X}_{t_{i_k}},\mathcal{D}_{t_{i_k}};\omega(t_{i_k},\beta_{i_k})\big)\xrightarrow{\mathrm{GH}}\big(Y,E;\omega_Y(\mathbf{T})\big)\quad\text{as } k\to\infty,$$

from which we obtain that there are $g_{i_k}\in\mathrm{U}(N+1)$ such that

$$g_{i_k}\cdot\mathrm{Hilb}\big(\mathcal{X}_{t_{i_k}},(1-\beta_{i_k})\mathcal{D}_{t_{i_k}}\big)\longrightarrow\mathrm{Hilb}\big(Y,(1-\mathbf{T})E\big),$$

where $\mathrm{Hilb}(Y,(1-\mathbf{T})E)$ is the Hilbert point corresponding to Tian's embedding of (Y,E) using the limiting conical Kähler–Einstein metric $\omega_Y(\mathbf{T})$ with cone angle $2\pi(1-(1-\mathbf{T})/m)$ along a $\mathbb{Q}$-Cartier divisor E. In particular, (Y,E) is $\mathbf{T}$-K-polystable by [4, Theorem 4.2]. On the other hand, by (22) we have

$$\mathrm{Hilb}\big(Y,(1-\mathbf{T})E\big)\in\overline{\mathrm{SL}(N+1)\cdot\mathrm{Hilb}(X,D)}\subset\mathbb{H}^{\mathcal{X};N}. \tag{23}$$

Suppose that $(Y,E)\ncong(X,D)$. Then by [17, Proposition 1], there is a test configuration of (X,D) with central fiber (Y,E) and vanishing generalized Futaki invariant since (Y,E) is $\mathbf{T}$-K-polystable. This contradicts our assumption that (X,D) is $\mathbf{T}$-K-polystable. Hence we must have $(Y,E)\cong(X,D)$. In particular, X admits a weak conical Kähler–Einstein metric with cone angle $2\pi(1-(1-\mathbf{T}))$ along D.

In conclusion, we have

$$\big(\mathcal{X}_{t_{i_k}},\mathcal{D}_{t_{i_k}};\omega(t_{i_k},\beta_{i_k})\big)\xrightarrow{\mathrm{GH}}\big(X,D;\omega_X(\mathbf{T})\big),$$

which implies that

$$\mathrm{dist}_{\mathbb{H}^{\mathcal{X};N}}(\mathrm{Hilb}\big(X,(1-\beta_i)D\big),\mathrm{U}(N+1)\cdot\mathrm{Hilb}\big(X,(1-\mathbf{T})D\big)\longrightarrow 0.$$

Combining with (22), the proof is completed. ☐

Remark 6.7
Note that in the above argument, the existence of the conical Kähler–Einstein metric on $\mathcal{X}_{t_i}$ is needed only for an angle $\beta_i < \mathbf{T}$ instead of $\mathbf{T}$. So the proof remains *valid* only by assuming that $\mathcal{X}_t$ is $\mathbf{T}$-*K-semistable* for any $t \in C^\circ$ instead of being $\mathbf{T}$-K-polystable.

An immediate consequence is the following.

COROLLARY 6.8
The automorphism group $\mathrm{Aut}(X, D)$ *is finite. If* $\mathbf{T} = 1$*, then* $\mathrm{Aut}(X)$ *is reductive.*

Proof
The first part is just Lemma 5.4. The second part follows from [11, Theorem 6] thanks to the existence of a weak Kähler–Einstein metric on X. □

Let

$$\overline{BO} := \lim_{t\to 0} \overline{\mathrm{SL}(N+1)\cdot \mathrm{Hilb}(\mathcal{X}_t, \mathcal{D}_t)} \subset \mathbb{H}^{\boldsymbol{\chi};N} \tag{24}$$

denote the limiting orbit, and let

$$O_{\mathrm{Hilb}(X,(1-\mathbf{T})D)} = \mathrm{SL}(N+1)\cdot \mathrm{Hilb}\big(X, (1-\mathbf{T})D\big) \qquad \text{and}$$
$$\overline{O_{\mathrm{Hilb}(X,(1-\mathbf{T})D)}} \subset \mathbb{H}^{\boldsymbol{\chi};N}$$

be the $\mathrm{SL}(N+1)$-orbit of $\mathrm{Hilb}(X, (1-\mathbf{T})D)$ and its closure. By Corollary 6.8, this allows us to construct an $\mathrm{SL}(N+1)$-*invariant Zariski-open* neighborhood

$$\mathrm{Hilb}\big(X, (1-\mathbf{T})D\big) \in U \subset \mathbb{H}^{\boldsymbol{\chi};N} \tag{25}$$

satisfying the condition (3) in Lemma 3.1. We point out that the open neighborhood U is *independent* of $\mathbf{T}$ (see Remark 4.7(1)). Then we have the following.

LEMMA 6.9
Let $\{t_i\} \subset C$ *be a sequence of points approaching* $0 \in C$*, and let*

$$\{\beta_i\}, \{\beta_i^*\}, \{\beta_i'\} \subset [\epsilon, 1]$$

be three sequences satisfying $\beta_i^* < \beta_i$ *for all* i*.*

(1) *Assume that* $\beta_i \to \mathbf{T}$, $\beta_i^* \to \mathbf{T}$ *and that there is a sequence* $\{(\mathcal{X}_{t_i}, \mathcal{D}_{t_i}) \mid (\mathcal{X}_{t_i}, \mathcal{D}_{t_i})$ *being* β_i*-K-polystable*$\}$ *with* $t_i \to 0$ *such that*

$$\mathrm{Hilb}\big(\mathcal{X}_{t_i}, (1-\beta_i^*)\mathcal{D}_{t_i}\big) \xrightarrow{i\to\infty} \mathrm{U}(N+1)\cdot \mathrm{Hilb}\big(X, (1-\mathbf{T})D\big), \tag{26}$$

and for $g_i \in \mathrm{U}(N+1)$

$$g_i \cdot \mathrm{Hilb}\big(\mathcal{X}_{t_i}, (1-\beta_i)\mathcal{D}_{t_i}\big) \xrightarrow{i\to\infty} \mathrm{Hilb}\big(Y, (1-\mathbf{T})E\big). \tag{27}$$

Then $\mathrm{Hilb}(Y,(1-\mathbf{T})E) = g \cdot \mathrm{Hilb}(X,(1-\mathbf{T})D)$ *for some* $g \in \mathrm{U}(N+1)$.

(2) *Assume that* $\beta_i' \nearrow \mathbf{T}$ *and that for any fixed* i*, there is a* $g_i \in \mathrm{U}(N+1)$ *such that*

$$\mathrm{Hilb}\big(\mathcal{X}_t, (1-\beta_i')\mathcal{D}_t\big) \xrightarrow{t\to 0} g_i \cdot \mathrm{Hilb}\big(X, (1-\beta_i')D\big) \tag{28}$$

and

$$\mathrm{Hilb}\big(\mathcal{X}_{t_i}, (1-\beta_i')\mathcal{D}_{t_i}\big) \xrightarrow{i\to\infty} \mathrm{Hilb}\big(Y, (1-\mathbf{T})E\big) \in \overline{BO} \setminus O_{\mathrm{Hilb}(X,(1-\mathbf{T})D)}. \tag{29}$$

If $(X,D) \not\cong (Y,E)$*, then there exists a sequence* $\{t_i'\}$ *satisfying* $0 < \mathrm{dist}_C(t_i',0) < \mathrm{dist}_C(t_i,0)$ *such that*

$$\begin{aligned}\mathrm{Hilb}\big(Y', (1-\mathbf{T})E'\big) &= \lim_{i\to\infty} \mathrm{Hilb}\big(\mathcal{X}_{t_i'}, (1-\beta_i')\mathcal{D}_{t_i'}\big)\\ &\in \big(\overline{O_{\mathrm{Hilb}(X,(1-\mathbf{T})D)}} \cup (U \cap \overline{BO})\big) \setminus O_{\mathrm{Hilb}(X,(1-\mathbf{T})D)}\\ &\subset \mathbb{H}^{\boldsymbol{\chi};N},\end{aligned} \tag{30}$$

where $\mathrm{dist}_C : C \times C \to \mathbb{R}$ *is a fixed continuous distance function on* C.

Proof

To prove Lemma 6.9(1), one first notes that (26), together with Lemma 4.9, implies that (X,D) is $\mathbf{T}$-K-polystable. We will show that under the above assumption and

$$\mathrm{Hilb}\big(Y,(1-\mathbf{T})E\big) \notin \mathrm{U}(N+1) \cdot \mathrm{Hilb}\big(X,(1-\mathbf{T})D\big),$$

one can then construct a *new* sequence $\{\beta_i''\}$ satisfying $\beta_i'' \in [\beta_i^*, \beta_i]$ such that

$$\begin{aligned}\mathrm{Hilb}\big(Y', (1-\mathbf{T})E'\big) &= \lim_{i\to\infty} \mathrm{Hilb}\big(\mathcal{X}_{t_i}, (1-\beta_i'')\mathcal{D}_{t_i}\big)\\ &\in \big(\overline{O_{\mathrm{Hilb}(X,(1-\mathbf{T})D)}} \cup (U \cap \overline{BO})\big) \setminus O_{\mathrm{Hilb}(X,(1-\mathbf{T})D)} \subset \mathbb{H}^{\boldsymbol{\chi};N}.\end{aligned}$$

On the other hand, Lemma 4.9 implies that

$$\big(\mathcal{X}_{t_i}, \mathcal{D}_{t_i}; \omega(t_i, \beta_i'')\big) \xrightarrow{\mathrm{GH}} \big(Y', E'; \omega_{Y'}(\mathbf{T})\big),$$

thus (Y',E') admits a weak conical Kähler–Einstein metric with cone angle $2\pi(1-(1-\mathbf{T})/m)$ along E' and hence is $\mathbf{T}$-K-polystable. These allow one to construct either a test configuration of (X, D) with central fiber (Y', E') and vanishing generalized Futaki invariant or a test configuration of (Y', E') with central fiber (X, D) and vanishing generalized Futaki invariant, contradicting the fact that both (X, D) and (Y', E') are $\mathbf{T}$-K-polystable. So we must have

$$\mathrm{Hilb}\big(Y, (1-\mathbf{T})E\big) = g \cdot \mathrm{Hilb}\big(X, (1-\mathbf{T})D\big)$$

for some $g \in \mathrm{U}(N+1)$.

Now we proceed to the construction of $\{\beta_i''\}$. Let

$$B\big(\mathrm{Hilb}\big(X, (1-\mathbf{T})D\big), \epsilon_1\big) \Subset U$$

be the *open* balls of radius ϵ_1 with respect to the distance function (21), and let U be given as in (25).

By shrinking the pointed curve $(0 \in C)$ if necessary, we may assume that

$$\mathrm{Hilb}\big(\mathcal{X}_{t_i}, (1-\beta_i^*)\mathcal{D}_{t_i}\big) \in \mathrm{U}(N+1) \cdot B\big(\mathrm{Hilb}\big(X, (1-\mathbf{T})D\big), \epsilon_1\big) \tag{31}$$

for all i thanks to our assumption (26). On the other hand, by our assumption that $(X, D) \ncong (Y, E)$—and we may assume that (Y, E) is not in the closure of the orbit of (X, D) (otherwise, we can just let $\beta_i'' = \beta_i$)—there is an $\epsilon_1 > 0$ such that

$$\mathrm{dist}_{\mathbb{H}_{\mathcal{X};N}}\big(\mathrm{Hilb}\big(\mathcal{X}_{t_i}, (1-\beta_i)\mathcal{D}_{t_i}\big), O_{\mathrm{Hilb}(X,(1-\mathbf{T})D)}\big) > \epsilon_1 \quad \text{for } i \gg 1.$$

By the β-continuity of $\tau(\cdot, t_i)$ for each fixed $i \gg 1$, for any $0 < \varepsilon < \epsilon_1$ there is

$$\begin{aligned}\beta_{i,k}'' = \sup\big\{\beta \in (\beta_i^*, \beta_i) \;\big|\; \tau(\cdot, t_i)\big|_{(\beta_i^*, \beta)} \subset B(O_{\mathrm{Hilb}(X,(1-\mathbf{T})D)}, \varepsilon/2^k) \\ \cup\, \mathrm{U}(N+1) \cdot B\big(\mathrm{Hilb}\big(X, (1-\mathbf{T})D\big), \epsilon_1\big)\big\},\end{aligned} \tag{32}$$

where $B(O_{\mathrm{Hilb}(X,(1-\mathbf{T})D)}, \varepsilon/2^k)$ is the $\varepsilon/2^k$-tubular neighborhood of $O_{\mathrm{Hilb}(X,(1-\mathbf{T})D)}$; that is, $\beta_{i,k}''$ is the smallest β such that $\tau(\cdot, t_i)$ escapes $B(O_{\mathrm{Hilb}(X,(1-\mathbf{T})D)}, \varepsilon/2^k) \cup \mathrm{U}(N+1) \cdot B(\mathrm{Hilb}(X, (1-\mathbf{T})D), \epsilon_1)$. Clearly, we have $\beta_{i,k+1}'' \le \beta_{i,k}''$. Now if

$$\tau(\beta_{i,0}'', t_i) \in \mathrm{SL}(N+1) \cdot B\big(\mathrm{Hilb}\big(X, (1-\mathbf{T})D\big), \epsilon_1\big),$$

then we let $\beta_i'' = \beta_{i,0}''$; otherwise, we let $\beta_i'' = \beta_{i,k}''$, where $\beta_{i,k}''$ is the *first* number satisfying

$$\tau(\beta_{i,k}'', t_i) \in \mathrm{SL}(N+1) \cdot B\big(\mathrm{Hilb}\big(X, (1-\mathbf{T})D\big), \epsilon_1\big).$$

Such a k exists because of (31). Now by our construction, there is a $g_i \in \mathrm{SL}(N+1)$ such that

$$\tau(\beta_i'', t_i) \in g_i \cdot B\big(\mathrm{Hilb}\big(X, (1-\mathbf{T})D\big), \epsilon_1\big). \tag{33}$$

We let

$$M_i = \inf\big\{\mathrm{Tr}(g^*g) \,\big|\, g \in \mathrm{SL}(N+1) \text{ such that (33) is satisfied}\big\} + 1,$$

and by passing through a subsequence we may assume that $\mathrm{Tr}(g_i^* g_i) \le M_i$. Then we have the following dichotomy.

Case 1: There is a subsequence $\{M_{i_l}\}$ such that $|M_{i_l}| < M$ for some constant M independent of i. Then we claim that

$$\big\{\tau(\beta''_{i_l}, t_{i_l}) = \mathrm{Hilb}\big(\mathcal{X}_{t_{i_l}}, (1-\beta''_{i_l})\mathcal{D}_{t_{i_l}}\big)\big\}$$

is the subsequence we want, and its limit $\mathrm{Hilb}(Y', (1-\mathbf{T})E')$ lies in

$$(U \cap \overline{BO}) \setminus O_{\mathrm{Hilb}(X,(1-\mathbf{T})D)}.$$

To see this, one only needs to note that it follows from our construction of β''_{i_l} that

$$\mathrm{dist}_{\mathbb{H}^{\boldsymbol{\chi};N}}\big(\tau(\beta''_{i_l}, t_i), O_{\mathrm{Hilb}(X,(1-\mathbf{T})D)}\big)$$

is *uniformly* bounded from below by some $\varepsilon/2^k$, since there is a $k = k(M)$ such that

$$\begin{aligned}&\big\{z \in \mathbb{H}^{\boldsymbol{\chi};N} \,\big|\, \mathrm{dist}_{\mathbb{H}^{\boldsymbol{\chi};N}}\big(z, g\cdot \mathrm{Hilb}\big(X,(1-\mathbf{T})D\big)\big) \le \varepsilon/2^{k(M)} \text{ and } |g| < M\big\}\\ &\quad \subset \mathrm{SL}(N+1)\cdot U.\end{aligned}$$

Case 2: $|M_i| \to \infty$. If this happens, we replace ε by $\varepsilon/2$ in (32) and repeat the above process. If for the new sequence $\{M_i^{[1]}\} \subset \mathbb{R}$ there is a bounded subsequence $\{M_{i_l}^{[1]}\}$, then we reduce to Case 1; otherwise, we keep on repeating this process. Then either we stop at a finite stage, or this becomes an infinite process. If we stop at a finite stage, then we obtain our subsequence as before; if the process never terminates, we claim that we are able to extract a subsequence whose limit $\mathrm{Hilb}(Y', (1-\mathbf{T})E')$ lands in the *boundary*

$$\partial\overline{O_{\mathrm{Hilb}(X,(1-\mathbf{T})D)}} = \overline{O_{\mathrm{Hilb}(X,(1-\mathbf{T})D)}} \setminus O_{\mathrm{Hilb}(X,(1-\mathbf{T})D)}.$$

This is because by choosing a diagonal sequence we will have

$$\mathrm{dist}_{\mathbb{H}^{\boldsymbol{\chi};N}}\big(\tau(\beta_{i_k}^{\prime\prime,[k]}, t_{i_k}), O_{\mathrm{Hilb}(X,(1-\mathbf{T})D)}\big) < \varepsilon/2^k \to 0,$$

so we know that

$$z := \lim_{k\to\infty} \tau(\beta_{i_k}^{\prime\prime,[k]}, t_{i_k}) \in \overline{O_{\mathrm{Hilb}(X,(1-\mathbf{T})D)}}.$$

On the other hand, if $z \in O_{\mathrm{Hilb}(X,(1-\mathbf{T})D}$, then

$$z = g \cdot \mathrm{Hilb}\big(X, (1-\mathbf{T})D\big)$$

for some $g \in \mathrm{SL}(N+1)$. In particular, $g \cdot B(\mathrm{Hilb}(X,(1-\mathbf{T})D), \epsilon_1)$ contains a neighborhood of z. However, this violates the assumption that $|M_{i_k}^{[k]}| \to \infty$ as $k \to \infty$. Hence our proof is completed.

The proof of Lemma 6.9(2) is similar. In contrast to part (1), we will vary t instead of β in $\tau(\beta, t)$. First by our assumption (29) together with Lemma 4.9, (Y, E) is $\mathbf{T}$-K-polystable, and hence

$$\mathrm{Hilb}\big(Y, (1-\mathbf{T})E\big) \notin \partial\overline{O_{\mathrm{Hilb}(X,(1-\mathbf{T})D)}}.$$

So there is an $\epsilon_1 > 0$ such that

$$\mathrm{dist}_{\mathbb{H}^{\chi;N}}\big(\mathrm{Hilb}\big(\mathcal{X}_{t_i}, (1-\beta_i')\mathcal{D}_{t_i}\big), O_{\mathrm{Hilb}(X,(1-\mathbf{T})D)}\big) > \epsilon_1 \quad \text{for } i \gg 1.$$

On the other hand, by our assumption (28) and Lemma 6.6 we have that, for any *fixed* β_i' with $i \gg 1$, there is a $0 < s_i \in \mathbb{R}$ such that

$$\mathrm{Hilb}\big(\mathcal{X}_{t_i'}, (1-\beta_i')\mathcal{D}_{t_i'}\big) \in \mathrm{U}(N+1) \cdot B\big(\mathrm{Hilb}\big(X, (1-\mathbf{T})D\big), \epsilon_1\big)$$

for any t satisfying $0 < \mathrm{dist}_C(t, 0) < s_i$, since

$$\mathrm{U}(N+1) \cdot \mathrm{Hilb}\big(X, (1-\beta_i')D\big) \xrightarrow{i\to\infty} \mathrm{U}(N+1) \cdot \mathrm{Hilb}\big(X, (1-\mathbf{T})D\big)$$

inside $\mathbb{H}^{\chi;N}/\mathrm{U}(N+1)$.

By the t-continuity of $\tau(\beta_i', \cdot)$ for each fixed $i \gg 1$, for any $\varepsilon < \epsilon_1/2$ there is

$$\begin{aligned} s_{i,k} := \sup\big\{s \in \big[0, |t_i|\big) \bigm| \tau(\beta_i', \cdot)\big|_{B_C(0,s)} &\subset B(O_{\mathrm{Hilb}(X,(1-\mathbf{T})D)}, \varepsilon/2^k) \\ &\cup \mathrm{U}(N+1) \cdot B\big(\mathrm{Hilb}\big(X, (1-\mathbf{T})D\big), \epsilon_1\big)\big\}, \end{aligned} \tag{34}$$

where $|t_i| := \mathrm{dist}_C(t_i, 0)$ and $B_C(0,s) := \{t \in C \mid \mathrm{dist}_C(t,0) \le s\}$. Then $s_{i,k} = |t_{i,k}|$ is the smallest distance needed for t so that $\tau(\beta_i', t)$ escapes $B(O_{\mathrm{Hilb}(X,(1-\mathbf{T})D)}, \varepsilon/2^k) \cup \mathrm{U}(N+1) \cdot B(\mathrm{Hilb}(X,(1-\mathbf{T})D), \epsilon_1)$. Clearly, we have $s_{i,k+1} < s_{i,k}$. Now if

$$\tau(\beta_i', t_{i,0}) \in \mathrm{SL}(N+1) \cdot B\big(\mathrm{Hilb}\big(X, (1-\mathbf{T})D\big), \epsilon_1\big),$$

then we let $t_i' = t_{i,0}$; otherwise, we let $t_i' = t_{i,k}'$, where $t_{i,k}'$ is the *first* point in C satisfying

$$\tau(\beta_i', t_{i,k}') \in \mathrm{SL}(N+1) \cdot B\big(\mathrm{Hilb}\big(X, (1-\mathbf{T})D\big), \epsilon_1\big).$$

Such a process must terminate in *finite* steps by (28). Now we define $M_i \in \mathbb{R}$ to be

$$M_i := \inf_{g_i}\big\{\mathrm{Tr}(g_i^* g_i) + 1 | \tau(\beta_i', t_i') \in g_i \cdot B\big(\mathrm{Hilb}\big(X, (1-\mathbf{T})D\big), \epsilon_1\big)\big\}.$$

Then again we have two situations exactly the same as in the proof of Lemma 6.9(1) depending on $\{M_i\}$ being bounded or not. Replacing β_i'' by t_i' in the argument for part (1), one see that the rest of the proof is a verbatim copy, which we omit. Thus the proof of Lemma 6.9 is completed. $\square$

Remark 6.10
Note that when $\mathbf{T} = 1$ and both $\beta_i, \beta_i^* \leq 1$, $\forall i$, then Lemma 6.9 and its proof imply a slight variation of the following form.

Let

$$\begin{aligned} \pi_1 : \mathbb{H}^{\boldsymbol{\chi};N} = \mathbb{H}^{\chi;N} \times \mathbb{P}^{\tilde{\chi};N} &\longrightarrow \mathbb{P}^{\chi,n;N} \\ \big(\mathrm{Hilb}(X), \mathrm{Hilb}(D)\big) &\longmapsto \mathrm{Hilb}(X) \end{aligned} \tag{35}$$

be the projection to the first factor.

(1) Assume that $\beta_i \to 1$, $\beta_i^* \to 1$ and assume that there is a sequence $\{(\mathcal{X}_{t_i}, \mathcal{D}_{t_i}) \mid (\mathcal{X}_{t_i}, \mathcal{D}_{t_i})$ being β_i-K-polystable$\}$ with $t_i \to 0$ such that

$$\pi_1\big(\mathrm{Hilb}\big(\mathcal{X}_{t_i}, (1-\beta_i^*)\mathcal{D}_{t_i}\big)\big) \xrightarrow{i\to\infty} \mathrm{U}(N+1)\cdot \mathrm{Hilb}(X) \subset \mathbb{H}^{\chi;N}, \tag{36}$$

and for $g_i \in U(N+1)$

$$\pi_1\big(g_i \cdot \mathrm{Hilb}\big(\mathcal{X}_{t_i}, (1-\beta_i)\mathcal{D}_{t_i}\big)\big) \xrightarrow{i\to\infty} \mathrm{Hilb}(Y) \in \mathbb{H}^{\chi;N}. \tag{37}$$

Then $\mathrm{Hilb}(Y) = g \cdot \mathrm{Hilb}(X)$ for some $g \in U(N+1)$.

(2) Assume that $\beta_i' \nearrow 1$ and suppose that for any fixed i there is a $g_i \in \mathrm{U}(N+1)$ such that

$$\mathrm{Hilb}\big(\mathcal{X}_t, (1-\beta_i')\mathcal{D}_t\big) \xrightarrow{t\to 0} g_i \cdot \mathrm{Hilb}\big(X, (1-\beta_i')D\big) \in \mathbb{H}^{\boldsymbol{\chi};N} \tag{38}$$

and

$$\pi_1\big(\mathrm{Hilb}\big(\mathcal{X}_{t_i}, (1-\beta_i')\mathcal{D}_{t_i}\big)\big) \xrightarrow{i\to\infty} \mathrm{Hilb}(Y) \in \overline{BO} \setminus O_{\mathrm{Hilb}(X)} \subset \mathbb{H}^{\chi;N}. \tag{39}$$

If $X \not\cong Y$, then there exists a sequence $\{t_i'\}$ satisfying $0 < \mathrm{dist}_C(t_i', 0) < \mathrm{dist}_C(t_i, 0)$ such that

$$\begin{aligned} \mathrm{Hilb}(Y') &= \lim_{i\to\infty} \pi_1\big(\mathrm{Hilb}\big(\mathcal{X}_{t_i'}, (1-\beta_i')\mathcal{D}_{t_i'}\big)\big) \\ &\in \big(\overline{O_{\mathrm{Hilb}(X)}} \cup (U \cap \overline{BO})\big) \setminus O_{\mathrm{Hilb}(X)} \subset \mathbb{H}^{\chi;N}, \end{aligned} \tag{40}$$

where $\mathrm{dist}_C : C \times C \to \mathbb{R}$ is a fixed continuous distance function on C.

Now we are ready to prove the openness.

PROPOSITION 6.11
Let $(\mathcal{X}, \mathcal{D}; \mathcal{L}) \to C$ *be a Kähler–Einstein degeneration of index* $(r, \mathfrak{B})$ *as in Definition 6.1 with* $r = r(\mathcal{X}, \mathcal{D})$ *being the uniform index as in Theorem 4.1(3). Then* $\mathbf{B}_r(\mathcal{X}, \mathcal{D}) \subset [\epsilon, \mathfrak{B}]$ *is an open set.*

Proof
Let us assume that $\mathbf{T} \in \mathbf{B}_r(\mathcal{X}, \mathcal{D})$. Then by fixing a local basis $\{s_i\}$ for $\pi_* \omega_{\mathcal{X}/C}^{-\otimes r}$, we have

$$\mathrm{dist}_{\mathbb{H}^{\mathcal{X};N}} \big(\mathrm{Hilb}\big(\mathcal{X}_t, (1-\mathbf{T})\mathcal{D}_t\big), \mathrm{U}(N+1) \cdot \mathrm{Hilb}\big(X, (1-\mathbf{T})D\big)\big) \longrightarrow 0 \quad \text{as } t \to 0. \tag{41}$$

Now we claim that there is a $\delta > 0$ such that $[\epsilon, \mathbf{T} + \delta) \subset \mathbf{B}_r(\mathcal{X}, \mathcal{D})$. Suppose that does not hold. For any k, there is a $\mathbf{T} < \beta_k < \mathbf{T} + 1/k$ and a sequence $\{t_{i,k}\}_{k=1}^{\infty}$ such that

$$\mathrm{Hilb}\big(\mathcal{X}_{t_{i,k}}, (1-\beta_k)\mathcal{D}_{t_{i,k}}\big) \stackrel{i\to\infty}{\longrightarrow} \mathrm{Hilb}\big(Y_k, (1-\beta_k)E_k\big) \notin U \subset \mathbb{H}^{\mathcal{X};N},$$

with $U \subset \mathbb{H}^{\mathcal{X};N}$ being the $\mathrm{SL}(N+1)$-invariant Zariski-open neighborhood of $\mathrm{Hilb}(X, (1-\mathbf{T})D)$ constructed in Lemma 3.1, since (X, D) is also β_k-K-polystable because of $\beta_k \in [\epsilon, \mathfrak{B}]$ and Lemma 2.4. For any fixed i, we can pick up $k_i \gg 0$ such that

$$\mathrm{Hilb}\big(\mathcal{X}_{t_{i,k_i}}, (1-\beta_{k_i})\mathcal{D}_{t_{i,k_i}}\big) \notin \mathrm{U}(N+1) \cdot B\big(\mathrm{Hilb}\big(X, (1-T)D\big), \epsilon_1\big).$$

Now let us introduce the *diagonal* sequence

$$\big\{\mathrm{Hilb}\big(\mathcal{X}_{t_i}, (1-\beta_i)\mathcal{D}_{t_i}\big) := \mathrm{Hilb}\big(\mathcal{X}_{t_{i,k_i}}, (1-\beta_{k_i})\mathcal{D}_{t_{i,k_i}}\big)\big\}_{i=0}^{\infty}.$$

Then by Theorem 4.1, after passing to a subsequence if necessary, we obtain a new sequence, which by abuse of notation will still be denoted by $\beta_i \searrow \mathbf{T}$ and $t_i \to 0$, such that

$$\mathrm{Hilb}\big(\mathcal{X}_{t_i}, (1-\beta_i)\mathcal{D}_{t_i}\big) \longrightarrow \mathrm{Hilb}\big(Y, (1-\mathbf{T})E\big) \notin O_{\mathrm{Hilb}(X,(1-\mathbf{T})D)}. \tag{42}$$

But this violates Lemma 6.9(1) with $\beta_i^* = \mathbf{T}\ \forall i$. □

Next, we prove the closedness.

PROPOSITION 6.12
Let $(\mathcal{X}, \mathcal{D}) \to C$ *be a family satisfying the condition of Proposition 6.11. Suppose further that* $\mathcal{X} \to C$ *is a family of* $\mathfrak{B}$-K-polystable *varieties. Then* $\mathbf{B}(\mathcal{X}, \mathcal{D}) \subset [\epsilon, \mathfrak{B}]$ *is also closed with respect to the induced topology, and hence* $\mathbf{B}(\mathcal{X}, \mathcal{D}) = [\epsilon, \mathfrak{B}]$.

Proof
By our assumption, for every $t \in C^\circ$, $(\mathcal{X}_t, \mathcal{D}_t)$ is a smooth Fano pair with $\mathcal{D}_t \in |-mK_{\mathcal{X}_t}|$. Since $\mathcal{X}_t$ is $\mathfrak{B}$-K-polystable, it is β-K-polystable for $\beta \in [\epsilon, \mathfrak{B}]$ by Lemma 2.4. As $(\mathcal{X}_t, \mathcal{D}_t)$ are smooth, by Theorem 4.1, [45, Proposition 2.2], and [31, Proposition 1.7] they admit a unique conical Kähler–Einstein metric ω_t solving

$$\mathrm{Ric}\big(\omega(t,\beta)\big) = \beta\omega(t,\beta) + \frac{1-\beta}{m}[\mathcal{D}_t]$$

with cone angle $2\pi(1-(1-\beta)/m)$ along $\mathcal{D}_t$ for any $\beta \in [\epsilon, \mathfrak{B}]$. By Theorem 4.1 and the definition of $\mathbf{T}$, for any *fixed* $\beta < \mathbf{T}$, we have

$$\big(\mathcal{X}_t, \mathcal{D}_t; \omega(t,\beta)\big) \xrightarrow{\mathrm{GH}} \big(\mathcal{X}_0, \mathcal{D}_0; \omega(0,\beta)\big) \quad \text{as } t \to 0.$$

By Lemma 6.6, for any sequence $\beta_i \nearrow \mathbf{T}$, we have

$$\mathrm{dist}_{\mathbb{H}_{\mathcal{X};N}}\Big(\mathrm{Hilb}\big(X,(1-\beta_i)D\big), \mathrm{U}(N+1)\cdot \mathrm{Hilb}\big(X,(1-\mathbf{T})D\big)\Big) \longrightarrow 0.$$

Our goal is to prove that

$$\mathrm{Hilb}\big(\mathcal{X}_t, (1-\mathbf{T})\mathcal{D}_t\big) \longrightarrow \mathrm{U}(N+1)\cdot \mathrm{Hilb}\big(X,(1-\mathbf{T})D\big)\big) \quad \text{as } t \to 0.$$

We will argue by contradiction. Suppose first that this is not the case. Then there is a subsequence $\{t_i\}_{i=1}^\infty \subset C$, $t_i \to 0$ as $i \to \infty$ such that

$$\mathrm{Hilb}\big(\mathcal{X}_{t_i}, (1-\mathbf{T})\mathcal{D}_{t_i}\big) \to \mathrm{Hilb}\big(Y,(1-\mathbf{T})E\big) \notin \mathrm{U}(N+1)\cdot \mathrm{Hilb}\big(X,(1-\mathbf{T})D\big).$$

By the continuity of $\tau(\cdot, t_i)$ at $\mathbf{T}$ for each fixed i (see Lemma 6.5), there is a consequence $\{\beta_i'\}_{i=1}^\infty \subset (\epsilon_0, \mathbf{T})$ such that $\beta_i' \nearrow \mathbf{T}$ and

$$\begin{aligned}\mathrm{Hilb}\big(\mathcal{X}_{t_i}, (1-\beta_i')\mathcal{D}_{t_i}\big) &\to \mathrm{Hilb}\big(Y,(1-\mathbf{T})E\big)\\ &\notin \mathrm{U}(N+1)\cdot \mathrm{Hilb}\big(X,(1-\mathbf{T})D\big) \quad \text{as } i \to \infty. \end{aligned} \tag{43}$$

We claim that $\mathrm{Hilb}(Y,(1-\mathbf{T})E) \in \overline{BO} \setminus \mathrm{SL}(N+1)\cdot U$. Otherwise, $\mathrm{Hilb}(Y,(1-\mathbf{T})E) \in U$. Then

$$\mathrm{Hilb}\big(X,(1-\mathbf{T})D\big) \in \overline{\mathrm{SL}(N+1)\cdot \mathrm{Hilb}\big(Y,(1-\mathbf{T})E\big)}.$$

But this violates the fact that (Y,E) is $\mathbf{T}$-K-polystable by [4, Theorem 4.2], since we can construct a test configuration of (Y,E) with central fiber (X,D) and vanishing generalized Futaki invariant. Hence our claim is proved.

Now we can apply Lemma 6.9(2) to obtain a *new* sequence $\{t_i'\} \subset C^\circ$ satisfying $t_i' \to 0 \in C$ and

$$
\begin{aligned}
\mathrm{Hilb}\big(Y', (1-\mathbf{T})E'\big) &= \lim_{i\to\infty} \mathrm{Hilb}\big(\mathcal{X}_{t_i'}, (1-\beta_i')\mathcal{D}_{t_i'}\big) \\
&\in \big(\overline{O_{\mathrm{Hilb}(X,(1-\mathbf{T})D)}} \cup (U \cap \overline{BO})\big) \setminus O_{\mathrm{Hilb}(X,(1-\mathbf{T})D)} \\
&\subset \mathbb{H}^{\chi;N}, \qquad (44)
\end{aligned}
$$

which contradicts the fact that both (Y', E') and (X, D) are $\mathbf{T}$-K-polystable by the same reason as above. Thus the proof is completed. □

Remark 6.13
We remark that, interestingly, in the proof of Proposition 6.11 we only used the continuity of $\tau(\cdot, t)$ for each fixed t. In particular, the continuity of τ with respect to the variable t is not used. Contrast this with the continuity of $\tau(\beta, \cdot)$ with respect to t, which is what we use in the proof of Proposition 6.12.

We note that, by this point, we have already established the following.

COROLLARY 6.14
Theorem 1.2 holds under an additional *assumption that $\mathcal{X}_t$ is β-K-polystable for all $t \in C^\circ$.*

7. K-semistability of the nearby fibers

7.1. Orbit of K-semistable points

In this section, we extend our continuity method to study the *uniqueness* of K-polystable Fano varieties that a K-semistable Fano manifold can specialize to, which will also be needed in the proof of our main theorem.

Let X be a smooth Fano manifold, and let $D \in |-mK_X|$ be a smooth divisor for $m \geq 2$. Assume that X is $\mathbf{T}$-K-semistable with respect to D. By Theorem 4.1, we know that for any sequence $\beta_i \nearrow \mathbf{T}$, after possibly passing to a subsequence (which by abuse of notation will still be denoted by $\beta_i \nearrow \mathbf{T}$), there exists a log $\mathbb{Q}$-Fano pair (X_0, D_0) which is the Gromov–Hausdorff limit of the conical Kähler–Einstein metric $(X, D; \omega(\beta_i))$, that is,

$$
\begin{aligned}
\mathrm{Hilb}\big(X, (1-\beta_i)D\big) &\longrightarrow \mathrm{U}(N+1) \cdot \mathrm{Hilb}\big(X_0, (1-\mathbf{T})D_0\big) \\
&\in \overline{O_{\mathrm{Hilb}(X,(1-\mathbf{T})D)}} \quad \text{as } i \to \infty
\end{aligned}
$$

with X_0 being $\mathbf{T}$-K-polystable, where

$$
\overline{O_{\mathrm{Hilb}(X,(1-\mathbf{T})D)}} = \text{the closure of } \mathrm{SL}(N+1) \cdot \mathrm{Hilb}\big(X, (1-\mathbf{T})D\big) \subset \mathbb{H}^{\chi;N}.
$$

In particular, (X_0, D_0) admits a weak conical Kähler–Einstein metric $\omega(\mathbf{T})$ with cone angle $2\pi(1-(1-\mathbf{T})/m)$ along the divisor $D_0 \subset X_0$.

LEMMA 7.1
The limit is independent of the choice of the sequence $\{\beta_i\}$ in the sense that for every sequence $\beta_i \nearrow \mathbf{T}$,

$$\big(X, D; \omega(\beta_i)\big) \xrightarrow{\text{GH}} \big(X_0, D_0; \omega(\mathbf{T})\big).$$

Proof
The existence of a weak conical Kähler–Einstein metric $\omega(\mathbf{T})$ on (X_0, D_0) allows us to construct a test configuration $(\mathcal{X}, \mathcal{D}; \mathcal{L})$ of (X, D) with central fiber (X_0, D_0) since $\operatorname{Aut}(X_0, D_0)$ is reductive by Theorem 4.1. Now our claim follows by applying Lemma 6.9(1) to the family $(\mathcal{X}, \mathcal{D}; \mathcal{L})$. $\square$

THEOREM 7.2
Suppose that X is a smooth K-semistable Fano manifold and that $D_0 \in |-m_0 K_X|$ and $D_1 \in |-m_1 K_X|$ are two smooth divisors. Let X_0 and X_1 be the limits defined as in Lemma 7.1 with $\mathbf{T} = 1$. Then $X_0 \cong X_1$.

Proof
By introducing a third divisor in $|-mK_X|$ with $m = \operatorname{lcm}(m_0, m_1)$, we may assume that $rm_0 = m_1$ for a positive integer r. By Bertini's theorem, we may choose $\{D_t\}_{t\in[0,1]} \subset |-mK_X|$ to be a continuous path joining rD_0 and D_1 such that

- the path $\{D_t\}$ lies in an algebraic arc $C \subset |-mK_X|$ with corresponding family $\mathcal{D} \to C$,
- D_t is smooth for all $t \neq 0$.

By assumption, X is K-semistable, and hence (X, D_t) are β-K-stable for all $(\beta, t) \in (0,1) \times (0,1]$. In particular, $\{(X, D_t)\}$ admit a conical Kähler–Einstein metric $\omega(t, \beta)$, $\forall (\beta, t) \in (0,1) \times [0,1]$ by Corollary 4.2. Using Tian's embedding, we can similarly define a map

$$\begin{aligned} \sigma : (0,1) \times (0,1] &\longrightarrow \mathbb{H}^{\boldsymbol{\chi};N}, \\ (\beta, t) &\longmapsto \operatorname{Hilb}\big(X, (1-\beta) D_t\big) \end{aligned} \tag{45}$$

using a prefixed basis of $H^0(X, \mathcal{O}_X(-rK_X))$. By Proposition 4.8 and [16, Theorem 2], σ is continuous on $(0,1) \times (0,1]$. We claim that $q \circ \sigma$ is continuous on $(0,1) \times [0,1]$ with $q\colon \mathbb{H}^{\boldsymbol{\chi};N} \to \mathbb{H}^{\boldsymbol{\chi};N}/\mathrm{U}(N+1)$. For fixed $\beta \in (0,1)$, we can deduce the continuity of $\sigma(\beta, \cdot)$ at zero by applying Corollary 6.14 to the product family $(\mathcal{X} = X \times C, \mathcal{D}) \to C$ with $(\mathcal{X}_t, \mathcal{D}_t) = (X, D_t)$. Thus, all we need to show is

$$\lim_{\beta\to 1} \mathrm{dist}_{\mathbb{H}^{\chi;N}}\big(\tilde{\sigma}(\beta,t), \mathrm{U}(N+1)\cdot \mathrm{Hilb}(X_0)\big) = 0, \quad \forall t \in [0,1], \tag{46}$$

where $\tilde{\sigma} := \pi_1 \circ \sigma$ with π_1 being given in (35). To achieve this, let $\tilde{q} : \mathbb{H}^{\chi;N} \to \mathbb{H}^{\chi;N}/\mathrm{U}(N+1)$. Then Lemma 7.1 allows us to introduce

$$\lim_{\beta\to 1} \tilde{q}\circ\tilde{\sigma}(\beta,t) = \mathrm{U}(N+1)\cdot \mathrm{Hilb}(X_t) \in \mathbb{H}^{\chi;N}/\mathrm{U}(N+1) \quad \text{for } t\in[0,1]$$

with X_t being a $\mathbb{Q}$-Fano variety admitting a weak Kähler–Einstein metric for each $t \in [0,1]$. Let $\mathcal{X}_1 \to \mathbb{A}^1$ be a test configuration with central fiber X_1, and let $\mathrm{Hilb}(X_1) \in U \subset \mathbb{H}^{\chi;N}$ be the open neighborhood constructed for the family $\mathcal{X}_1 \to \mathbb{A}^1$ via Lemma 3.1.

Now suppose that (46) does not hold; that is, there is a $t_0 \in [0,1]$ such that

$$\lim_{\beta\to 1} \tilde{\sigma}(\beta,t_0) = \mathrm{Hilb}(X_{t_0}) \notin U\cdot \mathrm{Hilb}(X_1).$$

Then by applying the continuity of $\tilde{q}\circ\tilde{\sigma}(\beta,\cdot)$ with respect to $t\in[0,1]$ for fixed β the same way as in the proof of Lemma 6.9(2), we can construct a new sequence $\{(\beta_i,t_i)\}_{i=1}^{\infty} \subset (0,1]\times[t_0,1]$ such that $\beta_i \nearrow 1$ as $i\to\infty$ and

$$\mathrm{Hilb}(Y) = \lim_{i\to\infty} \tilde{\sigma}(\beta_i,t_i) \in \big(\overline{O_{\mathrm{Hilb}(X_1)}} \cup (U\cap \partial\overline{O_{\mathrm{Hilb}(X)}})\big)\setminus O_{\mathrm{Hilb}(X_1)} \subset \mathbb{H}^{\chi;N},$$

with both X_1 and Y ($\ncong X_1$) being K-polystable, which is impossible. Hence our proof is completed. ☐

7.2. Zariski openness of K-semistable varieties

In this section, we will study the Zariski openness of the locus of the $\mathbb{Q}$-Gorenstein smoothable, K-semistable varieties inside Hilbert schemes. This needs a combination of the continuity method with the algebraic result in Appendix A.1.

Let

$$\begin{array}{ccc} (\mathcal{X},\mathcal{D}) & \xrightarrow{\ \iota\ } & \mathbb{P}^N\times\mathbb{P}^N\times S \\ \big\downarrow{\scriptstyle\pi} & & \big\downarrow \\ S & = & S \end{array}$$

be a flat family of $\mathbb{Q}$-*Fano varieties* over a smooth (not necessarily complete) base S, and let $\mathcal{D} \in |-mK_{\mathcal{X}}|$ be an irreducible divisor defined by a section $s_{\mathcal{D}} \in \Gamma(S,\mathcal{O}_{\mathcal{X}}(-mK_{\mathcal{X}}))$. Let us assume further that $\mathcal{O}_{\mathcal{X}}(-rK_{\mathcal{X}})$ is relatively very ample and that ι is the embedding induced by a prefixed basis $\{s_i(t)\}_{i=0}^N \subset \Gamma(S, \pi_*\mathcal{O}_{\mathcal{X}}(-rK_{\mathcal{X}/S}))$, and in particular, that $\iota^*\mathcal{O}_{\mathbb{P}^N}(1) \cong \mathcal{O}_{\mathcal{X}}(-rK_{\mathcal{X}/S})$. Then we have the following.

THEOREM 7.3
Let $(\mathcal{X}, \mathcal{D}) \to C$ *be the family over a* smooth curve *such that* $(\mathcal{X}_t, \mathcal{D}_t)$ *is smooth for* $t \in C^\circ$ *and* $(\mathcal{X}_t, \frac{1}{m}\mathcal{D}_t)$ *is klt for all* $t \in C$. *Assume that* $(\mathcal{X}_0, \mathcal{D}_0)$ *is* $\mathfrak{B}$*-K-semistable. Then there is a Zariski-open neighborhood* $0 \in C^* \subset C$ *such that* $(\mathcal{X}_t, \mathcal{D}_t)$ *is* $\mathfrak{B}$*-K-semistable for* $t \in C^*$. *Furthermore, if* $(\mathcal{X}_0, \mathcal{D}_0)$ *is* $\mathfrak{B}$*-K-polystable and has only finitely many automorphisms, then* $(\mathcal{X}_t, \mathcal{D}_t)$ *is* $\mathfrak{B}$*-K-polystable after a possible further shrinking of* C^*.

Definition 7.4
For every $t \in S$, we define the kst as

$$\operatorname{kst}(\mathcal{X}_t, \mathcal{D}_t) := \sup\bigl\{\beta \in [0, \mathfrak{B}] \bigm| (\mathcal{X}_t, \mathcal{D}_t) \text{ is } \beta\text{-K-semistable}\bigr\}.$$

By Theorem 4.1, testing β-K-semistability for $\mathcal{X}_t$, $\forall t \in S$, is reduced to testing for all 1-PS's inside $\mathrm{SL}(N+1)$ for a fixed sufficiently large $\mathbb{P}^N$. This implies that $\operatorname{kst}(\mathcal{X}_t, \mathcal{D}_t)$ is a constructible function of t (see Proposition 7.5 below). By Remark 6.4, we know that $(\mathcal{X}_t, \mathcal{D}_t)$ is β-K-stable for all $\beta \in (0, \beta_0]$. This, together with Lemma 2.4, in particular implies that $\operatorname{kst}(\mathcal{X}_t, \mathcal{D}_t)$ is actually a *maximum* for every $t \in S$.

Then we have the following proposition which essentially follows from Paul's work, especially his theory on stability of pairs (see [41, Theorem 1.3]). For the reader's convenience, a proof is included in Proposition A.4 in the Appendix.

PROPOSITION 7.5
The K-semistable threshold $\operatorname{kst}(\mathcal{X}_t, \mathcal{D}_t)$ *defines a constructible function on* S*; that is,* $S = \bigsqcup_i S_i$ *is a union of* finite *constructible sets* $\{S_i\}$ *on which* $\operatorname{kst}(\mathcal{X}_t, \mathcal{D}_t)$ *is constant.*

Proof of Theorem 7.3
By Proposition 7.5, $\operatorname{kst}(\mathcal{X}_t, \mathcal{D}_t)$ is constant when restricted to each stratum S_i. So all we need to show is that if $t_i \to 0$ and $(\mathcal{X}_{t_i}, \mathcal{D}_{t_i})$ is *strictly* $\mathbf{T}$-K-semistable, then

$$\mathbf{T} = \operatorname{kst}(\mathcal{X}_{t_i}, \mathcal{D}_{t_i}) \geq \operatorname{kst}(X, D) = \mathfrak{B}.$$

Suppose that this is not the case. Then we have $\mathfrak{B} > \mathbf{T}$, and we look for a contradiction. First, we claim that for any sequence $t_i \to 0$, after passing to a subsequence which by abuse of notation is still denoted by $\{t_i\}$, we can find a sequence $\{\beta_i^*\} \nearrow \mathbf{T}$ such that

$$\operatorname{dist}_{\mathbb{H}_{\mathcal{X}};N}\Bigl(\operatorname{Hilb}\bigl(\mathcal{X}_{t_i}, (1-\beta_i^*)\mathcal{D}_{t_i}\bigr), \mathrm{U}(N+1)\cdot\operatorname{Hilb}\bigl(X, (1-\mathbf{T})D\bigr)\Bigr) \longrightarrow 0. \qquad (47)$$

In fact, since we have already established Theorem 1.2 under the extra assumption that the nearby points are all β-K-polystable (see Corollary 6.14), for any fixed $\beta < \mathbf{T}$ we have

$$\operatorname{dist}_{\mathbb{H}_{\mathcal{X}};N}\big(\operatorname{Hilb}\big(\mathcal{X}_t,(1-\beta)\mathcal{D}_t\big),\mathrm{U}(N+1)\cdot\operatorname{Hilb}\big(X,(1-\beta)D\big)\big)\longrightarrow 0 \quad \text{as } t\to 0,$$

and thus Lemma 6.6 implies that

$$\operatorname{dist}_{\mathbb{H}_{\mathcal{X}};N}\big(\operatorname{Hilb}\big(X,(1-\beta_i')D\big),\mathrm{U}(N+1)\cdot\operatorname{Hilb}\big(X,(1-\mathbf{T})D\big)\big)\longrightarrow 0$$

for any sequence $\beta_i' \nearrow \mathbf{T} < \mathfrak{B}$. Since $t_i \to 0$, for any fixed β_i' there is a $k_i \geq i$ such that

$$\operatorname{dist}_{\mathbb{H}_{\mathcal{X}};N}\big(\operatorname{Hilb}\big(\mathcal{X}_{t_{k_i}},(1-\beta_i)\mathcal{D}_{t_{k_i}}\big),\mathrm{U}(N+1)\cdot\operatorname{Hilb}\big(X,(1-\beta_i)D\big)\big) < 1/i.$$

Now we pick the subsequence $\{t_{k_i}\}$ and define $\beta_{k_i}^* := \beta_i'$. Then the sequence $\{\beta_{k_i}^*\}_{i\to\infty} \nearrow \mathbf{T}$ is a sequence satisfying (47); hence our claim is justified.

On the other hand, for each fixed t_i, let $\beta \nearrow \mathbf{T}$. By Theorem 4.1, we have

$$\operatorname{dist}_{\mathbb{H}_{\mathcal{X}};N}\big(\operatorname{Hilb}\big(\mathcal{X}_{t_i},(1-\beta)\mathcal{D}_{t_i}\big),\mathrm{U}(N+1)\cdot\operatorname{Hilb}\big(\tilde{\mathcal{X}}_{t_i},(1-\mathbf{T})\tilde{\mathcal{D}}_{t_i}\big)\big)\longrightarrow 0 \tag{48}$$

with $\operatorname{Hilb}(\tilde{\mathcal{X}}_{t_i},(1-\mathbf{T})\tilde{\mathcal{D}}_{t_i}) \in \partial\overline{O_{\operatorname{Hilb}(\mathcal{X}_{t_i},\mathcal{D}_{t_i})}}$ and $(\tilde{\mathcal{X}}_{t_i},(1-\mathbf{T})\tilde{\mathcal{D}}_{t_i})$ being a $\mathbf{T}$-K-polystable variety. Now we claim that

$$\operatorname{Hilb}\big(\tilde{\mathcal{X}}_{t_i},(1-\mathbf{T})\tilde{\mathcal{D}}_{t_i}\big)\longrightarrow g\cdot\operatorname{Hilb}\big(X,(1-\mathbf{T})D\big) \quad \text{for some } g\in\mathrm{U}(N+1). \tag{49}$$

To see this, one notes that by Theorem 4.1 and Lemma 4.9, after passing to a subsequence, there is a sequence $\beta_i \nearrow \mathbf{T}$ such that

$$\operatorname{Hilb}\big(\tilde{\mathcal{X}}_{t_i},(1-\beta_i)\tilde{\mathcal{D}}_{t_i}\big)\longrightarrow\operatorname{Hilb}\big(Y,(1-\mathbf{T})E\big),$$

and such that (Y,E) is $\mathbf{T}$-K-polystable. Moreover, we may assume that $\beta_i^* < \beta_i$, $\forall i$ after rearranging. Combining (48) and Lemma 4.9, we have

$$\big(\mathcal{X}_{t_i},\mathcal{D}_{t_i};\omega(t_i,\beta_i)\big)\xrightarrow{\text{GH}}\big(Y,E;\omega_Y(\mathbf{T})\big),$$

where (Y,E) is a log $\mathbb{Q}$-Fano pair admitting a weak conical Kähler–Einstein metric $\omega_Y(\mathbf{T})$ with cone angle $2\pi(1-(1-\mathbf{T})/m)$ along E. In particular, (Y,E) is $\mathbf{T}$-K-polystable. By Lemma 6.9(1), we conclude that

$$\operatorname{Hilb}\big(Y,(1-\mathbf{T})E\big)=g\cdot\operatorname{Hilb}\big(X,(1-\mathbf{T})D\big) \quad \text{for some } g\in\mathrm{U}(N+1).$$

Hence our claim is proved.

To conclude the proof, we note that the stabilizer group of $\operatorname{Hilb}(\mathcal{X}_{t_i}',(1-\mathbf{T})\mathcal{D}_{t_i}')$ is of positive dimension for each i. Let $\mathfrak{g}=\mathfrak{sl}(N+1)$ be the Lie algebra. By the upper semicontinuity of the dimension of the stabilizer $\mathfrak{g}_{\mathrm{Hilb}(\mathcal{X}'_{t_i},(1-\mathbf{T})\mathcal{D}'_{t_i})}$, we must have $\dim \mathfrak{g}_{\mathrm{Hilb}(X,(1-\mathbf{T})D)} > 0$, contradicting the fact that the automorphism group of (X, D) is finite for $\mathbf{T} < \mathfrak{B} \le 1$ (see Corollary 6.8). To prove the last part of the statement, we just note that, under our assumption, $(\mathcal{X}'_t, \mathcal{D}'_t)$ has to have finite automorphism groups, which implies that

$$(\mathcal{X}'_t, \mathcal{D}'_t) \cong (\mathcal{X}_t, \mathcal{D}_t).$$

Hence our proof is completed for this case. □

7.3. Proofs of Theorems 1.1 and 1.2

Before we start the proofs, let us fix a divisor $\mathcal{D} \sim_C -mK_{\mathcal{X}}$ in *general position* for the flat family $\mathcal{X} \to C$ satisfying the assumption of Theorem 5.2 and with $(\mathcal{X}_t, \mathcal{D}_t)$ being smooth for all $t \in C^\circ$.

Proof of Theorem 1.1

First, we note that (i) is proved in Section 7.2. To prove (ii), one notes that Theorem 4.1 implies that there exists an r such that the Gromov–Hausdorff limit of the family $(\mathcal{X}_t, \mathcal{D}_t; \omega(t, \beta_t))$ for any $t \in C$ and $\beta < 1$ can all be embedded into $\mathbb{P}^N$ for $N = N(r, d)$. By combining Propositions 6.11 and 6.12, we obtain that for every $\mathfrak{B} < 1$,

$$\mathbf{B}_r(\mathcal{X}, \mathcal{D}) = [\epsilon, \mathfrak{B}]$$

for $(\mathcal{X}, \mathcal{D})$ (see Corollary 6.14). Therefore, their union will contain $[\epsilon, 1)$. In particular, it follows from Lemma 6.6 and Remark 6.7 for $\mathfrak{B} = 1$ that $X = \mathcal{X}_0$ admits a Kähler–Einstein metric. This in particular verifies the first part of (iii).

Now we finish the proof of part (ii). By part (i), after a possible shrinking of C, we may assume that $\mathcal{X}_t$ is K-semistable for every $t \in C^\circ$. For any $t \ne 0$, there is a *unique* K-polystable $\mathbb{Q}$-Fano $\tilde{\mathcal{X}}_t$ such that $\mathrm{Hilb}(\tilde{\mathcal{X}}_t) \in \overline{O_{\mathrm{Hilb}(\mathcal{X}_t)}}$ by Theorem 7.2, which is the Gromov–Hausdorff limit of $(\mathcal{X}_t, \mathcal{D}_t; \omega(\beta))$ as $\beta \to 1$, and hence admits a weak Kähler–Einstein metric $\tilde{\omega}(t)$ by Theorem 1.2.

We claim that

$$\mathrm{dist}_{\mathbb{H}^{\chi;N}}\big(\mathrm{U}(N+1)\cdot \mathrm{Hilb}(\tilde{\mathcal{X}}_t), \mathrm{U}(N+1)\cdot \mathrm{Hilb}(X)\big) \longrightarrow 0 \quad \text{as } i \to \infty, \tag{50}$$

and hence part (ii) follows. To prove this, let $t_i \to 0$ be *any* sequence. It follows from the *compactness* of the Hilbert scheme of $\mathbb{P}^N$ that, after passing to a subsequence if necessary, we may assume that

$$\mathrm{Hilb}(\tilde{\mathcal{X}}_{t_i}) \longrightarrow \mathrm{Hilb}(Y) \quad \text{as } t_i \to 0.$$

Since

$$\big(\mathcal{X}_{t_i}, \mathcal{D}_{t_i}; \omega(t_i, \beta)\big) \xrightarrow{\mathrm{GH}} \big(\tilde{\mathcal{X}}_{t_i}; \tilde{\omega}(t_i)\big) \quad \text{as } \beta \nearrow 1,$$

by Theorem 7.2, there is a sequence $\beta_i \nearrow 1$ such that

$$\mathrm{dist}_{\mathbb{H}\chi;N}\big(\pi_1 \circ \mathrm{Hilb}\big(\mathcal{X}_{t_i}, (1-\beta_i)\mathcal{D}_{t_i}\big), \mathrm{U}(N+1)\cdot \mathrm{Hilb}(\tilde{\mathcal{X}}_{t_i})\big) < 1/i,$$

where π_1 is given in (35). In particular, by passing to another subsequence if necessary, we may assume that

$$\big(\mathcal{X}_{t_i}, (1-\beta_i)\mathcal{D}_{t_i}; \omega(t_i, \beta_i)\big) \xrightarrow{\mathrm{GH}} (Y, \omega_Y)$$

by Lemma 4.9, where Y is a $\mathbb{Q}$-Fano variety admitting a weak Kähler–Einstein metric ω_Y. This implies that

$$\mathrm{dist}_{\mathbb{H}\chi;N}\big(\pi_1 \circ \mathrm{Hilb}\big(\mathcal{X}_{t_i}, (1-\beta_i)\mathcal{D}_{t_i}\big), \mathrm{U}(N+1)\cdot \mathrm{Hilb}(Y)\big) \longrightarrow 0 \quad \text{as } i \to \infty. \tag{51}$$

On the other hand, by Lemma 2.4 we know that $(\mathcal{X}_{t_i}, \mathcal{D}_{t_i})$ is β-K-polystable for any $\beta < 1$. This, together with Corollary 6.14, implies that for every fixed $\beta < 1$,

$$\mathrm{dist}_{\mathbb{H}\chi;N}\big(\mathrm{Hilb}\big(\mathcal{X}_{t_i}, (1-\beta)\mathcal{D}_{t_i}\big), \mathrm{U}(N+1)\cdot \mathrm{Hilb}\big(X, (1-\beta)D\big)\big) \longrightarrow 0 \quad \text{as } i \to \infty.$$

Therefore, for any fixed β_i there is a $k_i > i$ such that

$$\mathrm{dist}_{\mathbb{H}\chi;N}\big(\mathrm{Hilb}\big(\mathcal{X}_{t_{k_i}}, (1-\beta_i)\mathcal{D}_{t_{k_i}}\big), \mathrm{U}(N+1)\cdot \mathrm{Hilb}\big(X, (1-\beta_i)D\big)\big) < 1/i.$$

On the other hand, Lemma 6.6 implies that

$$\mathrm{dist}_{\mathbb{H}\chi;N}\big(\pi_1 \circ \mathrm{Hilb}\big(X, (1-\beta)D\big), \mathrm{U}(N+1)\cdot \mathrm{Hilb}(X)\big) \longrightarrow 0 \quad \text{as } \beta \to 1.$$

This implies that if we define $\beta^*_{k_i} := \beta_i < \beta_{k_i}$, then $\beta^*_{k_i} \to 1$ and

$$\mathrm{dist}_{\mathbb{H}\chi;N}\big(\pi_1 \circ \mathrm{Hilb}\big(\mathcal{X}_{t_{k_i}}, (1-\beta^*_{k_i})\mathcal{D}_{t_{k_i}}\big), \mathrm{U}(N+1)\cdot \mathrm{Hilb}(X)\big) \longrightarrow 0 \quad \text{as } i \to \infty. \tag{52}$$

By combining (52) and (51) and applying Remark 6.10(1), we conclude that $\mathrm{Hilb}(Y) \in \mathrm{U}(N+1)\cdot \mathrm{Hilb}(X)$, and (50) is established. Thus the proof of part (ii) is completed.

Finally, to finish the proof of part (iii), we can assume that $\mathcal{X}_t$ is K-polystable for all $t \in C$ by Theorem 7.3. Then by taking $\mathfrak{B} = 1$, we can conclude that $\mathbf{B}_r(\mathcal{X}, \mathcal{D}) = [\epsilon, 1]$. In particular, $(\mathcal{X}_{t_i}; \omega(t_i)) \xrightarrow{\mathrm{GH}} (\mathcal{X}_0; \omega_{\mathcal{X}_0})$. Hence our proof is completed. ☐

Proof of Theorem 1.2

Choose a sequence $\beta \nearrow \mathfrak{B}$. By applying Propositions 6.11 and 6.12, we obtain that $\mathbf{B}_r(\mathcal{X}, \mathcal{D}) = [\epsilon, \mathfrak{B}]$. Then by repeating the argument in a manner completely parallel to the one given above, we obtain the conclusion. $\square$

Remark 7.6

We call a $\mathbb{Q}$-Fano variety *$\mathbb{Q}$-Gorenstein smoothable* if there is a projective flat family $\mathcal{X}$ over a smooth curve C such that $K_{\mathcal{X}}$ is $\mathbb{Q}$-Cartier and antiample over C, a general fiber $\mathcal{X}_t$ is smooth, and $X \cong \mathcal{X}_0$ for some $0 \in C$. We note that by a standard argument, we can generalize Theorems 1.1, 7.2, and 7.3 to the case that the base is of higher dimension. As a consequence, we can just assume in these theorems that the general fibers are $\mathbb{Q}$-Gorenstein smoothable instead of smooth. We will make frequent use of these extensions in Section 8.

8. Local geometry near a $\mathbb{Q}$-Gorenstein smoothable, K-polystable $\mathbb{Q}$-Fano variety

In this section, we will devote to the proof of Theorem 1.3 based on Theorem 1.1, the results in Section 7, and the following criterion.

THEOREM 8.1 ([2, Theorem 1.2])

Let $\mathcal{X}$ be an algebraic stack of finite type over $\mathbb{C}$. Suppose that:

(1) *for every closed point $x \in \mathcal{X}$, there exists a* local quotient presentation $f : \mathcal{W} \longrightarrow \mathcal{X}$ *(see* [2, Definition 2.1]*) around x such that:*
  (a) *the morphism f is* stabilizer preserving *(see* [2, Definition 2.5]*) at closed points of $\mathcal{W}$, and*
  (b) *the morphism f sends closed points to closed points; and*

(2) *for any point $x \in X(\mathbb{C})$, the closed substack $\overline{\{x\}}$ admits a good moduli space.*

Then $\mathcal{X}$ admits a good moduli space as an algebraic space.

Let us fix our notation.

Definition 8.2

We define

$$Z := \left\{ \mathrm{Hilb}(Y) \;\middle|\; \begin{array}{l} Y \subset \mathbb{P}^N \text{ be a smooth Fano manifold with} \\ N = \dim H^0\big(Y, \mathcal{O}_Y(-rK_Y)\big), \\ \mathcal{O}_{\mathbb{P}^N}(1)|_Y \cong \mathcal{O}_Y(-rK_Y) \text{ and } \chi\big(Y, \mathcal{O}_{\mathbb{P}^N}(k)|_Y\big) = \chi(k) \end{array} \right\}$$
$$\subset \mathbb{H}^{\chi;N} \subset \mathbb{P}^M, \tag{53}$$

where the last inclusion is the Plücker embedding. By the boundedness of smooth Fano manifolds with fixed dimension (see [28]), we may choose $r \gg 1$ such that Z includes all such Fano manifolds. Now let $\overline{Z} \subset \mathbb{H}^{\chi;N}$ be the closure of $Z \subset \mathbb{H}^{\chi;N}$, and let Z° be the *open* set of $\overline{Z}$ that parameterizes the K-semistable $\mathbb{Q}$-Fano subvariety Y such that $\mathcal{O}_Y(-rK_Y) \sim \mathcal{O}_{\mathbb{P}^N}(1)|_Y$ (see [54, Lemma 1.19]). Let Z^* be the seminormalization of Z°_{red}, which is the reduction of Z°.

Remark 8.3
By Theorem 4.1, the Gromov–Hausdorff limit of Fano Kähler–Einstein manifolds is automatically in Z°, and hence, so are the $\mathbb{Q}$-Gorenstein smoothable, K-polystable $\mathbb{Q}$-Fano varieties.

Then we have a commutative diagram

$$\begin{tikzcd} \mathcal{X}^* \arrow[r, "i"] \arrow[d, "\pi"'] & \mathbb{P}^N \times Z^* \arrow[r] \arrow[d] & \mathbb{P}^N \times Z^\circ_{\mathrm{red}} \arrow[d] \\ Z^* \arrow[r] & Z^* \arrow[r] & Z^\circ_{\mathrm{red}} \end{tikzcd} \tag{54}$$

where $\mathcal{X}^*$ is the universal family over Z^* (see [26, Section I.3]). Before we state the main result of this section, let us first deduce the following boundedness result which is a consequence of our Theorem 1.1.

LEMMA 8.4
The K-semistable $\mathbb{Q}$-Fano varieties admitting a $\mathbb{Q}$-Gorenstein smoothing with a fixed dimension form a bounded family.

Proof
We first prove the statement for K-polystable $\mathbb{Q}$-Fano varieties. Let X be an n-dimensional, $\mathbb{Q}$-Gorenstein smoothable, K-polystable $\mathbb{Q}$-Fano variety, and let $\mathcal{X} \to C$ be a smoothing of X with $\mathcal{X}_0 = X$. It follows from Theorem 1.1 that the nearby fibers $\mathcal{X}_t$ are all K-semistable, and we can take a $\mathcal{D} \sim_C -mK_{\mathcal{X}/C}$ such that $\mathcal{X}_0$ is the Gromov–Hausdorff limit of $(\mathcal{X}_{t_i}, (1-\beta_i)\mathcal{D}_t)$ for any sequences $t_i \to 0$ and $\beta_i \to 1$.

On the other hand, by the boundedness of smooth Fano varieties, we know that there exists m_0 depending only on n, and a divisor

$$\mathcal{D}^* \sim_{C^\circ} -m_0 K_{\mathcal{X}^\circ/C^\circ},$$

such that $\mathcal{D}^*_t$ is smooth for any $t \in C^\circ$ after a possible shrinking of the base. Since all the $\mathcal{X}_t$'s are K-semistable, they admit conical Kähler–Einstein metrics $\omega(t, \beta_i)$ with cone angle $2\pi(1-(1-\beta_i)/m)$ along $\mathcal{D}^*_t$. By applying Theorem 1.2(iii) for

$(\mathcal{X}_t, (1-\beta_i)\mathcal{D}_t^*)$, we know that the Gromov–Hausdorff limit for this family as $t \to 0$ is also $\mathcal{X}_0$. Thus it is a subvariety of a fixed $\mathbb{P}^N$ for some $N \gg 0$ by Theorem 4.1.

In general, if X is a $\mathbb{Q}$-Gorenstein smoothable, K-semistable $\mathbb{Q}$-Fano variety, then we know that the closure of its orbit contains a unique K-polystable $\mathbb{Q}$-Fano variety X_0 (see Theorem 7.2, Remark 7.6). And as a consequence of volume convergence for the Gromov–Hausdorff limit, we obtain that

$$(-K_{X_0})^n = (-K_X)^n$$

are bounded from above; on the other hand, the Cartier index of K_X divides the Cartier index of K_{X_0}, which is also bounded from above thanks to work of [19, Theorem 1.2]. Therefore, X is contained in a bounded family (see, e.g., [23, Corollary 1.8]). □

Fix a K-polystable $\mathbb{Q}$-Fano variety X parameterized by a point in Z^*, which is $\mathbb{Q}$-Gorenstein smoothable by the definition of Z^*, so it admits a weak Kähler–Einstein metric by Theorem 1.1 from which we deduce that $\mathrm{Aut}(X) \subset \mathrm{SL}(N+1)$ is reductive. Let $\mathrm{Hilb}(X)$ be the Hilbert point for Tian's embedding of $X \subset \mathbb{P}^N$ after we fix a basis of $H^0(\mathcal{O}_X(-rK_X))$. Let $\mathbb{H}^{\chi;N} \subset \mathbb{P}^M$ be the Plücker embedding which is clearly $\mathrm{SL}(N+1)$-equivariant. Then by [17, Proposition 1] or the proof of Lemma 3.1, there is an $\mathrm{Aut}(X)$-invariant linear subspace $z_0 := \mathrm{Hilb}(X) \in \mathbb{P}W \subset \mathbb{P}^M$ such that

$$\mathbb{P}^M = \mathbb{P}\big(W \oplus \mathbb{C}z_0 \oplus \mathfrak{aut}(X)^\perp\big) \quad \text{with } \mathfrak{aut}(X)^\perp \oplus \mathfrak{aut}(X) = \mathfrak{sl}(N+1), \tag{55}$$

where $W \oplus \mathbb{C}\cdot z_0 \oplus \mathfrak{aut}(X)^\perp = \mathbb{C}^{M+1}$ is a decomposition as $\mathrm{Aut}(X)$-invariant subspaces.

In particular, this induces a representation $\rho : \mathrm{Aut}(X) \to \mathrm{SL}(W)$. On the other hand, $\mathrm{Hilb}(X)$ is fixed by $\mathrm{Aut}(X)$. We let $\rho_X : \mathrm{Aut}(X) \to \mathbb{G}_m$ denote the character corresponding to the linearization of $\mathrm{Aut}(X)$ on $\mathcal{O}_{\mathbb{H}^{\chi;N}}(1)|_{\mathrm{Hilb}(X)}$ induced from the embedding $\mathrm{Aut}(X) \subset \mathrm{SL}(N+1)$. Then we can introduce the following.

Definition 8.5
A point $z \in \mathbb{P}W$ is *GIT-polystable* (resp., *GIT-semistable*) if z is *polystable* (resp., *semistable*) with respect to the linearization $\rho \otimes \rho_X^{-1}$ on $\mathcal{O}_{\mathbb{P}W}(1) \to \mathbb{P}W$ in the GIT sense.

Before we apply the results in the Appendix, notably Theorem A.10 and Lemma A.15, to finish our proof of Theorem 1.3, let us review the geometric consequences we have obtained so far.

Summary 8.6
Let us consider the set $\Sigma \subset \mathbb{H}^{\chi,N}$ of Hilbert points corresponding to $\mathbb{Q}$-Gorenstein

smoothable, Kähler–Einstein $\mathbb{Q}$-Fano varieties via Tian's embedding. By [19], Σ is compact, and Theorem 1.1 implies that it is Hausdorff. Thus it yields a *proper* $\mathrm{U}(N+1)$-invariant slice

$$\begin{tikzcd} \Sigma \subset (Z^*)^{\mathrm{kps}} \arrow[r, hook] \arrow[d] & \mathbb{H}^{\chi;N} \arrow[r, hook, "\text{Plücker}"] & \mathbb{P}^M \arrow[d] \\ \Sigma/\mathrm{U}(N+1) \arrow[rr, hook] & & \mathbb{P}^M/\mathrm{U}(N+1) \end{tikzcd} \tag{56}$$

where $(Z^*)^{\mathrm{kps}} \subset Z^*$ denotes the locus of K-polystable points in Z^*. In other words, $\Sigma \subset (Z^*)^{\mathrm{kps}}$ is a $\mathrm{U}(N+1)$-invariant *closed* subset such that there is a bijection between the quotient $\Sigma(N+1)/U(N+1)$ and all isomorphic classes of $\mathbb{Q}$-Gorenstein smoothable, Kähler–Einstein $\mathbb{Q}$-Fano varieties. Moreover, we have $\mathrm{Aut}(X) = (\mathrm{Aut}(X) \cap \mathrm{U}(N+1))^{\mathbb{C}}$ for all $\mathrm{Hilb}(X) \in \Sigma$ (see Lemma 8.7).

LEMMA 8.7
Let X be a $\mathbb{Q}$-Gorenstein smoothable, $\mathbb{Q}$-Fano variety admitting a weak Kähler–Einstein metric. Then $\mathrm{Aut}(X) = (\mathrm{Isom}(X))^{\mathbb{C}}$. *In particular,* $\mathrm{Aut}(X) = (\mathrm{Aut}(X) \cap \mathrm{U}(N+1))^{\mathbb{C}}$.

Proof
This follows from the proof of [11, Theorem 4]. □

Our first main result of this section is the following.

THEOREM 8.8
There is an $\mathrm{Aut}(X)$*-invariant linear subspace* $\mathbb{P}W \subset \mathbb{P}^M$ *and a Zariski-open neighborhood* $\mathrm{Hilb}(X) \in U_W \subset \mathbb{P}W \times_{\mathbb{P}^M} Z^*$ *such that for any* $\mathrm{Hilb}(Y) \in U_W$*, Y is K-polystable if and only if* $\mathrm{Hilb}(Y)$ *is GIT-polystable with respect to* $\mathrm{Aut}(X)$*-action on* $\mathbb{P}W \times_{\mathbb{P}^M} Z^*$*. Moreover, for all GIT-polystable* $\mathrm{Hilb}(Y) \in U_W$*, we have* $\mathrm{Aut}(Y) < \mathrm{Aut}(X)$*; that is, the local GIT presentation* $U_W /\!\!/ \mathrm{Aut}(X)$ *is* stabilizer-preserving *in the sense of [2, Definition 2.5].*

Remark 8.9
As we will see in Corollary 8.14, we are able to establish the stabilizer-preserving property for all *GIT-semistable* $\mathrm{Hilb}(Y) \in U_W$. This property is *stronger* than the condition of being *strongly étale* introduced in [2, Definition 2.5].

Let

$$\begin{aligned} \Delta : Z^* &\longrightarrow \mathbb{H}^{\chi;N} \times Z^*, \\ z &\longmapsto (z, z). \end{aligned} \tag{57}$$

be the diagonal morphism. We define $O_{Z^*} := \mathrm{SL}(N+1) \cdot \Delta(Z^*) \subset \mathbb{H}^{\chi;N} \times Z^*$, where $\mathrm{SL}(N+1)$ acts *trivially* on Z^* and acts on $\mathbb{H}^{\chi;N}$ via the action induced from $\mathbb{P}^N$. This allows us to construct the family of limiting orbit spaces associated to the family (54) as

$$\begin{tikzcd} \overline{BO}_z \subset \overline{BO}_{Z^*} \arrow[r, "i"] \arrow[d] & \mathbb{H}^{\chi;N} \times Z^* \arrow[d, "\pi_{Z^*}"] \\ z \in Z^* \arrow[r, equal] & Z^* \end{tikzcd} \tag{58}$$

where $\overline{BO}_{Z^*} \subset \mathbb{H}^{\chi;N} \times Z^*$ is the closure of O_{Z^*} and $\overline{BO}_z$ is the union of limiting *broken orbits*. Then by Theorem 1.1, we know that there is a *unique* K-polystable orbit inside $\overline{BO}_z$. To see this, one only needs to note that for any $z \in Z^*$, we can always find a smooth curve $f : C \to Z^*$ that passes through z and where the image $f(C)$ meets the dense open locus inside of Z^* corresponding to *smooth K-polystable Fano manifolds* with the *maximal* dimension of its $\mathrm{SL}(N+1)$-orbit space. Then our claim follows by applying Theorem 1.1 to the pullback family over C.

For a K-polystable point $\mathrm{Hilb}(X) \in Z^*$ (corresponding to Tian's embedding of $X \subset \mathbb{P}^N$ with respect to the Kähler–Einstein metric), by Lemma 3.1, we can find a Zariski open neighborhood $\mathrm{Hilb}(X) \in U \subset Z^*$, and (after a possible shrinking) we may assume that

$$\begin{aligned} &U \cap \overline{BO}_{\mathrm{Hilb}(X)} \text{ contains a unique minimal (see Lemma 3.1)} \\ &\qquad \text{orbit } \mathrm{SL}(N+1) \cdot \mathrm{Hilb}(X). \end{aligned} \tag{59}$$

By Theorem 7.2 (and its extension in Remark 7.6), every $z \in U$ can be specialized to a K-polystable point $\hat{z}$ unique up to $\mathrm{SL}(N+1)$-translation. Moreover, we have the following.

LEMMA 8.10
Let $\mathrm{Hilb}(X) \in U \subset Z^*$ *be as above. Then there is an analytic-open neighborhood* $\mathrm{Hilb}(X) \in U^{\mathrm{ks}}$ *such that for any K-semistable point* $z \in U^{\mathrm{ks}}$*, we can specialize it to a K-polystable point* $\hat{z} \in U$ *via a 1-PS* $\lambda \subset \mathrm{SL}(N+1)$. *Moreover, if* $\lim_{i\to\infty} z_i = \mathrm{Hilb}(X)$*, then*

$$\lim_{i\to\infty} \mathrm{dist}_{\mathbb{H}^{\chi;N}} \big(\mathrm{Hilb}\big(\mathcal{X}_{\hat{z}_i}, \omega_{\mathrm{KE}}(\hat{z}_i)\big), \mathrm{U}(N+1) \cdot \mathrm{Hilb}(X)\big) = 0,$$

where $\mathrm{Hilb}(\mathcal{X}_{\hat{z}_i}, \omega_{\mathrm{KE}}(\hat{z}_i))$ *is the Hilbert point corresponding to Tian's embedding of* $\mathcal{X}_{\hat{z}_i}$ *with respect to the weak Kähler–Einstein metric* $\omega_{\mathrm{KE}}(\hat{z}_i)$.

Proof
Suppose that this is not the case. Then there is a sequence $z_i = \mathrm{Hilb}(\mathcal{X}_{z_i}) \stackrel{i\to\infty}{\longrightarrow} \mathrm{Hilb}(X)$ and

$$O_{\hat{z}_j} \cap U = \varnothing \quad \text{with } O_{\hat{z}} := \mathrm{SL}(N+1)\cdot \hat{z}.$$

In particular, by equipping each $\mathcal{X}_{\hat{z}_i}$ with a weak Kähler–Einstein metric $\omega_{\mathrm{KE}}(\hat{z}_i)$, and taking the Gromov–Hausdorff limit Y, which is still embedded in $\mathbb{P}^N$ by Lemma 8.4, we obtain

$$\mathrm{Hilb}\big(\mathcal{X}_{\hat{z}_i}, \omega_{\mathrm{KE}}(\hat{z}_i)\big) \stackrel{i\to\infty}{\longrightarrow} g\cdot \mathrm{Hilb}(Y) \in \overline{BO}_{\mathrm{Hilb}(X)} \setminus U \quad \text{for some } g \in \mathrm{U}(N+1),$$

contradicting the fact that the limiting broken orbits $\overline{BO}_z$ contain a unique K-polystable orbit. □

Now we are ready to prove Theorem 8.8.

Proof of Theorem 8.8
Let U be the open set constructed above satisfying (59), and let (see Lemma 8.10)

$$U_W^{\mathrm{an}} = (U^{\mathrm{ks}} \cap \mathbb{P}W) \times_{\mathbb{P}^M} Z^*.$$

After a possible shrinking, we may assume that all the points in U_W^{an} are GIT-semistable and that every GIT-semistable point can be degenerated to a GIT-polystable point in U_W^{an}.

Suppose that $\mathrm{Hilb}(Y) \in U_W^{\mathrm{an}}$ is GIT-polystable and strictly K-semistable. Then by Lemma 8.10, we can degenerate it to a variety $Y' \subset \mathbb{P}^N$ which is K-polystable such that

$$\mathrm{Hilb}(Y') \in U \cap \overline{\mathrm{SL}(N+1)\cdot \mathrm{Hilb}(Y)} \subset Z^\circ \subset \mathbb{H}^{\chi;N},$$

where $\mathrm{Hilb}(Y')$ is *close* to $\mathrm{Hilb}(Y)$ in $\mathbb{H}^{\chi;N}$ in the sense that there is a *short* (with respect to the metric $\mathrm{dist}_{\mathbb{H}^{\chi;N}}$) path inside $\overline{\mathrm{SL}(N+1)\cdot \mathrm{Hilb}(Y)}$ joining $\mathrm{Hilb}(Y)$ and $\mathrm{Hilb}(Y')$.

Using the transversality of the action of $\mathfrak{aut}(X)^\perp \subset \mathfrak{sl}(N+1)$ on $\mathbb{P}W \subset \mathbb{P}^M$, one can always find a $g \in \mathrm{SL}(N+1)$ *close* to the identity such that

$$\mathrm{Hilb}(Y'') := g\cdot \mathrm{Hilb}(Y') \in \mathbb{P}W \times_{\mathbb{P}^M} Z^*,$$

where $Y'' \cong Y'$ is GIT-semistable. This allows us to find a *short* path inside $\overline{\mathrm{SL}(N+1)\cdot \mathrm{Hilb}(Y)}$ joining $\mathrm{Hilb}(Y)$ and $\mathrm{Hilb}(Y'')$, which by transversality we may assume to be entirely contained in $\mathbb{P}W$ and which satisfies $\mathrm{Hilb}(Y'') \in$

$\overline{\mathrm{Aut}(X)\cdot\mathrm{Hilb}(Y)}$. But this is absurd, since $\mathrm{Hilb}(Y)$ is already GIT-polystable, and no point on the boundary of $\overline{\mathrm{Aut}(X)\cdot\mathrm{Hilb}(Y)}$ is semistable.

Conversely, suppose that $\mathrm{Hilb}(Y)\in U_W^{\mathrm{an}}$ and Y is K-polystable, but $\mathrm{Hilb}(Y)$ is not GIT-polystable. Then there is a 1-PS $\lambda\subset\mathrm{Aut}(X)$ degenerating $\mathrm{Hilb}(Y)$ to a nearby GIT-polystable

$$\mathrm{Hilb}(Y')\in\overline{\mathrm{Aut}(X)\cdot\mathrm{Hilb}(Y)}\cap U_W^{\mathrm{an}}$$

by the classical GIT. Thus Y' is K-polystable by the preceding paragraph, contradicting the assumption of Y being K-polystable. Hence our proof is completed.

To pass from an analytic neighborhood to a Zariski neighborhood, we need to investigate the geometry of $\mathrm{Aut}(X)$-orbits. Let $U_W^{\mathrm{ss}}\subset\mathbb{P}W$ containing $\mathrm{Hilb}(X)$ be the Zariski-open set of *GIT-semistable points*. By [35, Chapter 2, Proposition 2.14] and [38, Lemmas 2.11, 2.12], we know that the set of GIT-polystable points in U_W^{ss} forms a *constructible* set. On the other hand, K-polystable points inside $U_W^{\mathrm{ss}}\cap Z_{\mathrm{red}}^{\circ}$ also form a constructible set (see Remark A.5) containing the point $\mathrm{Hilb}(X)$. These two constructible sets coincide along U_W^{an} after lifting to $\mathbb{P}W\times_{\mathbb{P}M}Z^*$ by the proof above, so they must coincide on a Zariski-open set.

Finally, we establish the last statement. The slice Σ obtained from Summary 8.6 satisfies Assumption A.9. Thus, by applying Theorem A.10 to our setting we can construct an analytic-open set $U_W\subset\mathbb{P}W\times_{\mathbb{P}M}Z^*$ that is stabilizer-preserving. To obtain the Zariski openness, one observes that

$$\mathrm{Aut}(Z^*):=\bigl\{(z,g)\in Z^*\times\mathrm{SL}(N+1)\bigm| g\cdot z=z\bigr\}=\phi_{\mathrm{SL}(N+1)}^{-1}(\Delta_{Z^*})$$

is a *closed* subset of $Z^*\times\mathrm{SL}(N+1)$, where

$$\begin{aligned} Z^*\times\mathrm{SL}(N+1)&\xrightarrow{\phi_{\mathrm{SL}(N+1)}} Z^*\times Z^* \\ (z,g)&\longmapsto (z,g\cdot z)\end{aligned}\qquad\text{and}$$

$$\Delta_{Z^*}=\bigl\{(z,z)\bigm| z\in Z^*\bigr\}\subset Z^*\times Z^*.$$

Next let

$$\mu:\begin{aligned}\mathrm{Aut}(Z^*)&\longrightarrow Z^*\\ (z,g)&\longmapsto g\cdot\mathrm{Hilb}(X).\end{aligned}$$

Then the locus of

$$\bigl\{\mathrm{Hilb}(Y)\in Z^*\bigm|\mathrm{Aut}(Y)<\mathrm{Aut}(X)\bigr\}$$

is precisely the complement of $\mathrm{pr}_1(\mu^{-1}(Z^*\setminus\{\mathrm{Hilb}(X)\}))$ which is constructible, where $\mathrm{pr}_1:\mathrm{Aut}(Z^*)\to Z^*$ is the projection to the first factor. So we can prolong U_W from an analytic-open subset to a Zariski-open one, and our proof is completed. $\square$

Remark 8.11
One notes that, in contrast to Theorem 8.8, there exist smooth Fano varieties admitting Kähler–Einstein metrics which are not asymptotically Chow stable (see [40]). On the other hand, Theorem 8.8 can be regarded as an extension of work in [48] to the case of $\mathbb{Q}$-Gorenstein smoothable, $\mathbb{Q}$-Fano varieties.

Finally, to prove Theorem 1.3, we need to show that for each $\mathbb{C}$-closed point $[z] \in [Z^*/\operatorname{SL}(N+1)]$, $\overline{\{z\}}$ has a good moduli space in the sense of [2, Theorem 1.2, Proposition 3.1]. To do this, let us first establish Assumption A.11 in the Appendix. Let $z = \operatorname{Hilb}(Y) \in U_W$ specialize to $z_0 = \operatorname{Hilb}(X) \in U_W \subset \mathbb{H}^{\chi;N}$ via a 1-PS $\lambda(t)$: $\mathbb{G}_m \to \operatorname{Aut}(X) < \operatorname{SL}(N+1)$. Let $(\mathcal{Y} = \mathcal{X}|_C, X) \to (C = \overline{\lambda(t) \cdot z}, z_0) \subset U_W$ be the restriction of the universal family $\mathcal{X} \to Z^*$ to the pointed curve (C, z_0), and we also prefix a basis $\{s_i\} \subset \mathcal{O}_{\mathcal{Y}}(-rK_{\mathcal{Y}/C})$.

LEMMA 8.12
Under the notation introduced above, we have $\operatorname{Aut}(Y) < \operatorname{Aut}(X)$ *for* $z := \operatorname{Hilb}(Y)$ *close to* $z_0 = \operatorname{Hilb}(X)$.

Proof
By property (3) in the proof of Lemma 3.1, for $z = \operatorname{Hilb}(Y) \in U_0$ we have $\mathfrak{aut}(Y) \subset \mathfrak{aut}(X)$; hence the identity component of $\operatorname{Aut}(Y)$ lies in $\operatorname{Aut}(X)$. We will assume from now on that $z = \operatorname{Hilb}(Y) \in \mathbb{P}W$ lies in a *small* analytic neighborhood of $z_0 = \operatorname{Hilb}(X) \in U_1$; that is, z is *very close* to z_0. This, together with the fact that there always exists a *finite* subgroup $H < \operatorname{Aut}(Y)$ that meets *every* connected component of $\operatorname{Aut}(Y) < \operatorname{SL}(N+1)$, implies that all we need is that: for any finite subgroup $H < \operatorname{Aut}(Y)$, we have $H < \operatorname{Aut}(X)$. To achieve this, let us choose an H-invariant *smoothable* divisor $E \in |-mK_Y|^H$ so that $(Y, \frac{E}{m})$ is klt. The existence of such $E \subset Y$ is guaranteed by the following result.

CLAIM 8.13
Let Y be a $\mathbb{Q}$-Gorenstein smoothable, $\mathbb{Q}$-Fano variety. Fix a finite group $H \subset \operatorname{Aut}(Y)$. *For m sufficiently divisible there is an invariant section $E \in |-mK_Y|^H$ such that $(Y, (1-\epsilon)E)$ is klt for any $0 < \epsilon \leq 1$ and $\mathbb{Q}$-Gorenstein smoothable. In particular, $(Y, \frac{1}{m}E)$ is $\mathbb{Q}$-Gorenstein smoothable and klt for $m > 1$. Moreover, m can be uniformly bounded provided Y is inside a bounded family.*

Proof
Let $\mu\colon Y \to \tilde{Y}$ be the quotient of Y by H, and let D be the branched divisor. So $\mu^*(K_{\tilde{Y}} + D) = K_Y$. In particular, $(\tilde{Y}, D)$ is klt (since klt is preserved under finite

quotient; see [29, Theorem 5.20]) and $-(K_{\tilde{Y}} + D)$ is ample. Thus, for a sufficiently divisible m satisfying $-m(K_{\tilde{Y}} + D)$ being very ample, we can choose a general section $F \in |-m(K_{\tilde{Y}} + D)|$ so that $(\tilde{Y}, D + (1-\epsilon)F)$ is klt for any $0 < \epsilon \leq 1$. Then $E := \mu^*(F)$ is H-invariant and $(Y, (1-\epsilon)E)$ is klt for any $0 < \epsilon \leq 1$. Finally, we justify that (Y, E) is actually$\mathbb{Q}$-Gorenstein smoothable as long as Y is. Since Y is a degeneration of a smooth family $\{Y_t\}_t$, and every element in $|-mK_Y|$ can be represented as a degeneration of general members of $|-mK_{Y_t}|$, we conclude that (Y, E) is a degeneration of smooth pairs $\{(Y_t, E_t)\}_t$. $\square$

Then by Theorems 1.2 and 5.2, $(Y, \frac{E}{m})$ admits a continuous family of Kähler metrics $\{\omega_Y(\beta)\}$ solving

$$\mathrm{Ric}\big(\omega_Y(\beta)\big) = \beta\omega_Y(\beta) + \frac{1-\beta}{m}[E] \quad \text{on } Y,$$

from which we obtain

$$\mathrm{Hilb}\big(Y, \omega_Y(\beta)\big) \xrightarrow{\beta \to 1} \mathrm{U}(N+1) \cdot \mathrm{Hilb}(X) \subset \mathbb{H}^{\chi;N} \tag{60}$$

thanks to Theorem 7.2 and the fact that m is uniformly bounded by Lemma 8.4, where $\mathrm{Hilb}(Y, \omega_Y(\beta))$ is the Hilbert point corresponding to Tian's embedding of $Y \subset \mathbb{P}^N$ with respect to the metric $\omega_Y(\beta)$ on $Y \subset \mathbb{P}^N$ and any prefixed basis $\{s_i\} \subset H^0(\mathcal{O}_Y(-rK_Y))$. This allows us to introduce a *continuous* family of Hermitian metrics $h_{\mathrm{KE}}(\beta(t))$ with $\beta(t) := 1 - |t|$ on $\mathcal{O}_{\mathcal{Y}_t}(-K_{\mathcal{Y}_t}) \to \mathcal{Y}_t$ for $0 < |t| := \mathrm{dist}_C(t, 0) < 1$ such that $\omega_Y(\beta(t)) = -\sqrt{-1}\partial\bar{\partial}\log h_{\mathrm{KE}}(\beta)$. By (60), the metrics $h_{\mathrm{KE}}(\beta(t))$ can be continuously extended to $0 \in C$. Now let $\{s_i\}$ be the local basis of $\pi_*\mathcal{O}_{\mathcal{Y}}(-rK_{\mathcal{Y}/C})|_{\{|t|<1\}\subset C} = \pi_*(\mathcal{O}_{\mathbb{P}^N}(1)|_{\mathcal{Y}})$ corresponding to the coordinate sections of $\mathcal{O}_{\mathbb{P}^N}(1)$ such that $\{s_i(0)\}$ induces Tian's embedding for $z_0 = \mathrm{Hilb}(X)$, and define

$$A_{\mathrm{KE}}\big(t, \beta(t)\big) = \big[(s_i, s_j)_{\mathrm{KE},\beta(t)}(t)\big]$$

with

$$(s_i, s_j)_{\mathrm{KE},\beta}(t) = \int_{\mathcal{Y}_t} \big\langle s_i(t), s_j(t)\big\rangle_{h_{\mathrm{KE}}^{\otimes r}(\beta(t))} \omega_Y^n\big(\beta(t)\big).$$

Then we obtain a family of Tian's embeddings

$$T : \big(\mathcal{Y}_t, \mathcal{E}_t; \omega_Y\big(\beta(t)\big)\big) \longrightarrow \mathbb{P}^N \quad \text{with } (\mathcal{Y}_t, \mathcal{E}_t) \cong (Y, E) \text{ for } t \neq 0, \tag{61}$$

given by $\{g(t) \circ s_j(t)\}_{j=0}^N$ with $g(t) = A_{\mathrm{KE}}^{-1/2}(\beta(t))$. The map T extends to $\mathcal{Y}_0 = X$ thanks to the continuity of the metric $h_{\mathrm{KE}}(\beta(t))$ at $0 \in C$.

Now by our choice of z_0 and basis $\{s_i(t)\}$, we have $A_{\mathrm{KE}}(0, 1) = I_{N+1} \in \mathrm{SL}(N+1)$, and hence

$$g(t) \sim I_{N+1} + O(t). \tag{62}$$

This implies that

$$\tilde{z}(t) := \mathrm{Hilb}\big(\mathcal{Y}_t, \omega_Y\big(\beta(t)\big)\big) = g(t) \cdot z_t \in U_{1,\epsilon} := \exp\big(\mathfrak{aut}(X)^{\perp}_{<\epsilon}\big) \cdot U_1 \tag{63}$$

for $0 < |t| \ll 1$, where $z_t = \lambda(t) \cdot \mathrm{Hilb}(Y)$. Since $\omega_Y(\beta(t))$ is a conical Kähler–Einstein metric on $\mathcal{Y}_t$, it follows from the log version of Lemma 8.7 (see [11, Theorem 4]) that

$$H_{\tilde{z}(t)} = g(t) \cdot H_{z(t)} \cdot g(t)^{-1} < U(N+1) \quad \text{where } H_{z(t)} = \lambda(t) \cdot H \cdot \lambda(t)^{-1}.$$

By Lemma A.8 and (62), we obtain that $H_{z(t)} < \mathrm{Aut}(X)$ and hence $H < \mathrm{Aut}(X)$ as $\lambda(t) < \mathrm{Aut}(X)$ by our choice. On the other hand, by transversality of $\mathfrak{aut}(X)^{\perp}$-action on $U_{1,\epsilon}$, for $0 < t \ll 1$ we have $\mathrm{Aut}_0(Y) < \mathrm{Aut}(X)$, where $\mathrm{Aut}_0(Y)$ is the identity component of $\mathrm{Aut}(Y)$. This implies that $\mathrm{Aut}(Y) = \langle \mathrm{Aut}_0(Y), H\rangle < \mathrm{Aut}(X)$, where $\langle \mathrm{Aut}(Y), H\rangle$ is the subgroup generated by H and $\mathrm{Aut}_0(Y)$, and our proof is completed. □

As a direct consequence of Lemmas 8.4, 8.10, and 8.12, we have the following statement, which implies Assumption A.11.

COROLLARY 8.14
After a possible shrinking of the Zariski-open neighborhood $z_0 \in U_W \subset \mathbb{P}W \times_{\mathbb{P}^M} Z^*$, we have

$$\mathrm{SL}(N+1)_z < \mathrm{Aut}(X), \quad \forall z \in U_W,$$

where $\mathrm{SL}(N+1)_z$ *is the stabilizer of z inside* $\mathrm{SL}(N+1)$. *In other words, Assumption A.11 holds in this case.*

Next, in order to apply Lemma A.15 in the Appendix, we now establish Assumption A.14. Let us fix $G = \mathrm{SL}(N+1)$ and $G_{z_0} = \mathrm{Aut}(X)$. Recall from Assumption A.14 that an analytic-open neighborhood of $z_0 \in U^{\mathrm{fd}} \subset \mathbb{P}W$ is of finite distance if there is a compact subset $G_{U^{\mathrm{fd}}} \Subset G/G_{z_0}$ depending only on U^{fd} and z_0 such that for any pair $(z, g) \in U^{\mathrm{fd}} \times G$ satisfying $g \cdot z \in U^{\mathrm{fd}}$, there is an $h \in G$, $[h] \in G_{U^{\mathrm{fd}}} \Subset G/G_{z_0}$ such that $g \cdot z = h \cdot z$.

LEMMA 8.15
Let $z_0 \in U_r \subset \mathbb{P}W$ be defined in Definition A.13, and let

$$U_{Z^*,r} := U_r \times_{\mathbb{P}^M} Z^*.$$

Then for $0 < r$ *sufficiently small,* $U_{Z^*,r}$ *is a* G_{z_0}*-invariant subset of* finite distance*; that is, Assumption A.14 is satisfied for* $U_{Z^*,r}$*.*

Proof

In order to better illustrate the idea, let us first deal with the case that z_0 is K-stable, and hence $G_{z_0} < \infty$. As we have seen in the proof of Theorem 8.8, there is a *proper* $\mathrm{U}(N+1)$-invariant slice $z_0 \in \Sigma \subset \mathbb{H}^{\chi;N}$ obtained via Tian's embedding. By the continuity of Σ and the transversality of the $\mathfrak{g}_{z_0}^{\perp}$-action on U_0 (see the proof Lemma 3.1), for some $0 < r'' < r' \ll 1$ and $0 < \epsilon \ll 1$ we have

$$B_{Z^*}(z_0, r'') \subset U_{r'} \cap \exp \mathfrak{g}_{z_0,<\epsilon}^{\perp} \cdot \Sigma, \tag{64}$$

where $\mathfrak{g}_{z_0,<\epsilon}^{\perp} := \{\xi \in \mathfrak{g}_{z_0}^{\perp} \mid |\xi| < \epsilon\}$ and $B_{Z^*}(z_0, r'')$ denotes the ball of radius ϵ centered at $z_0 \in Z^*$ with respect to a prefixed continuous metric on Z^*. Moreover, by choosing a small r if necessary, we may assume that $\mathcal{X}_z$ is K-stable for all $z \in B_{Z^*}(z_0, r'')$.

To see the lemma, let $\{s_i\}$ be the local basis of $\pi_*(\mathcal{O}_{\mathbb{P}^N}(1)|_{\mathcal{X}})$ corresponding to the coordinate sections of $\mathbb{P}^N$ such that the induced embedding of $X = \mathcal{X}_{z_0} \subset \mathbb{P}^N$ gives rise to $\mathrm{Hilb}(X)$. Now let us equip the line bundle $\mathcal{O}_{\mathcal{X}}(-rK_{\mathcal{X}/Z^{*,\mathrm{kps}}}) \cong \mathcal{O}_{\mathbb{P}^N}(1)|_{\mathcal{X}}$ with a Hermitian metric which gives rise to the *unique* Kähler–Einstein metric when restricted to each $\mathcal{X}_z$ with $z \in B_{Z^*}(z_0, r'')$, and we can introduce the matrix $A_{\mathrm{KE}}(z)$ as in the proof of Lemma 8.12. Then (64) follows from the continuity of $A_{\mathrm{KE}}(z)$ with respect to $z \in Z^*$ and $A_{\mathrm{KE}}(z_0) = I_{N+1}$ (as $X \subset \mathbb{P}^N$ is Tian's embedding).

As a consequence, for any pair $(z, g) \in B_{Z^*}(z_0, r'') \times G$ satisfying $g \cdot z \in B_{Z^*}(z_0, r'')$, there are $h', h'' \in G$ such that under the quotient map

$$[\cdot] : G \to G/G_{z_0},$$

$[h'], [h''] \in G/G_{z_0}$ are perturbations of $[1] \in G/G_{z_0}$ and $h' \cdot z, h'' \cdot g \cdot z \in \Sigma$. Since both $h \cdot z$ and $h' \cdot g \cdot z$ are the Hilbert points of Tian's embedding of the *same* $\mathbb{Q}$-Fano variety, we know that $u := h'^{-1} \cdot h'' \cdot g \in \mathrm{U}(N+1)$. This implies that $g \cdot z = h \cdot z$ with $h = h''^{-1} \cdot h' \cdot u$ and $[h]$ being *uniformly bounded* (with the bound depending only on $B_{Z^*}(z_0, r'')$ and z_0) in G/G_{z_0}. Since the property describing whether or not z lies in $U_{Z^*,r}$ is independent of the G_{z_0}-translation, we conclude that Assumption A.14 holds for all points in $U_{Z^*,r} \subset G_{z_0} \cdot B_{Z^*}(z_0, r'')$ for some $0 < r < r''$.

For the general case, let us introduce a general divisor $\mathcal{D} \in |-mK_{\mathcal{X}}|$ for a sufficiently divisible m such that

(1) $(\mathcal{X}, \mathcal{D})|_{U_W}$ (where U_W is given in the proof of Theorem 8.8) are a family of $\mathbb{Q}$-Fano varieties;

(2) $\mathcal{D}_z$ is smooth whenever $\mathcal{X}_z$ is for $z \in U_W$.

Then by Theorem 1.2, we can construct a *proper* $\mathrm{U}(N+1)$-invariant slice $\Sigma_{\frac{1-\beta}{m}\mathcal{D}} \subset \mathbb{H}^{\chi;N}$ using Tian's embedding of $\mathcal{X}_z \subset \mathbb{P}^N$ with respect to the *unique* conical Kähler–Einstein metric

$$\mathrm{Ric}\big(\omega_{\mathcal{X}_z}(\beta)\big) = \beta\omega_{\mathcal{X}_z}(\beta) + \frac{1-\beta}{m}[\mathcal{D}_z] \quad \text{on } \mathcal{X}_z$$

for all $z \in U_W$ near z_0. In particular, Theorems 1.2 and 7.2 imply that $\Sigma_{\frac{1-\beta}{m}\mathcal{D}} \to \Sigma$ in the sense that $\forall\epsilon > 0$, $\Sigma_{\frac{1-\beta}{m}\mathcal{D}}$ falls into a ϵ-tubular neighborhood of Σ as $\beta \to 1$. This implies that for $0 < r' \ll 1$ and $z, z' \in B_{Z^*}(z_0, r')$ that are contained in (see (74))

$$\big(G \cdot \mathrm{Hilb}(\mathcal{X}_z)\big) \cap \big(\mathrm{U}(N+1) \cdot \exp\sqrt{-1}\mathfrak{g}^{\perp}_{z_0,<\epsilon}\big) \cdot B_{Z^*}(z_0, r')$$

with $\mathfrak{g}^{\perp}_{z_0,<\epsilon} := \{\xi \in \mathfrak{g}_{z_0} \mid |\xi| < \epsilon\}$, the $\mathrm{U}(N+1)$-orbits for Tian's embedding of $(\mathcal{X}_z, \mathcal{D}_z)$ and $(\mathcal{X}_{z'}, \mathcal{D}_{z'})$ are very close in the sense that they can be translated to each other by an element $h \in \mathrm{U}(N+1) \cdot \exp\sqrt{-1}\mathfrak{g}^{\perp}_{z_0,<\epsilon} \cdot G_{z_0} \subset G$ (i.e., $[h] \in G/G_{z_0}$ is bounded in the sense of (74)). In particular, this allows us to treat these two $\mathrm{U}(N+1)$-orbits as *almost identical*, and we argue in exactly the same way as in the K-stable case. This completes the justification of Assumption A.14 for a neighborhood of $z_0 \in U_{Z^*,r}$ for some sufficiently small $r > 0$. $\square$

Finally, with all the above preparations in hand, we are now ready to finish our main construction of this section.

Proof of Theorem 1.3
By [2, Theorem 1.2, Proposition 3.1], proving our statement boils down to establishing the following. For any $\mathbb{C}$-closed point $[z_0] \in [Z^*/\mathrm{SL}(N+1)]$ there is an *affine* neighborhood $z_0 \in U_W \subset \mathbb{P}W$ determined in (55) such that

(1) the morphism $[U_W/G_{z_0}] \to [Z^*/G]$ is *affine and strongly étale* (i.e., stabilizer-preserving and sending closed point to closed point), and
(2) for any $z \in Z^*$ specializing to z_0 under G-action, the closure of substack $[z]$ inside $[Z^*/G]$, $\overline{\{[z]\}} \subset [Z^*/G]$ admits a good moduli space.

Here we fix $G = \mathrm{SL}(N+1)$ and $G_{z_0} = \mathrm{Aut}(X)$.

We have shown that the morphism is strongly étale by Theorem 8.8. Next we confirm the affineness. Since $Z^* \to [Z^*/\mathrm{SL}(N+1)]$ is faithfully flat, it suffices to show that

$$\phi\colon G \times_{G_{z_0}} U_W \to Z^*$$

is affine. Since ϕ is quasifinite and Z^* is separated, it suffices to choose U_W such that $G \times_{G_{z_0}} U_W$ is affine. Let $U_W \subset Z^* \cap \mathbb{P}(W)$ be a G_{z_0}-invariant affine open set. Then we know that $G \times_{G_{z_0}} U_W$ is *affine* since it is a quotient of the affine scheme $G \times U_W$ by the free action of the reductive group G_{z_0}. Furthermore, we have an isomorphism

$$(G \times_{G_{z_0}} U_W) /\!\!/ G \cong U_W /\!\!/ G_{z_0},$$

where $G \times_{G_{z_0}} U_W$ is the inverse image of the affine neighborhood

$$\pi_W|_{U_W}(z_0) = 0 \in U_W /\!\!/ G_{z_0}$$

with π_W defined in (73) under the GIT quotient by G.

Now we establish the second condition. Since we have already established the *uniqueness* of a minimal orbit contained in $\overline{BO}_{z_0}$ (stated after diagram (58)), all we need is the affineness of $G \cdot \pi_W^{-1}(0)$ as it implies that, for any $z \in Z^*$ satisfying $\overline{G \cdot z} \ni z_0$, the closure of $[z] \in [Z^*/G]$ is a closed substack of $[G \cdot \pi_W^{-1}(0)/G]$, which can be written as $[\mathrm{Spec}(A)/G]$ for some affine scheme $\mathrm{Spec}(A)$; hence $\overline{[z]}$ admits a good moduli space.

To obtain the affineness, one notes that Theorem 8.8 and Corollary 8.14 guarantee Assumption A.11. Additionally, we have already established Assumption A.14 by Lemma 8.15. Thus

$$\phi|_{G \times_{G_{z_0}} U_{Z^*,r}} : G \times_{G_{z_0}} U_{Z^*,r} \to G \cdot U_{Z^*,r} \tag{65}$$

is a finite morphism for $0 < r \ll 1$ by Lemma A.15 in the Appendix. By choosing $0 < r$ even smaller, we may conclude that $\phi|_{G \times_{G_{z_0}} U_r}$ is an *analytic isomorphism*, since $\phi|_{G \cdot z_0}$ is an isomorphism and an immersion near $G \cdot z_0$. Now we restrict ϕ to the fiber over $[z_0] \in [Z^*/G]$. We have a finite morphism

$$G \times_{G_{z_0}} \pi_W^{-1}(0) \longrightarrow G \cdot \pi_W^{-1}(0).$$

Since $G \times_{G_{z_0}} \pi_W^{-1}(0)$ is a fiber of a GIT quotient morphism, we conclude that $G \cdot \pi_W^{-1}(0)$ is affine.

As a consequence, the étale chart $\phi/G : (G \times_{G_{z_0}} U_W) /\!\!/ G \to G \cdot U_W/G$ is actually a *finite* morphism, which implies that $G \cdot U_W/G$ is affine. This gives an *affine* neighborhood of $[z_0] \in \mathcal{KF}_N$, and proves that the algebraic space $\mathcal{KF}_N$ is actually a *scheme*. Finally, to prove the last statement of Theorem 1.3, we observe that Lemma 8.4 implies that the closed points of $\mathcal{KF}_N$ stabilize. However, since $\mathcal{KF}_N$ is seminormal, we indeed know that they are isomorphic (see [26, Chapter I, 7.2]). ☐

Remark 8.16

We want to point out that by shrinking U_W if necessary, the map $\phi : G \times_{G_{z_0}} U_W \to G \cdot U_W$ is actually *strongly étale* in the sense of [35, p. 198]; that is, U_W is Luna's étale slice. To see this, one notes that we have already established in the above that the categorical quotient $(G \cdot U_W)/G$ is in fact a *good quotient* (see also [20, Definition 2.12]); moreover, the map ϕ induces an étale morphism

$$\phi/G : (G \times_{G_{z_0}} U_W)/G \to (G \cdot U_W)/G.$$

So all we need to show is that

$$(\phi, \pi_{G \times_{G_{z_0}} U_W}) : \begin{array}{rcl} G \times_{G_{z_0}} U_W & \xrightarrow{\phi} & G \cdot U_W \times_{(G \cdot U_W)/G} (G \times_{G_{z_0}} U_W)/G \\ (g, w) & \longmapsto & \big(g \cdot w, [g \cdot w]\big) \end{array}$$

is an isomorphism, where $\pi_{G \times_{G_{z_0}} U_W} : G \times_{G_{z_0}} U_W \to (G \times_{G_{z_0}} U_W)/G \cong U_W /\!\!/ G_{z_0}$ is the GIT quotient map. But this follows from the fact that $\phi|_{G \times_{G_{z_0}} U_r}$ in (65) is an analytic *isomorphism* for small r and ϕ is *finite*.

Remark 8.17
Note that we can take the local GIT quotient of a similarly defined Z°_{red} for each $N_r = \chi(X, \mathcal{O}_X(-rK_X)) - 1$. Although we are unable to conclude that those local GIT quotients we constructed in this section will be stabilized for $N \gg 1$, their seminormalizations indeed will be. Another reason why we work over a seminormal base is that the condition of being smoothable does not yield a reasonable moduli functor for schemes; for example, in general, there is no good definition of smoothable varieties over an Artinian ring.

We also remark that in our case it makes no difference whether one works on the Hilbert scheme or the Chow variety, at least after the seminormalization. This is because by our definition of Z° (see Definition 8.2), the closed points correspond to $\mathbb{Q}$-Fano varieties (see Remark 4.3) which in particular are geometrically reduced, hence the Hilbert-to-Chow morphism is a bijection (see [26, Chapter I, Theorem 6.3]) when restricted to Z°, and thus they share the same seminormalization (see [26, Section 3.15]).

Appendix

A.1. *Constructibility of* kst

In this section, we will prove Proposition 7.5 in a more general setting. First, let us recall some basics from [35, Section 2, Chapter 2]. Let G be a reductive group acting on a (quasi)projective variety (Z, L) polarized by a G-linearized very ample line bundle L.

Definition A.1
The *rational flag complex* $\Delta(G)$ is the set of nontrivial 1-PS's λ of G modulo the

equivalence relation $\lambda_1 \sim \lambda_2$ if there are positive integers n_1 and n_2 and a point $\gamma \in P(\lambda_1)$ such that

$$\lambda_2(t^{n_2}) = \gamma^{-1}\lambda_1(t^{n_1})\gamma \quad \text{for all } t \in \mathbb{G}_m,$$

where

$$P(\lambda) := \big\{\gamma \in G \;\big|\; \lim_{t\to 0} \lambda(t)\gamma\lambda(t^{-1}) \text{ exists}\big\} \subset G$$

is the unique *parabolic subgroup* associated to λ. The point of $\Delta(G)$ defined by λ will be denoted by $\Delta(\lambda)$. In particular, for a maximal torus $T \subset G$, $\Delta(T) = \mathrm{Hom}_{\mathbb{Q}}(\mathbb{G}_m, T)$.

Then we have the following.

LEMMA A.2 ([35, Chapter 2, Proposition 2.7])
For any 1-PS $\lambda : \mathbb{G}_m \to G$, let $\mu^L(z,\lambda)$ denote the λ-weight *of $z \in Z$ with respect to the G-linearization of L. Then for any $(\gamma, z) \in G \times Z$, we have*

$$\mu^L(z,\lambda) = \mu^L(\gamma z, \gamma\lambda\gamma^{-1}).$$

Moreover, if $\gamma \in P(\lambda)$, then $\mu^L(z,\lambda) = \mu^L(z, \gamma\lambda\gamma^{-1})$.

The next lemma is a slight extension of [35, Chapter 2, Proposition 2.14] essentially contained in [38, proof of Lemma 2.11], and hence the proof will be omitted.

LEMMA A.3
Let $T \subset G$ be a maximal torus, and let L_i, $i = 1,2$ be two G-linearized ample *line bundles over Z. Then there is a finite set of* linear functionals $l_1^{L_i}, \dots, l_{r_{L_i}}^{L_i}$, $i = 1,2$, *which are rational on* $\mathrm{Hom}_{\mathbb{Q}}(\mathbb{G}_m, T)$ *with the following property:*

$$\forall z \in Z, \quad \exists I(z, L_i) \subset \{1,\dots,r_{L_i}\}, I(z, L_2) \subset \{1,\dots,r_{L_2}\} \tag{66}$$

such that the λ-weight *of $z \in Z$ with respect to the linearization of G on $L_1 \otimes L_2^{-1}$ is given by*

$$\mu^{L_1}(z,\lambda) - \mu^{L_2}(z,\lambda) = \max\big\{l_i^{L_1}(\lambda) \;\big|\; i \in I(z,L_1)\big\} - \max\big\{l_i^{L_2}(\lambda) \;\big|\; i \in I(z,L_2)\big\}$$

for all 1-PS's $\lambda \subset T$. Moreover, the functions

$$\begin{aligned} \psi^{L_1,L_2} : Z &\longrightarrow 2^{\{1,\dots,r_{L_1}\}} \sqcup 2^{\{1,\dots,r_{L_2}\}} \\ z &\longmapsto I(z; L_1, L_2) := I(z, L_1) \sqcup I(z, L_2) \end{aligned}$$

are constructible *in the sense that* $\forall I \in 2^{\{1,\dots,r_{L_1}\}} \sqcup 2^{\{1,\dots,r_{L_2}\}}$*, the set* $\psi^{-1}(I) \subset Z$ *is constructible.*

For any line bundle that can be written as $L_1 \otimes L_2^{-1}$ with L_1 and L_2 both being G-linearized and very ample, we can similarly show that Z can be decomposed into a union of finitely many constructible sets indexed by $2^{\{1,\dots,r_{L_1}\}} \sqcup 2^{\{1,\dots,r_{L_2}\}}$ such that, restricted on each piece,

$$\mu^{L_1 \otimes L_2^{-1}}(z,\lambda) = \mu^{L_1}(z,\lambda) - \mu^{L_2}(z,\lambda)$$

is a rational function on $\mathrm{Hom}_{\mathbb{Q}}(\mathbb{G}_m, T)$.

PROPOSITION A.4

Let G *act on an polarized variety* (Z, L)*. Let* $M_i, i = 1, 2$ *be two* G*-linearized (not necessarily ample) line bundles on* Z*. For* $z \in Z$ *and* $\delta \in \Delta(G)$*, we define*

$$\nu_{1-\beta}^{M_1,M_2}(z,\delta) := \frac{\mu^{M_1}(z,\lambda) - (1-\beta)\mu^{M_2}(z,\lambda)}{|\lambda|} \quad \text{with } \Delta(\lambda) = \delta,$$

and we define

$$\varpi_G^{M_1,M_2}(z) := \sup\bigl\{\beta \in (0,1] \bigm| \inf_{\delta \in \Delta(G)} \nu_{1-\beta'}^{L,M}(z,\delta) \geq 0, \forall \beta' \in [0,\beta)\bigr\}$$

or zero if the set on the right-hand side is an empty set. Suppose that $S \subset Z$ *is a* constructible *set such that* $\varpi_G^{M_1,M_2}|_S > 0$*. Then* $\varpi(M_1, M_2)$ *defines a* $\mathbb{Q}$*-valued constructible function on* S*; that is,* $S = \bigsqcup_i S_i$ *is a union of finite constructible sets with* $\varpi(M_1, M_2)$ *being constant on each* S_i*.*

Proof

We replace $L \to Z$ by its power such that $L_1 := L \otimes M_1$ and $L_2 := L \otimes M_2$ are both ample. Then we fix a maximal torus $T \subset G$ and let $\{l_i^{L_1}\}$ and $\{l_i^{L_2}\}$ be the *rational* linear functionals on $\mathrm{Hom}_{\mathbb{Q}}(\mathbb{G}_m, T)$ associated to $L_i, i = 1, 2$. By Lemma A.3, for any $I \in 2^{\{1,\dots,r_{L_1}\}} \sqcup 2^{\{1,\dots,r_{L_2}\}}$, $S_I^T := \psi^{-1}(I) \cap S$ is a constructible set. Now we define

$$\varpi_T^{M_1,M_2}(z) := \sup\bigl\{\beta \in (0,1] \bigm| \inf_{\delta \in \Delta(T)} \nu_{1-\beta'}^{L,M}(z,\delta) \geq 0, \forall \beta' \in [0,\beta)\bigr\}$$

or zero if the right-hand side is an empty set. In other words, it is the *first time* such that the difference of two *rational* piecewise linear *convex* functions

$$\mu^{L_1}(z,\cdot) - (1-\beta)\mu^{L_2}(z,\cdot) - \beta\mu^{L}(z,\cdot) = \mu^{M_1}(z,\cdot) - (1-\beta)\mu^{M_2}(z,\cdot)$$

vanishes along a ray in $\mathrm{Hom}_{\mathbb{Q}}(\mathbb{G}_m, T)$ or in $\{0,1\}$. Clearly, we have $\beta_I \in \mathbb{Q}$ and they are independent of the choice of L.

Now in order to pass from $\varpi_T^{M_1,M_2}$ to $\varpi_G^{M_1,M_2}$, let us recall Chevalley's lemma in [25, Chapter II, Exercise 3.19] which states that the image of a constructible set under an algebro-geometric morphism is again constructible. By applying it to the group action morphism

$$G \times Z \longrightarrow Z,$$

we obtain that $S_I^G := G \cdot (\psi^{-1}(I) \cap S) \supset S_I^T$ are all constructible $\forall I \in 2^{\{1,\ldots,r_{L_1}\}} \sqcup 2^{\{1,\ldots,r_{L_2}\}}$. Now for any 1-PS λ, there is a $\gamma \in G$ such that $\gamma\lambda\gamma^{-1} \subset T$. By Lemma A.2, we have $\mu^{L_i}(z,\lambda) = \mu^{L_i}(\gamma z, \gamma\lambda\gamma^{-1})$, $i = 1,2$, which implies that

$$\varpi_G^{M_1,M_2}(z) = \min\{\beta_J \mid S_J^T \cap G \cdot z \neq \varnothing \text{ for } J \in 2^{\{1,\ldots,r_{L_1}\}} \sqcup 2^{\{1,\ldots,r_{L_2}\}}\}.$$

To see that it is a constructible function on the constructible set $G \cdot S$, one notes that all possible finite intersections of $\{S_J^G\}_J$ form a stratification of $G \cdot S$ into constructible sets and $\varpi_G^{M_1,M_2}$ is constant on each stratum. $\square$

Now we proceed to apply the above setup to the β-K-stability of $(X,D) \subset \mathbb{P}^N$ with respect to the $\mathrm{SL}(N+1)$ action. Let $N+1 = \dim H^0(X, K_X^{\otimes(-r)})$, and define an open subscheme

$$Z := \left\{ \mathrm{Hilb}(X,D) \;\middle|\; \begin{array}{l} (X,D) \subset \mathbb{P}^N \times \mathbb{P}^N \text{ be a } klt \text{ pair with} \\ \text{Hilbert polynomial } \boldsymbol{\chi} = (\chi, \tilde{\chi}) \text{ satisfying:} \\ D \subset X, D \in |-mK_X| \text{ and } \mathcal{O}_{\mathbb{P}^N}(1)|_X \cong K_X^{-\otimes r} \end{array} \right\} \subset \mathbb{H}^{\boldsymbol{\chi};N}. \tag{67}$$

Let $\lambda_{\mathrm{CM}} \to Z$ (see [22], [21, Definition 2.3], [42, (2.4)]) be the CM-line bundle over Z normalized in such a way that the corresponding weight for any 1-PS of $\mathrm{SL}(N+1)$ is exactly the DF introduced in Definition 2.3, and let

$$\lambda_{\mathrm{Chow}}(\mathcal{X}) := \lambda_{\mathrm{CH}}\big(\mathcal{X}, \mathcal{O}_{\mathcal{X}}(-rK_{\mathcal{X}})\big) \to Z$$
$$\lambda_{\mathrm{Chow}}(\mathcal{D}) := \lambda_{\mathrm{CH}}\big(\mathcal{D}, \mathcal{O}_{\mathcal{X}}(-rK_{\mathcal{X}})|_{\mathcal{D}}\big) \to Z$$

be the Chow line bundles introduced in [21, (3.3)]) for the flat families $\mathcal{X} \to Z$ and $\mathcal{D} \to Z$, respectively.

Proof of Proposition 7.5
Let us introduce (see (2))

$$M_1 := \lambda_{\mathrm{CM}} \qquad \text{and} \qquad M_2 := \big(\lambda_{\mathrm{Chow}}(\mathcal{X})^{\otimes \frac{nm}{(n+1)r}} \otimes \lambda_{\mathrm{Chow}}(\mathcal{D})^{-1}\big)^{\otimes \frac{1}{2mr^n \cdot (-K_X)^n}}.$$

By Theorem 5.2, we know that $(\mathcal{X}_t, \mathcal{D}_t)$ is β-K-stable $\forall t \in C$ and $\beta \in (0, \beta_0]$. After removing a finite number of points from C, we obtain a quasiprojective $0 \in S \subset C$ over which $\pi_* \mathcal{O}_{\mathcal{X}}(-rK_{\mathcal{X}/C})|_S \cong \mathcal{O}_S^{\oplus N+1}$. By fixing a basis of $\pi_* \mathcal{O}_{\mathcal{X}}(-rK_{\mathcal{X}/C})|_S$, we obtain an embedding

$$\iota : \big(\mathcal{X}, \mathcal{D}; \mathcal{O}_{\mathcal{X}}(-rK_{\mathcal{X}/C})\big) \times_C S \longrightarrow \mathbb{P}^N \times \mathbb{P}^N \times S,$$

which in turn induces an embedding $S \subset Z$ with S being constructible and $\varpi^{M_1,M_2}_{\mathrm{SL}(N+1)} \geq \beta_0 > 0$. By applying Proposition A.4 to $S \subset Z$, we obtain that

$$\mathrm{kst}(\mathcal{X}_t, \mathcal{D}_t) = \varpi^{M_1,M_2}_{\mathrm{SL}(N+1)}(t), \quad \forall t \in S$$

is a constructible function. Our proof is completed. □

Remark A.5
The above argument first appeared in Paul [41] and Odaka [38] independently, where both authors observed that one can conclude that the K-polystable locus in S is also constructible.

A.2. *Stabilizer-preserving and finite distance properties*
In this section, we will establish the criteria that guarantee the *stabilizer preserving* condition and ingredients needed to prove the *existence of good moduli* for the closed substack $\overline{\{[z]\}}$ for any $\mathbb{C}$-point $[z] \in [Z^*/\,\mathrm{SL}(N+1)]$.

A.2.1. *Stabilizer preserving*
The following example indicates that stabilizer preserving condition is a condition of *properness* and cannot be deduced from the reductivity of stabilizer alone.

Example A.6 (Richardson's example)
Consider the $\mathrm{SL}(2,\mathbb{C})$-action on

$$\mathrm{Sym}^{\otimes 3}\mathbb{C}^2 = H^0\big(\mathcal{O}_{\mathbb{P}^1}(3)\big) = \mathrm{Span}_{\mathbb{C}}\{X^3, X^2Y, XY^2, Y^3\}$$

induced by the standard action on $\mathbb{C}^2$. Then the stabilizer of $p_0(X,Y) = (X-Y) \times (X+Y)^2$ is trivial, and the stabilizer of $p(X,Y) = (X-Y)(X-\omega Y)(X-\omega^2 Y)$ is given by

$$\begin{bmatrix} \omega & 0 \\ 0 & \omega^{-1} \end{bmatrix} \in \mathrm{SL}(2,\mathbb{C}) \quad \text{with } \omega^3 = 1.$$

Let

$$\alpha(t) = \frac{1}{2}\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} t^2 & 0 \\ 0 & t^{-1} \end{bmatrix}\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \in \mathrm{GL}(2,\mathbb{C})$$
$$= \frac{1}{2}\begin{bmatrix} t^2+1/t & -t^2+1/t \\ -t^2+1/t & t^2+1/t \end{bmatrix} \in \mathrm{GL}(2,\mathbb{C}).$$

Then

$$\alpha(t) : \begin{cases} X - Y \longrightarrow t^2(X-Y) \\ X + Y \longrightarrow t^{-1}(X+Y) \end{cases}$$

fixes $p_0(X,Y) = \frac{3}{4}(X-Y)(X+Y)^2 \in \mathrm{Sym}^{\otimes 3}\mathbb{C}^2$. Now let us define

$$\begin{aligned} p_t(X,Y) &= p\big(\alpha(t)\cdot X, \alpha(t)\cdot Y\big) \\ &= \frac{1}{4}t^2(X-Y)\Big(\big(t^2(1+\omega)+t^{-1}(1-\omega)\big)X \\ &\quad + \big(-t^2(1+\omega)+t^{-1}(1-\omega)\big)Y\Big) \\ &\quad \cdot \Big(\big(t^2(1+\omega^2)+t^{-1}(1-\omega^2)\big)X + \big(-t^2(1+\omega^2)+t^{-1}(1-\omega^2)\big)Y\Big) \\ &= \frac{1}{4}(X-Y)\Big(\big(t^3(1+\omega)+(1-\omega)\big)X + \big(-t^3(1+\omega)+(1-\omega)\big)Y\Big) \\ &\quad \cdot \Big(\big(t^3(1+\omega^2)+(1-\omega^2)\big)X + \big(-t^3(1+\omega^2)+(1-\omega^2)\big)Y\Big). \end{aligned}$$

Then we have

$$\lim_{t\to 0} p_t(X,Y) = \frac{3}{4}(X-Y)(X+Y)^2,$$

and the stabilizer of p_t is the subgroup $\langle \zeta_t := \zeta_{p_t}\rangle \subset \mathrm{SL}(2)$ with

$$\begin{aligned} \zeta_{p_t} &:= \alpha(t^{-1})\begin{bmatrix} \omega & 0 \\ 0 & \omega^{-1} \end{bmatrix}\alpha(t) \\ &= \frac{1}{2}\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} \omega+\omega^{-1} & t^{-3}(\omega-\omega^{-1}) \\ t^3(\omega-\omega^{-1}) & \omega+\omega^{-1} \end{bmatrix}\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \xrightarrow{t\to 0} \infty. \end{aligned}$$

In particular, the family of stabilizers $\langle \zeta_t \rangle \subset \mathrm{SL}(2,\mathbb{C})$ is *unbounded* as $t \to 0$ unless $\omega = 1$.

Our goal here is to find conditions preventing the existence of pathological examples like above. Let us collect some basic facts on compact Lie groups acting on $\mathbb{P}^M$. Although our main application is to the situation in Summary 8.6, we will proceed in a more general fashion as it might be valuable for future applications.

Let K be a compact Lie group, let $\rho : K \to \mathrm{SU}(M+1)$ be a linear representation, and let $\rho^{\mathbb{C}} : G = K^{\mathbb{C}} \to \mathrm{SL}(M+1)$ be its complexification. Let $z_0 \in \mathbb{P}^N$ with stabilizer $G_{z_0} = (K_{z_0})^{\mathbb{C}} := (G_{z_0} \cap K)^{\mathbb{C}}$. Let $\mathfrak{k}_{z_0} = \mathrm{Lie}(K_{z_0})$ be the Lie algebra. Fix a bi-invariant inner product $\langle \cdot, \cdot \rangle_{\mathfrak{k}}$ on $\mathfrak{k}$, and let $\mathfrak{k}_{z_0}^{\perp} \subset \mathfrak{k}$ be its orthogonal complement with respect to $\langle \cdot, \cdot \rangle_{\mathfrak{k}}$. Then the infinitesimal action $\sigma_{z_0} : \mathfrak{g} \longrightarrow T_{z_0}\mathbb{P}^M$ is G_{z_0}-equivariant in the sense that

$$\sigma_{z_0}(\mathrm{Ad}_g\, \xi) = g \cdot \sigma_{z_0}(\xi) \quad \text{for all } g \in G_{z_0},$$

and there is a G_{z_0}-invariant linear subspace $W' \subset \mathbb{C}^{M+1}$ such that

$$\mathbb{C}^{M+1} = W \oplus (\mathfrak{k}_{z_0}^{\perp})^{\mathbb{C}} := W' \oplus \mathbb{C}\hat{z}_0 \oplus (\mathfrak{k}_{z_0}^{\perp})^{\mathbb{C}} \quad \text{with } (\mathfrak{k}_{z_0}^{\perp})^{\mathbb{C}} := \mathfrak{k}_{z_0}^{\perp} \otimes \mathbb{C}$$

is a decomposition as a G_{z_0}-module. Hence we have

$$\mathbb{P}^M = \mathbb{P}\big(W \oplus (\mathfrak{k}_{z_0}^{\perp})^{\mathbb{C}}\big) = \mathbb{P}\big(W' \oplus \mathbb{C}\hat{z}_0 \oplus (\mathfrak{k}_{z_0}^{\perp})^{\mathbb{C}}\big), \tag{68}$$

where $0 \neq \hat{z}_0 \in \mathbb{C}^{M+1}$ is a lift of $z_0 \in \mathbb{P}^M$.

Consider the map

$$\begin{aligned} G \times \mathbb{P}W &\xrightarrow{\quad \phi \quad} G \cdot \mathbb{P}W \subset \mathbb{P}^M, \\ (g, w) &\longmapsto g \cdot w. \end{aligned} \tag{69}$$

Then for $\xi \in \mathfrak{g}_{z_0}$ and $\delta w \in T_{z_0}\mathbb{P}W$, we have

$$d\phi|_{(e,z_0)}(\xi, \delta w) = \sigma_{z_0}(\xi) + \delta w \in T_{z_0}\mathbb{P}^N \cong (\mathfrak{k}_{z_0}^{\perp})^{\mathbb{C}} \oplus T_{z_0}\mathbb{P}W,$$

where $\sigma_{z_0} : \mathfrak{g} = \mathfrak{k}^{\mathbb{C}} \to T_{z_0}\mathbb{P}^M$ denotes the infinitesimal action, $e \in G$ denotes the identity, and as a consequence $\ker d\phi|_{(e,z_0)} = \mathfrak{g}_{z_0}$. Now let us define an open set

$$\begin{aligned} U_0 := \big\{ w \in \mathbb{P}W \;\big|\; \mathrm{rk}\big(q \circ d\phi\big|_{\{1\}\times\mathbb{P}W} : \mathfrak{g} \times T\mathbb{P}W \to (T\mathbb{P}^N|_{\mathbb{P}W})/T\mathbb{P}W\big) = \dim \mathfrak{g}_{z_0}^{\perp} \big\} \\ \subset \mathbb{P}W \end{aligned}$$

with $q : T\mathbb{P}^N|_{\mathbb{P}W} \to (T\mathbb{P}^N|_{\mathbb{P}W})/T\mathbb{P}W$ being the quotient morphism between vector bundles over $\mathbb{P}W$. Then we have the following.

LEMMA A.7
*We have that $U_0 \subset \mathbb{P}W$ is a G_{z_0}-*invariant *Zariski-open set.*

Proof
Note that the Zariski openness follows from the fact that $q \circ d\phi \in H^0(\mathbb{P}W, T(G \times \mathbb{P}W)|_{\{1\}\times\mathbb{P}W}^{\vee} \otimes (T\mathbb{P}^N|_{\mathbb{P}W})/T\mathbb{P}W)$. So all we need is the G_{z_0}-invariance. To achieve this, one notes that for any $g \in G_{z_0}$, $\xi \in \mathfrak{g}$, and $w \in \mathbb{P}W$, we have

$$(g\cdot)_*\sigma_w(\xi) = \sigma_{g\cdot w}(\mathrm{Ad}_g\,\xi),$$

which implies that

$$\sigma_w(\xi) \in T_w\mathbb{P}W \Longleftrightarrow \sigma_{g\cdot w}(\mathrm{Ad}_g\,\xi) \in T_{g\cdot w}\mathbb{P}W.$$

Now $w \in U_0$ can be characterized as $q \circ d\phi$ being of full rank, which is also equivalent to

$$\sigma_w(\xi) \in T_w\mathbb{P}W \Longleftrightarrow \xi \in \mathfrak{g}_{z_0}. \tag{70}$$

If $g \cdot w \notin U_0$, then there is a $0 \neq \mathrm{Ad}_g\,\xi \in \mathfrak{g}_{z_0}^{\perp}$ such that $\sigma_{g\cdot w}(\mathrm{Ad}_g\,\xi) \in T_{g\cdot w}\mathbb{P}W$, and hence $\sigma_w(\xi) \in T_w\mathbb{P}W$. On the other hand, we have decomposition $\mathfrak{g} = \mathfrak{g}_{z_0} \oplus \mathfrak{g}_{z_0}^{\perp}$ as a G_{z_0}*-module* via the adjoint action thanks to the *reductivity* of G_{z_0}. This implies that $0 \neq \xi \in \mathfrak{g}_{z_0}^{\perp}$, contradicting (70) and the assumption that $w \in U_0$. Thus our proof is completed. □

Now ϕ is G_{z_0}*-invariant* with respect to the action $h \cdot (g, w) = (gh^{-1}, h \cdot w)$, and hence it descends to a K-invariant map, which by abuse of notation is still denoted by

$$\begin{aligned} G \times_{G_{z_0}} \mathbb{P}W &\xrightarrow{\ \phi\ } G \cdot \mathbb{P}W \subset \mathbb{P}^M, \\ (g, w) &\longmapsto g \cdot w. \end{aligned} \tag{71}$$

Moreover, it is a *biholomorphism* (see the proof of [44, Theorem 1.12]) from a K-invariant tubular neighborhood

$$\begin{aligned} U_\epsilon &:= \left\{(g \exp \sqrt{-1}\xi, w) \in G \times_{G_{z_0}} V \;\middle|\; g \in K, \xi \in \mathfrak{k}_{<\epsilon}\right\} \\ &\text{with } \mathfrak{k}_{<\epsilon} := \left\{\xi \in \mathfrak{k} \;\middle|\; |\xi| < \epsilon\right\} \end{aligned} \tag{72}$$

of the orbit $K \cdot z_0 \cong K/K_{z_0}$ onto $\phi(U_\epsilon) = K \cdot \exp \mathfrak{k}_{<\epsilon} \cdot V$ for $0 < \epsilon \ll 1$, where $z_0 \in V \subset \mathbb{P}W$ is a K-invariant analytic-open neighborhood.

Now suppose that $\tilde{g} = g \cdot \exp \sqrt{-1}\xi$ satisfies $g \in K$ and $\xi \in \mathfrak{k}$ with $|\xi| < \epsilon$ such that $\tilde{g} \cdot w = w$. Then

$$\phi(g \cdot \exp \sqrt{-1}\xi, w) = \phi(\tilde{g}, w) = \tilde{g} \cdot w = w = \phi(e, w) \qquad \text{and} \qquad (\tilde{g}, w) \in U_\epsilon.$$

This, together with the fact that $\phi|_{U_\epsilon}$ is biholomorphic, implies that

$$(\tilde{g}, w) \overset{G_{z_0}}{\sim} (e, w) \in G \times \mathbb{P}W;$$

that is, there is an $h \in G_{z_0}$ such that $(\tilde{g}h^{-1}, hw) = (e, w)$, hence $\tilde{g} = h \in G_{z_0} \cap G_w$. In conclusion, we obtain the following.

LEMMA A.8 (Local rigidity)
Let $w \in V \subset \mathbb{P}W$ *(defined in (72)), and suppose that* $\tilde{g} \in G_w$ *is of the form* $\tilde{g} = g \cdot \exp \xi$ *with* $g \in K$ *and that* $\xi \in \mathfrak{g}$ *satisfies* $|\xi| < \epsilon$*. Then* $\tilde{g} \in G_{z_0}$.

Assumption A.9 (Properness)
There is a *closed* K-invariant subset

$$\Sigma \hookrightarrow \mathbb{P}^M$$

satisfying:

(1) $\forall z \in \mathbb{P}^M$, $(G \cdot z) \cap \Sigma$ consists of *at most one* K-orbit. Σ is *continuous* in the sense that for any sequence $\{z_i\}_{i=1}^{\infty} \subset \mathbb{P}^M$ satisfying $(G \cdot z_i) \cap \Sigma \neq \varnothing$, $\forall i$ and $\lim_{i \to \infty} z_i = z_\infty \in \Sigma$, we have

$$\lim_{i \to \infty} \operatorname{dist}_{\mathbb{P}^M}((G \cdot z_i) \cap \Sigma, K \cdot z_\infty) = 0.$$

(2) $G_z = (G_z \cap K)^{\mathbb{C}}$ for all $z \in \Sigma$.

THEOREM A.10
Let K *be a compact Lie group acting on* $\mathbb{P}^M$ *via a representation* $K \to \mathrm{U}(M+1)$*, and let* $G = K^{\mathbb{C}}$ *be its complexification. Let* $z_0 \in \mathbb{P}^M$ *with its stabilizer* G_{z_0} *satisfying* $G_{z_0} = (G_{z_0} \cap K)^{\mathbb{C}}$ *and* $z_0 \in \Sigma \subset \mathbb{P}^M$ *satisfying Assumption A.9. Then there is a* G_{z_0}*-invariant* Zariski-open *neighborhood* $z_0 \in U^{\mathrm{sp}} \subset \mathbb{P}W$ *such that for* $\forall w \in U^{\mathrm{sp}} \cap G \cdot \Sigma$*, we have* $G_w < G_{z_0}$.

Proof
We will first prove that our statement holds true for an *analytic* neighborhood, then we can pass from an analytic-open to a Zariski-open neighborhood by the constructibility.

Suppose that Assumption A.9 holds. Then the continuity of the slice Σ implies that there is a sufficiently small analytic K_{z_0}-invariant neighborhood $z_0 \in \tilde{V} \subset V \subset \mathbb{P}W$ such that for any $w \in \tilde{V}$, there is a $\xi \in (\mathfrak{k}_{z_0}^{\perp})^{\mathbb{C}}$ satisfying $|\xi| < \delta < \epsilon$ and $z \in \Sigma$ such that $w = \exp \xi \cdot z$. In particular, $\exp \xi \cdot K_z \cdot \exp(-\xi) \subset G_w$ is a maximal compact subgroup of G_w. Since $K_z < K$ is compact, we have

$$\begin{aligned}\exp \xi \cdot K_z \cdot \exp(-\xi) &= \left\{h \cdot \exp(\mathrm{Ad}_{h^{-1}} \xi) \cdot \exp(-\xi) \;\middle|\; h \in K_z\right\} \\ &\subset \left\{g \cdot \exp \sqrt{-1}\zeta | \zeta \in \mathfrak{g}, |\zeta| < \epsilon \text{ and } g \in K\right\}.\end{aligned}$$

By Lemma A.8, we must have $\exp(-\xi) \cdot K_z \cdot \exp \xi \subset G_{z_0}$. Hence

$$G_{z_0} \supset \left(\exp(-\xi) \cdot K_z \cdot \exp \xi\right)^{\mathbb{C}} = G_w,$$

since G_{z_0} is reductive. Finally, we note that the set

$$\{w \in \mathbb{P}W \mid G_w < G_{z_0}\} \supset G_{z_0} \cdot \tilde{V}$$

is G_{z_0}-invariant and constructible. This allows us to choose a G_{z_0}-invariant Zariski-open subset $U^{\mathrm{sp}} \supset G_{z_0} \cdot \tilde{V}$, and our proof is completed. $\square$

Assumption A.11 (Stabilizer-preserving)
There is a G_{z_0}-invariant Zariski-open neighborhood of $z_0 \in U^{\mathrm{sp}} \subset \mathbb{P}W$ such that $G_w < G_{z_0}$ for all $w \in U^{\mathrm{sp}}$.

Example A.12
Note that Assumption A.11 does not hold in general, even in the situation of a 1-PS $\alpha(t)$ degenerating $\lim_{t\to 0} \alpha(t) \cdot z = z_0$, so we cannot conclude that $G_{z_t} < G_{z_0}$. Consider the SL(2)-action on $\mathbb{P}(\mathrm{Sym}^{\otimes 3} \mathbb{C}^2)$ as in Example A.6. The 1-PS

$$\alpha(t) = \frac{1}{2}\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} t & 0 \\ 0 & t^{-1} \end{bmatrix}\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} t + 1/t & -t + 1/t \\ -t + 1/t & t + 1/t \end{bmatrix} \in \mathrm{SL}(2, \mathbb{C})$$

degenerates $p(X,Y)$ to $p_0(X,Y) \in \mathbb{P}(\mathrm{Sym}^{\otimes 3} \mathbb{C}^2)$. Then $\mathbb{Z}/3\mathbb{Z} \cong \mathrm{SL}(2)_{p_t} \not\subset \mathrm{SL}(2)_{p_0} = \langle \alpha(t) \rangle \cong \mathbb{G}_m$, and the map

$$\mathrm{SL}(2) \times_{\mathbb{G}_m} \mathbb{P}W \longrightarrow \mathrm{SL}(2) \cdot \mathbb{P}W$$

is *not* finite.

A.2.2. Finite distance property
In this subsection, we establish a criteria that guarantees the properness of the map ϕ (defined in (71)) near z_0, which is crucial to prove the existence of a good moduli space of $\overline{\{[z]\}} \subset [Z^*/G]$ for any $[z]$ specializing to $[z_0] \in [Z^*/G]$ in Section 8.

Twisting the linearization of G_{z_0} on $\mathcal{O}_{\mathbb{P}M}(1)|_{\mathbb{P}W}$ by the inverse of the character corresponding to the action $G_{z_0} \curvearrowright \mathcal{O}_{\mathbb{P}M}(1)|_{z_0}$ as in the proof of Lemma 3.1, we obtain that $z_0 \in \mathbb{P}W$ is GIT-polystable with respect to the new G_{z_0}-linearization on $\mathcal{O}_{\mathbb{P}M}(1)|_{\mathbb{P}W}$. Let $U^{\mathrm{ss}} \subset \mathbb{P}W$ denote the GIT-semistable points with respect to this linearization, and let

$$\pi_W : \mathbb{P}W \supset U^{\mathrm{ss}} \longrightarrow \mathcal{M} := \mathbb{P}W /\!\!/ G_{z_0} \quad \text{with } \pi_W(z_0) = 0 \in \mathcal{M} \tag{73}$$

denote the GIT quotient map. Let $0 \in B_{\mathcal{M}}(0,r) \subset \mathcal{M}$ be the *open* ball of radius r with respect to a prefixed continuous metric. Then for each $r > 0$, we introduce the following.

Definition A.13
Let U_r be the *connected component* of

$$\big(G\cdot\pi_W^{-1}\big(B(0,r)\big)\big)\cap\mathbb{P}W\subset U^{\mathrm{ss}}$$

containing z_0. In particular, U_r is G_{z_0}*-invariant.*

Let $[\cdot]:G\to G/G_{z_0}$ denote the quotient map. We say that a sequence $\{h_i\}\subset G$ is *bounded* in G/G_{z_0} if and only if $\{\psi^{-1}([h_i])\}$ is contained in a *bounded* subset of $K\times_{K_{z_0}}(\sqrt{-1}\mathfrak{k}_{z_0}^{\perp})$, where ψ is the Cartan decomposition (see [44, (1.8)])

$$\begin{aligned}\psi: K\times_{K_{z_0}}(\sqrt{-1}\mathfrak{k}_{z_0}^{\perp}) &\longrightarrow G/G_{z_0},\\ (g,\sqrt{-1}\xi) &\longmapsto (g\cdot\exp\sqrt{-1}\xi)\cdot G_{z_0},\end{aligned}\tag{74}$$

which is a K-equivariant *diffeomorphism.*

Assumption A.14 (Finite distance)
An analytic-open neighborhood of $z_0\in U^{\mathrm{fd}}\subset\mathbb{P}W$ is of finite distance if there is a bounded (in the above sense) set $G_{U^{\mathrm{fd}}}\Subset G/G_{z_0}$ depending only on U^{fd} and z_0 such that for any pair $(z,g)\in U^{\mathrm{fd}}\times G$ satisfying $g\cdot z\in U^{\mathrm{fd}}$, there is an $h\in G$, $[h]\in G_{U^{\mathrm{fd}}}\Subset G/G_{z_0}$ such that $g\cdot z=h\cdot z$, where $[\cdot]:G\to G/G_{z_0}$ is the quotient map, and $\Subset$ stands for the compact embedding with respect to the analytic topology. It follows from the definition that U^{fd} is G_{z_0}-invariant.

LEMMA A.15
Suppose that Assumptions A.11 and A.14 are satisfied. Then there is a positive $\epsilon>0$ such that for any $0<r<\epsilon$, U_r (defined in Definition A.13) satisfies the following: for any sequence $\{(g_i,y_i)\}\in G\times_{G_{z_0}}U_r$ satisfying $z_i=g_i\cdot y_i\to z_\infty\in G\cdot U_r$, as $i\to\infty$, after passing to a subsequence, there is a

$$(g_\infty,y_\infty)\in\overline{\big\{(g_i,y_i)\big\}_i}\subset G\times_{G_{z_0}}U_r\quad\textit{such that } g_\infty\cdot y_\infty=z_\infty.$$

In particular, the map $\phi|_{G\times_{G_{z_0}}U_r}:G\times_{G_{z_0}}U_r\to G\cdot U_r$ is a finite morphism.

Proof
First, we note that after translating z_∞ by a $g\in G$ if necessary, we may assume that $z_\infty\in U_r$. Since $\overline{U}_r\subset\mathbb{P}W$ is compact by Definition A.13, by passing to a subsequence if necessary we may and will assume $y_i\xrightarrow{i\to\infty}z_\infty\in U_r$ after a possible shrinking of r.

By Assumption A.14, we may choose $0<r\ll 1$ such that $U_r\subset U^{\mathrm{fd}}$. Then there is a sequence $\{h_i\}\subset G$, with $\{[h_i]\}$ being bounded in G/G_{z_0} and satisfying $g_i\cdot y_i=$

$h_i \cdot y_i$. Hence $h_i^{-1} \cdot g_i \in G_{y_i}$, $\forall i$. Now by Assumption A.11, we have

$$h_i^{-1} \cdot g_i \in G_{y_i} < G_{z_0}, \quad \forall i,$$

from which we conclude that $\{[g_i]\}$ is *bounded* in G/G_{z_0}, and hence the set $\{(g_i, y_i)\} \subset G \times_{G_{z_0}} U_r$ is precompact. Thus the morphism $\phi|_{G \times_{G_{z_0}} U_r} : G \times_{G_{z_0}} U_r \to G \cdot U_r$ is a proper and étale morphism and hence finite. □

Remark A.16
Assumption A.14 is introduced to guarantee that the multiplication morphism

$$\phi|_{G \times_{G_{z_0}} U_r} : G \times_{G_{z_0}} U_r \to G \cdot U_r$$

is *proper*. For that purpose, we want to make sure that for $0 < r \ll 1$, any point $z \in U_r$, and any infinite sequence $\{g_i\}_{i=1}^{\infty} \subset G$ satisfying $G/G_{z_0} \ni [g_i] \to \infty$ (with respect to the analytic topology), there is no infinite recurrence of points $g_i \cdot z$ inside U_r. As we have seen in the proof of Lemma 8.15, the *properness* of slice Σ obtained via Tian's embedding guarantees Assumption A.14.

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